12
The Classical Lie Algebras

Our aim in this chapter is to study the classical Lie algebras $\mathsf{sl}(n, \mathbf{C})$, $\mathsf{so}(n, \mathbf{C})$, and $\mathsf{sp}(n, \mathbf{C})$ for $n \geq 2$. We shall show that, with two exceptions, all these Lie algebras are simple. We shall also find their root systems and the associated Dynkin diagrams and describe their Killing forms. The main result we prove is the following theorem.

Theorem 12.1

If L is a classical Lie algebra other than $\mathsf{so}(2, \mathbf{C})$ and $\mathsf{so}(4, \mathbf{C})$, then L is simple.

We also explain how the root systems we have determined can be used to rule out most isomorphisms between different classical Lie algebras (while suggesting the presence of those that do exist). This will lead us to a complete classification of the classical Lie algebras up to isomorphism.

In the following section, we describe a programme that will enable us to deal with each of the families of classical Lie algebras in a similar way. We then carry out this programme for each family in turn.

K. Erdmann et al., *Introduction to Lie Algebras*

12.1 General Strategy

Let L be a classical Lie algebra. In each case, it follows from the definitions given in §4.3 that L has a large subalgebra H of diagonal matrices. The maps $\operatorname{ad} h$ for $h \in H$ are diagonalisable, as was first seen in Exercise 1.17, so H consists of semisimple elements.

We can immediately say a bit more about the action of H. The subspace $L \cap \operatorname{Span}\{e_{ij} : i \neq j\}$ of off-diagonal matrices in L is also invariant under $\operatorname{ad} h$ for $h \in H$ and hence the action of $\operatorname{ad} H$ on this space is diagonalisable. Let

$$L \cap \operatorname{Span}\{e_{ij} : i \neq j\} = \bigoplus_{\alpha \in \Phi} L_\alpha,$$

where for $\alpha \in H^\star$, L_α is the α-eigenspace of H on the off-diagonal part of L and

$$\Phi = \{\alpha \in H^* : \alpha \neq 0, L_\alpha \neq 0\}.$$

This gives us the decomposition

$$(\star) \qquad L = L_0 \oplus \bigoplus_{\alpha \in \Phi} L_\alpha,$$

which looks very much like a root space decomposition. We shall first show that $H = L_0$, from which it will follow that H is a Cartan subalgebra of L.

Lemma 12.2

Let $L \subseteq \mathsf{gl}(n, \mathbf{C})$ and H be as in $(\star)$ above. Suppose that for all non-zero $h \in H$ there is some $\alpha \in \Phi$ such that $\alpha(h) \neq 0$. Then H is a Cartan subalgebra of L.

Proof

We know already that H is abelian and that all the elements of H are semisimple. It remains to show that H is maximal with these properties. Suppose that $x \in L$ and that $[H, x] = 0$. (Equivalently, $x \in L_0$.)

Using the direct sum decomposition $(\star)$, we may write x as $x = h_x + \sum_{\alpha \in \Phi} c_\alpha x_\alpha$, where $x_\alpha \in L_\alpha$, $c_\alpha \in \mathbf{C}$, and $h_x \in H$. For all $h \in H$, we have

$$0 = [h, x] = \sum_\alpha c_\alpha \alpha(h) x_\alpha.$$

By the hypothesis, for every $\alpha \in \Phi$ there is some $h \in H$ such that $\alpha(h) \neq 0$, so $c_\alpha = 0$ for each α and hence $x \in H$. □

To show that the classical Lie algebras (with the two exceptions mentioned in Theorem 12.1) are simple, we first need to show that they are semisimple. We shall use the following criterion.

Proposition 12.3

Let L be a complex Lie algebra with Cartan subalgebra H. Let

$$L = H \oplus \bigoplus_{\alpha \in \Phi} L_\alpha$$

be the direct sum decomposition of L into simultaneous eigenspaces for the elements of $\operatorname{ad} H$, where Φ is the set of non-zero $\alpha \in H^\star$ such that $L_\alpha \neq 0$. (So we assume that $H = L_0$.) Suppose that the following conditions hold:

(i) For each $0 \neq h \in H$, there is some $\alpha \in \Phi$ such that $\alpha(h) \neq 0$.

(ii) For each $\alpha \in \Phi$, the space L_α is 1-dimensional.

(iii) If $\alpha \in \Phi$, then $-\alpha \in \Phi$, and if L_α is spanned by x_α, then $[[x_\alpha, x_{-\alpha}], x_\alpha] \neq 0$.

Then L is semisimple.

Proof

By Exercise 4.6, it is enough to show that L has no non-zero abelian ideals. Let A be an abelian ideal of L. By hypothesis, H acts diagonalisably on L and $[H, A] \subseteq A$, so H also acts diagonalisably on A. We can therefore decompose A as

$$A = (A \cap H) \oplus \bigoplus_{\alpha \in \Phi} (A \cap L_\alpha)\,.$$

Suppose for a contradiction that $A \cap L_\alpha \neq 0$ for some $\alpha \in \Phi$. Then, because L_α is 1-dimensional, we must have $L_\alpha \subseteq A$. Since A is an ideal, this implies that $[L_\alpha, L_{-\alpha}] \subseteq A$, so A contains an element h of the form $h = [x_\alpha, x_{-\alpha}]$, where x_α spans L_α and $x_{-\alpha}$ spans $L_{-\alpha}$. Since A is abelian and both x_α and h are known to lie in A, we deduce that $[h, x_\alpha] = 0$. However, condition (iii) says that $[h, x_\alpha] \neq 0$, a contradiction.

We have therefore proved that $A = A \cap H$; that is, $A \subseteq H$. If A contains some non-zero element h, then, by condition (i), we know that there is some $\alpha \in \Phi$ such that $\alpha(h) \neq 0$. But then $[h, x_\alpha] = \alpha(h) x_\alpha \in L_\alpha$ and also $[h, x_\alpha] \in A$, so $x_\alpha \in L_\alpha \cap A$, which contradicts the previous paragraph. Therefore $A = 0$. □

Note that since $[L_\alpha, L_{-\alpha}] \subseteq L_0 = H$, condition (iii) holds if and only if $\alpha([L_\alpha, L_{-\alpha}]) \neq 0$. Therefore, to show that this condition holds, it is enough to verify that $[[L_\alpha, L_{-\alpha}], L_\alpha] \neq 0$ for one member of each pair of roots $\pm\alpha$; this will help to reduce the amount of calculation required.

Having found a Cartan subalgebra of L and shown that L is semisimple, we will then attempt to identify the root system. We must find a base for Φ, and then for β, γ in the base we must find the Cartan number $\langle \beta, \gamma \rangle$. To do this, we shall use the identity

$$\langle \beta, \gamma \rangle = \beta(h_\gamma),$$

where h_γ is part of the standard basis of the subalgebra $\mathsf{sl}(\gamma)$ associated to the root γ (see §10.4). To find h_γ will be an easy calculation for which we can use the work done in checking condition (iii) of Proposition 12.3.

Now, to show that L is simple, it is enough, by the following proposition, to show that Φ is irreducible, or equivalently (by Exercise 11.7) that the Dynkin diagram of Φ is connected.

Proposition 12.4

Let L be a complex semisimple Lie algebra with Cartan subalgebra H and root system Φ. If Φ is irreducible, then L is simple.

Proof

By the root space decomposition, we may write L as

$$L = H \oplus \bigoplus_{\alpha \in \Phi} L_\alpha.$$

Suppose that L has a proper non-zero ideal I. Since H consists of semisimple elements, it acts diagonalisably on I, and so I has a basis of common eigenvectors for the elements of $\operatorname{ad} H$. As we know that each root space L_α is 1-dimensional, this implies that

$$I = H_1 \oplus \bigoplus_{\alpha \in \Phi_1} L_\alpha$$

for some subspace H_1 of $H = L_0$ and some subset Φ_1 of Φ. Similarly, we have

$$I^\perp = H_2 \oplus \bigoplus_{\alpha \in \Phi_2} L_\alpha,$$

where $I^\perp$ is the perpendicular space to I with respect to the Killing form. As $I \oplus I^\perp = L$, we must have $H_1 \oplus H_2 = H$, $\Phi_1 \cap \Phi_2 = \emptyset$, and $\Phi_1 \cup \Phi_2 = \Phi$.

If Φ_2 is empty, then $L_\alpha \subseteq I$ for all $\alpha \in \Phi$. As L is generated by its root spaces, this implies that $I = L$, a contradiction. Similarly, Φ_1 is non-empty. Now, given $\alpha \in \Phi_1$ and $\beta \in \Phi_2$, we have

$$\langle \alpha, \beta \rangle = \alpha(h_\beta) = 0$$

as $\alpha(h_\beta)e_\alpha = [h_\beta, e_\alpha] \in I^\perp \cap I = 0$, so $(\alpha, \beta) = 0$ for all $\alpha \in \Phi_1$ and $\beta \in \Phi_2$, which shows that Φ is reducible. □

In summary, our programme is:

(1) Find the subalgebra H of diagonal matrices in L and determine the decomposition $(\star)$. This will show directly that conditions (i) and (ii) of Proposition 12.3 hold.

(2) Check that $[[L_\alpha, L_{-\alpha}], L_\alpha] \neq 0$ for each root $\alpha \in \Phi$.

By Lemma 12.2 and Proposition 12.3, we now know that L is semisimple and that H is a Cartan subalgebra of L.

(3) Find a base for Φ.

(4) For γ, β in the base, find h_γ and e_β and hence $\langle \beta, \gamma \rangle = \beta(h_\gamma)$. This will determine the Dynkin diagram of our root system, from which we can verify that Φ is irreducible and L is simple.

12.2 $\mathsf{sl}(\ell+1,\mathbf{C})$

For this Lie algebra, most of the work has already been done.

(1) We saw at the start of Chapter 10 that the root space decomposition of $L = \mathsf{sl}(\ell+1,\mathbf{C})$ is

$$L = H \oplus \bigoplus_{i \neq j} L_{\varepsilon_i - \varepsilon_j},$$

where $\varepsilon_i(h)$ is the i-th entry of h and the root space $L_{\varepsilon_i-\varepsilon_j}$ is spanned by e_{ij}. Thus $\Phi = \{\pm(\varepsilon_i - \varepsilon_j) : 1 \leq i < j \leq l+1\}$.

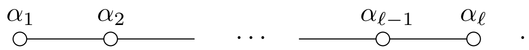
(2) If $i < j$, then $[e_{ij}, e_{ji}] = e_{ii} - e_{jj} = h_{ij}$ and $[h_{ij}, e_{ij}] = 2e_{ij} \neq 0$.

(3) We know from Exercise 11.4 that the root system Φ has as a base $\{\alpha_i : 1 \leq i \leq \ell\}$, where $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$.

(4) From (2) we see that standard basis elements for the subalgebras $\mathsf{sl}(\alpha_i)$ can be taken as $e_{\alpha_i} = e_{i,i+1}$, $f_{\alpha_i} = e_{i+1,i}$, $h_{\alpha_i} = e_{ii} - e_{i+1,i+1}$. Calculation shows that

$$\langle \alpha_i, \alpha_j \rangle = \alpha_i(h_{\alpha_j}) = \begin{cases} 2 & i = j \\ -1 & |i-j| = 1 \\ 0 & \text{otherwise,} \end{cases}$$

so the Cartan matrix of Φ is as calculated in Example 11.17(1) and the Dynkin diagram is



This diagram is connected, so L is simple. We say that the root system of $\mathsf{sl}(\ell+1, \mathbf{C})$ has *type* A_ℓ.

12.3 so$(2\ell + 1, \mathbf{C})$

Let $L = \mathsf{gl}_S(2\ell+1, \mathbf{C})$ for $\ell \geq 1$, where

$$S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & I_\ell \\ 0 & I_\ell & 0 \end{pmatrix}.$$

Recall that this means

$$L = \left\{ x \in \mathsf{gl}(2\ell+1, \mathbf{C}) : x^t S = -Sx \right\}.$$

We write elements of L as block matrices, of shapes adapted to the blocks of S. Calculation shows, using Exercise 2.12, that

$$L = \left\{ \begin{pmatrix} 0 & c^t & -b^t \\ b & m & p \\ -c & q & -m^t \end{pmatrix} : p = -p^t \text{ and } q = -q^t \right\}.$$

As usual, let H be the set of diagonal matrices in L. It will be convenient to label the matrix entries from 0 to 2ℓ. Let $h \in H$ have diagonal entries $0, a_1, \ldots, a_\ell, -a_1, \ldots, -a_\ell$, so with our numbering convention,

$$h = \sum_{i=1}^{\ell} a_i (e_{ii} - e_{i+\ell, i+\ell}).$$

(1a) We start by finding the root spaces for H. Consider the subspace of L spanned by matrices whose non-zero entries occur only in the positions labelled by b and c. This subspace has as a basis $b_i = e_{i,0} - e_{0,\ell+i}$ and $c_i = e_{0,i} - e_{\ell+i,0}$ for $1 \le i \le \ell$. (Note that b_i and c_i are matrices, not scalars!) We calculate that

$$[h, b_i] = a_i b_i, \quad [h, c_i] = -a_i c_i.$$

(1b) We extend to a basis of L by the matrices

$$\begin{aligned} m_{ij} &= e_{ij} - e_{\ell+j,\ell+i} \text{ for } 1 \le i \ne j \le \ell, \\ p_{ij} &= e_{i,\ell+j} - e_{j,\ell+i} \text{ for } 1 \le i < j \le l, \\ q_{ji} = p_{ij}^t &= e_{\ell+j,i} - e_{\ell+i,j} \text{ for } 1 \le i < j \le l. \end{aligned}$$

Again we are fortunate that the obvious basis elements are in fact simultaneous eigenvectors for the action of H. Calculation shows that

$$\begin{aligned} [h, m_{ij}] &= (a_i - a_j) m_{ij}, \\ [h, p_{ij}] &= (a_i + a_j) p_{ij}, \\ [h, q_{ji}] &= -(a_i + a_j) q_{ji}. \end{aligned}$$

We can now list the roots. For $1 \le i \le \ell$, let $\varepsilon_i \in H^\star$ be the map sending h to a_i, its entry in position i.

root	ε_i	$-\varepsilon_i$	$\varepsilon_i - \varepsilon_j$	$\varepsilon_i + \varepsilon_j$	$-(\varepsilon_i + \varepsilon_j)$
eigenvector	b_i	c_i	m_{ij} $(i \ne j)$	p_{ij} $(i < j)$	q_{ji} $(i < j)$

(2) We check that $[h, x_\alpha] \ne 0$, where $h = [x_\alpha, x_{-\alpha}]$.

(2a) For $\alpha = \varepsilon_i$, we have

$$h_i := [b_i, c_i] = e_{ii} - e_{\ell+i,\ell+i}$$

and, by (1a), $[h_i, b_i] = b_i$.

(2b) For $\alpha = \varepsilon_i - \varepsilon_j$ and $i < j$, we have

$$h_{ij} := [m_{ij}, m_{ji}] = (e_{ii} - e_{\ell+i,\ell+i}) - (e_{jj} - e_{\ell+j,\ell+j})$$

and, by (1b), $[h_{ij}, m_{ij}] = 2m_{ij}$.

(2c) Finally, for $\alpha = \varepsilon_i + \varepsilon_j$, for $i < j$, we have

$$k_{ij} := [p_{ij}, q_{ji}] = (e_{ii} - e_{\ell+i,\ell+i}) + (e_{jj} - e_{\ell+j,\ell+j})$$

and, by (1b), $[k_{ij}, p_{ij}] = 2p_{ij}$.

(3) We claim that a base for our root system is given by

$$B = \{\alpha_i : 1 \le i < \ell\} \cup \{\beta_\ell\},$$

where $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ and $\beta_\ell = \varepsilon_\ell$. To see this, note that when $1 \le i < \ell$,

$$\varepsilon_i = \alpha_i + \alpha_{i+1} + \ldots + \alpha_{\ell-1} + \beta_\ell,$$

and that when $1 \le i < j \le \ell$,

$$\begin{aligned} \varepsilon_i - \varepsilon_j &= \alpha_i + \alpha_{i+1} + \ldots + \alpha_{j-1}, \\ \varepsilon_i + \varepsilon_j &= \alpha_i + \ldots \alpha_{j-1} + 2(\alpha_j + \alpha_{j+1} + \ldots + \alpha_{\ell-1} + \beta_\ell). \end{aligned}$$

Going through the table of roots shows that if $\gamma \in \Phi$ then either γ or $-\gamma$ appears above as a non-negative linear combination of elements of B. Since B has ℓ elements and $\ell = \dim H$, this is enough to show that B is a base of Φ.

(4) We now determine the Cartan matrix. For $i < \ell$, we take $e_{\alpha_i} = m_{i,i+1}$, and then $h_{\alpha_i} = h_{i,i+1}$ follows from (2b). We take $e_{\beta_\ell} = b_\ell$, and then from (2a) we see that $h_\beta = 2(e_{\ell,\ell} - e_{2\ell,2\ell})$.

For $1 \le i, j < \ell$, we calculate that

$$[h_{\alpha_j}, e_{\alpha_i}] = \begin{cases} 2e_{\alpha_j} & i = j \\ -e_{\alpha_j} & |i-j| = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Hence

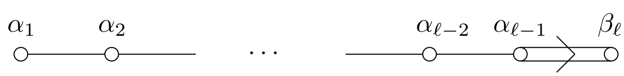
$$\langle \alpha_i, \alpha_j \rangle = \begin{cases} 2 & i = j \\ -1 & |i-j| = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Similarly, by calculating $[h_{\beta_\ell}, e_{\alpha_i}]$ and $[h_{\alpha_i}, e_{\beta_\ell}]$, we find that

$$\langle \alpha_i, \beta_\ell \rangle = \begin{cases} -2 & i = \ell - 1 \\ 0 & \text{otherwise,} \end{cases}$$

$$\langle \beta_\ell, \alpha_i \rangle = \begin{cases} -1 & i = \ell - 1 \\ 0 & \text{otherwise.} \end{cases}$$

This shows that the Dynkin diagram of Φ is



As the Dynkin diagram is connected, Φ is irreducible and so L is simple. The root system of $\mathsf{so}(2\ell+1, \mathbf{C})$ is said to have *type* B_ℓ.

12.4 $\mathsf{so}(2\ell, \mathbf{C})$

Let $L = \mathsf{gl}_S(2\ell, \mathbf{C})$, where

$$S = \begin{pmatrix} 0 & I_\ell \\ I_\ell & 0 \end{pmatrix}.$$

We write elements of L as block matrices, of shapes adapted to the blocks of S. Calculation shows that

$$L = \left\{ \begin{pmatrix} m & p \\ q & -m^t \end{pmatrix} : p = -p^t \text{ and } q = -q^t \right\}.$$

We see that if $\ell = 1$ then the Lie algebra is 1-dimensional, and so, by definition, not simple or semisimple. For this reason, we assumed in the statement of Theorem 12.1 that $\ell \geq 2$.

As usual, we let H be the set of diagonal matrices in L. We label the matrix entries from 1 up to 2ℓ. This means that we can use the calculations already done for $\mathsf{so}(2\ell+1, \mathbf{C})$ by ignoring the row and column of matrices labelled by 0.

(1) All the work needed to find the root spaces in $\mathsf{so}(2\ell, \mathbf{C})$ is done for us by step (1b) for $\mathsf{so}(2\ell + 1, \mathbf{C})$. Taking the notation from this part, we get the following roots:

root	$\varepsilon_i - \varepsilon_j$	$\varepsilon_i + \varepsilon_j$	$-(\varepsilon_i + \varepsilon_j)$
eigenvector	m_{ij} $(i \neq j)$	p_{ij} $(i < j)$	q_{ji} $(i < j)$

(2) The work already done in steps (2b) and (2c) for $\mathsf{sl}(2\ell + 1, \mathbf{C})$ shows that $[[L_\alpha, L_{-\alpha}], L_\alpha] \neq 0$ for each root α.

(3) We claim that a base for our root system is given by

$$B = \{\alpha_i : 1 \leq i < \ell\} \cup \{\beta_\ell\},$$

where $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ and $\beta_\ell = \varepsilon_{\ell-1} + \varepsilon_\ell$. To see this, note that when $1 \leq i < j < \ell$,

$$\varepsilon_i - \varepsilon_j = \alpha_i + \alpha_{i+1} + \ldots + \alpha_{j-1},$$
$$\varepsilon_i + \varepsilon_j = (\alpha_i + \alpha_{i+1} + \ldots + \alpha_{\ell-2}) + (\alpha_j + \alpha_{j+1} + \ldots + \alpha_{\ell-1} + \beta_\ell).$$

This shows that if $\gamma \in \Phi$, then either γ or $-\gamma$ is a non-negative linear combination of elements of B with integer coefficients, so B is a base for our root system.

(4) We calculate the Cartan integers. The work already done for $\mathsf{so}(2\ell+1,\mathbf{C})$ gives us the Cartan numbers $\langle \alpha_i, \alpha_j \rangle$ for $i, j < \ell$. For the remaining ones, we take $e_{\beta_\ell} = p_{\ell-1,\ell}$ and then we find from step (2c) for $\mathsf{so}(2\ell+1,\mathbf{C})$ that

$$h_{\beta_\ell} = (e_{\ell-1,\ell-1} - e_{2\ell-1,2\ell-1}) + (e_{\ell,\ell} - e_{2\ell,2\ell}).$$

Hence

$$\langle \alpha_j, \beta_\ell \rangle = \begin{cases} -1 & j = \ell - 2 \\ 0 & \text{otherwise,} \end{cases}$$

$$\langle \beta_\ell, \alpha_j \rangle = \begin{cases} -1 & j = \ell - 2 \\ 0 & \text{otherwise.} \end{cases}$$

If $\ell = 2$, then the base has only the two orthogonal roots α_1 and β_2, so in this case, Φ is reducible. In fact, $\mathsf{so}(4,\mathbf{C})$ is isomorphic to $\mathsf{sl}(2,\mathbf{C}) \oplus \mathsf{sl}(2,\mathbf{C})$, as you were asked to prove in Exercise 10.8. This explains the other Lie algebra excluded from the statement of Theorem 12.1.

If $\ell \geq 3$, then our calculation shows that the Dynkin diagram of Φ is

As this diagram is connected, the Lie algebra is simple. When $\ell = 3$, the Dynkin diagram is the same as that of A_3, the root system of $\mathsf{sl}(3,\mathbf{C})$, so we might expect that $\mathsf{so}(6,\mathbf{C})$ should be isomorphic to $\mathsf{sl}(4,\mathbf{C})$. This is indeed the case; see Exercise 14.1. For $\ell \geq 4$, the root system of $\mathsf{so}(2\ell,\mathbf{C})$ is said to have *type* D_ℓ.

12.5 $\mathsf{sp}(2\ell,\mathbf{C})$

Let $L = \mathsf{gl}_S(2\ell,\mathbf{C})$, where S is the matrix

$$S = \begin{pmatrix} 0 & I_\ell \\ -I_\ell & 0 \end{pmatrix}.$$

We write elements of L as block matrices, of shapes adapted to the blocks of S. Calculation shows that

$$L = \left\{ \begin{pmatrix} m & p \\ q & -m^t \end{pmatrix} : p = p^t \text{ and } q = q^t \right\}.$$

We see that when $\ell = 1$, this is the same Lie algebra as $\mathsf{sl}(2, \mathbf{C})$. In what follows, we shall assume that $\ell \geq 2$.

Let H be the set of diagonal matrices in L. We label the matrix entries in the usual way from 1 to 2ℓ. Let $h \in H$ have diagonal entries $a_1, \ldots, a_\ell, -a_1, \ldots, -a_\ell$, that is,

$$h = \sum_{i=1}^{\ell} a_i(e_{ii} - e_{i+\ell,i+\ell}).$$

(1) We take the following basis for the root spaces of L:

$$\begin{aligned} m_{ij} &= e_{ij} - e_{\ell+j,\ell+i} \text{ for } 1 \leq i \neq j \leq \ell, \\ p_{ij} &= e_{i,\ell+j} + e_{j,\ell+i} \text{ for } 1 \leq i < j \leq \ell, \quad p_{ii} = e_{i,\ell+i} \text{ for } 1 \leq i \leq \ell, \\ q_{ji} = p_{ij}^t &= e_{\ell+j,i} + e_{\ell+i,j} \text{ for } 1 \leq i < j \leq \ell, \quad q_{ii} = e_{\ell+i,i} \text{ for } 1 \leq i \leq \ell. \end{aligned}$$

Calculation shows that

$$\begin{aligned} [h, m_{ij}] &= (a_i - a_j)m_{ij}, \\ [h, p_{ij}] &= (a_i + a_j)p_{ij}, \\ [h, q_{ji}] &= -(a_i + a_j)q_{ji}. \end{aligned}$$

Notice that for p_{ij} and q_{ji} it is allowed that $i = j$, and in these cases we get the eigenvalues $2a_i$ and $-2a_i$, respectively.

We can now list the roots. Write ε_i for the element in H^* sending h to a_i.

root	$\varepsilon_i - \varepsilon_j$	$\varepsilon_i + \varepsilon_j$	$-(\varepsilon_i + \varepsilon_j)$	$2\varepsilon_i$	$-2\varepsilon_i$
eigenvector	m_{ij} $(i \neq j)$	p_{ij} $(i < j)$	q_{ji} $(i < j)$	p_{ii}	q_{ii}

(2) For each root α, we must check that $[h, x_\alpha] \neq 0$, where $h = [x_\alpha, x_{-\alpha}]$. When $\alpha = \varepsilon_i - \varepsilon_j$, this has been done in step (2b) for $\mathsf{so}(2\ell + 1, \mathbf{C})$. If $\alpha = \varepsilon_i + \varepsilon_j$, then $x_\alpha = p_{ij}$ and $x_{-\alpha} = q_{ji}$ and

$$h = (e_{ii} - e_{\ell+i,\ell+i}) + (e_{jj} - e_{\ell+j,\ell+j})$$

if $i \neq j$, and $h = e_{ii} - e_{\ell+i,\ell+i}$ if $i = j$. Hence $[h, x_\alpha] = 2x_\alpha$ in both cases.

(3) Let $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $1 \leq i \leq \ell - 1$ as before, and let $\beta_\ell = 2\varepsilon_\ell$. We claim that $\{\alpha_1, \ldots, \alpha_{\ell-1}, \beta_\ell\}$ is a base for Φ. By the same argument as used before, this follows once we observe that for $1 \leq i < j \leq \ell$

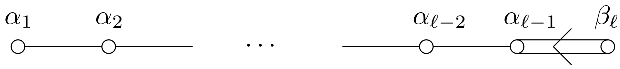
$$\begin{aligned} \varepsilon_i - \varepsilon_j &= \alpha_i + \alpha_{i+1} + \ldots + \alpha_{j-1}, \\ \varepsilon_i + \varepsilon_j &= \alpha_i + \alpha_{i+1} + \ldots + \alpha_{j-1} + 2(\alpha_j + \ldots + \alpha_{\ell-1}) + \beta_\ell, \\ 2\varepsilon_i &= 2(\alpha_i + \alpha_{i+1} + \ldots + \alpha_{\ell-1}) + \beta_\ell. \end{aligned}$$

(4) We calculate the Cartan integers. The numbers $\langle \alpha_i, \alpha_j \rangle$ are already known. Take $e_{\beta_\ell} = p_{\ell\ell}$, then we find that $h_{\beta_\ell} = e_{\ell,\ell} - e_{2\ell,2\ell}$ and so

$$\langle \alpha_i, \beta_\ell \rangle = \begin{cases} -1 & i = \ell - 1 \\ 0 & \text{otherwise,} \end{cases}$$

$$\langle \beta_\ell, \alpha_j \rangle = \begin{cases} -2 & i = \ell - 1 \\ 0 & \text{otherwise.} \end{cases}$$

The Dynkin diagram of Φ is



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which is connected, so L is simple. The root system of $\mathsf{sp}(2\ell, \mathbf{C})$ is said to have *type* C_ℓ. Since the root systems C_2 and B_2 have the same Dynkin diagram, we might expect that the Lie algebras $\mathsf{sp}(4, \mathbf{C})$ and $\mathsf{so}(5, \mathbf{C})$ would be isomorphic. This is the case, see Exercise 13.1.

12.6 Killing Forms of the Classical Lie Algebras

Now that we know that (with two exceptions) the classical Lie algebras are simple, we can use some of our earlier work to compute their Killing forms. We shall see that they can all be given by a simple closed formula.

Lemma 12.5

Let $L \subseteq \mathsf{gl}(n, \mathbf{C})$ be a simple classical Lie algebra. Let $\beta : L \times L \to \mathbf{C}$ be the symmetric bilinear form

$$\beta(x, y) := \operatorname{tr}(xy).$$

Then β is non-degenerate.

Proof

Let $J = \{x \in L : \beta(x, y) = 0 \text{ for all } y \in L\}$. It follows from the associative property of trace, as in Exercise 9.3, that J is an ideal of L. Since L is simple, and clearly β is not identically zero, we must have $J = 0$. Therefore β is non-degenerate. □

In Exercise 9.11, we showed that any two non-degenerate symmetric associative bilinear forms on a simple Lie algebra are scalar multiples of one another.

Hence, by Cartan's Second Criterion (Theorem 9.9), if κ is the Killing form on L, then $\kappa = \lambda\beta$ for some non-zero scalar $\lambda \in \mathbf{C}$. To determine the scalar λ, we use the root space decomposition to compute $\kappa(h, h')$ for $h, h' \in H$. For example, for $\mathsf{sl}(\ell+1, \mathbf{C})$ let $h \in H$, with diagonal entries $a_1, \ldots, a_{\ell+1}$, and similarly let $h' \in H$ with diagonal entries $a'_1, \ldots, a'_{\ell+1}$. Then, using the root space decomposition given in step (1) of §12.2, we get

$$\kappa(h, h') = \sum_{\alpha \in \Phi} \alpha(h)\alpha(h') = 2\sum_{i<j}(a_i - a_j)(a'_i - a'_j).$$

Putting $h = h'$, and $a_1 = 1$, $a_2 = -1$ and all other entries zero, we get $\kappa(h, h) = 8 + 4(\ell - 1) = 4(\ell + 1)$. Since $\operatorname{tr} h^2 = 2$, this implies that $\lambda = 2(\ell + 1)$.

For the remaining three families, see Exercise 12.3 below (or its solution in Appendix E). We get $\kappa(x, y) = \lambda \operatorname{tr}(xy)$, where

$$\lambda = \begin{cases} 2(\ell+1) & L = \mathsf{sl}(\ell+1, \mathbf{C}) \\ 2\ell - 1 & L = \mathsf{so}(2\ell+1, \mathbf{C}) \\ 2(\ell+1) & L = \mathsf{sp}(2\ell, \mathbf{C}) \\ 2(\ell-1) & L = \mathsf{so}(2\ell, \mathbf{C}). \end{cases}$$

12.7 Root Systems and Isomorphisms

Let L be a complex semisimple Lie algebra. We have seen how to define the root system associated to a Cartan subalgebra of L. Could two different Cartan subalgebras of L give different root systems? The following theorem, whose proof may be found in Appendix C, shows that the answer is no.

Theorem 12.6

Let L be a complex semisimple Lie algebra. If Φ_1 and Φ_2 are the root systems associated to two Cartan subalgebras of L, then Φ_1 is isomorphic to Φ_2. □

Suppose now that L_1 and L_2 are complex semisimple Lie algebras that have non-isomorphic root systems (with respect to some Cartan subalgebras). Then, by the theorem, L_1 and L_2 cannot be isomorphic. Thus we can use root systems to rule out isomorphisms between the classical Lie algebras. This does most of the work needed to prove the following proposition.

Proposition 12.7

The only isomorphisms between classical Lie algebras are:

(1) $\mathsf{so}(3, \mathbf{C}) \cong \mathsf{sp}(2, \mathbf{C}) \cong \mathsf{sl}(2, \mathbf{C})$; root systems of type A_1,

(2) $\mathsf{so}(4, \mathbf{C}) \cong \mathsf{sl}(2, \mathbf{C}) \oplus \mathsf{sl}(2, \mathbf{C})$; root systems of type $A_1 \times A_1$,

(3) $\mathsf{so}(5, \mathbf{C}) \cong \mathsf{sp}(4, \mathbf{C})$; root systems of types B_2 and C_2,

(4) $\mathsf{so}(6, \mathbf{C}) \cong \mathsf{sl}(4, \mathbf{C})$; root systems of types D_3 and A_3.

Note that we have not yet proved the existence of all these isomorphisms. However, we have already seen the first two (see Exercises 1.14 and 10.8). The third isomorphism appears in Exercise 12.2 below and the last is discussed in Chapter 15. We are therefore led to conjecture that the converse of Theorem 12.6 also holds; that is, if two complex semisimple Lie algebras have isomorphic root systems, then they are isomorphic as Lie algebras.

We shall see in Chapter 14 that this is a corollary of Serre's Theorem. Thus isomorphisms of root systems precisely reflect isomorphisms of complex semisimple Lie algebras. To classify the complex semisimple Lie algebras, we should therefore first classify root systems. This is the subject of the next chapter.

EXERCISES

12.1. Show that the dimensions of the classical Lie algebras are as follows

$$\begin{aligned} \dim \mathsf{sl}(\ell+1, \mathbf{C}) &= \ell^2 + 2\ell, \\ \dim \mathsf{so}(2\ell+1, \mathbf{C}) &= 2\ell^2 + \ell, \\ \dim \mathsf{sp}(2\ell, \mathbf{C}) &= 2\ell^2 + \ell, \\ \dim \mathsf{so}(2\ell, \mathbf{C}) &= 2\ell^2 - \ell. \end{aligned}$$

12.2.⋆ Show that the Lie algebras $\mathsf{sp}(4, \mathbf{C})$ and $\mathsf{so}(5, \mathbf{C})$ are isomorphic. (For instance, use the root space decomposition to show that they have bases affording the same structure constants.)

12.3.† This exercise gives a way to establish the semisimplicity of the classical Lie algebras using the Killing form.

(i) Let L be a classical Lie algebra, and let H be the subalgebra of diagonal matrices, with eigenspace decomposition

$$L = H \oplus \bigoplus_{\alpha \in \Phi} L_\alpha,$$

so H is self-centralising. Assume also that the following conditions hold

(a) For each $\alpha \in \Phi$, the space L_α is 1-dimensional. If $\alpha \in \Phi$, then $-\alpha \in \Phi$.

(b) For each $\alpha \in \Phi$, the space $[L_\alpha, L_{-\alpha}]$ is non-zero.

(c) The Killing form restricted to H is non-degenerate, and for $h \in H$, if $\kappa(h,h) = 0$ then $h = 0$.

Show that the Killing form of L is then non-degenerate.

In the earlier sections of this chapter, we have found the roots with respect to H explicitly. We can make use of this and find the Killing form restricted to H explicitly.

(ii) Use the root space decomposition of $\mathsf{sl}(\ell+1, \mathbf{C})$ to show that if κ is the Killing form of $\mathsf{sl}(\ell+1, \mathbf{C})$, then

$$\kappa(h, h') = 2n \operatorname{tr}(hh') \quad \text{for all } h, h' \in H.$$

Hence, show that condition (c) above holds for the restriction of κ to H and deduce that $\mathsf{sl}(\ell+1, \mathbf{C})$ is semisimple.

(iii) Use similar methods to prove that the orthogonal and symplectic Lie algebras are semisimple.

12.4.†⋆ Let L be a Lie algebra with a faithful irreducible representation. Show that either L is semisimple or $Z(L)$ is 1-dimensional and $L = Z(L) \oplus L'$, where the derived algebra L' is semisimple. (This gives yet another way to prove the semisimplicity of the classical Lie algebras.)