4
Solvable Lie Algebras and a Rough Classification

Abelian Lie algebras are easily understood. There is a sense in which some of the low-dimensional Lie algebras we studied in Chapter 3 are close to being abelian. For example, the 3-dimensional Heisenberg algebra discussed in §3.2.1 has a 1-dimensional centre. The quotient algebra modulo this ideal is also abelian. We ask when something similar might hold more generally. That is, to what extent can we "approximate" a Lie algebra by abelian Lie algebras?

4.1 Solvable Lie Algebras

To start, we take an ideal I of a Lie algebra L and ask when the factor algebra L/I is abelian. The following lemma provides the answer.

Lemma 4.1

Suppose that I is an ideal of L. Then L/I is abelian if and only if I contains the derived algebra L'.

K. Erdmann et al., *Introduction to Lie Algebras*

Proof

The algebra L/I is abelian if and only if for all $x, y \in L$ we have

$$[x + I, y + I] = [x, y] + I = I$$

or, equivalently, for all $x, y \in L$ we have $[x, y] \in I$. Since I is a subspace of L, this holds if and only if the space spanned by the brackets $[x, y]$ is contained in I; that is, $L' \subseteq I$. □

This lemma tells us that the derived algebra L' is the smallest ideal of L with an abelian quotient. By the same argument, the derived algebra L' itself has a smallest ideal whose quotient is abelian, namely the derived algebra of L', which we denote $L^{(2)}$, and so on. We define the *derived series* of L to be the series with terms

$$L^{(1)} = L' \quad \text{and} \quad L^{(k)} = [L^{(k-1)}, L^{(k-1)}] \text{ for } k \geq 2.$$

Then $L \supseteq L^{(1)} \supseteq L^{(2)} \supseteq \ldots$.

As the product of ideals is an ideal, $L^{(k)}$ is an ideal of L (and not just an ideal of $L^{(k-1)}$).

Definition 4.2

The Lie algebra L is said to be *solvable* if for some $m \geq 1$ we have $L^{(m)} = 0$.

The Heisenberg algebra is solvable. Similarly, the algebra of upper triangular matrices is solvable (see Exercise 4.5 below). Furthermore, the classification of 2-dimensional Lie algebras in §3.1 shows that any 2-dimensional Lie algebra is solvable. On the other hand, if $L = \mathsf{sl}(2, \mathbf{C})$, then we have seen in Exercise 2.2 that $L = L'$ and therefore $L^{(m)} = L$ for all $m \geq 1$, so $\mathsf{sl}(2, \mathbf{C})$ is not solvable.

If L is solvable, then the derived series of L provides us with an "approximation" of L by a finite series of ideals with abelian quotients. This also works the other way around.

Lemma 4.3

If L is a Lie algebra with ideals

$$L = I_0 \supseteq I_1 \supseteq \ldots \supseteq I_{m-1} \supseteq I_m = 0$$

such that I_{k-1}/I_k is abelian for $1 \leq k \leq m$, then L is solvable.

Proof

We shall show that $L^{(k)}$ is contained in I_k for k between 1 and m. Putting $k = m$ will then give $L^{(m)} = 0$.

Since L/I_1 is abelian, we have from Lemma 4.1 that $L' \subseteq I_1$. For the inductive step, we suppose that $L^{(k-1)} \subseteq I_{k-1}$, where $k \geq 2$. The Lie algebra I_{k-1}/I_k is abelian. Therefore by Lemma 4.1, this time applied to the Lie algebra I_{k-1}, we have $[I_{k-1}, I_{k-1}] \subseteq I_k$. But $L^{(k-1)}$ is contained in I_{k-1} by our inductive hypothesis, so we deduce that

$$L^{(k)} = [L^{(k-1)}, L^{(k-1)}] \subseteq [I_{k-1}, I_{k-1}],$$

and hence $L^{(k)} \subseteq I_k$. □

This proof shows that if $L^{(k)}$ is non-zero then I_k is also non-zero. Hence the derived series may be thought of as the fastest descending series whose successive quotients are abelian.

Lie algebra homomorphisms are linear maps that preserve Lie brackets, and so one would expect that they preserve the derived series.

Exercise 4.1

Suppose that $\varphi : L_1 \to L_2$ is a surjective homomorphism of Lie algebras. Show that

$$\varphi(L_1^{(k)}) = (L_2)^{(k)}.$$

This exercise suggests that the property of being solvable should be inherited by various constructions.

Lemma 4.4

Let L be a Lie algebra.

(a) If L is solvable, then every subalgebra and every homomorphic image of L are solvable.

(b) Suppose that L has an ideal I such that I and L/I are solvable. Then L is solvable.

(c) If I and J are solvable ideals of L, then $I + J$ is a solvable ideal of L.

Proof

(a) If L_1 is a subalgebra of L, then for each k it is clear that $L_1^{(k)} \subseteq L^{(k)}$, so if $L^{(m)} = 0$, then also $L_1^{(m)} = 0$. For the second part, apply Exercise 4.1.

(b) We have $(L/I)^{(k)} = \left(L^{(k)} + I\right)/I$. (Either apply Exercise 4.1 to the canonical homomorphism $L \to L/I$ or prove this directly by induction on k.) If L/I is solvable then for some $m \geq 1$ we have $(L/I)^{(m)} = 0$; that is, $L^{(m)} + I = I$ and therefore $L^{(m)} \subseteq I$. If I is also solvable, then $I^{(k)} = 0$ for some $k \geq 1$ and hence $(L^{(m)})^{(k)} \subseteq I^{(k)} = 0$. Now one can convince oneself that by definition

$$(L^{(m)})^{(k)} = L^{(m+k)}.$$

(c) By the second isomorphism theorem $(I + J)/I \cong J/I \cap J$, so it is solvable by Lemma 4.4(a). Since I is also solvable, part (b) of this lemma implies that $I + J$ is solvable. □

Corollary 4.5

Let L be a finite-dimensional Lie algebra. There is a unique solvable ideal of L containing every solvable ideal of L.

Proof

Let R be a solvable ideal of largest possible dimension. Suppose that I is any solvable ideal. By Lemma 4.4(c), we know that $R + I$ is a solvable ideal. Now $R \subseteq R + I$ and hence $\dim R \leq \dim(R + I)$. We chose R of maximal possible dimension and therefore we must have $\dim R = \dim(R+I)$ and hence $R = R+I$, so I is contained in R. □

This largest solvable ideal is said to be the *radical* of L and is denoted $\operatorname{rad} L$. The radical will turn out to be an essential tool in helping to describe the finite-dimensional Lie algebras. It also suggests the following definition.

Definition 4.6

A non-zero finite-dimensional Lie algebra L is said to be *semisimple* if it has no non-zero solvable ideals or equivalently if $\operatorname{rad} L = 0$.

For example, by Exercise 1.13, $\mathsf{sl}(2, \mathbf{C})$ is semisimple. The reason for the word "semisimple" is revealed in §4.3 below.

Lemma 4.7

If L is a Lie algebra, then the factor algebra $L/\operatorname{rad} L$ is semisimple.

Proof

Let $\bar{J}$ be a solvable ideal of $L/\operatorname{rad} L$. By the ideal correspondence, there is an ideal J of L containing $\operatorname{rad} L$ such that $\bar{J} = J/\operatorname{rad} L$. By definition, $\operatorname{rad} L$ is solvable, and $J/\operatorname{rad} L = \bar{J}$ is solvable by hypothesis. Therefore Lemma 4.4 implies that J is solvable. But then J is contained in $\operatorname{rad} L$; that is, $\bar{J} = 0$. $\square$

4.2 Nilpotent Lie Algebras

We define the *lower central series* of a Lie algebra L to be the series with terms

$$L^1 = L' \quad \text{and} \quad L^k = [L, L^{k-1}] \text{ for } k \geq 2.$$

Then $L \supseteq L^1 \supseteq L^2 \supseteq \ldots$. As the product of ideals is an ideal, L^k is even an ideal of L (and not just an ideal of L^{k-1}). The reason for the name "central series" comes from the fact that L^k/L^{k+1} is contained in the centre of L/L^{k+1}.

Definition 4.8

The Lie algebra L is said to be *nilpotent* if for some $m \geq 1$ we have $L^m = 0$.

The Lie algebra $\mathsf{n}(n, F)$ of strict upper triangular matrices over a field F is nilpotent (see Exercise 4.4). Furthermore, any nilpotent Lie algebra is solvable. To see this, show by induction on k that $L^{(k)} \subseteq L^k$. There are solvable Lie algebras which are not nilpotent; the standard example is the Lie algebra $\mathsf{b}(n, F)$ of upper triangular matrices over a field F for $n \geq 2$ (see Exercise 4.5). Another is the two-dimensional non-abelian Lie algebra (see §3.1).

Lemma 4.9

Let L be a Lie algebra.

(a) If L is nilpotent, then any Lie subalgebra of L is nilpotent.

(b) If $L/Z(L)$ is nilpotent, then L is nilpotent.

Proof

Part (a) is clear from the definition. By induction, or by a variation of Exercise 4.1, one can show that $(L/Z(L))^k$ is equal to $\left(L^k + Z(L)\right)/Z(L)$. So if $(L/Z(L))^m$ is zero, then L^m is contained in $Z(L)$ and therefore $L^{m+1} = 0$. $\square$

Remark 4.10

The analogue of Lemma 4.4(b) does not hold; that is, if I is an ideal of a Lie algebra L, then it is possible that both L/I and I are nilpotent but L is not. An example is given by the 2-dimensional non-abelian Lie algebra. This perhaps suggests that solvability is more fundamental to the structure of Lie algebras than nilpotency.

4.3 A Look Ahead

The previous section suggests that we might have a chance to understand all finite-dimensional Lie algebras. The radical $\operatorname{rad} L$ of any Lie algebra L is solvable, and $L/\operatorname{rad} L$ is semisimple, so to understand L it is necessary to understand

(i) an arbitrary solvable Lie algebra and

(ii) an arbitrary semisimple Lie algebra.

Working over $\mathbf{C}$, an answer to (i) was found by Lie, who proved (in essence) that every solvable Lie algebra appears as a subalgebra of a Lie algebras of upper triangular matrices. We give a precise statement of Lie's Theorem in §6.4 below.

For (ii) we shall show that every semisimple Lie algebra is a direct sum of *simple* Lie algebras.

Definition 4.11

The Lie algebra L is *simple* if it has no ideals other than 0 and L and it is not abelian.

The restriction that a simple Lie algebra may not be abelian removes only the 1-dimensional abelian Lie algebra. Without this restriction, this Lie algebra would be simple but not semisimple: This is obviously undesirable.

We then need to find all simple Lie algebras over $\mathbf{C}$. This is a major theorem; the proof is based on work by Killing, Engel, and Cartan. We shall eventually prove most of the following theorem.

Theorem 4.12 (Simple Lie algebras)

With five exceptions, every finite-dimensional simple Lie algebra over $\mathbf{C}$ is isomorphic to one of the *classical Lie algebras*:

$$\mathsf{sl}(n,\mathbf{C}), \quad \mathsf{so}(n,\mathbf{C}), \quad \mathsf{sp}(2n,\mathbf{C}).$$

The five exceptional Lie algebras are known as e_6, e_7, e_8, f_4, and g_2.

We have already introduced the family of special linear Lie algebras, $\mathsf{sl}(n,\mathbf{C})$. The remaining families can be defined as certain subalgebras of $\mathsf{gl}(n,\mathbf{C})$ using the construction introduced in Exercise 1.15. Recall that if $S \in \mathsf{gl}(n,\mathbf{C})$, then we defined a Lie subalgebra of $\mathsf{gl}(n,\mathbf{C})$ by

$$\mathsf{gl}_S(n,\mathbf{C}) := \left\{x \in \mathsf{gl}(n,\mathbf{C}) : x^t S = -Sx\right\}.$$

Assume first of all that $n = 2\ell$. Take S to be the matrix with $\ell \times \ell$ blocks:

$$S = \begin{pmatrix} 0 & I_\ell \\ I_\ell & 0 \end{pmatrix}.$$

We define $\mathsf{so}(2\ell,\mathbf{C}) = \mathsf{gl}_S(2\ell,\mathbf{C})$. When $n = 2\ell+1$, we take

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_\ell \\ 0 & I_\ell & 0 \end{pmatrix}$$

and define $\mathsf{so}(2\ell+1,\mathbf{C}) = \mathsf{gl}_S(2\ell+1,\mathbf{C})$. These Lie algebras are known as the *orthogonal Lie algebras*.

The Lie algebras $\mathsf{sp}(n,\mathbf{C})$ are only defined for even n. If $n = 2\ell$, we take

$$S = \begin{pmatrix} 0 & I_\ell \\ -I_\ell & 0 \end{pmatrix}$$

and define $\mathsf{sp}(2\ell,\mathbf{C}) = \mathsf{gl}_S(2\ell,\mathbf{C})$. These Lie algebras are known as the *symplectic Lie algebras*.

It follows from Exercise 2.12 that $\mathsf{so}(n,\mathbf{C})$ and $\mathsf{sp}(n,\mathbf{C})$ are subalgebras of $\mathsf{sl}(n,\mathbf{C})$. (This also follows from the explicit bases given in Chapter 12.)

We postpone discussion of the exceptional Lie algebras until Chapter 14.

Exercise 4.2

Let $x \in \mathsf{gl}(2\ell,\mathbf{C})$. Show that x belongs to $\mathsf{sp}(2\ell,\mathbf{C})$ if and only if it is of the form

$$x = \begin{pmatrix} m & p \\ q & -m^t \end{pmatrix},$$

where p and q are symmetric. Hence find the dimension of $\mathsf{sp}(2\ell,\mathbf{C})$. (See Exercise 12.1 for the other families.)

EXERCISES

4.3. Use Lemma 4.4 to show that if L is a Lie algebra then L is solvable if and only if $\operatorname{ad} L$ is a solvable subalgebra of $\mathsf{gl}(L)$. Show that this result also holds if we replace "solvable" with "nilpotent."

4.4. Let $L = \mathsf{n}(n, F)$, the Lie algebra of strictly upper triangular $n \times n$ matrices over a field F. Show that L^k has a basis consisting of all the matrix units e_{ij} with $j - i > k$. Hence show that L is nilpotent. What is the smallest m such that $L^m = 0$?

4.5. Let $L = \mathsf{b}(n, F)$ be the Lie algebra of upper triangular $n \times n$ matrices over a field F.

(i) Show that $L' = \mathsf{n}(n, F)$.

(ii) More generally, show that $L^{(k)}$ has a basis consisting of all the matrix units e_{ij} with $j - i \geq 2^{k-1}$. (The commutator formula for the e_{ij} given in §1.2 will be helpful.)

(iii) Hence show that L is solvable. What is the smallest m such that $L^{(m)} = 0$?

(iv) Show that if $n \geq 2$ then L is not nilpotent.

4.6. Show that a Lie algebra is semisimple if and only if it has no non-zero abelian ideals. (This was the original definition of semisimplicity given by Wilhelm Killing.)

4.7. Prove directly that $\mathsf{sl}(n, \mathbf{C})$ is a simple Lie algebra for $n \geq 2$.

4.8.† Let L be a Lie algebra over a field F such that $[[a, b], b] = 0$ for all $a, b \in L$, (or equivalently, $(\operatorname{ad} b)^2 = 0$ for all $b \in L$).

(i) Suppose the characteristic of F is not 3. Show that then $L^3 = 0$.

(ii)⋆ Show that if F has characteristic 3 then $L^4 = 0$. *Hint*: show first that the Lie brackets $[[x, y], z]$ are alternating; that is,

$$[[x, y], z] = -[[y, x], z], \quad [[x, y], z] = -[[x, z], y]$$

for all $x, y, z \in L$.

4.9.⋆ The purpose of this exercise is to give some idea why the families of Lie algebras are given the names that we have used. We shall not need to refer to this exercise later; some basic group theory is needed.

We begin with the Lie algebra $\mathsf{sl}(n, \mathbf{C})$. Recall that the $n \times n$ matrices with determinant 1 form a group under matrix multiplication, denoted $\mathrm{SL}(n, \mathbf{C})$. Let I denote the $n \times n$ identity matrix. We ask: when is $I + \varepsilon X \in \mathrm{SL}(n, \mathbf{C})$ for X an $n \times n$ matrix?

(i) Show that $\det(I + \varepsilon X)$ is a polynomial in ε of degree n with the first two terms

$$\det(I + \varepsilon X) = 1 + (\operatorname{tr} X)\varepsilon + \ldots.$$

If we neglect all powers of ε except 1 and ε, then we obtain the statement

$$I + \varepsilon X \in \mathrm{SL}(n, \mathbf{C}) \iff X \in \mathsf{sl}(n, \mathbf{C}).$$

This could have been taken as the definition of $\mathsf{sl}(n, \mathbf{C})$. (This is despite the fact that, interpreted literally, it is false!)

(ii) (a) Let S be an $n \times n$ matrix. Let $(-,-)$ denote the complex bilinear form with matrix S. Show that if we let $G_S(n, \mathbf{C})$ be the set of invertible matrices A such that $(Av, Av) = (v, v)$ for all $v \in \mathbf{C}^n$, then $G_S(n, \mathbf{C})$ is a group.

(b) Show that if we perform the construction in part (i) with $G_S(n, \mathbf{C})$ in place of $\mathrm{SL}(n, \mathbf{C})$, we obtain $\mathsf{gl}_S(n, \mathbf{C})$.

(iii) (a) An invertible matrix A is customarily said to be *orthogonal* if $A^t A = AA^t = I$; that is, if $A^{-1} = A^t$. Show that the set of $n \times n$ orthogonal matrices with coefficients in $\mathbf{C}$ is the group $G_I(n, \mathbf{C})$ and that the associated Lie algebra, $g_I(n, \mathbf{C})$, is the space of all anti-symmetric matrices.

(b) Prove that $g_I(n, \mathbf{C}) \cong \mathsf{so}(n, \mathbf{C})$. *Hint*: Use Exercise 2.11. (The reason for not taking this as the definition of $\mathsf{so}(n, \mathbf{C})$ will emerge.)

(iv) A bilinear form (see Appendix A) on a vector space v is said to be *symplectic* if $(v, v) = 0$ for all $v \in V$. Show that

$$S = \begin{pmatrix} 0 & I_\ell \\ -I_\ell & 0 \end{pmatrix}$$

is the matrix of a non-degenerate symplectic bilinear form on a 2ℓ-dimensional space. The associated Lie algebra is $\mathsf{gl}_S(2\ell, \mathbf{C}) = \mathsf{sp}(2\ell, \mathbf{C})$.

The reader is entitled to feel rather suspicious about our cavalier treatment of the powers of ε. A rigorous and more general treatment is given in books on matrix groups and Lie groups, such as *Matrix Groups* by Baker [3] in the SUMS series. We shall not attempt to go any further in this direction.

4.10.* Let F be a field. Exercise 2.11 shows that if $S, T \in \mathsf{gl}(n, F)$ are congruent matrices (that is, there exists an invertible matrix P such that $T = P^t S P$), then $\mathsf{gl}_S(n, F) \cong \mathsf{gl}_T(n, F)$. Does the converse hold when $F = \mathbf{C}$? For a challenge, think about other fields.