13
The Classification of Root Systems

In §11.4, we saw how to define the Dynkin diagram of a root system. By Theorem 11.16, which states that any two bases of a root system are conjugate by an element of the Weyl group, this diagram is unique up to the labelling of the vertices. (The labels merely indicate our notation for elements of the base; and so have no essential importance.) Conversely, we saw in §11.4.1 that a root system is determined up to isomorphism by its Dynkin diagram.

From the point of view of classifying complex semisimple Lie algebras, there is no need to distinguish between isomorphic root systems. Hence the problem of finding all root systems can be reduced to the problem of finding all Dynkin diagrams; this gives us a very convenient way to organise the classification.

We shall prove that apart from the four infinite families of root systems associated to the classical Lie algebras there are just five more root systems, the so-called exceptional root systems. We end this chapter by saying a little about how they may be constructed.

K. Erdmann et al., *Introduction to Lie Algebras*

13.1 Classification of Dynkin Diagrams

Our aim in this section is to prove the following theorem.

Theorem 13.1

Given an irreducible root system R, the unlabelled Dynkin diagram associated to R is either a member of one of the four families

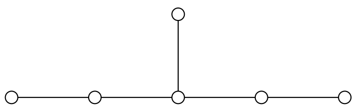
A_ℓ for $\ell \geq 1$: 

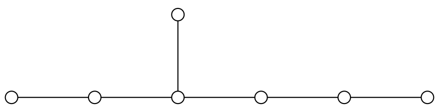
B_ℓ for $\ell \geq 2$: 

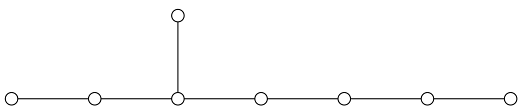
C_ℓ for $\ell \geq 3$: 

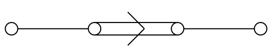
D_ℓ for $\ell \geq 4$: 

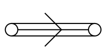
where each of the diagrams above has ℓ vertices, or one of the five exceptional diagrams

E_6: 

E_7: 

E_8: 

F_4: 

G_2: 

Note that there are no repetitions in this list. For example, we have not included C_2 in the list, as it is the same diagram as B_2, and so the associated root systems are isomorphic. (Exercise 13.1 at the end of this chapter asks you to construct an explicit isomorphism.)

Let $\varDelta$ be a connected Dynkin diagram. As a first approximation, we shall determine the possible underlying graphs for $\varDelta$, ignoring for the moment any arrows that may appear. To find these graphs, we do not need to know that they come from root systems. Instead it is convenient to work with more general sets of vectors.

Definition 13.2

Let E be a real inner-product space with inner product $(-,-)$. A subset A of E consisting of linearly independent vectors $v_1, v_2, \ldots, v_n$ is said to be *admissible* if it satisfies the following conditions:

(a) $(v_i, v_i) = 1$ for all i and $(v_i, v_j) \leq 0$ if $i \neq j$.

(b) If $i \neq j$, then $4(v_i, v_j)^2 \in \{0, 1, 2, 3\}$.

To the admissible set A, we associate the graph Γ_A with vertices labelled by the vectors $v_1, \ldots, v_n$, and with $d_{ij} := 4(v_i, v_j)^2 \in \{0, 1, 2, 3\}$ edges between v_i and v_j for $i \neq j$.

Example 13.3

Suppose that B is a base of a root system. Set $A := \{\alpha/\sqrt{(\alpha, \alpha)} : \alpha \in B\}$. Then A is easily seen to be an admissible set. Moreover, the graph Γ_A is the Coxeter graph of B, as defined in §11.4.

We now find all the connected graphs that correspond to admissible sets. Let A be an admissible set in the real inner-product space E with connected graph $\Gamma = \Gamma_A$. The first easy observation we make is that any subset of A is also admissible. We shall use this several times below.

Lemma 13.4

The number of pairs of vertices joined by at least one edge is at most $|A| - 1$.

Proof

Suppose $A = \{v_1, \ldots, v_n\}$. Set $v = \sum_{i=1}^{n} v_i$. As A is linearly independent, $v \neq 0$. Hence $(v, v) = n + 2\sum_{i<j}(v_i, v_j) > 0$ and so

$$n > \sum_{i<j} -2(v_i, v_j) = \sum_{i<j} \sqrt{d_{ij}} \geq N,$$

where N is the number of pairs $\{v_i, v_j\}$ such that $d_{ij} \geq 1$; this is the number that interests us. $\square$

Corollary 13.5

The graph Γ does not contain any cycles.

Proof

Suppose that Γ does have a cycle. Let A' be the subset of A consisting of the vectors involved in this cycle. Then A' is an admissible set with the same number (or more) edges as vertices, in contradiction to the previous lemma. □

Lemma 13.6

No vertex of Γ is incident to four or more edges.

Proof

Take a vertex v of Γ, and let $v_1, v_2, \ldots, v_k$ be all the vertices in Γ joined to v. Since Γ does not contain any cycles, we must have $(v_i, v_j) = 0$ for $i \neq j$. Consider the subspace U with basis $v_1, v_2, \ldots, v_k, v$. The Gram–Schmidt process allows us to extend $v_1, \ldots, v_k$ to an orthonormal basis of U, say by adjoining v_0; necessarily $(v, v_0) \neq 0$. We may express v in terms of this orthonormal basis as

$$v = \sum_{i=0}^{k} (v, v_i) v_i.$$

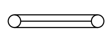
By assumption, v is a unit vector, so expanding (v, v) gives $1 = (v, v) = \sum_{i=0}^{k} (v, v_i)^2$. Since $(v, v_0)^2 > 0$, this shows that

$$\sum_{i=1}^{k} (v, v_i)^2 < 1.$$

Now, as A is admissible and $(v, v_i) \neq 0$, we know that $(v, v_i)^2 \geq \frac{1}{4}$ for $1 \leq i \leq k$. Hence $k \leq 3$. □

An immediate corollary of this lemma is the following.

Corollary 13.7

If Γ is connected and has a triple edge, then $\Gamma =$  . □

Lemma 13.8 (Shrinking Lemma)

Suppose Γ has a subgraph which is a *line*, that is, of the form

where there are no multiple edges between the vertices shown. Define $A' = (A \setminus \{v_1, v_2, \ldots, v_k\}) \cup \{v\}$ where $v = \sum_{i=1}^{k} v_i$. Then A' is admissible and the graph $\Gamma_{A'}$ is obtained from Γ_A by shrinking the line to a single vertex.

Proof

Clearly A' is linearly independent, so we need only verify the conditions on the inner products. By assumption, we have $2(v_i, v_{i+1}) = -1$ for $1 \leq i \leq k-1$ and $(v_i, v_j) = 0$ for $i \neq j$ otherwise. This allows us to calculate (v, v). We find that

$$(v, v) = k + 2\sum_{i=1}^{k-1}(v_i, v_{i+1}) = k - (k-1) = 1.$$

Suppose that $w \in A$ and $w \neq v_i$ for $1 \leq i \leq k$. Then w is joined to at most one of $v_1, \ldots, v_k$ (otherwise there would be a cycle). Therefore either $(w, v) = 0$ or $(w, v) = (w, v_i)$ for some $1 \leq i \leq k$ and then $4(w, v)^2 \in \{0, 1, 2, 3\}$, so A' satisfies the defining conditions for an admissible set. These remarks also determine the graph $\Gamma_{A'}$. $\square$

Say that a vertex of Γ is a *branch vertex* if it is incident to three or more edges; by Lemma 13.6 such a vertex is incident to exactly three edges.

Lemma 13.9

The graph Γ has

(i) no more than one double edge;

(ii) no more than one branch vertex; and

(iii) not both a double edge and a branch vertex.

Proof

Suppose Γ has two (or more) double edges. Since Γ is connected, it has a subgraph consisting of two double edges connected by a line of the form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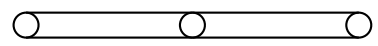
By the Shrinking Lemma, we obtain an admissible set with graph



which contradicts Lemma 13.6. The proofs of the remaining two parts are very similar, so we leave them to the reader. □

For the final steps of the proof of Theorem 13.1, we shall need the following calculation of an inner product.

Lemma 13.10

Suppose that Γ has a line as a subgraph:

Let $v = \sum_{i=1}^{p} i v_i$. Then $(v, v) = \frac{p(p+1)}{2}$.

Proof

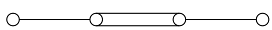
The shape of the subgraph tells us that $2(v_i, v_{i+1}) = -1$ for $1 \leq i \leq p-1$ and that $(v_i, v_j) = 0$ for $i \neq j$ otherwise, so

$$(v, v) = \sum_{i=1}^{p} i^2 + 2\sum_{i=1}^{p-1}(v_i, v_{i+1}) i(i+1) = \sum_{i=1}^{p} i^2 - \sum_{i=1}^{p-1} i(i+1) = p^2 - \sum_{i=1}^{p-1} i,$$

which is equal to $p(p+1)/2$. □

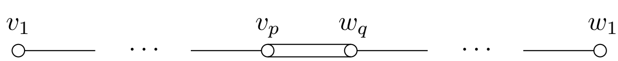
Proposition 13.11

If Γ has a double edge, then Γ is one of



Proof

By Lemma 13.9, any such Γ has the form



where, without loss of generality, $p \geq q$. Let $v = \sum_{i=1}^{p} i v_i$ and $w = \sum_{i=1}^{q} i w_i$. By the calculation above, we have

$$(v, v) = \frac{p(p+1)}{2}, \quad (w, w) = \frac{q(q+1)}{2}.$$

We see from the graph that $4(v_p, w_q)^2 = 2$ and $(v_i, w_j) = 0$ in all other cases. Hence

$$(v, w)^2 = (pv_p, qw_q)^2 = \frac{p^2q^2}{2}.$$

As v and w are linearly independent, the Cauchy–Schwarz inequality implies that $(v, w)^2 < (v, v)(w, w)$. Substituting, we get $2pq < (p+1)(q+1)$, and hence

$$(p-1)(q-1) = pq - p - q + 1 < 2$$

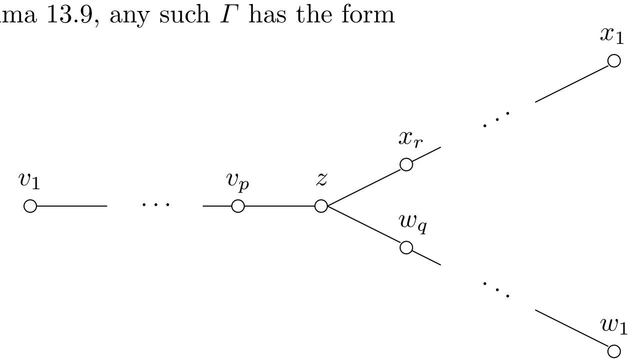
so either $q = 1$ or $p = q = 2$. □

Proposition 13.12

If Γ has a branch point, then either Γ is D_n for some $n \geq 4$ or Γ is E_6, E_7, or E_8.

Proof

By Lemma 13.9, any such Γ has the form



where, without loss of generality, $p \geq q \geq r$. We must show that either $q = r = 1$ or $q = 2$, $r = 1$, and $p \leq 4$.

As in the proof of the last proposition, we let $v = \sum_{i=1}^{p} iv_i$, $w = \sum_{i=1}^{q} iw_i$, and $x = \sum_{i=1}^{r} ix_i$. Then v, w, x are pairwise orthogonal. Let $\hat{v} = v/\|v\|$, $\hat{w} = w/\|w\|$, and $\hat{x} = x/\|x\|$. The space U spanned by v, w, x, z has as an orthonormal basis

$$\{\hat{v}, \hat{w}, \hat{x}, z_0\}$$

for some choice of z_0 which will satisfy $(z, z_0) \neq 0$. We may write

$$z = (z, \hat{v})\hat{v} + (z, \hat{w})\hat{w} + (z, \hat{x})\hat{x} + (z, z_0)z_0.$$

As z is a unit vector and $(z, z_0) \neq 0$, we get

$$(z, \tilde{v})^2 + (z, \tilde{w})^2 + (z, \tilde{x})^2 < 1.$$

We know the lengths of v, w, x from Lemma 13.10. Furthermore, $(z,v)^2 = (z, pv_p)^2 = p^2/4$, and similarly $(z,w)^2 = q^2/4$ and $(z,x)^2 = r^2/4$. Substituting these into the previous inequality gives

$$\frac{2p^2}{4p(p+1)} + \frac{2q^2}{4q(q+1)} + \frac{2r^2}{4r(r+1)} < 1.$$

By elementary steps, this is equivalent to

$$\frac{1}{p+1} + \frac{1}{q+1} + \frac{1}{r+1} > 1.$$

Since $\frac{1}{p+1} \leq \frac{1}{q+1} \leq \frac{1}{r+1} \leq \frac{1}{2}$, we have $1 < \frac{3}{r+1}$ and hence $r < 2$, so we must have $r = 1$. Repeating this argument gives that $q < 3$, so $q = 1$ or $q = 2$. If $q = 2$, then we see that $p < 5$. On the other hand, if $q = 1$, then there is no restriction on p. □

We have now found all connected graphs which come from admissible sets. We return to our connected Dynkin diagram Δ. We saw in Example 13.3 that the Coxeter graph of Δ, say $\bar{\Delta}$, must appear somewhere in our collection. If Δ has no multiple edges, then, by Proposition 13.12, $\Delta = \bar{\Delta}$ is one of the graphs listed in Theorem 13.1.

If Δ has a double edge, then Proposition 13.11 tells us that there are two possibilities for $\bar{\Delta}$. In the case of B_2 and F_4, we get essentially the same graph whichever way we put the arrow; otherwise there are two different choices, giving B_n and C_n for $n \geq 3$. Finally, if Δ has a triple edge, then Corollary 13.7 tells us that $\Delta = G_2$. This completes the proof of Theorem 13.1.

13.2 Constructions

We now want to show that all the Dynkin diagrams listed in Theorem 13.1 actually occur as the Dynkin diagram of some root system.

Our analysis of the classical Lie algebras $\mathsf{sl}_{\ell+1}$, $\mathsf{so}_{2\ell+1}$, $\mathsf{sp}_{2\ell}$, and $\mathsf{so}_{2\ell}$ in Chapter 12 gives us constructions of root systems of types A, B, C, and D respectively. We discuss the Weyl groups of these root systems in Appendix D. For the exceptional Dynkin diagrams G_2, F_4, E_6, E_7, and E_8, we have to do more work. For completeness, we give constructions of all the corresponding root systems, but as those of type E are rather large and difficult to work with, we do not go into any details for this type.

In each case, we shall take for the underlying space E a subspace of a Euclidean space $\mathbf{R}^m$. Let ε_i be the vector with 1 in position i and 0 elsewhere.

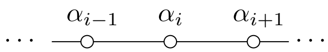
When describing bases, we shall follow the pattern established in Chapter 12 by taking as many simple roots as possible from the set $\{\alpha_1, \dots, \alpha_{m-1}\}$, where

$$\alpha_i := \varepsilon_i - \varepsilon_{i+1}.$$

For these elements, we have

$$\langle \alpha_i, \alpha_j \rangle = \begin{cases} 2 & i = j \\ -1 & |i - j| = 1 \\ 0 & \text{otherwise,} \end{cases}$$

so the corresponding part of the Dynkin diagram is a line,



and the corresponding part of the Cartan matrix is

$$\begin{pmatrix} & \vdots & \vdots & \vdots & \\ \dots & 2 & -1 & 0 & \dots \\ \dots & -1 & 2 & -1 & \dots \\ \dots & 0 & -1 & 2 & \dots \\ & \vdots & \vdots & \vdots & \end{pmatrix}.$$

Both of these will be familiar from root systems of type A.

13.2.1 Type G_2

We have already given one construction of a root system of type G_2 in Example 11.6(c). We give another here, which is more typical of the other constructions that follow. Let $E = \left\{ v = \sum_{i=1}^{3} c_i \varepsilon_i \in \mathbf{R}^3 : \sum c_i = 0 \right\}$, let

$$I = \left\{ m_1\varepsilon_1 + m_2\varepsilon_2 + m_3\varepsilon_3 \in \mathbf{R}^3 : m_1, m_2, m_3 \in \mathbf{Z} \right\},$$

and let

$$R = \{\alpha \in I \cap E : (\alpha, \alpha) = 2 \text{ or } (\alpha, \alpha) = 6\}.$$

This is motivated by noting that the ratio of the length of a long root to the length of a short root in a root system of type G_2 is $\sqrt{3}$. By direct calculation, one finds that

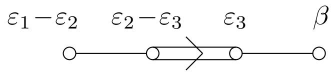
$$R = \left\{ \pm(\varepsilon_i - \varepsilon_j), i \neq j \right\} \cup \left\{ \pm(2\varepsilon_i - \varepsilon_j - \varepsilon_k), \{i, j, k\} = \{1, 2, 3\} \right\}.$$

This gives 12 roots in total, as expected from the diagram in Example 11.6(c). To find a base, we need to find $\alpha, \beta \in R$ of different lengths, making an angle of $5\pi/6$. One suitable choice is $\alpha = \varepsilon_1 - \varepsilon_2$ and $\beta = \varepsilon_2 + \varepsilon_3 - 2\varepsilon_1$.

The Weyl group for G_2 is generated by the simple reflections s_α and s_β. By Exercise 11.14(ii), it is the dihedral group of order 12.

13.2.2 Type F_4

Since the Dynkin diagram F_4 contains the Dynkin diagram B_3, we might hope to construct the corresponding root system by extending the root system of B_3. Therefore we look for $\beta \in \mathbf{R}^4$ so that $B = (\varepsilon_1 - \varepsilon_2, \varepsilon_2 - \varepsilon_3, \varepsilon_3, \beta)$ is a base with Cartan numbers given by the labelled Dynkin diagram:



It is easy to see that the only possible choices for β are $\beta = -\frac{1}{2}(\varepsilon_1 + \varepsilon_2 + \varepsilon_3) \pm \frac{1}{2}\varepsilon_4$. Therefore it seems hopeful to set

$$R = \{\pm\varepsilon_i : 1 \le i \le 4\} \cup \{\pm\varepsilon_i \pm \varepsilon_j : 1 \le i \ne j \le 4\} \cup \{\tfrac{1}{2}(\pm\varepsilon_1 \pm \varepsilon_2 \pm \varepsilon_3 \pm \varepsilon_4)\}.$$

One can check directly that axioms (R1) up to (R4) hold; see Exercise 13.3. It remains to check that

$$\begin{aligned} \beta_1 &= \varepsilon_1 - \varepsilon_2, \\ \beta_2 &= \varepsilon_2 - \varepsilon_3, \\ \beta_3 &= \varepsilon_3, \\ \beta_4 &= \frac{1}{2}(-\varepsilon_1 - \varepsilon_2 - \varepsilon_3 + \varepsilon_4), \end{aligned}$$

defines a base for R. Note that R has 48 elements, so we need to find 24 positive roots. Each ε_i is a positive root, and if $1 \le i < j \le 3$ then so are $\varepsilon_i - \varepsilon_j$ and $\varepsilon_i + \varepsilon_j$. Furthermore, for $1 \le i \le 3$, also $\varepsilon_4 \pm \varepsilon_i$ are positive roots.

This already gives us 16 roots. In total, there are 16 roots of the form $\frac{1}{2}(\sum \pm\varepsilon_i)$. As one would expect, half of these turn out to be positive roots. Obviously each must have a summand equal to β_4. There are 3 positive roots of the form $\beta_4 + \varepsilon_j$, and also 3 of the form $\beta_4 + \varepsilon_j + \varepsilon_k$. Then there is β_4 itself, and finally $\beta_4 + \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = \frac{1}{2}\sum \varepsilon_i$.

The Weyl group is known to have order $2^7\,3^2$, but its structure is too complicated to be discussed here.

13.2.3 Type E

To construct the root systems of types E, it will be convenient to first construct a root system of type E_8 and then to find root systems of types E_6 and E_7 inside it.

Let $E = \mathbf{R}^8$ and let

$$R = \left\{\pm\varepsilon_i \pm \varepsilon_j : i < j\right\} \cup \left\{\frac{1}{2}\sum_{i=1}^{8} \pm\varepsilon_i\right\},$$

where in the second set an even number of + signs are chosen.

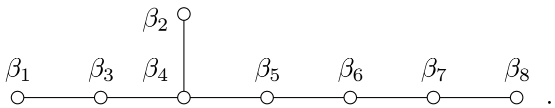
The first set in the union contributes 112 roots and the second 128, giving 240 roots in all. Assuming that R is a root system, we claim that a base for R is given by $B = \{\beta_1, \beta_2, \ldots, \beta_8\}$, where

$$\beta_1 = \frac{1}{2}\Big(-\varepsilon_1 - \varepsilon_8 + \sum_{i=2}^{7} \varepsilon_i\Big),$$
$$\beta_2 = -\varepsilon_1 - \varepsilon_2,$$
$$\beta_i = \varepsilon_{i-2} - \varepsilon_{i-1} \text{ if } 3 \le i \le 8.$$

To see that B is a base, one first verifies that the roots $\pm\varepsilon_i - \varepsilon_j$ for $i < j$ can be written as linear combinations of the elements of B with positive coefficients. The remaining positive roots are those of the form

$$\frac{1}{2}\Big(-\varepsilon_8 + \sum_{i=1}^{7} \pm\varepsilon_i\Big),$$

where there are an odd number of + signs chosen in the sum. To check this, subtract off β_1, and then verify that the result is a positive integral linear combination of the remaining roots. (This is where the condition on the signs comes in.) The labelled Dynkin diagram is



Omitting the root β_8 gives a base for a root system of type E_7, and omitting the roots β_7 and β_8 gives a base for a root system of type E_6. We leave it to the keen reader to explicitly construct these root systems. (Details may be found in Bourbaki, *Lie Groups and Lie Algebras* [6], Chapter 5, Section 4, Number 4.)

EXERCISES

13.1. Find an explicit isomorphism between the root systems of types B_2 and C_2. That is, find a linear map between the vector spaces for B_2 and C_2, respectively, which interchanges a long root with a short root, and preserves the Cartan numbers.

13.2. Show that the root systems of types B_n and C_n are dual to one another in the sense defined in Exercise 11.13.

13.3. Check that the construction of F_4 given in §13.2.2 really does give a root system. This can be simplified by noting that R contains

$$\{\pm\varepsilon_i : 1 \leq i \leq 3\} \cup \{\pm\varepsilon_i \pm \varepsilon_j : 1 \leq i \neq j \leq 3\},$$

which is the root system of type B_3 we constructed in Chapter 12.