On the Equivalence Between Liveness and Deadlock-Freeness in Petri Nets

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Abstract. This paper deals with the structure theory of Petri nets. We define the class of P/T systems namely K-systems for which the equivalence between controlled-siphon property (cs property), deadlock freeness, and liveness holds. Using the new structural notions of ordered transitions and root places, we revisit the non liveness characterization of P/T systems satisfying the cs property and we define by syntactical manner new and more expressive subclasses of K-systems where the interplay between conflict and synchronization is relaxed.

Keywords: structure theory, liveness, deadlock-freeness, cs-property.

1 Introduction

Place / Transition (P/T) systems are a mathematical tool well suited for the modelling and analyzing systems exhibiting behaviours such as concurrency, conflict and causal dependency among events. The use of structural methods for the analysis of such systems presents two major advantages with respect to other approaches: the state explosion problem inherent to concurrent systems is avoided, and the investigation of the relationship between the behaviour and the structure (the graph theoretic and linear algebraic objects and properties associated with the net and initial marking) usually leads to a deep understanding of the system. Here we deal with liveness of a marking, i.e. , the fact that every transition can be enabled again and again. It is well known that this behavioural property is as important as formally hard to treat. Although some structural techniques can be applied to general nets, the most satisfactory results are obtained when the interplay between conflicts and synchronization is limited. An important theoretical result is the controlled siphon property[3]. Indeed this property is a condition which is necessary for liveness and sufficient for deadlock-freeness. The aim of this work is to define and recognize structurally a class of P/T systems, as large as possible, for which the equivalence between liveness and deadlock freeness holds. In order to reach such a goal, a deeper understanding of the causes of the non equivalence between liveness and deadlock-freeness is required.

This paper is organized as follows. In section 2, we recall the basic concepts and notations of P/T systems. In section 3, we first define a class of P/T systems, namely

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K-systems, for which the equivalence between controlled-siphon property (cs property), deadlock freeness, and liveness holds. In section 4, we revisit the structural conditions for the non liveness under the cs property hypothesis. In section 5, we define by a syntactical manner several new subclasses of K-systems where the interplay between conflict and synchronization is relaxed. Such subclasses are characterized using the new structural notions of ordered transitions and root places. In section 6, we define two other subclasses of K-systems based on T-invariants. We conclude in section 5 with a summary of our results and a discussion of an open question.

2 Basic Definitions and Notations

This section contains the basic definitions and notations of Petri nets' theory [11] which will be needed in the rest of the paper.

2.1 Place/Transition Nets

Definition 1. *A P/T net is a weighted bipartite digraph* $N = \langle P, T, F, V \rangle$ *where:*

- $P \neq \emptyset$ *is a finite set of node* places*;*
- $T \neq \emptyset$ *is a finite set of node* transitions*;*
- $F \subseteq (P \times T) \cup (T \times P)$ *is the* flow *relation;*
- $V : F \rightarrow \mathbb{N}^+$ *is the* weight function *(valuation).*

Definition 2. *Let* $N = \langle P, T, F, V \rangle$ *be a P/T net.*
The preset *of a node* $x \in (P \cup T)$ *is defined as* ${}^\bullet x = \{y \in (P \cup T) s.t. (y, x) \in F\}$,
The postset *of a node* $x \in (P \cup T)$ *is defined as* $x^\bullet = \{y \in (P \cup T) s.t. (x, y) \in F\}$,
The preset (resp. postset) of a set of nodes is the union of the preset (resp. postset) of its elements.
The sub-net *induced by a sub-set of places* $P' \subseteq P$ *is the net* $N' = \langle P', T', F', V' \rangle$ *defined as follows:*

- $T' = {}^\bullet P' \cup P'^\bullet$,
- $F' = F \cap ((P' \times T') \cup (T' \times P'))$,
- V *is the restriction of* V *on* F'.

The sub-net induced by a sub-set of transitions $T' \subseteq T$ *is defined analogously.*

Definition 3. *Let* $N = \langle P, T, F, V \rangle$ *be a P/T net.*

- *A shared place* p $(\mid p^\bullet \mid \geq 2)$ *is said to be* homogenous *iff:* $\forall t,\, t' \in p^\bullet$, $V(p, t) = V(p, t')$.
- *A place* $p \in P$ *is said to be* non-blocking *iff:* $p^\bullet \neq \emptyset \Rightarrow Min_{t \in {}^\bullet p}\{V(t, p)\} \geq Min_{t \in p^\bullet}\{V(p, t)\}$.
- *If all shared places of* P *are homogenous, then the valuation* V *is said to be homogenous.*

The valuation V *of a P/T net* N *can be extended to the application* W *from* $(P \times T) \cup (T \times P) \rightarrow \mathbb{N}$ *defined by:*
$\forall u \in (P \times T) \cup (T \times P)$, $W(u) = V(u)$ *if* $u \in F$ *and* $W(u) = 0$ *otherwise.*

Definition 4. *The matrix* C *indexed by* $P \times T$ *and defined by* $C(p,t) = W(t,p) - W(p,t)$ *is called the incidence matrix of the net.*
An integer vector $f \neq 0$ *indexed by* P ($f \in \mathbf{Z}^P$) *is a* P-invariant *iff* $f^t.C = 0^t$.
An integer vector $g \neq 0$ *indexed by* T ($g \in \mathbf{Z}^T$) *is a* T-invariant *iff* $C.g = 0$.
$\|f\| = \{p \in P / f(p) \neq 0\}$ *(resp.* $\|g\| = \{t \in t / g(t) \neq 0\}$*) is called the support of* f *(resp. of* g*).*
We denote by $\|f\|^+ = \{p \in P / f(p) > 0\}$ *and by* $\|f\|^- = \{p \in P / f(p) < 0\}$.
N *is said to be conservative iff there exists a* P-invariant f *such that* $\|f\| = \|f\|^+ = P$.

2.2 Place/Transition Systems

Definition 5. *A* marking M *of a P/T net* $N = \langle P, T, F, V \rangle$ *is a mapping* $M : P \rightarrow \mathbb{N}$ *where* $M(p)$ *denotes the number of tokens contained in place* p.
The pair $\langle N, M_0 \rangle$ *is called a P/T system with* M_0 *as initial marking.*
A transition $t \in T$ *is said to be* enabled *under* M, *in symbols* $M \xrightarrow{t}$, *iff* $\forall p \in {}^\bullet t$: $M(p) \geq V(p,t)$. *If* $M \xrightarrow{t}$, *the transition* t *may* occur, *resulting in a new marking* M', *in symbols* $M \xrightarrow{t} M'$, *with:* $M'(p) = M(p) - W(p,t) + W(t,p)$, $\forall p \in P$.
The set of all reachable markings, in symbols $R(M_0)$, *is the smallest set such that* $M_0 \in R(M_0)$ *and* $\forall M \in R(M_0)$, $t \in T$, $M \xrightarrow{t} M' \Rightarrow M' \in R(M_0)$.
If $M_0 \xrightarrow{t_1} M_1 \xrightarrow{t_2} \ldots M_{n-1} \xrightarrow{t_n}$, *then* $\sigma = t_1 t_2 \ldots t_n$ *is called an* occurrence sequence.

In the following, we recall the definition of some basic behavioural properties.

Definition 6. *Let* $\langle N, M_0 \rangle$ *be a P/T system.*
A transition $t \in T$ *is said to be dead for a marking* $M \in R(M_0)$ *iff* $\nexists\, M^* \in R(M)$ *s.t.* $M^* \xrightarrow{t}$.
A marking $M \in R(M_0)$ *is said to be a dead marking iff* $\forall t \in T$, t *is dead for* M.
$\langle N, M_0 \rangle$ *is* weakly *live (or deadlock-free) for* M_0 *iff* $\forall M \in R(M_0)$, $\exists t \in T$ *such that* $M \xrightarrow{t}$ *(*$\langle N, M_0 \rangle$ *has no dead marking).*
A transition $t \in T$ *is said to be* live *for* M_0 *iff* $\forall M \in R(M_0)$, $\exists M' \in R(M)$ *such that* $M' \xrightarrow{t}$ *(*t *is not live iff* $\exists M' \in R(M_0)$ *for which* t *is dead).*
$\langle N, M_0 \rangle$ *is* live *for* M_0 *iff* $\forall t \in T$, t *is live for* M_0.
A place $p \in P$ *is said to be marked for* $M \in R(M_0)$ *iff* $M(p) \geq Min_{t \in p^\bullet}\{V(p,t)\}$.
A place $p \in P$ *is said to be* bounded *for* M_0 *iff* $\exists k \in \mathbb{N}$ *s.t.* $\forall M \in R(M_0)$, $M(p) \leq k$. $\langle N, M_0 \rangle$ *is* bounded *iff* $\forall p \in P$, p *is bounded for* M_0.
If N *is conservative then* $\langle N, M_0 \rangle$ *is bounded for any initial marking* M_0.

2.3 Controlled Siphon Property

A key concept of structure theory is the siphon.

Definition 7. *Let* $\langle N, M_0 \rangle$ *be a P/T system.*
A nonempty set $S \subseteq P$ *is called a* siphon *iff* ${}^\bullet S \subseteq S^\bullet$. *Let* S *be a siphon,* S *is called* minimal *iff it contains no other siphon as a proper subset.*

In the following, we assume that all P/T nets have *homogeneous valuation*, and $V(p)$ denotes $V(p,t)$ for a any $t \in p^\bullet$.

Definition 8. *A siphon S of a P/T system $N = \langle P, T, F, V \rangle$ is said to be* controlled *iff:*
S is marked at any reachable marking i.e. $\forall M \in R(M_0)$, $\exists p \in S$ s.t. p is marked.

Definition 9. *A P/T system $\langle N, M_0 \rangle$ is said to be satisfying the* controlled-siphon property *(cs-property) iff each minimal siphon of $\langle N, M_0 \rangle$ is controlled.*

In order to check the cs-property, two main structural conditions (*sufficient but not necessary*) permitting to determine whether a given siphon is controlled are developed in [3, 9]. These conditions are recalled below.

Proposition 1. *Let $\langle N, M_0 \rangle$ be a P/T system and S a siphon of $\langle N, M_0 \rangle$. If one of the two following conditions holds, then S is controlled:*

1 $\exists R \subseteq S$ such that $R^\bullet \subseteq {}^\bullet R$, R is marked at M_0 and places of R are non-blocking (siphon S is said to be containing a trap R).
2 $\exists$ a P-invariant $f \in \mathbf{Z}^P$ such that $S \subseteq \|f\|$ and $\forall p \in (\|f\|^- \cap S)$, $V(p) = 1$, $\|f\|^+ \subseteq S$ and $\sum_{p \in P}[f(p).M_0(p)] > \sum_{p \in S}[f(p).(V(p) - 1)]$.

A siphon controlled by the first (resp. second) mechanism is said to be trap-controlled (resp. invariant controlled).
Now, we recall two well-known basic relations between liveness and the cs-property [3]. The first states that the cs-property is a sufficient deadlock-freeness condition, the second states that the cs-property is a necessary liveness condition.

Proposition 2. *Let $\langle N, M_0 \rangle$ be a P/T system. The following property holds: $\langle N, M_0 \rangle$ satisfies the cs-property $\Rightarrow$ $\langle N, M_0 \rangle$ is weakly live (deadlock-free).*

Proposition 3. *Let $\langle N, M_0 \rangle$ be a P/T system. The following property holds: $\langle N, M_0 \rangle$ is live $\Rightarrow$ $\langle N, M_0 \rangle$ satisfies the cs-property.*

Hence, for P/T systems where the cs-property is a sufficient liveness condition, there is an equivalence between liveness and deadlock freeness. In the following section, we define such systems and propose basic notions helping for their recognition.

3 K-Systems

In this section, we first introduce a new class of P/T systems, namely *K-systems*, for which the equivalence between liveness and deadlock freeness holds. Before, let us establish some new concepts and properties related to the causality relationship among dead transitions.

Definition 10. *Let $\langle N, M_0 \rangle$ be a P/T system. A reachable marking $M^* \in R(M_0)$ is said to be stable iff $\forall t \in T$, t is either live ore dead for M^*. Hence, T is is partitioned into two subsets $T_D(M^*)$ and $T_L(M^*)$, and for which all transitions of $T_L(M^*)$ are live and all transitions of $T_D(M^*)$ are dead.*

Proposition 4. *Let $\langle N, M_0 \rangle$ be a weakly live but not live P/T system. There exists a reachable stable marking M^* for which $T_D \neq \emptyset$ and $T_L \neq \emptyset$.*

Proof. trivial, otherwise the net is live ($T = T_L$) or not weakly live ($T = T_D$).

Remark: This partition is not necessarily unique but there exists at least one. It is important to note that T_D is maximal in the sense that all transitions that do not belong to T_D, will never become dead.

Definition 11. *Let $N = \langle P, T, F, V \rangle$ be a P/T net, $r \in P$, $t \in r^\bullet$. r is said to a be a root place for t iff $r^\bullet \subseteq p^\bullet$, $\forall p \in {}^\bullet t$.*

An important feature of root places is highlighted in the following proposition.

Proposition 5. *Let $N = \langle P, T, F, V \rangle$ be a P/T net, $r \in P$, $t \in r^\bullet$. If r is a root place for t then $\forall t' \in r^\bullet$, ${}^\bullet t \subseteq {}^\bullet t'$.*

Proof. Let t be a transition having r as a root place and let t' be a transition in $r^\bullet$. Now, let p be a place in ${}^\bullet t$ and let as show that $p \in {}^\bullet t'$:
Since r is a root place for t and $p \in {}^\bullet t$ then we have $r^\bullet \subseteq p^\bullet$ and hence $t' \in r^\bullet$ implies that $t' \in p^\bullet$, equivalently $p \in {}^\bullet t'$.

Given a transition t, $Root(t)_N$ denotes the set of its root places in N. When the net is clear from the context, this set is simply denoted by $Root(t)$.

Definition 12. *Let t be a transition of T. If $Root(t) \neq \emptyset$, t is said to be an ordered transition iff $\forall p, q \in {}^\bullet t, p^\bullet \subseteq q^\bullet$ or $q^\bullet \subseteq p^\bullet$.*

Remark: An ordered transition has necessarily a root but one transition admitting a root is not necessarily ordered. P/T Systems where all transitions are ordered are called ordered systems. Consider the Figure 1, one can check that $Root(t_1) = \{a\}$, $Root(t_2) = \{b\}$, $Root(t_3) = \{e\}$ and $Root(t_4) = \{d\}$. Transitions t_1, t_3, t_4 are ordered but not t_2.

Proposition 6. *Let $\langle N, M_0 \rangle$ be a not live P/T system. Let r be a root of a transition t: $t \in T_D \Rightarrow r^\bullet \cap T_L = \emptyset$ (i.e. $r^\bullet \subseteq T_D$).*

Proof. As ${}^\bullet t \subset {}^\bullet t'$ for every t' of $r^\bullet$: t, dead for M, can never be enabled, a fortiori t' can not be enabled.

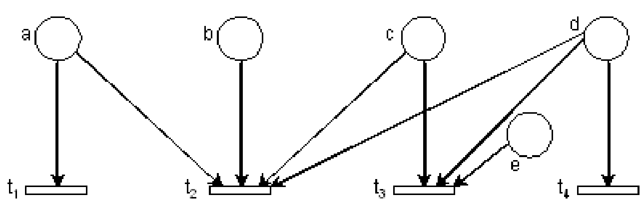

Fig. 1. Illustration: a not ordered transition

Also, we can state the following : if all input transitions of a place are dead, then all its output transitions are dead.

Proposition 7. *Let $\langle N, M_0\rangle$ be a not live P/T system. Let p be a place of P: $^\bullet p \cap T_L = \emptyset \Rightarrow p^\bullet \cap T_L = \emptyset$.*

Proof. Suppose that the proposition is not true. In this case, there exists a place p with all input transitions in T_D ($^\bullet p \cap T_L = \emptyset$) and at least one output transition t_v in T_L ($p^\bullet \cap T_D \neq \emptyset$). Since t_v is live, after a finite number of firings, place p becomes non marked because all its input transitions are dead. So t_v becomes dead. This contradicts that $t \in T_L$ (and maximality of T_D).

Proposition 8. *Let $\langle N, M_0\rangle$ be a not live P/T system. Let p be a bounded place of P: $p^\bullet \cap T_L = \emptyset \Rightarrow {}^\bullet p \cap T_L = \emptyset$.*

Definition 13. *Let $\langle N, M_0\rangle$ be a P/T system. $\langle N, M_0\rangle$ is a K-system iff for all stable markings M^*, $T_D(M^*) = T$ or $T_L = T$. The above property is called the K-property.*

Remark:
According to the previous definition, one can say that the K-systems contain all the live systems and a subclass of not deadlock-free systems. One can then deduce the following theorem.

Theorem 1. *Let $\langle N, M_0\rangle$ be a P/T system. $\langle N, M_0\rangle$ is a K-system. Then the three following assertions are equivalent:*

- *(1) $\langle N, M_0\rangle$ is deadlock free,*
- *(2) $\langle N, M_0\rangle$ satisfies the cs-property,*
- *(3) $\langle N, M_0\rangle$ is live.*

Proof. $\Rightarrow$ Note first that we immediately have (3) $\Rightarrow$ (2) $\Rightarrow$ (1) using proposition 2 and 3. The proof is then reduced to show that deadlock freeness is a sufficient liveness condition for K-systems. Assume that the K-system $\langle N, M_0\rangle$ is not live then by definition it is not deadlock free (since $T_D(M^*) = T$ for each stable marking M^*).
$\Leftarrow$ The converse consists to prove the following implication:
$((1) \Rightarrow (3)) \Rightarrow \langle N, M_0\rangle$ is a K-system
Assume that $\langle N, M_0\rangle$ is not a K-system. Then, by definition, there exists a stable marking $m*$ for which $T_D \neq \emptyset$ and $T_L \neq \emptyset$. Hence, $\langle N, M_0\rangle$ is deadlock free but not live, which contradicts $((1) \Rightarrow (3))$.

The Definition13 of K-systems is a behavioural one. In the following part of this paper, we deal with the problem of recognizing, in a structural manner, the membership of a given P/T system in the class of K-systems.

4 Structural Non-liveness Characterization

In this section, we highlight some intrinsical properties of systems satisfying the cs-property but not live. Our idea is to characterize a "topological construct" making possible the simultaneous existence of dead and live transitions for such systems.

Lemma 1. *Let $\langle N, M_0 \rangle$ be a P/T system satisfying the cs-property but not live. Let M^* be a reachable stable marking. There exists $t^* \in T_D$ such that:*
$\forall p \in {}^\bullet t^*$ such that ${}^\bullet p \cap T_L = \emptyset$, $M(p) = M^*(p) \geq V(p, t^*)$ $\forall M \in R(M^*)$.

Proof. Suppose that $\forall t \in T_D$, there exists $p_t \in {}^\bullet t$ with ${}^\bullet p \cap T_L = \emptyset$ and $M^*(p_t) < V(p_t, t^*)$. Let $S = \{p_t,\ t \in T_D\}$. By construction, ${}^\bullet S \subseteq T_D$ and $T_D \subseteq S^\bullet$ (for all $p_t \in S$, ${}^\bullet p_t \cap T_L = \emptyset$). So S is a siphon. Since $\forall p_t \in S$, $M^*(p_t) < V(p_t, t)$, S is non marked for M^* ($M^* \in R(M^*)$) and hence the cs-property hypothesis is denied. Using now the proposition 7, (if a place p has no live input transition then all output transitions of p are dead), one can deduce that the marking of such places does not change for all reachable markings from M^*.

Theorem 2. *Let $\langle N, M_0 \rangle$ be a P/T system satisfying the cs-property but not live. Let M^* be a reachable stable marking.*
There exists a non ordered transition $t^* \in T_D$ and $\forall M \in R(M^*)$, $\exists p \in {}^\bullet t^*$ s.t. $M(p) < V(p, t^*)$.

Proof. Let t^* be a transition satisfying the previous lemma 1. Let us denote by $L_P(t^*)$ the subset of shared places included in ${}^\bullet t^*$ and defined as follows:
$L_P(t^*) = \{p \in {}^\bullet(t^*) \text{ s.t. } {}^\bullet p \cap T_L \neq \emptyset \text{ and } p^\bullet \cap T_L \neq \emptyset\}$.
We first prove that $L_P(t^*) \neq \emptyset$ ($L_p(t^*) \subseteq {}^\bullet t^*$). Suppose that $L_P = \emptyset$: any input place of t^* having a live input transition (there exists at least one otherwise t^* will be enabled at M^* using proposition 7). As the other input places of t^* are such that their pre-conditions on t^* are satisfied at M^* and remain satisfied (proposition 7), we can reach a marking M from M^* such that $t*$ would be enabled at M. This contradicts that t^* is dead for M^*. Moreover, t^* is not ordered otherwise $L_P(t^*) = \{p_1, \ldots, p_m\}$ ($\mid L_P \mid = m$) can be linearly ordered. Without loss of generality we may assume that ${p_1}^\bullet \subseteq \cdots \subseteq {p_m}^\bullet$. Then there exists a marking M' reachable from M^* for which a transition $t \in {p_1}^\bullet \cap T_L$ and t^* are enabled (homogenous valuation). This contradicts that t^* is dead for M^*. Since $L_P(t^*)$ ($\subset {}^\bullet t^*$) has no root place we deduce that t^* is not ordered.
Finally, $\forall M \in R(M^*)$, $L_P(t^*)$ contains a non marked place otherwise t^* would be not dead for M^*.

From the previous theorem (theorem 2) one can derive easily the following result.

Theorem 3. *Let $\langle N, M_0 \rangle$ be an ordered P/T system. The two following statements are equivalent:*

- *(1) $\langle N, M_0 \rangle$ satisfies the cs-property,*
- *(2) $\langle N, M_0 \rangle$ is live.*

This last result permits us to highlight the structural and behavioural unity between subclasses of ordered P/T systems i.e. (not necessarily bounded) asymmetric choice systems [3] (AC), Join Free (JF) systems, Equal Conflict (EC) systems[13], and Extended Free Choice (EFC) nets. Let us recall that, for theses subclasses, except AC nets, the cs-property is reduced to the well-known Commoner's property [2], [1], [5], [6] and the liveness monotonicity [3]) holds.

In the following, we show how to exploit this material in order to recognize structurally other subclasses of K-systems, with non ordered transition, for which the equivalence between deadlock-freeness and liveness hold. Such structural extensions are based on the two following concepts: the notion of root places as a relaxation of the strong property of ordered transitions and the covering of non ordered transitions by invariants.

5 Dead-Closed Systems

From our better understanding of requirements which are at the heart of non equivalence between deadlock-freeness and liveness, we shall define new subclasses of K-systems for which membership problem is always reduced to examining the net without requiring any exploration of the behaviour.
Let t be a transition of a P/T system, we denote by $D(t)$ the set of transitions defined as follows: $D(t) = \{t' \in T \text{ s.t. } t \in T_D \Rightarrow t' \in T_D\}$
This set is called the *dead closure* of the transition t. In fact, $D(t)$ contains all transitions that are dead once t is assumed to be dead.
In the following, we show how one can compute structurally a subset $D_{Sub}(t)$ of $D(t)$ for any transition t.
Given a transition t_0, we set $D_{Sub}(t_0) = \{t_0\}$ and enlarge it using the three following structural rules related to propositions 6, 7, and 8 respectively:

R_1. Let p be a root place of t, $t \in D_{Sub}(t_0) \Rightarrow p^\bullet \subseteq D_{Sub}(t_0)$
R_2. Let p be a place of P, ${}^\bullet p \subseteq D_{Sub}(t_0) \Rightarrow p^\bullet \subseteq D_{Sub}(t_0)$
R_3. Let p be a bounded place of P, $p^\bullet \subseteq D_{Sub}(t_0) \Rightarrow {}^\bullet p \subseteq D_{Sub}(t_0)$.

Formally, $D_{Sub}(t_0)$ is defined as the smallest subset of T containing t_0 and fulfilling rules R_i $(i = 1 \ldots 3)$. When the computed subsets $D_{Sub}(t)$ are all equal to T, we deduce that the system is a K-system.

Definition 14. *Let $\langle N, M_0\rangle$ be a P/T system. $\langle N, M_0\rangle$ is said to be a dead-closed system if for every transition t of N: $D_{Sub}(t) = T$.*

The algorithm5 [4] computes the subset $D_{Sub}(t)$ for a given transition t. Its complexity is similar to classical graph traversal algorithms. An overall worst-case complexity bound is $\emptyset(\mid P \mid \times \mid T \mid)$.

Theorem 4. *Let $\langle N, M_0\rangle$ be a dead-closed system. Then $\langle N, M_0\rangle$ is a K-system.*

Proof. The proof is obvious since the computed set $D_{Sub}(t)$ for every transition t is a subset of $D(t)$.

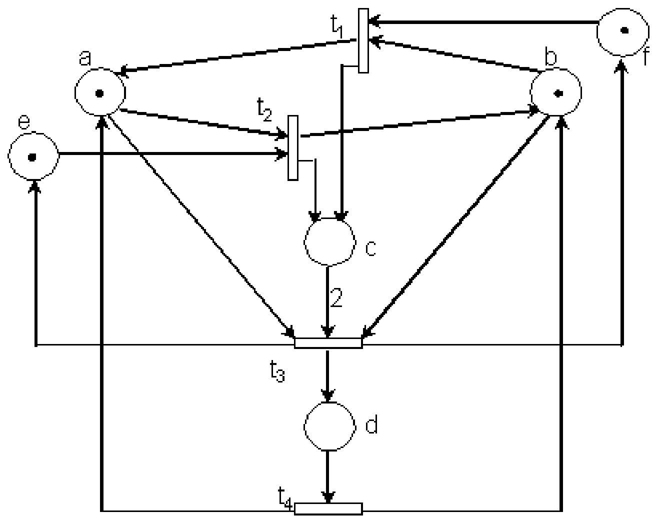

Fig. 2. An example of K system

Using theorem 1 one can deduce the following result.

Corollary 1. *Let $\langle N, M_0 \rangle$ be a dead-closed system. The three following statements are equivalent:*

- *(1) $\langle N, M_0 \rangle$ is deadlock free,*
- *(2) $\langle N, M_0 \rangle$ satisfies the cs-property,*
- *(3) $\langle N, M_0 \rangle$ is live.*

Consider the net of Figure2, note first that it is a conservative net. One can check, by applying the algorithm *computing D(t)*, that it is a dead-closed system. It contains the four following minimal siphons: $S_1 = \{a, b, d\}$, $S_2 = \{e, c, f\}$, $S_3 = \{e, b, d\}$ and $S_4 = \{a, f, d\}$. For any initial marking (e.g. $M_0 = a+b+e+f$) satisfying the four following conditions: $a+b+d > 0$, $e+c+f > 0$, $e+b+d-f > 0$ and $a + f + d - e > 0$, this net satisfies the cs-property and hence is live.

5.1 Root Systems

Here, we define a subclass of dead-closed systems called *Root Systems* exploiting in particular the causality relationships among output transitions of root places. Before, we define a class of P/T nets where each transition admits a root place, such nets are called *Root nets.*

Definition 15. *Let $N = \langle P, T, F, V \rangle$ be a P/T net. N is a root net iff $\forall t \in T$, $\exists$ a place $r \in P$ which is a root for t.*

Every transition t of N has (at least) a root place, but it is not necessarily ordered. Thus, ordered nets are strictly included in root nets.

Algorithm 5.1 Computing $D_{Sub}(t)$

Input: a transition t; // t is assumed to be dead
Output: $D_{Sub}(t)$, a set of transitions; // $D(t)$
Variable $D_t marked$: a set of transitions//
Begin
$D_{Sub}(t) \leftarrow \{t\}$;
$D_t marked \leftarrow \emptyset$
for $(D_t marked \leftarrow \emptyset;\ (D_{Sub}(t) \setminus D_t marked) \neq \emptyset; D_t marked \leftarrow D_t marked \cup \{t\})$ **do**
    get t from $D_{Sub}(t) \setminus D_t marked$;
    if r is root place **then**
        $D_{Sub}(t) \leftarrow D_{Sub}(t) \cup r^{\bullet}$; //application of R_1
        for each $(p \in t^{\bullet})$ **do**
            if $({}^{\bullet}p \subseteq D_{Sub}(t))$ **then**
                $D_{Sub}(t) \leftarrow D_{Sub}(t) \cup p^{\bullet}$; //application of R_2
            end if
        end for
        for for each $(p \in {}^{\bullet}t)$ **such that** (p is bounded) **do**
            if $p^{\bullet} \subseteq D_{Sub}(t)$ **then**
                $D_{Sub}(t) \leftarrow D_{Sub}(t) \cup {}^{\bullet}p$;// application of R_3
            end if
        end for
    end if
end for
End

The class of *Root nets* is extremely large, we have to add some structural constraints in order to recognize structurally their membership in the class of dead-closed systems.

Given a root net N, we first define a particular subnet called *Root component* based on the set of the root places of N. The *Root component* is slightly different from the subnet induced by the root places: It contains all root places and adjacent transitions. But, a (root) place p admits an output transition t in the *root subnet* if and only if p is a root place for t.

Definition 16. *Let $N = \langle P, T, F, V \rangle$ be a root net and $Root_N$ be the set of its root places.*
The Root component *of N is the net $N'^{*} = \langle Root_N, T^*, F^*, V^* \rangle$ defined as follows:*

- $T^* = {Root_N}^{\bullet} = T$,
- $F^* \subseteq (F \cap ((Root_N \times T^*) \cup (T^* \times Root_N)))$, *s.t.* $(p, t) \in F^*$ *iff* $(p, t) \in F$ *and p is a root place for t, and* $(t, p) \in F^*$ *iff* $(t, p) \in F$
- *V' is the restriction of V on F^*.*

Definition 17. *Let $\langle N, M_0 \rangle$ be P/T system.*
$\langle N, M_0 \rangle$ is called a Root System iff N is a root net and its root component N^* is conservative and strongly connected.

Theorem 5. *Let $\langle N, M_0 \rangle$ be a Root-system.*
$\langle N, M_0 \rangle$ is a dead-closed system.

Proof. Note first that the subnet N^* contains all transitions of N (N is weakly ordered). Let us show that $D(t) = T$ for all transition $t \in T$ in N^*. Let t and t' be two transitions and suppose that t is dead. Since N^* is strongly connected, there exists a path $\mathcal{P}_{t' \to t} = t' r_1 t_1 \ldots t_n r_n t$ leading from t' to t s.t. all the places r_i ($i \in \{1 \ldots\}$) are root places. Let us reason by recurrence on the length $\mid \mathcal{P}_{t' \to t} \mid$ of $\mathcal{P}_{t' \to t}$.

- $\mid \mathcal{P}_{t' \to t} \mid = 1$: Obvious
- Suppose that the proposition is true for each path $\mathcal{P}_{t' \to t}$ with $\mid \mathcal{P}_{t' \to t} \mid = n$.
- Let $\mathcal{P}_{t' \to t}$ be a $n+1$-length path leading from t' to t.
  Using proposition 6 (or rule R_1), one can deduce that all output transitions of r_n are dead. Now, since r is a bounded place, we use proposition 8 (or R_3) to deduce that all its input transition are dead and fortiori the transition t_n (the last transition before t in the path) is dead. Now the path $\mathcal{P}_{t' \to t_n}$ satisfies the recurrence hypothesis. Consequently, one can deduce that t' is dead as soon as t_n is dead.

The following corollary is a direct consequence of theorem 5, theorem 4 and theorem 1 respectively.

Corollary 2. *Let $\langle N, M_0 \rangle$ be a Root-system. The three following assertions are equivalent:*

- *(1) $\langle N, M_0 \rangle$ is deadlock free,*
- *(2) $\langle N, M_0 \rangle$ satisfies the cs-property,*
- *(3) $\langle N, M_0 \rangle$ is live.*

Example: The K-system (dead-closed) of figure 2 is not a Root system. Indeed, its root component N^* (Figure 3) is not strongly connected.

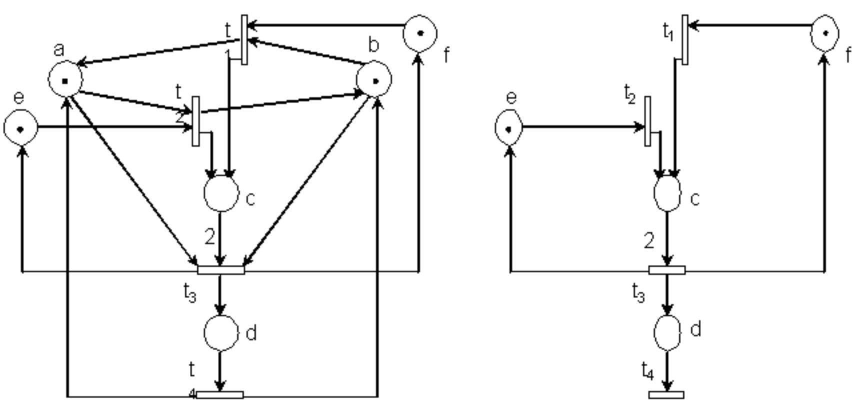

Fig. 3. An example of non Root but K system

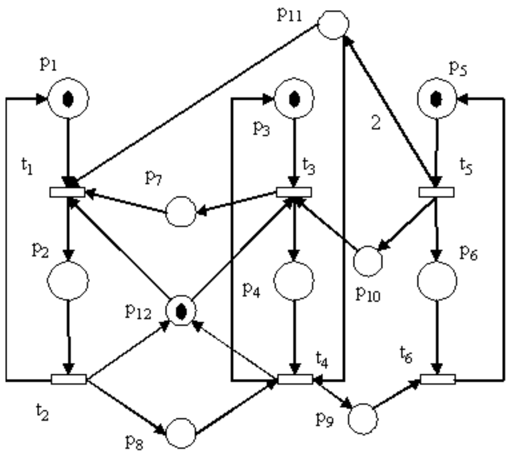

Fig. 4. An example of Root system

However, the non ordered system $\langle N, M_0\rangle$ of Figure4 (t_1 is not ordered and p_{11}, p_{12} are not root places) is a Root system. In fact, the corresponding root component is conservative and strongly connected.

Let us analyze structurally the corresponding net N. One can check that N admits the eight following minimal siphons: $S_1 = \{p_5, p_6\}$, $S_2 = \{p_3, p_4\}$, $S_3 = \{p_1, p_2\}$, $S_4 = \{p_5, p_{10}, p_4, p_9\}$, $S_5 = \{p_5, p_{10}, p_7, p_2, p_8, p_9\}$, $S_6 = \{p_{12}, p_2, p_4\}$, $S_7 = \{p_3, p_7, p_2, p_8\}$ and $S_8 = \{p_5, p_{11}, p_9\}$. These siphons are invariant-controlled for any initial marking satisfying the following conditions: $p_5+p_6 > 0, p_3+p_4 > 0$, $p_1+p_2 > 0, p_5+p_{10}+p_4+p_9 > 0, p_5+p_{10}+p_7+p_2+p_8+p_9 > 0, p_{12}+p_2+p_4 > 0$, $p_3 + p_7 + p_2 + p_8 > 0$ and $p_{11} + 2.p_9 + 2.p_5 - p_3 - p_7 > 0\}$. Such conditions hold for the chosen initial marking $M_0 = p_1 + p_3 + p_5 + p_{12}$. Consequently, $\langle N, M_0\rangle$ satisfies the cs-property i.e. live (according to theorem 5).

Obviously, the structure of N^* is a sufficient but not a necessary condition to ensure the K-property (and its membership in the class of K-systems). However, by adding structure to the subnet induced by the Root places (considered as modules) one can provide methods for synthesis of live K-systems.

In the following section, we first prove that the class of dead-closed systems is closed by a particular synchronization through asynchronous buffers. Then, this result will be used to extend the subclass of dead-closed systems structurally analyzable.

5.2 SDCS: Synchronized Dead-Closed Systems

In this section we prove that the class of dead-closed systems admits an interesting feature: it is closed by a particular synchronization through asynchronous buffers. The obtained class is a modular subclass of P/T nets called *Synchronized dead-closed Systems* (SDCS). By *modular* we emphasize that their definition is oriented to a bottom-up modelling methodology or structured view: individual agents, or modules, in the system are identified and modelled independently by means of live (i.e. cs-property) dead-closed systems (for example root systems), and the global model is obtained by synchronizing these modules through a set of places, the *buffers*. Such building process was already be used to define the class of Deterministically Synchronized Sequential Processes (DSSP) (see [10], [7] [12] for successive generalization) where elementary modules are simply live and safe state machines and where the interplay between conflict and synchronization is limited compared to dead-closed systems.

Definition 18. *A P/T system $\langle N, M_0\rangle$, with $N = \langle P, T, F, V\rangle$, is a Synchronized dead-closed System (or simply an SDCS) if and only if P is the disjoint union $P_1, \ldots, P_n$ and B, T is the disjoint union $T_1, \ldots, T_n$, and the following holds:*

- *(1) For every $i \in \{1, \ldots, n\}$, let $N_i = \langle P_i, T_i, F_{\lfloor((P_i\times T_i)\cup(T_i\times P_i))}, V_{\lfloor((P_i\times T_i)\cup(T_i\times P_i))}\rangle$. Then $\langle N_i, m_{0_{\lfloor P_i}}\rangle$ is a live dead-closed system.*
- *(2) For every $i, j \in \{1, \ldots, n\}$, if $i \neq j$ then $V_{\lfloor((P_i\times T_i)\cup(T_i\times P_i))} = \mathbf{0}$.*
- *(3) For each module N_i, $i \in \{1, \ldots, n\}$:*
  - *(a) $\exists$ (a buffer) $b \in B$ s.t. $b^\bullet \subseteq T_i$ (a private output buffer),*
  - *(b) $\forall b \in B$, b preserves the sets of root places of N_i (i.e., $\forall t \in T_i$, $Root(t)_{N_i} \subseteq Root(t)_N$).*
- *(4) Let $B' \subseteq B$ denotes the set of the output private buffers of N, then there exists a subset $B'' \subseteq B'$ such that the subnet induced by the dead-closed systems (N_i, $i \in \{1, \ldots, n\}$) and the buffers of B'' is conservative and strongly connected.*

Actually, we synchronize dead-closed system in such a way that we preserve the K-property (i.e. the equivalence between deadlock-freeness and liveness). Contrary to the DSSP modules, competition between those of an SDCS system is allowed, as long as the sets of root places of modules are preserved by composition($3.b$) (but not necessarily the set of equal conflicts). After composition, a buffer can be a root place in the composed net but it cannot take the place of another one. Moreover, no restriction is imposed on the connection nature of the buffers. This allows modules to compete for resources. A second feature of SDCS class which enlarge the description power of DSSP is the fact that a given buffer does not have to be a output (destination) private as long as it exists such a buffer for each module ($3.a$).

Hence, one can easily prove that the class of SDCS represents a strict generalization of conservative and strongly connected DSSP systems. Moreover, when we compose dead-closed systems, or even root systems, the obtained system remains dead-closed.

Figure 5 illustrates an example of SDCS system. This system is composed of two modules, N_1 and N_2 (enclosed by the dashed lines) communicating through three buffers b_1, b_2 and b_3. Each module is represented by a Root system (N_1 is not a state machine). Also, each buffer is not restrained to respect internal modules conflict as long as it preserves their root places. For instance, the buffer b_1 doesn't respect the conflict between transitions t_1 and t_3 of N_1 ($V(b_1, t_1) = 1$

but $V(b_1, t_3) = 0$) but it preserves the root place p_1 of t_1. This system is not a Root-system since its root component N^*, induced here by N_1, N_2 and the buffers b_2 and b_3, (the buffer b_1 is not a root place), is strongly connected but not conservative (the buffer b_3 is not structurally bounded). However, this system is an SDCS since, with notations of definition 18 (4), the subset $B'' = \{b_1, b_2\}$ allows the condition (4) of to be satisfied.

The following theorem states that the class of SDCS is a subclass of dead-closed Systems. This means that when we synchronize several dead-closed systems as described in definition18 we obtain a dead-closed system.

Theorem 6. *Let $\langle N, M_0 \rangle$ be an SDCS system. Then $\langle N, M_0 \rangle$ is a dead-closed system.*

Proof. Let t and t' be two transitions of N and suppose that t is dead. Let N_n and N_1 be the modules containing t and t' respectively. Since the subnet induced by modules and output private buffers is strongly connected, there exists an (elementary) path $\mathcal{P}_{N_1 \to N_n} = N_1 b_1 \dots b_{n-1} N_n$ leading from N_1 to N_n and each b_i ($i \in \{1, \dots, n-1\}$) is a buffer having N_{i+1} as output private. Let us reason by induction on the number of modules N_i ($i \in \{1, \dots, n\}$) involved in the path, Let us note $\mid \mathcal{P}_{N_i} \mid$ such a number.

- $\mid \mathcal{P}_{N_i} \mid = 0$: i.e. t and t' belong to the same module N_1. Since N_1 is a dead-closed system, one can use Theorem 4 (N_1 is also a K-system) to deduce that t' is dead.

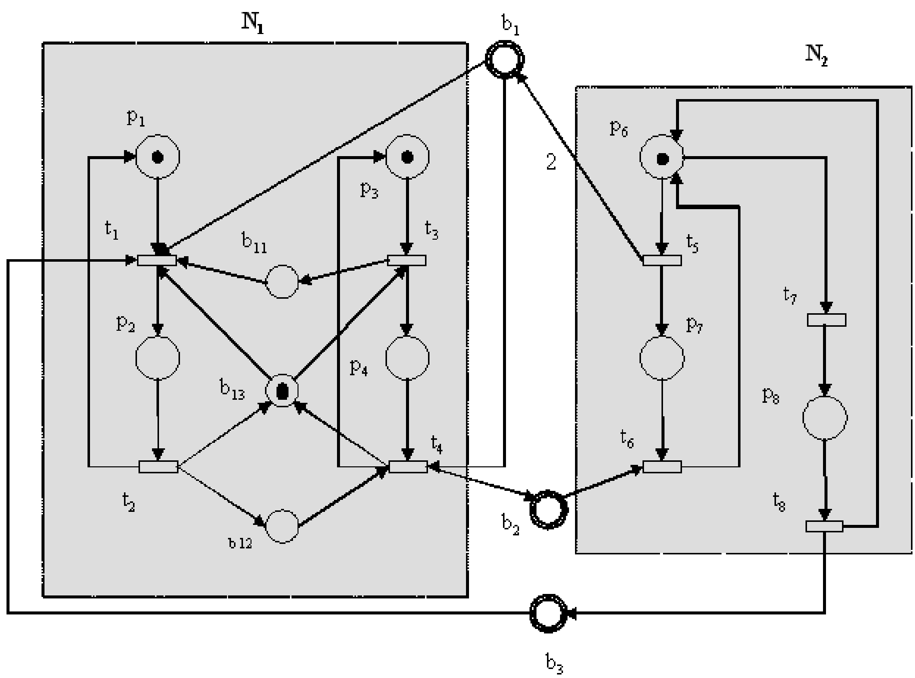

Fig. 5. An example of SKS system

– Suppose that the proposition is true for each path $\mathcal{P}_{N_1 \to N_n}$ involving less than n modules.
– Let $\mathcal{P}_{N_1 \to N_n}$ be a path leading from N_1 to N_n and passing through n modules.
  Consider b_{n-1}, the output private buffer of the module N_n. Using Theorem 5 one can deduce that all transitions in N_n are dead. Now, since b_{n-1} is a bounded place, we use proposition 8 to deduce that all its input transitions are dead and fortiori those of the module N_{n-1} (the module that appears before N_n in the path) are dead. The subpath leading from N_1 to N_{n-1} involves $n-1$ modules and hence satisfies the recurrence hypothesis. Consequently, one can deduce that t' is dead as soon as an input transition of b_{n-1} (belonging to N_{n-1}) is dead.

Corollary 3. *Let $\langle N, M_0 \rangle$ be an synchronized dead-closed system. The three following assertions are equivalent:*

– *(1) $\langle N, M_0 \rangle$ is deadlock free,*
– *(2) $\langle N, M_0 \rangle$ satisfies the cs-property,*
– *(3) $\langle N, M_0 \rangle$ is live.*

The previous corollary is a direct consequence of theorem 6, theorem4 and theorem1 respectively.

The main practical advantage of the definition of the SDCS class systems is that the equivalence between deadlock freeness and liveness can be preserved when we properly synchronize several dead-closed systems. A larger subclass based on the root nets structure can be obtained by applying the basic building process of the SDCS in a recursive way, i.e. modules can be Root systems, SDCS (or simply synchronized root systems) or more complex systems defined in this way. We are then able to revisit and extend the building process of the class of modular systems called multi-level deterministically synchronized processes (DS)*SP systems proposed in [8] which generalizes DSSP. Such a result will permit to enlarge the subclass of K-systems, structurally recognizable, for which the cs-property is a sufficient liveness condition. One can follow the same building process of (DS)*SP by taking live root systems as elementary modules (instead of safe and live state machines). We synchronize these root systems leading to an (root-based system) SDCS which is not necessarily a Root system. Then, one can take several (root-based system based) SDCS and synchronize them in a the same way. The resulting net, that is dead-closed system, can be considered as an agent in a further interconnection with other agents, etc. Doing so, a multi-level synchronization structure is built: the obtained system is composed of several agents that are coupled through buffers; these agents may also be a set of synchronized agents, etc. This naturally corresponds to systems with different levels of coupling: low level agents are tightly coupled to form an agent in a higher level, which is coupled with other agents, and so on. The class of systems thus obtained is covered by dead-closed systems, but largely generalizes strongly connected and conservative (DS)*SP (for which the deadlock freeness is a sufficient liveness condition [8]). From this view, the system of Figure5 can be viewed as "multilevel" SDCS where the module N_1 is composed of two submodules (Root systems) communicating through three buffers b_{11}, b_{12} and b_{13} (which is not output private but it preserves the set of the root places).

6 Other Subclasses Based on T-Invariants

Finally, we define two other subclasses of live K-systems, exploiting the fact that in every infinite occurrence sequence there must be a repetition of markings under boundedness hypothesis. We denote by T_{no} the subset of non ordered transitions. Nets of the first class are bounded and satisfy the following structural condition : the support of each T-invariant contains all non-ordered transitions. This class includes the one T-invariants nets from which (ordinary) bounded nets covered by T-invariants can be approximated as proved in [9].

Theorem 7. *Let $\langle N, M_0 \rangle$ be a P/T system such that:*

(i) N is conservative
(ii) $\forall$ T-invariant j: $T_{no} \subseteq \|j\|$.

$\langle N, M_0 \rangle$ is live if and only if $\langle N, M_0 \rangle$ satisfies the controlled-siphon property.

Proof. Assume that $\langle N, M_0 \rangle$ satisfies the cs-property but is not live. According to theorem 2, $T_D \neq \emptyset$ and $T_L \neq \emptyset$. Consider the subnet induced by T_L. This subnet is live and bounded for M^*. There exists necessarily an occurrence sequence for which count-vector is a T-invariant j and $T_{no} \not\subseteq \|j\|$. This contradicts condition (ii).

Now, we define a last subclass of non-ordered systems (systems having a non ordered transition) where the previous structural condition (ii) is refined as follows: for any non-ordered transition t, we can not get a T-invariant on the subnet induced by $T \setminus D(t)$.

Theorem 8. *Let $\langle N, M_0 \rangle$ be a non-ordered system satisfying the two following conditions:*

(i) N is conservative
(ii) $\forall$ T-invariant j and $\forall t \in T_{no}$: $(\|j\| \cap D(t)) \neq \emptyset$

$\langle N, M_0 \rangle$ is live if and only if $\langle N, M_0 \rangle$ satisfies the controlled-siphon property.

Proof. Let $\langle N, M_0 \rangle$ be satisfying the cs-property but not live. Consider the subnet induced by T_L. This subnet is live and bounded. Hence, there exists a T-invariant j corresponding to an occurrence sequence in the subnet and do not cover neither the (not ordered) transition t^* nor any transition in $D(t^*)$ $((\|j\| \cap D(t^*)) = \emptyset)$. This contradicts condition (ii).

Remark: Note that the non ordered system (t_3 is not ordered) of Figure 2 can also be recognized structurally as a K-system since conditions (i) and (ii) of Theorem 8 ($D(t_3) = T$) are satisfied.

7 Conclusion

The aim of this paper was to deepen into the structure theory on P/T systems,namely K-systems, for which the equivalence between controlled-siphon property, deadlock freeness, and liveness holds. Using the new structural concepts of ordered transitions and root places, we present a refined characterization of the non-liveness condition under cs property hypothesis. Such result permits us to revisit from a new perspective some well known results and to structurally characterize new and more expressive subclasses of K-systems. This work poses a challenging question: What are the structural mechanisms ensuring a siphon to be controlled other than based on trap or p-invariant concept? The interest of a positive answer is a broader decision power of controlled siphon property in particular for systems where the purely algebraic methods such rank theorem [5] are important.

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