14
Simple Lie Algebras

In this chapter, we shall show that for each isomorphism class of irreducible root systems there is a unique simple Lie algebra over **C** (up to isomorphism) with that root system. Moreover, we shall prove that every simple Lie algebra has an irreducible root system, so every simple Lie algebra arises in this way. These results mean that the classification of irreducible root systems in Chapter 13 gives us a complete classification of all complex simple Lie algebras.

We have already shown in Proposition 12.4 that if the root system of a Lie algebra is irreducible, then the Lie algebra is simple. We now show that the converse holds; that is, the root system of a simple Lie algebra is irreducible. We need the following lemma concerning reducible root systems.

Lemma 14.1

Suppose that Φ is a root system and that $\Phi = \Phi_1 \cup \Phi_2$ where $(\alpha, \beta) = 0$ for all $\alpha \in \Phi_1$, $\beta \in \Phi_2$.

(a) If $\alpha \in \Phi_1$ and $\beta \in \Phi_2$, then $\alpha + \beta \notin \Phi$.

(b) If $\alpha, \alpha' \in \Phi_1$ and $\alpha + \alpha' \in \Phi$, then $\alpha + \alpha' \in \Phi_1$.

Proof

For (a), note that $(\alpha, \alpha+\beta) = (\alpha, \alpha) \neq 0$, so $\alpha+\beta \notin \Phi_2$. Similarly, $(\beta, \alpha+\beta) = (\beta, \beta) \neq 0$, so $\alpha + \beta \notin \Phi_1$.

To prove (b), we suppose for a contradiction that $\alpha+\alpha' \in \Phi_2$. Remembering that $-\alpha' \in \Phi_1$, we have $\alpha = -\alpha' + (\alpha + \alpha')$, so α can be expressed as the sum of a root in Φ_1 and a root in Φ_2. This contradicts the previous part. □

Proposition 14.2

Let L be a complex semisimple Lie algebra with Cartan subalgebra H and root system Φ. If L is simple, then Φ is irreducible.

Proof

By the root space decomposition, we may write L as

$$L = H \oplus \bigoplus_{\alpha \in \Phi} L_\alpha.$$

Suppose that Φ is reducible, with $\Phi = \Phi_1 \cup \Phi_2$, where Φ_1 and Φ_2 are non-empty and $(\alpha, \beta) = 0$ for all $\alpha \in \Phi_1$ and $\beta \in \Phi_2$. We shall show that the root spaces L_α for $\alpha \in \Phi_1$ generate a proper ideal of L, and so L is not simple.

For each $\alpha \in \Phi_1$ we have defined a Lie subalgebra $\mathsf{sl}(\alpha) \cong \mathsf{sl}(2, \mathbf{C})$ of L with standard basis $\{e_\alpha, f_\alpha, h_\alpha\}$. Let

$$I := \operatorname{Span}\{e_\alpha, f_\alpha, h_\alpha : \alpha \in \Phi_1\}.$$

The root space decomposition shows that I is a non-zero proper subspace of L.

We claim that I is an ideal of L; it is a subspace by definition, so we only have to show that $[x, a] \in L$ for all $x \in L$ and $a \in I$. For this it suffices to take $a = e_\alpha$ and $a = f_\alpha$ for $\alpha \in \Phi_1$ since these elements generate I. Moreover, we may assume that x lies in one of the summands of the root space decomposition of L.

If $x \in H$, then $[x, e_\alpha] = \alpha(x)e_\alpha \in I$ and similarly $[x, f_\alpha] = -\alpha(x)e_\alpha \in I$. Suppose that $x \in L_\beta$. Then, for any $\alpha \in \Phi_1$, $[x, e_\alpha] \in L_{\alpha+\beta}$ by Lemma 10.1(i). If $\beta \in \Phi_2$, then by Lemma 14.1(a) above, we know that $\alpha + \beta$ is not a root, so $L_{\alpha+\beta} = 0$, and hence $[x, e_\alpha] \in I$. Otherwise $\beta \in \Phi_1$, and then by Lemma 14.1(b) we know that $\alpha+\beta \in \Phi_1$, so $L_{\alpha+\beta} \subseteq I$, by the definition of I. Similarly, one shows that $[x, f_\alpha] \in I$. (Alternatively, one may argue that as f_α is a scalar multiple of $e_{-\alpha}$, it is enough to look at the elements e_α.) □

14.1 Serre's Theorem

Serre's Theorem is a way to describe a complex semisimple Lie algebra by generators and relations that depend only on data from its Cartan matrix. The reader will probably have seen examples of groups, such as the dihedral groups, given by specifying a set of generators and the relations that they satisfy. The situation for Lie algebras is analogous.

14.1.1 Generators

Let L be a complex semisimple Lie algebra with Cartan subalgebra H and root system Φ. Suppose that Φ has as a base $\{\alpha_1, \ldots, \alpha_\ell\}$. For each i between 1 and ℓ let e_i, f_i, h_i be a standard basis of $\mathsf{sl}(\alpha_i)$. We ask whether the e_i, f_i, h_i for $1 \leq i \leq \ell$ might already generate L; that is, can every element of L be obtained by repeatedly taking linear combinations and Lie brackets of these elements?

Example 14.3

Let $L = \mathsf{sl}(\ell + 1, \mathbf{C})$. We shall show that the elements $e_{i,i+1}$ and $e_{i+1,i}$ for $1 \leq i \leq \ell$ already generate L as a Lie algebra. By taking the commutators $[e_{i,i+1}, e_{i+1,i}]$ we get a basis for the Cartan subalgebra H of diagonal matrices. For $i + 1 < j$, we have $[e_{i,i+1}, e_{i+1,j}] = e_{ij}$, and hence by induction we get all e_{ij} with $i < j$. Similarly, we may obtain all e_{ij} with $i > j$.

It is useful to look at these in terms of roots. Recall that the root system of L with respect to H has as a base $\alpha_1, \ldots, \alpha_\ell$, where $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$. For $i < j$, we have $\mathrm{Span}\{e_{ij}\} = L_\beta$, where $\beta = \alpha_i + \gamma$ and $\gamma = \alpha_{i+1} + \ldots + \alpha_{j-1}$. This can be expressed neatly using reflections since

$$s_{\alpha_i}(\gamma) = \gamma - \langle \gamma, \alpha_i \rangle \alpha_i = \gamma + \alpha_i = \beta.$$

In fact, this method gives a general way to obtain any non-zero root space. To show this, we only need to remind ourselves of some earlier results.

Lemma 14.4

Let L be a complex semisimple Lie algebra, and let $\{\alpha_1, \ldots, \alpha_\ell\}$ be a base of the root system. Suppose $\{e_i, f_i, h_i\}$ is a standard basis of $\mathsf{sl}(\alpha_i)$. Then L can be generated, as a Lie algebra, by $\{e_1, \ldots, e_\ell, f_1, \ldots, f_\ell\}$.

Proof

We first show that every element of H can be obtained. Since $h_i = [e_i, f_i]$, it is sufficient to prove that H is spanned by $h_1, \ldots, h_\ell$. Recall that we have identified H with H^*, via the Killing form κ, so that $\alpha_i \in H^*$ corresponds to the element $t_{\alpha_i} \in H$. As $H^\star$ is spanned by the roots $\alpha_1, \ldots, \alpha_\ell$, H has as a basis $\{t_{\alpha_i} : 1 \le i \le \ell\}$. By Lemma 10.6, h_i is a non-zero scalar multiple of t_{α_i}. Hence $\{h_1, \ldots, h_\ell\}$ is a basis for H.

Now let $\beta \in \Phi$. We want to show that L_β is contained in the Lie subalgebra generated by the e_i and the f_i. Call this subalgebra $\tilde{L}$. By Proposition 11.14, we know that $\beta = w(\alpha_j)$, where w is a product of reflections s_{α_i} for some base elements α_i. Hence, by induction on the number of reflections, it is enough to prove the following: If $\beta = s_{\alpha_i}(\gamma)$ for some $\gamma \in \Phi$ with $L_\gamma \subseteq \tilde{L}$, then $L_\beta \subseteq \tilde{L}$.

By hypothesis, $\beta = \gamma - \langle \gamma, \alpha_i \rangle \alpha_i$. In Proposition 10.10, we looked at the $\mathsf{sl}(\alpha_i)$-submodule of L defined by

$$\bigoplus_k L_{\gamma + k\alpha_i},$$

where the sum is over all $k \in \mathbf{Z}$ such that $\gamma + k\alpha_i \in \Phi$, and the module structure is given by the adjoint action of $\mathsf{sl}(\alpha_i)$. We proved that this is an irreducible $\mathsf{sl}(\alpha_i)$-module. If $0 \neq e_\gamma \in L_\gamma$, then by applying powers of $\operatorname{ad} e$ or $\operatorname{ad} f$ we may obtain $e_{\gamma + k\alpha_i}$ whenever $\gamma + k\alpha_i \in \Phi$. Hence, if we take $k = \langle \gamma, \alpha_i \rangle$, then we will obtain e_β. Hence L_β is contained in $\tilde{L}$. □

14.1.2 Relations

Next, we search for relations satisfied by the e_i, f_i, and h_i. These should only involve information which can be obtained from the Cartan matrix. We write $c_{ij} = \langle \alpha_i, \alpha_j \rangle$. Note that since the angle between any two base elements is obtuse (see Exercise 11.3), $c_{ij} \le 0$ for all $i \neq j$.

Lemma 14.5

The elements e_i, f_i, h_i for $1 \le i \le \ell$ satisfy the following relations.

(S1) $[h_i, h_j] = 0$ for all i, j;

(S2) $[h_i, e_j] = c_{ji} e_j$ and $[h_i, f_j] = -c_{ji} f_j$ for all i, j;

(S3) $[e_i, f_i] = h_i$ for each i and $[e_i, f_j] = 0$ if $i \neq j$;

(S4) $(\text{ad } e_i)^{1 - c_{ji}}(e_j) = 0$ and $(\text{ad } f_i)^{1 - c_{ji}}(f_j) = 0$ if $i \neq j$.

Proof

We know H is a Cartan subalgebra and hence it is abelian, so (S1) holds. Condition (S2) follows from

$$[h_i, e_j] = \alpha_j(h_i) e_j = \langle \alpha_j, \alpha_i \rangle e_j = c_{ji} e_j,$$

while the first part of (S3) follows from the isomorphism of $\mathsf{sl}(\alpha_i)$ with $\mathsf{sl}(2, \mathbf{C})$. If $i \neq j$, we have $[e_i, f_j] \in L_{\alpha_i - \alpha_j}$; see Lemma 10.1(i). But since $\alpha_1, \ldots, \alpha_\ell$ form a base for Φ, $\alpha_i - \alpha_j \notin \Phi$. Therefore $L_{\alpha_i - \alpha_j} = 0$. This proves the second part of (S3).

To prove (S4), we fix α_i, α_j in the base and consider

$$M = \bigoplus_k L_{\alpha_j + k\alpha_i},$$

where the sum is taken over all $k \in \mathbf{Z}$ such that $\alpha_j + k\alpha_i \in \Phi$. As before, this is an $\mathsf{sl}(\alpha_i)$-module. Since $\alpha_j - \alpha_i \notin \Phi$, the sum only involves $k \geq 0$ and $k = 0$ does occur. Thus the smallest eigenvalue of $\operatorname{ad} h_i$ on M is $\langle \alpha_j, \alpha_i \rangle = c_{ji}$. By the classification of irreducible $\mathsf{sl}(2, \mathbf{C})$-modules in Chapter 8, the largest eigenvalue of $\operatorname{ad} h_i$ must be $-c_{ji}$.

An $\operatorname{ad} h_i$ eigenvector with eigenvalue $-c_{ji}$ is given by $x = (\operatorname{ad} e_i)^{-c_{ji}}(e_j)$, so applying $\operatorname{ad} e_i$ to x gives zero. This proves the first part of (S4). In fact, we have even proved that $1 - c_{ji}$ is the minimal integer $r \geq 0$ such that $(\operatorname{ad} e_i)^r(e_j) = 0$.

The other part of (S4) is proved by the same method. (Alternatively, one might note that the set of $-\alpha_j$ also is a base for the root system with standard basis $f_i, e_i, -h_i$.) $\square$

Serre's Theorem says that these relations completely determine the Lie algebra.

Theorem 14.6 (Serre's Theorem)

Let C be the Cartan matrix of a root system. Let L be the complex Lie algebra which is generated by elements e_i, f_i, h_i for $1 \leq i \leq \ell$, subject to the relations (S1) to (S4). Then L is finite-dimensional and semisimple with Cartan subalgebra H spanned by $\{h_1, \ldots, h_\ell\}$, and its root system has Cartan matrix C.

We immediately give our main application. Suppose that L is a complex semisimple Lie algebra with Cartan matrix C. By Lemma 14.5 this Lie algebra satisfies the Serre relations, so we can deduce that it must be isomorphic to the Lie algebra in Serre's Theorem with Cartan matrix C. Hence, up to isomorphism, there is just one Lie algebra for each root system. (We remarked at the end of Chapter 12 on some examples that support this statement.)

Serre's Theorem also solves the problem of constructing Lie algebras with the exceptional root systems G_2, F_4, E_6, E_7, and E_8: Just apply it with the Cartan matrix for the type required! Moreover, it shows that, up to isomorphism, there is just one exceptional Lie algebra for each type.

One might like to know whether the exceptional Lie algebras occur in any natural way. They had not been encountered until the classification. But subsequently, after looking for them, they have all been found as algebras of derivations of suitable algebras. See Exercise 14.4 below for an indication of the approaches used.

14.2 On the Proof of Serre's Theorem

We will now give an outline of the proof of Serre's Theorem. The full details are quite involved; they are given for example in Humphreys, *Introduction to Lie Algebras and Representation Theory*, [14].

Step 1. One first considers the Lie algebra $\mathcal{L}$ generated by the elements e_i, f_i, h_i for $1 \leq i \leq \ell$ which satisfies the relations (S1) to (S3) but where (S4) is not yet imposed. This Lie algebra is (usually) infinite-dimensional. Its structure had been determined before Serre by Chevalley, Harish-Chandra, and Jacobson.

One difficulty of studying $\mathcal{L}$ is that one cannot easily see how large it is, and therefore one needs some rather advanced technology: Just defining a Lie algebra by generators and relations may well produce something which is either much smaller or larger than one intended — see Exercise 14.2 for a small illustration of this.

The structure of $\mathcal{L}$ is as follows. Let $\mathcal{E}$ be the Lie subalgebra of $\mathcal{L}$ generated by $\{e_1, \ldots, e_\ell\}$, and let $\mathcal{F}$ be the Lie subalgebra of $\mathcal{L}$ generated by $\{f_1, \ldots, f_\ell\}$. Let H be the span of $\{h_1, \ldots, h_\ell\}$. Then, as a vector space,

$$\mathcal{L} = \mathcal{F} \oplus H \oplus \mathcal{E}.$$

We pause to give two examples.

Example 14.7

Consider the root system of type $A_1 \times A_1$ shown in Example 11.6(d). Here $\mathcal{E}$ is the Lie algebra generated by e_1 and e_2, with the only relations being those coming from the Jacobi identity and the anticommutativity of the Lie bracket. A Lie algebra of this kind is known as a *free Lie algebra* and, as long as it has at least two generators, it is infinite-dimensional.

If instead we take the root system of type A_1, then each of $\mathcal{E}$ and $\mathcal{F}$ is 1-dimensional and $\mathcal{L}$ is just $\mathsf{sl}(2, \mathbf{C})$. This is the only case where $\mathcal{L}$ is finite-dimensional.

Step 2. Now we impose the relations (S4) onto $\mathcal{L}$. Let $\mathcal{U}^+$ be the ideal of $\mathcal{E}$ generated by all θ_{ij}, where

$$\theta_{ij} := (\operatorname{ad} e_i)^{1-c_{ji}}(e_j).$$

Similarly, let $\mathcal{U}^-$ be the ideal of $\mathcal{F}$ generated by all θ_{ij}^-, where

$$\theta_{ij}^- := (\operatorname{ad} f_i)^{1-c_{ji}}(f_j).$$

Let $\mathcal{U} := \mathcal{U}^+ \oplus \mathcal{U}^-$, and let

$$N^+ := \mathcal{E}/\mathcal{U}^+, \quad N^- := \mathcal{F}/\mathcal{U}^-.$$

One shows that $\mathcal{U}^+$, $\mathcal{U}^-$, and hence $\mathcal{U}$ are actually ideals of $\mathcal{L}$. Hence the Lie algebra L in Serre's Theorem, which by definition is $\mathcal{L}/\mathcal{U}$, decomposes as

$$L = \ N^- \oplus H \oplus N^+.$$

By definition, $\mathcal{U}^+$ and $\mathcal{U}^-$ are invariant under $\operatorname{ad} h_i$ for each i and therefore $\operatorname{ad} h_i$ acts diagonally on L. One now has to show that L is finite-dimensional, with Cartan subalgebra H, and that the corresponding root space decomposition has a base giving the prescribed Cartan matrix.

Example 14.8

For the root system $A_1 \times A_1$, we have, by definition, $c_{12} = c_{21} = 0$ and hence $\mathcal{U}^+$ is the ideal generated by $(\operatorname{ad} e_1)(e_2)$ and $(\operatorname{ad} e_2)(e_1)$; that is, by $[e_1, e_2]$. This produces a very small quotient $\mathcal{E}/\mathcal{U}^+$, which is spanned by the cosets of e_1, e_2 and is 2-dimensional.

This is not quite obvious, so we sketch a proof. Given $x \in \mathcal{E}$, we can subtract off an element in the span of e_1 and e_2 to leave x as a sum of elements of the form $[u, v]$ for $u, v \in \mathcal{E}$. Now, as $[e_1, e_1] = [e_2, e_2] = 0$, the bracket $[e_1, e_2]$ must appear in every element in $\mathcal{E}'$ (when expressed in terms of e_1 and e_2), so $\mathcal{E}' = \mathcal{U}^+$ and $x \in \operatorname{Span}\{e_1, e_2\} + \mathcal{U}^+$.

Similarly, $\mathcal{F}/\mathcal{U}^-$ is 2-dimensional, spanned by the cosets of f_1 and f_2. Write $\bar{x}$ for the coset of x in L. We see directly that $L = \mathcal{L}/\mathcal{U}$ has a direct sum decomposition

$$\operatorname{Span}\{\bar{e}_1, \bar{f}_1, h_1\} \oplus \operatorname{Span}\{\bar{e}_2, \bar{f}_2, h_2\},$$

where $\bar{e}_1$ denote the coset $e + \mathcal{U}^+$, and so on. These are ideals in L, and each is isomorphic to $\mathsf{sl}(2, \mathbf{C})$, so in this case we get that L is the direct sum of two copies of $\mathsf{sl}(2, \mathbf{C})$, as we should expect.

For the general proof, more work is needed. The reader might like to try to construct a Lie algebra of type B_2 by this method to get some flavour of what is required.

14.3 Conclusion

The definition of a semisimple Lie algebra does not, on the face of it, seem very restrictive, so the fact that the complex semisimple Lie algebras are determined, up to isomorphism, by their Dynkin diagrams should seem quite remarkable.

In Appendix C, we show that the root system of a semisimple Lie algebra is uniquely determined (up to isomorphism). Thus complex semisimple Lie algebras with different Dynkin diagrams are not isomorphic. This is the last ingredient we need to establish a bijective correspondence between isomorphism classes of complex semisimple Lie algebras and the Dynkin diagrams listed in Theorem 13.1.

This classification theorem is one of the most important and far-reaching in mathematics; we look at some of the further developments it has motivated in the final chapter.

EXERCISES

14.1. Use Serre's Theorem to show that the Lie algebra $\mathsf{so}(6, \mathbf{C})$ is isomorphic to $\mathsf{sl}(4, \mathbf{C})$. (This isomorphism can also be shown by geometric arguments; see Chapter 15.)

14.2. Let L be the Lie algebra generated by x, y, z subject to the relations

$$[x, y] = z, \ [y, z] = x, \ [z, x] = x.$$

Show that L is one-dimensional.

14.3. Let L be a Lie algebra generated by x, y, with no relations other than the Jacobi identity, and $[u, v] = -[v, u]$ for $u, v \in L$. Show that any Lie algebra $\mathcal{G}$ generated by two elements occurs as a homomorphic image of L. (So if you could establish that there are such $\mathcal{G}$ of arbitrary large dimensions, then you could deduce that L must be infinite-dimensional.)

14.4. Let H be the algebra of quaternions. Thus H is the 4-dimensional real associative algebra with basis $1, i, j, k$ and multiplication described by $i^2 = j^2 = k^2 = ijk = -1$. (These are the equations famously carved in 1843 by Hamilton on Brougham Bridge in Dublin.)

(i) Let $\delta \in \operatorname{Der} H$, the Lie algebra of derivations of H. Show that δ preserves the subspace of H consisting of purely imaginary quaternions (that is, those elements of the form $xi+yj+zk$) and that $\delta(1) = 0$. Hence show that $\operatorname{Der} H$ is isomorphic to the

Lie algebra of antisymmetric 3×3 real matrices. (In particular, it has a faithful 3-dimensional representation.)

(ii) Show that if we complexify $\operatorname{Der} H$ by taking the algebra of antisymmetric 3×3 complex matrices, we obtain $\mathsf{sl}(2, \mathbf{C})$.

One step up from the quaternions lies the 8-dimensional Cayley algebra of octonions. One can construct the exceptional Lie algebra g_2 of type G_2 by taking the algebra of derivations of the octonions and then complexifying; this construction also gives its smallest faithful representation. The remaining exceptional Lie algebras can also be constructed by related techniques. For details, we refer the reader to either Schafer, *An Introduction to Nonassociative Algebras* [21] or Baez, "The Octonions" [2].