9
Cartan's Criteria

How can one decide whether a complex Lie algebra is semisimple? Working straight from the definition, one would have to test every single ideal for solvability, seemingly a daunting task. In this chapter, we describe a practical way to decide whether a Lie algebra is semisimple or, at the other extreme, solvable, by looking at the traces of linear maps.

We have already seen examples of the usefulness of taking traces. For example, we made an essential use of the trace map when proving the Invariance Lemma (Lemma 5.5). An important identity satisfied by trace is

$$\operatorname{tr}([a,b]c) = \operatorname{tr}(a[b,c])$$

for linear transformations a, b, c of a vector space. This holds because $\operatorname{tr} b(ac) = \operatorname{tr}(ac)b$; we shall see its usefulness in the course of this chapter. Furthermore, note that a nilpotent linear transformation has trace zero.

From now on, we work entirely over the complex numbers.

9.1 Jordan Decomposition

Working over the complex numbers allows us to consider the Jordan normal form of linear transformations. We use this to define for each linear transformation x of a complex vector space V a unique *Jordan decomposition.* The Jordan decomposition of x is the unique expression of x as a sum $x = d + n$ where $d : V \to V$ is diagonalisable, $n : V \to V$ is nilpotent, and d and n commute.

Very often, a diagonalisable linear map of a complex vector space is also said to be *semisimple*.

We review the Jordan normal form and prove the existence and uniqueness of the Jordan decomposition in Appendix A. The lemma below is also proved in this Appendix; see §16.6.

Lemma 9.1

Let x be a linear transformation of the complex vector space V. Suppose that x has Jordan decomposition $x = d + n$, where d is diagonalisable, n is nilpotent, and d and n commute.

(a) There is a polynomial $p(X) \in \mathbf{C}[X]$ such that $p(x) = d$.

(b) Fix a basis of V in which d is diagonal. Let $\bar{d}$ be the linear map whose matrix with respect to this basis is the complex conjugate of the matrix of d. There is a polynomial $q(X) \in \mathbf{C}[X]$ such that $q(x) = \bar{d}$. □

Using Jordan decomposition, we can give a concise reinterpretation of two earlier results (see Exercise 1.17 and Lemma 5.1).

Exercise 9.1

Let V be a vector space, and suppose that $x \in \mathsf{gl}(V)$ has Jordan decomposition $d+n$. Show that $\operatorname{ad} x : \mathsf{gl}(V) \to \mathsf{gl}(V)$ has Jordan decomposition $\operatorname{ad} d + \operatorname{ad} n$.

9.2 Testing for Solvability

Let V be a complex vector space and let L be a Lie subalgebra of $\mathsf{gl}(V)$. Why might it be reasonable to expect solvability to be visible from the traces of the elements of L? The following exercise (which repeats part of Exercise 6.5) gives one indication.

Exercise 9.2

Suppose that L is solvable. Use Lie's Theorem to show that there is a basis of V in which every element of L' is represented by a strictly upper triangular matrix. Conclude that $\operatorname{tr} xy = 0$ for all $x \in L$ and $y \in L'$.

Thus we have a necessary condition, in terms of traces, for L to be solvable. Remarkably, this condition is also sufficient. Before proving this, we give a small example.

Example 9.2

Let L be the 2-dimensional non-abelian Lie algebra with basis x, y such that $[x, y] = x$, which we constructed in §3.1. In this basis we have

$$\operatorname{ad} x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \operatorname{ad} y = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}.$$

As expected, $\operatorname{tr} \operatorname{ad} x = 0$.

Proposition 9.3

Let V be a complex vector space and let L be a Lie subalgebra of $\mathsf{gl}(V)$. If $\operatorname{tr} xy = 0$ for all $x, y \in L$, then L is solvable.

Proof

We shall show that every $x \in L'$ is a nilpotent linear map. It will then follow from Engel's Theorem (Theorem 6.1) that L' is nilpotent, and so, by the 'if' part of Exercise 6.5(ii), L is solvable.

Let $x \in L'$ have Jordan decomposition $x = d + n$, where d is diagonalisable, n is nilpotent, and d and n commute. We may fix a basis of V in which d is diagonal and n is strictly upper triangular. Suppose that d has diagonal entries $\lambda_1, \ldots, \lambda_m$. Since our aim is to show that $d = 0$, it will suffice to show that

$$\sum_{i=1}^{m} \lambda_i \bar{\lambda}_i = 0.$$

The matrix of $\bar{d}$ is diagonal, with diagonal entries $\bar{\lambda}_i$ for $1 \le i \le m$. A simple computation shows that

$$\operatorname{tr} \bar{d}x = \sum_{i=1}^{m} \lambda_i \bar{\lambda}_i.$$

Now, as $x \in L'$, we may express x as a linear combination of commutators $[y, z]$ with $y, z \in L$, so we need to show that $\operatorname{tr}(\bar{d}[y, z]) = 0$. By the identity mentioned at the start of this chapter, this is equivalent to

$$\operatorname{tr}([\bar{d}, y]z) = 0.$$

This will hold by our hypothesis, provided we can show that $[\bar{d}, y] \in L$. In other words, we must show that $\operatorname{ad} \bar{d}$ maps L into L.

By Exercise 9.1, the Jordan decomposition of $\operatorname{ad} x$ is $\operatorname{ad} d + \operatorname{ad} n$. Therefore, by part (b) of Lemma 9.1, there is a polynomial $p(X) \in \mathbf{C}[X]$ such that $p(\operatorname{ad} x) = \overline{\operatorname{ad} d} = \operatorname{ad} \bar{d}$. Now $\operatorname{ad} x$ maps L into itself, so $p(\operatorname{ad} x)$ does also. □

To apply this proposition to an abstract Lie algebra L, we need a way to regard L as a subalgebra of some $\mathsf{gl}(V)$. The adjoint representation of L is well-suited to this purpose, as L is solvable if and only if $\operatorname{ad} L$ is solvable.

Theorem 9.4

Let L be a complex Lie algebra. Then L is solvable if and only if $\operatorname{tr}(\operatorname{ad} x \circ \operatorname{ad} y) = 0$ for all $x \in L$ and all $y \in L'$.

Proof

Suppose that L is solvable. Then $\operatorname{ad} L \subseteq \mathsf{gl}(L)$ is a solvable subalgebra of $\mathsf{gl}(L)$, so the result now follows from Exercise 9.2.

Conversely, if $\operatorname{tr}(\operatorname{ad} x \circ \operatorname{ad} y) = 0$ for all $x \in L$ and all $y \in L'$, then Proposition 9.3 implies that $\operatorname{ad} L'$ is solvable. So L' is solvable, and hence L is solvable. □

9.3 The Killing Form

Definition 9.5

Let L be a complex Lie algebra. The *Killing form* on L is the symmetric bilinear form defined by

$$\kappa(x, y) := \operatorname{tr}(\operatorname{ad} x \circ \operatorname{ad} y) \quad \text{for } x, y \in L.$$

The Killing form is bilinear because ad is linear, the composition of maps is bilinear, and tr is linear. (The reader may wish to write out a more careful proof of this.) It is symmetric because $\operatorname{tr} ab = \operatorname{tr} ba$ for linear maps a and b. Another very important property of the Killing form is its *associativity*, which states that for all $x, y, z \in L$ we have

$$\kappa([x, y], z) = \kappa(x, [y, z]).$$

This follows from the identity for trace mentioned at the start of this chapter.

Using the Killing form, we can state Theorem 9.4 as follows.

Theorem 9.6 (Cartan's First Criterion)

The complex Lie algebra L is solvable if and only if $\kappa(x, y) = 0$ for all $x \in L$ and $y \in L'$. □

Example 9.7

Let L be the 2-dimensional non-abelian Lie algebra with basis x, y such that $[x, y] = x$. The matrices computed in Example 9.2 show that $\kappa(x, x) = \kappa(x, y) = \kappa(y, x) = 0$ and $\kappa(y, y) = 1$. The matrix of κ in the basis x, y is therefore

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

The Killing form is compatible with restriction to ideals. Suppose that L is a Lie algebra and I is an ideal of L. We write κ for the Killing form on L and κ_I for the Killing form on I, considered as a Lie algebra in its own right.

Lemma 9.8

If $x, y \in I$, then $\kappa_I(x, y) = \kappa(x, y)$.

Proof

Take a basis for I and extend it to a basis of L. If $x \in I$, then $\operatorname{ad} x$ maps L into I, so the matrix of $\operatorname{ad} x$ in this basis is of the form

$$\begin{pmatrix} A_x & B_x \\ 0 & 0 \end{pmatrix},$$

where A_x is the matrix of $\operatorname{ad} x$ restricted to I.

If $y \in I$, then $\operatorname{ad} x \circ \operatorname{ad} y$ has matrix

$$\begin{pmatrix} A_x A_y & A_x B_y \\ 0 & 0 \end{pmatrix},$$

where $A_x \circ A_y$ is the matrix of $\operatorname{ad} x \circ \operatorname{ad} y$ restricted to I. Only the block $A_x A_y$ contributes to the trace of this matrix, so

$$\kappa(x, y) \ = \operatorname{tr}(A_x B_x) = \kappa_I(x, y). \qquad \square$$

9.4 Testing for Semisimplicity

Recall that a Lie algebra is said to be semisimple if its radical is zero; that is, if it has no non-zero solvable ideals. Since we can detect solvability by using the Killing form, it is perhaps not too surprising that we can also use the Killing form to decide whether or not a Lie algebra is semisimple.

We begin by recalling a small part of the general theory of bilinear forms; for more details, see Appendix A. Let β be a symmetric bilinear form on a finite-dimensional complex vector space V. If S is a subset of V, we define the *perpendicular space* to S by

$$S^{\perp} = \{x \in V : \beta(x, s) = 0 \text{ for all } s \in S\}.$$

This is a vector subspace of V. We say that β is *non-degenerate* if $V^{\perp} = 0$; that is, there is no non-zero vector $v \in V$ such that $\beta(v, x) = 0$ for all $x \in V$.

If β is non-degenerate and W is a vector subspace of V, then

$$\dim W + \dim W^{\perp} = \dim V.$$

Note that even if β is non-degenerate it is possible that $W \cap W^{\perp} \neq 0$. For example, if κ is the Killing form of $\mathsf{sl}(2, \mathbf{C})$, then $\kappa(e, e) = 0$. (You are asked to compute the Killing form of $\mathsf{sl}(2, \mathbf{C})$ in Exercise 9.4 below.)

Now we specialise to the case where L is a Lie algebra and κ is its Killing form, so perpendicular spaces are taken with respect to κ. We begin with a simple observation which requires the associativity of κ.

Exercise 9.3

Suppose that I is an ideal of L. Show that $I^{\perp}$ is an ideal of L.

By this exercise, $L^{\perp}$ is an ideal of L. If $x \in L^{\perp}$ and $y \in \left(L^{\perp}\right)'$, then, as in particular $y \in L$, we have $\kappa(x, y) = 0$. Hence it follows from Cartan's First Criterion that $L^{\perp}$ is a solvable ideal of L. Therefore, if L is semisimple, then $L^{\perp} = 0$ and κ is non-degenerate.

Again the converse also holds.

Theorem 9.9 (Cartan's Second Criterion)

The complex Lie algebra L is semisimple if and only if the Killing form κ of L is non-degenerate.

Proof

We have just proved the "only if" direction. Suppose that L is not semisimple, so $\operatorname{rad} L$ is non-zero. By Exercise 4.6, L has a non-zero abelian ideal, say A. Take a non-zero element $a \in A$, and let $x \in L$. The composite map

$$\operatorname{ad} a \circ \operatorname{ad} x \circ \operatorname{ad} a$$

sends L to zero, as the image of $\operatorname{ad} x \circ \operatorname{ad} a$ is contained in the abelian ideal A. Hence $(\operatorname{ad} a \circ \operatorname{ad} x)^2 = 0$. Nilpotent maps have trace 0, so $\kappa(a, x) = 0$. This holds for all $x \in L$, so a is a non-zero element in $L^{\perp}$. Thus κ is degenerate. $\square$

It is possible that $L^{\perp}$ is properly contained in $\operatorname{rad} L$. For example, Exercise 9.2 shows that this is the case if L is the 2-dimensional non-abelian Lie algebra.

Cartan's Second Criterion is an extremely powerful characterisation of semisimplicity. In our first application, we shall show that a semisimple Lie algebra is a direct sum of simple Lie algebras; this finally justifies the name *semi*simple which we have been using. The following lemma contains the main idea needed.

Lemma 9.10

If I is a non-trivial proper ideal in a complex semisimple Lie algebra L, then $L = I \oplus I^{\perp}$. The ideal I is a semisimple Lie algebra in its own right.

Proof

As usual, let κ denote the Killing form on L. The restriction of κ to $I \cap I^{\perp}$ is identically 0, so by Cartan's First Criterion, $I \cap I^{\perp} = 0$. It now follows by dimension counting that $L = I \oplus I^{\perp}$.

We shall show that I is semisimple using Cartan's Second Criterion. Suppose that I has a non-zero solvable ideal. By the "only if" direction of Cartan's Second Criterion, the Killing form on I is degenerate. We have seen that the Killing form on I is given by restricting the Killing form on L, so there exists $a \in I$ such that $\kappa(a, x) = 0$ for all $x \in I$. But as $a \in I$, $\kappa(a, y) = 0$ for all $y \in I^{\perp}$ as well. Since $L = I \oplus I^{\perp}$, this shows that κ is degenerate, a contradiction. $\square$

We can now prove the following theorem.

Theorem 9.11

Let L be a complex Lie algebra. Then L is semisimple if and only if there are simple ideals $L_1, \ldots, L_r$ of L such that $L = L_1 \oplus L_2 \oplus \ldots \oplus L_r$.

Proof

We begin with the "only if" direction, working by induction on $\dim L$. Let I be an ideal in L of the smallest possible non-zero dimension. If $I = L$, we are done. Otherwise I is a proper simple ideal of L. (It cannot be abelian as by hypothesis L has no non-zero abelian ideals.) By the preceding lemma, $L = I \oplus I^{\perp}$, where, as an ideal of L, $I^{\perp}$ is a semisimple Lie algebra of smaller dimension than L.

So, by induction, $I^\perp$ is a direct sum of simple ideals,

$$I^\perp = L_2 \oplus \ldots \oplus L_r.$$

Each L_i is also an ideal of L, as $[I, L_i] \subseteq I \cap I^\perp = 0$, so putting $L_1 = I$ we get the required decomposition.

Now for the "if" direction. Suppose that $L = L_1 \oplus \ldots \oplus L_r$, where the L_r are simple ideals. Let $I = \operatorname{rad} L$; our aim is to show that $I = 0$. For each ideal L_i, $[I, L_i] \subseteq I \cap L_i$ is a solvable ideal of L_i. But the L_i are simple, so

$$[I, L] \subseteq [I, L_1] \oplus \ldots \oplus [I, L_r] = 0.$$

This shows that I is contained in $Z(L)$. But by Exercise 2.6(ii)

$$Z(L) = Z(L_1) \oplus \ldots \oplus Z(L_r).$$

We know that $Z(L_1) = \ldots = Z(L_r) = 0$ as the L_i are simple ideals, so $Z(L) = 0$ and $I = 0$. □

Using very similar ideas, we can prove the following.

Lemma 9.12

If L is a semisimple Lie algebra and I is an ideal of L, then L/I is semisimple.

Proof

We have seen that $L = I \oplus I^\perp$, so L/I is isomorphic to $I^\perp$, which we have seen is a semisimple Lie algebra in its own right. □

9.5 Derivations of Semisimple Lie Algebras

In our next application of Cartan's Second Criterion, we show that the only derivations of a complex semisimple Lie algebra are those of the form $\operatorname{ad} x$ for $x \in L$. More precisely, we have the following.

Proposition 9.13

If L is a finite-dimensional complex semisimple Lie algebra, then $\operatorname{ad} L = \operatorname{Der} L$.

Proof

We showed in Example 1.2 that for each $x \in L$ the linear map $\operatorname{ad} x$ is a derivation of L, so ad is a Lie algebra homomorphism from L to $\operatorname{Der} L$. Moreover, if δ is a derivation of L and $x, y \in L$, then

$$\begin{aligned}[\delta, \operatorname{ad} x]y &= \delta[x, y] - \operatorname{ad} x(\delta y) \\ &= [\delta x, y] + [x, \delta y] - [x, \delta y] \\ &= \operatorname{ad}(\delta x)y.\end{aligned}$$

Thus the image of $\operatorname{ad} : L \to \operatorname{Der} L$ is an ideal of $\operatorname{Der} L$. This much is true for any Lie algebra.

Now we bring in our assumption that L is complex and semisimple. First, note that $\operatorname{ad} : L \to \operatorname{Der} L$ is one-to-one, as $\ker \operatorname{ad} = Z(L) = 0$, so the Lie algebra $M := \operatorname{ad} L$ is isomorphic to L and therefore it is semisimple as well.

To show that $M = \operatorname{Der} L$, we exploit the Killing form on the Lie algebra $\operatorname{Der} L$. If M is properly contained in $\operatorname{Der} L$ then $M^\perp \neq 0$, so it is sufficient to prove that $M^\perp = 0$. As M is an ideal of $\operatorname{Der} L$, the Killing form κ_M of M is the restriction of the Killing form on $\operatorname{Der} L$. By Cartan's Second Criterion, κ_M is non-degenerate, so $M^\perp \cap M = 0$ and hence $[M^\perp, M] = 0$. Thus, if $\delta \in M^\perp$ and $\operatorname{ad} x \in M$, then $[\delta, \operatorname{ad} x] = 0$. But we saw above that

$$[\delta, \operatorname{ad} x] = \operatorname{ad}(\delta x),$$

so, for all $x \in L$, we have $\delta(x) = 0$; in other words, $\delta = 0$. $\square$

In Exercise 9.17, this proposition is used to give an alternative proof that a semisimple Lie algebra is a direct sum of simple Lie algebras. Another important application occurs in the following section.

9.6 Abstract Jordan Decomposition

Given a representation $\varphi : L \to \mathsf{gl}(V)$ of a Lie algebra L, we may consider the Jordan decomposition of the linear maps $\varphi(x)$ for $x \in L$.

For a general Lie algebra there is not much that can be said about this decomposition without knowing more about the representation φ. For example, if L is the 1-dimensional abelian Lie algebra, spanned, say by x, then we may define a representation of L on a vector space V by mapping x to *any* element of $\mathsf{gl}(V)$. So the Jordan decomposition of $\varphi(x)$ is essentially arbitrary.

However, representations of a complex semisimple Lie algebra are much better behaved. To demonstrate this, we use derivations to define a Jordan decomposition for elements of an arbitrary complex semisimple Lie algebra. We need the following proposition.

Proposition 9.14

Let L be a complex Lie algebra. Suppose that δ is a derivation of L with Jordan decomposition $\delta = \sigma + \nu$, where σ is diagonalisable and ν is nilpotent. Then σ and ν are also derivations of L.

Proof

For $\lambda \in \mathbf{C}$, let

$$L_\lambda = \{x \in L : (\delta - \lambda 1_L)^m x = 0 \text{ for some } m \geq 1\}$$

be the generalised eigenspace of δ corresponding to λ. Note that if λ is not an eigenvalue of δ, then $L_\lambda = 0$. By the Primary Decomposition Theorem, L decomposes as a direct sum of generalised eigenspaces, $L = \bigoplus_\lambda L_\lambda$, where the sum runs over the eigenvalues of δ. In Exercise 9.8 below, you are asked to show that

$$[L_\lambda, L_\mu] \subseteq L_{\lambda+\mu}.$$

We shall use this to show that σ and ν are derivations.

As σ acts diagonalisably, the λ-eigenspace of σ is L_λ. Take $x \in L_\lambda$ and $y \in L_\mu$. Then, by the above, $[x, y] \in L_{\lambda+\mu}$, so

$$\sigma([x, y]) = (\lambda + \mu)[x, y],$$

which is the same as

$$[\sigma(x), y] + [x, \sigma(y)] = [\lambda x, y] + [x, \mu y].$$

Thus σ is a derivation, and so $\delta - \sigma = \nu$ is also a derivation. □

Theorem 9.15

Let L be a complex semisimple Lie algebra. Each $x \in L$ can be written uniquely as $x = d + n$, where $d, n \in L$ are such that $\operatorname{ad} d$ is diagonalisable, $\operatorname{ad} n$ is nilpotent, and $[d, n] = 0$. Furthermore, if $y \in L$ commutes with x, then $[d, y] = 0$ and $[n, y] = 0$.

Proof

Let $\operatorname{ad} x = \sigma + \nu$ where $\sigma \in \mathsf{gl}(L)$ is diagonalisable, $\nu \in \mathsf{gl}(L)$ is nilpotent, and $[\sigma, \nu] = 0$. By Proposition 9.14, we know that σ and ν are derivations of the semisimple Lie algebra L. In Proposition 9.13, we saw that $\operatorname{ad} L = \operatorname{Der} L$, so there exist $d, n \in L$ such that $\operatorname{ad} d = \sigma$ and $\operatorname{ad} n = \nu$. As ad is injective and

$$\operatorname{ad} x = \sigma + \nu = \operatorname{ad} d + \operatorname{ad} n = \operatorname{ad}(d + n),$$

we get that $x = d + n$. Moreover, $\operatorname{ad}[d, n] = [\operatorname{ad} d, \operatorname{ad} n] = 0$ so $[d, n] = 0$. The uniqueness of d and n follows from the uniqueness of the Jordan decomposition of $\operatorname{ad} x$.

Suppose that $y \in L$ and that $(\operatorname{ad} x)y = 0$. By Lemma 9.1, σ and ν may be expressed as polynomials in $\operatorname{ad} x$. Let

$$\nu = c_0 1_L + c_1 \operatorname{ad} x + \ldots + c_r (\operatorname{ad} x)^r.$$

Applying ν to y, we see that $\nu(y) = c_0 y$. But ν is nilpotent and $\nu(x) = c_0 x$, so $c_0 = 0$. Thus $\nu(y) = 0$ and so $\sigma(y) = (\operatorname{ad} x - \nu)y = 0$ also. □

We say that x has *abstract Jordan decomposition* $x = d + n$. If $n = 0$, then we say that x is *semisimple*.

There is a potential ambiguity in the terms "Jordan decomposition" and "semisimple" which arises when $L \subseteq \mathsf{gl}(V)$ is a semisimple Lie algebra. In this case, as well as the abstract Jordan decomposition just defined, we may also consider the usual Jordan decomposition, given by taking an element of L and regarding it as a linear map on V. It is an important property of the abstract Jordan decomposition that the two decompositions agree; in particular, an element of L is diagonalisable if and only if it is semisimple.

Take $x \in L$. Suppose that the usual Jordan decomposition of x, as an element of $\mathsf{gl}(V)$, is $d + n$. By Exercise 9.1, the Jordan decomposition of the map $\operatorname{ad} x : L \to L$ is $\operatorname{ad} d + \operatorname{ad} n$, so by definition $d + n$ is also the abstract Jordan decomposition of x.

We are now ready to prove the main result about the abstract Jordan decomposition.

Theorem 9.16

Let L be a semisimple Lie algebra and let $\theta : L \to \mathsf{gl}(V)$ be a representation of L. Suppose that $x \in L$ has abstract Jordan decomposition $x = d + n$. Then the Jordan decomposition of $\theta(x) \in \mathsf{gl}(V)$ is $\theta(x) = \theta(d) + \theta(n)$.

Proof

By Lemma 9.12, $\operatorname{im}\theta \cong L/\ker\theta$ is a semisimple Lie algebra. It therefore makes sense to talk about the abstract Jordan decomposition of elements of $\operatorname{im}\theta$.

Let $x \in L$ have abstract Jordan decomposition $d+n$. It follows from Exercise 9.16 below that the abstract Jordan decomposition of $\theta(x)$, considered as an element of $\operatorname{im}\theta$, is $\theta(d)+\theta(n)$. By the remarks above, this is also the Jordan decomposition of $\theta(x)$, considered as an element of $\mathsf{gl}(V)$. □

The last theorem is a very powerful result, which we shall apply several times in the next chapter. For another application, see Exercise 9.15 below.

EXERCISES

9.4.† (i) Compute the Killing form of $\mathsf{sl}(2,\mathbf{C})$. This is a symmetric bilinear form on a 3-dimensional vector space, so you should expect it to be described by a symmetric 3×3 matrix. Check that the Killing form is non-degenerate.

(ii) Is the Killing form of $\mathsf{gl}(2,\mathbf{C})$ non-degenerate?

9.5. Suppose that L is a nilpotent Lie algebra over a field F. Show by using the ideals L^m, or otherwise, that the Killing form of L is identically zero. Does the converse hold? (The following exercise may be helpful.)

9.6.† For each of the 3-dimensional complex Lie algebras studied in Chapter 3, find its Killing form with respect to a convenient basis.

9.7. Let $L = \mathsf{gl}(n,\mathbf{C})$. Show that the Killing form of L is given by

$$\kappa(a,b) = 2n\operatorname{tr}(ab) - 2(\operatorname{tr} a)(\operatorname{tr} b).$$

For instance, start with $(\operatorname{ad} b)e_{rs}$, apply $\operatorname{ad} a$, and then express the result in terms of the basis and find the coefficient of e_{rs}. Hence prove that if $n \geq 2$ then $\mathsf{sl}(n,\mathbf{C})$ is semisimple.

9.8. Let δ be a derivation of a Lie algebra L. Show that if $\lambda, \mu \in \mathbf{C}$ and $x, y \in L$, then

$$(\delta - (\lambda+\mu)1_L)^n[x,y] = \sum_{k=0}^{n} \binom{n}{k} \left[(\delta - \lambda 1_L)^k x, (\delta - \mu 1_L)^{n-k} y\right].$$

Hence show that if the primary decomposition of L with respect to δ is $L = \bigoplus_\lambda L_\lambda$ (as in the proof of Proposition 9.14), then

$$[L_\lambda, L_\mu] \subseteq L_{\lambda+\mu}.$$

9.9. (i) Show that if L is a semisimple Lie algebra then $L' = L$.

(ii) Suppose that L is the direct sum of simple ideals $L = L_1 \oplus L_2 \oplus \ldots \oplus L_k$. Show that if I is a simple ideal of L, then I is equal to one of the L_i. *Hint*: Consider the ideal $[I, L]$.

(iii)* If $L' = L$, must L be semisimple?

9.10. Suppose that L is a Lie algebra over $\mathbf{C}$ and that β is a symmetric, associative bilinear form of L. Show that β induces a linear map

$$\theta : L \to L^*, \quad \theta(x) = \beta(x, -),$$

where by $\beta(x, -)$ we mean the map $y \mapsto \beta(x, y)$. Viewing both L and $L^\star$ as L-modules, show that θ is an L-module homomorphism. (The L-module structure of $L^\star$ is given by Exercise 7.12.) Deduce that if β is non-degenerate, then L and L^* are isomorphic as L-modules.

9.11.† Let L be a simple Lie algebra over $\mathbf{C}$ with Killing form κ. Use Exercise 9.10 to show that if β is any other symmetric, associative, non-degenerate bilinear form on L, then there exists $0 \neq \lambda \in \mathbf{C}$ such that $\kappa = \lambda\beta$.

9.12. Assuming that $\mathsf{sl}(n, \mathbf{C})$ is simple, use Exercise 9.11 to show that

$$\kappa(x, y) = 2n \operatorname{tr}(xy), \quad x, y \in \mathsf{sl}(n, \mathbf{C}).$$

To identify the scalar λ, it might be useful to take as a standard basis for the Lie algebra; $\{e_{ij} : i \neq j\} \cup \{e_{ii} - e_{i+1,i+1} : 1 \leq i < n\}$.

9.13. Give an example to show that the condition $[d, n] = 0$ in the Jordan decomposition is necessary. That is, find a matrix x which can be written as $x = d + n$ with d diagonalisable and n nilpotent but where this is not the Jordan decomposition of x.

9.14.† Let L be a complex semisimple Lie algebra. Suppose L has a faithful representation in which $x \in L$ acts diagonalisably. Show that x is a semisimple element of L (in the sense of the abstract Jordan decomposition) and hence that x acts diagonalisably in *any* representation of L.

9.15.†* Suppose that M is an $\mathsf{sl}(2, \mathbf{C})$-module. Use the abstract Jordan decomposition to show that M decomposes as a direct sum of h-eigenspaces. Hence use Exercise 8.6 to show that M is completely reducible.

9.16.† Suppose that L_1 and L_2 are complex semisimple Lie algebras and that $\theta : L_1 \to L_2$ is a surjective homomorphism. Show that if $x \in L_1$ has abstract Jordan decomposition $x = d + n$, then $\theta(x) \in L_2$ has abstract Jordan decomposition $\theta(x) = \theta(d) + \theta(n)$. *Hint*: Exercise 2.8 is relevant.

9.17. Use Exercise 2.13 and Proposition 9.13 (that if L is a complex semisimple Lie algebra, then $\operatorname{ad} L = \operatorname{Der} L$) to give an alternative proof of Theorem 9.11 (that a complex semisimple Lie algebra is a direct sum of simple ideals).

9.18.⋆ Some small-dimensional examples suggest that if L is a Lie algebra and I is an ideal of L, then one can always find a basis of I and extend it to a basis of L in such a way that the Killing form of L has a matrix of the form

$$\begin{pmatrix} \kappa_I & 0 \\ 0 & \star \end{pmatrix}.$$

Is this always the case?