8
Representations of $\mathsf{sl}(2, \mathbf{C})$

In this chapter, we study the finite-dimensional irreducible representations of $\mathsf{sl}(2, \mathbf{C})$. In doing this, we shall see, in a stripped-down form, many of the ideas needed to study representations of an arbitrary semisimple Lie algebra. Later we will see that representations of $\mathsf{sl}(2, \mathbf{C})$ control a large part of the structure of all semisimple Lie algebras.

We shall use the basis of $\mathsf{sl}(2, \mathbf{C})$ introduced in Exercise 1.12 throughout this chapter. Recall that we set

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \ h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

8.1 The Modules V_d

We begin by constructing a family of irreducible representations of $\mathsf{sl}(2, \mathbf{C})$.

Consider the vector space $\mathbf{C}[X, Y]$ of polynomials in two variables X, Y with complex coefficients. For each integer $d \geq 0$, let V_d be the subspace of homogeneous polynomials in X and Y of degree d. So V_0 is the 1-dimensional vector space of constant polynomials, and for $d \geq 1$, the space V_d has as a basis the monomials $X^d, X^{d-1}Y, \ldots, XY^{d-1}, Y^d$. This basis shows that V_d has dimension $d+1$ as a $\mathbf{C}$-vector space.

We now make V_d into an $\mathsf{sl}(2, \mathbf{C})$-module by specifying a Lie algebra homomorphism $\varphi : \mathsf{sl}(2, \mathbf{C}) \to \mathsf{gl}(V_d)$. Since $\mathsf{sl}(2, \mathbf{C})$ is linearly spanned by the

K. Erdmann et al., *Introduction to Lie Algebras*

matrices e, f, h, the map φ will be determined once we have specified $\varphi(e)$, $\varphi(f)$, $\varphi(h)$.

We let

$$\varphi(e) := X\frac{\partial}{\partial Y};$$

that is, $\varphi(e)$ is the linear map which first differentiates a polynomial with respect to Y and then multiplies it with X. This preserves the degrees of polynomials and so maps V_d into V_d. Similarly, we let

$$\varphi(f) := Y\frac{\partial}{\partial X}.$$

Finally, we let

$$\varphi(h) := X\frac{\partial}{\partial X} - Y\frac{\partial}{\partial Y}.$$

Notice that

$$\varphi(h)(X^aY^b) = (a-b)X^aY^b,$$

so h acts diagonally on V_d with respect to our chosen basis.

Theorem 8.1

With these definitions, φ is a representation of $\mathsf{sl}(2, \mathbf{C})$.

Proof

By construction, φ is linear. Thus, all we have to check is that φ preserves Lie brackets. By linearity, it is enough to check this on the basis elements of $\mathsf{sl}(2, \mathbf{C})$, so there are just three equations we need to verify.

(1) We begin by showing $[\varphi(e), \varphi(f)] = \varphi([e, f]) = \varphi(h)$. If we apply the left-hand side to a basis vector X^aY^b with $a, b \geq 1$ and $a + b = d$, we get

$$\begin{aligned}[\varphi(e), \varphi(f)](X^aY^b) &= \varphi(e)\left(\varphi(f)(X^aY^b)\right) - \varphi(f)\left(\varphi(e)(X^aY^b)\right) \\ &= \varphi(e)\left(aX^{a-1}Y^{b+1}\right) - \varphi(f)\left(bX^{a+1}Y^{b-1}\right) \\ &= a(b+1)X^aY^b - b(a+1)X^aY^b \\ &= (a-b)X^aY^b.\end{aligned}$$

This is the same as $\varphi(h)(X^aY^b)$. We check separately the action on X^d,

$$\begin{aligned}[\varphi(e), \varphi(f)](X^d) &= \varphi(e)\left(\varphi(f)(X^d)\right) - \varphi(f)\left(\varphi(e)(X^d)\right) \\ &= \varphi(e)\left(dX^{d-1}Y\right) - \varphi(f)(0) = dX^d,\end{aligned}$$

which is the same as $\varphi(h)(X^d)$. Similarly, one checks the action on Y^d, so $[\varphi(e), \varphi(f)]$ and $\varphi(h)$ agree on a basis of V_d and so are the same linear map.

(2) We also need $[\varphi(h), \varphi(e)] = \varphi([h, e]) = \varphi(2e) = 2\varphi(e)$. Again we can prove this by applying the maps to basis vectors of V_d. For $b \geq 1$, we get

$$\begin{aligned}
[\varphi(h), \varphi(e)](X^a Y^b) &= \varphi(h)\left(\varphi(e)(X^a Y^b)\right) - \varphi(e)\left(\varphi(h)(X^a Y^b)\right) \\
&= \varphi(h)\left(bX^{a+1}Y^{b-1}\right) - \varphi(e)\left((a-b)X^a Y^b\right) \\
&= b\left((a+1)-(b-1)\right)X^{a+1}Y^{b-1} - (a-b)bX^{a+1}Y^{b-1} \\
&= 2bX^{a+1}Y^{b-1}.
\end{aligned}$$

This is the same as $2\varphi(e)(X^a Y^b)$. If $b = 0$ and $a = d$, then a separate verification is needed. We leave this to the reader.

(3) Similarly, one can check that $[\varphi(h), \varphi(f)] = -2\varphi(f)$. Again, we leave this to the reader.

□

8.1.1 Matrix Interpretation

It can be useful to know the matrices that correspond to the action of e, f, h on V_d; these give the matrix representation corresponding to φ.

As usual, we take the basis $X^d, X^{d-1}Y, \ldots, Y^d$ of V_d. The calculations in the proof of Theorem 8.1 show that the matrix of $\varphi(e)$ with respect to this basis is

$$\begin{pmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & d \\ 0 & 0 & 0 & \ldots & 0 \end{pmatrix},$$

the matrix of $\varphi(f)$ is

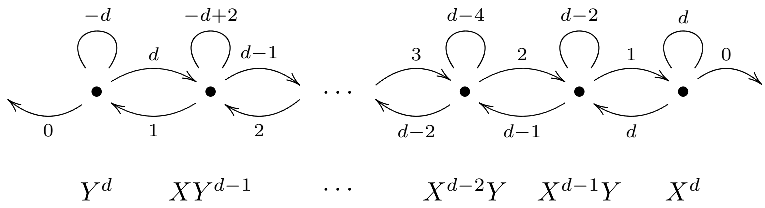
$$\begin{pmatrix} 0 & 0 & \ldots & 0 & 0 \\ d & 0 & \ldots & 0 & 0 \\ 0 & d-1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & 1 & 0 \end{pmatrix},$$

and $\varphi(h)$ is diagonal:

$$\begin{pmatrix} d & 0 & \ldots & 0 & 0 \\ 0 & d-2 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & -d+2 & 0 \\ 0 & 0 & \ldots & 0 & -d \end{pmatrix}$$

where the diagonal entries are the numbers $d - 2k$, where $k = 0, 1, \ldots, d$. By explicitly computing the commutators of these matrices, we can give another (but equivalent) way to prove that φ is a representation of sl$(2, \mathbf{C})$.

Another way to represent the action of h, e, f is to draw a diagram like



where loops represent the action of h, arrows to the right represent the action of e, and arrows to the left represent the action of f.

8.1.2 Irreducibility

One virtue of the diagram above is that it makes it almost obvious that the sl$(2, \mathbf{C})$-submodule of V_d generated by any particular basis element $X^a Y^b$ contains all the basis elements and so is all of V_d.

Exercise 8.1

Check this assertion.

A possible disadvantage of our diagram is that it may blind us to the existence of the *many other* vectors in V_d, which, while linear combinations of the basis vectors, are not basis vectors themselves.

Theorem 8.2

The sl$(2, \mathbf{C})$-module V_d is irreducible.

Proof

Suppose U is a non-zero sl$(2, \mathbf{C})$-submodule of V_d. Then $h \cdot u \in U$ for all $u \in U$. Since h acts diagonalisably on V_d, it also acts diagonalisably on U, so there is an eigenvector of h which lies in U. We have seen that all eigenspaces of h on V_d are one-dimensional, and each eigenspace is spanned by some monomial $X^a Y^b$, so the submodule U must contain some monomial, and by the exercise above, U contains a basis for V_d. Hence $U = V_d$. □

8.2 Classifying the Irreducible $\mathsf{sl}(2,\mathbf{C})$-Modules

It is clear that for different d the $\mathsf{sl}(2,\mathbf{C})$-modules V_d cannot be isomorphic, as they have different dimensions. In this section, we prove that any finite-dimensional irreducible $\mathsf{sl}(2,\mathbf{C})$-module is isomorphic to one of the V_d. Our strategy will be to look at the eigenvectors and eigenvalues of h. For brevity, we shall write $e^2 \cdot v$ rather than $e \cdot (e \cdot v)$, and so on.

Lemma 8.3

Suppose that V is an $\mathsf{sl}(2,\mathbf{C})$-module and $v \in V$ is an eigenvector of h with eigenvalue λ.

(i) Either $e \cdot v = 0$ or $e \cdot v$ is an eigenvector of h with eigenvalue $\lambda + 2$.

(ii) Either $f \cdot v = 0$ or $f \cdot v$ is an eigenvector of h with eigenvalue $\lambda - 2$.

Proof

As V is a representation of $\mathsf{sl}(2,\mathbf{C})$, we have

$$h \cdot (e \cdot v) = e \cdot (h \cdot v) + [h, e] \cdot v = e \cdot (\lambda v) + 2e \cdot v = (\lambda + 2)e \cdot v.$$

The calculation for $f \cdot v$ is similar. □

Lemma 8.4

Let V be a finite-dimensional $\mathsf{sl}(2,\mathbf{C})$-module. Then V contains an eigenvector w for h such that $e \cdot w = 0$.

Proof

As we work over $\mathbf{C}$, the linear map $h : V \to V$ has at least one eigenvalue and so at least one eigenvector. Let $h \cdot v = \lambda v$. Consider the vectors

$$v,\ e \cdot v,\ e^2 \cdot v, \ldots .$$

If they are non-zero, then by Lemma 8.3 they form an infinite sequence of h-eigenvectors with distinct eigenvalues. Eigenvectors with different eigenvalues are linearly independent, so V would contain infinitely many linearly independent vectors, a contradiction.

Therefore there exists $k \geq 0$ such that $e^k \cdot v \neq 0$ and $e^{k+1} \cdot v = 0$. If we set $w = e^k \cdot v$, then $h \cdot w = (\lambda + 2k)w$ and $e \cdot w = 0$. □

We are now ready to prove our main result.

Theorem 8.5

If V is a finite-dimensional irreducible $\mathsf{sl}(2, \mathbf{C})$-module, then V is isomorphic to one of the V_d.

Proof

By Lemma 8.4, V has an h-eigenvector w such that $e \cdot w = 0$. Suppose that $h \cdot w = \lambda w$. Consider the sequence of vectors

$$w, \ f \cdot w, \ f^2 \cdot w, \ldots.$$

By the proof of Lemma 8.4, there exists $d \geq 0$ such that $f^d \cdot w \neq 0$ and $f^{d+1} \cdot w = 0$.

Step 1: We claim that the vectors w, $f \cdot w$, $\ldots$, $f^d \cdot w$ form a basis for a submodule of V. They are linearly independent because, by Lemma 8.3, they are eigenvectors for h with distinct eigenvalues. By construction, the span of $w, f \cdot w, \ldots, f^d \cdot w$ is invariant under h and f. To show that it is invariant under e, we shall prove by induction on k that

$$e \cdot (f^k \cdot w) \in \operatorname{Span}\{f^j \cdot w : 0 \leq j < k\}.$$

If $k = 0$, then we know that $e \cdot w = 0$. For the inductive step, note that

$$e \cdot (f^k \cdot w) = (fe + h) \cdot (f^{k-1} \cdot w).$$

By the inductive hypothesis, $e \cdot (f^{k-1} \cdot w)$ is in the span of the $f^j \cdot w$ for $j < k-1$ and therefore $fef^{k-1} \cdot w$ is in the span of all $f^j \cdot w$ for $j < k$. Moreover $hf^{k-1} \cdot w$ is a scalar multiple of $f^{k-1} \cdot w$. This gives the inductive step.

Now, since V is irreducible, the submodule spanned by the $f^k \cdot w$ for $0 \leq k \leq d$ is equal to V.

Step 2: In this step, we shall show that $\lambda = d$. The matrix of h with respect to the basis $w, f \cdot w, \ldots, f^d \cdot w$ of V is diagonal, with trace

$$\lambda + (\lambda - 2) + \ldots + (\lambda - 2d) = (d + 1)\lambda - (d + 1)d.$$

Since $[e, f] = h$, the matrix of h is equal to the commutator of the matrices of e and f, so it has trace zero and $\lambda = d$.

Step 3: To finish, we produce an explicit isomorphism $V \cong V_d$. As we have seen, V has basis $\{w, f \cdot w, \dots, f^d \cdot w\}$. Furthermore, V_d has basis

$$\{X^d, f \cdot X^d, \dots, f^d \cdot X^d\},$$

where $f^k \cdot X^d$ is a scalar multiple of $X^{d-k}Y^k$. Moreover, the eigenvalue of h on $f^k \cdot w$ is the same as the eigenvalue of h on $f^k \cdot X^d$. Clearly, to have a homomorphism, we must have a map which takes h-eigenvectors to h-eigenvectors for the same eigenvalue. So we may set

$$\psi(w) = X^d$$

and then we must define ψ by

$$\psi(f^k \cdot w) := f^k \cdot X^d.$$

This defines a vector space isomorphism, which commutes with the actions of f and h. To show that it also commutes with the action of e, we use induction on k and a method similar to Step 1. Explicitly, for $k = 0$ we have $\psi(e \cdot w) = 0$ and $e\psi(w) = e \cdot X^d = 0$. For the inductive step,

$$\psi(ef^k \cdot w) = \psi((fe + h) \cdot (f^{k-1} \cdot w)) = f \cdot \psi(ef^{k-1} \cdot w) + h \cdot \psi(f^{k-1} \cdot w)$$

using that ψ commutes with f and h. We use the inductive hypothesis to take e out and obtain that the expression can be written as

$$(fe + h) \cdot \psi(f^{k-1} \cdot w) = ef \cdot \psi(f^{k-1} \cdot w) = e \cdot \psi(f^k \cdot w). \qquad \square$$

Corollary 8.6

If V is a finite-dimensional representation of sl$(2, \mathbf{C})$ and $w \in V$ is an h-eigenvector such that $e \cdot w = 0$, then $h \cdot w = dw$ for some non-negative integer d and the submodule of V generated by w is isomorphic to V_d.

Proof

Step 1 in the previous proof shows that for some $d \geq 0$ the vectors $w, f \cdot w, \dots, f^d \cdot w$ span a submodule of V. Now apply steps 2 and 3 to this submodule to get the required conclusions. $\square$

A vector v of the type considered in this corollary is known as a *highest-weight vector*. If d is the associated eigenvalue of h, then d is said to be a *highest weight*. (See §15.1 for a more general setting.)

8.3 Weyl's Theorem

In Exercise 7.6, we gave an example of a module for a Lie algebra that was not completely reducible; that is, it could not be written as a direct sum of irreducible submodules. Finite-dimensional representations of complex semisimple Lie algebras however are much better behaved.

Theorem 8.7 (Weyl's Theorem)

Let L be a complex semisimple Lie algebra. Every finite-dimensional representation of L is completely reducible.

The proof of Weyl's Theorem is fairly long, so we defer it to Appendix B. Weyl's Theorem tells us that to understand the finite-dimensional representations of a semisimple Lie algebra it is sufficient to understand its irreducible representations. We give an introduction to this topic in §15.1.

In the main part of this book, we shall only need to apply Weyl's Theorem to representations of $\mathsf{sl}(2, \mathbf{C})$, in which case a somewhat easier proof, exploiting properties of highest-weight vectors, is possible. Exercise 8.6 in this chapter does the groundwork, and the proof is finished in Exercise 9.15. (Both of these exercises have solutions in Appendix E.)

EXERCISES

8.2. Find explicit isomorphisms between

(i) the trivial representation of $\mathsf{sl}(2, \mathbf{C})$ and V_0;

(ii) the natural representation of $\mathsf{sl}(2, \mathbf{C})$ and V_1;

(iii) the adjoint representation of $\mathsf{sl}(2, \mathbf{C})$ and V_2.

8.3. Show that the subalgebra of $\mathsf{sl}(3, \mathbf{C})$ consisting of matrices of the form

$$\begin{pmatrix} \star & \star & 0 \\ \star & \star & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

is isomorphic to $\mathsf{sl}(2, \mathbf{C})$. We may therefore regard $\mathsf{sl}(3, \mathbf{C})$ as a module for $\mathsf{sl}(2, \mathbf{C})$, with the action given by $x \cdot y = [x, y]$ for $x \in \mathsf{sl}(2, \mathbf{C})$ and $y \in \mathsf{sl}(3, \mathbf{C})$. Show that as an $\mathsf{sl}(2, \mathbf{C})$-module

$$\mathsf{sl}(3, \mathbf{C}) \cong V_2 \oplus V_1 \oplus V_1 \oplus V_0.$$

8.4. Suppose that V is a finite-dimensional module for $\mathsf{sl}(2, \mathbf{C})$. Show, by using Weyl's Theorem and the classification of irreducible representations in this chapter, that V is determined up to isomorphism by the eigenvalues of h. In particular, prove that if V is the direct sum of k irreducible modules, then

$$k = \dim W_0 + \dim W_1,$$

where $W_r = \{v \in V : h \cdot v = rv\}$.

8.5. Let V be an $\mathsf{sl}(2, \mathbf{C})$-module, not necessarily finite-dimensional. Suppose $w \in V$ is a highest-weight vector of weight λ; that is, $e \cdot w = 0$ and $h \cdot w = \lambda w$ for some $\lambda \in \mathbf{C}$, and $w \neq 0$. Show that

(i) for $k = 1, 2, \ldots$ we have $e \cdot (f^k \cdot w) = k(\lambda - k + 1) f^{k-1} \cdot w$, and

(ii) $e^k f^k \cdot w = (k!)^2 \binom{\lambda}{k} w$.

Deduce that if $\binom{\lambda}{k} \neq 0$ then the set of all $f^j \cdot w$ for $0 \leq j \leq k$ is linearly independent. Hence show that if V is finite-dimensional, then λ must be a non-negative integer.

8.6.† Let M be a finite-dimensional $\mathsf{sl}(2, \mathbf{C})$-module. Define a linear map $c : M \longrightarrow M$ by

$$c(v) = \left(ef + fe + \frac{1}{2}h^2\right) \cdot v \quad \text{for } v \in M.$$

(i) Show that c is a homomorphism of $\mathsf{sl}(2, \mathbf{C})$-modules. *Hint*: For example, to show that c commutes with the action of e, show that $(efe + fe^2 + \frac{1}{2}h^2e) \cdot v$ and $(e^2f + efe + \frac{1}{2}eh^2) \cdot v$ are both equal to $(2efe + \frac{1}{2}heh) \cdot v$.

(ii) By Schur's Lemma, c must act as a scalar, say λ_d, on the irreducible module V_d. Show that $\lambda_d = \frac{1}{2}d(d+2)$, and deduce that d is determined by λ_d.

(iii) Let $\lambda_1, \ldots, \lambda_r$ be the distinct eigenvalues of c acting on M. Let the primary decomposition of M be

$$M = \bigoplus_{i=1}^{r} \ker(c - \lambda_i 1_M)^{m_i}.$$

Show that the summands are $\mathsf{sl}(2, \mathbf{C})$-submodules.

So, to express the module as a direct sum of simple modules, we therefore may assume that M has just one generalised eigenspace, where c has, say, eigenvalue λ.

(iv) Let U be an irreducible submodule of M. Suppose that U is isomorphic to V_d. Show by considering the action of c on V_d that $\lambda = \frac{1}{2}d(d+2)$ and hence that *any* irreducible submodule of M is isomorphic to V_d.

(v) Show more generally that if N is a submodule of M, then any irreducible submodule of M/N is isomorphic to V_d.

The linear map c is known as the *Casimir operator*. The following exercise gives an indication of how it was first discovered; it will appear again in the proof of Weyl's Theorem (see Appendix B).

8.7. Exercise 1.14 gives a way to embed the real Lie algebra $\mathbf{R}^3_\wedge$ into $\mathsf{sl}(2, \mathbf{C})$. With the given solution, we would take

$$\psi(x) = \begin{pmatrix} 0 & 1/2 \\ -1/2 & 0 \end{pmatrix}, \psi(y) = \begin{pmatrix} 0 & -i/2 \\ -i/2 & 0 \end{pmatrix}, \psi(z) = \begin{pmatrix} -i/2 & 0 \\ 0 & i/2 \end{pmatrix}$$

Check that $\psi(x)^2 + \psi(y)^2 + \psi(z)^2 = -3/4I$, where I is the 2×2 identity matrix. By expressing x, y, z in terms of e, f, h, recover the description of the Casimir operator given above.

The interested reader might like to look up "angular momentum" or "Pauli matrices" in a book on quantum mechanics to see the physical interpretation of the Casimir operator.