11
Root Systems

The essential properties of the roots of complex semisimple Lie algebras may be captured in the idea of an abstract "root system". In this chapter, we shall develop the basic theory of root systems. Our eventual aim, achieved in Chapters 13 and 14, will be to use root systems to classify the complex semisimple Lie algebras.

Root systems have since been discovered to be important in many other areas of mathematics, so while this is probably your first encounter with root systems, it may well not be your last! In MathSciNet, the main database for research papers in mathematics, there are, at the time of writing, 297 papers whose title contains the words "root system", and many thousands more in which root systems are mentioned in the text.

11.1 Definition of Root Systems

Let E be a finite-dimensional real vector space endowed with an inner product written $(-,-)$. Given a non-zero vector $v \in E$, let s_v be the reflection in the hyperplane normal to v. Thus s_v sends v to $-v$ and fixes all elements y such that $(y,v) = 0$. As an easy exercise, the reader may check that

$$s_v(x) = x - \frac{2(x,v)}{(v,v)}v \quad \text{for all } x \in E$$

K. Erdmann et al., *Introduction to Lie Algebras*

and that s_v preserves the inner product, that is,

$$(s_v(x), s_v(y)) = (x, y) \quad \text{for all } x, y \in E.$$

As it is a very useful convention, we shall write

$$\langle x, v\rangle := \frac{2(x, v)}{(v, v)},$$

noting that the symbol $\langle x, v\rangle$ is only linear with respect to its first variable, x. With this notation, we can now define root systems.

Definition 11.1

A subset R of a real inner-product space E is a *root system* if it satisfies the following axioms.

(R1) R is finite, it spans E, and it does not contain 0.

(R2) If $\alpha \in R$, then the only scalar multiples of α in R are $\pm\alpha$.

(R3) If $\alpha \in R$, then the reflection s_α permutes the elements of R.

(R4) If $\alpha, \beta \in R$, then $\langle \beta, \alpha\rangle \in \mathbf{Z}$.

The elements of R are called *roots*.

Example 11.2

The root space decomposition gives our main example. Let L be a complex semisimple Lie algebra, and suppose that Φ is the set of roots of L with respect to some fixed Cartan subalgebra H. Let E denote the real span of Φ. By Proposition 10.15, the symmetric bilinear form on E induced by the Killing form $(-,-)$ is an inner product.

We can use the results of §10.5 and §10.6 to show that Φ is a root system in E. By definition, $0 \notin \Phi$ and, as we observed early on, Φ is finite. We showed that (R2) holds in Proposition 10.9. To show that (R3) holds, we note that if $\alpha, \beta \in \Phi$ then

$$s_\alpha(\beta) = \beta - \frac{2(\beta, \alpha)}{(\alpha, \alpha)}\alpha = \beta - \beta(h_\alpha)\alpha,$$

which lies in Φ by Proposition 10.10. To get the second equality above, we used the identity of Exercise 10.4, which may be proved as follows:

$$\beta(h_\alpha) = \kappa(t_\beta, h_\alpha) = \kappa\left(t_\beta, \frac{2t_\alpha}{(t_\alpha, t_\alpha)}\right) = \frac{2(\beta, \alpha)}{(\alpha, \alpha)} = \langle \beta, \alpha\rangle\,.$$

As the eigenvalues of h_α are integers, this identity also establishes (R4).

Exercise 11.1

We work in $\mathbf{R}^{\ell+1}$, with the Euclidean inner product. Let ε_i be the vector in $\mathbf{R}^{\ell+1}$ with i-th entry 1 and all other entries zero. Define

$$R := \{\pm(\varepsilon_i - \varepsilon_j) : 1 \leq i < j \leq \ell + 1\}$$

and let $E = \operatorname{Span} R = \{\sum \alpha_i \varepsilon_i : \sum \alpha_i = 0\}$. Show that R is a root system in E.

Remark 11.3

We shall see that our axioms isolate all the essential properties of roots of Lie algebras. For this reason, there is no need in this chapter to keep the full body of theory we have developed in mind — doing so would burden us with extraneous notions while needlessly reducing the applicability of our arguments. In any case, we shall see later that every root system *is* the set of roots of a complex semisimple Lie algebra, so our problem is no more general than is necessary: "*It is the mark of the educated mind to use for each subject the degree of exactness which it admits*" (Aristotle).

11.2 First Steps in the Classification

The following lemma gives the first indication that the axioms for root systems are quite restrictive.

Lemma 11.4 (Finiteness Lemma)

Suppose that R is a root system in the real inner-product space E. Let $\alpha, \beta \in R$ with $\beta \neq \pm\alpha$. Then

$$\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle \in \{0, 1, 2, 3\}.$$

Proof

Thanks to (R4), the product in question is an integer: We must establish the bounds. For any non-zero $v, w \in E$, the angle θ between v and w is such that $(v, w)^2 = (v, v)(w, w) \cos^2 \theta$. This gives

$$\langle \alpha, \beta \rangle \langle \beta, \alpha \rangle = 4 \cos^2 \theta \leq 4.$$

Suppose we have $\cos^2 \theta = 1$. Then θ is an integer multiple of π and so α and β are linearly dependent, contrary to our assumption. □

We now use this lemma to show that there are only a few possibilities for the integers $\langle \alpha, \beta \rangle$. Take two roots α, β in a root system R with $\alpha \neq \pm\beta$. We may choose the labelling so that $(\beta, \beta) \geq (\alpha, \alpha)$ and hence

$$|\langle \beta, \alpha \rangle| = \frac{2\,|(\beta, \alpha)|}{(\alpha, \alpha)} \geq \frac{2\,|(\alpha, \beta)|}{(\beta, \beta)} = |\langle \alpha, \beta \rangle| \,.$$

By the Finiteness Lemma, the possibilities are:

$\langle \alpha, \beta \rangle$	$\langle \beta, \alpha \rangle$	θ	$\dfrac{(\beta,\beta)}{(\alpha,\alpha)}$
0	0	$\pi/2$	undetermined
1	1	$\pi/3$	1
-1	-1	$2\pi/3$	1
1	2	$\pi/4$	2
-1	-2	$3\pi/4$	2
1	3	$\pi/6$	3
-1	-3	$5\pi/6$	3

Given roots α and β, we would like to know when their sum and difference lie in R. Our table gives some information about this question.

Proposition 11.5

Let $\alpha, \beta \in R$.

(a) If the angle between α and β is strictly obtuse, then $\alpha + \beta \in R$.

(b) If the angle between α and β is strictly acute and $(\beta, \beta) \geq (\alpha, \alpha)$, then $\alpha - \beta \in R$.

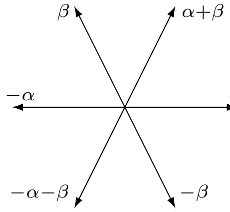
Proof

In either case, we may assume that $(\beta, \beta) \geq (\alpha, \alpha)$. By (R3), we know that $s_\beta(\alpha) = \alpha - \langle \alpha, \beta \rangle\, \beta$ lies in R. The table shows that if θ is strictly acute, then $\langle \alpha, \beta \rangle = 1$, and if θ is strictly obtuse, then $\langle \alpha, \beta \rangle = -1$. □

Example 11.6

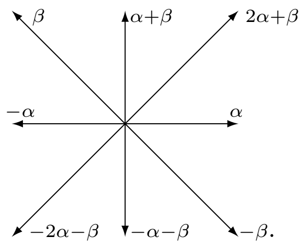
Let $E = \mathbf{R}^2$ with the Euclidean inner product. We shall find all root systems R contained in E. Take a root α of the shortest possible length. Since R spans E, it must contain some root $\beta \neq \pm\alpha$. By considering $-\beta$ if necessary, we may assume that β makes an obtuse angle with α. Moreover, we may assume that this angle, say θ, is as large as possible.

(a) Suppose that $\theta = 2\pi/3$. Using Proposition 11.5, we find that R contains the six roots shown below.



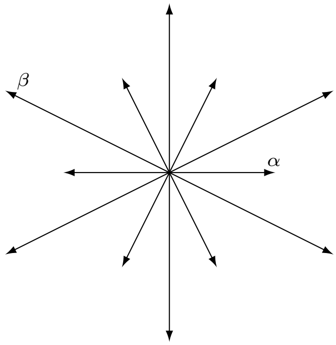
One can check that this set is closed under the action of the reflections $s_\alpha, s_\beta, s_{\alpha+\beta}$. As $s_{-\alpha} = s_\alpha$, and so on, this is sufficient to verify (R3). We have therefore found a root system in E. This root system is said to have *type* A_2. (The 2 refers to the dimension of the underlying space.)

(b) Suppose that $\theta = 3\pi/4$. Proposition 11.5 shows that $\alpha + \beta$ is a root, and applying s_α to β shows that $2\alpha + \beta$ is a root, so R must contain



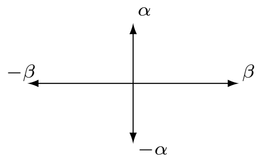
This root system is said to have *type* B_2. A further root would make an angle of at most $\pi/8$ with one of the existing roots, so this must be all of R.

(c) Suppose that $\theta = 5\pi/6$. We leave it to the reader to show that R must be



and to determine the correct labels for the remaining roots. This root system is said to have *type* G_2.

(d) Suppose that β is perpendicular to α. This gives us the root system of *type* $A_1 \times A_1$.



Here, as $(\alpha, \beta) = 0$, the reflection s_α fixes the roots $\pm\beta$ lying in the space perpendicular to α, so there is no interaction between the roots $\pm\alpha$ and $\pm\beta$. In particular, knowing the length of α tells us nothing about the length of β. These considerations suggest the following definition.

Definition 11.7

The root system R is *irreducible* if R cannot be expressed as a disjoint union of two non-empty subsets $R_1 \cup R_2$ such that $(\alpha, \beta) = 0$ for $\alpha \in R_1$ and $\beta \in R_2$.

Note that if such a decomposition exists, then R_1 and R_2 are root systems in their respective spans. The next lemma tells us that it will be enough to classify the irreducible root systems.

Lemma 11.8

Let R be a root system in the real vector space E. We may write R as a disjoint union

$$R = R_1 \cup R_2 \cup \ldots \cup R_k,$$

where each R_i is an irreducible root system in the space E_i spanned by R_i, and E is a direct sum of the orthogonal subspaces $E_1, \ldots, E_k$.

Proof

Define an equivalence relation $\sim$ on R by letting $\alpha \sim \beta$ if there exist $\gamma_1, \gamma_2, \ldots, \gamma_s$ in R with $\alpha = \gamma_1$ and $\beta = \gamma_s$ such that $(\gamma_i, \gamma_{i+1}) \neq 0$ for $1 \leq i < s$. Let the R_i be the equivalence classes for this relation. It is clear that they satisfy axioms (R1), (R2), and (R4); you are asked to check (R3) in the following exercise. That each R_i is irreducible follows immediately from the construction.

As every root appears in some E_i, the sum of the E_i spans E. Suppose that $v_1 + \ldots + v_k = 0$, where $v_i \in E_i$. Taking inner products with v_j, we get

$$0 = (v_1, v_j) + \ldots + (v_j, v_j) + \ldots + (v_k, v_j) = (v_j, v_j)$$

so each $v_j = 0$. Hence $E = E_1 \oplus \ldots \oplus E_k$. $\square$

Exercise 11.2

Show that if $(\alpha, \beta) \neq 0$, then $(\alpha, s_\alpha(\beta)) \neq 0$. Deduce that the equivalence classes defined in the proof of the lemma satisfy (R3).

11.3 Bases for Root Systems

Let R be a root system in the real inner-product space E. Because R spans E, any maximal linearly independent subset of R is a vector space basis for R. Proposition 11.5 suggests that it might be convenient if we could find such a subset where every pair of elements made an obtuse angle. In fact, we can ask for something stronger, as in the following.

Definition 11.9

A subset B of R is a *base* for the root system R if

(B1) B is a vector space basis for E, and

(B2) every $\beta \in R$ can be written as $\beta = \sum_{\alpha \in B} k_\alpha \alpha$ with $k_\alpha \in \mathbf{Z}$, where all the non-zero coefficients k_α have the same sign.

Exercise 11.3

Show that if B is a base for a root system, then the angle between any two distinct elements of B is obtuse (that is, at least $\pi/2$).

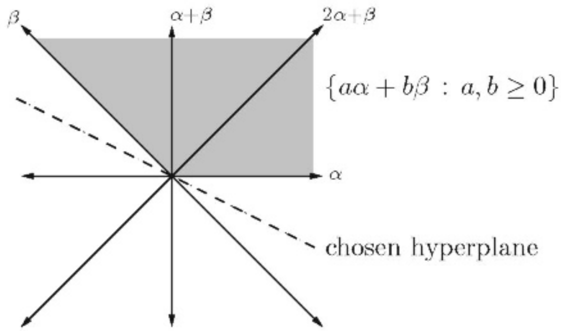
We say that a root $\beta \in R$ is *positive with respect to* B if the coefficients given in (B2) are positive, and otherwise it is *negative with respect to* B.

Exercise 11.4

Let $R = \{\pm(\varepsilon_i - \varepsilon_j) : 1 \leq i < j \leq \ell + 1\}$ be the root system in Exercise 11.1. Let $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$ for $1 \leq i \leq \ell$. Show that $B = \{\alpha_1, \ldots, \alpha_\ell\}$ is a base for R and find the positive roots.

A natural way to label the elements of R as positive or negative is to fix a hyperplane of codimension 1 in E which does not contain any element of R and then to label the roots of one side of the hyperplane as positive and those on the other side as negative. Suppose that R has a base B compatible with this labelling. Then the elements of B must lie on the positive side of the hyperplane. For example, the diagram below shows a possible base for the root system in Example 11.6(b).

$\{\alpha, \beta\}$ is a base of the root system of type B_2



Note that the roots in the base are those nearest to the hyperplane. This observation motivates the proof of our next theorem.

Theorem 11.10

Every root system has a base.

Proof

Let R be a root system in the real inner-product space E. We may assume that E has dimension at least 2, as the case $\dim E = 1$ is obvious. We may choose a vector $z \in E$ which does not lie in the perpendicular space of any of the roots. Such a vector must exist, as E has dimension at least 2, so it is not the union of finitely many hyperplanes (see Exercise 11.8 or Exercise 11.12).

Let R^+ be the set of $\alpha \in R$ which lie on the positive side of z, that is, those α for which $(z, \alpha) > 0$. Let

$$B := \{\alpha \in R^+ : \alpha \text{ is not the sum of two elements in } R^+\}.$$

We claim that B is a base for R.

We first show that (B2) holds. If $\beta \in R$, then either $\beta \in R^+$ or $-\beta \in R^+$, so it is sufficient to prove that every $\beta \in R^+$ can be expressed as $\beta = \sum_{\alpha \in B} k_\alpha \alpha$ for some $k_\alpha \in \mathbf{Z}$ with each $k_\alpha \geq 0$. If this fails, then we may pick, from the elements of R^+ that are not of this form, an element $\beta \in R^+$ such that the inner product (z, β) is as small as possible. As $\beta \notin B$, there exist $\beta_1, \beta_2 \in R^+$ such that $\beta = \beta_1 + \beta_2$. By linearity,

$$(z, \beta) = (z, \beta_1) + (z, \beta_2)$$

is the sum of two positive numbers, and therefore $0 < (z, \beta_i) < (z, \beta)$ for

$i = 1, 2$. Now at least one of β_1, β_2 cannot be expressed as a positive integral linear combination of the elements of B; this contradicts the choice of β.

It remains to show that B is linearly independent. First note that if α and β are distinct elements of B, then by Exercise 11.3 the angle between them must be obtuse. Suppose that $\sum_{\alpha \in B} r_\alpha \alpha = 0$, where $r_\alpha \in \mathbf{R}$. Collecting all the terms with positive coefficients to one side gives an element

$$x := \sum_{\alpha : r_\alpha > 0} r_\alpha \alpha = \sum_{\beta : r_\beta < 0} (-r_\beta)\beta.$$

Hence

$$(x, x) = \sum_{\substack{\alpha : r_\alpha > 0 \\ \beta : r_\beta < 0}} r_\alpha(-r_\beta)(\alpha, \beta) \leq 0$$

and so $x = 0$. Therefore

$$0 = (x, z) = \sum_{\alpha : r_\alpha > 0} r_\alpha(\alpha, z),$$

where each $(\alpha, z) > 0$ as $\alpha \in R^+$, so we must have $r_\alpha = 0$ for all α, and similarly $r_\beta = 0$ for all β. □

Let R^+ denote the set of all positive roots in a root system R with respect to a base B, and let R^- be the set of all negative roots. Then $R = R^+ \cup R^-$, a disjoint union. The set B is contained in R^+; the elements of B are called *simple roots*. The reflections s_α for $\alpha \in B$ are known as *simple reflections*.

Remark 11.11

A root system R will usually have many possible bases. For example, if B is a base then so is $\{-\alpha : \alpha \in B\}$. In particular, the terms "positive" and "negative" roots are always taken with reference to a fixed base B.

Exercise 11.5

Let R be a root system with a base B. Take any $\gamma \in R$. Show that the set $\{s_\gamma(\alpha) : \alpha \in B\}$ is also a base of R.

11.3.1 The Weyl Group of a Root System

For each root $\alpha \in R$, we have defined a reflection s_α which acts as an invertible linear map on E. We may therefore consider the group of invertible linear transformations of E generated by the reflections s_α for $\alpha \in R$. This is known as the *Weyl group* of R and is denoted by W or $W(R)$.

Lemma 11.12

The Weyl group W associated to R is finite.

Proof

By axiom (R3), the elements of W permute R, so there is a group homomorphism from W into the group of all permutations of R, which is a finite group because R is finite. We claim that this homomorphism is injective, and so W is finite.

Suppose that $g \in W$ is in the kernel of this homomorphism. Then, by definition, g fixes all the roots in R. But E is spanned by the roots, so g fixes all elements in a basis of E, and so g must be the identity map. $\square$

11.3.2 Recovering the Roots

Suppose that we are given a base B for a root system R. We shall show that this alone gives us enough information to recover R. To do this, we use the Weyl group and prove that every root β is of the form $\beta = g(\alpha)$ for some $\alpha \in B$ and some g in the subgroup $W_0 := \langle s_\gamma : \gamma \in B \rangle$ of W. (We shall also see that $W = W_0$.) Thus, if we repeatedly apply reflections in the simple roots, we will eventually recover the full root system.

Some evidence for this statement is given by Example 11.6: in each case we started with a pair of roots $\{\alpha, \beta\}$, and knowing only the positions of α and β, we used repeated reflections to construct the unique root system containing these roots as a base.

Lemma 11.13

If $\alpha \in B$, then the reflection s_α permutes the set of positive roots other than α.

Proof

Suppose that $\beta \in R^+$ and $\beta \neq \alpha$. We know that $\beta = \sum_{\gamma \in B} k_\gamma \gamma$ for some $k_\gamma \geq 0$. Since $\beta \neq \alpha$ and $\beta \in R$, there is some $\gamma \in B$ with $k_\gamma \neq 0$ and $\gamma \neq \alpha$. We know that $s_\alpha(\beta) \in R$; and from $s_\alpha(\beta) = \beta - \langle \beta, \alpha \rangle \alpha$ we see that the coefficient of γ in $s_\alpha(\beta)$ is k_γ, which is positive. As all the non-zero coefficients in the expression of $s_\alpha(\beta)$ as a linear combination of base elements must have the same sign, this tells us that $s_\alpha(\beta)$ lies in R^+. $\square$

Proposition 11.14

Suppose that $\beta \in R$. There exists $g \in W_0$ and $\alpha \in B$ such that $\beta = g(\alpha)$.

Proof

Suppose first of all that $\beta \in R^+$ and that $\beta = \sum_{\gamma \in B} k_\gamma \gamma$ with $k_\gamma \in \mathbf{Z}$, $k_\gamma \geq 0$. We shall proceed by induction on the height of β defined by

$$\mathrm{ht}(\beta) = \sum_{\gamma \in B} k_\gamma .$$

If $\mathrm{ht}(\beta) = 1$, then $\beta \in B$, so we may take $\alpha = \beta$ and let g be the identity map. For the inductive step, suppose that $\mathrm{ht}(\beta) = n \geq 2$. By axiom (R2), at least two of the k_γ are strictly positive.

We claim that there is some $\gamma \in B$ such that $(\beta, \gamma) > 0$. If not, then $(\beta, \gamma) \leq 0$ for all $\gamma \in B$ and so

$$(\beta, \beta) = \sum_{\gamma} k_\gamma (\beta, \gamma) \leq 0,$$

which is a contradiction because $\beta \neq 0$. We may therefore choose some $\gamma \in B$ with $(\beta, \gamma) > 0$. Then $\langle \beta, \gamma \rangle > 0$ and so

$$\mathrm{ht}(s_\gamma(\beta)) = \mathrm{ht}(\beta) - \langle \beta, \gamma \rangle < \mathrm{ht}(\beta).$$

(We have $s_\gamma(\beta) \in R^+$ by the previous lemma.) The inductive hypothesis now implies that there exists $\alpha \in B$ and $h \in W_0$ such that $s_\gamma(\beta) = h(\alpha)$. Hence $\beta = s_\gamma(h(\alpha))$ so we may take $g = s_\gamma h$, which lies in W_0.

Now suppose that $\beta \in R^-$, so $-\beta \in R^+$. By the first part, $-\beta = g(\alpha)$ for some $g \in W_0$ and $\alpha \in B$. By linearity of g, we get $\beta = g(-\alpha) = g(s_\alpha(\alpha))$, where $gs_\alpha \in W_0$. □

We end this section by proving that a base for a root system determines its full Weyl group. We need the following straightforward result.

Exercise 11.6

Suppose that α is a root and that $g \in W$. Show that $g s_\alpha g^{-1} = s_{g\alpha}$.

Lemma 11.15

We have $W_0 = W$; that is, W is generated by the s_α for $\alpha \in B$.

Proof

By definition, W is generated by the reflections s_β for $\beta \in R$, so it is sufficient to prove that $s_\beta \in W_0$ for any $\beta \in R$. By Proposition 11.14, we know that given β there is some $g \in W_0$ and $\alpha \in B$ such that $\beta = g(\alpha)$. Now the reflection s_β is equal to $g s_\alpha g^{-1} \in W_0$ by the previous exercise. □

11.4 Cartan Matrices and Dynkin Diagrams

Although in general a root system can have many different bases, the following theorem shows that from a geometric point of view they are all essentially the same. As the proof of this theorem is slightly technical, we postpone it to Appendix D.

Theorem 11.16

Let R be a root system and suppose that B and B' are two bases of R, as defined in Definition 11.9. Then there exists an element g in the Weyl group $W(R)$ such that $B' = \{g(\alpha) : \alpha \in B\}$. □

Let B be a base in a root system R. Fix an order on the elements of B, say $(\alpha_1, \ldots, \alpha_\ell)$. The *Cartan matrix* of R is defined to be the $\ell \times \ell$ matrix with ij-th entry $\langle \alpha_i, \alpha_j \rangle$. Since for any root β we have

$$\langle s_\beta(\alpha_i), s_\beta(\alpha_j) \rangle = \langle \alpha_i, \alpha_j \rangle \,,$$

it follows from Theorem 11.16 that the Cartan matrix depends only on the ordering adopted with our chosen base B and not on the base itself. Note that by (R4) the entries of the Cartan matrix are integers.

Example 11.17

(1) Let R be as in Exercise 11.4(ii). Calculation shows that the Cartan matrix with respect to the ordered base $(\alpha_1, \ldots, \alpha_\ell)$ is

$$\begin{pmatrix} 2 & -1 & 0 & \ldots & 0 & 0 \\ -1 & 2 & -1 & \ldots & 0 & 0 \\ 0 & -1 & 2 & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & 2 & -1 \\ 0 & 0 & 0 & \ldots & -1 & 2 \end{pmatrix}.$$

(2) Let R be the root system which we have drawn in Example 11.6(b). This has ordered base (α, β), and the corresponding Cartan matrix is

$$C = \begin{pmatrix} 2 & -1 \\ -2 & 2 \end{pmatrix}.$$

Another way to record the information given in the Cartan matrix is in a graph $\Delta = \Delta(R)$, defined as follows. The vertices of Δ are labelled by the simple roots of B. Between the vertices labelled by simple roots α and β, we draw $d_{\alpha\beta}$ lines, where

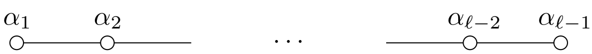
$$d_{\alpha\beta} := \langle \alpha, \beta \rangle \langle \beta, \alpha \rangle \in \{0, 1, 2, 3\}.$$

If $d_{\alpha\beta} > 1$, which happens whenever α and β have different lengths and are not orthogonal, we draw an arrow pointing from the longer root to the shorter root. This graph is called the *Dynkin diagram* of R. By Theorem 11.16, the Dynkin diagram of R is independent of the choice of base.

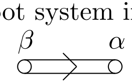
The graph with the same vertices and edges, but without the arrows, is known as the *Coxeter graph* of R.

Example 11.18

Using the base given in Exercise 11.4(ii), the Dynkin diagram of the root system introduced in Exercise 11.1 is



The Dynkin diagram for the root system in Example 11.6(b) is



Exercise 11.7

Show that a root system is irreducible if and only if its Dynkin diagram is connected; that is, given any two vertices, there is a path joining them.

Given a Dynkin diagram, one can read off the numbers $\langle \alpha_i, \alpha_j \rangle$ and so recover the Cartan matrix. In fact, more is true: The next section shows that a root system is essentially determined by its Dynkin diagram.

11.4.1 Isomorphisms of Root Systems

Definition 11.19

Let R and R' be root systems in the real inner-product spaces E and E', respectively. We say that R and R' are isomorphic if there is a vector space isomorphism $\varphi : E \to E'$ such that

(a) $\varphi(R) = R'$, and

(b) for any two roots $\alpha, \beta \in R$, $\langle \alpha, \beta \rangle = \langle \varphi(\alpha), \varphi(\beta) \rangle$.

Recall that if θ is the angle between roots α and β, then $4\cos^2\theta = \langle \alpha, \beta \rangle \langle \beta, \alpha \rangle$, so condition (b) says that φ should preserve angles between root vectors. For irreducible root systems, a stronger geometric characterisation is possible — see Exercise 11.15 at the end of this chapter.

Example 11.20

Let R be a root system in the inner-product space E. We used that the reflection maps s_α for $\alpha \in R$ are isomorphisms (from R to itself) when we defined the Cartan matrix of a root system.

An example of an isomorphism that is not distance preserving is given by scaling: For any non-zero $c \in \mathbf{C}$, the set $cR = \{c\alpha : \alpha \in R\}$ is a root system in E, and the map $v \mapsto cv$ induces an isomorphism between R and cR.

It follows immediately from the definition of isomorphism that isomorphic root systems have the same Dynkin diagram. We now prove that the converse holds.

Proposition 11.21

Let R and R' be root systems in the real vector spaces E and E', respectively. If the Dynkin diagrams of R and R' are the same, then the root systems are isomorphic.

Proof

We may choose bases $B = \{\alpha_1, \ldots, \alpha_\ell\}$ in R and $B' = \{\alpha'_1, \ldots, \alpha'_\ell\}$ in R' so that for all i, j one has

$$\langle \alpha_i, \alpha_j \rangle = \langle \alpha'_i, \alpha'_j \rangle .$$

Let $\varphi : E \to E'$ be the linear map which maps α_i to α'_i. By definition, this is a vector space isomorphism satisfying condition 11.19(b). We must show that $\varphi(R) = R'$.

Let $v \in E$ and $\alpha_i \in B$. We have

$$\begin{aligned}\varphi\left(s_{\alpha_i}(v)\right) &= \varphi\left(v - \langle v, \alpha_i \rangle \alpha_i\right) \\ &= \varphi(v) - \langle v, \alpha_i \rangle \alpha'_i.\end{aligned}$$

We claim that $\langle v, \alpha_i \rangle = \langle \varphi(v), \alpha_i' \rangle$. To show this, express v as a linear combination of $\alpha_1, \ldots, \alpha_n$ and then use that $\langle -, - \rangle$ is linear in its first component. Therefore the last equation may be written as

$$\varphi(s_{\alpha_i}(v)) = s_{\alpha_i'}(\varphi(v)).$$

By Lemma 11.15, the simple reflections s_{α_i} generate the Weyl group of R. Hence the image under φ of the orbit of $v \in E$ under the Weyl group of R is contained in the orbit of $\varphi(v)$ under the Weyl group of R'. Now Proposition 11.14 tells us that $\{g(\alpha) : g \in W_0, \alpha \in B\} = R$ so, since $\varphi(B) = B'$, we must have $\varphi(R) \subseteq R'$.

The same argument may be applied to the inverse of φ to show that $\varphi^{-1}(R') \subseteq R$. Hence $\varphi(R) = R'$, as required. □

EXERCISES

11.8. Let E be a real inner-product space of dimension $n \geq 2$. Show that E is not the union of finitely many hyperplanes of dimension $n-1$. For a more general result, see Exercise 11.12 below.

11.9.† Let E be a finite-dimensional real inner-product space. Let $b_1, \ldots, b_n$ be a vector space basis of E. Show that there is some $z \in E$ such that $(z, b_i) > 0$ for all i.

11.10. Let R be as in Exercise 11.1 with $\ell = 2$. We may regard the reflections s_{α_j} for $j = 1, 2$ as linear maps on $\mathbf{R}^3$. Determine $s_{\alpha_j}(\varepsilon_i)$ for $1 \leq i \leq 3$ and $1 \leq j \leq 2$. Hence show that $W(R)$ is isomorphic to the symmetric group $\mathcal{S}_3$.

11.11. Suppose that R is a root system in E, that R' is a root system in E', and that $\varphi : E \to E'$ is a linear map which induces an isomorphism of root systems. Show that for $\alpha \in R$ one then has

$$s_\alpha = \varphi^{-1} \circ s_{\varphi(\alpha)} \circ \varphi.$$

Prove that the Weyl group associated to R is isomorphic to the Weyl group associated to R'. (If you know what it means, prove that the pairs $(R, W(R))$ and $(R', W(R'))$ are isomorphic as G-spaces.)

11.12.† Suppose that E is a finite-dimensional vector space over an infinite field. Suppose $U_1, U_2, \ldots, U_n$ are proper subspaces of E of the same dimension. Show that the set-theoretic union $\bigcup_{i=1}^n U_i$ is not a subspace. In particular, it is a proper subset of E.

11.13. Suppose that R is a root system in the real inner-product space E. Show that

$$\check{R} := \left\{ \frac{2\alpha}{(\alpha, \alpha)} : \alpha \in R \right\}$$

is also a root system in E. Show that the Cartan matrix of $\check{R}$ is the transpose of the Cartan matrix of $\check{R}$ (when each is taken with respect to suitable ordering of the roots) and that the Weyl groups of R and $\check{R}$ are isomorphic. One says $\check{R}$ is the *dual root system* to R.

11.14.† Show that if R is a root system and $\alpha, \beta \in R$ are roots with $\alpha \neq \pm\beta$ then the subgroup of the Weyl group $W(R)$ generated by s_α, s_β is a dihedral group with rotational subgroup generated by $s_\alpha s_\beta$. Hence, or otherwise, find the Weyl groups of the root systems in Example 11.6.

Hint: A group generated by two elements x and y, each of order 2, is dihedral of order $2m$, where m is the order of xy.

11.15.⋆ Let R and R' be irreducible root systems in the real inner-product spaces E and E'. Prove that R and R' are isomorphic if and only if there exist a scalar $\lambda \in \mathbf{R}$ and a vector space isomorphism $\varphi : E \to E'$ such that $\varphi(R) = R'$ and

$$(\varphi(u), \varphi(v)) = \lambda(u, v) \text{ for all } u, v \in E.$$