10
The Root Space Decomposition

We are now ready to embark on the classification of the complex semisimple Lie algebras. So far we have proved the simplicity of only one family of Lie algebras, namely the algebras $\mathsf{sl}(n, \mathbf{C})$ for $n \geq 2$ (see Exercise 9.7). There is, however, a strong sense in which their behaviour is typical of all complex semisimple Lie algebras. We therefore begin by looking at the structures of $\mathsf{sl}(2, \mathbf{C})$ and $\mathsf{sl}(3, \mathbf{C})$ in the belief that this will motivate the strategy adopted in this chapter.

In §3.2.4, we proved that $\mathsf{sl}(2, \mathbf{C})$ was the unique 3-dimensional semisimple complex Lie algebra by proceeding as follows:

(1) We first showed that if L was a 3-dimensional Lie algebra such that $L = L'$, then there was some $h \in L$ such that $\operatorname{ad} h$ was diagonalisable.

(2) We then took a basis of L consisting of eigenvectors for $\operatorname{ad} h$ and by finding the structure constants with respect to this basis showed that L was isomorphic to $\mathsf{sl}(2, \mathbf{C})$.

In the case of $\mathsf{sl}(3, \mathbf{C})$, a suitable replacement for the element $h \in \mathsf{sl}(2, \mathbf{C})$ is the 2-dimensional subalgebra H of diagonal matrices in $\mathsf{sl}(3, \mathbf{C})$. One can see directly that $\mathsf{sl}(3, \mathbf{C})$ decomposes into a direct sum of common eigenspaces for the elements of $\operatorname{ad} H$. Suppose $h \in H$ has diagonal entries a_1, a_2, a_3. Then

$$[h, e_{ij}] = (a_i - a_j)e_{ij}$$

so the elements e_{ij} for $i \neq j$ are common eigenvectors for the elements of $\operatorname{ad} H$. Moreover, as H is abelian, H is contained in the kernel of every element of $\operatorname{ad} H$.

It will be helpful to express this decomposition using the language of weights and weight spaces introduced in Chapter 5. Define $\varepsilon_i : H \to \mathbf{C}$ by $\varepsilon_i(h) = a_i$. We have

$$(\operatorname{ad} h)e_{ij} = (\varepsilon_i - \varepsilon_j)(h)e_{ij}.$$

Here $\varepsilon_i - \varepsilon_j$ is a weight and e_{ij} is in its associated weight space. In fact one can check that if L_{ij} is the weight space for $\varepsilon_i - \varepsilon_j$, that is

$$L_{ij} = \{x \in \mathsf{sl}(3, \mathbf{C}) : (\operatorname{ad} h)x = (\varepsilon_i - \varepsilon_j)(h)x \text{ for all } h \in H\},$$

then we have $L_{ij} = \operatorname{Span}\{e_{ij}\}$ for $i \neq j$. Hence there is a direct sum decomposition

$$\mathsf{sl}(3, \mathbf{C}) = H \oplus \bigoplus_{i \neq j} L_{ij}.$$

The existence of this decomposition can be seen in a more abstract way. Let L be a complex semisimple Lie algebra and let H be an abelian subalgebra of L consisting of semisimple elements. By definition, $\operatorname{ad} h$ is diagonalisable for every $h \in H$. Moreover, as commuting linear maps may be simultaneously diagonalised, H acts diagonalisably on L in the adjoint representation. We may therefore decompose L into a direct sum of weight spaces for the adjoint action of H.

Our strategy is therefore:

(1) to find an abelian Lie subalgebra H of L that consists entirely of semisimple elements; and

(2) to decompose L into weight spaces for the action of $\operatorname{ad} H$ and then exploit this decomposition to determine information about the structure constants of L.

In the following section, we identify the desirable properties of the subalgebra H and prove some preliminary results about the decomposition. We then show that subalgebras H with these desirable properties always exist and complete step (2).

10.1 Preliminary Results

Suppose that L is a complex semisimple Lie algebra containing an abelian subalgebra H consisting of semisimple elements. What information does this give us about L?

We have seen that L has a basis of common eigenvectors for the elements of $\operatorname{ad} H$. Given a common eigenvector $x \in L$, the eigenvalues are given by the associated weight, $\alpha : H \to \mathbf{C}$, defined by

$$(\operatorname{ad} h)x = \alpha(h)x \text{ for all } h \in H.$$

Weights are elements of the dual space $H^\star$. For each $\alpha \in H^\star$, let

$$L_\alpha := \{x \in L : [h, x] = \alpha(h)x \text{ for all } h \in H\}$$

denote the corresponding weight space. One of these weight spaces is the zero weight space:

$$L_0 = \{z \in L : [h, z] = 0 \text{ for all } h \in H\}.$$

This is the same as the centraliser of H in L, $C_L(H)$. As H is abelian, we have $H \subseteq L_0$.

Let Φ denote the set of non-zero $\alpha \in H^\star$ for which L_α is non-zero. We can write the decomposition of L into weight spaces for H as

$$L = L_0 \oplus \bigoplus_{\alpha \in \Phi} L_\alpha. \tag{$\star$}$$

Since L is finite-dimensional, this implies that Φ is finite.

Lemma 10.1

Suppose that $\alpha, \beta \in H^\star$. Then

(i) $[L_\alpha, L_\beta] \subseteq L_{\alpha+\beta}$.

(ii) If $\alpha + \beta \neq 0$, then $\kappa(L_\alpha, L_\beta) = 0$.

(iii) The restriction of κ to L_0 is non-degenerate.

Proof

(i) Take $x \in L_\alpha$ and $y \in L_\beta$. We must show that $[x, y]$, if non-zero, is an eigenvector for each $\operatorname{ad} h \in H$, with eigenvalue $\alpha(h) + \beta(h)$. Using the Jacobi identity we get

$$\begin{aligned} [h, [x, y]] = [[h, x], y] + [x, [h, y]] &= [\alpha(h)x, y] + [x, \beta(h)y] \\ &= \alpha(h)[x, y] + \beta(h)[x, y] \\ &= (\alpha + \beta)(h)[x, y]. \end{aligned}$$

(ii) Since $\alpha + \beta \neq 0$, there is some $h \in H$ such that $(\alpha + \beta)(h) \neq 0$. Now, for any $x \in L_\alpha$ and $y \in L_\beta$, we have, using the associativity of the Killing form,

$$\alpha(h)\kappa(x, y) = \kappa([h, x], y) = -\kappa([x, h], y) = -\kappa(x, [h, y]) = -\beta(h)\kappa(x, y),$$

and hence
$$(\alpha+\beta)(h)\kappa(x,y)=0.$$
Since by assumption $(\alpha+\beta)(h)\neq 0$, we must have $\kappa(x,y)=0$.

(iii) Suppose that $z \in L_0$ and $\kappa(z,x)=0$ for all x in L_0. By (ii), we know that $L_0 \perp L_\alpha$ for all $\alpha \neq 0$. If $x \in L$, then by $(\star)$ we can write x as
$$x = x_0 + \sum_{\alpha\in\Phi} x_\alpha$$
with $x_\alpha \in L_\alpha$. By linearity, $\kappa(z,x)=0$ for all $x \in L$. Since κ is non-degenerate, it follows that $z=0$, as required. □

Exercise 10.1

Show that if $x \in L_\alpha$ where $\alpha \neq 0$, then $\operatorname{ad} x$ is nilpotent.

If H is small, then the decomposition $(\star)$ is likely to be rather coarse, with few non-zero weight spaces other than L_0. Furthermore, if H is properly contained in L_0, then we get little information about how the elements in L_0 that are not in H act on L. This is illustrated by the following exercise.

Exercise 10.2

Let $L = \mathsf{sl}(n,\mathbf{C})$, where $n \geq 2$, and let $H = \operatorname{Span}\{h\}$, where $h = e_{11}-e_{22}$. Find $L_0 = C_L(H)$, and determine the direct sum decomposition $(\star)$ with respect to H.

We conclude that for the decomposition $(\star)$ of L into weight spaces to be as useful as possible, H should be as large as possible, and ideally we would have $H = L_0 = C_L(H)$.

Definition 10.2

A Lie subalgebra H of a Lie algebra L is said to be a *Cartan subalgebra* (or CSA) if H is abelian and every element $h \in H$ is semisimple, and moreover H is maximal with these properties.

Note that we do not assume L is semisimple in this definition. For example, the subalgebra H of $\mathsf{sl}(3,\mathbf{C})$ considered in the introduction to this chapter is a Cartan subalgebra of $\mathsf{sl}(3,\mathbf{C})$. One straightforward way to see this is to show that $C_{\mathsf{sl}(3,\mathbf{C})}(H) = H$; thus H is not contained in any larger abelian subalgebra of $\mathsf{sl}(3,\mathbf{C})$.

We remark that some texts use a “maximal toral subalgebra” in place of what we have called a Cartan subalgebra. The connection is discussed at the end of Appendix C, where we establish that the two types of algebras are the same.

10.2 Cartan Subalgebras

Let L be a complex semisimple Lie algebra. We shall show that L has a non-zero Cartan subalgebra. We first note that L must contain semisimple elements. If $x \in L$ has Jordan decomposition $x = s + n$, then by Theorem 9.15 both s and n belong to L. If the semisimple part s were always zero, then by Engel's Theorem (in its second version), L would be nilpotent and therefore solvable. Hence we can find a non-zero semisimple element $s \in L$. We can now obtain a non-zero Cartan subalgebra of L by taking any subalgebra which contains s and which is maximal subject to being abelian and consisting of semisimple elements. (Such a subalgebra must exist because L is finite-dimensional.)

We shall now show that if H is a Cartan subalgebra then $H = C_L(H)$. The proof of this statement is slightly technical, so the reader may prefer to defer or skip some of the details. In this case, she should continue reading at §10.3.

Lemma 10.3

Let H be a Cartan subalgebra of L. Suppose that $h \in H$ is such that the dimension of $C_L(h)$ is minimal. Then every $s \in H$ is central in $C_L(h)$, and so $C_L(h) \subseteq C_L(s)$. Hence $C_L(h) = C_L(H)$.

Proof

We shall show that if s is not central in $C_L(h)$, then there is a linear combination of s and h whose centraliser has smaller dimension than $C_L(h)$.

First we construct a suitable basis for L. We start by taking a basis of $C_L(h) \cap C_L(s)$, $\{c_1, \ldots, c_n\}$, say. As s is semisimple and $s \in C_L(h)$, $\operatorname{ad} s$ acts diagonalisably on $C_L(h)$. We may therefore extend this basis to a basis of $C_L(h)$ consisting of $\operatorname{ad} s$ eigenvectors, say by adjoining $\{x_1, \ldots, x_p\}$. Similarly we may extend $\{c_1, \ldots, c_n\}$ to a basis of $C_L(s)$ consisting of $\operatorname{ad} h$ eigenvectors, say by adjoining $\{y_1, \ldots, y_q\}$. We leave it to the reader to check that

$$\{c_1, \ldots, c_n, x_1, \ldots, x_p, y_1, \ldots, y_q\}$$

is a basis of $C_L(h) + C_L(s)$. Finally, as $\operatorname{ad} h$ and $\operatorname{ad} s$ commute and act diagonalisably on L, we may extend this basis to a basis of L by adjoining simultaneous eigenvectors for $\operatorname{ad} h$ and $\operatorname{ad} s$, say $\{w_1, \ldots, w_r\}$.

Note that if $[s, x_j] = 0$ then $x_j \in C_L(s) \cap C_L(h)$, a contradiction. Similarly, one can check that $[h, y_k] \neq 0$. Let $[h, w_l] = \theta_l w_l$ and $[s, w_l] = \sigma_l w_l$. Again we have $\theta_l, \sigma_l \neq 0$ for $1 \leq l \leq r$. The following table summarises the eigenvalues of $\operatorname{ad} s$, $\operatorname{ad} h$, and $\operatorname{ad} s + \lambda \operatorname{ad} h$, where $\lambda \neq 0$:

	c_i	x_j	y_k	w_l
$\operatorname{ad} s$	0	$\neq 0$	0	σ_l
$\operatorname{ad} h$	0	0	$\neq 0$	θ_l
$\operatorname{ad} s + \lambda \operatorname{ad} h$	0	$\neq 0$	$\neq 0$	$\sigma_l + \lambda\theta_l$

Thus, if we choose λ so that $\lambda \neq 0$ and $\lambda \neq -\sigma_l/\theta_l$ for any l, then we will have

$$C_L(s + \lambda h) = C_L(s) \cap C_L(h).$$

By hypothesis, $C_L(h) \not\subseteq C_L(s)$, so this subspace is of smaller dimension than $C_L(h)$; this contradicts the choice of h.

Now, since $C_L(H)$ is the intersection of the $C_L(s)$ for $s \in H$, it follows that $C_L(h) \subseteq C_L(H)$. The other inclusion is obvious, so we have proved that $C_L(h) = C_L(H)$. □

Theorem 10.4

If H is a Cartan subalgebra of L and $h \in H$ is such that $C_L(h) = C_L(H)$, then $C_L(h) = H$. Hence H is self-centralising.

Proof

Since H is abelian, H is certainly contained in $C_L(h)$. Suppose $x \in C_L(h)$ has abstract Jordan decomposition $x = s+n$. As x commutes with h, Theorem 9.15 implies that both s and n lie in $C_L(h)$, so we must show that $s \in H$ and $n = 0$.

We almost know already that $s \in H$. Namely, since $C_L(h) = C_L(H)$, we have that s commutes with every element of H and therefore $H + \operatorname{Span}\{s\}$ is an abelian subalgebra of L consisting of semisimple elements. It contains the Cartan subalgebra H and hence by maximality $s \in H$.

To show that the only nilpotent element in $C_L(H)$ is 0 takes slightly more work.

Step 1: $C_L(h)$ is nilpotent. Take $x \in C_L(h)$ with $x = s + n$ as above. Since $s \in H$, it must be central in $C_L(h)$, so, regarded as linear maps from $C_L(h)$ to itself, we have $\operatorname{ad} x = \operatorname{ad} n$. Thus for every $x \in C_L(h)$, $\operatorname{ad} x : C_L(h) \to C_L(h)$ is nilpotent. It now follows from the second version of Engel's Theorem (Theorem 6.3) that $C_L(h)$ is a nilpotent Lie algebra.

Step 2: Every element in $C_L(h)$ is semisimple. Let $x \in C_L(h)$ have abstract Jordan decomposition $x = s+n$ as above. As $C_L(h)$ is nilpotent, it is certainly solvable, so by Lie's Theorem (Theorem 6.5) there is a basis of L in which the maps $\operatorname{ad} x$ for $x \in C_L(h)$ are represented by upper triangular matrices. As $\operatorname{ad} n : L \to L$ is nilpotent, its matrix must be strictly upper triangular. Therefore

$$\kappa(n, z) = \operatorname{tr}(\operatorname{ad} n \circ \operatorname{ad} z) = 0$$

for all $z \in C_L(h)$. By Lemma 10.1(iii), the restriction of κ to $C_L(H)$ is non-degenerate, so we deduce $n = 0$, as required. □

10.3 Definition of the Root Space Decomposition

Let H be a Cartan subalgebra of our semisimple Lie algebra L. As $H = C_L(H)$, the direct sum decomposition of L into weight spaces for H considered in §10.1 may be written as

$$L = H \oplus \bigoplus_{\alpha \in \Phi} L_\alpha,$$

where Φ is the set of $\alpha \in H^\star$ such that $\alpha \neq 0$ and $L_\alpha \neq 0$. Since L is finite-dimensional, Φ is finite.

If $\alpha \in \Phi$, then we say that α is a *root* of L and L_α is the associated *root space*. The direct sum decomposition above is the *root space decomposition*. It should be noted that the roots and root spaces depend on the choice of Cartan subalgebra H.

10.4 Subalgebras Isomorphic to $\mathsf{sl}(2, \mathbf{C})$

We shall now associate to each root $\alpha \in \Phi$ a Lie subalgebra of L isomorphic to $\mathsf{sl}(2, \mathbf{C})$. These subalgebras will enable us to apply the results in Chapter 8 on representations of $\mathsf{sl}(2, \mathbf{C})$ to deduce several strong results on the structure of L. Chapters 11 and 12 give many examples of the theory we develop in the next three sections. See also Exercise 10.6 for a more immediate example.

Lemma 10.5

Suppose that $\alpha \in \Phi$ and that x is a non-zero element in L_α. Then $-\alpha$ is a root and there exists $y \in L_{-\alpha}$ such that $\operatorname{Span}\{x, y, [x, y]\}$ is a Lie subalgebra of L isomorphic to $\mathsf{sl}(2, \mathbf{C})$.

Proof

First we claim that there is some $y \in L_{-\alpha}$ such that $\kappa(x,y) \neq 0$ and $[x,y] \neq 0$. Since κ is non-degenerate, there is some $w \in L$ such that $\kappa(x,w) \neq 0$. Write $w = y_0 + \sum_{\beta \in \Phi} y_\beta$ with $y_0 \in L_0$ and $y_\beta \in L_\beta$. When we expand $\kappa(x,y)$, we find by Lemma 10.1(ii) that the only way a non-zero term can occur is if $-\alpha$ is a root and $y_{-\alpha} \neq 0$, so we may take $y = y_{-\alpha}$. Now, since α is non-zero, there is some $t \in H$ such that $\alpha(t) \neq 0$. For this t, we have

$$\kappa(t,[x,y]) = \kappa([t,x],y) = \alpha(t)\kappa(x,y) \neq 0$$

and so $[x,y] \neq 0$.

Let $S := \operatorname{Span}\{x, y, [x,y]\}$. By Lemma 10.1(i), $[x,y]$ lies in $L_0 = H$. As x and y are simultaneous eigenvectors for all elements of $\operatorname{ad} H$, and so in particular for $\operatorname{ad}[x,y]$, this shows that S is a Lie subalgebra of L. It remains to show that S is isomorphic to $\mathsf{sl}(2,\mathbf{C})$.

Let $h := [x,y] \in S$. We claim that $\alpha(h) \neq 0$. If not, then $[h,x] = \alpha(h)x = 0$; similarly $[h,y] = -\alpha(h)y = 0$, so $\operatorname{ad} h : L \to L$ commutes with $\operatorname{ad} x : L \to L$ and $\operatorname{ad} y : L \to L$. By Proposition 5.7, $\operatorname{ad} h : L \to L$ is a nilpotent map. On the other hand, because H is a Cartan subalgebra, h is semisimple. The only element of L that is both semisimple and nilpotent is 0, so $h = 0$, a contradiction.

Thus S is a 3-dimensional complex Lie algebra with $S' = S$. By §3.2.4, S is isomorphic to $\mathsf{sl}(2,\mathbf{C})$. $\square$

Using this lemma, we may associate to each $\alpha \in \Phi$ a subalgebra $\mathsf{sl}(\alpha)$ of L isomorphic to $\mathsf{sl}(2,\mathbf{C})$. The following exercise gives a standard basis for this Lie algebra.

Exercise 10.3

Show that for each $\alpha \in \Phi$, $\mathsf{sl}(\alpha)$ has a basis $\{e_\alpha, f_\alpha, h_\alpha\}$ such that

(i) $e_\alpha \in L_\alpha$, $f_\alpha \in L_{-\alpha}$, $h_\alpha \in H$, and $\alpha(h_\alpha) = 2$.

(ii) The map $\theta : \mathsf{sl}(\alpha) \to \mathsf{sl}(2,\mathbf{C})$ defined by $\theta(e_\alpha) = e$, $\theta(f_\alpha) = f$, $\theta(h_\alpha) = h$ is a Lie algebra isomorphism.

Hint: With the notation used in the statement of the lemma, one can take $e_\alpha = x$ and $f_\alpha = \lambda y$ for a suitable choice of $\lambda \in \mathbf{C}$.

10.5 Root Strings and Eigenvalues

We can use the Killing form to define an isomorphism between H and $H^\star$ as follows. Given $h \in H$, let θ_h denote the map $\theta_h \in H^\star$ defined by

$$\theta_h(k) = \kappa(h, k) \text{ for all } k \in H.$$

By Lemma 10.1(iii) the Killing form is non-degenerate on restriction to H, so the map $h \mapsto \theta_h$ is an isomorphism between H and $H^\star$. (If you did Exercise 9.10 then you will have seen this before; the proof is outlined in Appendix A.) In particular, associated to each root $\alpha \in \Phi$ there is a unique element $t_\alpha \in H$ such that

$$\kappa(t_\alpha, k) = \alpha(k) \text{ for all } k \in H.$$

One very useful property of this correspondence is the following lemma.

Lemma 10.6

Let $\alpha \in \Phi$. If $x \in L_\alpha$ and $y \in L_{-\alpha}$, then $[x, y] = \kappa(x, y)t_\alpha$. In particular, $h_\alpha = [e_\alpha, f_\alpha] \in \mathrm{Span}\{t_\alpha\}$.

Proof

For $h \in H$, we have

$$\kappa(h, [x, y]) = \kappa([h, x], y) = \alpha(h)\kappa(x, y) = \kappa(t_\alpha, h)\kappa(x, y).$$

Now we view $\kappa(x, y)$ as a scalar and rewrite the right-hand side to get

$$\kappa(h, [x, y]) \ = \kappa(h, \kappa(x, y)t_\alpha).$$

This shows that $[x, y] - \kappa(x, y)t_\alpha$ is perpendicular to all $h \in H$, and hence it is zero as κ restricted to H is non-degenerate. □

We are now in a position to apply the results of Chapter 8 on the representation theory of $\mathsf{sl}(2, \mathbf{C})$. Let α be a root. We may regard L as an $\mathsf{sl}(\alpha)$-module via restriction of the adjoint representation. Thus, if $a \in \mathsf{sl}(\alpha)$ and $y \in L$, then the action is given by

$$a \cdot y = (\mathrm{ad}\, a)y = [a, y].$$

Note that the $\mathsf{sl}(\alpha)$-submodules of L are precisely the vector subspaces M of L such that $[s, m] \in M$ for all $s \in \mathsf{sl}(\alpha)$ and $m \in M$. Of course, it is enough to check this when s is one of the standard basis elements $h_\alpha, e_\alpha, f_\alpha$. We shall also need the following lemma.

Lemma 10.7

If M is an $\mathsf{sl}(\alpha)$-submodule of L, then the eigenvalues of h_α acting on M are integers.

Proof

By Weyl's Theorem, M may be decomposed into a direct sum of irreducible $\mathsf{sl}(\alpha)$-modules; for irreducible $\mathsf{sl}(2, \mathbf{C})$-modules, the result follows from the classification of Chapter 8. □

Example 10.8

(1) If you did Exercise 8.3, then you will have seen how $\mathsf{sl}(3, \mathbf{C})$ decomposes as an $\mathsf{sl}(\alpha)$-module where $\alpha = \varepsilon_1 - \varepsilon_2$ is a root of the Cartan subalgebra of $\mathsf{sl}(3, \mathbf{C})$ consisting of all diagonal matrices.

(2) Let $U = H + \mathsf{sl}(\alpha)$. Let $K = \ker \alpha \subseteq H$. By the rank-nullity formula, $\dim K = \dim H - 1$. (We know that $\dim \operatorname{im} \alpha = 1$ as $\alpha(h_\alpha) \neq 0$.) As H is abelian, $[h_\alpha, x] = 0$ for all $x \in K$. Moreover, if $x \in K$, then

$$[e_\alpha, x] = -[x, e_\alpha] = -\alpha(x)e_\alpha = 0$$

and similarly $[f_\alpha, x] = 0$. Thus every element of $\mathsf{sl}(\alpha)$ acts trivially on K. It follows that $U = K \oplus \mathsf{sl}(\alpha)$ is a decomposition of U into $\mathsf{sl}(\alpha)$-modules. By Exercise 8.2(iii), the adjoint representation of $\mathsf{sl}(\alpha)$ is isomorphic to V_2, so U is isomorphic to the direct sum of $\dim H - 1$ copies of the trivial representation, V_0, and one copy of the adjoint representation, V_2.

(3) If $\beta \in \Phi$ or $\beta = 0$, let

$$M := \bigoplus_c L_{\beta + c\alpha},$$

where the sum is over all $c \in \mathbf{C}$ such that $\beta + c\alpha \in \Phi$. It follows from Lemma 10.1(i) that M is an $\mathsf{sl}(\alpha)$-submodule of L. This module is said to be the *α-root string through β*. Analysing these modules will give the main results of this section.

Proposition 10.9

Let $\alpha \in \Phi$. The root spaces $L_{\pm\alpha}$ are 1-dimensional. Moreover, the only multiples of α which lie in Φ are $\pm\alpha$.

Proof

If $c\alpha$ is a root, then h_α takes $c\alpha(h_\alpha) = 2c$ as an eigenvalue. As the eigenvalues of h_α are integral, either $c \in \mathbf{Z}$ or $c \in \mathbf{Z} + \frac{1}{2}$. To rule out the unwanted values for c, we consider the root string module

$$M := H \oplus \bigoplus_{c\alpha \in \Phi} L_{c\alpha}.$$

Let $K = \ker \alpha \subseteq H$. By Example 10.8(2) above, $K \oplus \mathsf{sl}(\alpha)$ is an $\mathsf{sl}(\alpha)$-submodule of M. By Weyl's Theorem, modules for $\mathsf{sl}(\alpha)$ are completely reducible, so we may write

$$M = K \oplus \mathsf{sl}(\alpha) \oplus W,$$

where W is a complementary submodule.

If either of the conclusions of the proposition are false, then W is non-zero. Let $V \cong V_s$ be an irreducible submodule of W. If s is even, then it follows from the classification of Chapter 8 that V contains an h_α-eigenvector with eigenvalue 0. Call this eigenvector v. The zero-eigenspace of h_α on M is H, which is contained in $K \oplus \mathsf{sl}(\alpha)$. Hence $v \in (K \oplus \mathsf{sl}(\alpha)) \cap V = 0$, which is a contradiction.

Before considering the case where s is odd, we pursue another consequence of this argument. Suppose that $2\alpha \in \Phi$. Then h_α has $2\alpha(h_\alpha) = 4$ as an eigenvalue. As the eigenvalues of h_α on $K \oplus \mathsf{sl}(\alpha)$ are 0 and ± 2, the only way this could happen is if W contains an irreducible submodule V_s with s even, which we just saw is impossible.

Now suppose that s is odd. Then V must contain an h_α-eigenvector with eigenvalue 1. As $\alpha(h_\alpha) = 2$, this implies that $\frac{1}{2}\alpha$ is a root of L. But then both $\frac{1}{2}\alpha$ and α are roots of L, which contradicts the previous paragraph. □

Proposition 10.10

Suppose that $\alpha, \beta \in \Phi$ and $\beta \neq \pm\alpha$.

(i) $\beta(h_\alpha) \in \mathbf{Z}$.

(ii) There are integers $r, q \geq 0$ such that if $k \in \mathbf{Z}$, then $\beta + k\alpha \in \Phi$ if and only if $-r \leq k \leq q$. Moreover, $r - q = \beta(h_\alpha)$.

(iii) If $\alpha + \beta \in \Phi$, then $[e_\alpha, e_\beta]$ is a non-zero scalar multiple of $e_{\alpha+\beta}$.

(iv) $\beta - \beta(h_\alpha)\alpha \in \Phi$.

Proof

Let $M := \bigoplus_k L_{\beta+k\alpha}$ be the root string of α through β. To prove (i), we note that $\beta(h_\alpha)$ is the eigenvalue of h_α acting on L_β, and so it lies in $\mathbf{Z}$.

We know from the previous proposition that $\dim L_{\beta+k\alpha} = 1$ whenever $\beta+k\alpha$ is a root, so the eigenspaces of $\operatorname{ad} h_\alpha$ on M are all 1-dimensional and, since $(\beta + k\alpha)h_\alpha = \beta(h_\alpha) + 2k$, the eigenvalues of $\operatorname{ad} h_\alpha$ on M are either all even or all odd. It now follows from Chapter 8 that M is an irreducible $\mathsf{sl}(\alpha)$-module. Suppose that $M \cong V_d$. On V_d, the element h_α acts diagonally with eigenvalues

$$\{d, d-2, \ldots, -d\}\,,$$

whereas on M, h_α acts diagonally with eigenvalues

$$\{\beta(h_\alpha) + 2k : \beta + k\alpha \in \Phi\}\,.$$

Equating these sets shows that if we define r and q by $d = \beta(h_\alpha) + 2q$ and $-d = \beta(h_\alpha) - 2r$, then (ii) will hold.

Suppose $v \in L_\beta$, so v belongs to the h_α-eigenspace where h_α acts as $\beta(h_\alpha)$. If $(\operatorname{ad} e_\alpha)e_\beta = 0$, then e_β is a highest-weight vector in the irreducible representation $M \cong V_d$, with highest weight $\beta(h_\alpha)$. If $\alpha + \beta$ is a root, then h_α acts on the associated root space as $(\alpha + \beta)h_\alpha = \beta(h_\alpha) + 2$. Therefore e_β is *not* in the highest weight space of the irreducible representation M, and so $(\operatorname{ad} e_\alpha)e_\beta \neq 0$. This proves (iii).

Finally, (iv) follows from part (ii) as

$$\beta - \beta(h_\alpha)\alpha = \beta - (r - q)\alpha$$

and $-r \leq -r + q \leq q$. $\square$

We now have a good idea about the structure constant of L (with respect to a basis given by the root space decomposition). The action of H on the root spaces of L is determined by the roots. Part (iii) of the previous proposition shows that (up to scalar factors) the set of roots also determines the brackets $[e_\alpha, e_\beta]$ for roots $\alpha \neq \pm\beta$. Lastly, by construction, $[e_\alpha, e_{-\alpha}]$ is in the span of $[e_\alpha, f_\alpha] = h_\alpha$. The reader keen to see a complete answer should read about Chevalley's Theorem in §15.3.

10.6 Cartan Subalgebras as Inner-Product Spaces

We conclude this chapter by showing that the roots of L all lie in a real vector subspace of $H^\star$ and that the Killing form induces an inner product on the space. This will enable us to bring some elementary geometric ideas to bear on the classification problem.

The two propositions in the previous section show that the set Φ of roots cannot be too big: For example, we saw that if $\alpha \in \Phi$ then the only multiples of $\alpha \in \Phi$ are $\pm\alpha$. On the other hand, there must be roots, as otherwise the root space decomposition would imply that $L = H$ was abelian. What more can be said?

Lemma 10.11

(i) If $h \in H$ and $h \neq 0$, then there exists a root $\alpha \in \Phi$ such that $\alpha(h) \neq 0$.

(ii) The set Φ of roots spans $H^\star$.

Proof

Suppose that $\alpha(h) = 0$ for all roots α. Then we have $[h, x] = \alpha(h)x = 0$ for all $x \in L_\alpha$ and for all roots α. Since H is abelian, it follows from the root space decomposition that $h \in Z(L)$, which is zero as L is semisimple.

In a sense, (ii) is just a reformulation of (i) in the language of linear algebra. Let $W \subseteq H^\star$ denote the span of Φ. Suppose that W is a proper subspace of $H^\star$. Then the annihilator of W in H,

$$W^\circ = \{h \in H : \theta(h) = 0 \text{ for all } \theta \in W\},$$

has dimension $\dim H - \dim W \neq 0$. (See Appendix A.) Therefore there is some non-zero $h \in H$ such that $\theta(h) = 0$ for all $\theta \in W$, so in particular $\alpha(h) = 0$ for all $\alpha \in \Phi$, in contradiction to part (i). $\square$

In the previous section, we found that the elements t_α and h_α spanned the same 1-dimensional subspace of L. More precisely, we have the following.

Lemma 10.12

For each $\alpha \in \Phi$, we have

(i) $t_\alpha = \dfrac{h_\alpha}{\kappa(e_\alpha, f_\alpha)}$ and $h_\alpha = \dfrac{2t_\alpha}{\kappa(t_\alpha, t_\alpha)}$;

(ii) $\kappa(t_\alpha, t_\alpha)\kappa(h_\alpha, h_\alpha) = 4$.

Proof

The expression for t_α follows from Lemma 10.6 applied with $x = e_\alpha$ and $y = f_\alpha$.

As $\alpha(h_\alpha) = 2$, we have

$$2 = \kappa(t_\alpha, h_\alpha) = \kappa(t_\alpha, \kappa(e_\alpha, f_\alpha)t_\alpha),$$

which implies that $\kappa(e_\alpha, f_\alpha)\kappa(t_\alpha, t_\alpha) = 2$. Now substitute for $\kappa(e_\alpha, f_\alpha)$ to get the second expression. Finally,

$$\kappa(h_\alpha, h_\alpha) = \kappa\left(\frac{2t_\alpha}{\kappa(t_\alpha, t_\alpha)}, \frac{2t_\alpha}{\kappa(t_\alpha, t_\alpha)}\right) = \frac{4}{\kappa(t_\alpha, t_\alpha)}$$

gives (ii). □

Corollary 10.13

If α and β are roots, then $\kappa(h_\alpha, h_\beta) \in \mathbf{Z}$ and $\kappa(t_\alpha, t_\beta) \in \mathbf{Q}$.

Proof

Using the root space decomposition to compute $\operatorname{tr}(\operatorname{ad} h_\alpha \circ \operatorname{ad} h_\beta)$, we get

$$\kappa(h_\alpha, h_\beta) = \sum_{\gamma \in \Phi} \gamma(h_\alpha)\gamma(h_\beta).$$

Since the eigenvalues of h_α and h_β are integers, this shows that $\kappa(h_\alpha, h_\beta) \in \mathbf{Z}$. We now use the previous lemma to get

$$\begin{aligned}\kappa(t_\alpha, t_\beta) &= \kappa\left(\frac{\kappa(t_\alpha, t_\alpha)h_\alpha}{2}, \frac{\kappa(t_\beta, t_\beta)h_\beta}{2}\right)\\ &= \frac{\kappa(t_\alpha, t_\alpha)\kappa(t_\beta, t_\beta)}{4}\kappa(h_\alpha, h_\beta) \in \mathbf{Q}.\end{aligned}$$

□

We can translate the Killing form on H to obtain a non-degenerate symmetric bilinear form on H^*, denoted $(-,-)$. This form may be defined by

$$(\theta, \varphi) = \kappa(t_\theta, t_\varphi),$$

where t_θ and t_φ are the elements of H corresponding to θ and φ under the isomorphism $H \equiv H^\star$ induced by κ. In particular, if α and β are roots, then

$$(\alpha, \beta) = \kappa(t_\alpha, t_\beta) \in \mathbf{Q}.$$

Exercise 10.4

Show that $\beta(h_\alpha) = \frac{2(\beta,\alpha)}{(\alpha,\alpha)}$.

We saw in Lemma 10.11 that the roots of L span $H^\star$, so $H^\star$ has a vector space basis consisting of roots, say $\{\alpha_1, \alpha_2, \ldots, \alpha_\ell\}$. We can now prove that something stronger is true as follows.

Lemma 10.14

If β is a root, then β is a linear combination of the α_i with coefficients in $\mathbf{Q}$.

Proof

Certainly we may write $\beta = \sum_{i=1}^{\ell} c_i \alpha_i$ with coefficients $c_i \in \mathbf{C}$. For each j with $1 \le j \le \ell$, we have

$$(\beta, \alpha_j) = \sum_{i=1}^{\ell} (\alpha_i, \alpha_j) c_i.$$

We can write these equations in matrix form as

$$\begin{pmatrix} (\beta, \alpha_1) \\ \vdots \\ (\beta, \alpha_\ell) \end{pmatrix} = \begin{pmatrix} (\alpha_1, \alpha_1) & \dots & (\alpha_\ell, \alpha_1) \\ \vdots & \ddots & \vdots \\ (\alpha_1, \alpha_\ell) & \dots & (\alpha_\ell, \alpha_\ell) \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_\ell \end{pmatrix}.$$

The matrix is the matrix of the non-degenerate bilinear form $(-,-)$ with respect to the chosen basis of roots, and so it is invertible (see Appendix A). Moreover, we have seen that its entries are rational numbers, so it has an inverse with entries in $\mathbf{Q}$. Since also $(\beta, \alpha_j) \in \mathbf{Q}$, the coefficients c_i are rational. □

By this lemma, the *real* subspace of $H^\star$ spanned by the roots $\alpha_1, \dots, \alpha_\ell$ contains all the roots of Φ and so does not depend on our particular choice of basis. Let E denote this subspace.

Proposition 10.15

The form $(-,-)$ is a real-valued inner product on E.

Proof

Since $(-,-)$ is a symmetric bilinear form, we only need to check that the restriction of $(-,-)$ to E is positive definite. Let $\theta \in E$ correspond to $t_\theta \in H$. Using the root space decomposition and the fact that $(\operatorname{ad} t_\theta) e_\beta = \beta(t_\theta) e_\beta$, we get

$$(\theta, \theta) = \kappa(t_\theta, t_\theta) = \sum_{\beta \in \Phi} \beta(t_\theta)^2 = \sum_{\beta \in \Phi} \kappa(t_\beta, t_\theta)^2 = \sum_{\beta \in \Phi} (\beta, \theta)^2.$$

As (β, θ) is real, the right-hand side is real and non-negative. Moreover, if $(\theta, \theta) = 0$, then $\beta(t_\theta) = 0$ for all roots β, so by Lemma 10.11(i), $\theta = 0$. □

EXERCISES

10.5. Suppose that L is a complex semisimple Lie algebra with Cartan subalgebra H and root system Φ. Use the results of this chapter to prove that

$$\dim L = \dim H + |\Phi|.$$

Hence show that there are no semisimple Lie algebras of dimensions 4, 5, or 7.

10.6.† Let $L = \mathsf{sl}(3, \mathbf{C})$. With the same notation as in the introduction, let $\alpha := \varepsilon_1 - \varepsilon_2$ and $\beta := \varepsilon_2 - \varepsilon_3$. Show that the set of roots is

$$\Phi = \{\pm\alpha, \pm\beta \pm (\alpha + \beta)\}.$$

Show that the angle between the roots α and β is $2\pi/3$ and verify some of the results in §10.5 and §10.6 for $\mathsf{sl}(3, \mathbf{C})$.

10.7. Suppose L is semisimple of dimension 6. Let H be a Cartan subalgebra of L and let Φ be the associated set of roots.

(i) Show that $\dim H = 2$ and that if $\alpha, \beta \in \Phi$ span $H^\star$, then $\Phi = \{\pm\alpha, \pm\beta\}$.

(ii) Hence show that

$$[L_\alpha, L_{\pm\beta}] = 0 \text{ and } [L_{\pm\beta}, [L_\alpha, L_{-\alpha}]] = 0$$

and deduce that the subalgebra $L_\alpha \oplus L_{-\alpha} \oplus [L_\alpha, L_{-\alpha}]$ is an ideal of L. Show that L is isomorphic to the direct sum of two copies of $\mathsf{sl}(2, \mathbf{C})$.

10.8. Show that the set of diagonal matrices in $\mathsf{so}(4, \mathbf{C})$ (as defined in Chapter 4) forms a Cartan subalgebra of $\mathsf{so}(4, \mathbf{C})$ and determine the corresponding root space decomposition. Hence show that $\mathsf{so}(4, \mathbf{C}) \cong \mathsf{sl}(2, \mathbf{C}) \oplus \mathsf{sl}(2, \mathbf{C})$. (The reader will probably now be able to guess the reason for choosing non-obvious bilinear forms in the definition of the symplectic and orthogonal Lie algebras.)

10.9. Let L be a semisimple Lie algebra with Cartan subalgebra H. Use the root space decomposition to show that $N_L(H) = H$. (The notation $N_L(H)$ is defined in Exercise 5.6.)

10.10. In Lemma 10.5, we defined for each $\alpha \in \Phi$ a subalgebra of L isomorphic to $\mathsf{sl}(2, \mathbf{C})$. In the main step in the proof, we showed that if $x \in L_\alpha$ and $y \in L_{-\alpha}$, and $h = [x, y] \neq 0$, then $\alpha(h) \neq 0$. Here is an alternative proof of this using root string modules.

Suppose that $\alpha(h) = 0$. Let $\beta \in \Phi$ be any root. Let M be the α root string module through β,

$$M = \bigoplus_c L_{\beta+c\alpha}.$$

By considering the trace of h on M, show that $\beta(h) = 0$ and hence get a contradiction.

10.11. Let L be a complex semisimple Lie algebra with Cartan subalgebra H and root space Φ. Let $\alpha \in \Phi$ and let $\mathsf{sl}(\alpha) = \mathrm{Span}\{e_\alpha, f_\alpha, h_\alpha\}$ be the corresponding subalgebra constructed in §10.4. Show that this subalgebra is unique up to

(1) scaling basis elements as $ce_\alpha, c^{-1}f_\alpha, h_\alpha$ for non-zero $c \in \mathbf{C}$; and

(2) swapping e_α and f_α and then replacing h_α with $-h_\alpha$.

10.12.† Let L be a semisimple Lie algebra and let Φ be its set of roots. Let $\alpha \in \Phi$ and let

$$N := \mathrm{Span}\{f_\alpha\} \oplus \mathrm{Span}\{h_\alpha\} \oplus L_\alpha \oplus L_{2\alpha} \oplus \dots\ .$$

Show that N is an $\mathsf{sl}(\alpha)$-submodule of L. By considering the trace of $h_\alpha : N \to N$, give an alternative proof of Proposition 10.9.