1
Introduction

We begin by defining Lie algebras and giving a collection of typical examples to which we shall refer throughout this book. The remaining sections in this chapter introduce the basic vocabulary of Lie algebras. The reader is reminded that the prerequisite linear and bilinear algebra is summarised in Appendix A.

1.1 Definition of Lie Algebras

Let F be a field. A *Lie algebra* over F is an F-vector space L, together with a bilinear map, the *Lie bracket*

$$L \times L \to L, \quad (x, y) \mapsto [x, y],$$

satisfying the following properties:

$$[x, x] = 0 \quad \text{for all } x \in L, \tag{L1}$$

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 \quad \text{for all } x, y, z \in L. \tag{L2}$$

The Lie bracket $[x, y]$ is often referred to as the *commutator* of x and y. Condition (L2) is known as the *Jacobi identity*. As the Lie bracket $[-, -]$ is bilinear, we have

$$0 = [x + y, x + y] = [x, x] + [x, y] + [y, x] + [y, y] = [x, y] + [y, x].$$

Hence condition (L1) implies

$$[x, y] = -[y, x] \quad \text{for all } x, y \in L. \tag{L1$'$}$$

K. Erdmann et al., *Introduction to Lie Algebras*

If the field F does not have characteristic 2, then putting $x = y$ in (L1$'$) shows that (L1$'$) implies (L1).

Unless specifically stated otherwise, all Lie algebras in this book should be taken to be finite-dimensional. (In Chapter 15, we give a brief introduction to the more subtle theory of infinite-dimensional Lie algebras.)

Exercise 1.1

(i) Show that $[v, 0] = 0 = [0, v]$ for all $v \in L$.

(ii) Suppose that $x, y \in L$ satisfy $[x, y] \neq 0$. Show that x and y are linearly independent over F.

1.2 Some Examples

(1) Let $F = \mathbf{R}$. The vector product $(x, y) \mapsto x \wedge y$ defines the structure of a Lie algebra on $\mathbf{R}^3$. We denote this Lie algebra by $\mathbf{R}^3_\wedge$. Explicitly, if $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$, then

$$x \wedge y = (x_2y_3 - x_3y_2, x_3y_1 - x_1y_3, x_1y_2 - x_2y_1).$$

Exercise 1.2

Convince yourself that $\wedge$ is bilinear. Then check that the Jacobi identity holds. *Hint*: If $x \cdot y$ denotes the dot product of the vectors $x, y \in \mathbf{R}^3$, then

$$x \wedge (y \wedge z) = (x \cdot z)y - (x \cdot y)z \quad \text{for all } x, y, z \in \mathbf{R}^3.$$

(2) Any vector space V has a Lie bracket defined by $[x, y] = 0$ for all $x, y \in V$. This is the *abelian* Lie algebra structure on V. In particular, the field F may be regarded as a 1-dimensional abelian Lie algebra.

(3) Suppose that V is a finite-dimensional vector space over F. Write $\mathsf{gl}(V)$ for the set of all linear maps from V to V. This is again a vector space over F, and it becomes a Lie algebra, known as the *general linear algebra*, if we define the Lie bracket $[-, -]$ by

$$[x, y] := x \circ y - y \circ x \quad \text{for } x, y \in \mathsf{gl}(V),$$

where $\circ$ denotes the composition of maps.

Exercise 1.3

Check that the Jacobi identity holds. (This exercise is famous as one that every mathematician should do at least once in her life.)

(3′) Here is a matrix version. Write $\mathsf{gl}(n, F)$ for the vector space of all $n \times n$ matrices over F with the Lie bracket defined by

$$[x, y] := xy - yx,$$

where xy is the usual product of the matrices x and y.

As a vector space, $\mathsf{gl}(n, F)$ has a basis consisting of the *matrix units* e_{ij} for $1 \leq i, j \leq n$. Here e_{ij} is the $n \times n$ matrix which has a 1 in the ij-th position and all other entries are 0. We leave it as an exercise to check that

$$[e_{ij}, e_{kl}] = \delta_{jk} e_{il} - \delta_{il} e_{kj},$$

where δ is the Kronecker delta, defined by $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise. This formula can often be useful when calculating in $\mathsf{gl}(n, F)$.

(4) Recall that the trace of a square matrix is the sum of its diagonal entries. Let $\mathsf{sl}(n, F)$ be the subspace of $\mathsf{gl}(n, F)$ consisting of all matrices of trace 0. For arbitrary square matrices x and y, the matrix $xy - yx$ has trace 0, so $[x, y] = xy - yx$ defines a Lie algebra structure on $\mathsf{sl}(n, F)$: properties (L1) and (L2) are inherited from $\mathsf{gl}(n, F)$. This Lie algebra is known as the *special linear algebra*. As a vector space, $\mathsf{sl}(n, F)$ has a basis consisting of the e_{ij} for $i \neq j$ together with $e_{ii} - e_{i+1,i+1}$ for $1 \leq i < n$.

(5) Let $\mathsf{b}(n, F)$ be the upper triangular matrices in $\mathsf{gl}(n, F)$. (A matrix x is said to be upper triangular if $x_{ij} = 0$ whenever $i > j$.) This is a Lie algebra with the same Lie bracket as $\mathsf{gl}(n, F)$.

Similarly, let $\mathsf{n}(n, F)$ be the strictly upper triangular matrices in $\mathsf{gl}(n, F)$. (A matrix x is said to be strictly upper triangular if $x_{ij} = 0$ whenever $i \geq j$.) Again this is a Lie algebra with the same Lie bracket as $\mathsf{gl}(n, F)$.

Exercise 1.4

Check the assertions in (5).

1.3 Subalgebras and Ideals

The last two examples suggest that, given a Lie algebra L, we might define a *Lie subalgebra* of L to be a vector subspace $K \subseteq L$ such that

$$[x, y] \in K \quad \text{for all } x, y \in K.$$

Lie subalgebras are easily seen to be Lie algebras in their own right. In Examples (4) and (5) above we saw three Lie subalgebras of $\mathsf{gl}(n, F)$.

We also define an *ideal* of a Lie algebra L to be a subspace I of L such that

$$[x, y] \in I \quad \text{for all } x \in L,\ y \in I.$$

By (L1′), $[x, y] = -[y, x]$, so we do not need to distinguish between left and right ideals. For example, $\mathsf{sl}(n, F)$ is an ideal of $\mathsf{gl}(n, F)$, and $\mathsf{n}(n, F)$ is an ideal of $\mathsf{b}(n, F)$.

An ideal is always a subalgebra. On the other hand, a subalgebra need not be an ideal. For example, $\mathsf{b}(n, F)$ is a subalgebra of $\mathsf{gl}(n, F)$, but provided $n \geq 2$, it is not an ideal. To see this, note that $e_{11} \in \mathsf{b}(n, F)$ and $e_{21} \in \mathsf{gl}(n, F)$. However, $[e_{21}, e_{11}] = e_{21} \notin \mathsf{b}(n, F)$.

The Lie algebra L is itself an ideal of L. At the other extreme, $\{0\}$ is an ideal of L. We call these the *trivial ideals* of L. An important example of an ideal which frequently is non-trivial is the *centre* of L, defined by

$$Z(L) := \{x \in L : [x, y] = 0 \text{ for all } y \in L\}.$$

We know precisely when $L = Z(L)$ as this is the case if and only if L is abelian. On the other hand, it might take some work to decide whether or not $Z(L) = \{0\}$.

Exercise 1.5

Find $Z(L)$ when $L = \mathsf{sl}(2, F)$. You should find that the answer depends on the characteristic of F.

1.4 Homomorphisms

If L_1 and L_2 are Lie algebras over a field F, then we say that a map $\varphi : L_1 \to L_2$ is a *homomorphism* if φ is a linear map and

$$\varphi([x, y]) = [\varphi(x), \varphi(y)] \quad \text{for all } x, y \in L_1.$$

Notice that in this equation the first Lie bracket is taken in L_1 and the second Lie bracket is taken in L_2. We say that φ is an *isomorphism* if φ is also bijective.

An extremely important homomorphism is the *adjoint homomorphism* . If L is a Lie algebra, we define

$$\mathrm{ad} : L \to \mathsf{gl}(L)$$

by $(\mathrm{ad}\, x)(y) := [x, y]$ for $x, y \in L$. It follows from the bilinearity of the Lie bracket that the map $\mathrm{ad}\, x$ is linear for each $x \in L$. For the same reason, the map $x \mapsto \mathrm{ad}\, x$ is itself linear. So to show that ad is a homomorphism, all we need to check is that

$$\mathrm{ad}([x, y]) = \mathrm{ad}\, x \circ \mathrm{ad}\, y - \mathrm{ad}\, y \circ \mathrm{ad}\, x \quad \text{for all } x, y \in L;$$

this turns out to be equivalent to the Jacobi identity. The kernel of ad is the centre of L.

Exercise 1.6

Show that if $\varphi : L_1 \to L_2$ is a homomorphism, then the kernel of φ, $\ker \varphi$, is an ideal of L_1, and the image of φ, $\operatorname{im} \varphi$, is a Lie subalgebra of L_2.

Remark 1.1

Whenever one has a mathematical object, such as a vector space, group, or Lie algebra, one has attendant homomorphisms. Such maps are of interest precisely because they are structure preserving — *homo*, same; *morphos*, shape. For example, working with vector spaces, if we add two vectors, and then apply a homomorphism of vector spaces (also known as a linear map), the result should be the same as if we had first applied the homomorphism, and then added the image vectors.

Given a class of mathematical objects one can (with some thought) work out what the relevant homomorphisms should be. Studying these homomorphisms gives one important information about the structures of the objects concerned. A common aim is to classify all objects of a given type; from this point of view, we regard isomorphic objects as essentially the same. For example, two vector spaces over the same field are isomorphic if and only if they have the same dimension.

1.5 Algebras

An *algebra* over a field F is a vector space A over F together with a bilinear map,

$$A \times A \to A, \quad (x, y) \mapsto xy.$$

We say that xy is the *product* of x and y. Usually one studies algebras where the product satisfies some further properties. In particular, Lie algebras are the algebras satisfying identities (L1) and (L2). (And in this case we write the product xy as $[x, y]$.)

The algebra A is said to be *associative* if

$$(xy)z = x(yz) \quad \text{for all } x, y, z \in A$$

and *unital* if there is an element 1_A in A such that $1_A x = x = x 1_A$ for all non-zero elements of A.

For example, $\mathsf{gl}(V)$, the vector space of linear transformations of the vector space V, has the structure of a unital associative algebra where the product is given by the composition of maps. The identity transformation is the identity element in this algebra. Likewise $\mathsf{gl}(n, F)$, the set of $n \times n$ matrices over F, is a unital associative algebra with respect to matrix multiplication.

Apart from Lie algebras, most algebras one meets tend to be both associative and unital. It is important not to get confused between these two types of algebras. One way to emphasise the distinction, which we have adopted, is to always write the product in a Lie algebra with square brackets.

Exercise 1.7

Let L be a Lie algebra. Show that the Lie bracket is associative, that is, $[x, [y, z]] = [[x, y], z]$ for all $x, y, z \in L$, if and only if for all $a, b \in L$ the commutator $[a, b]$ lies in $Z(L)$.

If A is an associative algebra over F, then we define a new bilinear operation $[-, -]$ on A by

$$[a, b] := ab - ba \quad \text{for all } a, b \in A.$$

Then A together with $[-, -]$ is a Lie algebra; this is not hard to prove. The Lie algebras $\mathsf{gl}(V)$ and $\mathsf{gl}(n, F)$ are special cases of this construction. In fact, if you did Exercise 1.3, then you will already have proved that the product $[-, -]$ satisfies the Jacobi identity.

1.6 Derivations

Let A be an algebra over a field F. A *derivation* of A is an F-linear map $D : A \to A$ such that

$$D(ab) = aD(b) + D(a)b \quad \text{for all } a, b \in A.$$

Let $\operatorname{Der} A$ be the set of derivations of A. This set is closed under addition and scalar multiplication and contains the zero map. Hence $\operatorname{Der} A$ is a vector subspace of $\mathsf{gl}(A)$. Moreover, $\operatorname{Der} A$ is a Lie subalgebra of $\mathsf{gl}(A)$, for by part (i) of the following exercise, if D and E are derivations then so is $[D, E]$.

Exercise 1.8

Let D and E be derivations of an algebra A.

(i) Show that $[D, E] = D \circ E - E \circ D$ is also a derivation.

(ii) Show that $D \circ E$ need not be a derivation. (The following example may be helpful.)

Example 1.2

(1) Let $A = C^\infty \mathbf{R}$ be the vector space of all infinitely differentiable functions $\mathbf{R} \to \mathbf{R}$. For $f, g \in A$, we define the product fg by pointwise multiplication: $(fg)(x) = f(x)g(x)$. With this definition, A is an associative algebra. The usual derivative, $Df = f'$, is a derivation of A since by the product rule

$$D(fg) = (fg)' = f'g + fg' = (Df)g + f(Dg).$$

(2) Let L be a Lie algebra and let $x \in L$. The map $\operatorname{ad} x : L \to L$ is a derivation of L since by the Jacobi identity

$$(\operatorname{ad} x)[y, z] = [x, [y, z]] = [[x, y], z] + [y, [x, z]] = [(\operatorname{ad} x)y, z] + [y, (\operatorname{ad} x)z]$$

for all $y, z \in L$.

1.7 Structure Constants

If L is a Lie algebra over a field F with basis $(x_1, \ldots, x_n)$, then $[-, -]$ is completely determined by the products $[x_i, x_j]$. We define scalars $a_{ij}^k \in F$ such that

$$[x_i, x_j] = \sum_{k=1}^{n} a_{ij}^k x_k.$$

The a_{ij}^k are the *structure constants* of L with respect to this basis. We emphasise that the a_{ij}^k depend on the choice of basis of L: Different bases will in general give different structure constants.

By (L1) and its corollary (L1′), $[x_i, x_i] = 0$ for all i and $[x_i, x_j] = -[x_j, x_i]$ for all i and j. So it is sufficient to know the structure constants a_{ij}^k for $1 \le i < j \le n$.

Exercise 1.9

Let L_1 and L_2 be Lie algebras. Show that L_1 is isomorphic to L_2 if and only if there is a basis B_1 of L_1 and a basis B_2 of L_2 such that the structure constants of L_1 with respect to B_1 are equal to the structure constants of L_2 with respect to B_2.

Exercise 1.10

Let L be a Lie algebra with basis $(x_1, \ldots, x_n)$. What condition does the Jacobi identity impose on the structure constants a_{ij}^k?

EXERCISES

1.11.† Let L_1 and L_2 be two abelian Lie algebras. Show that L_1 and L_2 are isomorphic if and only if they have the same dimension.

1.12.† Find the structure constants of $\mathsf{sl}(2, F)$ with respect to the basis given by the matrices

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \; f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \; h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

1.13. Prove that $\mathsf{sl}(2, \mathbf{C})$ has no non-trivial ideals.

1.14.† Let L be the 3-dimensional *complex* Lie algebra with basis (x, y, z) and Lie bracket defined by

$$[x, y] = z, \; [y, z] = x, \; [z, x] = y.$$

(Here L is the "complexification" of the 3-dimensional real Lie algebra $\mathbf{R}^3_\wedge$.)

(i) Show that L is isomorphic to the Lie subalgebra of $\mathsf{gl}(3, \mathbf{C})$ consisting of all 3×3 antisymmetric matrices with entries in $\mathbf{C}$.

(ii) Find an explicit isomorphism $\mathsf{sl}(2, \mathbf{C}) \cong L$.

1.15. Let S be an $n \times n$ matrix with entries in a field F. Define

$$\mathsf{gl}_S(n, F) = \{x \in \mathsf{gl}(n, F) : x^t S = -Sx\}.$$

(i) Show that $\mathsf{gl}_S(n, F)$ is a Lie subalgebra of $\mathsf{gl}(n, F)$.

(ii) Find $\mathsf{gl}_S(2, \mathbf{R})$ if $S = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

(iii) Does there exist a matrix S such that $\mathsf{gl}_S(2, \mathbf{R})$ is equal to the set of all diagonal matrices in $\mathsf{gl}(2, \mathbf{R})$?

(iv) Find a matrix S such that $\mathsf{gl}_S(3, \mathbf{R})$ is isomorphic to the Lie algebra $\mathbf{R}^3_\wedge$ defined in §1.2, Example 1.

Hint: Part (i) of Exercise 1.14 is relevant.

1.16.† Show, by giving an example, that if F is a field of characteristic 2, there are algebras over F which satisfy (L1$'$) and (L2) but are not Lie algebras.

1.17. Let V be an n-dimensional complex vector space and let $L = \mathsf{gl}(V)$. Suppose that $x \in L$ is diagonalisable, with eigenvalues $\lambda_1, \ldots, \lambda_n$. Show that $\operatorname{ad} x \in \mathsf{gl}(L)$ is also diagonalisable and that its eigenvalues are $\lambda_i - \lambda_j$ for $1 \le i, j \le n$.

1.18. Let L be a Lie algebra. We saw in §1.6, Example 1.2(2) that the maps $\operatorname{ad} x : L \to L$ for $x \in L$ are derivations of L; these are known as *inner derivations*. Show that if $\operatorname{IDer} L$ is the set of inner derivations of L, then $\operatorname{IDer} L$ is an ideal of $\operatorname{Der} L$.

1.19. Let A be an algebra and let $\delta : A \to A$ be a derivation. Prove that δ satisfies the Leibniz rule

$$\delta^n(xy) = \sum_{r=0}^{n} \binom{n}{r} \delta^r(x)\delta^{n-r}(y) \quad \text{for all } x, y \in A.$$