Statistical Aspects of Acausal Pulses in Physics and Wavelets Applications

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Abstract. In the mathematical theory of distributions are widely used test-functions (which differ to zero only on a limited interval and have continuous derivatives of any order on the whole real axis). The use of such functions is also recommended in Fourier analysis of wavelets. However, less attention was given to connections between test-functions and equations used in mathematical physics (as wave equation). This paper shows that test-functions, considered at the macroscopic scale (that means not as δ -functions) can represent solutions for the wave-equation, under the form of acausal pulses (which appear under initial null conditions and without any source-term to exist). This implies the necessity for some supplementary requirements to be added to the wave-equation, so as the possibility of appearing such pulses to be rejected. It will be shown that such a possibility represents in fact a kind of bifurcation point, and a statistic interpretation (based on probability for state-variables to make certain jumps) is presented for justifying the fact that such pulses are not observed. Finally the advantage of using practical test function for wavelets processing is presented.

1 Introduction

As it is known, in Fourier analysis based on wavelets the user wants to obtain the mean value of the received signal multiplied by certain alternating functions over a limited time interval. Usually this operation is performed by a direct integration of the signal on this time interval. However, such structures are very sensitive at random variations of the integration period, due to stochastic phenomena appearing when an electric current is interrupted. For this reason, a multiplication of the received signal with a test-function - a function which differs to zero only on this time interval and with continuous derivatives of any order on the whole real axis - is recommended. Yet such test functions, similar to the Dirac functions, can't be generated by a differential equation. The existence of such an equation of evolution, beginning to act at an initial moment of time, would imply the necessity for a derivative of certain order to make a jump at this initial moment of time from the zero value to a nonzero value. But this aspect is in contradiction with the property of test-functions

O. Gervasi et al. (Eds.): ICCSA 2005, LNCS 3482, pp. 598–603, 2005.

to have continuous derivatives of any order on the whole real axis, represented in this case by the time axis. So it results that an ideal test-function can't be generated by a differential equation (see also [1]); the analysis has to be restricted at possibilities of generating practical test-functions (functions similar to test-functions, but having a finite number of continuous derivatives on the whole real axis) useful for wavelets analysis. Due to the exact form of the derivatives of test-functions, we can't apply derivative free algorithms [2] or algorithms which can change in time [3]. Starting from the exact mathematical expressions of a certain test-function and of its derivatives, we must use specific differential equations for generating such practical test-functions.

For example, the bump-like function

$$\varphi(\tau) = \begin{cases} \exp\left(\frac{1}{\tau^2-1}\right) & \text{if} \quad \tau \in (-1,1) \\ 0 \text{ otherwise} \end{cases} \tag{1}$$

is a test-function on $[-1,1]$. We are looking for an initial value problem for generating a practical test-function f on $[-1,1]$ by considering differential equations satisfied by the exact form of the amplitude and of the derivatives of the bump function φ. Such equations are

$$f^{(1)} = \frac{-2\tau}{(\tau^2-1)^2} f, \quad f(-0.99) = \varphi(-0.99) \tag{2}$$

$$f^{(2)} = \frac{6\tau^4-2}{(\tau^2-1)^4} f, \quad f(-0.99) = \varphi(-0.99),\ f^{(1)}(-0.99) = \varphi^{(1)}(-0.99) \tag{3}$$

Numerically integrations give solutions similar to φ, but having a very small amplitude.

2 Utility of Test-Functions in Mathematical Physics

Test-functions are known as having as limit the Dirac function when the interval on which they differ to zero decreases toward zero. However, less attention was given to the fact that such test-functions, considered at the macroscopic scale (that means not as Dirac-functions) can represent solutions for certain equations in mathematical physics (an example being the wave-equation). The main consequence of this consists in the possibility of certain pulses to appear as solutions of the wave-equation under initial null conditions for the function and for all its derivatives and without any free-term (a source-term) to exist. In order to prove the possibility of appearing acausal pulses as solutions of the wave-equation (not determined by the initial conditions or by some external forces) we begin by writing the wave-equation

$$\frac{\partial^2\phi}{\partial x^2} - \frac{1}{v^2}\frac{\partial^2\phi}{\partial t^2} = 0 \tag{4}$$

for a free string defined on the length interval $(0,l)$ (an open set), where ϕ represents the amplitude of the string oscillations and v represents the velocity of the waves inside the string medium. At the initial moment of time (the zero moment) the amplitude ϕ together with all its derivatives of first and second order are equal to zero. From the mathematical theory of the wave-equation we know that any solution of this equation must be a superposition of a direct wave and of a reverse wave. We shall restrict our analyze at direct waves and consider a supposed extension of the string on the whole Ox axis, ϕ being defined by the function

$$\phi(\tau) = \begin{cases} \exp\left(\frac{1}{(x-vt-1)^2-1}\right) \text{ for } x - vt < 1 \\ 0 \text{ for } x - vt \geq 1 \end{cases} \tag{5}$$

where $t \geq 0$. This function for the extended string satisfies the wave-equation (being a function of x-vt , a direct wave). It is a continuous function, having continuous partial derivatives of any order for $x \in (-\infty, \infty)$ and for $x \geq 0$. For $x \in (0, l)$ (the real string)the amplitude ϕ and all its derivatives are equal to zero at the zero moment of time, as required by the initial null conditions for the real string. We can notice that for $t = 0$ the amplitude ϕ and its partial derivatives differ to zero only on a finite space interval, this being a property of the functions defined on a compact set (test functions). But the argument of the exponential function is $x - vt$; this implies that the positive amplitude existing on the length interval $(-2, 0)$ at the zero moment of time will move along the Ox axis in the direction $x = +\infty$. So at some time moments $t_1 < t_2 < t_3 < t_4 < \ldots$ after the zero moment the amplitude ϕ will be present inside the string, moving from one edge to the other. It can be noticed that the pulse passes through the real string and at a certain time moment t_{fin} (when the pulse existing at the zero moment of time on the length interval $(-2, 0)$ has moved into the length interval $(l, l + 2)$) its action upon the real string ceases. We must point the fact that the limit points $x = 0$ and $x = l$ are not considered to belong to the string; but this is in accordance with the rigorous definition of derivatives (for this limit points can't be defined derivatives as related to any direction around them). The problem that a classical equation (such as the wave-equation) admits acausal solutions (for initial null conditions and without any external forces to exist) can be solved using deterministic methods, such as adding supplementary mathematical requirements to the wave-equation (the principle of least action, for example) or considering a causal chain:

a) external force (free-term) $\Longrightarrow$

$\Longrightarrow$ b) changes in the value of partial derivatives as related to space coordinates

$\Longrightarrow$ c) changes in the partial derivatives of the amplitude as related to time

$\Longrightarrow$ d) changes in the value of the function

so as the possibility of appearing acausal pulses (not yet observed) to be rejected. Such a causal chain can be represented in a mathematical form only as a differential equation able to generate functions similar to test functions, defined as practical test functions.

Another kind of method, based on statistical physics, is also available. Taking into account the fact that at the zero moment of time all derivatives of the amplitude of the real string are equal to zero on the whole length of the string and after a very small time interval, at moment t' close to zero they may become different to zero in a small area inside it, we can consider the zero moment of time as a bifurcation point. At this moment of time there are several branches in the phase-space which satisfy the wave equation (the zero amplitude and the acausal pulse, for example). We consider the hypothesis that the string can choose a branch due to some stochastic jumps of the state-variables (the amplitude and some of its derivatives) around the zero moment of time, in a certain point. This implies that small changes in a small number of state-variables at the zero moment of time, imply a higher probability for that branch of evolution to appear. In the case we have presented, the acausal pulse (the test function) possess an infinite number of derivatives different to zero for any value of the argument for which the function differs to zero. This imply that a jump on this trajectory requires an infinite number of changes in the state-variables (the amplitude and its derivatives) at the edge $x = 0$ of the real string at a time moment t' very close to the zero moment. These changes have a very small module, but they establish in a very short time Δt the shape of the amplitude ϕ on a very small length Δx around the point $x = 0$ (the edge of the real string) along the positive part of the Ox axis (the real string). This nonzero amplitude appearing on length Δx can be considered as *part of an acausal pulse starting to move through the real string* .By noting these state-variables (the amplitude and its derivatives of different order at the point $x = 0$) with $a_0, a_1, \ldots a_k \ldots$ and by noting the state-variables of the acausal pulse at a moment of time t' very close to zero with $b_0, b_1, \ldots b_k \ldots$,we may write the probability of appearing a trajectory representing an acausal pulse as a consequence of such jumps under the form:

$$P_{ac} = P_0 \cap P_1 \cap P_2 \cap \ldots P_k \cap \ldots \tag{6}$$

In the previous equation P_0 is defined as

$$P_0 = P\left(a_0\left(t'\right) = b_0 \mid a_0\left(0\right) = 0\right) \tag{7}$$

and it represents the probability of the state-variable a_0 to become equal to b_0 at the time moment t' close to the zero moment, taking into account the fact that this state-variable was equal to zero at the zero moment of time. P_1 is defined as

$$P_1 = P\left(a_1\left(t'\right) = b_1 \mid a_1\left(0\right) = 0\right) \tag{8}$$

and it represents the probability of the state-variable a_1 to become equal to b_1 at the time moment t' close to the zero moment, taking into account the fact that this state-variable was equal to zero at the zero moment of time,.. P_k is defined as

$$P_k = P\left(a_k\left(t'\right) = b_k \mid a_k\left(0\right) = 0\right) \tag{9}$$

and it represents the probability of a state-variable a_k to become equal to b_k at the time moment t' close to the zero moment, taking into account the fact that this state-variable was equal to zero at the zero moment of time.

Considering possible independent jumps for each state-variable a_k and considering also that each factor P_k appearing in expression of P_{ac} is less than a certain value $m < 1$ (P_k corresponding to a probability) , we may write:

$$P_{ac} = P_0 \cap P_1 \cap P_2 \ldots \cap P_k \ldots \Rightarrow \tag{10}$$

$$P_{ac} = P_0 \cdot P_1 \cdot P_2 \ldots \cdot P_k \ldots \Rightarrow \tag{11}$$

(the probabilities P_k are considered to be independent, so P_{ac} is represented by the product of all P_k)

$$P_{ac} < P_0 \cdot P_1 \cdot P_2 \ldots \cdot P_n \Rightarrow \tag{12}$$

(all factors P_k, with $k > n$, are less than unity, so the right part of the previous equality increases if these factors are removed)

$$P_{ac} < m^n \Rightarrow \tag{13}$$

(because each factor is considered to be less than m, where $m < 1$)

$$P_{ac} \to 0 \; for \; n \to \infty \tag{14}$$

(the number of state variables a_k trends to infinite, and so n can be chosen as great as we want).

So $Pac \to 0$, the probability of appearing an acausal pulse being equal to zero. On the contrary, the probability for the string to keep its initial trajectory (the zero trajectory, which means that no changes in the amplitude appear) is very high, while this implies that the initial state-variables do not vary.

Another statistic method of solving this aspect consist in considering that at each moment the probability of appearing such an acausal pulse is equal to the amplitude of appearing an acausal pulse having the same amplitude, but with an opposite sign. So the resulting amplitude is be equal to zero and no motion appears.

3 Applications for Generating Wavelets

As shown in previous paragraph, acausal pulses similar to test-functions can't be generated by equations with partial derivatives (such as in [4]) such as the wave equation, due to the changes appearing at a certain moment of time, on a very small length Δx , for an infinite number of state-variables. The statistic method presented for justifying the fact that such acausal pulses are not observed implies also the fact that statistic computer methods for generating different functions using differential equations (by varying the initial conditions) are not adequate for test-functions. So for wavelets processing applications, we must use practical test-functions, generated by differential equations of evolution.

A first choice would be the use of a practical test-function for a primary multiplication of the received signal before multiplying this signal with an alternating function (the wavelet); yet this would imply two operations to be performed upon this received signal. This can be avoided if we use the associative property of multiplication. By noting the received signal with f, the wavelet with w and the practical test function with $\varphi(t)$, we can write the results z of both operations (the function which must be integrated) under the form

$$z(t) = w(t)\left[\varphi(t) f(t)\right] \tag{15}$$

or under the form

$$z(t) = [w(t)\varphi(t)](t) = w_{\varphi}(t) f(t) . \quad (16)$$

The function $w_{\varphi}(t)$ can be obtained by multiplying the usual wavelet with a practical test function; it is used further for processing the received signal f (by multiplying and integrating the result on a limited time interval). Due to the fact that the values of the practical test function $\varphi(t)$ and of certain number of its derivatives are also equal to zero at the beginning and the end of the integrating period, the values of the function $w_{\varphi}(t)$ and of a certain number of its derivatives will be also equal to zero at these moments of time. Thus $w_{\varphi}(t)$ possess properties similar to test functions. Moreover, if the usual wavelet $w(t)$ is asymmetrical as related to the middle of the working period, than the function $w_{\varphi}(t)$ is also asymmetrical as related to this moment ($\varphi(t)$ being symmetrical as related to this moment) and the integral of $w_{\varphi}(t)$ on the whole real axis will be equal to zero. By adjusting the magnitude of the practical test function the integral of $[w_{\varphi}(t)]^2$ can be made equal to unity, and thus $w_{\varphi}(t)$ becomes also a wavelet. In this manner wavelets similar to test functions can be generated.

4 Conclusions

This paper has presented the possibility of some acausal pulses to appear as solutions of the wave-equation for a free string considered on the length interval $(0, l)$. Such pulses are in fact extended Dirac functions which can be imagined as coming from outside the string. It is shown that the possibility of appearing such pulses represents in fact a bifurcation in the phase-space of the string. This study tries to apply this concept to stochastic jumps on trajectories determined by test functions (having an infinite number of derivatives different to zero inside a limited open set and equal to zero outside it). Then the utility of using practical test-functions (functions similar to such extended Dirac-functions, which can be generated by a differential equation of evolution) in wavelets analysis is presented.

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