On the Equivalence of Algebraic Approaches to the Minimization of Forms on the Simplex

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We consider the problem of minimizing a form on the standard simplex [equivalently, the problem of minimizing an even form on the unit sphere]. Two converging hierarchies of approximations for this problem can be constructed, that are based, respectively, on results by Schmüdgen-Putinar and by Pólya about representations of positive polynomials in terms of sums of squares. We show that the two approaches yield, in fact, the same approximations.

1 Introduction

1.1 Representations of positive forms on the simplex

We consider the problem of minimizing a form (i.e., homogeneous polynomial) p of degree d on the standard simplex; that is, the problem of computing

$$p_{\min} := \min \; p(x) \;\; \text{s.t.} \; x \in \Delta := \left\{ x \in \mathbb{R}^n_+ \mid \sum_{i=1}^n x_i = 1 \right\}. \tag{1}$$

The polynomial

$$\tilde{p}(x) := p(x_1^2, \ldots, x_n^2)$$

is an even form of degree $2d$ and problem (1) can be reformulated as the problem of minimizing $\tilde{p}$ on the unit sphere:

$$p_{\min} = \min \; \tilde{p}(x) \;\; \text{s.t.} \; x \in S := \left\{ x \in \mathbb{R}^n \mid \sum_{i=1}^n x_i^2 = 1 \right\}. \tag{2}$$

Equivalently, this is the problem of finding the maximum scalar t for which

$$\tilde{p}(x) - t \geq 0 \; \forall x \in S; \; \text{equivalently,} \; \tilde{p}(x) - t\|x\|^{2d} \geq 0 \; \forall x \in \mathbb{R}^n. \tag{3}$$

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Here, $\|x\|^2 = \sum_{i=1}^n x_i^2$. Hence, lower bounds for the optimum value can be obtained by replacing the condition (3) by some stronger conditions. Instances of such stronger conditions are given below, for any integer $r \geq 0$:

$$\left(\tilde{p}(x) - t\|x\|^{2d}\right)\|x\|^{2r} \in \mathbb{R}^+_{ev}[X] \tag{4}$$

$$(\tilde{p}(x) - t\|x\|^{2d})\|x\|^{2r} \in \Sigma^2 \tag{5}$$

$$\tilde{p}(x) - t \in \mathbb{R}^+_{ev,2(r+d)}[X] + (1 - \|x\|^2)\mathbb{R}[X] \tag{6}$$

$$\tilde{p}(x) - t \in \Sigma^2_{2(r+d)} + (1 - \|x\|^2)\mathbb{R}[X] \tag{7}$$

Here, $\mathbb{R}[X]$ denotes the set of polynomials in the n variables $x_1, .., x_n$, $\mathbb{R}^+_{ev}[X]$ is the set of even polynomials with nonnegative coefficients, Σ^2 is the set of polynomials that are sums of squares, and a subscript $2(r+d)$ indicates the bound $2(r+d)$ on the degree. (See section 1.2 for definitions and notation.)

Note that, in (4), one could replace $\mathbb{R}^+_{ev}[X]$ by $\mathbb{R}^+[X]$, since the polynomial is even by construction.

Condition (4) can be equivalently reformulated in terms of the initial polynomial p as

$$\left(p(x) - t\left(\sum_{i=1}^n x_i\right)^d\right)\left(\sum_{i=1}^n x_i\right)^r \in \mathbb{R}^+[X]. \tag{8}$$

One can also reformulate condition (5) in terms of the original polynomial p, using the following result of Zuluaga et al. [16].

Proposition 1 (Zualaga et al. [16]). *Let p be a form of degree d and $\tilde{p}(x) := p(x_1^2, \dots, x_n^2)$ the associated even form. Then,*

$$\tilde{p} \in \Sigma^2 \Longleftrightarrow p(x) = \sum_{\substack{I \subseteq \{1,\dots,n\} \\ |I| \equiv d \bmod 2}} \left(\prod_{i \in I} x_i\right) p_I, \quad \text{where } p_I \in \Sigma^2$$
$$\text{and } p_I \text{ is a form of degree } d - |I|$$

The following implications obviously hold:

$$(4) \Longrightarrow (5) \Longrightarrow (3), \quad (6) \Longrightarrow (7) \Longrightarrow (3).$$

Each of the conditions (4)-(7) permits to formulate a hierarchy of lower bounds for $p_{\min}$ depending on r. For instance, the (linear) bound:

$$p_L^{(r)} := \max\ t \text{ s.t. (4) (or (8)) holds,} \tag{9}$$

and the (semidefinite) bound:

$$p^{(r)} := \max\ t \text{ s.t. (5) holds.} \tag{10}$$

Obviously,

$$p_L^{(r)} \le p_L^{(r+1)},\ p^{(r)} \le p^{(r+1)},\ p_L^{(r)} \le p^{(r)} \le p_{\min}. \tag{11}$$

Asymptotic convergence of the bounds $p_L^{(r)}$ to $p_{\min}$ as r goes to infinity, follows from the following theorem of Pólya about representations of positive forms on the simplex. .

Theorem 1 (Pólya [10]). *Let p be a form which is positive on the standard simplex $\Delta = \{x \in \mathbb{R}^n \mid \sum_i x_i = 1\}$. Then there exists an $r \in \mathbb{N}$ such that*

$$p(x)\left(\sum_{i-1}^{n} x_i\right)^r \in \mathbb{R}^+[X].$$

Two other hierarchies of lower bounds can be defined analogously, using (6) and (7), and they satisfy the analogue of (11). Their asymptotic convergence to $p_{\min}$ follows from the following theorem of Schmüdgen (or its refinement by Putinar) about representations of positive polynomials on compact semi-algebraic sets.

Theorem 2. *Let F be a semi-algebraic set of the form:*

$$F = \{x \in \mathbb{R}^n \mid p_1(x) \ge 0, \dots, p_k(x) \ge 0\},\ \textit{where } p_1, \dots, p_k \in \mathbb{R}[X].$$

(i) **(Schmüdgen [15])** *If F is compact, then every polynomial which is positive on F belongs to* $\sum_{I \subseteq \{1,\dots,k\}} \left(\prod_{i \in I} p_i\right) \Sigma^2$.

(ii) **(Putinar [13])** *Assume that F is compact and that there exists a polynomial $p_0 \in \Sigma^2 + p_1\Sigma^2 + \ldots + p_k\Sigma^2$ for which the set $\{x \mid p_0(x) \ge 0\}$ is compact. Then every polynomial which is positive on F belongs to $\Sigma^2 + p_1\Sigma^2 + \ldots + p_k\Sigma^2$.*

Corollary 1. *Every polynomial which is positive on the unit sphere belongs to $\Sigma^2 + (1 - \sum_{i=1}^n x_i^2)\mathbb{R}[X]$.*

This idea of constructing hierarchies of bounds for optimization over semi-algebraic sets, based on real algebraic results about representations of positive polynomials, has been explored by several authors.

In particular, Pólya's result led Parrilo [8, 9] to introduce hierarchies of conic relaxations for the cone of copositive matrices. These relaxations were used by De Klerk and Pasechnik [6] for approximating the stable set problem in graphs, and by Bomze and De Klerk [1] for constructing a PTAS for the minimization of degree 2 forms on the simplex. Hierarchies of conic relaxations were introduced, more generally, for the cone of positive semidefinite forms, in particular, by Faybusovich [2] (who also gives estimations on the quality of the approximations) and by Zuluaga et al. [16]. These relaxations have been used in the recent paper by De Klerk, Laurent and Parrilo [5] for giving a PTAS for the minimization of a form of degree d on the simplex.

On the other hand, Putinar's result led Lasserre [7] to define converging hierarchies of semidefinite bounds for the approximation of polynomials on (special) compact semi-algebraic sets.

The main contribution of this paper is to show that these two approaches, based on Pólya's and Schmüdgen-Putinar's theorems, are in fact equivalent, when applied to the problem of minimizing a form on the standard simplex (or, equivalently, minimizing an even form on the unit sphere). More precisely, we prove the following result in Section 2, showing that the assertions (4) and (6) (resp., (5) and (7)) are equivalent.

Theorem 3. *Let p be a form of degree d and let $\tilde{p}(x) := p(x_1^2, \ldots, x_n^2)$ be the associated even form of degree $2d$. For every integer $r \geq 0$, consider the linear bound $p_L^{(r)}$ (defined by (9)) and the semidefinite bound $p^{(r)}$ (defined by (10)) for the minimum value $p_{\min}$ of p over the standard simplex. Then,*

$$p_L^{(r)} \leq p^{(r)} \leq p_{\min},$$

$$\begin{aligned} p_L^{(r)} &= \max\ t \quad s.t. \quad \left(\tilde{p}(x) - t\left(\sum_{i=1}^n x_i^2\right)^d\right)\left(\sum_{i=1}^n x_i^2\right)^r \in \mathbb{R}^+[X] \\ &= \max\ t \quad s.t.\ \tilde{p}(x) - t \in \mathbb{R}^+_{ev,2(r+d)}[X] + \left(1 - \sum_{i=1}^n x_i^2\right)\mathbb{R}[X], \end{aligned} \tag{12}$$

$$\begin{aligned} p^{(r)} &= \max\ t \quad s.t. \quad \left(\tilde{p}(x) - t\left(\sum_{i=1}^n x_i^2\right)^d\right)\left(\sum_{i=1}^n x_i^2\right)^r \in \Sigma^2 \\ &= \max\ t \quad s.t.\ \tilde{p}(x) - t \in \Sigma^2_{2(r+d)} + \left(1 - \sum_{i=1}^n x_i^2\right)\mathbb{R}[X]. \end{aligned} \tag{13}$$

We conclude with a 'negative result' in Section 3, concerning representations of polynomials positive on the unit sphere, namely

$$q \in \Sigma^2 + \left(1 - \sum_{i=1}^n x_i^2\right)\Sigma^2 \Longrightarrow q \in \Sigma^2.$$

Compare this to the representation $p \in \Sigma^2 + (1 - \sum_{i=1}^n x_i^2)\mathbb{R}[X]$ in Corollary 1 that holds for any p positive on the unit sphere.

1.2 Notation

The following notation will be used throughout the paper.

$\mathbb{R}[x_1, \ldots, x_n]$, also abbreviated as $\mathbb{R}[X]$, is the set of polynomials in n variables. Write $p \in \mathbb{R}[X]$ as $\sum_{\alpha \in \mathbb{N}^n} p_\alpha x^\alpha$, where $x^\alpha := x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. Then, $p_\alpha x^\alpha$ is a *term* of p if $p_\alpha \neq 0$; $|\alpha| := \sum_{i=1}^n \alpha_i$ is the *degree* of the term $p_\alpha x^\alpha$, and the *degree* of p is the maximum degree of its terms. A polynomial p is a *form* if all its terms have the same degree; p is an *even* polynomial if $\alpha_1, \ldots, \alpha_n$ are even for every term $p_\alpha x^\alpha$ of p.

$\mathbb{R}_d[X]$ is the set of polynomials with degree $\leq d$; $\mathbb{R}^+[X]$ is the set of polynomials with nonnegative coefficients: $p = \sum_\alpha p_\alpha x^\alpha$ with $p_\alpha \geq 0$ for all α; $\mathbb{R}_{ev}[X]$ is the set of even polynomials: $p = \sum_\alpha p_\alpha x^{2\alpha}$. Moreover, $\mathbb{R}_d^+[X] := \mathbb{R}^+[X] \cap \mathbb{R}_d[X]$, $\mathbb{R}_{ev}^+[X] := \mathbb{R}^+[X] \cap \mathbb{R}_{ev}[X]$, $\mathbb{R}_{ev,d}^+[X] := \mathbb{R}_{ev}^+[X] \cap \mathbb{R}_d[X]$.

Σ^2 is the set of polynomials that can be written as a sum of squares of polynomials: $p = \sum_\ell f_\ell^2$ for some $f_\ell \in \mathbb{R}[X]$, and $\Sigma_d^2 := \Sigma^2 \cap \mathbb{R}_d[X]$. Obviously, $\mathbb{R}_{ev}^+[X] \subseteq \Sigma^2$.

2 Pólya's and Putinar's Theorems Give the Same Bounds for Optimization on the Simplex

We prove here a slightly more general version of Theorem 3, which holds for forms of even degree. We begin with some preliminary results.

Proposition 2. *Let q be a form of even degree $2d \geq 2$. The following assertions are equivalent:*

$$q(x) \left(\sum_{i=1}^n x_i^2 \right)^r \in \mathcal{P} \tag{14}$$

$$q \in \mathcal{P} + \left(1 - \|x\|^2\right) \mathbb{R}[X] \tag{15}$$

where $\mathcal{P}$ stands for $\mathbb{R}_{ev,2(r+d)}^+[X]$ or $\Sigma_{2(r+d)}^2$.

Proof. Suppose first that (14) holds. Then, the polynomial

$$f(x) := q(x) \left(\sum_{i=1}^n x_i^2 \right)^r$$

belongs to $\mathcal{P}$ and

$$f(x) = q(x) \left(1 - 1 + \sum_{i=1}^n x_i^2 \right)^r = q(x) + \sum_{s=1}^r \binom{r}{s} q(x) \left(\sum_{i=1}^n x_i^2 - 1 \right)^s,$$

which implies that

$$q(x) = f(x) + \left(1 - \|x\|^2\right) \sum_{s=1}^{r} \binom{r}{s} q(x) \left(\sum_{i=1}^{n} x_i^2 - 1\right)^{s-1}$$

and, thus, (15) holds.

Suppose now that (15) holds; that is,

$$q(x) = s(x) + (1 - \|x\|^2)r(x)$$

where $s \in \mathcal{P}$ and $r \in \mathbb{R}[X]$. Then, $q\left(\frac{x}{\|x\|}\right) = s\left(\frac{x}{\|x\|}\right)$ and, thus,

$$q(x)\|x\|^{2r} = s\left(\frac{x}{\|x\|}\right)\|x\|^{2(r+d)} \quad \text{for all } x \in \mathbb{R}^n \setminus \{0\}. \tag{16}$$

In what follows, we show that

$$f(x) := s\left(\frac{x}{\|x\|}\right)\|x\|^{2(r+d)}$$

is a polynomial belonging to $\mathcal{P}$. This implies that the polynomial $q(x)\|x\|^{2r}$ coincides with $f(x)$ (by continuity) and, thus, belongs to $\mathcal{P}$, which shows that (14) holds.

Suppose first that $\mathcal{P} = \mathbb{R}^+_{ev,2(r+d)}[X]$. Then, $s(x) = \sum_{|\alpha| \le r+d} s_\alpha x^{2\alpha}$, with all $s_\alpha \ge 0$. Therefore, $f(x) = \sum_{|\alpha| \le r+d} s_\alpha x^{2\alpha} \|x\|^{2(r+d-|\alpha|)}$, which is an even polynomial with nonnegative coefficients and, thus, belongs to $\mathcal{P}$.

Suppose now that $\mathcal{P} = \Sigma^2_{2(r+d)}$. We begin with observing that one can assume that each term of s has an even degree. To see it, write $s = s_0 + s_1$, where each term of s_0 (resp., of s_1) has even (resp., odd) degree. Then, $s_0(-x) = s_0(x)$ and $s_1(-x) = -s_1(x)$ for all x. As q is a form of even degree, $q(-x) = q(x)$ for all x. In view of (16), this implies that $s(-x) = s(x)$ for all x with $\|x\| = 1$. Therefore, $s_1(x) = 0$ and, thus, $s(x) = s_0(x)$ for all x with $\|x\| = 1$. Hence, one can replace s by s_0 in the definition of f.

As $s \in \Sigma^2_{2(r+d)}$, write

$$s = \sum_\ell (s_\ell)^2, \;\; s_\ell = u_\ell + v_\ell$$

where s_ℓ are polynomials of degree $\le r + d$, u_ℓ consists of the terms of s_ℓ whose degree has the same parity as $r + d$, and $v_\ell := s_\ell - u_\ell$. Thus,

$$s = \sum_\ell (u_\ell)^2 + (v_\ell)^2 + 2\sum_\ell u_\ell v_\ell.$$

As each term of s, $(u_\ell)^2$, and $(v_\ell)^2$ has even degree, while each term of $u_\ell v_\ell$ has odd degree, we deduce that $\sum_\ell u_\ell v_\ell = 0$, implying that $s = \sum_\ell (u_\ell)^2 + (v_\ell)^2$. Therefore,

$$f(x) = s\left(\frac{x}{\|x\|}\right)\|x\|^{2(r+d)} = \sum_{\ell}\left(u_\ell\left(\frac{x}{\|x\|}\right)\|x\|^{r+d}\right)^2 + \left(v_\ell\left(\frac{x}{\|x\|}\right)\|x\|^{r+d}\right)^2.$$

Observe now that $u_\ell\left(\frac{x}{\|x\|}\right)\|x\|^{r+d} = \varphi_\ell(x)$ and $v_\ell\left(\frac{x}{\|x\|}\right)\|x\|^{r+d} = \|x\|\psi_\ell(x)$ where φ_ℓ and ψ_ℓ are polynomials in x. Indeed, say, $u_\ell(x) = \sum_\alpha u_{\ell,\alpha}x^\alpha$. Then, $u_\ell\left(\frac{x}{\|x\|}\right)\|x\|^{r+d}$ is equal to $\sum_\alpha u_{\ell,\alpha}x^\alpha\|x\|^{r+d-|\alpha|}$, which is a polynomial in x since all $r+d-|\alpha|$ are even integers. Analogously for v_ℓ. This shows that

$$f(x) = \sum_\ell \varphi_\ell(x)^2 + \sum_\ell \psi_\ell(x)^2(\sum_{i=1}^n x_i^2)$$

belongs to $\mathcal{P}$, thus concluding the proof. ∎

Lemma 1. *Let q be a form of even degree $2d$ and let t be a real number. The following assertions are equivalent:*

$$q(x) - t\|x\|^{2d} \in \mathcal{P} + \left(1 - \sum_{i=1}^n x_i^2\right)\mathbb{R}[X] \tag{17}$$

$$q(x) - t \in \mathcal{P} + \left(1 - \sum_{i=1}^n x_i^2\right)\mathbb{R}[X], \tag{18}$$

where $\mathcal{P}$ stands for $\mathbb{R}^+_{ev,2(r+d)}[X]$ or $\Sigma^2_{2(r+d)}$.

Proof. If (17) holds, then $q(x) - t\|x\|^{2d} = s + \left(1 - \sum_{i=1}^n x_i^2\right)r$, where $s \in \mathcal{P}$ and $r \in \mathbb{R}[X]$. Therefore, $q(x) - t = s + \left(1 - \sum_{i=1}^n x_i^2\right)r + t\left(\|x\|^{2d} - 1\right)$. Now, $\|x\|^{2d} - 1 = \left(\sum_{i=1}^n x_i^2\right)^d - 1 = (\sum_{i=1}^n x_i^2 - 1)u(x)$, for some polynomial u. Therefore, (18) holds.

Conversely, if (18) holds, then $q(x) - t = s + \left(1 - \sum_{i=1}^n x_i^2\right)r$, where $s \in \mathcal{P}$ and $r \in \mathbb{R}[X]$. Then, $q(x) - t\|x\|^{2d} = s + \left(1 - \sum_{i=1}^n x_i^2\right)r - t\left(\|x\|^{2d} - 1\right)$ and, thus, (17) holds. ∎

Theorem 4. *Let q be a form of even degree $2d$, $q_{\min}$ the minimum of $q(x)$ over the unit sphere, and $r \geq 0$ an integer. Then,*

$$q_L^{(r)} \leq q^{(r)} \leq q_{\min}, \quad \textit{where}$$

$$\begin{aligned} q_L^{(r)} &:= \max\ t\ \ s.t.\ \left(q(x) - t\left(\sum_{i=1}^n x_i^2\right)^d\right)\left(\sum_{i=1}^n x_i^2\right)^r \in \mathbb{R}^+_{ev}[X] \\ &= \max\ t\ \ s.t.\ q(x) - t \in \mathbb{R}^+_{ev,2(r+d)}[X] + \left(1 - \sum_{i=1}^n x_i^2\right)\mathbb{R}[X], \end{aligned} \tag{19}$$

$$\begin{aligned} q^{(r)} &:= \max\ t \quad s.t. \quad \left(q(x) - t\left(\sum_{i=1}^n x_i^2\right)^d\right)\left(\sum_{i=1}^n x_i^2\right)^r \in \Sigma^2 \\ &= \max\ t \quad s.t.\ q(x) - t \in \Sigma^2_{2(r+d)} + \left(1 - \sum_{i=1}^n x_i^2\right)\mathbb{R}[X]. \end{aligned} \tag{20}$$

Proof. Follows directly from Proposition 2 (applied to the form $q(x) - t\|x\|^{2d}$) and from Lemma 1. ∎

Therefore, Theorem 3 follows from Theorem 4 applied to the (even) form $q(x) := \tilde{p}(x)$.

We have formulated in Theorem 4 two bounds for the minimum of a form q of even degree on the unit sphere: a linear bound $q_L^{(r)}$ and a semidefinite bound $q^{(r)}$ using, respectively, representations in terms of even polynomials and sums of squares of polynomials. At that point, one should point out that the hierarchy of linear bounds is interesting *only* when q is an *even* form. Indeed, if the form q is not even, then $q_L^{(r)} = -\infty$ for all $r \geq 0$; this follows from the following facts.

Lemma 2. *A polynomial $p \in \mathbb{R}[X]$ is even if and only if*

$$p(x_1, \ldots, x_n) = p(-x_1, x_2, \ldots, x_n) = \ldots = p(x_1, \ldots, x_{n-1}, -x_n). \tag{21}$$

Proof. Necessity is obvious. Conversely, assume that (21) holds; we show that p is even. For this, let p_1 be the sum of the even terms of p and set $q := p - p_1$. Then, $q = \sum_\alpha q_\alpha x^\alpha$ where α has some odd component whenever $q_\alpha \neq 0$. As p_1 is an even form, it satisfies (21) and thus q too satisfies (21). We show that $q = 0$, which implies that $p = p_1$ is even. For this, write $q = q_1 + q_2$, where $q_1 := \sum_{\alpha | \alpha_1 \text{ odd}} q_\alpha x^\alpha$. Then, $q(x) = q(-x_1, x_2, \ldots, x_n)$, $q_1(-x_1, x_2, \ldots, x_n) = -q_1(x)$, $q_2(-x_1, x_2, \ldots, x_n) = q_2(x)$; hence,

$$q_1(x) + q_2(x) = q_1(-x_1, x_2, \ldots, x_n) + q_2(-x_1, x_2, \ldots, x_n) = -q_1(x) + q_2(x),$$

which implies that $q_1(x) = 0$. From this follows that $q_\alpha = 0$ whenever α_1 is odd. The same reasoning applied to the other coordinates shows that all q_α are equal to 0. ∎

Corollary 2. *Given $p \in \mathbb{R}[X]$, the polynomial $p(x)(\sum_{i=1}^n x_i^2)^r$ is even for some $r \geq 0$ if and only if p is even.* ∎

3 A Negative Result

Let us now turn to the question of existence of a stronger type of decomposition. Let q be a form of even degree $2d$ which is positive on the unit sphere.

Then, $q(x) > 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$. In particular, q is positive on the unit ball $F := \{x \in \mathbb{R}^n \mid 1 - \sum_{i=1}^n x_i^2 \geq 0\}$ except at the origin where it is zero. One may wonder whether an extension of Putinar's result might still hold, permitting to conclude that

$$q \in \Sigma^2 + \left(1 - \sum_{i=1}^{n} x_i^2\right) \Sigma^2. \tag{22}$$

Scheiderer [14] has recently investigated such extensions of Putinar's result (see Corollary 3.17 in [14]).

Proposition 3 (Example 3.18 in [14]). *Let $p \in \mathbb{R}[X]$ be a polynomial for which the level set*

$$K := \{x \in \mathbb{R}^n \mid p(x) \geq 0\}$$

is compact. Let $q \in \mathbb{R}[X]$ be nonnegative on K. Assume that the following conditions hold:

1. *q has only finitely many zeros in K, each lying in the interior of K.*
2. *the Hessian $\nabla^2 q$ is positive definite at each of these zeroes.*

Then $q \in \Sigma^2 + p\Sigma^2$.

Unfortunately, in the case where K is the unit ball and q a positive semidefinite form of degree at least 4, this theorem does not apply (since the Hessian of q is zero at the origin). In fact, one can show that in this case such a decomposition (22) exists *only* when q itself is a sum of squares.

Proposition 4. *Let q be a form of degree $2d$. Then,*

$$q \in \Sigma^2 + \left(1 - \sum_{i=1}^{n} x_i^2\right) \Sigma^2 \Longleftrightarrow q \in \Sigma^2.$$

Proof. The 'if' part being trivial, we prove the 'only if' part. Assume that $q = f + (1 - \sum_{i=1}^n x_i^2)g$, where $f, g \in \Sigma^2$; we show that $q \in \Sigma^2$. Write $f = \sum_\ell f_\ell^2$ and $g = \sum_k g_k^2$. Let $s \geq 0$ be the largest integer for which each term of f_ℓ, g_k has degree $\geq s$; that is, $f_\ell(x) = \sum_{|\alpha|\geq s} f_{\ell,\alpha} x^\alpha$, $g_k(x) = \sum_{|\alpha|\geq s} g_{k,\alpha} x^\alpha$ for all ℓ, k and at least one of the polynomials f_ℓ, g_k has a term of degree s. Define f'_ℓ as the sum of the terms of degree s in f_ℓ and $f''_\ell := f_\ell - f'_\ell$; then,

$$f'_\ell(x) = \sum_{|\alpha|=s} f_{\ell,\alpha} x^\alpha, \;\; f''_\ell(x) = \sum_{|\alpha|\geq s+1} f_{\ell,\alpha} x^\alpha.$$

Analogously, define

$$g'_k(x) := \sum_{|\alpha|=s} g_{k,\alpha} x^\alpha, \quad g''_k(x) := \sum_{|\alpha|\geq s+1} g_{k,\alpha} x^\alpha.$$

We have that

$$q = q_1 + q_2, \quad \text{where } q_1 := \sum_\ell (f'_\ell)^2 + \sum_k (g'_k)^2, \quad \text{and}$$

$$q_2 := 2\sum_\ell f'_\ell f''_\ell + 2\sum_k g'_k g''_k + \sum_\ell (f''_\ell)^2 + \sum_k (g''_k)^2 - \left(\sum_{i=1}^n x_i^2\right) g.$$

Therefore, q_1 is a (nonzero) form of degree $2s$, while each term of q_2 has degree $\geq 2s+1$. If $s \leq d-1$, then q is a form of degree $2d \geq 2s+2$, which implies that $q_1 = 0$, a contradiction. Hence, $s \geq d$ and, in fact, $s = d$. From this follows that $q_2 = 0$ and, thus, $q = q_1$ is a sum of squares. ∎

4 Conclusion

We conclude with some comments on the computational implications of Theorem 4 where we showed that

$$\begin{aligned} q^{(r)} &:= \max\ t \ \text{ s.t. } \left(q(x) - t\left(\sum_{i=1}^n x_i^2\right)^d\right)\left(\sum_{i=1}^n x_i^2\right)^r \in \Sigma^2 \\ &= \max\ t \ \text{ s.t. } q(x) - t \in \Sigma^2_{2(r+d)} + \left(1 - \sum_{i=1}^n x_i^2\right)\mathbb{R}[X]. \end{aligned}$$

The first representation of $q^{(r)}$ corresponds to various relaxations introduced in the literature for different special cases of the problem

$$q_{\min} = \min\ q(x) \ \text{ s.t. } x \in S := \left\{x \in \mathbb{R}^n \mid \sum_{i=1}^n x_i^2 = 1\right\}, \tag{23}$$

by

1. De Klerk and Pasechnik [6] for obtaining the stability number of a graph;
2. Parrilo [9], Bomze and De Klerk [1], Faybusovich [2], and De Klerk, Laurent and Parrilo [5] for minimization of forms on the simplex.

The difficulty with these approaches up to now was that — once an exact relaxation was obtained — it was not clear how to extract a globally optimal solution of problem (23).

The second representation of $q^{(r)}$ in Theorem 4 corresponds exactly to the dual form of the SDP relaxation obtained by applying the general methodology introduced by Lasserre [7] to problem (23).

The approach of Lasserre [7] has now been implemented in the software package Gloptipoly [3] by Henrion and Lasserre.

The authors have also described sufficient conditions for the relaxation of order r to be exact, and have implemented an algorithm for extracting an optimal solution if it is known that the relaxation of order r is exact. The extraction procedure only involves linear algebra on the primal optimal solution of the relaxation; see [4] for details.

Theorem 4 therefore shows how to apply the solution extraction procedure implemented in Gloptipoly to the relaxations studied by De Klerk and Pasechnik [6], Parrilo [9], Bomze and De Klerk [1] and Faybusovich [2].

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