Schnorr Dimension

Rodney Downey[1], Wolfgang Merkle[2], and Jan Reimann[2]

[1] School of Mathematics, Statistics and Computer Science,
Victoria University P. O. Box 600,
Wellington, New Zealand
[2] Arbeitsgruppe Mathematische Logik und Theoretische Informatik,
Institut für Informatik,
Fakultät für Mathematik und Informatik,
Ruprecht-Karls-Universität Heidelberg,
Im Neuenheimer Feld 294,
D-69120 Heidelberg, Germany

Abstract. Following Lutz's approach to effective (constructive) dimension, we define a notion of dimension for individual sequences based on Schnorr's concept(s) of randomness. In contrast to computable randomness and Schnorr randomness, the dimension concepts defined via computable martingales and Schnorr tests coincide. Furthermore, we give a machine characterization of Schnorr dimension, based on prefix free machines whose domain has computable measure. Finally, we show that there exist computably enumerable sets which are Schnorr irregular: while every c.e. set has Schnorr Hausdorff dimension 0 there are c.e. sets of Schnorr packing dimension 1, a property impossible in the case of effective (constructive) dimension, due to Barzdin's Theorem.

1 Introduction

Martin-Löf's concept of individual random sequences was recently generalized by Lutz [10, 11], who introduced an effective notion of Hausdorff dimension. As (classical) Hausdorff dimension can be seen as a refinement of Lebesgue measure on $2^{\mathbb{N}}$, in the sense that it further distinguishes between classes of measure 0, the effective Hausdorff dimension of an individual sequence can be interpreted as a *degree of randomness* of the sequence. This viewpoint is supported by a series of results due to Ryabko [15, 16], Staiger [21, 20], Cai and Hartmanis [3], and Mayordomo [12], which establish that the effective Hausdorff dimension of a sequence equals its lower asymptotic Kolmogorov complexity (plain or prefix-free).

Criticizing Martin-Löf's approach to randomness as not being truly algorithmic, Schnorr [18] presented two alternative randomness concepts, one based one computable martingales, nowadays referred to as *computable randomness*, the other based on stricter effectivity requirements for Martin-Löf tests. This concept is known as *Schnorr randomness*.

S.B. Cooper, B. Löwe, and L. Torenvliet (Eds.): CiE 2005, LNCS 3526, pp. 96–105, 2005.

In this paper we will generalize and extend Schnorr's randomness concepts to Hausdorff dimension. Like in the case of randomness, the approach suffers from some technical difficulties like the absence of a universal test/martingale. We will see that for dimension, Schnorr's two approaches coincide, in contrast to Schnorr randomness and computable randomness. Furthermore, it turns out that, with respect to Schnorr dimension, computably enumerable sets can expose a complex behavior, to some extent. Namely, we will show that there are c.e. sets of high Schnorr packing dimension, which is impossible in the effective case, due to a result by Barzdin' [2]. On the other hand, we prove that the Schnorr Hausdorff dimension of the characteristic sequence of a c.e. set is 0. Thus, the class of computably enumerable sets contains *irregular* sequences – sequences for which Hausdorff and packing dimension do not coincide.

The paper is structured is as follows. In Section 2 we give a short introduction to the classical theory of Hausdorff measures and dimension, as well as packing dimension. In Section 3 we will define algorithmic variants of these concepts based on Schnorr's approach to randomness. In Section 4 we prove that the approach to Schnorr dimension via coverings and via computable martingales coincide, in contrast to Schnorr randomness and computable randomness. In Section 5 we derive a machine characterization of Schnorr Hausdorff and packing dimension. Finally, in Section 6, we study the Schnorr dimension of computably enumerable sets. The main result here will be that on those sets Schnorr Hausdorff dimension and Schnorr packing dimension can differ as largely as possible.

We will use fairly standard notation. $2^{\mathbb{N}}$ will denote the set of infinite binary sequences. Sequences will be denoted by upper case letters like A, B, C, or X, Y, Z. We will refer to the nth bit ($n \geq 0$) in a sequence B by either B_n or $B(n)$, i.e. $B = B_0 B_1 B_2 \ldots = B(0)\, B(1)\, B(2) \ldots$.

Strings, i.e. finite sequences of 0s and 1s will be denoted by lower case letters from the end of the alphabet, u, v, w, x, y, z along with some lower case Greek letters like σ and τ. $\{0,1\}^*$ will denote the set of all strings. The *initial segment of length* n, $A\restriction_n$, of a sequence A is the string of length n corresponding to the first n bits of A.

Given two strings v, w, v is called a *prefix* of w, $v \sqsubseteq w$ for short, if there exists a string x such that $vx = w$, where vx is the concatenation of v and x. Obviously, this relation can be extended to hold between strings and sequences as well. A set of strings is called *prefix free* if all its elements are pairwise incomparable.

Initial segments induce a standard topology on $2^{\mathbb{N}}$. The basis of the topology is formed by the *basic open cylinders* (or just *cylinders*, for short). Given a string $w = w_0 \ldots w_{n-1}$ of length n, these are defined as $[w] = \{A \in 2^{\mathbb{N}} : A\restriction_n = w\}$. For $C \subseteq \{0,1\}^*$, we define $[C] = \bigcup_{w \in C} [w]$.

Throughout the paper we assume familiarity with the basic concepts of computability theory such as Turing machines, computably enumerable sets, computable and left-computable (c.e.) reals. Due to space consideration, formal proofs of the results are omitted. (Some ideas are sketched.)

2 Hausdorff Measures and Dimension

The basic idea behind Hausdorff dimension is to generalize the process of measuring a set by approximating (covering) it with sets whose measure is already known. Especially, the size of the sets used in the measurement process will be manipulated by certain transformations, thus making it harder (or easier) to approximate a set with a covering of small accumulated measure. This gives rise to the notion of *Hausdorff measures.*

Definition 1. *Let $\mathcal{X} \subseteq 2^{\mathbb{N}}$. Given $\delta > 0$ and a real number $s \geq 0$, define*

$$\mathcal{H}^s_\delta(\mathcal{X}) = \inf \left\{ \sum_{w \in C} 2^{-|w|s} :\ (\forall w \in C)[2^{-|w|} \leq \delta] \ \wedge\ \mathcal{X} \subseteq [C] \right\}$$

The s-dimensional Hausdorff measure *of $\mathcal{X}$ is defined as*

$$\mathcal{H}^s(\mathcal{X}) = \lim_{\delta \to 0} \mathcal{H}^s_\delta(\mathcal{X}).$$

Note that $\mathcal{H}^s(\mathcal{X})$ is well defined, since, as δ decreases, there are fewer δ-covers available, hence $\mathcal{H}^s_\delta$ is non-decreasing. However, the value may be infinite. For $s = 1$, one obtains Lebesgue measure on $2^{\mathbb{N}}$.

The outer measures $\mathcal{H}^s$ have an important property.

Proposition 2. *Let $\mathcal{X} \subseteq 2^{\mathbb{N}}$. If, for some $s \geq 0$, $\mathcal{H}^s(\mathcal{X}) < \infty$, then $\mathcal{H}^t(\mathcal{X}) = 0$ for all $t > s$.*

This means that there can exist only one point $s \geq 0$ where a given class might have finite positive s-dimensional Hausdorff measure. This point is the *Hausdorff dimension* of the class.

Definition 3. *For a class $\mathcal{X} \subseteq 2^{\mathbb{N}}$, define the* Hausdorff dimension *of $\mathcal{X}$ as*

$$\dim_{\mathrm{H}}(\mathcal{X}) = \inf\{s \geq 0 :\ \mathcal{H}^s(\mathcal{X}) = 0\}.$$

For more on Hausdorff measures and dimension refer to the book by Falconer [7]. A presentation of Hausdorff measures and dimension in $2^{\mathbb{N}}$ can be found in [13].

2.1 Packing Dimension

We say that a prefix free set $P \subseteq \{0,1\}^*$ is a *packing* for $\mathcal{X} \subseteq 2^{\mathbb{N}}$, if for every $\sigma \in P$, there is some $A \in \mathcal{X}$ such that $\sigma \sqsubset A$.

Given $s \geq 0$, $\delta > 0$, let

$$\mathcal{P}^s_\delta(\mathcal{X}) = \sup \left\{ \sum_{w \in P} 2^{-|w|s} :\ P \text{ packing for } \mathcal{X} \text{ and } (\forall w \in P)[2^{-|w|} \leq \delta \right\}. \quad (1)$$

$\mathcal{P}^s_\delta(\mathcal{X})$ decreases with δ, so the limit $\mathcal{P}^s_0(\mathcal{X}) = \lim_{\delta\to 0}\mathcal{P}^s_\delta(\mathcal{X})$ exists. Finally, define

$$\mathcal{P}^s(\mathcal{X}) = \inf\left\{\sum \mathcal{P}^s_0(\mathcal{X}_i) : \mathcal{X} \subseteq \bigcup_{i\in\mathbb{N}} \mathcal{X}_i\right\}. \tag{2}$$

(The infimum is taken over arbitrary countable covers of $\mathcal{X}$.) $\mathcal{P}^s$ is called, in correspondence to Hausdorff measures, the *s-dimensional packing measure* on $2^{\mathbb{N}}$. Packing measures were introduced by Tricot [24] and Taylor and Tricot [23]. They can be seen as a dual concept to Hausdorff measures, and behave in many ways similar to them. In particular, one may define *packing dimension* in the same way as Hausdorff dimension.

Definition 4. *The* packing dimension *of a class $\mathcal{X} \subseteq 2^{\mathbb{N}}$ is defined as*

$$\dim_{\mathrm{P}} \mathcal{X} = \inf\{s : \mathcal{P}^s(\mathcal{X}) = 0\} = \sup\{s : \mathcal{P}^s(\mathcal{X}) = \infty\}. \tag{3}$$

Again, we refer to Falconer's book [7] for details on packing measures and dimension.

2.2 Martingales

It is possible do characterize Hausdorff and packing dimension via *martingales*, too. Martingales have become a fundamental tool in probability theory. In Cantor space $2^{\mathbb{N}}$, they can be described very conveniently.

Definition 5. *A* martingale *on $2^{\mathbb{N}}$ is a function $d : \{0,1\}^* \to [0,\infty)$ which satisfies*

$$d(w) = \frac{d(w01) + d(w1)}{2} \quad \text{for all } w \in \{0,1\}^*.$$

Martingales can be interpreted as capital functions of an accordant betting strategy, when applied to a binary sequence. The value $d(w)$ reflects the player's capital after bits $w(0), \dots, w(|w|-1)$ have been revealed to him.

Definition 6. *Let $g : \mathbb{N} \to [0,\infty)$ be a positive, unbounded function. A martingale is g-successful (or g-*succeeds*) on a sequence $B \in 2^{\mathbb{N}}$ if*

$$d(B\restriction_n) \geq g(n) \quad \text{for infinitely many } n.$$

d is strongly g-successful *(or g-*succeeds strongly*) on a sequence $B \in 2^{\mathbb{N}}$ if*

$$d(B\restriction_n) \geq g(n) \quad \text{for all but finitely many } n. \tag{4}$$

It turns out that, in terms of Hausdorff dimension, the relation between $\mathcal{H}^s$-nullsets and $2^{(1-s)n}$-successful martingales is very close.

Theorem 7. *Let $\mathcal{X} \subseteq 2^{\mathbb{N}}$. Then it holds that*

$$\dim_{\mathrm{H}} \mathcal{X} = \inf\{s : \exists \text{ martingale } d\ 2^{(1-s)n}\text{-successful on all } B \in \mathcal{X}\}. \tag{5}$$

$$\dim_{\mathrm{P}} \mathcal{X} = \inf\{s : \exists \text{ martingale } d \text{ strongly } 2^{(1-s)n}\text{-successful on all } B \in \mathcal{X}\} \tag{6}$$

In the form presented here, equation (5) was first proven by Lutz [9]. However, close connections between Hausdorff dimension and winning conditions on martingales have been observed by Ryabko [17] and Staiger [22]. Equation (6) is due to Athreya, Lutz, Hitchcock, and Mayordomo [1].

Note that, if a martingale $2^{(1-s)n}$-succeeds on a sequence A, for any $t > s$ it will hold that

$$\limsup_{n\to\infty} \frac{d(A\restriction_n)}{2^{(1-t)n}} = \infty. \tag{7}$$

So, when it comes to dimension, we will, if convenient, use (7) and the original definition interchangeably. Furthermore, a martingale which satisfies (7) for $t = 1$ is simply called *successful on* A.

3 Schnorr Null Sets and Schnorr Dimension

We now define a notion of dimension based on Schnorr's approach to randomness.

Definition 8. *Let* $s \in [0,1]$ *be a rational number.*

(a) A Schnorr s-test *is a computable sequence* $(S_n)_{n\in\mathbb{N}}$ *of c.e. sets of finite strings such that, for all* n, $\sum_{w\in S_n} 2^{-|w|s} \le 2^{-n}$, *and* $\sum_{w\in S_n} 2^{-|w|s}$ *is a uniformly computable real number.*
(b) A class $\mathcal{A} \subseteq 2^{\mathbb{N}}$ *is* Schnorr s-null *if there exists a Schnorr* s*-test* (S_n) *such that* $\mathcal{A} \subseteq \bigcap_{n\in\mathbb{N}}[S_n]$.

The *Schnorr random sequences* are those which are (as a singleton class in $2^{\mathbb{N}}$) not Schnorr 1-null.

Downey and Griffiths [4] observe that, by adding elements, one can replace any Schnorr 1-test by an equivalent one (i.e., one defining the same Schnorr nullsets) where each level of the test has measure exactly 2^{-n}. We can apply the same argument in the case of arbitrary rational s, and hence we may, if appropriate, assume that $\sum_{w\in S_n} 2^{-|w|s} = 2^{-n}$, for all n.

Note further that, for rational s, each set S_n in a Schnorr s-test is actually computable, since to determine whether $w \in S_n$ it suffices to enumerate S_n until the accumulated sum given by $\sum 2^{-|v|s}$ exceeds $2^{-n} - 2^{|w|s}$ (assuming the measure of the n-th level of the test is in fact 2^{-n}). If w has not been enumerated so far, it cannot be in S_n. (Observe, too, that the converse does not hold.)

One can describe Schnorr s-nullsets also in terms of *Solovay tests*. Solovay tests were introduced by Solovay [19] and allowed for a characterization of Martin-Löf nullsets via a single test set, instead of a uniformly computable sequence of test sets.

Definition 9. *Let* $s \in [0,1]$ *be rational.*

(a) An Solovay s-test *is a c.e. set* $C \subseteq \{0,1\}^*$ *such that* $\sum_{w\in C} 2^{-|w|s} \le 1$.
(b) An Solovay s*-test is* total *if* $\sum_{w\in C} 2^{-|w|s}$ *is a computable real number.*
(c) An Solovay s*-test* C covers *a sequence* $A \in 2^{\mathbb{N}}$ *if it contains infinitely many initial segments of* A. *In this case we also say that* A fails *the test* C.

Theorem 10. *For any rational $s \in [0,1]$, a class $\mathcal{X} \subseteq 2^{\mathbb{N}}$ is Schnorr s-null if and only if there is a total Solovay s-test which covers every sequence $A \in \mathcal{X}$.*

3.1 Schnorr Dimension

Like in the classical case, each class has a critical value a critical value with respect to Schnorr s-measure.

Proposition 11. *Let $\mathcal{X} \subseteq 2^{\mathbb{N}}$. For any rational $s \geq 0$, if $\mathcal{X}$ is Schnorr s-null then it is also Schnorr t-null for any rational $t \geq s$.*

This follows from the fact that every Schnorr s-test is also a t-test. The definition of Schnorr Hausdorff dimension can now be given in a straightforward way.

Definition 12. *The* Schnorr Hausdorff dimension *of a class $\mathcal{X} \subseteq 2^{\mathbb{N}}$ is defined as*

$$\dim_{\mathrm{H}}^{\mathrm{S}}(\mathcal{X}) = \inf\{s \geq 0 : \mathcal{X} \text{ is Schnorr } s\text{-null}\}.$$

For a sequence $A \in 2^{\mathbb{N}}$, we write $\dim_{\mathrm{H}}^{\mathrm{S}} A$ for $\dim_{\mathrm{H}}^{\mathrm{S}}\{A\}$ and refer to $\dim_{\mathrm{H}}^{\mathrm{S}} A$ as the Schnorr Hausdorff dimension of A.

3.2 Schnorr Packing Dimension

Due to the more involved definition of packing dimension, it is not immediately clear how to define a Schnorr-type version of packing dimension. However, we will see in the next section that Schnorr dimension allows an elegant characterization in terms of martingales, building on Theorem 7. This will also make it possible to define a Schnorr version of packing dimension.

4 Schnorr Dimension and Martingales

In view of his unpredictability paradigm for algorithmic randomness, Schnorr [18] suggested a notion of randomness based on *computable* martingales. According to this notion, nowadays referred to as *computable randomness*, a sequence is computably random if no computable martingale succeeds on it.

Schnorr [18] himself proved that a sequence is Martin-Löf random if and only if some *left-computable* martingale succeeds on it. Therefore, one might be tempted to derive a similar relation between Schnorr null sets and *computable* martingales. However, Schnorr [18] pointed out that the increase in capital of a successful computable martingale can be so slow it cannot be computably detected. Therefore, he introduced *orders* ("Ordnungsfunktionen"), which allow to ensure an effective control over the capital.

In general, any positive, real, unbounded function g is called an *order*. (It should be remarked that, in Schnorr's terminology, an "Ordnungsfunktion" is always computable.)

Schnorr showed that Schnorr nullsets can be characterized via computable martingales successful against computable orders.

Theorem 13 (Schnorr). *A set $\mathcal{X} \subseteq 2^{\mathbb{N}}$ is Schnorr 1-null if and only if there exists a computable martingale d and a computable order g such that d is g-successful on all $B \in \mathcal{X}$.*

Schnorr calls the functions $g(n) = 2^{(1-s)n}$ *exponential orders*, so much of the theory of effective dimension is already, though apparently without explicit reference, present in Schnorr's treatment of algorithmic randomness [18].

If one drops the requirement of being g-successful for some computable g, one obtains the concept of *computable randomness*. Wang [26] showed that the concepts of computable randomness and Schnorr randomness do not coincide. There are Schnorr random sequences on which some computable martingale succeeds. However, the differences vanish if it comes to dimension.

Theorem 14. *For any sequence $B \in 2^{\mathbb{N}}$,*

$$\dim_{\mathrm{H}}^{\mathrm{S}} B = \inf\{s \in \mathbb{Q} : \text{ some computable martingale } d \text{ is } s\text{-successful on } B\}.$$

So, in contrast to randomness, the approach via Schnorr tests and the approach via computable martingales to dimension yield the same concept.

Besides, we can build on Theorem 14 to introduce *Schnorr packing dimension.*

Definition 15. *Given a sequence $A \in 2^{\mathbb{N}}$, we define the* Schnorr packing dimension *of A, $\dim_{\mathrm{P}}^{\mathrm{S}} A$, as*

$$\dim_{\mathrm{P}}^{\mathrm{S}} A = \inf\{s \in \mathbb{Q} : \text{ some comp. martingale is strongly } s\text{-successful on } A\}$$

Schnorr packing dimension is implicitly introduced as a computable version of *strong dimension* in [1]. It follows from the definitions that for any sequence $A \in 2^{\mathbb{N}}$, $\dim_{\mathrm{H}}^{\mathrm{S}} A \leq \dim_{\mathrm{P}}^{\mathrm{S}} A$. We call sequences for which Schnorr Hausdorff and Schnorr packing dimension coincide *Schnorr regular* (see [24] and [1]). It is easy to construct a non-Schnorr regular sequence, however, in Section 6 we will see that such sequences already occur within the class of c.e. sets.

5 A Machine Characterization of Schnorr Dimension

One of the most powerful arguments in favor of Martin-Löf's approach to randomness is the coincidence of the Martin-Löf random sequences with the sequences that are incompressible in terms of (prefix free) Kolmogorov complexity.

Such an elegant characterization via machine compressibility is possible neither for Schnorr randomness nor Schnorr dimension. To obtain a machine characterization of Schnorr dimension, we have to restrict the admissible machines to those with domains having computable measure.

Definition 16. *A prefix free machine M is* computable *if $\sum_{w \in \mathrm{dom}(M)} 2^{-|w|}$ is a computable real number.*

Note that, as in the case of Schnorr tests, if a machine is computable, then its domain is computable (but not vice versa). To determine whether $M(w)\downarrow$, enumerate dom(M) until $\sum_{w\in\text{dom}(M)} 2^{-|w|}$ is approximated by a precision of 2^{-N}, where $N > |w|$. If $M(w)\downarrow$, w must have been enumerated up to this point.

Furthermore, we sometimes assume that the measure of the domain of a computable machine is 1. This can be justified, as in the case of Schnorr tests, by adding superfluous strings to the domain.

The definition of machine complexity follows the standard scheme. We restrict ourselves to prefix free machines.

Definition 17. *Given a Turing machine M with prefix free domain, the M-* complexity *of a string x is defined as*

$$\mathrm{K}_M(x) = \min\{|p| : M(p) = x\},$$

where $\mathrm{K}_M(x) = \infty$ if there does not exist a $p \in \{0,1\}^*$ such that $M(p) = x$.

We refer to the books by Li and Vitanyi [8] and Downey and Hirschfeldt [5] for comprehensive treatments on machine (Kolmogorov) complexity.

Downey and Griffiths [4] show that a sequence A is Schnorr random if and only if for every computable machine M, there exists a constant c such that $\mathrm{K}_M(A\restriction_n) \geq n - c$. Building on this characterization, we can go on to describe Schnorr dimension as asymptotic entropy with respect to computable machines.

Theorem 18. *For any sequence A it holds that*

$$\dim_{\mathrm{H}}^{\mathrm{S}} A = \inf_M \underline{\mathrm{K}}_M(A) \stackrel{\text{def}}{=} \inf_M \left\{ \liminf_{n\to\infty} \frac{\mathrm{K}_M(A\restriction_n)}{n} \right\},$$

where the infimum is taken over all computable prefix free machines M.

One can use an analogous argument to obtain a machine characterization of Schnorr packing dimension.

Theorem 19. *For any sequence A it holds that*

$$\dim_{\mathrm{P}}^{\mathrm{S}} A = \inf_M \overline{\mathrm{K}}_M(A) \stackrel{\text{def}}{=} \inf_M \left\{ \limsup_{n\to\infty} \frac{\mathrm{K}_M(A\restriction_n)}{n} \right\},$$

where the infimum is taken over all computable prefix free machines M.

6 Schnorr Dimension and Computable Enumerability

Usually, when studying algorithmic randomness, interest focuses on *c.e. reals* (i.e. left-computable real numbers) rather than on *c.e. sets*. The reason is that c.e. sets exhibit a trivial behavior with respect to most randomness notions, while there are c.e. reals which are Martin-Löf random, such as Chaitin's Ω.

As regards c.e. reals, we can extend the result of Downey and Griffiths [4] that every Schnorr random c.e. real is of high Turing degree.

Theorem 20. *Every sequence of positive Schnorr Hausdorff dimension has high Turing degree. That is, if* $\dim_{\mathrm{H}}^{\mathrm{S}} A > 0$*, then* $S' \equiv_{\mathrm{T}} 0''$*.*

Using the fact that every noncomputable c.e. set contains an infinite computable subset, and the fact that, for all n, $\lambda\{A \in 2^{\mathbb{N}} : A(n) = 1\} = 1/2$, it is not hard to show that no c.e. set can be Schnorr random.

It does not seem immediately clear how to improve this to Schnorr dimension zero. Indeed, defining coverings from the enumeration of a set directly might not work, because due to the dimension factor in Hausdorff measures, longer strings will be weighted higher. Depending on how the enumeration is distributed, this might not lead to a Schnorr s-covering at all.

However, one might exploit the somewhat predictable nature of a c.e. set to define a computable martingale which is, for any $s > 0$, s-successful on the characteristic sequence of the enumerable set, thereby ensuring that each c.e. set has Schnorr Hausdorff dimension 0.

Theorem 21. *Every computably enumerable set* $A \subseteq \mathbb{N}$ *has Schnorr Hausdorff dimension zero.*

On the other hand, concerning upper entropy, c.e. sets may exhibit a rather complicated structure, in sharp contrast to the case of effective (constructive) dimension, where Barzdin's Theorem [2] ensures that all c.e. sets have effective packing dimension 0. As the proof of the following theorem shows, this is due to the requirement that all machines involved in the determination of Schnorr dimension are total.

Theorem 22. *There exists a computably enumerable set* $A \subseteq \mathbb{N}$ *such that*

$$\dim_{\mathrm{P}}^{\mathrm{S}} A = 1.$$

References

1. K. B. Athreya, J. M. Hitchcock, J. H. Lutz, and E. Mayordomo. Effective strong dimension in algorithmic information and computational complexity. In *Proceedings of the Twenty-First Symposium on Theoretical Aspects of Computer Science (Montpellier, France, March 25–27, 2004)*, pages 632–643. Springer-Verlag, 2004.
2. Ja. M. Barzdin'. Complexity of programs which recognize whether natural numbers not exceeding n belong to a recursively enumerable set. *Sov. Math., Dokl.*, 9:1251–1254, 1968.
3. J.-Y. Cai and J. Hartmanis. On Hausdorff and topological dimensions of the Kolmogorov complexity of the real line. *J. Comput. System Sci.*, 49(3):605–619, 1994.
4. R. G. Downey and E. J. Griffiths. Schnorr randomness. *J. Symbolic Logic*, 69(2): 533–554, 2004.
5. R. G. Downey and D. R. Hirschfeldt. Algorithmic randomness and complexity. book, in preparation, 2004.
6. H. G. Eggleston. The fractional dimension of a set defined by decimal properties. *Quart. J. Math., Oxford Ser.*, 20:31–36, 1949.

7. K. Falconer. *Fractal Geometry: Mathematical Foundations and Applications.* Wiley, 1990.
8. M. Li and P. Vitányi. *An introduction to Kolmogorov complexity and its applications.* Graduate Texts in Computer Science. Springer-Verlag, New York, 1997.
9. J. H. Lutz. Dimension in complexity classes. In *Proceedings of the Fifteenth Annual IEEE Conference on Computational Complexity*, pages 158–169. IEEE Computer Society, 2000.
10. J. H. Lutz. Gales and the constructive dimension of individual sequences. In *Automata, languages and programming (Geneva, 2000)*, volume 1853 of *Lecture Notes in Comput. Sci.*, pages 902–913. Springer, Berlin, 2000.
11. J. H. Lutz. The dimensions of individual strings and sequences. *Inform. and Comput.*, 187(1):49–79, 2003.
12. E. Mayordomo. A Kolmogorov complexity characterization of constructive Hausdorff dimension. *Inform. Process. Lett.*, 84(1):1–3, 2002.
13. J. Reimann. *Computability and fractal dimension.* Doctoral dissertation, Universität Heidelberg, 2004.
14. J. Reimann and F. Stephan. Effective Hausdorff dimension. In *Logic Colloquium '01 (Vienna).* Assoc. Symbol. Logic. to appear.
15. B. Y. Ryabko. Coding of combinatorial sources and Hausdorff dimension. *Dokl. Akad. Nauk SSSR*, 277(5):1066–1070, 1984.
16. B. Y. Ryabko. Noise-free coding of combinatorial sources, Hausdorff dimension and Kolmogorov complexity. *Problemy Peredachi Informatsii*, 22(3):16–26, 1986.
17. B. Y. Ryabko. An algorithmic approach to the prediction problem. *Problemy Peredachi Informatsii*, 29(2):96–103, 1993.
18. C.-P. Schnorr. *Zufälligkeit und Wahrscheinlichkeit. Eine algorithmische Begründung der Wahrscheinlichkeitstheorie.* Springer-Verlag, Berlin, 1971.
19. R. M. Solovay. Lecture notes on algorithmic randomness. unpublished manuscript, UCLA, 1975.
20. L. Staiger. Constructive dimension equals Kolmogorov complexity. Inform. Proc. Letters, to appear.
21. L. Staiger. Kolmogorov complexity and Hausdorff dimension. *Inform. and Comput.*, 103(2):159–194, 1993.
22. L. Staiger. A tight upper bound on Kolmogorov complexity and uniformly optimal prediction. *Theory of Computing Systems*, 31(3):215–229, 1998.
23. S. J. Taylor and C. Tricot. Packing measure, and its evaluation for a Brownian path. *Trans. Amer. Math. Soc.*, 288(2):679–699, 1985.
24. C. Tricot, Jr. Two definitions of fractional dimension. *Math. Proc. Cambridge Philos. Soc.*, 91(1):57–74, 1982.
25. J. Ville. *Etude critique de la notion de collectif.* Gauthier-Villars, 1939.
26. Y. Wang. A separation of two randomness concepts. *Inform. Process. Lett.*, 69(3): 115–118, 1999.