2
Ideals and Homomorphisms

In this chapter we explore some of the constructions in which ideals are involved. We shall see that in the theory of Lie algebras ideals play a role similar to that played by normal subgroups in the theory of groups. For example, we saw in Exercise 1.6 that the kernel of a Lie algebra homomorphism is an ideal, just as the kernel of a group homomorphism is a normal subgroup.

2.1 Constructions with Ideals

Suppose that I and J are ideals of a Lie algebra L. There are several ways we can construct new ideals from I and J. First we shall show that $I \cap J$ is an ideal of L. We know that $I \cap J$ is a subspace of L, so all we need check is that if $x \in L$ and $y \in I \cap J$, then $[x, y] \in I \cap J$: This follows at once as I and J are ideals.

Exercise 2.1

Show that $I + J$ is an ideal of L where

$$I + J := \{x + y : x \in I,\ y \in J\}.$$

We can also define a product of ideals. Let

$$[I, J] := \operatorname{Span}\{[x, y] : x \in I, y \in J\}.$$

K. Erdmann et al., *Introduction to Lie Algebras*

We claim that $[I, J]$ is an ideal of L. Firstly, it is by definition a subspace. Secondly, if $x \in I$, $y \in J$, and $u \in L$, then the Jacobi identity gives

$$[u, [x, y]] = [x, [u, y]] + [[u, x], y].$$

Here $[u, y] \in J$ as J is an ideal, so $[x, [u, y]] \in [I, J]$. Similarly, $[[u, x], y] \in [I, J]$. Therefore their sum belongs to $[I, J]$.

A general element t of $[I, J]$ is a linear combination of brackets $[x, y]$ with $x \in I$, $y \in J$, say $t = \sum c_i[x_i, y_i]$, where the c_i are scalars and $x_i \in I$ and $y_i \in J$. Then, for any $u \in L$, we have

$$[u, t] = \Big[u, \sum c_i[x_i, y_i]\Big] = \sum c_i[u, [x_i, y_i]],$$

where $[u, [x_i, y_i]] \in [I, J]$ as shown above. Hence $[u, t] \in [I, J]$ and so $[I, J]$ is an ideal of L.

Remark 2.1

It is necessary to define $[I, J]$ to be the *span* of the commutators of elements of I and J rather than just the *set* of such commutators. See Exercise 2.14 below for an example where the set of commutators is not itself an ideal.

An important example of this construction occurs when we take $I = J = L$. We write L' for $[L, L]$: Despite being an ideal of L, L' is usually known as the *derived algebra* of L'. The term *commutator algebra* is also sometimes used.

Exercise 2.2

Show that $\mathsf{sl}(2, \mathbf{C})' = \mathsf{sl}(2, \mathbf{C})$.

2.2 Quotient Algebras

If I is an ideal of the Lie algebra L, then I is in particular a subspace of L, and so we may consider the cosets $z + I = \{z + x : x \in I\}$ for $z \in L$ and the quotient vector space

$$L/I = \{z + I : z \in L\}.$$

We review the vector space structure of L/I in Appendix A. We claim that a Lie bracket on L/I may be defined by

$$[w + I, z + I] := [w, z] + I \quad \text{for } w, z \in L.$$

Here the bracket on the right-hand side is the Lie bracket in L. To be sure that the Lie bracket on L/I is well-defined, we must check that $[w, z] + I$

depends only on the cosets containing w and z and not on the particular coset representatives w and z. Suppose $w + I = w' + I$ and $z + I = z' + I$. Then $w - w' \in I$ and $z - z' \in I$. By bilinearity of the Lie bracket in L,

$$\begin{aligned}[w', z'] &= [w' + (w - w'), z' + (z - z')] \\ &= [w, z] + [w - w', z'] + [w', z - z'] + [w - w', z - z'],\end{aligned}$$

where the final three summands all belong to I. Therefore $[w' + I, z' + I] = [w, z] + I$, as we needed. It now follows from part (i) of the exercise below that L/I is a Lie algebra. It is called the *quotient* or *factor algebra* of L by I.

Exercise 2.3

(i) Show that the Lie bracket defined on L/I is bilinear and satisfies the axioms (L1) and (L2).

(ii) Show that the linear transformation $\pi : L \to L/I$ which takes an element $z \in L$ to its coset $z + I$ is a homomorphism of Lie algebras.

The reader will not be surprised to learn that there are isomorphism theorems for Lie algebras just as there are for vector spaces and for groups.

Theorem 2.2 (Isomorphism theorems)

(a) Let $\varphi : L_1 \to L_2$ be a homomorphism of Lie algebras. Then $\ker \varphi$ is an ideal of L_1 and $\operatorname{im} \varphi$ is a subalgebra of L_2, and

$$L_1/\ker \varphi \cong \operatorname{im} \varphi.$$

(b) If I and J are ideals of a Lie algebra, then $(I + J)/J \cong I/(I \cap J)$.

(c) Suppose that I and J are ideals of a Lie algebra L such that $I \subseteq J$. Then J/I is an ideal of L/I and $(L/I)/(J/I) \cong L/J$.

Proof

That $\ker \varphi$ is an ideal of L_1 and $\operatorname{im} \varphi$ is a subalgebra of L_2 were proved in Exercise 1.6. All the isomorphisms we need are already known for vector spaces and their subspaces (see Appendix A): By part (ii) of Exercise 2.3, they are also homomorphisms of Lie algebras. $\square$

Parts (a), (b), and (c) of this theorem are known respectively as the *first*, *second*, and *third isomorphism theorems*.

Example 2.3

Recall that the trace of an $n \times n$ matrix is the sum of its diagonal entries. Fix a field F and consider the linear map $\operatorname{tr} : \mathsf{gl}(n, F) \to F$ which sends a matrix to its trace. This is a Lie algebra homomorphism, for if $x, y \in \mathsf{gl}(n, F)$ then

$$\operatorname{tr}[x, y] = \operatorname{tr}(xy - yx) = \operatorname{tr} xy - \operatorname{tr} yx = 0,$$

so $\operatorname{tr}[x, y] = [\operatorname{tr} x, \operatorname{tr} y] = 0$. Here the first Lie bracket is taken in $\mathsf{gl}(n, F)$ and the second in the abelian Lie algebra F.

It is not hard to see that tr is surjective. Its kernel is $\mathsf{sl}(n, F)$, the Lie algebra of matrices with trace 0. Applying the first isomorphism theorem gives

$$\mathsf{gl}(n, F)/\mathsf{sl}(n, F) \cong F.$$

We can describe the elements of the factor Lie algebra explicitly: The coset $x + \mathsf{sl}_n(F)$ consists of those $n \times n$ matrices whose trace is $\operatorname{tr} x$.

Exercise 2.4

Show that if L is a Lie algebra then $L/Z(L)$ is isomorphic to a subalgebra of $\mathsf{gl}(L)$.

2.3 Correspondence between Ideals

Suppose that I is an ideal of the Lie algebra L. There is a bijective correspondence between the ideals of the factor algebra L/I and the ideals of L that contain I. This correspondence is as follows. If J is an ideal of L containing I, then J/I is an ideal of L/I. Conversely, if K is an ideal of L/I, then set $J := \{z \in L : z + I \in K\}$. One can readily check that J is an ideal of L and that J contains I. These two maps are inverses of one another.

Example 2.4

Suppose that L is a Lie algebra and I is an ideal in L such that L/I is abelian. In this case, the ideals of L/I are just the subspaces of L/I. By the ideal correspondence, the ideals of L which contain I are exactly the subspaces of L which contain I.

EXERCISES

2.5.† Show that if $z \in L'$ then $\operatorname{tr} \operatorname{ad} z = 0$.

2.6. Suppose L_1 and L_2 are Lie algebras. Let $L := \{(x_1, x_2) : x_i \in L_i\}$ be the direct sum of their underlying vector spaces. Show that if we define

$$[(x_1, x_2), (y_1, y_2)] := ([x_1, y_1], [x_2, y_2])$$

then L becomes a Lie algebra, the *direct sum* of L_1 and L_2. As for vector spaces, we denote the direct sum of Lie algebras L_1 and L_2 by $L = L_1 \oplus L_2$.

(i) Prove that $\mathsf{gl}(2, \mathbf{C})$ is isomorphic to the direct sum of $\mathsf{sl}(2, \mathbf{C})$ with $\mathbf{C}$, the 1-dimensional complex abelian Lie algebra.

(ii) Show that if $L = L_1 \oplus L_2$ then $Z(L) = Z(L_1) \oplus Z(L_2)$ and $L' = L_1' \oplus L_2'$. Formulate a general version for a direct sum $L_1 \oplus \ldots \oplus L_k$.

(iii) Are the summands in the direct sum decomposition of a Lie algebra uniquely determined? *Hint*: If you think the answer is yes, now might be a good time to read §16.4 in Appendix A on the "diagonal fallacy". The next question looks at this point in more detail.

2.7. Suppose that $L = L_1 \oplus L_2$ is the direct sum of two Lie algebras.

(i) Show that $\{(x_1, 0) : x_1 \in L_1\}$ is an ideal of L isomorphic to L_1 and that $\{(0, x_2) : x_2 \in L_2\}$ is an ideal of L isomorphic to L_2. Show that the projections $p_1(x_1, x_2) = x_1$ and $p_2(x_1, x_2) = x_2$ are Lie algebra homomorphisms.

Now suppose that L_1 and L_2 do not have any non-trivial proper ideals.

(ii) Let J be a proper ideal of L. Show that if $J \cap L_1 = 0$ and $J \cap L_2 = 0$, then the projections $p_1 : J \to L_1$ and $p_2 : J \to L_2$ are isomorphisms.

(iii) Deduce that if L_1 and L_2 are not isomorphic as Lie algebras, then $L_1 \oplus L_2$ has only two non-trivial proper ideals.

(iv) Assume that the ground field is infinite. Show that if $L_1 \cong L_2$ and L_1 is 1-dimensional, then $L_1 \oplus L_2$ has infinitely many different ideals.

2.8. Let L_1 and L_2 be Lie algebras, and let $\varphi : L_1 \to L_2$ be a surjective Lie algebra homomorphism. True or false:

(a)† $\varphi(L_1') = L_2'$;

(b) $\varphi(Z(L_1)) = Z(L_2)$;

(c) if $h \in L_1$ and $\operatorname{ad} h$ is diagonalisable then $\operatorname{ad} \varphi(h)$ is diagonalisable.

What is different if φ is an isomorphism?

2.9. For each pair of the following Lie algebras over $\mathbf{R}$, decide whether or not they are isomorphic:

(i) the Lie algebra $\mathbf{R}^3_\wedge$ where the Lie bracket is given by the vector product;

(ii) the upper triangular 2×2 matrices over $\mathbf{R}$;

(iii) the strictly upper triangular 3×3 matrices over $\mathbf{R}$;

(iv) $L = \{x \in \mathsf{gl}(3, \mathbf{R}) : x^t = -x\}$.

Hint: Use Exercises 1.15 and 2.8.

2.10. Let F be a field. Show that the derived algebra of $\mathsf{gl}(n, F)$ is $\mathsf{sl}(n, F)$.

2.11.† In Exercise 1.15, we defined the Lie algebra $\mathsf{gl}_S(n, F)$ over a field F where S is an $n \times n$ matrix with entries in F.

Suppose that $T \in \mathsf{gl}(n, F)$ is another $n \times n$ matrix such that $T = P^t S P$ for some invertible $n \times n$ matrix $P \in \mathsf{gl}(n, F)$. (Equivalently, the bilinear forms defined by S and T are congruent.) Show that the Lie algebras $\mathsf{gl}_S(n, F)$ and $\mathsf{gl}_T(n, F)$ are isomorphic.

2.12. Let S be an $n \times n$ invertible matrix with entries in $\mathbf{C}$. Show that if $x \in \mathsf{gl}_S(n, \mathbf{C})$, then $\operatorname{tr} x = 0$.

2.13. Let I be an ideal of a Lie algebra L. Let B be the centraliser of I in L; that is,

$$B = C_L(I) = \{x \in L : [x, a] = 0 \text{ for all } a \in I\}.$$

Show that B is an ideal of L. Now suppose that

(1) $Z(I) = 0$, and

(2) if $D : I \to I$ is a derivation, then $D = \operatorname{ad} x$ for some $x \in I$.

Show that $L = I \oplus B$.

2.14.†⋆ Recall that if L is a Lie algebra, we defined L' to be the subspace spanned by the commutators $[x, y]$ for $x, y \in L$. The purpose of this exercise, which may safely be skipped on first reading, is to show that the *set* of commutators may not even be a vector space (and so certainly not an ideal of L).

Let $\mathbf{R}[x, y]$ denote the ring of all real polynomials in two variables. Let L be the set of all matrices of the form

$$A(f(x), g(y), h(x, y)) = \begin{pmatrix} 0 & f(x) & h(x,y) \\ 0 & 0 & g(y) \\ 0 & 0 & 0 \end{pmatrix}.$$

(i) Prove that L is a Lie algebra with the usual commutator bracket. (In contrast to all the Lie algebras seen so far, L is infinite-dimensional.)

(ii) Prove that

$$[A(f_1(x), g_1(y), h_1(x, y)), A(f_2(x), g_2(y), h_2(x, y))] = \\ A(0, 0, f_1(x)g_2(y) - f_2(x)g_1(y)).$$

Hence describe L'.

(iii) Show that if $h(x, y) = x^2 + xy + y^2$, then $A(0, 0, h(x, y))$ is not a commutator.