Filtering Aspects of Practical Test-Functions and the Ergodic Hypothesis

Flavia Doboga[1], Ghiocel Toma[2], Stefan Pusca[2], Mihaela Ghelmez[2], and Cristian Moraraescu[3]

[1] ITT Industries, Washington, U.S.A.
[2] Politehnica University, Department of Physics, Bucharest, Romania
[3] Poliethnica University, Department of Computers, Bucharest, Romania

Abstract. This papers presents properties of dynamical systems able to generate practical test-functions (defined as functions which differ to zero on a certain interval and possess only a finite number of continuous derivatives on the whole real axis) when the free-term of the differential equation (corresponding to the received input signal) is represented by alternating functions. The shape of the output signal (obtained by numerical simulations in Matlab based on Runge-Kutta functions) is analyzed, being shown that for high-frequency inputs an external observer could notice (in certain condition) the generation of two different pulses corresponding to two distinct envelopes. Such as aspect differs to the oscillations of unstable type second order systems studied using difference equations.

1 Introduction

In the ideal mathematical case, suddenly emerging pulses should be simulated using test-functions (functions which differ to zero only on a limited time interval and possessing an infinite number of continuous derivatives on the whole real axis. However, such test functions, similar to the Dirac functions, can't be generated by a differential equation. The existence of such an equation of evolution, beginning to act at an initial moment of time, would imply the necessity for a derivative of certain order to make a jump at this initial moment of time from the zero value to a nonzero value. But this aspect is in contradiction with the property of test-functions to have continuous derivatives of any order on the whole real axis, represented in this case by the time axis. So it results that an ideal test-function can't be generated by a differential equation. For this reason, the analysis must be restricted at practical test-functions [1], defined as functions which differ to zero on a certain interval and possess only a finite number of continuous derivatives on the whole real axis. Mathematical methods based on difference equations are well known [2], but for a higher accuracy of the computer simulation we shall use Runge-Kutta methods in Matlab. The properties of dynamical systems able to generate such practical test-functions will be studied, for the case when the free-term of the differential equation (corresponding to the

O. Gervasi et al. (Eds.): ICCSA 2005, LNCS 3482, pp. 563–568, 2005.

received input signal) is represented by alternating functions. The shape of the output signal (obtained by numerical simulations in Matlab based on Runge-Kutta functions) will be analyzed, being shown that for high-frequency inputs an external observer could notice (in certain condition) the generation of two different pulses corresponding to two distinct envelopes. Such as aspect differs to the oscillations of unstable type second order systems studied using difference equations [3].

2 Equations Suitable for Generating Symmetrical Pulses

As it is known, a test-function on $[a, b]$ is a function which is nonzero on this interval and which possess an infinite number of continuous derivatives on the whole real axis. For example, the function

$$\varphi(\tau) = \begin{cases} \exp\left(\frac{1}{\tau^2-1}\right) & \text{if} \quad \tau \in (-1,1) \\ 0 \text{ otherwise} \end{cases}$$

is a test-function on $[-1, 1]$. If the graph of the test-function is similar to the rectangular pulse (a unity-pulse), it is considered to be an ideal test-function. An example is the case of the function

$$\varphi(\tau) = \begin{cases} \exp\left(\frac{0.1}{\tau^2-1}\right) & \text{if} \quad \tau \in (-1,1) \\ 0 \text{ otherwise} \end{cases}$$

is close to being an ideal test-function.

Using the expression of $\varphi(\tau)$ and of its derivatives of first and second order, a differential equation which admits as solution the function φ can be obtained. However, a test-function can't be the solution of a differential equation. Such an equation of evolution implies a jump at the initial moment of time for a derivative of certain order, and test-function must possess continuous derivatives of any order on the whole real axis. So it results that a differential equation which admits a test-function φ as solution can generate only a practical test-function f similar to φ, but having a finite number of continuous derivatives on the whole real axis. In order to do this, we must add initial conditions for the function f (generated by the differential equations) and for some of its derivatives $f^{(1)}$, and/or $f^{(2)}$ etc. equal to the values of the test-function φ and of some of its derivatives $\varphi^{(1)}$, and/or $\varphi^{(2)}$ etc. at an initial moment of time t_{in} very close to the beginning of the working interval. This can be written under the form

$$f_{t_{in}} = \varphi_{t_{in}}, \;\; f_{t_{in}}^{(1)} = \varphi_{t_{in}}^{(1)} \text{ and/or } f_{t_{in}}^{(2)} = \varphi_{t_{in}}^{(2)} \text{ etc.} \tag{1}$$

If we want to generate practical test-functions f which are symmetrical as related to the middle of the working interval, we can choose as origin the middle of this interval, and so it results that the function f should be invariant under the transformation

$$\tau \rightarrow -\tau$$

Functions invariant under this transformation can be written in the form $f(\tau^2)$, and so the form of a general second order differential equation generating such functions must be

$$a_2\left(\tau^2\right)\frac{d^2 f}{d\left(\tau^2\right)^2}+a_1\left(\tau^2\right)\frac{df}{d\tau^2}+a_0\left(\tau^2\right)f=0 \tag{2}$$

However, for studying the filtering properties of practical test-functions we must add a free-term, corresponding to the received signal (the input of the system). Thus, a model for generating a practical test-function using a received signal $u=u(\tau),\ \tau\in[-1,1]$, is

$$a_2\left(\tau^2\right)\frac{d^2 f}{d\left(\tau^2\right)^2}+a_1\left(\tau^2\right)\frac{df}{d\tau^2}+a_0\left(\tau^2\right)f=u \tag{3}$$

subject to

$$\lim_{\tau\to\pm1} f^k(\tau)=0 \text{ for } k=0,1,\ldots,n. \tag{4}$$

which are the boundary conditions of a practical test-function.

The previous equation is linear as related to the input function u. So we can study independently the output of the system for an input represented by an alternating signal and for an input represented by a constant signal (a step-input). The two outputs signals can be joined together for obtaining the output signal for the case when the input is represented by a mix of a constant and of an alternating function.

3 Filtering Aspects of Practical Test-Functions

When coefficients a_k in (3) are set to

$$a_2=0, a_1=1 \text{ and } a_0=-1, \tag{5}$$

a first order system is obtained under the form

$$\frac{df}{d\left(\tau^2\right)}=f+u \tag{6}$$

which converts to

$$\frac{df}{d\tau}=2\tau f+2\tau u \tag{7}$$

representing a damped first order dynamical system. For the an alternating input $u=\sin 10\tau$, numerical simulations performed using Runge-Kutta functions in Matlab show an attenuation of about $A=3$. In figure 1 is represented the output f of this system for $u=\sin(10\tau)$, and in figure 2 is represented the output f of this system for $u=\cos(10\tau)$. It can be noticed that the mean value of the output oscillations generated in these circumstances is a function of the phase of the input signal. Similar aspects have been noticed for an alternating input

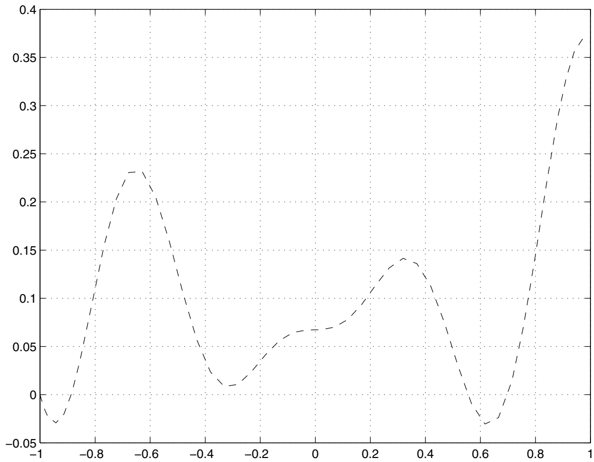

Fig. 1. f versus time for first order damped system, input $u = \sin(10\tau)$

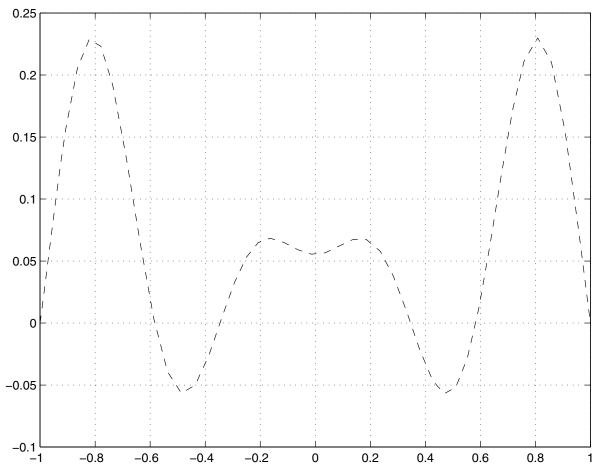

Fig. 2. f versus time for first order damped system, input $u = \cos(10\tau)$

of the form $u = \sin(100\tau)$ the output f of the dynamical system for this case being represented in figure 3, or of the form $u = \cos(100\tau)$ the output f of the dynamical system for this case being represented in figure 4.

When $a_2 = 0, a_1 = 1$ and $a_0 = 0$, another first order model is

$$\frac{df}{d\left(\tau^2\right)} = u \tag{8}$$

which converts to

$$\frac{df}{d\tau} = 2\tau u \tag{9}$$

representing an undamped dynamical system. The outputs f of this undamped system are quite similar to the outputs of the previous damped system , for the same inputs u (the differences are less than 15 %).

4 Connection with the Ergodic Hypothesis

Studying graphics presented in figure 3 and figure 4, we can notice the presence of two distinct envelopes. Their shape depends on the phase of the input alternating component. At first sight, an external observer could notice two different pulses generated by the dynamical system (each one corresponding to an envelope). In a more rigorous manner, we can consider that that at a certain moment of time can be detected, with equal probability, one of the two branches of evolution (corresponding to certain intervals around the two envelopes). Thus the mean value of the output f on a small time interval can be considered not as a mean value in time, but also as a mean value for two distinct internal states of the system which *exist together on this time interval*. This is an aspect similar to the ergodic hypothesis used in thermodynamics. By replacing the sinusoidal alternating input u with rectangular alternating functions, the existence of two different branches of evolution would become more obvious (the transition time from one branch to the other would become very short, and so the probability of measuring values different to the two envelopes for the output f .would decrease). This aspect can be put in correspondence with aspects in quantum mechanics, were distinct states can be measured at a certain moment of time, for certain external interactions. The statistical aspects of measuring different values can be studied using the dependence of the envelopes on the phase of the input alternating component.

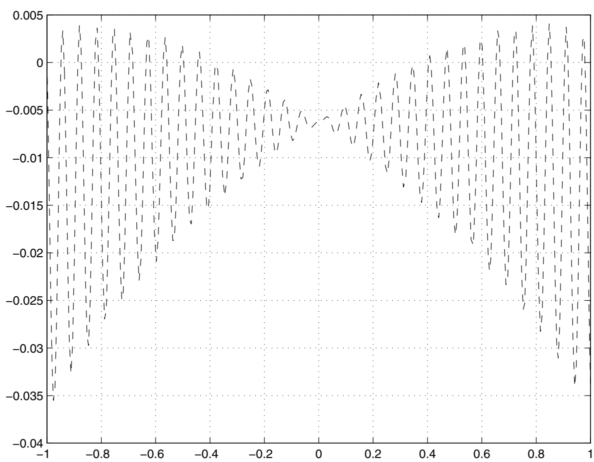

Fig. 3. f versus time for first order damped system, input $u = \sin(100\tau)$

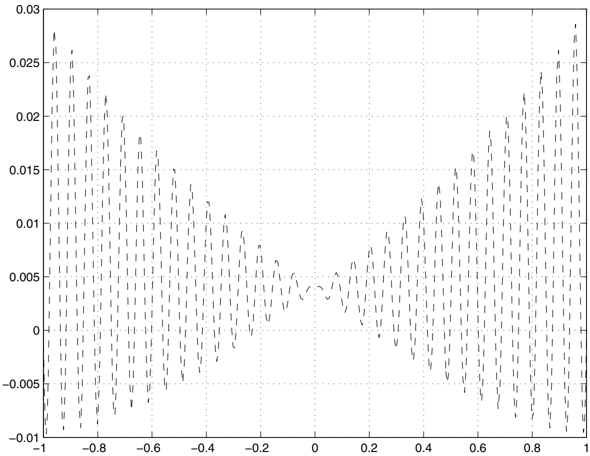

Fig. 4. f versus time for first order damped system, input $u = \cos(100\tau)$

5 Conclusions

This study has presented filtering properties of practical test-functions, the input being represented by alternating sinusoidal functions. The shape of the output signal (obtained by numerical simulations in Matlab based on Runge-Kutta functions) has been analyzed, being shown that for high-frequency inputs an external observer could notice (in certain condition) the generation of two different pulses corresponding to two distinct envelopes. This aspect has been put in correspondence with aspects in quantum mechanics, were distinct states can be measured at a certain moment of time, for certain external interactions. The statistical aspects of measuring different values can be studied using the dependence of the envelopes on the phase of the input alternating component.

References

1. Toma, C. : An extension of the notion of observability at filtering and sampling devices, Proceedings of the International Symposium on Signals, Circuits and Systems Iasi SCS 2001, Romania 233–236
2. Dzurina, J. : Oscillation of second order differential equations with advanced argument, Math. Slovaca, **45** 3 (1995) 263–268
3. Zhang, Zh., Ping, B., Dong, W. : Oscillation of unstable type second order nonlinear difference equation, Korean J. Computer and Appl. Math. **9** 1 (2002) 87–99