15
Further Directions

Now that we have a good understanding of the complex semisimple Lie algebras, we can also hope to understand their representation theory. This is the first of the topics we shall discuss. By Weyl's Theorem, every finite-dimensional representation is a direct sum of irreducible representations; we shall outline their construction. An essential tool in the representation theory of Lie algebras is the universal enveloping algebra associated to a Lie algebra. We explain what this is and why it is important.

The presentation of complex semisimple Lie algebras by generators and relations given in Serre's Theorem has inspired the definition of new families of Lie algebras. These include the Kac–Moody Lie algebras and their generalisations, which also have been important in the remarkable "moonshine" conjectures.

The theory of complex simple Lie algebras was used by Chevalley to construct simple groups of matrices over any field. The resulting groups are now known as *Chevalley groups* or as *groups of Lie type*. We briefly explain the basic idea and give an example.

Going in the other direction, given a group with a suitable 'smooth' structure, one can define an associated Lie algebra and use it to study the group. It was in fact in this way that Lie algebras were first discovered. We have given a very rough indication of this process in Exercise 4.9; as there are already many accessible books in this area, for example *Matrix Groups* by Baker [3] in the SUMS series, we refer the reader to them for further reading.

A very spectacular application of the theory of Lie algebra to group theory occurs in the restricted Burnside problem, which we discuss in §15.5. This involves Lie algebras defined over fields with prime characteristic. Lie algebras defined over fields of prime characteristic occur in several other contexts; we shall mention restricted Lie algebras and give an example of a simple Lie algebra that does not have an analogue in characteristic zero.

As well as classifying complex semisimple Lie algebras, Dynkin diagrams also appear in the representation theory of associative algebras. We shall explain some of the theory involved. Besides the appearance of Dynkin diagrams, one reason for introducing this topic is that there is a surprising connection with the theory of complex semisimple Lie algebras.

The survey in this chapter is certainly not exhaustive, and in places it is deliberately informal. Our purpose is to describe the main ideas; more detailed accounts exist and we give references to those that we believe would be accessible to the interested reader. For accounts of the early history of Lie algebras we recommend *Wilhelm Killing and the Structure of Lie algebras*, by Hawkins [12] and *The Mathematician Sophus Lie*, by Stubhaug [23].

15.1 The Irreducible Representations of a Semisimple Lie Algebra

We begin by describing the classification of the finite-dimensional irreducible representations of a complex semisimple Lie algebra L. By Weyl's Theorem, we may then obtain all finite-dimensional representations by taking direct sums of irreducible representations.

Let L have Cartan subalgebra H and root system Φ. Choose a base $\Pi = \{\alpha_1, \ldots, \alpha_\ell\}$ of Φ and let Φ^+ and Φ^- denote respectively the positive and negative roots with respect to Π. It will be convenient to use the *triangular decomposition*

$$L = N^- \oplus H \oplus N^+.$$

Here $N^+ = \bigoplus_{\alpha\in\Phi^+} L_\alpha$ and $N^- = \bigoplus_{\alpha\in\Phi^+} L_{-\alpha}$. Note that the summands H, N^-, and N^+ are subalgebras of L.

15.1.1 General Properties

Suppose that V is a finite-dimensional representation of L. Each element of H is semisimple, so it acts diagonalisably on L (see Exercise 9.14). Since finitely many commuting linear maps can be simultaneously diagonalised, V has a basis of simultaneous eigenvectors for H.

We can therefore decompose V into weight spaces for H. For $\lambda \in H^\star$, let

$$V_\lambda = \{v \in V : h \cdot v = \lambda(h)v \text{ for all } h \in H\}$$

and let Ψ be the set of $\lambda \in H^\star$ for which $V_\lambda \neq 0$. The *weight space decomposition* of V is then

$$V = \bigoplus_{\lambda \in \Psi} V_\lambda.$$

Example 15.1

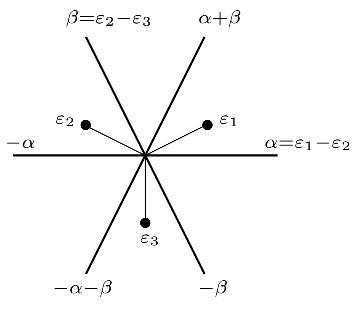
(1) Let $V = L$ with the adjoint representation. Then weights are the same thing as roots, and the weight space decomposition is just the root space decomposition.

(2) Let $L = \mathsf{sl}(3, \mathbf{C})$, let H be the Cartan subalgebra of diagonal matrices, and let $V = \mathbf{C}^3$ be its natural representation. The weights that appear are $\varepsilon_1, \varepsilon_2, \varepsilon_3$, where $\varepsilon_i(h)$ is the i-th entry of the diagonal matrix h.

For each $\alpha \in \Phi$, we may regard V as a representation of $\mathsf{sl}(\alpha)$. In particular, this tells us that the eigenvalues of h_α acting on V are integers, and hence the weights in Ψ lie in the real span of the roots. We saw in §10.6 that this space is an inner-product space.

Example 15.2

For example, the following diagram shows the weights of the natural and adjoint representations of $\mathsf{sl}(3, \mathbf{C})$ with respect to the Cartan subalgebra H of diagonal matrices projected onto a plane. The weight spaces of the natural representation are marked. To locate ε_1 we note that restricted to H, $\varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 0$, and hence ε_1 is the same map on H as $\frac{1}{3}(2\varepsilon_1 - \varepsilon_2 - \varepsilon_3) = \frac{1}{3}(2\alpha + \beta)$.



We now look at the action of e_α and f_α for $\alpha \in \Phi$. Let $v \in V_\lambda$. We leave it to the reader to check that $e_\alpha \cdot v \in V_{\lambda+\alpha}$ and $f_\alpha \cdot v \in V_{\lambda-\alpha}$; note that this generalises Lemma 8.3 for $\mathsf{sl}(2,\mathbf{C})$. Since the set Ψ of weights of V is finite, there must be some $\lambda \in \Psi$ such that for all $\alpha \in \Phi^+$, $\lambda + \alpha \notin \Psi$. We call such a λ a *highest weight*, and if $v \in V_\lambda$ is non-zero, then we say v is a highest-weight vector.

This agrees with our usage of these words in Chapter 8 since for representations of $\mathsf{sl}(2,\mathbf{C})$ a weight of the Cartan subalgebra spanned by h is essentially the same thing as an eigenvalue of h.

In the previous example, the positive roots of $\mathsf{sl}(3,\mathbf{C})$ with respect to the base $\Pi = \{\alpha, \beta\}$ are $\alpha, \beta, \alpha+\beta$, and so the (unique) highest weight of the natural representation of $\mathsf{sl}(3,\mathbf{C})$ is ε_1.

Lemma 15.3

Let V be a simple L-module. The set Ψ of weights of V contains a unique highest weight. If λ is this highest weight then V_λ is 1-dimensional and all other weights of V are of the form $\lambda - \sum_{\alpha_i \in \Pi} a_i \alpha_i$ for some $a_i \in \mathbf{Z}$, $a_i \geq 0$.

Proof

Take $0 \neq v \in V_\lambda$ and let W be the subspace of V spanned by elements of the form

$$f_{\alpha_{i_1}} f_{\alpha_{i_2}} \dots f_{\alpha_{i_k}} \cdot v, \qquad (\star)$$

where the α_{i_j} are not necessarily distinct elements of Π. Note that each element of the form $(\star)$ is an H-eigenvector. We claim that W is an L-submodule of V.

By Lemma 14.4, L is generated by the elements e_α, f_α for $\alpha \in \Pi$, so it is enough to check that W is closed under their action. For the f_α, this follows at once from the definition. Let $w = f_{\alpha_{i_1}} f_{\alpha_{i_2}} \dots f_{\alpha_{i_k}} \cdot v$. To show that $e_\alpha \cdot w \in W$, we shall use induction on k.

If $k = 0$ (that is, $w = v$), then we know that $e_\alpha \cdot v = 0$. For $k \geq 1$, let $w_1 = f_{\alpha_{i_2}} \dots f_{\alpha_{i_k}} v$ so that $w = f_{\alpha_{i_1}} w_1$ and

$$e_\alpha \cdot w = e_\alpha \cdot (f_{\alpha_{i_1}} \cdot w_1) = f_{\alpha_{i_1}} \cdot (e_\alpha \cdot w_1) + [e_\alpha, f_{\alpha_{i_1}}] \cdot w_1.$$

Now $[e_\alpha, f_{\alpha_{i_1}}] \in [L_\alpha, L_{-\alpha_{i_1}}] \subseteq L_{\alpha - \alpha_{i_1}}$. Both α and α_{i_1} are elements of the base Π, so $L_{\alpha-\alpha_{i_1}} = 0$, unless $\alpha = \alpha_{i_1}$, in which case $L_{\alpha - \alpha_{i_1}} \subseteq L_0 = H$. So in either case w_1 is an eigenvector for $[f_{\alpha_{i_1}}, e_\alpha]$. Moreover, by the inductive hypothesis, $e_\alpha \cdot w_1$ lies in W, so by the definition of W we have $f_{\alpha_{i_1}} \cdot (e_\alpha \cdot w_1) \in W$.

Since V is simple and W is non-zero, we have $V = W$. We can see from $(\star)$ that the weights of V are of the form $\lambda - \sum_i a_i \alpha_i$ for $\alpha_i \in \Pi$ and $a_i \geq 0$, so λ is the unique highest weight. $\square$

Example 15.4

Let $L = \mathsf{sl}(\ell + 1, \mathbf{C})$ and let $V = L$, with the adjoint representation. By Example 7.4, V is a simple L-module. We have seen above that the root space decomposition of L is the same as the weight space decomposition of this module. The unique highest weight is $\alpha_1 + \alpha_2 + \ldots + \alpha_\ell$, and for the highest-weight vector v in the lemma we can take $e_{1,\ell+1}$.

Suppose that λ is a weight for a finite-dimensional representation V. Let $\alpha \in \Pi$. Suppose that $\lambda(h_\alpha)$, the eigenvalue of h_α on the λ-weight space, is negative. Then, by the representation theory of $\mathsf{sl}(2, \mathbf{C})$, $e_\alpha \cdot L_\lambda \neq 0$, and so $\alpha + \lambda \in \Psi$. Thus, if λ is the highest weight for a finite-dimensional representation V, then $\lambda(h_\alpha) \geq 0$ for all $\alpha \in \Pi$.

This motivates the main result, given in the following theorem.

Theorem 15.5

Let Λ be the set of all $\lambda \in H^*$ such that $\lambda(h_\alpha) \in \mathbf{Z}$ and $\lambda(h_\alpha) \geq 0$ for all $\alpha \in \Pi$. For each $\lambda \in \Lambda$, there is a finite-dimensional simple L-module, denoted by $V(\lambda)$, which has highest weight λ. Moreover, any two simple L-modules with the same highest weight are isomorphic, and every simple L-module may be constructed in this way.

To describe Λ in general, one uses the *fundamental dominant weights*. These are defined to be the unique elements $\lambda_1, \ldots, \lambda_\ell \in H^*$ such that

$$\lambda_i(h_{\alpha_j}) = \delta_{ij}.$$

By the theorem above, Λ is precisely the set of linear combinations of the λ_i with non-negative integer coefficients. One would also like to relate the λ_i to the elements of our base of $H^\star$. Recall that $\lambda(h_\alpha) = \langle \lambda, \alpha \rangle$; so if we write $\lambda_i = \sum_{k=1}^{\ell} d_{ik}\alpha_k$, then

$$\lambda_i(h_{\alpha_j}) = \sum_k d_{ik} \langle \alpha_k, \alpha_j \rangle,$$

so the coefficients d_{ik} are given by the inverse of the Cartan matrix of L.

Example 15.6

Let $L = \mathsf{sl}(3, \mathbf{C})$. Then the inverse of the Cartan matrix is

$$\frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix},$$

and the fundamental dominant weights are $\frac{1}{3}(2\alpha+\beta)=\varepsilon_1$ and $\frac{1}{3}(\alpha+2\beta)=-\varepsilon_3$. The diagram in 15.2 shows that ε_1 is the highest weight of the natural representation; ε_3 appears as the highest weight of the dual of the natural representation.

So far, for a general complex simple Lie algebra L, the only irreducible representations we know are the trivial and adjoint representations. If L is a classical Lie algebra, then we can add the natural representation to this list. The previous theorem says there are many more representations. How can they be constructed?

15.1.2 Exterior Powers

Several general methods of constructing new modules from old ones are known. Important amongst these are tensor products and the related symmetric and exterior powers.

Let V be a finite-dimensional complex vector space with basis $v_1,\ldots,v_n$. For each i,j with $1\le i,j\le n$, we introduce a symbol $v_i\wedge v_j$, which satisfies $v_j\wedge v_i=-v_i\wedge v_j$. The *exterior square* $V\bigwedge V$ is defined to be the complex vector space of dimension $\binom{n}{2}$ with basis given by $\{v_i\wedge v_j:\ 1\le i<j\le n\}$. Thus, a general element of $V\bigwedge V$ has the form

$$\sum_{i<j} c_{ij}v_i\wedge v_j \quad \text{for scalars } c_{ij}\in\mathbf{C}.$$

For $v=\sum a_iv_i$ and $w=\sum b_jv_j$, define $v\wedge w$ by

$$v\wedge w=\sum_{i,j} a_ib_jv_i\wedge v_j.$$

This shows that the map $(v,w)\to v\wedge w$ is bilinear. One can show that the definition does not depend on the choice of basis. That is, if $w_1,\ldots,w_n$ is some other basis of V, then the set of all $w_i\wedge w_j$ for $1\le i<j\le n$ is a basis for $V\bigwedge V$ with the same properties as the previous basis.

Now suppose that L is a Lie algebra and $\rho: L\to\mathsf{gl}(V)$ is a representation. We may define a new representation $\wedge^2\rho: L\to\mathsf{gl}(V\bigwedge V)$ by

$$(\wedge^2\rho)(x)(v_i\wedge v_j)=\rho(x)v_i\wedge v_j+v_i\wedge\rho(x)v_j \quad \text{for } x\in L$$

and extending it to linear combinations of basis elements. (The reader might care to check that this really does define a representation of L.)

More generally, for any integer $r \leq n$, one introduces similarly symbols $v_{i_1} \wedge v_{i_2} \wedge \ldots \wedge v_{i_r}$ satisfying

$$v_{i_1} \wedge \ldots \wedge v_{i_k} \wedge v_{i_{k+1}} \wedge \ldots \wedge v_{i_r} = -v_{i_1} \wedge \ldots \wedge v_{i_{k+1}} \wedge v_{i_k} \wedge \ldots \wedge v_{i_r}.$$

The r-fold exterior power of V, denoted by $\bigwedge^r V$ is the vector space over $\mathbf{C}$ of dimension $\binom{n}{r}$ with basis

$$v_{i_1} \wedge v_{i_2} \wedge \ldots \wedge v_{i_r}, \quad 1 \leq i_1 < \ldots < i_r \leq n.$$

The action of L generalises so that

$$(\wedge^r \rho)(x)(v_{i_1} \wedge \ldots \wedge v_{i_r}) = \sum_{s=1}^{r} v_{i_1} \wedge \ldots \wedge \rho(x) v_{i_s} \wedge \ldots \wedge v_{i_r}.$$

It is known that if V is the natural module of a classical Lie algebra, then all the exterior powers V are irreducible. This is very helpful when constructing the irreducible representations of the classical Lie algebras.

We shall now use exterior powers to give a direct proof that $\mathsf{so}(6, \mathbf{C})$ and $\mathsf{sl}(4, \mathbf{C})$ are isomorphic. (In Chapter 14, we noted that this follows from Serre's Theorem, but we did not give an explicit isomorphism.)

Let $L = \mathsf{sl}(4, \mathbf{C})$, and let V be the 4-dimensional natural L-module. Then $\bigwedge^2 V$ is a 6-dimensional L-module. Now $\bigwedge^4 V$ has dimension $\binom{4}{4} = 1$. If we fix a basis $v_1, \ldots, v_4$ of V, then $\bigwedge^4 V$ is spanned by $\tilde{v} := v_1 \wedge v_2 \wedge v_3 \wedge v_4$. We may define a bilinear map

$$\bigwedge^2 V \times \bigwedge^2 V \to \mathbf{C}$$

by setting $(v, w) = c$ if $v \wedge w = c\tilde{v}$ for $c \in \mathbf{C}$.

Exercise 15.1

Find the matrix describing this bilinear form on $\bigwedge^2 V$ with respect to the basis $\{v_i \wedge v_j : i < j\}$. Show that it is congruent to the bilinear form defined by the matrix S, where

$$S = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

The module $\bigwedge^4 V$ is a 1-dimensional module for a semisimple Lie algebra, so it must be the trivial module for L. So for $x \in L$ and $v, w \in \bigwedge^2 V$, we have $x \cdot (v \wedge w) = 0$. But by the definition of the action of L, we get

$$x \cdot (v \wedge w) = v \wedge (xw) + (xv) \wedge w.$$

Hence, if we translate this into the bilinear form, we have

$$(v, xw) = -(xv, w).$$

Thus the image of $\varphi : L \to \mathsf{gl}(\bigwedge^2 V)$ is contained in $\mathsf{gl}_S(6, \mathbf{C}) = \mathsf{so}(6, \mathbf{C})$, where S is as above. Since L is simple and φ is non-zero, φ must be one-to-one, so by dimension counting it gives an isomorphism between $\mathsf{sl}(4, \mathbf{C})$ and $\mathsf{so}(6, \mathbf{C})$.

15.1.3 Tensor Products

Let V and W be finite-dimensional complex vector spaces with bases $v_1, \ldots, v_m$ and $w_1, \ldots, w_n$, respectively. For each i, j with $1 \leq i \leq m$ and $1 \leq j \leq n$, we introduce a symbol $v_i \otimes w_j$. The tensor product space $V \otimes W$ is defined to be the mn-dimensional complex vector space with basis given by $\{v_i \otimes w_j : 1 \leq i \leq n, 1 \leq j \leq m\}$. Thus a general element of $V \otimes W$ has the form

$$\sum_{i,j} c_{ij} v_i \otimes w_j \quad \text{for scalars } c_{ij} \in \mathbf{C}.$$

For $v = \sum_i a_i v_i \in V$ and $w = \sum_j b_j w_j \in W$, we define $v \otimes w \in V \otimes W$ by

$$v \otimes w = \sum_{i,j} a_i b_j (v_i \otimes w_j).$$

This shows that $(v, w) \to v \otimes w$ is bilinear. Again one can show that this definition of $V \otimes W$ does not depend on the choice of bases.

Suppose we have representations $\rho_1 : L \to \mathsf{gl}(V), \quad \rho_2 : L \to \mathsf{gl}(W)$. We may define a new representation $\rho : L \to \mathsf{gl}(V \otimes W)$ by

$$\rho(x)(v \otimes w) = \rho_1(x)(v) \otimes w + v \otimes \rho_2(x)(w).$$

Example 15.7

Let $L = \mathsf{sl}(2, \mathbf{C})$, and let $V = \mathbf{C}^2$ be the natural module with standard basis v_1, v_2. Let $W = \mathbf{C}^2$ be another copy of the natural module, with basis w_1, w_2. With respect to the basis $v_1 \otimes v_1, v_1 \otimes v_2, v_2 \otimes v_1, v_2 \otimes v_2$, one finds that the matrices of e, f, and h are

$$\rho(e) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \ \rho(f) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}, \ \rho(h) = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -2 \end{pmatrix}.$$

By Exercise 8.4, an $\mathsf{sl}(2, \mathbf{C})$-module is determined up to isomorphism by the eigenvalues of h. Here the highest eigenvalue appearing is 2, so V_2 is a submodule of $V \otimes W$. This leaves only an eigenvalue of 0, so we must have

$$V \otimes W \cong V_0 \oplus V_2.$$

Exercise 15.2

Find an explicit direct sum decomposition of $V \otimes W$ into irreducible submodules.

The book *Representation Theory* by Fulton and Harris [10] works out many more examples of this type.

For a general semisimple Lie algebra L, there is a more efficient and unified construction of the simple L-modules, which also allows one to construct certain infinite-dimensional representations. This uses the universal enveloping algebra of L. We shall now introduce this algebra and explain how to use it to construct the simple L-modules. We also explain the main idea in the proof of Theorem 15.5 above.

15.2 Universal Enveloping Algebras

Given a Lie algebra L over a field F, one can define its *universal enveloping algebra*, denoted by $U(L)$. This is an associative algebra (see §1.5) over F, which is always infinite-dimensional unless L is zero.

Assume that L is finite-dimensional with vector space basis $\{x_1, x_2, \ldots, x_n\}$. The structure constants with respect to this basis are the scalars a_{ij}^k given by

$$[x_i, x_j] = \sum_k a_{ij}^k x_k \quad \text{for } 1 \leq i, j \leq n.$$

Then $U(L)$ can be defined as the unital associative algebra, generated by $X_1, X_2, \ldots, X_n$, subject to the relations

$$X_i X_j - X_j X_i = \sum_{k=1}^{n} a_{ij}^k X_k \quad \text{for } 1 \leq i, j \leq n.$$

It can be shown (see Exercise 15.8) that the algebra $U(L)$ does not depend on the choice of the basis. That is, if we start with two different bases for L, then the algebras we get by this construction are isomorphic.

Example 15.8

(1) Let $L = \text{Span}\{x\}$ be a 1-dimensional abelian Lie algebra over a field F. The only structure constants come from $[x, x] = 0$. This gives us the relation $XX - XX = 0$, which is vacuous. Hence $U(L)$ is the associative algebra generated by the single element X. In other words, $U(L)$ is the polynomial algebra $F[X]$.

(2) More generally, let L be the n-dimensional abelian Lie algebra with basis $\{x_1, x_2, \ldots, x_n\}$. As before, all structure constants are zero, and hence $U(L)$ is isomorphic to the polynomial algebra in n variables.

We now consider a more substantial example. Let $L = \mathsf{sl}(2, \mathbf{C})$ with its usual basis, f, h, e. We know the structure constants and therefore we can calculate in the algebra $U(L)$. We should really write F, H, E for the corresponding generators of $U(L)$, but unfortunately this creates an ambiguity as H is already used to denote Cartan subalgebras. So instead we also write f, h, e for the generators of $U(L)$; the context will make clear the algebra in which we are working.

The triangular decomposition of L,

$$L = N^- \oplus H \oplus N^+,$$

where $N^- = \text{Span}\{f\}$, $H = \text{Span}\{h\}$, and $N^+ = \text{Span}\{e\}$, gives us three subalgebras of $U(L)$. For example, $U(L)$ contains all polynomials in e; this subalgebra can be thought of as the universal enveloping algebra $U(N^+)$. Similarly, $U(L)$ contains all polynomials in f and in h. But, in addition, $U(L)$ contains products of these elements. Using the relations $ef - fe = h$, $he - eh = 2e$, and $hf - fh = -2f$, valid in $U(L)$, one can show the following.

Lemma 15.9

Let $L = \mathsf{sl}(2, \mathbf{C})$. The associative algebra $U(L)$ has as a vector space basis

$$\{f^a h^b e^c : a, b, c \geq 0\}.$$

To show that this set spans the universal enveloping algebra, it suffices to verify that every monomial in the generators can be expressed as a linear combination of monomials of the type appearing in the lemma. The reader might, as an exercise, express the monomial hef as a linear combination of the given set; this should be enough to show the general strategy.

Proving linear independence is considerably harder, so we shall not go into the details. Indeed it is not even obvious that the elements $e, f \in U(L)$ are linearly independent, but this much at least will follow from Exercise 15.8.

In general, if the Lie algebra L has basis $x_1, \ldots, x_n$, then the algebra $U(L)$ has basis

$$\{X_1^{a_1} X_2^{a_2} \ldots X_n^{a_n} : a_1, \ldots a_n \geq 0\}.$$

This is known as a Poincaré–Birkhoff–Witt-basis or PBW-basis of $U(L)$. The previous lemma is the special case where $L = \mathsf{sl}(2, \mathbf{C})$ and $X_1 = f$, $X_2 = h$, and $X_3 = e$. We could equally well have taken the basis elements in a different order.

An important corollary is that the elements $X_1, X_2, \ldots, X_n$ are linearly independent, and so L can be found as a subspace of $U(L)$. Furthermore, if L_1 is a Lie subalgebra of L, then $U(L_1)$ is an associative subalgebra of $U(L)$; this justifies our earlier assertions about polynomial subalgebras of $U(\mathsf{sl}(2, \mathbf{C}))$.

15.2.1 Modules for $U(L)$

We now explain the sense in which the universal enveloping algebra of a Lie algebra L is "universal". We first need to introduce the idea of a representation of an associative algebra.

Let A be a unital associative algebra over a field F. A *representation* of A on an F-vector space V is a homomorphism of associative algebras

$$\varphi : A \to \mathrm{End}_F(V),$$

where $\mathrm{End}_F(V)$ is the associative algebra of linear maps on V. Thus φ is linear, φ maps the multiplicative identity of A to the identity map of V, and

$$\varphi(ab) = \varphi(a) \circ \varphi(b) \quad \text{for all } a, b \in A.$$

Unlike in earlier chapters, we now allow V to be infinite-dimensional. Note that the underlying vector space of $\mathrm{End}_F(V)$ is the same as that of $\mathsf{gl}(V)$; we write $\mathrm{End}_F(V)$ if we are using its associative structure and $\mathsf{gl}(V)$ if we are using its Lie algebra structure.

In what follows, it is most convenient to use the language of modules, so we shall indicate the action of L implicitly by writing $a \cdot v$ rather than $\varphi(a)(v)$.

Lemma 15.10

Let L be a Lie algebra and let $U(L)$ be its universal enveloping algebra. There is a bijective correspondence between L-modules and $U(L)$-modules. Under this correspondence, an L-module is simple if and only if it is simple as a module for $U(L)$.

Proof

Let V be an L-module. Since the elements X_i generate $U(L)$ as an associative algebra, the action $U(L)$ on V is determined by the action of the X_i. We let $X_i \in U(L)$ act on V in the same way as $x_i \in L$ acts on V. To verify that this defines an action of $U(L)$, one only needs to check that it satisfies the defining relations for $U(L)$. Consider the identity in L

$$[x_i, x_j] = \sum_k a_{ij}^k x_k.$$

For the action to be well-defined, we require that on V

$$(X_i X_j - X_j X_i)v = \sum_k a_{ij}^k X_k v.$$

By definition, the left-hand side is equal to $(x_i x_j - x_j x_i)v$; that is,

$$[x_i, x_j]v = \sum_k a_{ij}^k x_k v.$$

Since X_k acts on V in the same way as x_k, this is equal to the right-hand side, as we required.

Conversely, suppose V is a $U(L)$-module. By restriction, V is also an L-module since $L \subseteq U(L)$. Furthermore, V is simple as an L-module if and only if it is simple as a module for $U(L)$. This is a simple change of perspective and can easily be checked formally. □

The proof of this lemma demonstrates a certain *universal property* of $U(L)$. See Exercise 15.8 for more details.

15.2.2 Verma Modules

Suppose that L is a complex semisimple Lie algebra and $U(L)$ is the universal enveloping algebra of L. We shall use the equivalence between modules for $U(L)$ and L to construct an important family of L-modules.

Let H be a Cartan subalgebra of L, let Φ be the corresponding root system, and let Π be a base of Φ. As usual, we write Φ^+ for the positive roots with respect to Π. We may choose a basis $h_1, \ldots, h_\ell$ of H such that $h_i = h_{\alpha_i}$ for $\alpha_i \in \Pi$. For $\lambda \in H^*$ let $I(\lambda)$ be the left ideal of $U(L)$ generated by the elements e_α for $\alpha \in \Phi$ and also $h_i - \lambda(h_i)1$ for $1 \leq i \leq \ell$. Thus $I(\lambda)$ consists of all elements

$$\sum u_\alpha e_\alpha + \sum y_i(h_i - \lambda(h_i)1),$$

where the u_α and the y_i are arbitrary elements of $U(L)$. We may consider $I(\lambda)$ as a left module for $U(L)$. Let $M(\lambda)$ be the quotient space

$$M(\lambda) := U(L)/I(\lambda).$$

This becomes a $U(L)$-module with the action $u \cdot (v + I(\lambda)) = uv + I(\lambda)$. We say $M(\lambda)$ is the *Verma module* associated to λ.

Proposition 15.11

If $\bar{v} = 1 + I(\lambda)$, then $\bar{v}$ generates $M(\lambda)$ as a $U(L)$-module. For $\alpha \in \Phi^+$ and $e_\alpha \in L_\alpha$, we have $e_\alpha \bar{v} = 0$; and for $h \in H$ we have $h\bar{v} = \lambda(h)\bar{v}$. The module $M(\lambda)$ has a unique maximal submodule, and the quotient of $M(\lambda)$ by this submodule is the simple module $V(\lambda)$ with highest weight λ.

The first part of the theorem is easy: We have

$$e_\alpha \cdot \bar{v} = e_\alpha + I(\lambda),$$

which is zero in $M(\lambda)$. Moreover,

$$h_i \cdot \bar{v} = h_i + I(\lambda) = \lambda(h_i)1 + I(\lambda)$$

since $h_i - \lambda(h_i)1 \in I(\lambda)$. Since

$$x + I(\lambda) = x \cdot (1 + I(\lambda) = x \cdot \bar{v} \quad \text{for all } x \in U(L),$$

the coset $\bar{v}$ generates $M(\lambda)$.

One can show that a vector space basis for $M(\lambda)$ is given by the elements $u \cdot \bar{v}$, where u runs through a basis of $U(N^-)$. By the PBW-Theorem, $U(N^-)$ has a basis consisting of monomials in the f_α for $\alpha \in \Phi$; this shows that $M(\lambda)$ decomposes as a direct sum of simultaneous H-eigenspaces. We can then see that $M(\lambda)$ has a unique maximal weight, namely λ.

Knowing this, one can complete the proof of the proposition. Details can be found in Humphreys [14] (Chapter 20), or Dixmier [9]. Note, however, that the labelling in Dixmier is slightly different.

Example 15.12

We give two examples of Verma modules for $L = \mathsf{sl}(2, \mathbf{C})$. First we construct one which is irreducible; this will show that L has infinite-dimensional irreducible representations.

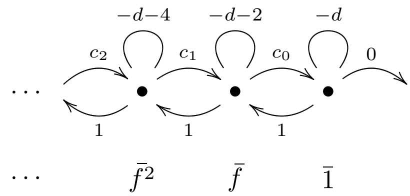
(1) Let $\lambda = -d$, where $d > 0$. Thus $M(\lambda) = U(L)/I(\lambda)$, where

$$I(\lambda) = U(L)e + U(L)(h + d1).$$

As a vector space, $M(\lambda)$ has basis

$$\left\{ \bar{f^a} = f^a + I_\lambda : a \geq 0 \right\}.$$

It follows by induction for each $a \geq 0$ that $\bar{f^a}$ is an eigenvector for h with eigenvalue $-d - 2a$. Furthermore, we have $e \cdot \bar{1} = 0$, $e \cdot \bar{f} = -d \cdot \bar{1}$, and inductively $e \cdot \bar{f}^a = c_{a-1}\bar{f}^{a-1}$, where c_{a-1} is a negative integer. As in §8.1.1, we can draw $M(\lambda)$ as



where loops represent the action of h, arrows to the right represent the action of e, and arrows to the left represent the action of f. Using this, one can check that for any non-zero $x \in M(\lambda)$ the span of

$$\{x, e \cdot x, e^2 \cdot x, \ldots\}$$

contains the generator $\bar{1} = 1 + I(\lambda)$ and hence $M(\lambda)$ is an infinite-dimensional simple module.

(2) We consider the Verma module $M(0)$. In this case, the span of all $\bar{f}^a$ where $a > 0$ is a proper submodule of $M(\lambda)$. For example,

$$e \cdot \bar{f} = ef + I(0) = (fe + h) + I(0),$$

which is zero, since $e, h \in I(0)$. The quotient of $M(0)$ by this submodule is the trivial L-module, $V(0)$.

Verma modules are the building blocks for the so-called *category* $\mathcal{O}$, which has recently been of major interest. Here the starting point is the observation that although each $M(\lambda)$ is infinite-dimensional, when viewed as $U(L)$-module it has finite length. That is, there are submodules

$$0 = M_0 \subseteq M_1 \subseteq M_2 \subseteq \ldots \subseteq M_k = M(\lambda)$$

such that M_i/M_{i-1} is simple for $1 \leq i \leq k$. A proof of this and more properties of Verma modules can be found in Dixmier [9] or Humphreys [14] (note, however, that Dixmier uses different labelling.)

In 1985, Drinfeld and Jimbo independently defined *quantum groups* by "deforming" the universal enveloping algebras of Lie algebras. (So contrary to what one might expect, quantum groups are really algebras!) Since then, quantum groups have found numerous applications in areas including theoretical physics, knot theory, and representations of algebraic groups. In 1990, Drinfeld was awarded a Fields Medal for his work. For more about quantum groups, see Jantzen, *Lectures on Quantum Groups* [16].

15.3 Groups of Lie Type

The theory of simple Lie algebras over $\mathbf{C}$ was used by Chevalley to construct simple groups of matrices over any field.

How can one construct invertible linear transformations from a complex Lie algebra? Let $\delta : L \to L$ be a derivation of L such that $\delta^n = 0$ for some $n \geq 1$. In Exercise 15.9, we define the exponential $\exp(\delta)$ and show that it is an automorphism of L.

Given a complex semisimple Lie algebra L, let x be an element in a root space. We know that $\operatorname{ad} x$ is a derivation of L, and by Exercise 10.1 $\operatorname{ad} x$ is nilpotent. Hence $\exp(\operatorname{ad} x)$ is an automorphism of L. One then takes the group generated by all the $\exp(\operatorname{ad} cx)$, for $c \in \mathbf{C}$, for x in a strategically chosen basis of L. This basis is known as the *Chevalley basis*; it is described in the following theorem.

Theorem 15.13

Let L be a simple Lie algebra over $\mathbf{C}$, with Cartan subalgebra H and associated root system Φ, and let Π be a base for Φ. For each $\alpha \in \Phi$, one may choose $h_\alpha \in H$ so that $h_\alpha \in [L_{-\alpha}, L_\alpha]$ and $\alpha(h_\alpha) = 2$. One may also choose an element $e_\alpha \in L_\alpha$ such that $[e_\alpha, e_{-\alpha}] = h_\alpha$ and $[e_\alpha, e_\beta] = \pm(p+1)e_{\alpha+\beta}$, where p is the greatest integer for which $\beta + p\alpha \in \Phi$.

The set $\{h_\alpha : \alpha \in \Pi\} \cup \{e_\beta : \beta \in \Phi\}$ is a basis for L. Moreover, for all $\gamma \in \Phi$, $[e_\gamma, e_{-\gamma}] = h_\gamma$ is an integral linear combination of the h_α for $\alpha \in \Pi$. The remaining structure constants of L with respect to this basis are as follows:

$$\begin{aligned}
[h_\alpha, h_\beta] &= 0, \\
[h_\alpha, e_\beta] &= \beta(h_\alpha)e_\beta, \\
[e_\alpha, e_\beta] &= \begin{cases} \pm(p+1)e_{\alpha+\beta} & \alpha+\beta \in \Phi \\ 0 & \alpha+\beta \notin \Phi \cup \{0\}. \end{cases}
\end{aligned}$$

In particular, they are all integers. □

Recall that in §10.4 we found for each $\alpha \in \Phi$ a subalgebra $\operatorname{Span}\{e_\alpha, f_\alpha, h_\alpha\}$ isomorphic to $\mathsf{sl}(2, \mathbf{C})$. Chevalley's Theorem asserts that the e_α and $f_\alpha = e_{-\alpha}$ can be chosen so as to give an especially convenient form for the structure constants of L.

Exercise 15.3

By using the calculations in Chapter 12, determine a Chevalley basis for the Lie algebra $\mathsf{so}(5, \mathbf{C})$ of type B_2.

Since the structure constants are integers, the $\mathbf{Z}$-span of such a basis, denoted by $L_{\mathbf{Z}}$, is closed under Lie brackets. If one now takes any field F, one can define a Lie algebra L_F over F as follows. Take as a basis

$$\{\bar{h}_\alpha : \alpha \in \Pi\} \cup \{\bar{e}_\beta, \beta \in \Phi\}$$

and define the Lie commutator by taking the structure constants for L_Z and interpreting them as elements in the prime subfield of F. For example, the standard basis e, h, f of $\mathsf{sl}(2, \mathbf{C})$ is a Chevalley basis, and applying this construction gives $\mathsf{sl}(2, F)$.

Now we can describe the automorphisms. First take the field $\mathbf{C}$. For $c \in \mathbf{C}$ and $\alpha \in \Phi$, define

$$x_\alpha(c) := \exp(c \operatorname{ad} e_\alpha).$$

As explained, this is an automorphism of L. One can show that it takes elements of the Chevalley basis to linear combinations of basis elements with coefficients of the form ac^i, where $a \in \mathbf{Z}$ and $i \geq 0$. Let $A_\alpha(c)$ be the matrix of $x_\alpha(c)$ with respect to the Chevalley basis of L. By this remark, the entries of $A_\alpha(c)$ have the form ac^i for $a \in \mathbf{Z}$ and $i \geq 0$. Define the *Chevalley group* associated to L by

$$G_{\mathbf{C}}(L) := \langle A_\alpha(c) : \alpha \in \Phi, c \in \mathbf{C} \rangle.$$

We can also define automorphisms of L_F. Take $t \in F$. Let $\tilde{A}_\alpha(t)$ be the matrix obtained from $A_\alpha(c)$ by replacing each entry ac^i by $\bar{a}t^i$, where $\bar{a}$ is a viewed as an element in the prime subfield of F. The Chevalley group of L over F is then defined to be the group

$$G_F(L) := \langle \tilde{A}_\alpha(t) : \alpha \in \Phi, t \in F \rangle.$$

Exercise 15.4

Let $L = \mathsf{sl}(2, \mathbf{C})$. Let $c \in \mathbf{C}$. Show that with respect to the Chevalley basis e, h, f, the matrix of $\exp(c \operatorname{ad} e)$ is

$$\begin{pmatrix} 1 & -2c & -c^2 \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}$$

and find the matrix of $\exp(c \operatorname{ad} f)$. Then describe the group $G_{\mathbf{F}_2}(L)$, where $\mathbf{F}_2$ is the field with 2-elements.

The structure of these groups is studied in detail in Carter's book *Simple Groups of Lie Type* [7], see also [13].

Remark 15.14

One reason why finite groups of Lie type are important is the Classification Theorem of Finite Simple Groups. This theorem, which is one of the greatest achievements of twentieth century mathematics (though to date not yet completely written down), asserts that there are two infinite families of finite simple groups, namely the alternating groups and the finite groups of Lie type, and that any finite simple group is either a member of one of these two families or is one of the 26 sporadic simple groups.

15.4 Kac–Moody Lie Algebras

The presentation of complex semisimple Lie algebras given by Serre's Theorem can be generalized to construct new families of Lie algebras. Instead of taking the Cartan matrix associated to a root system, one can start with a more general matrix and then use its entries, together with the Serre relations, to define a new Lie algebra. These Lie algebras are usually infinite-dimensional; in fact the finite-dimensional Lie algebras given by this construction are precisely the Lie algebras of types A, B, C, D, E, F, G which we have already seen.

We shall summarize a small section from the introduction of the book *Infinite Dimensional Lie Algebras* by Kac [18]. One defines a *generalised Cartan matrix* to be an $n \times n$ matrix $A = (a_{ij})$ such that

(a) $a_{ij} \in \mathbf{Z}$ for all i, j;

(b) $a_{ii} = 2$, and $a_{ij} \leq 0$ for $i \neq j$;

(c) if $a_{ij} = 0$ then $a_{ji} = 0$.

The associated *Kac–Moody Lie algebra* is the complex Lie algebra over $\mathbf{C}$ generated by the $3n$ elements e_i, f_i, h_i, subject to the Serre relations, as stated in §14.1.2.

When the rank of the matrix A is $n-1$, this construction gives the so-called *affine Kac–Moody Lie algebras*. Modifications of such algebras can be proved to be simple; there is much interest in their representation theory, and several new applications have been discovered.

15.4.1 The Moonshine Conjecture

The largest of the 26 sporadic simple groups is known (because of its enormous size) as the *monster group*. Its order is

$$2^{46}.3^{20}.5^{9}.7^{6}.11^{2}.13^{3}.17.19.23.29.31.41.47.59.71 \approx 8 \times 10^{53}.$$

Like Lie algebras and associative algebras, groups also have representations. The three smallest representations of the monster group over the complex numbers have dimensions 1, 196883 (it was through this representation that the monster was discovered), and 21296876.

In 1978, John MacKay noticed a near coincidence with the coefficients of the Fourier series expansion of the elliptic modular function j,

$$j(\tau) = q^{-1} + 744 + 196884q + 21493760q^2 + \dots,$$

where $q = e^{2\pi i\tau}$. As well as noting that $196884 = 196883 + 1$ and $21493760 = 21296876 + 196883 + 1$, he showed that (with a small generalisation) this connection persisted for *all* the coefficients of the j-function.

That there could be an underlying connection between the monster group and modular functions seemed at first so implausible that this became known as the *Moonshine Conjecture*. Yet in 1998 Borcherds succeeded in establishing just such a connection, thus proving the Moonshine Conjecture. A very important part of his work was a further generalisation of the Kac–Moody Lie algebras connected with the exceptional root system of type E_8.

Borcherds was awarded a Fields Medal for his work. A survey can be found in Ray [19]. The reader might also like to read the article by Carter [8].

15.5 The Restricted Burnside Problem

In 1902, William Burnside wrote "A still undecided point in the theory of discontinuous groups is whether the order of a group may not be finite, while the order of every operation it contains is finite." Here we shall consider a variation on his question which can be answered using techniques from Lie algebras.

We must first introduce two definitions: a group G has *exponent* n if $g^n = 1$ for all $g \in G$, and, moreover, n is the least number with this property. A group is r*-generated* if all its elements can be obtained by repeatedly composing a fixed subset of r of its elements. The *restricted Burnside problem* asks: Given $r, n \geq 1$, is there an upper bound on the orders of the finite r-generated groups of exponent n?

Since there are only finitely many isomorphism classes of groups of any given order, the restricted Burnside problem has an affirmative answer if and only if there are (up to isomorphism) only finitely many finite r-generated groups of exponent n.

The reader may well have already seen this problem in the case $n = 2$.

Exercise 15.5

Suppose that G has exponent 2 and is generated by $g_1, \dots, g_r$. Show that G is abelian and that $|G| \leq 2^r$.

So for $n = 2$, our question has an affirmative answer. In 1992, Zelmanov proved that this is the case whenever n is a prime power. Building on earlier work of Hall and Higman, this was enough to show that the answer is affirmative for all n and r. In 1994, Zelmanov was awarded a Fields Medal for his work.

We shall sketch a proof for the case $n = 3$, which shows some of the ideas in Zelmanov's proof.

Let G be a finitely generated group of exponent p, where p is prime. We define the *lower central series* of G by $G^0 = G$ and $G^i = [G, G^{i-1}]$ for $i \geq 1$. Here $[G, G^{i-1}]$ is the group generated by all group commutators $[x, y] = x^{-1}y^{-1}xy$ for $x \in G, y \in G^{i-1}$. We have

$$G = G^0 \geq G^1 \geq G^2 \geq \dots .$$

If for some $m \geq 1$ we have $G^m = 1$, then we say G is *nilpotent*.

The notation for group commutators used above is standard; x and y are group elements and the operations are products and inverses in a group. It should not be confused with a commutator in a Lie algebra.

Remark 15.15

It is no accident that the definition of nilpotency for groups mirrors that for Lie algebras. Indeed, nilpotency was first considered for Lie algebras and only much later for groups. This is in contrast to solvability, which was first considered for groups by Galois in his 1830s work on the solution of polynomial equations by radicals.

Each G^i/G^{i+1} is a finitely generated abelian group all of whose non-identity elements have order p. In other words, it is a vector space over $\mathbf{F}_p$, the field with p elements. We may make the (potentially infinite-dimensional) vector space

$$B = \bigoplus_{i=0}^{\infty} G^i/G^{i+1}$$

into a Lie algebra by defining

$$[xG^i, yG^j] = [x, y]G^{i+j}$$

and extending by linearity to arbitrary elements of B. Here on the left we have a commutator in the Lie algebra B and on the right a commutator taken in the group G. It takes some work to see that with this definition the Lie bracket is well defined and satisfies the Jacobi identity — see Vaughan-Lee, *The Restricted Burnside Problem* [24] §2.3, for details. Anticommutativity is more easily seen since if $x, y \in G$, then $[x, y]^{-1} = [y, x]$.

If G is nilpotent (and still finitely generated) then it must be finite, for each G^i/G^{i+1} is a finitely generated abelian group of exponent p, and hence finite. Moreover, if G is nilpotent, then B is a nilpotent Lie algebra. Unfortunately, the converse does not hold because the lower central series might terminate with $G^i = G^{i+1}$ still being an infinite group. However, one can still say something: For the proof of the following theorem, see §2.3 of Vaughan-Lee [24].

Theorem 15.16

If B is nilpotent, then there is an upper bound on the orders of the finite r-generated groups of exponent n. □

The general proof that B is nilpotent is hard. When $p = 3$, however, there are some significant simplifications. By Exercise 4.8, it is sufficient to prove that $[x, [x, y]] = 0$ for all $x, y \in B$. By the construction of B, this will hold if and only if $[g, [g, h]] = 1$ for all $g, h \in G$, now working with group commutators. We now show that this follows from the assumption that G has exponent 3:

$$\begin{aligned}
[g, [g, h]] &= g^{-1}[g, h]^{-1}g[g, h] \\
&= g^{-1}h^{-1}g^{-1}hggg^{-1}h^{-1}gh \\
&= g^{-1}h^{-1}(g^{-1}hg^{-1})ggh^{-1}gh \\
&= g^{-1}h^{-1}h^{-1}gh^{-1}g^{-1}h^{-1}gh \\
&= g^{-1}hg(h^{-1}g^{-1}h^{-1})gh \\
&= g^{-1}hgghggh \\
&= (g^{-1}hg^{-1})hg^{-1}h \\
&= h^{-1}gh^{-1}hg^{-1}h \\
&= 1,
\end{aligned}$$

where the bracketing indicates that in the coming step the "rewriting rule" $aba = b^{-1}a^{-1}b^{-1}$ for $a, b \in G$ will be used; this identity holds because $ababab = (ab)^3 = 1$. The reader might like to see if there is a shorter proof.

We must use the elementary argument of Exercise 4.8 rather than Engel's Theorem to prove that B is nilpotent since we have only proved Engel's Theorem for finite-dimensional Lie algebras. In fact, one of Zelmanov's main achievements was to prove an infinite-dimensional version of Engel's Theorem.

15.6 Lie Algebras over Fields of Prime Characteristic

Many Lie algebras over fields of prime characteristic occur naturally; for example, the Lie algebras just seen in the context of the restricted Burnside problem. We have already seen that such Lie algebras have a behaviour different from complex Lie algebras; for example, Lie's Theorem does not hold — see Exercise 6.4. However, other properties appear. For example, let A be an algebra defined over a field of prime characteristic p. Consider the Lie algebra $\operatorname{Der} A$ of derivations of A. The Leibniz formula (see Exercise 1.19) tells us that

$$D^p(xy) = \sum_{k=0}^{p} \binom{p}{k} D^k(x) D^{p-k}(y) = xD^p(y) + D^p(x)y$$

for all $x, y \in A$. Thus the p-th power of a derivation is again a derivation. This was one of the examples that led to the formulation of an axiomatic definition of *p-maps* on Lie algebras. A Lie algebra with a p-map is known as a p-Lie algebra. Details of this may be found in Jacobson's book *Lie Algebras* [15] and also in Strade and Farnsteiner, *Modular Lie Algebras and their Representations* [22] or, especially for the representation theory, Jantzen [17].

What can be said about simple Lie algebras over fields of prime characteristic p? Since Lie's Theorem fails in this context, one might expect that the classification of simple Lie algebras over the complex numbers would not generalise. For example, Exercise 15.11 shows that $\mathsf{sl}(n, F)$ is not simple when the characteristic of F divides n. Moreover, new simple Lie algebras have been discovered over fields of prime characteristic that do not have any analogues in characteristic zero.

As an illustration, we shall define the Witt algebra $W(1)$. Fix a field F of characteristic p. The Witt algebra $W(1)$ over F is p-dimensional, with basis

$$e_{-1}, e_0, \ldots, e_{p-2}$$

and Lie bracket

$$[e_i, e_j] = \begin{cases} (j-i)e_{i+j} & -1 \le i+j \le p-2 \\ 0 & \text{otherwise.} \end{cases}$$

When $p = 2$, this algebra is the 2-dimensional non-abelian Lie algebra.

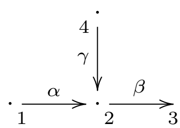
Exercise 15.6

Show that $W(1)$ is simple for $p \geq 3$. Show that if $p = 3$, the Lie algebra $W(1)$ is isomorphic to $\mathsf{sl}(2, F)$. Show that if $p > 3$ then $W(1)$ is not isomorphic to any classical Lie algebra defined over F. *Hint*: The dimensions of the classical Lie algebras (defined over any field) are as given in Exercise 12.1.

A classification of the simple Lie algebras over prime characteristic p is work currently in progress by Premet and Strade.

15.7 Quivers

A *quiver* is another name for a directed graph, for instance,



is a quiver with vertices labelled 1, 2, 3, 4 and arrows labelled α, β, γ. The *underlying graph* of a quiver is obtained by ignoring the direction of the arrows.

A *path* in a quiver is a sequence of arrows which can be composed. In the example above, $\beta\alpha$ is a path (we read paths from right to left as this is the order in which we compose maps), but $\alpha\beta$ and $\alpha\gamma$ are not.

Let $\mathcal{Q}$ be a quiver and let F be a field. The path algebra $F\mathcal{Q}$ is the vector space which has as basis all paths in $\mathcal{Q}$, including the vertices, regarded as paths of length zero. For example, the path algebra of the quiver above has basis

$$\{e_1, e_2, e_3, e_4, \alpha, \beta, \gamma, \beta\alpha, \beta\gamma\}.$$

If two basis elements can be composed to make a path, then their product is defined to be that path. Otherwise, their product is zero. For example, the product of β and α is $\beta\alpha$ since $\beta\alpha$ is a path, whereas the product of α and γ is zero. The behaviour of the vertices is illustrated by $e_1^2 = e_1$, $e_2\alpha = \alpha e_1 = \alpha$, $e_1e_2 = 0$. This turns $F\mathcal{Q}$ into an associative algebra, which is finite-dimensional precisely when $\mathcal{Q}$ has no oriented cycles.

One would like to understand the representations of $F\mathcal{Q}$. Let V be an $F\mathcal{Q}$-module. The vertices e_i are idempotents whose sum is the identity of the algebra

FQ and $e_i e_j = 0$ if $i \neq j$, so we can use them to decompose V as a direct sum of subspaces

$$V = \bigoplus e_i V.$$

The arrows act as linear maps between the $e_i V$. For example, in the quiver above, $\alpha = e_2 \alpha e_1$ so $\alpha(e_1 V) \subseteq e_2 V$. This allows us to draw a module pictorially: For instance,

$$\begin{array}{ccccc} & & 0 & & \\ & & \downarrow{\scriptstyle 0} & & \\ F & \xrightarrow{1} & F & \xrightarrow{0} & 0 \end{array}$$

shows a 2-dimensional module V, where $e_1 V \cong e_2 V \cong F$ and α acts as an isomorphism between $e_1 V$ and $e_2 V$ (and β and γ act as the zero map).

The simple FQ-modules are all 1-dimensional, with one for each vertex. For example,

$$\begin{array}{ccccc} & & 0 & & \\ & & \downarrow{\scriptstyle 0} & & \\ 0 & \xrightarrow{0} & F & \xrightarrow{0} & 0 \end{array}$$

shows the simple module corresponding to vertex 2. In this module, $e_2 V = V$ and all the other basis elements act as 0.

Usually there will be FQ-modules which are not direct sums of simple modules. For example, the first module defined above has $e_2 V$ as its unique non-trivial submodule, and so it does not split up as a direct sum of simple modules. Thus there are indecomposable FQ-modules which are not irreducible. One can measure the extent to which complete reducibility fails to hold by asking how many indecomposable FQ-modules there are.

If there are only finitely indecomposable modules (up to isomorphism), the algebra FQ is said to have *finite type*. In the 1970s, Gabriel found a necessary and sufficient condition for a quiver algebra to have finite type. He proved the following theorem.

Theorem 15.17 (Gabriel's Theorem)

The path algebra FQ has finite type if and only if the underlying graph of Q is a disjoint union of Dynkin diagrams of types A, D, E. Moreover, the indecomposable KQ-modules are parametrized naturally by the positive roots of the associated root system.

Example 15.18

Consider the quiver of type A_4

$$\underset{1}{\cdot} \xrightarrow{\alpha_1} \underset{2}{\cdot} \xrightarrow{\alpha_2} \underset{3}{\cdot} \xrightarrow{\alpha_3} \underset{4}{\cdot}$$

By Gabriel's Theorem, the indecomposable representations of this quiver are in bijection with the positive roots in the root system of type A_4. The simple roots $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ correspond to the simple modules. The positive root $\alpha_1+\alpha_2$ corresponds to the module

$$F \xrightarrow{1} F \xrightarrow{0} 0 \xrightarrow{0} 0$$

and so on.

One might wonder whether this connection with Dynkin diagrams is merely an accident. Not long ago, Ringel discovered a deep connection between quivers and the theory of Lie algebras. He showed that, when F is a finite field, one may define an algebra which encapsulates all the representations of $F\mathcal{Q}$. This algebra is now known as the *Ringel–Hall algebra*; Ringel proved that this algebra is closely related to the quantum group of the same type as the underlying graph of the quiver.

EXERCISES

15.7. Tensor products can also be used to construct representations of a direct sum of two Lie algebras. Let L_1 and L_2 be isomorphic copies of $\mathsf{sl}(2, \mathbf{C})$ and let $L = L_1 \oplus L_2$. Let $V(a)$ and $V(b)$ be irreducible modules for $\mathsf{sl}(2, \mathbf{C})$ with highest weights a and b, respectively.

(i) Show that we may make $V(a) \otimes V(b)$ into a module for L by setting

$$(x, y) \cdot v \otimes w = ((x \cdot v) \otimes w) + (v \otimes (y \cdot w))$$

for $x \in L_1$, $y \in L_2$, $v \in V(a)$, and $w \in V(b)$.

(ii) Show that $V(a) \otimes V(b)$ is an irreducible representation of L with highest weight λ defined by

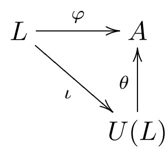
$$\lambda(h, 0) = a \quad \lambda(0, h) = b.$$

It can be shown that this construction gives every irreducible L-module. By Exercise 10.8, $\mathsf{sl}(2, \mathbf{C}) \oplus \mathsf{sl}(2, \mathbf{C}) \cong \mathsf{so}(4, \mathbf{C})$, so we have constructed all the finite-dimensional representations of $\mathsf{so}(4, \mathbf{C})$. Generalising these ideas, one can show that any semisimple Lie algebra has a faithful irreducible representation; from this it is not hard to prove a (partial) converse of Exercise 12.4.

15.8. Let L be a Lie algebra and let $U(L)$ be its universal enveloping algebra as defined above. Let $\iota : L \to U(L)$ be the linear map defined by $\iota(x_i) = X_i$.

Let A be an associative algebra; we may also view A as a Lie algebra with Lie bracket $[x, y] = xy - yx$ for $x, y \in A$ (see §1.5).

(i) Show that $U(L)$ has the following *universal property*: Given a Lie algebra homomorphism $\varphi : L \to A$, there exists a *unique* homomorphism of associative algebras $\theta : U(L) \to A$ such that $\theta \circ \iota = \varphi$. In other words, the following diagram commutes:



(ii) Suppose that V is an associative algebra and $\iota' : L \to V$ is a Lie algebra homomorphism (where we regard V as a Lie algebra) such that if we replace ι with ι' and $U(L)$ with V in the commutative diagram above then V has the universal property of $U(L)$. Show that V and $U(L)$ are isomorphic. In particular, this shows that $U(L)$ does not depend on the choice of basis of L.

(iii) Let $x_1, \ldots, x_k \in L$. Suppose that L has a representation $\varphi : L \to \mathsf{gl}(V)$ such that $\varphi(x_1), \ldots, \varphi(x_k)$ are linearly independent. Show that $X_1, \ldots, X_k$ are linearly independent elements of $U(L)$. Hence prove that if L is semisimple then ι is injective.

15.9. Let $\delta : L \to L$ be a derivation of a complex finite-dimensional Lie algebra L. Suppose that $\delta^n = 0$ where $n \geq 1$. Define $\exp(\delta) : L \to L$ by

$$\exp(\delta)(x) = \left(1 + \delta + \frac{\delta^2}{2!} + \ldots\right) x.$$

(By hypothesis the sum is finite.) Prove that $\exp(\delta)$ is an *automorphism* of L; that is, $\exp(\delta) : L \to L$ is an invertible linear map such that

$$[\exp \delta(x), \exp \delta(y)] = \exp \delta([x, y]) \quad \text{for all } x, y \in L.$$

15.10. Let L be a finite-dimensional complex Lie algebra and let α be an automorphism of L. For $\nu \in \mathbf{C}$, let

$$L_\nu = \{x \in L : \alpha(x) = \nu(x)\}.$$

Show that $[L_\lambda, L_\mu] \subseteq L_{\lambda\mu}$. Now suppose that we have $\alpha^3 = 1$, and that α fixes no non-zero element of L. Prove that L is nilpotent.

15.11. Let F be a field of prime characteristic p. Show that if p divides n then $\mathsf{sl}(n, F)$ is not simple.