Practical Test-Functions Generated by Computer Algorithms

Ghiocel Toma

Politehnica University, Department of Physics, Bucharest, Romania

Abstract. As it is known, Runge-Kutta methods are widely used for numerical simulations [1]. This paper presents an application of such methods (performed using MATLAB procedures) for generating practical test functions. First it is shown that differential equations can generate only functions similar to test functions (defined as practical test functions); then invariance properties of these practical test functions are used for obtaining a standard form for a differential equation able to generate such a function. Further this standard form is used for computer aided generation of practical test-functions; a heuristic algorithm (based on MATLAB simulations) is used so as to establish the most simple and robust expression for the differential equation. Finally it is shown that we obtain an oscillating system (a system working at the limit of stability, from initial null conditions, on limited time intervals) which can be built as an analog circuit using standard electrical components and amplifiers, in an easy manner.

1 Introduction

Many times the analysis of signals requires an integration on a limited time interval, which can't be performed in a robust manner (with sampling procedures) without using functions similar to test-functions (for multiplying the received signal before the integration, so as the result of the integration to be practically constant at the end of the integration period, at the sampling moment of time). Usually this operation is performed by an integration of the signal on this time interval, using an electric current charging a capacitor - the result of the integration being proportional to the mean value of the signal. However, such structures are very sensitive at random variations of the integration period. Even when devices with higher accuracy are used for establishing this time interval some random variations will appear due to the stochastic switching phenomena - when the electric current charging the capacitor is interrupted. For this reason, a multiplication of the received signal with a test-function - a function which differs to zero only on this time interval and with continuous derivatives of any order on the whole real axis - is recommended. In the ideal case, such a test-function should have a form similar to a rectangular pulse - a unity pulse - considered on this time interval. However, such test functions, similar to the Dirac functions, can't be generated by a differential equation. The existence of such an equation

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of evolution, beginning to act at an initial moment of time, would imply the necessity for a derivative of certain order to make a jump at this initial moment of time from the zero value to a nonzero value. But this aspect is in contradiction with the property of test-functions to have continuous derivatives of any order on the whole real axis, represented in this case by the time axis. So it results that an ideal test-function can't be generated by a differential equation. For this reason, we must restrict our analysis at the possibilities of generating practical test-functions. This practical or truncated test-functions differ to zero only on a certain interval and possess only a finite number of continuous derivatives on the whole real axis. We must find out what properties should be satisfied by a differential equation of evolution, so as starting from certain initial conditions such a practical test-function to be generated.

2 Preliminaries

In previous section has been shown that an ideal test function can't be generated by an equation of evolution (see also [2]). Besides, the problem of generating truncated test functions can't be solved by studying aspects connected with solitary waves [3] or by studying period doubling and chaos generated by thermal instability [4], because we must restrict restrictt restrict our analysis at a certain time interval and we must study only differential equations. So we must study equations of evolution able to generate pulses available for our task - the multiplication with the received signal - so as the average procedure to be insensitive at random variations of the integration period. The function which is integrated must be as possible zero at the end of the integration period; this result can be obtained only when the function which multiplies the received signal is a practical test function, how it has been shown. Finally the advantage of using such practical test function for wavelets processing is presented.

3 Differential Equations Able to Generate Practical Test-Functions

As it is known, a *test-function* on [a, b] is a C^∞ function on $\mathbf{R}$ which is nonzero on (a, b) and zero elsewhere. For example, the bump-like function

$$\varphi(\tau) = \begin{cases} \exp\left(\frac{1}{\tau^2-1}\right) & \text{if} \quad \tau \in (-1,1) \\ 0 \text{ otherwise} \end{cases} \tag{1}$$

is a test-function on $[-1, 1]$.

Definition 1. *An ideal test-function is a test-function that has a graph similar to a rectangular pulse (a unity-pulse) which is 1 on (a, b) and 0 elsewhere.*

For example, the bump-like function

$$\varphi(\tau) = \begin{cases} \exp\left(\frac{0.1}{\tau^2-1}\right) & \text{if} \quad \tau \in (-1,1) \\ 0 \text{ otherwise} \end{cases} \tag{2}$$

is close to being an ideal test-function. Such ideal test functions are recommended for multiplying the received signal.

Definition 2. *A practical test-function on [a, b] is a* C^n *function f on R (for a finite n) such that*
a) f is nonzero on (a, b)
b) f satisfies the boundary conditions $f^{(k)}(a) = f^{(k)}(b) = 0$ *for* $k = 0, 1, ..., n$ *and*
c) f restricted to (a, b) *is the solution of an initial value problem (i.e. an ordinary differential equation on* (a, b) *with initial conditions given at some point in this interval).*

The primary task of artificial intelligence consists in generating practical test-functions by numerical integration, using the expressions of a certain test function and of some of its derivatives. An initial value problem will be established, and the solution will be find using a Runge-Kutta method of order 4 or 5 in MATLAB.

Subroutine 1. Identifying an initial value problem to generate practical test-functions on $[-1, 1]$ begins with considering differential equations satisfied by the bump function φ; the first and second derivatives of φ are obtained (using standard program for derivatives) under the form
Step A

$$\varphi^{(1)}(\tau) = \frac{-2\tau}{(\tau^2-1)^2}\exp\left(\frac{1}{\tau^2-1}\right) \tag{3}$$

$$\varphi^{(2)}(\tau) = \frac{6\tau^4-2}{(\tau^2-1)^4}\exp\left(\frac{1}{\tau^2-1}\right) \tag{4}$$

Step B
A special algorithm tries to obtain a correspondence between the expressions of test function and of its derivatives, by replacing the exponential function. By simply dividing the function $\varphi(\tau)$ at $\varphi^{(1)}(\tau)$ we obtain the correspondence between $\varphi(\tau)$ and $\varphi^{(1)}(\tau)$ under the form

$$\varphi^{(1)} = \frac{-2\tau}{(\tau^2-1)^2}\varphi \tag{5}$$

Then the special algorithm replaces the functions $\varphi(\tau)$ and $\varphi^{(1)}(\tau)$ with functions $f(\tau)$ and $f^{(1)}(\tau)$; as initial conditions, it considers the values of $\varphi(\tau)$ at a moment of time $\tau = -0.99$ (close to the moment of time $\tau = -1$). Thus it results for generating a practical test function f the first order initial value problem

$$f^{(1)} = \frac{-2\tau}{(\tau^2-1)^2}f, \quad f(-0.99) = \varphi(-0.99) \tag{6}$$

The initial condition is roughly 1.5×10^{-22}, which means approximately zero. Numerical integration gives a solution having the form of φ but with a very small amplitude of 10^{-12}. The same way this special algorithm obtains a correspondence between the expressions of $\varphi, \varphi^{(2)}$; this results under the form

$$\varphi^{(2)} = \frac{6\tau^4 - 2}{(\tau^2 - 1)^4}\varphi \tag{7}$$

The replacement of functions $\varphi, \varphi^{(2)}$ with functions $f, f^{(2)}$ and the initial condition under the form $f(-0.99) = \varphi(-0.99),\ f^{(1)}(-0.99) = \varphi^{(1)}(-0.99)$ leads to the second order initial value problem for generating a practical test function f

$$f^{(2)} = \frac{6\tau^4 - 2}{(\tau^2 - 1)^4}f, \quad f(-0.99) = \varphi(-0.99),\ f^{(1)}(-0.99) = \varphi^{(1)}(-0.99) \tag{8}$$

Numerically integration gives a solution similar to φ, but with an amplitude that is only four times greater than that obtained from the first order initial value problem.

Step C

The algorithm analyzes the possibilities of generating a practical test function similar to an ideal unitary pulse. For this purpose, it replaces the bump-like function $\varphi(\tau)$ with the almost ideal test function

$$\varphi_a(\tau) = \begin{cases} \exp\left(\frac{0.1}{\tau^2 - 1}\right) & \text{if} \quad \tau \in (-1, 1) \\ 0 \text{ otherwise} \end{cases} \tag{9}$$

In the same way used for studying possibilities of generating practical test-functions similar to test-function $\varphi(\tau)$, the algorithm analyzes possibilities of generating practical test-functions similar to test-function $\varphi_a(\tau)$. Taking into account the expressions of $\varphi_a, \varphi_a^{(2)}$ (obtained using standard algorithms for derivatives), the correspondence between φ_a and $\varphi_a^{(2)}$ results now under the form

$$\varphi_a^{(2)} = \frac{0.6\tau^4 - 0.36\tau^2 - 0.2}{(\tau^2 - 1)^4}\varphi_a \tag{10}$$

By replacing the functions φ_a and $\varphi_a^{(2)}$ with f and $f^{(2)}$ and considering similar initial conditions $f(-0.99) = \varphi_a(-0.99), f^{(1)}(-0.99) = \varphi_a^{(1)}(-0.99)$ it results the second order initial value problem under the form

$$f^{(2)} = \frac{0.6\tau^4 - 0.36\tau^2 - 0.2}{(\tau^2 - 1)^4}f, f(-0.99) = \varphi_a(-0.99), f^{(1)}(-0.99) = \varphi_a^{(1)}(-0.99) \tag{11}$$

Numerical integration (performed using MATLAB functions) gives a solution that is nearly ideal; its amplitude is close to 1 for more than 2/3 of the interval $[-1, 1]$.

The whole heuristic program will continue by trying to design the most simple differential equation able to generate a practical test function. The algorithm is based on the fact that all practical test-functions numerically generated by the initial value problems considered so far are symmetric about $\tau = 0$, which means that they are invariant under the transformation

$$\tau \to -\tau.$$

It results

Subroutine 2

Step A: Functions invariant under this transformation can be written in the form $f(\tau^2)$ and the second order differential equations generating such functions must necessarily have the form

$$a_2\left(\tau^2\right)\frac{d^2 f}{d\left(\tau^2\right)^2} + a_1\left(\tau^2\right)\frac{df}{d\tau^2} + a_0\left(\tau^2\right) f = 0 \tag{12}$$

Step B: Using standard algorithms, all derivatives presented in the previous relation are replaced by derivatives having the form

$$b_1(\tau)\frac{df}{d\tau}, \quad b_2(\tau)\frac{d^2 f}{d\tau^2}$$

Because

$$\frac{df}{d\tau} = 2\tau\frac{df}{d\left(\tau^2\right)} \quad \text{and} \quad \frac{d^2 f}{d\tau^2} = 4\tau^2\frac{d^2 f}{d\left(\tau^2\right)^2} + 2\frac{df}{d\left(\tau^2\right)} \tag{13}$$

the previous differential equation results now under the form

$$\frac{a_2\left(\tau^2\right)}{4\tau^2}\frac{d^2 f}{d\tau^2} + \left(\frac{a_1\left(\tau^2\right)}{2\tau} - \frac{a_2\left(\tau^2\right)}{4\tau^3}\right)\frac{df}{d\tau} + a_0\left(\tau^2\right) f = 0 \tag{14}$$

Step C: Adding a possible free term in previous differential equation, it results a model for generating a practical test-function using a received signal $u = u(\tau), \tau \in [-1, 1]$, under the form

$$\frac{a_2\left(\tau^2\right)}{4\tau^2}\frac{d^2 f}{d\tau^2} + \left(\frac{a_1\left(\tau^2\right)}{2\tau} - \frac{a_2\left(\tau^2\right)}{4\tau^3}\right)\frac{df}{d\tau} + a_0\left(\tau^2\right) f = u \tag{15}$$

subject to

$$\lim_{\tau \to \pm 1} f^k(\tau) = 0 \text{ for } k = 0, 1, \ldots, n,$$

which are the boundary conditions of a practical test-function. While we are looking at the most simple solutions, the free term u is set by the algorithm to a constant value.

Step D: The coefficient

$$\frac{a_2\left(\tau^2\right)}{4\tau^2}$$

which multiplies

$$\frac{d^2 f}{d\tau^2}$$

is analyzed so as to result a constant expression for it. While the denominator is $4\tau^2$, for the constant set to unity results $a_2 = 4\tau^2$.

Step E: The coefficient

$$\left(\frac{a_1\left(\tau^2\right)}{2\tau} - \frac{a_2\left(\tau^2\right)}{4\tau^3}\right)$$

which multiplies

$$\frac{df}{d\tau}$$

is then analyzed so as to result a constant expression for it. While $a_2 = 4\tau^2$ (from step D), starting an algorithm for polynomial expressions so as to obtain a null coefficient (because we try to obtain the most simple differential equation), it results that $a_1 = 2$. Under these circumstances, the second term in the differential equation vanishes.

Step F: For obtaining the coefficient $a_0\left(\tau^2\right)$ which multiplies the term f in the differential equation the polynomial algorithm tries first to set a_0 to zero. But numerical simulation shows that the response does not satisfy the boundary conditions. So the polynomial algorithm will set the coefficient a_0 to unity, and the differential equations results under the form

$$4\tau^2 \frac{d^2 f}{d\left(\tau^2\right)^2} + 2\frac{df}{d\left(\tau^2\right)} + f = u \tag{16}$$

which converts to

$$\frac{d^2 f}{d\tau^2} + f = u \tag{17}$$

This is an autonomous differential equation; the form being invariant at time translation, so the point $\tau = -1$ is translated to $\tau = 0$. Thus we obtain the differential equation of an oscillating second order system, described by the transfer function

$$H(s) = \frac{1}{\left(T_0\right)^2 s^2 + 1} \tag{18}$$

where $T_0 = 1$ in our case.

Subroutine 3. We continue our study by analyzing the behavior of oscillating systems in the most general case (when $T_0 \neq 1$). This implies a differential equation under the form

$$\left(T_0\right)^2 \frac{d^2 f}{d\tau^2} + f = u \tag{19}$$

Step A. The behavior of this oscillating system can be obtained using standard algorithms (being a linear second order system). When u is represented by a constant, the output of the system consists of oscillations around a constant value. Analyzing the coefficients of the linear differential equation obtained, the existence of the oscillating linear system can be easy noticed, and the working interval can be set to the time interval corresponding to an oscillation, this means the time interval $[0, 2\pi T_0]$. When $u = 1$, the model generates the practical test-function

$$f(\tau) = 1 - \cos(\tau / T_0) \tag{20}$$

Step B. The algorithm generates the output of the oscillating system under the influence of a continuous useful signal u_1 (supposed to be constant) with an alternating noise u_2 of angular frequency ω added. Using the property of linearity, the equations

$$a_2\left(\tau^2\right)\frac{d^2 f_1}{d\left(\tau^2\right)^2} + a_1\left(\tau^2\right)\frac{df_1}{d\tau^2} + a_0\left(\tau^2\right) f_1 = u_1 \tag{21}$$

and

$$a_2\left(\tau^2\right)\frac{d^2 f_2}{d\left(\tau^2\right)^2} + a_1\left(\tau^2\right)\frac{df_2}{d\tau^2} + a_0\left(\tau^2\right) f = u_2 \tag{22}$$

imply that $f = f_1 + f_2$ is a solution of

$$a_2\left(\tau^2\right)\frac{d^2 f}{d\left(\tau^2\right)^2} + a_1\left(\tau^2\right)\frac{df}{d\tau^2} + a_0\left(\tau^2\right) f = u_1 + u_2 \tag{23}$$

(this aspect can be noticed by an algorithm by identifying constant values for the coefficients of the differential equation used). This reduces the study of the model when the input u is a mix of continuous useful signal and noise (an alternating input for example) to two cases: that of a continuous useful signal, and that of the noise. Then we can add the results to obtain the output when the noise overlaps the useful signal.

Step C. By checking the performances of this system under the influence of an external constant input $u = 1$, an averaging procedure on the working time interval $[0, 2\pi T_0]$ shows that it recovers the mean value of the useful signal $u = 1$ over this interval:

$$\frac{1}{2\pi T_0}\int_0^{2\pi T_0} \left(1 - \cos(\tau / T_0)\right) d\tau = 1 \tag{24}$$

The human user can also notice that the integration of this practical test-function on $[0, 2\pi T_0]$ is practically insensitive to the switching phenomena appearing at the sampling moment of time $2\pi T_0$ because

$$f(2\pi T_0) = 0 \text{ and } f^{(1)}(2\pi T_0) = 0 \tag{25}$$

Step D. By the other hand, an analysis of the oscillating system for an alternating input of $u = \sin \omega\tau$ with frequency $\omega > \sqrt{2}/T_0$ shows that the system attenuates this input:

$$\left|\frac{\text{input}}{\text{output}}\right| = (T_0\omega)^2 - 1 > 1 \tag{26}$$

The human observer can notice that oscillations of the form

$$a \sin\left(t/T_0\right) + b \cos\left(t/T_0\right) \tag{27}$$

generated by the alternating component of the input account for all the solutions of the associated homogeneous system

$$\left(T_0\right)^2 \frac{d^2 f}{d\tau^2} + f = 0 \tag{28}$$

These oscillations give null result by an integration over the working interval $[0, 2\pi T_0]$.

Final Conclusion. By comparing the fact that the mean value of the received useful signal (supposed to be constant) can be obtained using a linear differential equation where the external signal appears as a free term (this being a task performed by artificial intelligence), the human user can replace the use of a practical test function in a multiplying procedure (when it multiplies the received signal) with an use of a practical test function represented by a linear oscillating system in a generating procedure (the external signal representing the free term in the differential equation corresponding to the oscillating system).

As a conclusion, the simplest model for generating practical test-functions on $[0, 2\pi T_0]$ when the continuous signal is $u = 1$ designed using computer generated practical test functions based on MATLAB procedures and standard algorithms and verified using the same MATLAB procedures consists of the second order oscillating system

$$\left(T_0\right)^2 \frac{d^2 f}{d\tau^2} + f = u \tag{29}$$

over the interval $[0, 2\pi T_0]$, in which test-functions are subject to the boundary conditions

$$f(0) = 0, \quad f\left(2\pi T_0\right) = 0 \tag{30}$$

these implying also

$$f^{(1)}\left(2\pi T_0\right) = 0, \quad f^{(1)}\left(2\pi T_0\right) = 0 \tag{31}$$

4 Conclusions

This paper has presented a heuristic algorithm for generating practical test functions using MATLAB procedures. First it has been shown that ideal test functions can't be generated by differential equations, being underlined the fact that differential equations can generate only functions similar to test functions (defined as practical test functions). Then a step by step algorithm for designing the most simple differential equation able to generate a practical test function is presented, base on the invariance properties of the differential equation and on standard MATLAB procedures. The result of this algorithm is represented by a system working at the stability limit from initial null conditions, on limited time intervals, the external signal representing the free term in the differential equation corresponding to the oscillating system. Such a system can be built using standard components and operational amplifiers, in an easy manner.

References

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