6. Universal Generating Function in Analysis and Optimization of Special Types of Multi-state System

6.1 Multi-state Systems with Bridge Structure

The bridge structure (Figure 6.1) is an example of a complex system for which the *u*-function cannot be evaluated by decomposing it into series and parallel subsystems. Each of the five bridge components can in turn be a complex composition of the elements. After obtaining the equivalent *u*-functions of these components one should apply the general composition operator in the form (1.20) over all five *u*-functions of the components in order to obtain the *u*-function of the entire bridge. The choice of the structure function in this composition operator depends on the type of system.

By having the *u*-function of the entire bridge system, one can use it either directly for evaluating the system performance measures (as shown in Section 3.3) or use it as a *u*-function of an equivalent element that replaces the bridge structure for evaluating the *u*-function of a higher level system when applying the block diagram method.

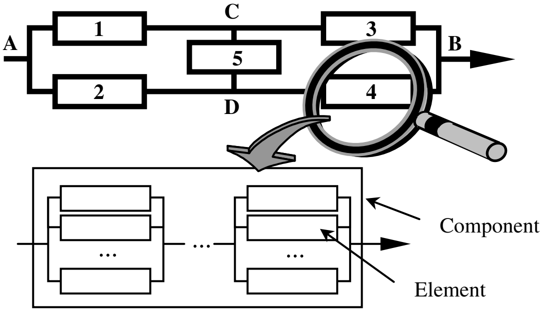


Figure 6.1. Bridge structure

6.1.1 *u*-function of Bridge Systems

6.1.1.1 Flow Transmission Multi-state Systems

In order to evaluate the performance of the flow transmission MSS with the flow dispersion, consider the flows through the bridge structure presented in Figure 6.1. First, there are two parallel flows through components 1, 3 and 2, 4. To determine the capacities of each of the parallel substructures composed from components connected in series, the function ϕ_{ser} (4.2) should be used. The function ϕ_{par} (4.9) should be used afterwards to obtain the total capacity of the two parallel substructures. Therefore, the structure function of the bridge, which does not contain diagonal component, is

$$\phi(G_1, G_2, G_3, G_4) = \phi_{\mathrm{par}}(\phi_{\mathrm{ser}}(G_1, G_3), \phi_{\mathrm{ser}}(G_2, G_4)) \tag{6.1}$$

and its total capacity for the flow transmission MSS with the flow dispersion is equal to $\min\{G_1, G_3\}+\min\{G_2, G_4\}$.

The surplus of the transferred product on one of end nodes of component 5 can be expressed as

$$s = \max\{(G_1-G_3),(G_2-G_4), 0\} \tag{6.2}$$

and the deficit of the transferred product on one of the end nodes of component 5 can be expressed as

$$d = \max\{(G_3-G_1), (G_4-G_2), 0\} \tag{6.3}$$

The necessary condition for the existence of the flow through component 5 is the simultaneous existence of a surplus on one end node and a deficit on the other end: $s \neq 0$, $d \neq 0$. This condition can be expressed as $(G_1-G_3)(G_2-G_4)<0$.

If the condition is met, the flow through the component 5 will transfer the amount of the product which cannot exceed the capacity of the component G_5 and the amount of the surplus product s. The deficit d on the second end of component 5 is the amount of the product that can be transferred by the component that follows the diagonal (component 3 or 4). Therefore, the flow through the diagonal component is also limited by d. Thus, the maximal flow through the diagonal component is $\min\{s, d, G_5\}$.

Now we can determine the total capacity of the bridge structure when the capacities of its five components are given:

$$\begin{aligned}\phi_{\mathrm{br}}(G_1, G_2, G_3, G_4, G_5) &= \min\{G_1, G_3\}+\min\{G_2, G_4\}\\ &+ \min\{|G_1-G_3|, |G_2-G_4|, G_5\} \times 1((G_1-G_3)(G_2-G_4)<0)\end{aligned} \tag{6.4}$$

Now consider the performance of the flow transmission MSS without flow dispersion. In such a system a single path between points A and B providing the greatest flow should be chosen. There exist four possible pathsconsisting of groups of components (1, 3), (2, 4), (1, 5, 4) and (2, 5, 3) connected in a series. The transmission capacity of each path is equal to the minimum transmission capacity of the elements belonging to this path. Therefore, the structure function of the entire bridge takes the form

$$\phi_{br}(G_1, G_2, G_3, G_4, G_5) = \max\{\min\{G_1,G_3\} \\ \min\{G_2,G_4\}, \min\{G_1,G_5,G_4\}, \min\{G_2,G_5,G_3\}\} \quad (6.5)$$

Note that the four parallel subsystems (paths) are not statistically independent, since some of them contain the same elements. Therefore, the bridge u-function cannot be obtained by system decomposition as for the series-parallel systems. Instead, one has to evaluate the structure function (6.5) for each combination of states of the five independent components.

6.1.1.2 Task Processing Multi-state Systems

In these types of system a task is executed consecutively by components connected in series. No stage of work execution can start until the previous stage is entirely completed. Therefore, the total processing time of the group of elements connected in series is equal to the sum of the processing times of the individual elements.

First, consider a system without work sharing in which the parallel components act in a competitive manner. There are four alternative sequences of task execution (paths) in a bridge structure. These paths consist of groups of components (1, 3), (2, 4), (1, 5, 4) and (2, 5, 3). The total task can be completed by the path with a minimal total processing time

$$T = \min\{t_1+t_3, t_2+t_4, t_1+t_5+t_4, t_2+t_5+t_3\} \quad (6.6)$$

where t_j and $G_j = 1/t_j$ are respectively the processing time and the processing speed of element j.

The entire bridge performance defined in terms of its processing speed can be determined as

$$G = 1/T = \phi_{br}(G_1, G_2, G_3, G_4, G_5) \\ = \max\{\phi_{ser}(G_1,G_3), \phi_{ser}(G_2,G_4), \phi_{ser}(G_1,G_4,G_5), \phi_{ser}(G_2,G_3,G_5))\} \quad (6.7)$$

where ϕ_{ser} is defined in Equation (4.5).

Now consider a system with work sharing for which the same three assumptions that were made for the parallel system with work sharing (Section 4.1.2) are made. There are two stages of work performing in the bridge structure. The first stage is performed by components 1 and 2 and the second stage is performed by components 3 and 4. The fifth component is necessary to transfer work between nodes C and D. Following these assumptions, the decision about work sharing can be made in the nodes of bridge A, C or D only when the entire amount of work is available in this node. This means that components 3 or 4 cannot start task processing before both the components 1 and 2 have completed their tasks and all of the work has been gathered at node C or D.

There are two ways to complete the first stage of processing in the bridge structure, depending on the node in which the completed work is gathered. To complete it in node C, the amount of work $(1-\alpha)x$ should be performed by component 1 with processing speed G_1 and the amount of work αx should be performed by component 2 with processing speed G_2 and then transferred from node D to node C with speed G_5 (α is the work sharing coefficient). The time the work performed by component 1 appears at node C is $t_1 = (1-\alpha)x/G_1$. The time the work performed by component 2 and transferred by component 5 appears at node C is t_2+t_5, where $t_2 = \alpha x/G_2$ and $t_5 = \alpha x/G_5$. The total time of the first stage of processing is $T_{1C} = \max\{t_1, t_2+t_5\}$. It can be easily seen that T_C is minimized when the α is chosen that provides equality $t_1 = t_2+t_5$. The work sharing coefficient obtained from this equality is $\alpha = G_2G_5/(G_1G_2+G_1G_5+G_2G_5)$ and the minimal processing time is

$$T_{1C} = x(G_2+G_5)/(G_1G_2+G_1G_5+G_2G_5) \tag{6.8}$$

To complete the first stage of processing in node D, the amount of work $(1-\beta)x$ should be performed by component 2 with processing speed G_2 and the amount of work βx should be performed by component 1 with processing speed G_1 and then transferred from node C to node D with speed G_5. The minimal possible processing time can be obtained in the same manner as T_{1C}. This time is

$$T_{1D} = x(G_1+G_5)/(G_1G_2+G_1G_5+G_2G_5) \tag{6.9}$$

If the first stage of processing is completed in the node C, then the amount of work $(1-\gamma)x$ should be performed by component 3 in the second stage of processing, which takes time $t_3 = (1-\gamma)x/G_3$. The rest of the work γx should be first transferred to node D by component 5 and then performed by component 4. This will take time $t_5+t_4 = \gamma x/G_5+\gamma x/G_4$. Using the optimal work sharing (when $t_3 = t_4+t_5$) with $\gamma = G_4G_5/(G_3G_4+G_3G_5+G_4G_5)$ we obtain the minimal time of the second stage of processing:

$$T_{2C} = x(G_4+G_5)/(G_3G_4+G_3G_5+G_4G_5) \tag{6.10}$$

Using the same technique we can obtain the minimal processing time when the second stage of processing starts from node D:

$$T_{2D} = x(G_3+G_5)/(G_3G_4+G_3G_5+G_4G_5) \tag{6.11}$$

Assuming that the optimal way of work performing can be chosen in node A, we obtain the total bridge processing time T as

$$T = \min\{T_{1C}+T_{2C}, T_{1D}+T_{2D}\} \tag{6.12}$$

where

$$T_{1C}+T_{2C} = x[(G_2+G_5)/\sigma+(G_4+G_5)/\pi]$$

$$T_{1D}+T_{2D} = x[(G_1+G_5)/\sigma+(G_3+G_5)/\pi]$$

$$\sigma = G_1G_2+G_1G_5+G_2G_5$$

$$\pi = G_3G_4+G_3G_5+G_4G_5$$

The condition $T_{1C}+T_{2C} \le T_{1D}+T_{2D}$ is satisfied when $(G_2-G_1)\pi \le (G_3-G_4)\sigma$.

The expressions obtained can be used to estimate the processing speed of the entire bridge:

$$G = 1/T = \phi_{br}(G_1, G_2, G_3, G_4, G_5)=\sigma\pi/[(f+G_5)\sigma+(e+G_5)\pi] \tag{6.13}$$

where

$$f = G_4,\ e = G_2 \ \text{ if } \ (G_2-G_1)\pi \le (G_3-G_4)\sigma$$

$$f = G_3,\ e = G_1 \ \text{ if } \ (G_2-G_1)\pi > (G_3-G_4)\sigma$$

6.1.1.3 Simplification Technique

Note that in the special case when one of the bridge elements is in a state of total failure, the bridge structure degrades to a series-parallel one. All five possible configurations of this degraded bridge are presented in Figure 6.2.

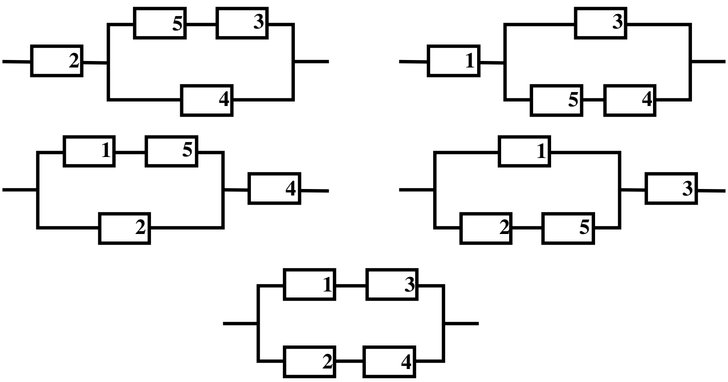


Figure 6.2. Degraded bridge structures in the case of single-element total failure

There is no need to use Equations (6.4), (6.5), (6.7) or (6.13) in order to evaluate the structure function of the bridge when one of the random values $G_1,\ldots,G_5$ is equal to zero. A simpler way to evaluate it is by using the reliability block diagram technique.

The following simplification rules can be used when more than one element is in a state of total failure:

1. If $G_1 = G_2 = 0$ or $G_3 = G_4 = 0$ the total bridge performance is equal to zero.

2. If $G_1 = G_4 = G_5 = 0$ or $G_2 = G_3 = G_5 = 0$ the total bridge performance is equal to zero.
3. If any two out of three random values composing groups $\{G_1, G_3, G_5\}$ or $\{G_2, G_4, G_5\}$ are equal to zero, the third value can also be zeroed. In this case, the bridge is reduced to two components connected in a series (2, 4 and 1, 3 respectively).

Example 6.1

Consider a bridge consisting of five elements with performance distributions presented in Table 6.1. The elements can have up to three states.

Table 6.1. Performance distributions for bridge elements

No. of element	Component performance distribution					
	State 0		State 1		State 2	
	g	*p*	*g*	*p*	*g*	*p*
1	0	0.10	6	0.60	8	0.30
2	0	0.05	7	0.95	-	-
3	0	0.10	4	0.10	6	0.80
4	0	0.05	6	0.20	9	0.75
5	0	0.15	2	0.85	-	-

The reliability and the performance deficiency for this bridge structure as functions of system demand are presented in Figure 6.3 for the structure interpreted as a system of four different types (numbered according to Table 4.4). The values of the expected performances obtained for these four different systems are ε = 8.23, ε = 6.33, ε = 3.87 and ε = 3.55.

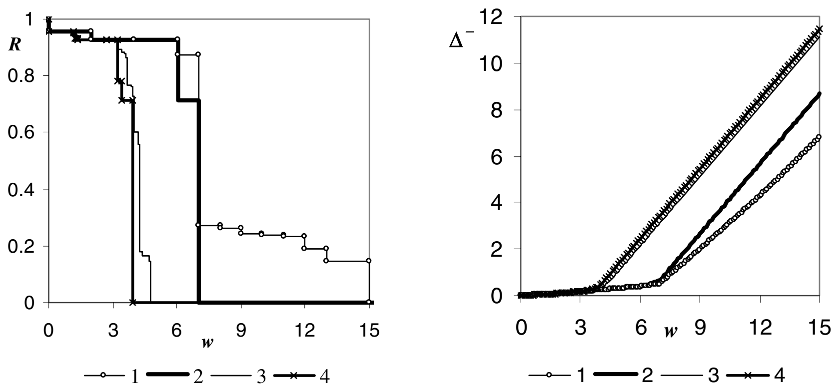


Figure 6.3. Reliability and performance deficiency for different types of bridge MSS

6.1.2 Structure Optimization of Bridge Systems

By having the technique for determining the *u*-function of bridge subsystems, one can apply it for solving structure optimization problems for systems with complex topology consisting of series-parallel and bridge subsystems. The formulation of the generalized structure optimization problem is similar to that presented in Section 5.1.2.1:

An MSS consists of N components connected in series, in parallel, or composing a bridge (according to a given reliability block diagram). Each component is a subsystem that can consist of parallel elements with the same functionality. For each component i, different versions of elements may be chosen from the list of element versions available in the market. The optimal solution corresponds to the minimal cost system configuration that provides the desired level of the given system performance measure.

When the system reliability is optimized and no constraints are imposed on the system configuration, the solutions for the structure optimization problem for the bridge system always produce a degraded series-parallel system. This happens because, from the reliability point of view, building a system with bridge topology is not justified. Indeed, when no allocation constraints are imposed on the system, the series-parallel solution is more reliable and less expensive than one based on the bridge architecture. One can see that for an arbitrary bridge structure (Figure 6.1) the less expensive and more reliable solution can be obtained by uniting elements of components 1 and 2 in component 1 and elements of components 3 and 4 in component 3 and by removing diagonal component 5. The existence of the bridge systems is justified only when some constraints are imposed on the allocation of the system elements or when the elements belonging to the same component are subject to CCFs.

6.1.2.1 Constrained Structure Optimization Problems

Connecting the system components in a bridge topology is widely used in design practice. There can be many different reasons for a system to take the bridge form. For example:

- the bridge configuration of the system is determined by factors not related to its reliability;

- in order to provide the redundancy on the component level the system should have parallel functionally equivalent components;

- the system contains parallel functionally equivalent but incompatible components;

- the number of elements that can be allocated within each component is limited.

In order to take into account such constraints, when the optimal system configuration is determined one has to modify the objective function in a way that penalizes the constraint violation. The methodology of solving the structure optimization problem for the systems containing series-parallel and bridge structures presumes using the optimization problem definition and GA implementation technique presented in Sections 5.1.2.1 and 5.1.2.2 and including the corresponding penalties to the solution fitness.

The solution decoding procedure, based on the UGF technique, performs the following steps:

1. Determines number of chosen elements n_{ib} for each system component and each element version from the string $\boldsymbol{a}$.

2. Determines u-functions $u_{ib}(z)$ of each version of elements according to their PD $\boldsymbol{g}_i(b)$, $\boldsymbol{p}_i(b)$.

3. Determines u-functions of each component i ($1\le i\le N$) by applying the composition operator $\otimes_{\varphi_{\text{par}}}$ over u-functions of the elements belonging to this component.

4. Determines the u-function of the entire MSS $U(z)$ by applying the corresponding composition operators using the reliability block diagram method and composition operators $\otimes_{\varphi_{\text{par}}}$, $\otimes_{\varphi_{\text{ser}}}$ and $\otimes_{\varphi_{\text{br}}}$.

5. Having the u-function of the entire system and its components, determines their performance measures as described in Section 3.3.

6. Determines the total system cost using Equation (5.2).

7. Determines the solution's fitness as a function of the MSS cost and performance measure as

$$M-C(\boldsymbol{a})-\pi_1(1+|O-O^*|)(1-f(O,O^*))-\pi_2\eta \tag{6.14}$$

where π_1 and π_2 are penalty coefficients, M is a constant value and η is a measure of constraint violation. For example, if it is important to provide the expected performance ε_j of each bridge component j ($1\le j\le 5$) at a level not less than ε_j^*, then η takes the form

$$\eta=\sum_{j=1}^{5}\max(\varepsilon_j^*-\varepsilon_j,0) \tag{6.15}$$

If no more than I_j elements can be allocated in each bridge component j, then η takes the form

$$\eta=\sum_{j=1}^{5}\max(\sum_{b=1}^{B_j}n_{jb}-I_j,0) \tag{6.16}$$

If a pair of parallel components j and i should provide an identical nominal performance (the components consist of two state elements with nominal performances $g_{j1}(b)$ and $g_{i1}(b)$, then η takes the form

$$\eta=\left|\sum_{b=1}^{B_j}n_{jb}g_{j1}(b)-\sum_{b=1}^{B_i}n_{ib}g_{i1}(b)\right| \tag{6.17}$$

Example 6.2

Consider a power station coal transportation system that receives the coal carried by sea [164]. There may be up to two separate piers where a ship with coal can be berthed. Each pier should be provided with a separate coal transportation line. The lines can be connected by an intermediate conveyor in order to balance their load.

This flow transmission system (with flow dispersion) contains six basic components:

1, 2. Coal unloading terminals, including a number of travelling rail cranes for the ship discharge with adjoining primary conveyors.

3, 4. Secondary conveyors that transport the coal to the stacker-reclaimer.

5. An intermediate conveyor that can be used for load balancing between the lines.

6. The stacker-reclaimer that transfers the coal to the boiler feeders.

Each element of the system is considered to be a two-state unit. Table 6.2 shows availability, nominal capacity, and unit cost for equipment available in the market. The system demand is $w = 1$.

First consider an optimization problem in which the number of cranes at any pier cannot exceed three because of allocation constraints. The penalty (6.16) with $I_1 = I_2 = 3$ and $I_3 = I_4 = I_5 = \infty$ is incorporated into the solution fitness function.

Table 6.2. Characteristics of available system elements

Component	Description	No. of version	g	p	c
1,2	Crane with primary conveyor	1	0.80	0.930	0.750
		2	0.60	0.920	0.590
		3	0.60	0.917	0.535
		4	0.40	0.923	0.400
		5	0.40	0.860	0.330
		6	0.25	0.920	0.230
3,4	Secondary conveyor	1	0.70	0.991	0.325
		2	0.70	0.983	0.240
		3	0.30	0.995	0.190
		4	0.25	0.936	0.150
5	Intermediate conveyor	1	0.70	0.971	0.1885
		2	0.60	0.993	0.1520
		3	0.40	0.981	0.1085
		4	0.20	0.993	0.1020
		5	0.10	0.990	0.0653
6	Stacker-reclaimer	1	1.30	0.981	1.115
		2	0.60	0.970	0.540
		3	0.30	0.990	0.320

The results obtained for different values of required availability A^* are presented in Figure 6.4 (each element is marked by its version number). One can see the modifications of the optimal structure of the system corresponding to different levels of the availability provided. The simple series-parallel system appears to be optimal when providing relatively little availability. In this case, the single-pier system satisfies the availability requirement (note that in this case there are no constraints on the interchangeability of the piers, on the number of piers, or on the number of ships to be discharged simultaneously).

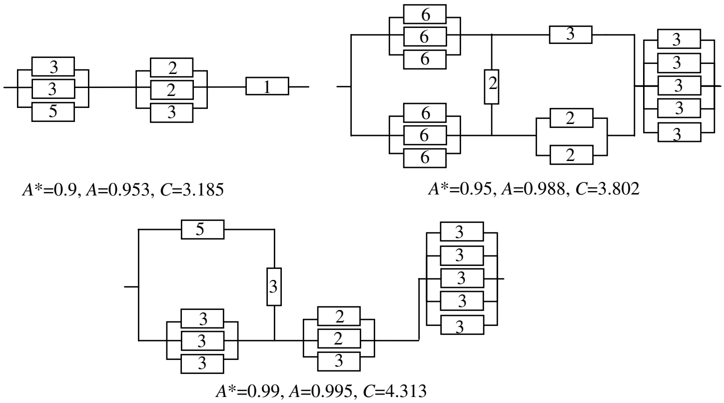


Figure 6.4. Optimal structures for a problem with allocation constraints

The bridge system becomes the best solution as A^* grows and, finally, for $A^* = 0.99$ the system returns to the series-parallel configuration. In this case the necessary redundancy of the ship discharge facilities is provided by the second pier connected with the single coal transportation line by the intermediate conveyor.

Now consider the problem in which the equality of the total installed capacities of components 1 and 2 is required to make the piers interchangeable (symmetry constraint). To meet this requirement, the additional penalty (6.17) with $j = 1$ and $i = 2$ is added to the solution fitness function.

The results obtained for different desired values of the system availability A^* are presented in Figure 6.5. One can see that the load-balancing diagonal element (intermediate conveyor) appears only in the solutions with relatively high availability.

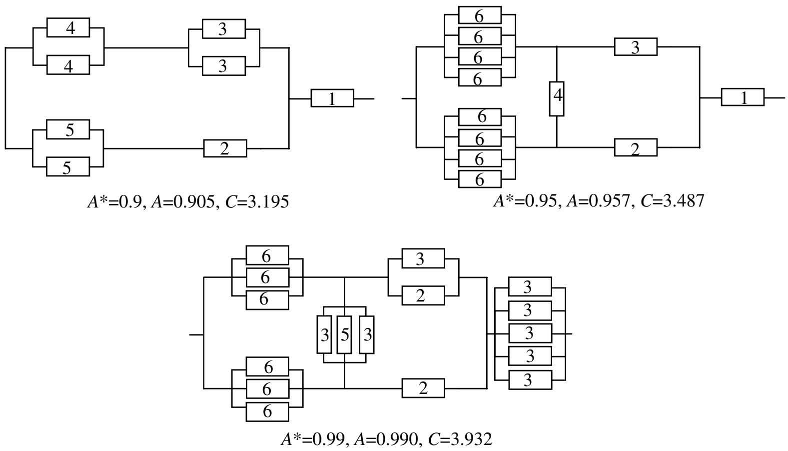


Figure 6.5. Optimal structures for a problem with symmetry constraints

For the next example consider the existing obsolete coal transportation system consisting of elements with low availability. This system can supply the boiler with availability A = 0.532. The parameters of the components of the existing system are presented in Table 6.3. The problem is to achieve a desired availability level A^* by including additional elements from Table 6.2 into the system.

Table 6.3. Structure of the obsolete system

Component	No. of parallel elements	Parameters of elements	
		g	p
1	2	0.30	0.918
2	2	0.30	0.918
3	1	0.60	0.907
4	1	0.60	0.903
5	0	-	-
6	1	1.00	0.911

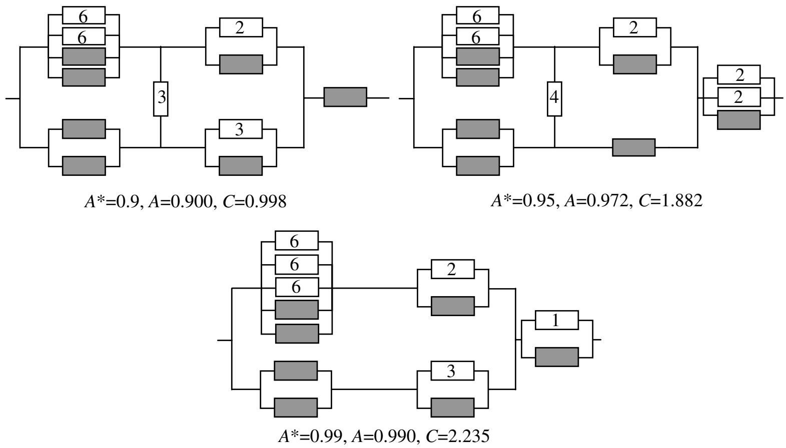


Figure 6.6. Optimal structures for the system extension problem

The minimal cost solutions of the system extension problem in which the cost of additional elements alone is considered obtained for this type of problem are presented in Figure 6.6 (the elements belonging to the initial system are depicted by grey rectangles).

Example 6.3

An alarm data-processing system has processing units of different types and data transmission/conversion facilities (see Figure 6.7). The system is composed of the following subsystems:

- primary data-processing subsystem containing components 1 and 2;
- alarm-type recognition subsystem containing components 3 and 4;
- data transmission subsystem (component 6);
- remote output data processing subsystem containing components 7 and 8;

- decision-support subsystem containing components 9 and 10.

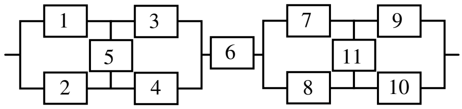


Figure 6.7. Structure of the alarm data-processing system

Table 6.4. Characteristics of available system elements

Component	Description	No. of version	g	p	c
1,2		1	8.0	0.830	0.750
		2	6.0	0.820	0.590
		3	6.0	0.807	0.535
		4	4.0	0.860	0.400
		5	4.0	0.825	0.330
		6	2.5	0.820	0.230
3,4		1	7.0	0.821	0.325
		2	7.0	0.803	0.240
		3	3.0	0.915	0.190
		4	2.5	0.806	0.150
5	Data transmission (conversion) units	1	20.0	0.871	0.1885
		2	18.0	0.893	0.1520
		3	14.0	0.881	0.1085
		4	12.0	0.893	0.1020
		5	11.0	0.890	0.0653
6	Data transmission line	1	11.3	0.881	1.115
		2	6.6	0.870	0.540
		3	4.3	0.890	0.320
7,8	Output data processing unit	1	3.2	0.801	0.750
		2	2.8	0.827	0.590
		3	2.8	0.801	0.535
9,10	Decision-support unit	1	5.7	0.891	0.325
		2	5.7	0.813	0.240
		3	5.3	0.925	0.220
		4	4.25	0.836	0.150
11	Data transmission (conversion) units	1	70.0	0.971	0.031
		2	60.0	0.993	0.025
		3	40.0	0.981	0.020

Pairs of components {1, 3}, {2, 4}, {7, 9} and {8, 10} have compatible data exchange protocols, whereas data transmission between pairs of components {1, 4}, {2, 3} and {7, 10}, {8, 9} requires its conversion, which can be performed by components 5 and 11 respectively. The set of available versions of two-state elements for each system component is presented in Table 6.4. The system performance (processing speed) should be no less than $w = 1$. The structure optimization problems were solved for two types of this task processing system: with and without work sharing [165].

First, the solutions of the unconstrained optimization problem were obtained for values of desired system reliability $R^* = 0.95$ and $R^* = 0.99$. The parameters of solutions obtained are presented in Table 6.5 and the optimal system structures are presented in Figures 6.8 and 6.9 (structures A and C). One can see that the simple series-parallel system appears to be optimal in all of the cases.

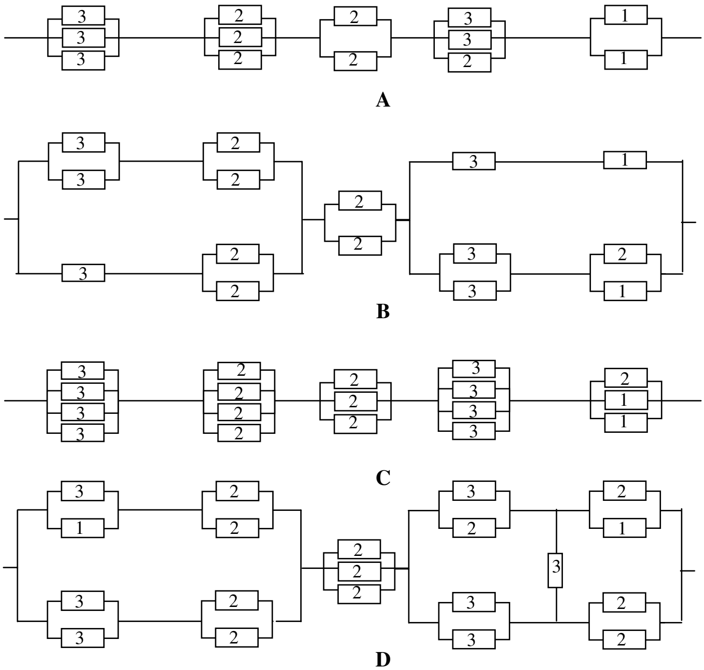


Figure 6.8. Optimal structures of system without work sharing.
A: R^* = 0.95, no constraints; B: R^* = 0.95, constraints;
C: R^* = 0.99, no constraints; D: R^* = 0.99, constraints

To force the GA to obtain the solution based on bridge structures, allocation constraints were imposed that forbid allocation of more than two parallel elements in all of the components except 5, 6 and 11. The parameters of the solutions obtained are also presented in Table 6.5 and the optimal system structures are presented in Figures 6.8 and 6.9 (structures B and D). The modification of the structure of the system without work sharing for different levels of required reliability is apparent. The transmission/conversion units appear unnecessary when providing $R>0.95$, but one such unit is included in the system providing $R>0.99$. For the system with work sharing, the use of a bridge diagonal element is more justified because its contribution to the total performance increases.

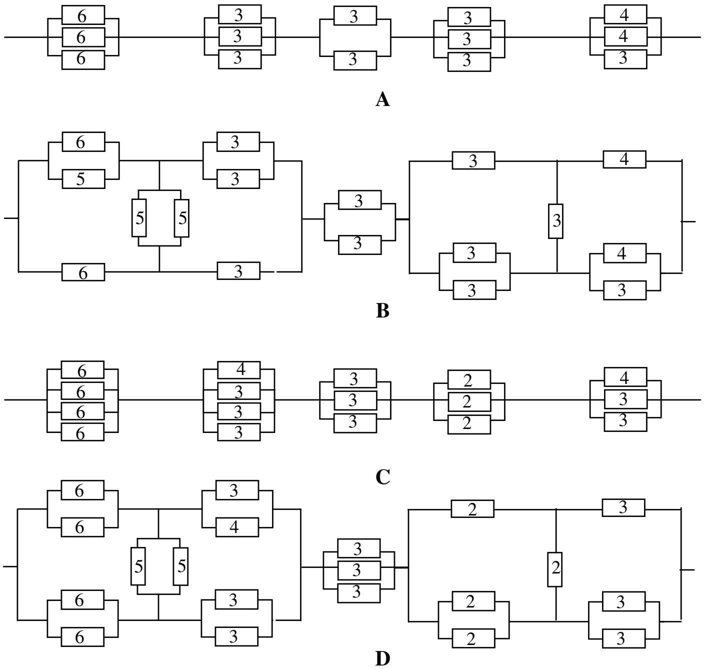


Figure 6.9. Optimal structures of system with work sharing.
A: $R^* = 0.95$, no constraints; B: $R^* = 0.95$, constraints;
C: $R^* = 0.99$, no constraints; D: $R^* = 0.99$, constraints

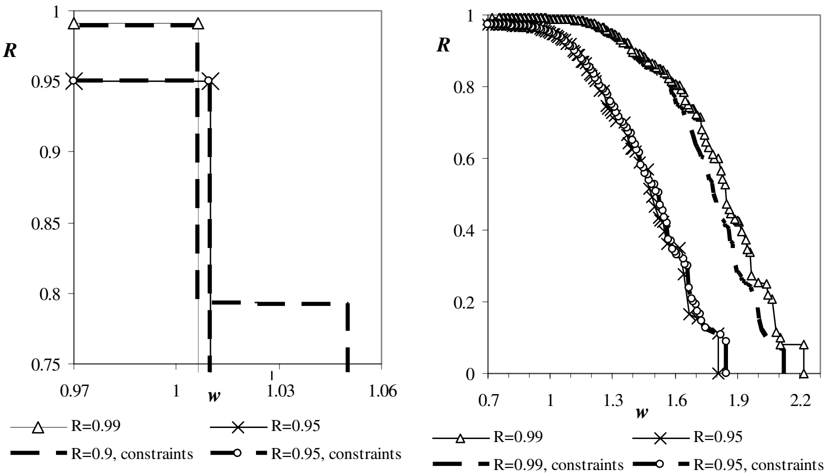
The work sharing allows the system to perform faster. Therefore, less expensive solutions were obtained for this type of system than for the system without work sharing.

Table 6.5. Parameters of the obtained solutions

R^*	0.99		0.95	
Constraints	No	Yes	No	Yes
	System without work sharing			
C	7.750	8.195	5.715	6.140
R	0.9911	0.9901	0.9505	0.9503
ε	0.997	1.032	0.960	0.960
	System with work sharing			
C	4.960	5.1856	4.02500	4.2756
R	0.9901	0.9901	0.9511	0.9518
ε	1.801	1.744	1.425	1.445

The systems without work sharing have few different possible levels of performance because the operator $\otimes_{\max}$ used in this case provides the same performance level for many different system states. In the solutions presented, only the system obtained for $R^* = 0.99$ with allocation constraints (Figure 6.8D) has different performance levels. The remainder have the single possible nonzero performance level because their components contain elements with the same nominal performance rates. The system performance distributions for systems without work sharing are presented in Figure 6.10A.

In contrast, the systems with work sharing have many possible performance values depending on their states, because the failure of each element affects the ability of the corresponding component to participate in the work sharing. The performance distributions for systems with work sharing are presented in Figure 6.10B.



A B

Figure 6.10. System reliability as a function of demand.
A: system without work sharing; B: system with work sharing

6.1.2.2 Structure Optimization in the Presence of Common Cause Failures

When the system components are subject to CCFs, the separation of elements among different parallel components can improve the overall system survivability (see Section 5.2.1). Such separation can justify the appearance of a bridge structure.

Consider, for example, an MSS containing two components A and B connected in series (Figure 6.11A). The components consist of M and L different elements respectively. All elements belonging to the same component are of the same functionality and are connected in parallel. The components A and B are subject to total CCF (they can be destroyed by hostile environments with the probabilities v_A and v_B respectively). The component destruction means that all of its elements are damaged and cannot perform their task. The system survivability is defined as the probability that a given demand w is met. This probability is affected by both the failures of the elements and the vulnerability of the components.

To enhance the system's survivability its components can be separated into two independent subcomponents. Let $\{1, \ldots, M\}$ and $\{1, \ldots, L\}$ be sets of numbers of elements belonging to components A and B respectively The elements' separation problem can be considered as a problem of partitioning these sets inttwo mutually disjoint subsets. The partition can be represented by the binary vectors $\boldsymbol{x}_A = \{x_{Aj}: 1\le j\le M\}$ and $\boldsymbol{x}_B = \{x_{Bj}: 1\le j\le L\}$, where $x_{Aj}, x_{Bj}\in\{0,1\}$ and two elements i and j belong to the same subset if and only if $x_{Ai} = x_{Aj}$ or $x_{Bi} = x_{Bj}$ for components A and B respectively.

Actually, the separation leads to the appearance of two independent parallel subsystems containing components connected in series: 1, 3 and 2, 4 (see Figure 6.11C). In these systems, components 1 and 2 have the same vulnerability as the basic component A, and components 3 and 4 have the same vulnerability as the basic component B. Usually, the separation disconnects component 1 from component 4 and component 2 from component 3 (for example, when components 1 and 3 are spatially separated from components 2 and 4). To provide a connection between these components, a diagonal component 5 can be included that delivers an output of component 1 to the input of component 4 or an output of component 2 to the input of component 3. The diagonal component can also consist of different multi-state elements. This component can also be characterized by its vulnerability v_5.

The problem of bridge structure optimization is to find the optimal separation $\boldsymbol{x}_A$, $\boldsymbol{x}_B$ of elements from components A and B which provides the maximal system survivability for the given demand w when the structure of the diagonal component is given:

$$S(\boldsymbol{x}_A, \boldsymbol{x}_B, v_A, v_B, v_5, w) \to \max \tag{6.18}$$

The separation solution can be represented in the GA by the binary string $\boldsymbol{a}$, which is a concatenation of the binary vectors $\boldsymbol{x}_A$ and $\boldsymbol{x}_B$. The following procedure determines the fitness value for an arbitrary solution defined by the string $\boldsymbol{a}$.

1. According to $\boldsymbol{a}$, determine lists of elements belonging to components 1, 2, 3 and 4.
2. For $1\le i\le 5$, determine the u-function of the entire component i using the composition operator $\otimes_{\varphi_{par}}$ over the u-functions of elements belonging to this component (the list of elements belonging to the diagonal component is given).
3. In order to incorporate the component vulnerability into its u-function, apply the ξ operator (4.58) with $v = v_i$ over the u-function of the component.
4. Determine the u-function of the entire bridge system using the composition operator $\otimes_{\varphi_{br}}$ over the u-functions of its components.
5. Determine the system survivability index S for the given demand w and evaluate the solution fitness as $S(\boldsymbol{a})$.

Example 6.4

Consider a system initially consisting of two components connected in series (see Figure 6.11A). The first component consists of 12 two-state elements and the second consists of eight two-state elements. All of the elements within each component are connected in parallel. The sets of elements belonging to the components consist of pairs of identical elements (six pairs for the first component and four pairs for the second one). This provides the possibility of symmetric separation. Parameters of elements of each pair are presented in Table 6.6.

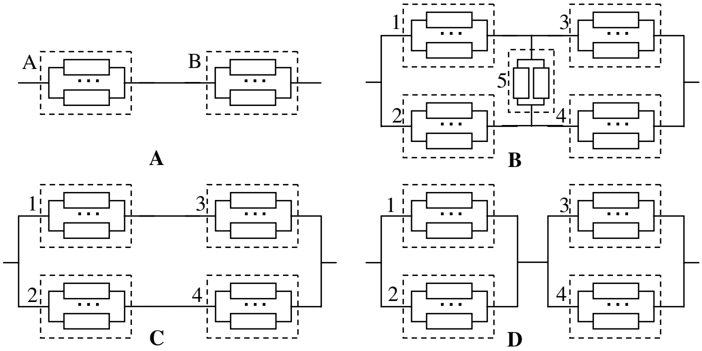


Figure 6.11. Special cases of bridge structure

It is assumed that the elements belonging to each component are subject to total CCFs (external impact) and the probability of the CCF (component vulnerability) is the same for each component.

Table 6.6. Parameters of MSS elements

No. of component	No. of element	G	p
1, 2	1	8.0	0.830
	2	6.0	0.820
	3	6.0	0.807
	4	4.0	0.860
	5	4.0	0.825
	6	2.0	0.820
3, 4	1	15.0	0.821
	2	10.0	0.803
	3	10.0	0.815
	4	5.0	0.806
5	1	6.0	0.871
	2	4.0	0.893

To enhance the system's survivability, each component can be divided into two subcomponents and a diagonal component consisting of two parallel elements can be included (Figure 6.11B). The parameters of the element belonging to diagonal component 5 are also presented in Table 6.6.

The system is interpreted first as a flow transmission one with flow dispersion and then as a task processing one without work sharing. The optimal solutions were obtained for both types of system for the same parameters of elements [166]. The system demands are $w = 25$ and $w = 4$ for the flow transmission system and the task processing system respectively. Solutions were obtained for three different values of component vulnerability: $v = 0$ (pure reliability optimization problem), $v = 0.05$ and $v = 0.5$.

The bridge structures providing the maximal survivability for the flow transmission system are presented in Table 6.7 and for the task processing system in Table 6.8. These solutions are compared with solutions having no component separation (Figure 6.11A) and with solutions having symmetric separation. The lists of elements belonging to each component, as well as the corresponding system survivability index $S(w)$ and mean performance ε, are presented for each solution.

Table 6.7. Solutions for flow transmission system (vulnerability variation)

Solution description	System structure		$v = 0$		$v = 0.05$		$v = 0.5$	
	No. of component	Elements included	S	ε	S	ε	S	ε
Optimal for $v = 0.05$	1	114466	0.9968	48.268	0.9333	43.957	0.2837	12.713
	2	223355						
	3	112						
	4	23344						
Symmetric	1	123456	0.9968	48.266	0.9244	43.969	0.2565	12.722
	2	123456						
	3	1234						
	4	1234						
Optimal for $v = 0$ (no separation)	1	112233445566	0.9974	48.518	0.9001	43.787	0.2493	12.129
	2	-						
	3	11223344						
	4	-						
Optimal for $v = 0.5$	1	23345566	0.9960	48.185	0.9319	43.870	0.2866	12.652
	2	1124						
	3	113						
	4	22344						

Table 6.8. Solutions for task processing system (vulnerability variation)

Solution description	System structure		$v = 0$		$v = 0.05$		$v = 0.5$	
	No. of component	Elements included	S	ε	S	ε	S	ε
Optimal for $v = 0.05$ and for $v = 0.5$	1	223346	0.9990	4.434	0.9841	4.381	0.4255	2.047
	2	114556						
	3	1144						
	4	2233						
Symmetric	1	123456	0.9983	5.135	0.9820	5.058	0.4221	2.325
	2	123456						
	3	1234						
	4	1234						
Optimal for $v = 0$ (no separation)	1	112233445566	0.9990	5.168	0.9016	4.664	0.2497	1.292
	2	-						
	3	11223344						
	4	-						

One can see that, when the probability of CCFs in the components is neglected ($v = 0$), the best solution is one without elements separation. The optimal solutions for different values of component vulnerability can differ (as in case of the flow transmission system, where the optimal solution for $v = 0.05$ is not optimal for $v = 0.5$ and *vice versa*). For both types of system the optimal solutions are not symmetric.

The solution that provides the maximal system survivability for a given demand w does not necessarily provide the greatest system mean performance. Indeed, the system resources are distributed in such a way that maximizes only the probability of demand w satisfaction, while the rest of the performance levels can be provided with probabilities lower than those obtained by an alternative solution. In Figure 6.12 one can see $S(w)$ functions for three different solutions for both types of system when $v = 0.05$. While the probability $\Pr\{G \geq w\}$ for the optimal solutions is maximal, the probabilities $\Pr\{G \geq w'\}$ for $w' > w$ is often greater for the symmetric solution and the solution without separation. Note that the greatest mean performance ε is achieved for the symmetric solutions for both types of system.

It is interesting that the optimal solution for the task processing system when $v = 0.05$ cannot even provide a processing speed $G > 4.44$, while the rest of the solutions provide a processing speed $G = 5.22$ with a probability close to 0.85. Indeed, the fastest elements in the optimal solution are located in components 1 and 3. Therefore, the path including only these two elements with nominal processing speeds $g_{11} = 8$ and $g_{31} = 15$ does not exist in this solution, since the two elements are not directly connected, though they are through the diagonal element. In the remainder of the solutions such a path exists. Creating the fastest path by exchanging elements between components 3 and 4 in the optimal solution improves the system's average processing speed (it grows from $\varepsilon = 4.381$ to $\varepsilon = 4.968$); however, this drastically decreases the system's survivability for the given demand $w = 4$ (from $S(4) = 0.984$ to $S(4) = 0.848$).

In order to estimate the effect of diagonal element parameters on the optimal separation, compare the optimal solution obtained above (Figure 6.11B) with two extreme cases. In the first case (Figure 6.11C) no diagonal component is available: $\Pr\{G_5 = 0\} = 1$. In the second case (Figure 6.11D), no capacity limitations are imposed on the fully reliable diagonal component: $\Pr\{G_5 = \infty\} = 1$. The solutions obtained for both types of system are presented in Tables 6.9 and 6.10.

Table 6.9. Solutions for flow transmission system (different diagonal elements)

Solution description	System structure		Case B		Case C		Case D	
	No. of component	Elements included	S	ε	S	ε	S	ε
Optimal for case B	1	114466	0.9333	43.957	0.9038	42.306	0.9432	44.505
	2	223355						
	3	112						
	4	23344						
Optimal for cases C and D	1	1124	0.9328	43.911	0.9053	42.202	0.9445	44.500
	2	23345566						
	3	112						
	4	23344						

One can see that the optimal separation solutions coincide in cases B and C for the flow transmission system and in cases C and D for the task processing system. Obviously, the highest system survivability is achieved for case D and the lowest for case C.

Table 6.10. Solutions for task processing system (different diagonal elements)

Solution description	System structure		Case B		Case C		Case D	
	No. of component	Elements included	S	ε	S	ε	S	ε
Optimal for cases B and C	1	223346	0.9841	4.381	0.9841	4.370	0.9862	5.026
	2	114556						
	3	1144						
	4	2233						
Optimal for case D	1	1234455	0.9820	5.057	0.9820	5.045	0.9914	5.113
	2	12366						
	3	123						
	4	12344						

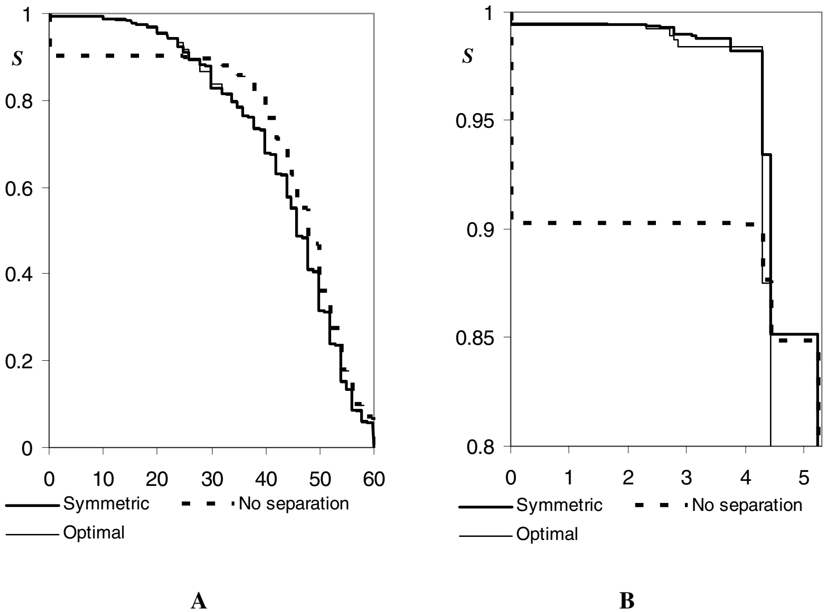


Figure 6.12. System survivability as function of demand for different element separation solutions in flow transmission system (A) and in task processing system (B)

6.2 Multi-state Systems with Two Failure Modes

Systems with two failure modes consist of statistically independent devices (elements) that are all to operate in the same two modes (the operation commands in each mode arrive at all the elements simultaneously). Each element can fail in either of two modes. A typical example of systems with two failure modes are switching systems that not only can fail to close when commanded to close but can also fail to open when commanded to open. Two different types of switching system can be distinguished:

- flow transmission systems, in which the main characteristic of each switching device is the flow controlled by this device (for example, fluid flow valve);
- task processing systems, in which the main characteristic of each switching device is the switching time of this device (for example, electronic diode).

The study of the systems with two failure modes started as early as in the 1950th [167-169] and still attracts interest of researchers [170-173]. In 1984, Barlow and Heidtman [14] suggested using a generating function method for computing k-out-of-n reliability of systems with two failure modes.

The aforementioned studies consider only reliability characteristics of elements composing the system. In many practical cases, some measures of element (system) performance must be taken into account. For example, fluid-transmitting capacity is the performance of fluid valves and of switching systems that consist of such valves, while operating time is the performance of electronic switches and of switching systems that consist of such switches. Each system element can be characterized in each mode by its nominal performance and the element fails if it is unable to provide its nominal performance. A system can have different levels of output performance depending on its structure and on the combination of elements available at any given moment. Therefore, the system is multi-state.

The system is considered to be in an operational state if its performance rate in open mode G_o satisfies condition $F_o(G_o, w_o) = 1$ and its performance rate in closed mode G_c satisfies condition $F_c(G_c, w_c) = 1$, where w_o and w_c are the required levels of system performance in the open and closed modes respectively and F_o and F_c are the acceptability functions in open and closed modes respectively.

Since the failures in open and closed modes, which have probabilities $Q_o = \Pr\{F_o(G_o, w_o) = 0\}$ and $Q_c = \Pr\{F_c(G_c, w_c) = 0\}$ respectively, are mutually exclusive events and the probabilities of both modes are equal to 0.5 (each command to close is followed by a command to open and *vice versa*), the entire system availability can be defined as

$$A = 1 - 0.5\,(Q_o + Q_c) \tag{6.19}$$

In order to characterize the expected performance of MSSs with two failure modes, one has to evaluate this index for both its modes: ε_o and ε_c.

Having u-functions $U_o(z)$ and $U_c(z)$ representing the system performance distributions in open and closed modes, one can obtain the probabilities Q_o and Q_c and the expected performance rates ε_o and ε_c using the technique presented in Section 3.3 over $U_o(z)$ and $U_c(z)$ respectively.

Usually, the switching systems consist of elements with total failures. Each element j has nominal performance rates g_{jo} and g_{jc}, performance rates in fault state $\tilde{g}_{jo}$ and $\tilde{g}_{jc}$, and probabilities of normal functioning p_{jo} and p_{jc} in open and closed modes respectively. The individual u-functions of the system element can be defined as

$$u_{jo}(z) = p_{jo}z^{g_{jo}} + (1-p_{jo})z^{\tilde{g}_{jo}}$$
$$u_{jc}(z) = p_{jc}z^{g_{jc}} + (1-p_{jc})z^{\tilde{g}_{jc}} \tag{6.20}$$

In order to obtain the system u-functions $U_o(z)$ and $U_c(z)$, one has to determine the parameters of the u-functions of individual system elements (6.20) and define the structure functions used in the composition operators for both modes. Having these u-functions one can easily evaluate the system availability using Equation (6.19) and operators (3.8) and (3.12):

$$A(w_o, w_c)=1-0.5[1-E(U_o(z)\underset{F_o}{\otimes} z^{w_o})+1-E(U_c(z)\underset{F_c}{\otimes} z^{w_c})]$$
$$= 0.5[E(U_o(z)\underset{F_o}{\otimes} z^{w_o})+E(U_c(z)\underset{F_c}{\otimes} z^{w_c})] \tag{6.21}$$

In the following sections we consider two typical switching systems.

6.2.1 Flow Transmission Multi-state System

In this model the performance of the switching element (flow valve) is defined as its transmitting capacity. To determine the u-function of an individual element j in the closed mode, note that in the operational state, which has the probability p_{jc}, the element should transmit a nominal flow f_j ($g_{jc} = f_j$) and in the failure state it fails to transmit any flow ($\tilde{g}_{jc} = 0$). Therefore, according to (6.20), the u-function of the element takes the form

$$u_{jc}(z) = p_{jc}z^{f_j} + (1-p_{jc})z^0 \tag{6.22}$$

In the open mode the element has to prevent the flow transmission through the system. If it succeeds in doing this (with probability p_{jo}), then the flow is zero ($g_{jo} = 0$), and if it fails to do so the flow is equal to its nominal value in the closed mode ($\tilde{g}_{jo} = f_j$). The u-function of the element in the open mode takes the form

$$u_{jo}(z) = p_{jo}z^0 + (1-p_{jo})z^{f_j} \tag{6.23}$$

The structure functions for subsystems of elements connected in a series, in parallel or composing a bridge structure for the flow transmission MSS with flow dispersion are defined by Equations (4.2), (4.9) and (6.4) respectively. Using the reliability block diagram method one can obtain the u-function of the arbitrary system by consecutively applying the corresponding composition operators.

Note that the u-function of a subsystem containing n identical parallel elements ($p_{jc} = p_c$, $p_{jo} = p_o$, $f_j = f$ for any j) can be obtained by applying the operator $\otimes_{+}(u(z),...,u(z))$ over n functions $u(z)$ of an individual element represented by (6.22) or (6.23). The u-function of this subsystem takes the form

$$U_c(z) = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} p_c^k (1-p_c)^{n-k} z^{kf} \tag{6.24}$$

for the closed mode and

$$U_o(z) = \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} p_o^{n-k} (1-p_o)^{k} z^{kf} \tag{6.25}$$

for the open mode. The u-function of a subsystem containing n identical elements connected in a series can be obtained by applying operator $\otimes_{\min}$ over n functions $u(z)$ of an individual element. The u-function of this subsystem takes the form

$$U_c(z) = p_c^n z^f + (1-p_c^n) z^0 \tag{6.26}$$

for the closed mode and

$$U_o(z) = [1-(1-p_o)^n] z^0 + (1-p_o)^n z^f \tag{6.27}$$

for the open mode.

To determine the system's reliability one has to define its acceptability function. For the flow transmission system, it is natural to require that in its closed mode the amount of flow should not be lower than the demand w_c, while in the open mode it should not exceed a value of w_o. Therefore, the conditions of the system's success are

$$F_c(G_c,w_c) = 1(G_c \ge w_c) \text{ and } F_o(G_o,w_o) = 1(G_o \le w_o) \tag{6.28}$$

6.2.2 Task Processing Multi-state Systems

In this type of switching system the task of each element is to connect (or disconnect) a circuit. Since the task processing in each mode is associated with a single switching action, the performance of element j is defined not as its processing speed but as its operation time.

To determine the u-function of an individual element with total failures (for example, an electronic diode) in closed and open modes, note that the element j operates in times $g_{jc} = t_{jc}$ and $g_{jo} = t_{jo}$ with the probabilities p_{jc} and p_{jo} respectively. If the element fails to operate, then its operation time is equal to infinity $(\tilde{g}_{jo} = \tilde{g}_{jc} = \infty)$. Therefore, according to (6.20), the u-functions of the element for the two modes take the form

$$\begin{aligned} u_{jo}(z) &= p_{jo} z^{t_{jo}} + (1 - p_{jo}) z^{\infty} \\ u_{jc}(z) &= p_{jc} z^{t_{jc}} + (1 - p_{jc}) z^{\infty} \end{aligned} \tag{6.29}$$

If several elements are connected in parallel within a subsystem, then the subsystem disconnection is completed only when all the elements including the slowest one are opened. Therefore, the operation time of n elements in the open mode is equal to the greatest of the operation times of the elements. The structure function for the open mode takes the form

$$\phi_{par}(G_1,...,G_n) = \max\{G_1,...,G_n\} \tag{6.30}$$

For n elements connected in series, the first disconnected element disconnects the subsystem in the open mode. Therefore, the structure function takes the form

$$\phi_{ser}(G_1,...,G_n) = \min\{G_1,...,G_n\} \tag{6.31}$$

If n elements are connected in parallel within a subsystem, then the first connected element makes the subsystem connected. Therefore, the operation time of the group of elements in closed mode is equal to the least of the operation times of the elements. The structure function for the closed mode takes the form

$$\phi_{par}(G_1,...,G_n) = \min\{G_1,...,G_n\} \tag{6.32}$$

For n elements connected in series, all of the elements, including the slowest one, should be connected to make the subsystem connected in the closed mode. Therefore, the structure function takes the form:

$$\phi_{ser}(G_1,...,G_n) = \max\{G_1,...,G_n\} \tag{6.33}$$

Combining the two operators one can obtain a u-function representing the performance distribution of an arbitrary series-parallel system in both modes. Note that the u-function of a subsystem containing n identical parallel elements is

$$U_c(z) = [1 - (1 - p_c)^n] z^{t_c} + (1 - p_c)^n z^{\infty} \tag{6.34}$$

for the closed mode and

$$U_o(z) = p_o^n z^{t_o} + (1 - p_o^n) z^\infty \tag{6.35}$$

for the open mode. The *u*-function of a subsystem containing *n* identical elements connected in series takes the form

$$U_c(z) = p_c^n z^{t_c} + (1 - p_c^n) z^\infty \tag{6.36}$$

for the closed mode and

$$U_o(z) = [1 - (1 - p_o)^n] z^{t_o} + (1 - p_o)^n z^\infty \tag{6.37}$$

for the open mode.

In order to evaluate the operation time of a bridge structure, notice that there are four possible parallel ways to connect input and output of the bridge (see Figure 6.1): through groups of elements {1, 3} or {2, 4} or {1, 5, 4} or {2, 5, 3} connected in series. Therefore, the entire bridge operation time can be obtained as

$$\phi_{br}(G_1, G_2, G_3, G_4, G_5)$$
$$= \phi_{par}(\phi_{ser}(G_1, G_3), \phi_{ser}(G_2, G_4), \phi_{ser}(G_1, G_5, G_4), \phi_{ser}(G_2, G_5, G_3)) \tag{6.38}$$

For the open mode this expression takes the form

$$\phi_{br}(G_1, G_2, G_3, G_4, G_5)$$

$$= \max(\min(G_1, G_3), \min(G_2, G_4), \min(G_1, G_5, G_4), \min(G_2, G_5, G_3)) \tag{6.39}$$

and for the closed mode it takes the form

$$\phi_{br}(G_1, G_2, G_3, G_4, G_5)$$

$$= \min(\max(G_1, G_3), \max(G_2, G_4), \max(G_1, G_5, G_4), \max(G_2, G_5, G_3)) \tag{6.40}$$

For a system in which operation time is the crucial factor, it is natural to require that in its closed and open modes the operation times should not exceed the values w_c and w_o respectively. The system's acceptability functions are

$$F_c(G_c, w_c) = 1(G_c \le w_c) \text{ and } F_o(G_o, w_o) = (G_o \le w_o) \tag{6.41}$$

Having these acceptability functions one can easily evaluate the system's availability using Equation (6.21).

Since, in the worst case, the operation time of the entire system is equal to infinity, determining the expected operation time makes no sense. A more natural way of evaluating expected performance is by using the conditional expected operation time (expected operation time given the system manages to operate). In this case, Equation (3.7) with the acceptability functions $F_o(G_o) = 1(G_o<\infty)$ and $F_c(G_c) = 1(G_c<\infty)$ should be used.

Example 6.5

Consider a switching series-parallel subsystem consisting of three elements 1-3 connected as depicted in Figure 6.13. The elements are characterized by their availability and performance level (transmitting capacity *f*) in open and closed modes. The parameters of the system elements are presented in Table 6.11 (first three rows). The *u*-functions of the individual elements according to (6.22) and (6.23) are

$$u_{1o}(z) = 0.87z^0+0.13z^{1.5};\ u_{1c}(z) = 0.89z^{1.5}+0.11z^0$$

$$u_{2o}(z) = 0.78z^0+0.22z^{3.5};\ u_{2c}(z) = 0.82z^{3.5}+0.18z^0$$

$$u_{3o}(z) = 0.82z^0+0.18z^{2.5};\ u_{3c}(z) = 0.91z^{2.5}+0.09z^0$$

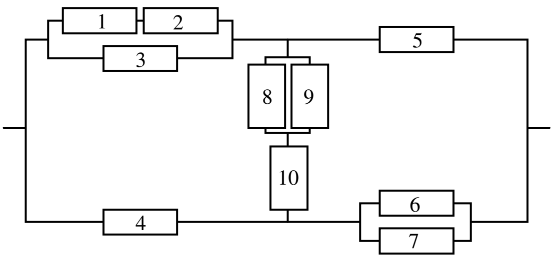


Figure 6.13. Reliability block diagram of MSS with two failure modes

In order to determine the system performance distribution in the open and closed modes we have to obtain the *u*-function of the entire system using composition operators over the *u*-functions of individual elements.

$$U_o(z) = [u_{1o}(z) \underset{\min}{\otimes} u_{2o}(z)] \underset{+}{\otimes} u_{3o}(z)$$

$$= [(0.87z^0+0.13z^{1.5}) \underset{\min}{\otimes} (0.78z^0+0.22z^{3.5})] \underset{+}{\otimes} (0.82z^0+0.18z^{2.5})$$

$$= (0.9714z^0+0.0286z^{1.5}) \underset{+}{\otimes} (0.82z^0+0.18z^{2.5})$$

$$= 0.7965z^0+0.0234z^{1.5}+0.1749z^{2.5}+0.0051z^4$$

$$U_c(z) = [u_{1c}(z) \underset{\min}{\otimes} u_{2c}(z)] \underset{+}{\otimes} u_{3c}(z) \; U_o(z)$$

$$= [(0.89z^{1.5}+0.11z^0) \underset{\min}{\otimes} (0.82z^{3.5}+0.18z^0)] \underset{+}{\otimes} (0.91z^{2.5}+0.09z^0)$$

$$= (0.7298z^{1.5}+0.2702z^0) \underset{+}{\otimes} (0.91z^{2.5}+0.09z^0)$$

$$= 0.6641z^4+0.0657z^{1.5}+0.2459z^{2.5}+0.0243z^0$$

Having the system u-functions for the open and closed modes one can determine the expected flows through the system in these modes by applying the operator (3.11):

$$\varepsilon_o = 0.7965\times0+0.0234\times1.5+0.1749\times2.5+0.0051\times4 = 0.4927$$

$$\varepsilon_c = 0.6641\times4+0.0657\times1.5+0.2459\times2.5+0.0243\times0 = 3.3697$$

Assume that in the closed mode the amount of flow should exceed $w_c = 2$, while in the open mode it should not exceed $w_o = 0.5$. Applying Equation (6.21) with acceptability functions (6.28) we obtain

$$E(U_c(z) \underset{F_c}{\otimes} 2) = 0.6641\times1(4\geq2)+0.0657\times1(1.5\geq2)+0.2459\times1(2.5\geq2)$$

$$+0.0243\times1(0\geq2) = 0.6641+0.2459 = 0.91$$

$$E(U_o(z) \underset{F_o}{\otimes} 0.5) = 0.7965\times1(0\leq0.5)+0.0234\times1(1.5\leq0.5)$$

$$+0.1749\times1(2.5\leq0.5)+0.0051\times1(4\leq0.5) = 0.7965$$

$$A(2, 0.5) = 0.5(E(U_c(z) \underset{F_c}{\otimes} 2) + E(U_o(z) \underset{F_o}{\otimes} 0.5))$$

$$= 0.5(0.91+0.7965) = 0.85325$$

Example 6.6

Consider a switching system with the configuration presented in Figure 6.13. Each one of the ten system elements is characterized by its availability and nominal performance rate in open and closed modes. In the case of a flow transmission system, the performance of an element is its transmitting capacity f. In the case of a system of electronic switches, the performance of an element is determined by its operation times in open mode t_o and in closed mode t_c. The parameters of the system elements are presented in Table 6.11.

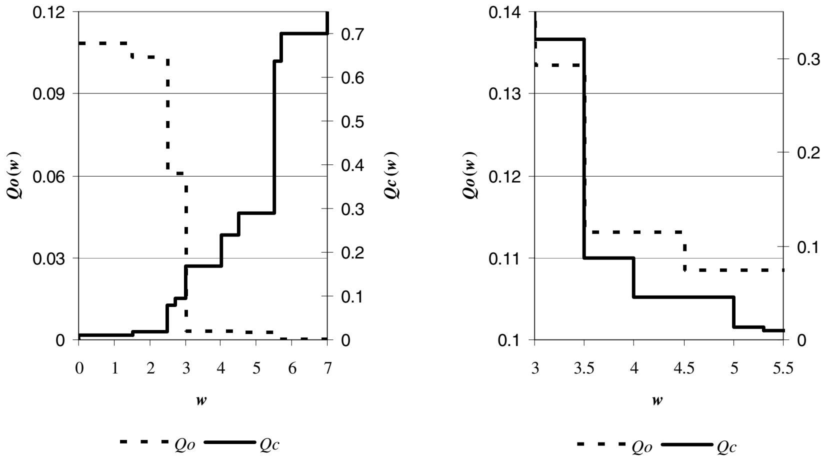
In order to determine the system PD in the open and closed modes one has to obtain the u-function of the entire system using the composition operators over u-functions of the individual elements $u_{o1}(z)$-$u_{o10}(z)$ and $u_{c1}(z)$-$u_{c10}(z)$ respectively.

Table 6.11. Parameters of MSS elements

No. of element	Flow transmission model			
		f	p_c	p_o
	Task processing model			
	t_c	t_o	p_c	p_o
1	3.0	1.5	0.89	0.87
2	5.0	3.5	0.82	0.78
3	3.5	2.5	0.91	0.82
4	2.5	3.0	0.85	0.82
5	3.0	2.5	0.80	0.76
6	3.0	3.0	0.80	0.78
7	4.0	3.0	0.91	0.85
8	4.0	4.5	0.84	0.79
9	5.3	2.5	0.93	0.91
10	5.0	2.7	0.92	0.90

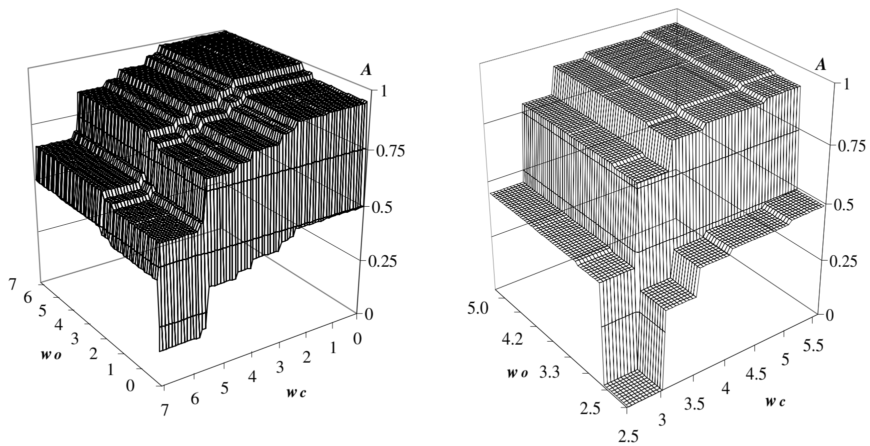
First, consider the system to be a combination of flow valves (flow transmission system with flow dispersion). The flow through the system can vary in the range of 0.0-7.0. In the closed mode the expected flow is $\varepsilon_c = 5.306$. The probability that the system provides the maximal flow in the closed mode is $\Pr\{G_c = 7\} = 0.30$. In the open mode the expected flow is $\varepsilon_o = 0.303$ and the probability that the system totally prevents the flow is $\Pr\{G_c = 0\} = 0.892$. The system failure is defined as its inability to provide at least the required constant level of flow w_c in its closed mode and to prevent the flow exceeding w_o in its open mode. The failure probabilities in both modes as functions of demand w are presented in Figure 6.14A. Note that $Q_c(w_c)$ is an increasing function (the greater the demand, the tougher the condition $G_c \geq w_c$), while $Q_o(w_o)$ is a decreasing function (the greater the demand, the easier the condition $G_o \leq w_o$). The entire system availability $A(w_o,w_c)$ as a function of maximal allowable flow in the open mode w_o and minimal required flow in the closed mode w_c is presented in Figure 6.15A.

Now consider the system to be a combination of electronic switches (task processing system). The probabilities that the system is able to operate in the open and closed modes (operation time is less than infinity) are $\Pr\{G_o<\infty\} = 0.892$ and $\Pr\{G_c<\infty\} = 0.990$. When $G_o<\infty$, the time needed by the system to disconnect its input from output in the open mode cannot be less than 3 and greater than 4.5. The conditional expected operation time is $\tilde{\varepsilon}_o = 3.02$. When $G_c<\infty$, the time needed by the system to connect its input with output in the closed mode cannot be less than 3 and greater than 5.3. The conditional expected operation time is $\tilde{\varepsilon}_c = 3.23$. When the system failure is defined as its inability to switch within the required time (w_o and w_c in open and closed modes respectively), the failure probabilities in both modes are functions of this time. The functions $Q_o(w_o)$ and $Q_c(w_c)$ are presented in Figure 6.14B. The entire system availability as a function of required switching times in the open and closed modes $A(w_o,w_c)$ is presented in Figure 6.15B.



A B

Figure 6.14. Failure probabilities as functions of demand.
A: flow transmission system; B: task processing system



A B

Figure 6.15. System availability as function of demands in open and closed modes.
A: flow transmission system; B: task processing system

The duality of roles of parallel and series connection of units in the two operation modes creates a situation in which any change in system configuration that increases system availability in an open mode can decrease it in a closed mode and *vice versa* [170, 171]. Therefore, the optimal system configuration should be found that provides the maximal overall system availability (6.21).

There exist two types of structure optimization problem when the systems with two failure modes are considered. The first one is an extension of the well-known redundancy optimization problem. In this problem, one has to determine the number of parallel elements (with identical functionality) for each system component when the system structure (topology of the reliability block diagram representing interaction of the components) is given. The algorithms for solving this problem were studied in [174, 175] for the binary-state systems without respect to the element performances. The second problem is to find the configuration (topology of the reliability block diagram) for a given set of elements that provides the greatest possible system availability. This problem was formulated and solved in [173] for the binary-state systems (also without respect to the element performances). In the following sections, algorithms for solving the two system optimization problems for MSSs with two failure nodes are presented.

6.2.3 Structure Optimization of Systems with Two Failure Modes

6.2.3.1 Problem Formulation

A system consists of N components connected according to a block diagram. Each component of type i contains a number of different switching elements connected in parallel. Different versions and numbers of elements may be chosen for any given system component. Element operation in open and closed modes is characterized by its availability and nominal performance rate.

For each component i there are B_i element versions available. A vector of parameters $g_{io}(b)$, $g_{ic}(b)$, $p_{io}(b)$ and $p_{ic}(b)$ can be specified for each version b of element of type i. The structure of system component i is defined by the numbers of parallel elements of each version n_{ib} for $1\le b\le B_i$. The vectors $\boldsymbol{n}_i=\{n(i,b)\}$ ($1\le i\le N$, $1\le b\le B_i$) define the entire system structure.

For a given set of vectors $\{\boldsymbol{n}_1, ..., \boldsymbol{n}_N\}$, the entire system fault probabilities $Q_o(w_o, \boldsymbol{n}_1, ..., \boldsymbol{n}_N)$ and $Q_c(w_c, \boldsymbol{n}_1, ..., \boldsymbol{n}_N)$ can be obtained for both modes. The requirement of providing the desired system availability in open and closed modes can be formulated as follows:

$$Q_o(w_o, \boldsymbol{n}_1, ..., \boldsymbol{n}_N)\le Q^*_o,\ \ Q_c(w_c, \boldsymbol{n}_1, ..., \boldsymbol{n}_N) \le Q^*_c, \tag{6.42}$$

where Q^*_o and Q^*_c are maximal allowable levels of system unavailability in open and closed modes respectively.

Having the given system structure, one can also determine the expected system performance in the both modes $\varepsilon_o(\boldsymbol{n}_1, ..., \boldsymbol{n}_N)$ and $\varepsilon_c(\boldsymbol{n}_1, ..., \boldsymbol{n}_N)$. While satisfying the availability requirements (6.42), one can desire to obtain expected system performance values as close to some specified values ε^*_o and ε^*_c as possible. The proximity between expected system performance and the desired level can be of different importance in open and closed modes.

Now consider two possible formulations of the problem of system structure optimization.

Formulation 1. Find system configuration $\{\boldsymbol{n}_1,\ldots, \boldsymbol{n}_N\}$ that provides maximal system availability:

$$A(w_{\mathrm{o}},w_{\mathrm{c}}, \boldsymbol{n}_1, \ldots, \boldsymbol{n}_N)$$
$$= 1-0.5(\mathrm{Q_o}(w_{\mathrm{o}},\boldsymbol{n}_1,\ldots,\boldsymbol{n}_N)+\mathrm{Q_c}(w_{\mathrm{c}},\boldsymbol{n}_1,\ldots,\boldsymbol{n}_N)) \rightarrow \max \tag{6.43}$$

Formulation 2: find system configuration $\{\boldsymbol{n}_1, \ldots, \boldsymbol{n}_N\}$ that provides the maximal proximity of expected system performance to the desired levels for both modes, while satisfying the availability requirements:

$$\begin{gathered}\alpha\,|\,\varepsilon_{\mathrm{o}}(\boldsymbol{n}_1,\ldots,\boldsymbol{n}_N)-\varepsilon_{\mathrm{o}}^*\,|+(1-\alpha)\,|\,\varepsilon_{\mathrm{c}}(\boldsymbol{n}_1,\ldots,\boldsymbol{n}_N)-\varepsilon_{\mathrm{c}}^*\,| \rightarrow \min \\ \text{subject to } \mathrm{Q_o}(w_{\mathrm{o}},\boldsymbol{n}_1,\ldots,\boldsymbol{n}_{\mathrm{N}}) \le Q_{\mathrm{o}}^*, \mathrm{Q_c}(w_{\mathrm{c}},\boldsymbol{n}_1,\ldots,\boldsymbol{n}_N) \le Q_{\mathrm{c}}^*\end{gathered} \tag{6.44}$$

where constant α reflects the relative importance of the open mode over the closed mode ($0\le\alpha\le1$). Note that for the task processing systems the measures ε_o and ε_c should be substituted by the corresponding conditional measures $\tilde{\varepsilon}_{\mathrm{o}}$ and $\tilde{\varepsilon}_{\mathrm{c}}$.

6.2.3.2 Implementing the Genetic Algorithm

The solution encoding is the same as described in Section 5.1.2.2, where the element a_j of the integer string $\boldsymbol{a}$ defines the number of parallel elements for each component i and version b (the relation between i, b and j is determined by Equation (5.10)).

In order to let the GA look for the solution meeting requirements (6.43) or (6.44), the following universal expression of solution quality (fitness) is used:

$$\begin{gathered}M-\alpha\,|\,\varepsilon_{\mathrm{o}}(\boldsymbol{a})\text{-}\varepsilon^*{}_{\mathrm{o}}\,|-(1-\alpha)\,|\,\varepsilon_{\mathrm{c}}(\boldsymbol{a})\text{-}\varepsilon^*{}_{\mathrm{c}}\,| \\ -\pi(\max\{0,\mathrm{Q_o}(\boldsymbol{a})-Q^*{}_{\mathrm{o}}\}+\max\{0,\mathrm{Q_c}(\boldsymbol{a})-Q^*{}_{\mathrm{c}}\})\end{gathered} \tag{6.45}$$

where π and M are constants much greater than the maximal possible value of system output performance.

The case when $Q^*{}_{\mathrm{o}}=Q^*{}_{\mathrm{c}}=0$ corresponds to formulation (6.43). Indeed, since π is sufficiently large, the value to be minimized in order to maximize the fitness is $\pi(Q_{\mathrm{o}}+Q_{\mathrm{c}})$. On the other hand, when $Q^*{}_{\mathrm{o}}=Q^*{}_{\mathrm{c}}=1$ all availability limitations are removed and expected performance becomes the only factor in determining the system structure.

The solution decoding procedure determines n_{ib} for each system component i and each element version b from the string $\boldsymbol{a}$ and determines the performance measures of the system separately for open and closed modes according to the algorithm described in Section 5.1.2.2. Then it determines the solution fitness using expression (6.45).

Example 6.7

Consider a system of electronic switches consisting of four components connected in series [176]. Each component can contain a number of switches connected in parallel. The elements in each component should belong to a certain type. (For example, each component can operate in a different medium, which causes specific requirements on the switches.) Each element version is characterized by element parameters: availability and performance rates (operation times in open mode t_o and in closed mode t_c). The problem is to find the optimal system configuration by choosing elements for each component from the lists of versions presented in Table 6.12.

Table 6.12. Parameters of electronic switches

No. of component	Version of element	t_o	t_c	p_o	p_c
1	1	5.80	3.00	0.81	0.76
	2	4.60	3.30	0.85	0.79
	3	4.50	3.50	0.86	0.75
	4	4.00	3.10	0.84	0.76
2	1	1.80	1.20	0.84	0.78
	2	1.81	1.30	0.86	0.72
	3	1.85	1.10	0.89	0.70
3	1	2.00	1.90	0.82	0.80
	2	2.10	1.92	0.87	0.81
	3	2.10	1.89	0.89	0.73
4	1	3.60	3.30	0.85	0.78
	2	4.00	2.80	0.87	0.77

Table 6.13 contains the results obtained for a system that is considered to be in normal condition if the switching time in both modes is not greater than $w_o = w_c = 5$; the desired operation time is $\varepsilon^*_o = \varepsilon^*_c = 0$. The conditional expected value $\tilde{\varepsilon}$ is estimated for switching time distributed in the range of allowable values (0, 5). It is assumed that the operation speed is equally important in both modes: $\alpha = 0.5$. Three solutions were obtained for different levels of desired availability in both modes $Q^*_o = Q^*_c = 0$, $Q^*_o = Q^*_c = 0.035$ and $Q^*_o = Q^*_c = 0.05$.

Table 6.13. Solutions obtained for the system of electronic switches

Component	$Q^*_o=Q^*_c=0$	$Q^*_o=Q^*_c=0.035$	$Q^*_o=Q^*_c=0.05$
1	3*2,1*3	4*1,1*3,1*4	7*1,6*3,1*4
2	5*3	3*3	3*3
3	3*2	3*2	3*3
4	4*2	5*2	7*2
Q_o	0.030	0.035	0.050
Q_c	0.014	0.034	0.046
A	0.978	0.965	0.952
$\tilde{\varepsilon}_o$	2.213	2.076	1.997
$\tilde{\varepsilon}_c$	3.301	3.001	3.000

Observe that the solution maximizing the system availability (first formulation corresponding to $Q^*_o = Q^*_c = 0$) has a relatively small number of elements. Further growth of the number of elements decreases the system availability. One can see that, with the growth of availability requirements, the system availability increases by the price of the increase of expected switching time. The failure probability distributions in closed and open modes for the solutions obtained are presented in Figure 6.16. Note that the requirement to improve the conditional expected values of system performance contradicts the requirement to maximize the system availability defined as its ability to reach a threshold level of the performance.

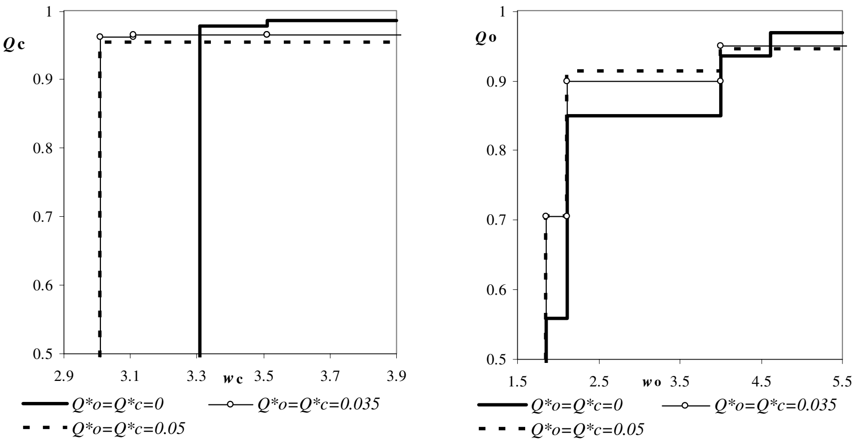


Figure 6.16. Failure probability distributions for the system of electronic switches

6.2.4 Optimal Topology of Systems with Two Failure Modes

6.2.4.1 Problem Formulation

The problem of optimizing a series-parallel MSS configuration is the following: find the series-parallel configuration of a given number of statistically independent units that provides maximal system availability when the units can experience two failure modes and are characterized by different performance rates and availability indices.

6.2.4.2 Implementing the Genetic Algorithm

According to its definition, any series-parallel system is either a single unit or it is two series-parallel subsystems connected in series or in parallel. Therefore, any such system can be represented by a binary tree. One of the possible representations was suggested in [173], where tree leaf nodes correspond to the primary units from which the configuration is built and the rest of the nodes are distinguished by the way the two children of the node are joined. As was shown in [173], such a binary tree can be easily represented by a symbolic string (post-order traversal) in which symbols from the set $\{1,\ldots,N\}$ correspond to unit numbers and symbols from the set {S, P} correspond to types of connection (S for series one and P for parallel one).

For example, the binary tree corresponding to the system presented in Figure6.17A can be represented by the following string: 12P3S45PP. The main disadvantage of this representation is that different strings can represent the same configuration, since the order of substrings representing the two child subtrees of any junction does not matter (observe that the system from Figure 6.17A can also be represented by the string 45P12P3SP). This causes situations in which the GA population is overwhelmed with different strings representing identical solutions. Such situations slow the algorithm convergence.

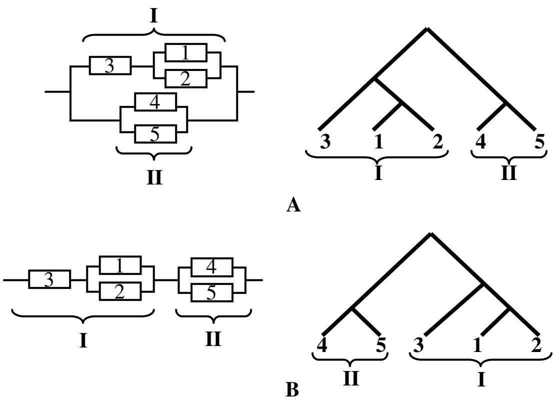


Figure 6.17. Binary tree representation of series-parallel system

In order to simplify the representation and to reduce the number of cases in which the same configuration is represented by different trees (strings), the following rule was introduced in [177] to determine the types of connection: for each node joining two child subtrees, determine the minimal number among the numbers of units belonging to the left child subtree x_L and right child subtree x_R. If $x_L<x_R$, then the subtrees are joined in parallel; if $x_L>x_R$, then they are joined in series.

Using this simplified representation one obtains a new configuration by swapping child subtrees of a given node (see Figure 6.17B) and does not need to distinguish nonleaf tree nodes (P or S), since they no longer determine the type of connection.

We use the following rule to represent the binary tree corresponding to a series-parallel configuration by a string: all numbers corresponding to units should appear in the string in the order that they appear in the tree from left to right. Each time that all of the numbers corresponding to subtrees connected by some node appear on the left-hand side of the given position in the string, the sign * representing the node should be inserted in this position.

Example 6.8

Consider the tree presented in Figure 6.17A. The corresponding string representation is 312**45**, where the underlined substring represents subsystem I and the double underlined substring represents subsystem II. Since for the root node $x_L = 1$ and $x_R = 4$, the subsystems I and II are connected in parallel. To make them connected in series, one just has to swap corresponding substrings and obtain string 45*312***, corresponding to the configuration presented in Figure 6.17B.

Note that, in the representation given, the last position of the string is always occupied by *, representing the root node of the tree. Therefore, this string element provides no information and can be removed. Since the total number of nonleaf nodes in the binary tree with N leafs is N–1, the string representing series-parallel configuration of N units should contain 2(N–1) elements.

Not every arbitrary string can represent a feasible solution. Indeed, consider string 3**1245*. Since each node sign * corresponds to two subtrees that the node connects, it should follow at least two unit numbers. In order to make an arbitrary solution feasible, in all cases where there are not enough numbers from the left-hand side of the node sign *, one has to find the closest number following the node sign * on the right-hand side and insert it immediately before the sign. (For the string given, such a procedure first produces string 31**245* and, when repeated, produces a feasible string 31*2*45*).

The simplest way to represent solution strings in the GA is by using permutations of integer numbers {1,…,2(N–1)} and by treating all the numbers greater than N as node signs * (for example, string 31827456 can be treated as 31*2*45*).

The following is a procedure for system availability evaluation based on decoding the system configuration from an arbitrary permutation $\boldsymbol{a}$ of integer numbers ranging from 1 to 2(N–1). The procedure enables the obtaining of u-functions $U_o(z)$ and $U_c(z)$ for the entire system, by applying composition operators over the corresponding u-functions of individual units in sequence determined by the system configuration encoded by a string $\boldsymbol{a} = (a_i,\ldots, a_{2N-2})$. To store the intermediate u-functions, a stack memory is used that allows binary subtrees (and corresponding u-functions) to be treated in order that is encoded by the string.

With each u-function $u(z)$ representing a subtree, we associate a number $x(u(z))$ equal to the smallest one from among the numbers of units belonging to the subtree.

The procedure performs the following steps:

1. Assigns number of string element $i = 1$.

2. If $a_i \le N$ (a_i corresponds to the number of the unit), assign $x(u_{a_i}(z)) = a_i$, place unit u-function $u_{a_i}(z)$ to the stack and go to step 5.

If $a_i > N$ (a_i corresponds to nonleaf node sign *) go to step 3.

3. If there are no fewer than two u-functions in the stack, go to step 4, else find string element a_j closest to a_i ($j>i$), which corresponds to the number of the unit ($a_j \le N$). Remove the element a_j from the string, shift all the elements a_i, …, a_{j-1} one position right and place element a_j into position i. Return to step 2.

4. Remove the upper u-function $u'(z)$ and second one from the top $u''(z)$ from the stack. Obtain the new u-function $u'''(z)$ either as $u'''(z) = u'(z) \otimes_{\varphi_{\text{ser}}} u''(z)$ (series connection) if $x(u'(z)) < x(u''(z))$ or as $u'''(z) = u'(z) \otimes_{\varphi_{\text{par}}} u''(z)$ (parallel connection) if $x(u'(z)) > x(u''(z))$.

Obtain index $x(u'''(z)) = \min\{x(u'(z)), x(u''(z))\}$. Place the new u-function $u'''(z)$ and the index $x(u'''(z))$ into the stack.

5. Increment i by one. If $i \le 2(N-1)$, return to step 2, else obtain the entire system u-function $U(z)$ as described in step 4.

Repeating steps 1-5 with the unit u-functions and composition operators corresponding to open and closed modes, one finally obtains the system u-functions $U_o(z)$ and $U_c(z)$ and determines the system's availability using Equation (6.21). The solution fitness is equal to the system's availability.

Since the solution of the optimization problem considered is represented by permutations of integer numbers (which corresponds to the sequencing problem), the corresponding fragment crossover operator and the mutation procedure that swaps two string elements (discussed in Section 1.3.2.6) are to be used. The following example illustrates the use of the fragment crossover operator, mutation procedure, and solution correction algorithm (used within the solution decoding procedure) for obtaining new feasible solutions from two parents.

Example 6.9

Consider two parent strings P1 and P2, representing configurations presented in Figure 6.18:

P1: 1 2 3 4 5 6 7 8
P2: 4 2 6 5 7 1 8 3

The fragment crossover operator is applied twice with the roles of the parents reversed. After applying the crossover with a randomly determined fragment, we obtain two offspring solutions O1 and O2 (the elements belonging to the fragment are underlined):

O1: 4 2 1 5 6 7 8 3
O2: 1 2 4 6 5 3 7 8

After applying the mutation procedure to O1 and O2, we obtain strings S1 and S2:

S1: 4 2 3 5 6 7 8 1
S2: 1 7 4 6 5 3 2 8

(the two randomly chosen positions are underlined).

Note that string S2 is infeasible (according to the feasibility rule presented in Section 3.2, the node sign 7 should follow at least two unit numbers representing two subtrees connected by the node). During the solution decoding procedure it is transformed into the string

S2: 1 4 7 5 6 3 2 8

The final solutions, as well as the intermediate ones, are also presented in Figure 6.18.

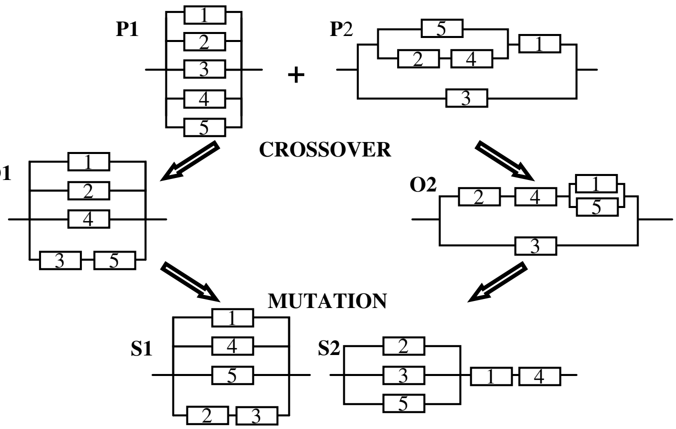


Figure 6.18. Examples of series-parallel configurations obtained by GA procedures

Example 6.10

Consider a set of 10 fluid flow valves. Each valve is characterized by its availability in open and closed modes (p_o, p_c), and by the nominal flow transmitting capacity f. These parameters are presented in Table 6.14. We want the flow to be not less than w_c in the closed mode and not greater than w_o in the open (disconnected) mode.

Three different system configurations were found by the GA for the different desired system transmitting capacities in open and closed modes: configuration A for $w_c = 5$, $w_o = 0.1$, configuration B for $w_c = 7$, $w_o = 0.1$, and configuration C for $w_c = 10$, $w_o = 3$. These configurations are presented in Figure 6.19. The probabilistic distributions of flows through the system in closed and open modes for the configurations obtained are presented in Figure 6.20 in the form of cumulative probabilities $\Pr\{G_c>w_c\}$ and $\Pr\{G_o<w_o\}$. Table 6.15 contains system fault probabilities Q_o and Q_c and availability index A obtained for each configuration for all the three demand combinations (w_c, w_o). Note that each configuration, though being the best for a certain combination (w_c, w_o), does not provide the greatest system availability for the two other combinations. Configuration A can not provide flow $f = w_c = 10$ in the closed mode even when all the units are available

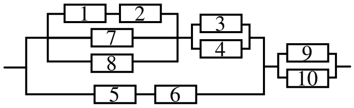
Note that while the configurations obtained seem to be not realistic for pure switching systems they are relevant when one considers the configuration of different types of flow transmission equipment (pumps, filters, *etc.*) having alarm valves aimed at preventing the flow in the case of contingency.

Table 6.14. Parameters of fluid flow valves

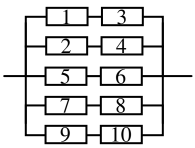
No. of unit	f	p_c	p_o
1	2.0	0.86	0.82
2	2.0	0.92	0.88
3	2.5	0.95	0.89
4	2.5	0.95	0.89
5	3.0	0.90	0.86
6	3.0	0.90	0.86
7	4.0	0.87	0.83
8	4.0	0.84	0.80
9	5.0	0.87	0.81
10	5.0	0.82	0.80

Table 6.15. Reliability characteristics of the obtained solutions

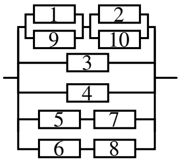
Solution		w_c=5.0, w_o=0.1	w_c=7.0, w_o=0.1	w_c=10.0, w_o=3.0
A	Q_c	0.059	0.489	1.000
	Q_o	0.032	0.032	0.008
	A	**0.955**	0.739	0.000
B	Q_c	0.017	0.045	0.208
	Q_o	0.119	0.119	0.089
	A	0.932	**0.918**	0.852
C	Q_c	0.002	0.004	0.042
	Q_o	0.323	0.323	0.110
	A	0.837	0.836	**0.924**



A



B



C

Figure 6.19. Optimal configurations obtained for the system of fluid flow valves

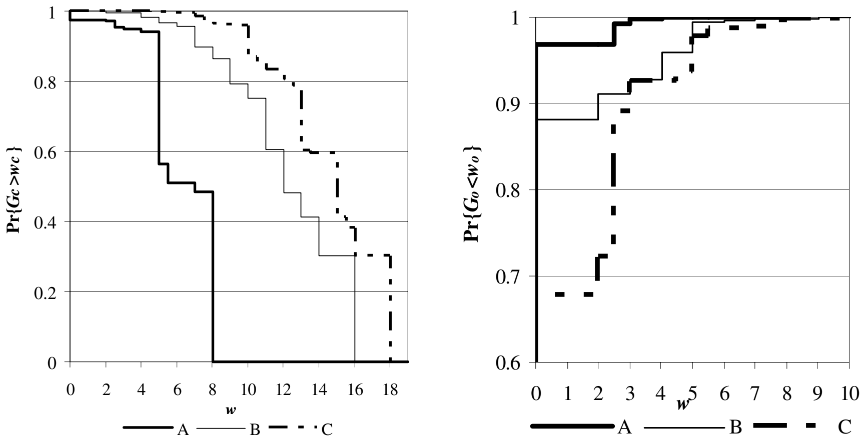


Figure 6.20. Cumulative probabilities Pr{G_c>w_c} and Pr{G_o<w_o}for the fluid flow valves configurations obtained

6.3 Weighted Voting Systems

The weighted voting system (WVS) consists of *n* independent voting units that provide a binary decision or abstain from voting. Each unit has its own individual weight. The system accepts the proposition *I* if the cumulative weight of the units supporting this proposition is at least the prespecified fraction τ of the cumulative weight of all non-abstaining units. The system abstains if all *n* units abstain. In all other cases, the system rejects proposition *I*. The system fails if it does not accept the proposition that should be accepted, does not reject the proposition that should be rejected, or abstains from voting.

This can be modelled by considering the system input *I* being either 1 (proposition to be accepted) or 0 (proposition to be rejected) which is supplied to each unit. Each unit *j* produces its decision (unit output) $d_j(I)$ which can be 1, 0, or *x* (in the case of abstention). Inequality $d_j(I) \neq I$ means that the decision made by the unit is wrong. The above listed errors can be expressed as

1. $d_j(0) = 1$ (unit fails stuck-at-1)
2. $d_j(1) = 0$ (unit fails stuck-at-0)
3. $d_j(I) = x$ (unit fails stuck-at-*x*)

Accordingly, the reliability of each unit *j* can be characterized by the probabilities of these errors: $q_{01}^{(j)}$ for the first one, $q_{10}^{(j)}$ for the second one, $q_{1x}^{(j)}$ and $q_{0x}^{(j)}$ for the third one, where $q_{im}^{(j)}$ is $\Pr\{d_j(I) = m \mid I = i\}$ (note that stuck-at-*x* probabilities can be different for inputs $i = 0$ and $i = 1$).

To make a decision about the proposition acceptance, the system incorporates all of the unit decisions into a unanimous system output *D* in the following manner:

$$D(I)=\begin{cases}1, & \text{if} \quad \sum\limits_{d_j(I)\neq x} \psi_j d_j(I) \geq \tau \sum\limits_{d_j(I)\neq x} \psi_j, \quad \sum\limits_{d_j(I)\neq x} \psi_j \neq 0 \\ 0, & \text{if} \quad \sum\limits_{d_j(I)\neq x} \psi_j d_j(I) < \tau \sum\limits_{d_j(I)\neq x} \psi_j, \quad \sum\limits_{d_j(I)\neq x} \psi_j \neq 0 \\ x, & \text{if} \quad \sum\limits_{d_j(I)\neq x} \psi_j = 0 \end{cases} \tag{6.46}$$

where ψ_j is the nonnegative weight of an individual unit j which expresses its relative importance in the WVS and τ is a threshold factor which determines what fraction of the overall weight of voted units should correspond to those that approve the proposition to make it accepted by the entire system.

The entire system output distribution is characterized by WVS output probabilities $Q_{im} = \Pr\{D(I) = m \mid I = i\}$, where $m \in \{0, 1, x\}$. The system fails if $D(I) \neq I$. The entire WVS reliability can be defined as $R = \Pr\{D(I) = I\}$. One can see that the system reliability is a function of the reliabilities of its units. The reliability characteristics of the WVS units, as well as the probability distribution of the propositions, $P_0 = \Pr\{I = 0\}$ and $P_1 = \Pr\{I = 1\}$, can be elicited from historical statistics. In technical systems, probabilities of different kinds of error can be obtained for each unit with a high precision by intensive testing. The entire WVS reliability also depends on the unit weights and the threshold. The proper choice of these parameters can improve the WVS reliability without improving the reliability of the voting units.

6.3.1 Evaluating the Weighted Voting System Reliability

Let us define the total weight of WVS units supporting proposition I as Ψ_I^1:

$$\Psi_I^1 = \sum_{d_j(I)\neq x} \psi_j d_j(I) \tag{6.47}$$

and the total weight of units voting for the proposition rejection as Ψ_I^0:

$$\Psi_I^0 = \sum_{d_j(I)\neq x} \psi_j (1 - d_j(I)) \tag{6.48}$$

The decision rule (6.46) can now be rewritten as follows:

$$D(I)=\begin{cases}1, & \text{if} \quad \Psi_I^1 \geq \tau(\Psi_I^1 + \Psi_I^0), \quad \Psi_I^1 + \Psi_I^0 \neq 0 \\ 0, & \text{if} \quad \Psi_I^1 < \tau(\Psi_I^1 + \Psi_I^0), \quad \Psi_I^1 + \Psi_I^0 \neq 0 \\ x, & \text{if} \quad \Psi_I^1 + \Psi_I^0 = 0 \end{cases} \tag{6.49}$$

Following this expression, the condition $D(I) = 0$ can be rewritten as

$$\Psi_I^1 < \tau(\Psi_I^1 + \Psi_I^0) \tag{6.50}$$

or

$$(1-\tau)\Psi_I^1 - \tau\Psi_I^0 < 0 \tag{6.51}$$

This gives one a simple way of tallying the units' votes: each unit j adds a value of $(1-\tau)\psi_j$ to the total WVS score if it votes for the proposition's acceptance, a value of $-\tau\psi_j$ if it votes for proposition's rejection, and nothing if it abstains. The proposition is rejected if the total score is negative.

6.3.1.1 Universal Generating Function Technique for Weighted Voting System Reliability Evaluation

Using the UGF approach one can describe the distributions of the random output G_{ij} of an individual three-state voting unit j for input i as

$$u_{ij}(z) = \sum_{k=0}^{2} p_{jk} z^{g_{jk}} \tag{6.52}$$

where for $i = 1$

$$\begin{aligned} p_{j0} &= q_{10}^{(j)}, \quad g_{j0} = -\tau\psi_j \\ p_{j1} &= q_{11}^{(j)} = (1 - q_{10}^{(j)} - q_{1x}^{(j)}), \quad g_{j1} = (1-\tau)\psi_j \\ p_{j2} &= q_{1x}^{(j)}, \quad g_{j2} = 0 \end{aligned} \tag{6.53}$$

and for $i = 0$

$$\begin{aligned} p_{j0} &= q_{01}^{(j)}, \quad g_{j0} = (1-\tau)\psi_j \\ p_{j1} &= q_{00}^{(j)} = (1 - q_{01}^{(j)} - q_{0x}^{(j)}), \quad g_{j1} = -\tau\psi_j \\ p_{j2} &= q_{0x}^{(j)}, \quad g_{j2} = 0 \end{aligned} \tag{6.54}$$

In each voting unit, state 0 corresponds to an incorrect decision, state 1 corresponds to a correct decision, and state 2 corresponds to an abstention.

The total random WVS score G_i for input $I = i$ is equal to the sum of the random outputs of n individual voting units:

$$G_i = \sum_{j=1}^{n} G_{ij} \tag{6.55}$$

Therefore, the u-function of the system score $U_i(z)$ can be obtained using the following composition operator:

$$U_i(z) = \underset{+}{\otimes}(u_{i1}(z), \ldots, u_{in}(z)) \tag{6.56}$$

Since the function (6.55) possesses commutative and associative properties, the u-function of the entire WVS can be obtained recursively by the consecutive determination of u-functions of the arbitrary subsets of the elements. For example it can be obtained by the recursive procedure

$$\tilde{U}_{i1}(z) = u_{i1}(z),\ \tilde{U}_{im}(z) = \tilde{U}_{im\text{-}1}(z) \underset{+}{\otimes} u_{im}(z) \text{ for } 1<m\leq n$$

$$U_i(z) = \tilde{U}_{in}(z) \tag{6.57}$$

In this procedure, $\tilde{U}_{im}(z)$ represents the score distribution of the WVS subsystem consisting of the first m voting units.

Note that, while the total number of different possible WVS states is 3^n, many of these states can result in the same values of score G_i. Therefore, the total number of terms $U_i(z)$ can be less than 3^n because of the like terms collection.

Using the criterion of proposition rejection as an acceptability function $F(G_i) = 1(G_i < 0)$ we obtain the proposition rejection probability Q_{i0}:

$$Q_{i0} = \Pr\{D(I) = 0 \mid I = i\} = E(F(G_i)) \tag{6.58}$$

which is equal to the sum of the coefficients of the terms with the negative exponents in $U_i(z)$.

Having Q_{10} one can easily obtain Q_{11} as

$$Q_{11}=1-Q_{10}-Q_{1x}, \quad \text{where } Q_{1x} = \prod_{j=1}^{n} q_{1x}^{(j)} \tag{6.59}$$

Events $I = 0$ and $I = 1$ are mutually exclusive. Therefore, the entire WVS reliability $\Pr\{D(I) = I\}$ can be defined as

$$\Pr\{D(I) = 0 \mid I = 0\} \Pr\{I = 0\}+\Pr\{D(I) = 1 \mid I = 1\} \Pr\{I = 1\} \tag{6.60}$$

and calculated as follows:

$$R = P_0Q_{00}+P_1Q_{11} = P_0Q_{00}+P_1(1-Q_{10}-Q_{1x}) \tag{6.61}$$

Example 6.11

Consider a WVS with

$$n = 2, P_0 = P_1 = 0.5,\ \tau = 0.6$$

$$q_{01}^{(1)}=0.02,\ q_{10}^{(1)}=0.02,\ q_{0x}^{(1)}=q_{1x}^{(1)}=0.01,\ \psi_1=5$$

$$q_{01}^{(2)}=0.02,\ q_{10}^{(2)}=0.05,\ q_{0x}^{(1)}=q_{1x}^{(2)}=0.02,\ \psi_2=3$$

Then

$$(1-\tau)\psi_1=2,\ (1-\tau)\psi_2=1.2,\ -\tau\psi_1=-3,\ -\tau\psi_2=-1.8$$

$$u_{01}(z)=10^{-2}(2z^2+97z^{-3}+z^0),\ u_{11}(z)=10^{-2}(2z^{-3}+97z^2+z^0)$$

$$u_{02}(z)=10^{-2}(2z^{1.2}+96z^{-1.8}+2z^0),\ u_{12}(z)=10^{-2}(5z^{-1.8}+93z^{1.2}+2z^0)$$

u-functions for the entire WVS are

$$\tilde{U}_{01}(z)=u_{01}(z),\ \tilde{U}_{02}(z)=u_{01}(z)\underset{+}{\otimes}u_{02}(z)$$

$$=10^{-4}(4z^{2.2}+4z^2+192z^{0.2}+2z^{1.2}+2z^0+\mathbf{96z^{-1.8}}+\mathbf{194z^{-0.2}}+\mathbf{194z^{-1.8}}+\mathbf{9312z^{-4.8}})$$

$$\tilde{U}_{11}(z)=u_{11}(z),\ \tilde{U}_{12}(z)=u_{11}(z)\underset{+}{\otimes}u_{12}(z)$$

$$=10^{-4}(\mathbf{10z^{-4.8}}+\mathbf{4z^{-3}}+\mathbf{186z^{-1.8}}+\mathbf{5z^{-1.8}}+2z^0+93z^{1.2}+485z^{0.2}+194z^2+9021z^{3.2})$$

The terms with negative exponents are marked in bold. To obtain Q_{10} and Q_{00} one should calculate the sums of the coefficients of the marked terms:

$$Q_{00}=10^{-4}(96+194+194+9312)=0.9796$$

$$Q_{10}=10^{-4}(10+4+186+5)=0.0205$$

In accordance with (6.59) $Q_{1x}=q_{1x}^{(1)}q_{1x}^{(2)}=0.01\times0.02=0.0002$. In accordance with (6.61) the WVS reliability is

$$R=P_0Q_{00}+P_1(1-Q_{10}-Q_{1x})=0.5\times0.9796+0.5\times(1-0.0205-0.0002)=0.97945$$

It should be noted that, owing to the additive property of the structure function, when adding a unit to an already evaluated system the new system does not need to be evaluated from scratch. Instead, the operator $\otimes_+$ should be applied to the u-functions of the evaluated system and to the new unit. Moreover, the associative property of the structure function allows the reliability of the WVS to be easily evaluated when it is combined from a number of subsystems for which corresponding u-functions are already obtained.

6.3.1.2 Simplification Technique

Consider the u-function $\tilde{U}_{im}(z)$ that represents the distribution of the score $\tilde{G}_{im}$ of the WVS subsystem λ_m consisting of first m voting units.

Let V_i be the sum of the weights of the units from i to n:

$$V_i = \sum_{j=i}^{n} \psi_j \tag{6.62}$$

One can see that V_{m+1} represents the sum of the weights of WVS units not belonging to λ_m.

The maximal possible value of the WVS score after the remainder of the units add their votes is $\tilde{G}_{im} + (1-\tau)V_{m+1}$ (if all of the units from m+1 to n vote for the proposition acceptance) and the minimal possible value of the WVS score is $\tilde{G}_{im} - \tau V_{m+1}$ (if all of the units from m+1 to n vote for the proposition rejection).

Therefore, if

$$\tilde{G}_{im} + (1-\tau)V_{m+1} < 0 \tag{6.63}$$

the proposition will be rejected independently of the states of the units m+1, …, n. (We will refer to the u-function terms corresponding to the realizations of the score $\tilde{G}_{im}$ meeting condition (6.63) as 0-terms). Indeed, in each 0-term the realization of the score $\tilde{G}_{im}$ is low enough to prevent the total system score from being positive. Therefore, there is no need to continue the calculations by combining the states of the remainder of the units with the states corresponding to 0-terms. The sum of the probabilities of all of the possible combinations of the units m+1, …, n is equal to unity. Therefore, the total overall probability of the unit state combinations in which the score $\tilde{G}_{im}$ guarantees the proposition rejection is equal to the sum of the coefficients of the 0-terms in the u-function $\tilde{U}_{im}(z)$.

If

$$\tilde{G}_{im} - \tau V_{m+1} \geq 0 \tag{6.64}$$

then there is no chance that WVS will reject the proposition even if the units m+1, …, n vote for its rejection. (We will refer to the u-function terms corresponding to the realizations of the score $\tilde{G}_{im}$ meeting condition (6.64) as 1-terms.) Combining any 1-term of $\tilde{U}_{im}(z)$ with any terms corresponding to the not-yet-considered units cannot produce a term with a negative score. Therefore, this term cannot participate in determining the Q_{i0}. This means that all of the 1-terms can be removed from the u-function without affecting the resulting value of Q_{i0}.

The technique described allows one to evaluate the entire WVS reliability using the following algorithm.

1. For each voting element j, define the two u-functions $u_{0j}(z)$ and $u_{1j}(z)$ in the form (6.52) using Equations (6.53) and (6.54).
2. Assign $Q_{10} = Q_{00} = 0$, $\tilde{U}_{01}(z) = u_{01}(z)$, $\tilde{U}_{11}(z) = u_{11}(z)$.
3. For $i = 0, 1$ and $m = 2, \ldots, n$ (voting units can be ordered arbitrarily):
   - remove 1-terms and 0-terms from $\tilde{U}_{\text{im-1}}(z)$;
   - add the coefficients of the removed 0-terms to Q_{i0};
   - obtain $\tilde{U}_{\text{im}}(z) = \tilde{U}_{\text{im-1}}(z) \otimes_{+} u_{\text{im}}(z)$.
3. Add the coefficients of the negative terms in $U_i(z) = \tilde{U}_{in}(z)$ to Q_{i0}.
4. Calculate the fault probability of Q_{11} using Equation (6.59).
5. Calculate the WVS reliability R using Equation (6.61).

Example 6.12

Consider the WVS from Example 6.11 and apply to it the suggested simplification technique. For the given WVS:

$$V_2 = \psi_2 = 3, -\tau V_2 = -1.8, \ (1-\tau)V_2 = 1.2$$

First, assign

$$Q_{10} = Q_{00} = 0$$

$$\tilde{U}_{01}(z) = u_{01}(z) = 10^{-2}(\underline{2z^2} + \mathbf{97z^{-3}} + z^0)$$

$$\tilde{U}_{11}(z) = u_{11}(z) = 10^{-2}(\mathbf{2z^{-3}} + \underline{97z^2} + z^0)$$

The 1-terms in the u-functions are underlined; the 0-terms are marked in bold. The coefficients of the 0-terms are added to Q_{00} and Q_{10}:

$$Q_{00} = 0.97, \ Q_{10} = 0.02$$

After removal of the 0-terms and the 1-terms one obtains

$$\tilde{U}_{01}(z) = 0.01z^0$$

$$\tilde{U}_{11}(z) = 0.01z^0$$

The u-functions for two voting units are

$$\tilde{U}_{02}(z) = \tilde{U}_{01}(z) \underset{+}{\otimes} u_{02}(z) = 10^{-4}z^0(2z^{1.2} + 96z^{-1.8} + 2z^0)$$

$$= 10^{-4}(2z^{1.2} + \mathbf{96z^{-1.8}} + 2z^0)$$

$$\tilde{U}_{12}(z) = \tilde{U}_{11}(z) \underset{+}{\otimes} u_{12}(z) = 10^{-4}z^0(5z^{-1.8}+93z^{1.2}+2z^0)$$

$$= 10^{-4}(\mathbf{5z^{-1.8}}+93z^{1.2}+2z^0)$$

After adding the coefficients of the negative terms (marked in bold) from $\tilde{U}_{I2}(z)$ to Q_{I0} one obtains the same values of Q_{00} and Q_{10} as in Example 6.11:

$$Q_{00} = 0.97+0.0096 = 0.9796,\ \ Q_{10} = 0.02+0.0005 = 0.0205$$

6.3.2 Optimization of Weighted Voting System Reliability

While the reliabilities of the voting units usually cannot be changed when the WVS is built, the weights and the threshold can be chosen in such a way that maximizes the entire system reliability. The WVS optimization problem is, therefore, formulated as follows.

Find the units' weights and the threshold value that maximize the reliability of the WVS consisting of units with the given fault probabilities:

$$R(\psi_1,\dots,\psi_n,\tau)\rightarrow\max \tag{6.65}$$

For an existing WVS with given weights the "tuning" problem can arise in which just the threshold value maximizing the system reliability should be found subject to changing conditions. For example, having information about the probability distribution between propositions that should be accepted or rejected (P_1 and P_0), one can modify the threshold value to achieve the greatest reliability.

Note that the reliability characteristics of WVS units, as well as the propositions' probability distributions, can be elicited from the historical statistics without respect to changes in WVS weights and threshold variation.

6.3.2.1 Implementing the Genetic Algorithm

The natural representation of a WVS weight distribution is by an n-length integer string in which the value in the jth position corresponds to the weight of the jth unit of the WVS. One can see that multiplying all the unit weights by the same value does not affect the WVS output defined by rule (6.46). Therefore, the unit weights can be normalized in such a way that the total weight V_1 is always equal to some constant c. The normalized weights from arbitrary integer string $\boldsymbol{a} = (a_1,\dots,a_n)$ are obtained as follows:

$$\psi_j = a_j c / \sum_{m=1}^{n} a_m \tag{6.66}$$

The range in which the integer numbers are generated affects the precision of weights determination.

For each given weight distribution (determined by the string $\boldsymbol{a}$), the solution decoding procedure obtains the optimal value of the WVS threshold $\tau(\boldsymbol{a}, P_0, P_1)$ by solving the single-variable optimization problem

$$R(\boldsymbol{a}, P_0, P_1, \tau) \rightarrow \max \quad (6.67)$$

and uses the optimal value of the system reliability obtained as the solution fitness.

Example 6.13

Consider a target identification WVS consisting of five voting units making their decisions based on different properties of the target [75]. The voting unit weights for the system can be represented in the GA by an arbitrary integer vector of length 5. In this example we used $c = 10$ and generated the elements of vector $\boldsymbol{a}$ in the range (0, 100). For such parameters, the vector (25 10 55 62 38) produces, according to (6.66), weights $\psi_1 = 1.316$, $\psi_2 = 0.526$, $\psi_3 = 2.895$, $\psi_4 = 3.263$, and $\psi_5 = 2.0$.

The reliability indices of the voting units $q_{i0}^{(j)}$ $q_{i1}^{(j)}$ and $q_{ix}^{(j)}$ are presented in Table 6.16.

Table 6.16. Parameters of WVS units

No. of unit	1	2	3	4	5
$q_{01}^{(j)}$	0.02	0.06	0.07	0.08	0.18
$q_{0x}^{(j)}$	0.08	0.00	0.05	0.16	0.12
$q_{10}^{(j)}$	0.15	0.18	0.07	0.12	0.16
$q_{1x}^{(j)}$	0.20	0.00	0.05	0.16	0.12
ψ_j	2.4862	1.8784	2.4033	1.6851	1.5470

The optimal weights of units obtained by the GA-based optimization procedure for $P_0 = P_1 = 0.5$ are also presented in this table.

The optimal value of threshold is $\tau = 0.412$, for which the system reliability is $R = 0.982$.

To estimate the effect of the input probability distribution on WVS reliability, the optimal threshold values and corresponding system reliabilities were obtained for the WVS with the obtained weights, and for WVS with equal weights as a functions of P_1 ($P_0 = 1-P_1$). These functions are presented in Figure 6.21, where $\tau(P_1)$ and $R(P_1)$ correspond to a WVS with optimal weights obtained for $P_1 = 0.5$, and $\tau^*(P_1)$ and $R^*(P_1)$ correspond to a WVS with equal weights. Note that optimal weights obtained for $P_1 = 0.5$ are not optimal for $P_1 \neq 0.5$, but they provide greater WVS reliability than equal weights on the whole range $0 \le P_1 \le 1$.

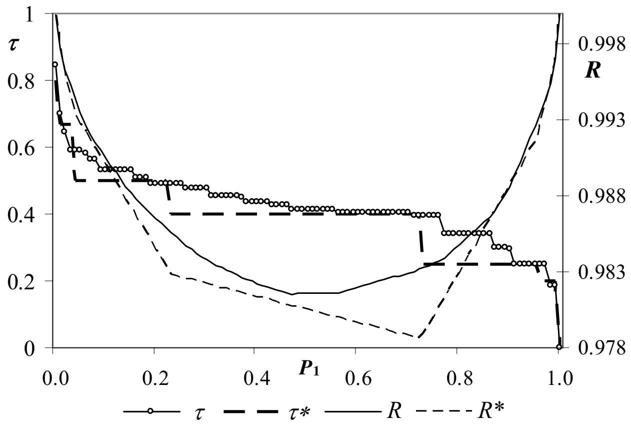


Figure 6.21. Optimal threshold value and WVS reliability as functions of P_1

6.3.3 Weighted Voting System Consisting of Voting Units with Limited Availability

In the WVS model considered in Section 6.3.1, all the voting units were assumed to be fully available (unit unavailability and stuck-at-x failure were not distinguished). In practice, one can deal with separate data concerning unit availability and probabilities of unit failure (wrong decision or abstention) when the unit is in its operating condition.

Two types of WVS can be defined with respect to their treatment of unavailable voting units. In the system of type 1, the unit stuck-at-x failure state and unit inoperable state cannot be distinguished by the system or the system cannot react to information about unit unavailability by changing its weights and threshold. The absence of a unit's output is interpreted by the WVS of this type as abstention from voting. In the system of type 2, the unavailable state of a unit and its abstention from voting can be distinguished and the WVS parameters can be adjusted to optimize its performance for each combination of available voting units.

In the system of type 1, the parameters (weights and threshold) can be chosen only once. The optimal WVS parameters obtained for the system with fully available units can be far away from optimality when the voting units have limited availability. In this section we demonstrate the incorporation of data about units availability into the procedure of parameters optimization for WVS of type 1. For the WVS of type 2 the parameter optimization procedure for fully available units presented in Section 6.3.2 should be applied each time the change of set of available units is detected.

Consider the voting unit j that can be in one of two states: $s_j = 1$ if the unit is available and $s_j = 0$ if it is unavailable. Let the operational availability of unit j be $\Pr\{s_j = 1\} = \alpha_j$. The unit can produce output $d_j(I) \neq x$ only if it is available

.

Therefore, for the given $I = i$ andfor each decision $m \in \{0, 1\}$ of the individualunit j

$$\Pr\{d_j(I) = m \mid I = i\} = \Pr\{ d_j(I) = m \mid I = i, s_j = 1\} \Pr\{s_j = 1\}$$
$$= \alpha_j q_{im}^{(j)} \tag{6.68}$$

When the unit is not available ($s_j = 0$), its output is interpreted by the WVS of type 1 as $d_j(I) = x$. The same output can also be produced by the unit when it is available but indecisive. Therefore:

$$\Pr\{d_j(I) = x \mid I = i \} = \Pr\{ s_j = 0\}$$
$$+\Pr\{d_j(I) = x \mid I = i, s_j = 1\} \Pr\{s_j = 1\} = 1 - \alpha_j + \alpha_j q_{ix}^{(j)} \tag{6.69}$$

Since the output distribution of the available unit is represented by the u-function (6.52)-(6.54), and since each unit j adds the value of zero to the total WVS score when $d_j(I) = x$, one can obtain the u-function $\hat{u}_{ij}(z)$ representing the output distribution of the unit, which has the availability α_j, using the following operator κ:

$$\hat{u}_{ij}(z) = \kappa(u_{ij}(z)) = \alpha_j u_{ij}(z) + (1 - \alpha_j) z^0 \tag{6.70}$$

The u-function $\hat{u}_{ij}(z)$ has the same form as the u-function $u_{ij}(z)$ (6.52)-(6.54), except that its coefficients are

$$\hat{p}_{j0} = \alpha_j p_{j0}$$
$$\hat{p}_{j1} = \alpha_j p_{j1}$$
$$\hat{p}_{j2} = \alpha_j p_{j2} + 1 - \alpha_j \tag{6.71}$$

Using the algorithm presented in Section 6.3.1.2 over u-functions $\hat{u}_{ij}(z)$, one can obtain the reliability of WVS consisting of units with limited availability.

Example 6.14

Consider the WVS of type 1 consisting of four voting units with reliability indices presented in Table 6.17. The optimal weights of units ψ_j obtained for the WVS with fully available units for $P_0 = P_1 = 0.5$ are also presented in this table. The optimal value of the threshold is $\tau = 0.58$, for which the system reliability is $R = 0.891$. Taking into account the limited availability of voting units (availability indices α_j for the units are also presented in Table 6.17), one obtains much lower reliability $R_\alpha = 0.815$ for a WVS with the same weights and threshold. The reliability of a WVS consisting of units with limited availability can be improved if the unit availability values are included in the reliability estimation procedure while the optimization problem is solved. The optimal weights ψ^*_j obtained for the system are presented in Table 6.17. The optimal value of the threshold is $\tau^* = 0.4$, for which the system reliability is $R^*_\alpha = 0.846$. Including information about voting unit availability into the WVS parameters optimization problem enables the system reliability to be improved.

Table 6.17. Parameters of WVS units with limited availability

Unit no.	α_j	$q_{01}^{(j)}$	$q_{0x}^{(j)}$	$q_{10}^{(j)}$	$q_{1x}^{(j)}$	ψ_j	ψ^*_j	I_Rb_j
1	0.76	0.00	0.35	0.35	0.00	2.759	3.459	0.071
2	0.80	0.34	0.10	0.23	0.10	0.172	1.541	0.044
3	0.82	0.11	0.07	0.36	0.06	3.060	2.444	0.065
4	0.78	0.30	0.12	0.07	0.00	4.009	2.556	0.056

In order to find weaknesses in the WVS design and to suggest modifications for system upgrade or to determine the optimal voting unit maintenance policy one has to perform the unit availability importance analysis. According to the definition (4.71), the Birnbaum importance index for the WVS element j can be obtained as

$$I_Rb_j = R_{j1} - R_{j0} \tag{6.72}$$

where R_{j1} is theWVS reliability when the voting unit j is fully available and the remainder of the units have their availability α_j; R_{j0} is the WVS reliability when the voting unit j is unavailable.

An improvement in availability of the unit with the highest importance I_Rb_j causes the greatest increase in WVS reliability.

To determine the voting unit importance in the WVS of type 1, one has to apply the algorithm presented in Section 6.3.1.2 twice: the first time substituting $\alpha_j = 1$ in (6.71) to obtain R_{j1}, and the second time substituting $\alpha_j = 0$ in (6.71) to obtain R_{j0} and then to use Equation (6.72).

Example 6.15

The availability importance indices I_Rb_j of voting units of the WVS from Example 6.14 were obtained for optimal weights ψ^*_j and threshold τ^* [178]. These indices are presented in Table 6.17. The unit availability importance does not depend on the availability of this unit, but it depends strongly on the availability of the rest of units. This dependence is linear. Figure 6.22 presents dependencies of unit importance indices on the availability of voting unit 1. It should be noted that the relative importance of units can vary with variation of unit availability. For example, unit 4 is the most important one in the WVS for $\alpha_1<0.6$, but it becomes the least important one when $\alpha_1>0.92$.

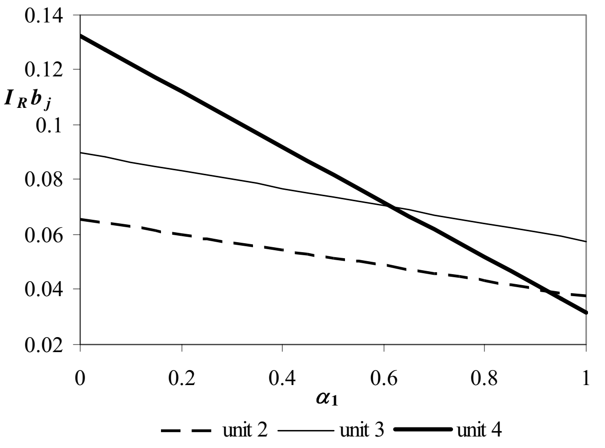


Figure 6.22. Voting unit availability importance as a function of availability α_1

In an adjustable WVS (WVS of type 2), the optimal weights and threshold are found for each combination of voting units available at the moment. Each *h*th combination is represented as the subset λ_h of the set Λ of all of the WVS units. The total number of possible combinations (subsets of Λ) in the WVS consisting of *n* units is 2^n. Let $\psi_j(\lambda_h)$ and $\tau(\lambda_h)$ be the optimal parameters of the WVS consisting of fully available voting units belonging to λ_h, and $R_\alpha(\lambda_h)$ is the reliability of this WVS with the optimal parameters. The entire reliability of the WVS of type 2 can be obtained as follows:

$$R = \sum_{h=1}^{2^n} [R_\alpha(\lambda_h) \prod_{e\in\lambda_h} \alpha_e \prod_{e\notin\lambda_h} (1-\alpha_e)] \tag{6.73}$$

In order to distinguish the availability of voting unit *j*, this expression can be rewritten as follows:

$$R = \alpha_j \sum_{h=1}^{2^{n-1}} [R_\alpha(\mu_h \cup \{j\}) \prod_{e\in\mu_h} \alpha_e \prod_{e\notin\mu_h} (1-\alpha_e)]$$
$$+ (1-\alpha_j) \sum_{h=1}^{2^{n-1}} [R_\alpha(\mu_h) \prod_{e\in\mu_h} \alpha_e \prod_{i\notin\mu_h} (1-\alpha_e)] \tag{6.74}$$

where $\mu_h = \lambda_h \setminus \{j\}$. Using (6.74), one can determine the availability importance of the voting unit *j* as

$$I_R b_j = \partial R / \partial \alpha_j = \sum_{h=1}^{2^{n-1}} [R_\alpha(\mu_h \cup \{j\}) \prod_{e \in \mu_h} \alpha_e \prod_{e \notin \mu_h} (1-\alpha_e)]$$

$$- \sum_{h=1}^{2^{n-1}} [R_\alpha(\mu_h) \prod_{e \in \mu_h} \alpha_e \prod_{e \notin \mu_h} (1-\alpha_e)] \quad (6.75)$$

which is the same as

$$I_R b_j = \sum_{h=1}^{2^n} [\xi_h R_\alpha(\lambda_h) \prod_{e \in \lambda_h} \alpha_e \prod_{e \notin \lambda_h} (1-\alpha_e)] \quad (6.76)$$

where

$$\xi_h = \begin{cases} 1/(\alpha_j - 1), & j \notin \lambda_h \\ 1/\alpha_j, & j \in \lambda_h \end{cases} \quad (6.77)$$

For a WVS in which voting units have high availability, the terms multiplied by $(1-\alpha_e)$ can be neglected and, therefore, Equation (6.76) can be approximated as follows:

$$\tilde{I}_R b_j = \frac{R_\alpha(\Lambda) - R_\alpha(\Lambda \setminus \{j\})}{\alpha_j} \prod_{e=1}^{n} \alpha_e \quad (6.78)$$

In some WVSs of type 2, only the threshold value can be adjusted according to different combinations of available units, whereas unit weights remain the same. Let ψ_1, …, ψ_n be the constant unit weights and $\tau(\lambda_h)$ be the optimal threshold obtained for the given weights for the WVS consisting of fully available voting units belonging to λ_h. For the fixed weights and the optimal threshold $\tau(\lambda_h)$ one can obtain the reliability $R_\alpha^\tau(\lambda_h)$ of subsystem λ_h. Substituting in Equations (6.76) and (6.78) $R_\alpha(\lambda_h)$ with $R_\alpha^\tau(\lambda_h)$, one obtains the availability importance index $I_R B_j^\tau$ for unit j.

Example 6.16

Consider the WVS from Example 6.14. The maximal reliability values obtained by the optimization procedure for each possible combination of WVS units are presented in Table 6.18. Note that $R_\alpha^\tau(\lambda_h)$ (obtained for fixed weights ψ_j from Table 6.17) is always not greater than $R_\alpha(\lambda_h)$. Indeed, optimizing both weights and threshold results in better reliability than that obtained by optimizing just the threshold.

Table 6.18. WVS reliabilities for possible combinations of available voting units

λ_h	$R_\alpha(\lambda_h)$	$R_\alpha^\tau(\lambda_h)$	λ_h	$R_\alpha(\lambda_h)$	$R_\alpha^\tau(\lambda_h)$
∅	0	0	{2,3}	0.740	0.740
{1}	0.650	0.650	{2,4}	0.795	0.789
{2}	0.615	0.615	{3,4}	0.808	0.804
{3}	0.700	0.700	{1,2,3}	0.867	0.866
{4}	0.755	0.755	{1,2,4}	0.868	0.829
{1,2}	0.772	0.755	{1,3,4}	0.889	0.889
{1,3}	0.872	0.859	{2,3,4}	0.843	0.837
{1,4}	0.838	0.817	{1,2,3,4}	0.891	0.891

The voting unit availability importance indices I_Rb_j and $I_Rb_j^\tau$ are presented in Table 6.19. Observe that the relative importance of units differ for different types of WVS adjustment. For example, availability of unit 1 is most important for a WVS with adjustable weights and threshold, whereas in a WVS with adjustable threshold the most important is availability of unit 3.

Table 6.19. Unit availability importance indices for WVS with adjustable parameters

Unit no.	I_Rb_j	$I_Rb_j^\tau$
1	0.080	0.078
2	0.022	0.020
3	0.059	0.082
4	0.063	0.061

One can use Equation (6.78) to estimate the unit's availability importance only when the unit's availability is very high. For example, consider the availability importance indices obtained for the given WVS when the availability of all of its units is 0.99. Table 6.20 contains unit availability importance indices obtained using the exact expression (6.76) and the approximate expression (6.78). The indices take similar values. Table 6.21 contains the same indices obtained for a WVS with the availability of all of its units equal to 0.95. In this case, the difference between the values obtained by the exact and approximate expressions is much greater. Observe that, in both cases, substituting the exact availability importance values with their approximations does not violate the order of units when they are arranged according to their relative importance. Therefore, Equation (6.78) can be used to identify the most important element in the WVS.

Table 6.20. Exact and approximate values of unit availability importance indices ($\alpha_j = 0.99$)

Unit no.	I_Rb_j	$\tilde{I}_Rb_j$	$I_Rb_j^\tau$	$\tilde{I}_Rb_j^\tau$
1	0.0492	0.0464	0.0552	0.0527
2	0.0029	0.0023	0.0028	0.0023
3	0.0249	0.0230	0.0632	0.0609
4	0.0253	0.0232	0.0264	0.0244

Table 6.21. Exact and approximate values of unit availability importance indices ($\alpha_j = 0.95$)

Unit no.	$I_R b_j$	$\tilde{I}_R b_j$	$I_R b_j^\tau$	$\tilde{I}_R b_j^\tau$
1	0.0547	0.0410	0.0588	0.0466
2	0.0053	0.0021	0.0049	0.0021
3	0.0301	0.0203	0.0652	0.0538
4	0.0314	0.0205	0.0317	0.0215

6.3.4 Optimization of Weighted Voting Systems in the Presence of Common Cause Failures

When the voting units of a WVS are subject to CCFs caused by external impacts, the system's survivability can be enhanced by the proper separation of the units. In this section we consider the optimal unit separation problem that is analogous to the one considered in Section 5.2.1 for series-parallel systems. We assume that the units not separated from one another belong to the same CCG and can be destroyed by the same impact (total CCF).

Since the voting units have different decision probability distributions, the way in which they are partitioned into CCGs strongly affects the system's survivability (defined as the probability of making correct decisions). The way the units are separated and the values of the adjustable parameters of the WVS (weights and threshold) are interdependent factors affecting WVS survivability. Therefore, the WVS survivability maximization problem is to find the optimal separation of units, their weights, and the system threshold value.

6.3.4.1 Problem formulation

A WVS consists of n voting units with the given decision probability distributions $q_{i0}^{(j)}$, $q_{i1}^{(j)}$ $q_{ix}^{(j)}$. The units can be separated into B independent groups (see, for example, Figure 6.23), where B can vary from 1 (all of the units are gathered within a single group) to n (all of the units are separated from one another). It is assumed that all of the units belonging to the same group can be destroyed by the total CCF with probability v, which characterizes the WVS vulnerability. The destroyed units cannot produce positive or negative decisions and, therefore, are considered as abstaining ones.

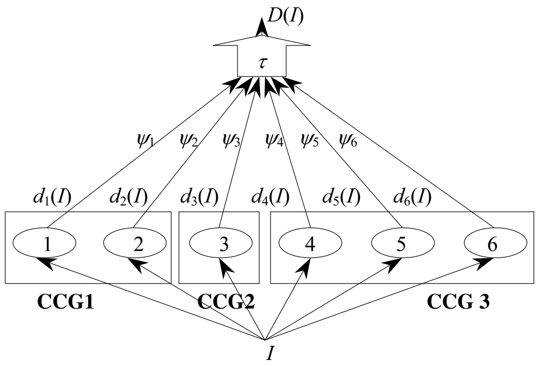


Figure 6.23. Example of WVS with separated voting units

The units' separation problem can be considered as a problem of partitioning a set Λ of n items into a collection of B mutually disjoint subsets λ_b ($1\le b\le B$). Each set can contain from 0 to n elements. The partition of set Λ can be represented by the vector $\boldsymbol{x} = \{x_j: 1\le j\le n\}$, where x_j is the number of the subset to which element j belongs. The weights of voting units in the WVS can be represented by the vector $\boldsymbol{\psi} = \{\psi_j: 1\le j\le n\}$.

The WVS survivability optimization problem is formulated as follows. Find the vectors $\boldsymbol{x}$ (when no more than B different CCGs are allowed) and $\boldsymbol{\psi}$ and the threshold value τ that maximizes the system's survivability $S = \Pr\{D(I) = I\}$.

6.3.4.2 Evaluating Survivability of Weighted Voting Systems with Separated Common Cause Groups

Consider a separated group of voting units λ_b. Let the score distribution for this group be represented by the u-function $U_i^{\lambda_b}(z)$. Note that, since all of the units belonging to λ_b can be destroyed with the probability v, the probability of each state of the group (corresponding to a realization of its random score) should be multiplied by the probability of the group survival: $1-v$. If the group is destroyed, then the entire WVS considers all of the units belonging to λ_b as abstaining. This corresponds to the total score of group $\lambda_b = 0$.

The score of zero can be obtained when all of the units of group λ_b are indecisive or unavailable (because of internal causes) or when they are destroyed by the total CCF. Therefore, the overall probability that the score of separated group $\lambda_b = 0$ for input $I = i$ is

$$\Pr\{\bigcap_{j\in\lambda_b} d_j(I) = x \mid I = i\} = v + (1-v)\prod_{j\in\lambda_b} q_{ix}^{(j)} \tag{6.79}$$

To incorporate the group vulnerability into its score distribution one has to apply the operator ξ (4.58) over the u-function $U_i^{\lambda_b}(z)$

$$\xi(U_i^{\lambda_b}(z)) = (1-v)U_i^{\lambda_b}(z) + vz^0 \tag{6.80}$$

For the given distribution of voting units among CCGs one has to obtain the *u*-functions $U_i^{\lambda_b}(z)$ for each group λ_b:

$$U_i^{\lambda_b}(z) = \underset{+}{\otimes}(\breve{u}_{i1}(z),...,\breve{u}_{in}(z)) \tag{6.81}$$

where

$$\breve{u}_{ij}(z) = \begin{cases} u_{ij}(z), & \text{if } x_j = i \\ 1, & \text{if } x_j \neq i \end{cases} \tag{6.82}$$

The *u*-functions $U_i(z)$ for the entire WVS can then be obtained as

$$U_i(z) = \underset{+}{\otimes}(\xi(U_i^{\lambda_1}(z)), \xi(U_i^{\lambda_2}(z)),..., \xi(U_i^{\lambda_B}(z))) \tag{6.83}$$

After obtaining $U_0(z)$ and $U_1(z)$ one has to determine the system's survivability following steps 3-5 of the algorithm presented in Section 6.3.1.2.

6.3.4.3 Implementing the Genetic Algorithm

Let the WVS have n units that can be distributed among B groups. The system parameters are represented by the n-length integer string $\boldsymbol{a} = (a_1, \ldots, a_n)$ with values of elements ranging in the interval (0,100B). In order to allow the value of a_j to represent both the weight of the jth unit and the number of the group to which it belongs, the following decoding procedure is used:

$$x_j = \lfloor a_j/100 \rfloor + 1,\ \psi_j' = \mathrm{mod}_{100}(a_j) \tag{6.84}$$

The unit weights are further normalized in such a way that their total weight is always equal to some constant c:

$$\psi_j = \psi'_j c / \sum_{k=1}^{N} \psi'_k \tag{6.85}$$

In our GA we used c =10.

Consider the example in which the parameters of a WVS consisting of n = 6 units and up to B = 5 groups are determined by the following string:

$\boldsymbol{a}$ = (264, 57, 74, 408, 221, 23)

Using (6.84) we obtain: $x_1 = x_5 = 3$, $x_2 = x_3 = x_6 = 1$, $x_4 = 5$, vector of the unit weights before normalization $\psi' = (64, 57, 74, 8, 21, 23)$ and $\sum_{j=1}^{6} \psi'_j = 247$.

Now, using (6.85) with constant $c = 10$ we obtain the vector of the normalized unit weights:

$$\psi = (2.59, 2.31, 3.00, 0.32, 0.85, 0.93)$$

The WVS threshold τ is not determined by the string of system parameters. For each set of parameters determined by a solution string $\boldsymbol{a}$, WVS survivability S remains a function of the single argument τ. When WVS survivability is evaluated for a given set of parameters by the solution decoding procedure, this procedure determines the value of τ maximizing S. The maximal S obtained is considered to be a solution fitness, which is used to compare different solutions.

Example 6.17

Consider a WVS from [179] consisting of five voting units with the failure probabilities presented in Table 6.22. The solutions obtained for $P_0 = P_1 = 0.5$ and for vulnerability $v = 0.2$ are presented in Table 6.23. The solutions were obtained for each possible number of separated groups $1 \le B \le 5$. Table 6.23 contains voting unit weights and the WVS threshold for each solution obtained. It also contains for each unit the number of the group the unit belongs to. The values of the system's survivability are presented for each solution.

Table 6.22. Parameters of voting units

No. of unit	$q_{01}^{(j)}$	$q_{0x}^{(j)}$	$q_{10}^{(j)}$	$q_{1x}^{(j)}$
1	0.25	0.27	0.06	0.21
2	0.06	0.40	0.15	0.23
3	0.24	0.08	0.19	0.30
4	0.26	0.31	0.24	0.20
5	0.35	0.13	0.04	0.22

Table 6.23. Parameters of obtained solutions

No. of voting unit	$B = 1$		$B = 2$		$B = 3$		$B = 4$		$B = 5$	
	No. of group	Unit weight	No. of group	Unit weight	No. of group	Unit weight	No. of group	Unit weight	No. of group	Unit weight
1	1	2.381	1	1.969	1	2.263	1	2.413	1	2.302
2	1	2.275	2	2.563	2	2.514	2	2.297	2	2.474
3	1	1.905	2	1.875	2	1.844	3	1.919	3	1.856
4	1	1.085	2	1.531	1	1.034	3	1.047	4	0.997
5	1	2.354	1	2.063	3	2.346	4	2.326	5	2.371
τ	0.560		0.553		0.564		0.563		0.563	
S	0.710		0.824		0.843		0.850		0.852	

The WVS survivability as a function of group vulnerability v is presented in Figure 6.24 for each of the solutions obtained. It can be seen that, for $B = 1$, S is a linear function of v. For $B>1$ the dependencies are polynomial. It can also be seen from Figure 6.24 that the separation into two groups has the greatest effect on the system's survivability, whereas further separation leads to a smaller improvement of *S*. The growth of the group's vulnerability makes the separation more beneficial from the survivability improvement standpoint.

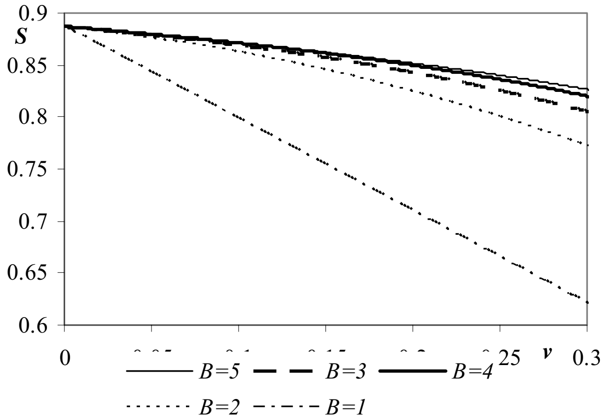


Figure 6.24. WVS survivability as a function of group vulnerability

6.3.5 Asymmetric Weighted Voting Systems

The reliability of a WVS consisting of a given set of voting units can be further improved by taking advantage of the knowledge about the statistical asymmetry of the units (asymmetric probabilities of making correct decisions with respect to the input *I*). In such a WVS, each voting unit *j* has two weights: ψ^0_j, which is assigned to the unit when it votes for the proposition rejection, and ψ^1_j, which is assigned to the unit when it votes for the proposition acceptance. As in the case of the regular (symmetric) voting systems, the proposition is rejected by the WVS if the total weight of the units voting for its acceptance is less than a prespecified fraction τ of the total weight of the non-abstaining units.

6.3.5.1 Evaluating the Reliability of Asymmetric Weighted Voting Systems

The decision rule in the asymmetric WVS takes the form

$$D(I) = x, \quad \text{if} \quad \sum_{d_j(I)\neq x} (\psi_j^0 + \psi_j^1) = 0 \tag{6.86}$$

otherwise:

$$D(I)=\begin{cases}1, & \text{if } \sum_{d_j(I)\neq x}\psi_j^1 d_j(I)\geq\tau\sum_{d_j(I)\neq x}[\psi_j^1 d_j(I)+\psi_j^0(1-d_j(I))]\\ 0, & \text{if } \sum_{d_j(I)\neq x}\psi_j^1 d_j(I)<\tau\sum_{d_j(I)\neq x}[\psi_j^1 d_j(I)+\psi_j^0(1-d_j(I))]\end{cases}\tag{6.87}$$

From this rule we can obtain the condition that $D(I) = 0$:

$$(1-\tau)\sum_{d_j(I)\neq x}\psi_j^1 d_j(I)-\tau\sum_{d_j(I)\neq x}\psi_j^0(1-d_j(I))<0\tag{6.88}$$

This provides a way for tallying the units' votes: each unit j adds the value of $(1-\tau)\psi^1_j$ to the total WVS score if it votes for proposition acceptance, a value of $-\tau\psi^0_j$ if it votes for proposition rejection, and nothing if it abstains. The proposition is rejected if the total score is negative.

One can define the terms of the u-functions (6.52) of the individual voting units as follows:

$$\begin{aligned}&p_{j0}=q_{10}^{(j)},\quad g_{j0}=-\tau\psi_j^0\\&p_{j1}=q_{11}^{(j)}=(1-q_{10}^{(j)}-q_{1x}^{(j)}),\quad g_{j1}=(1-\tau)\psi_j^1\\&p_{j2}=q_{1x}^{(j)},\quad g_{j2}=0\end{aligned}\tag{6.89}$$

for $i = 1$ and

$$\begin{aligned}&p_{j0}=q_{01}^{(j)},\quad g_{j0}=(1-\tau)\psi_j^1\\&p_{j1}=q_{00}^{(j)}=(1-q_{01}^{(j)}-q_{0x}^{(j)}),\quad g_{j1}=-\tau\psi_j^0\\&p_{j2}=q_{0x}^{(j)},\quad g_{j2}=0\end{aligned}\tag{6.90}$$

for $i = 0$ and obtain the system reliability applying steps 2-5 of the algorithm presented in Section 6.3.1.2.

It can be easily seen that the conditions (6.63) and (6.64) in the simplification technique should be replaced in the case of the asymmetric WVS by the conditions

$$\widetilde{G}_{im}+(1-\tau)V^1{}_{m+1}<0\tag{6.91}$$

and

$$\widetilde{G}_{im}-\tau V^0{}_{m+1}\geq 0\tag{6.92}$$

respectively, where

$$V^i{}_m = \sum_{j=m}^{n} \psi^i_m \text{ , for } i = 0, 1 \tag{6.93}$$

Example 6.18

Given

$$n = 2, P_0 = P_1 = 0.5, \tau = 0.6$$

and that the parameters of the first voting unit are

$$q_{01}{}^{(1)} = 0.02, q_{10}{}^{(1)} = 0.02, q_{0x}{}^{(1)} = q_{1x}{}^{(1)} = 0.01, \psi^0{}_1 = 2, \psi^1{}_1 = 4$$

and the parameters of the second voting unit are

$$q_{01}{}^{(2)} = 0.02, q_{10}{}^{(2)} = 0.05, q_{0x}{}^{(2)} = q_{1x}{}^{(2)} = 0.02, \psi^0{}_2 = 3, \psi^1{}_2 = 1$$

then:

$$(1-\tau)\psi^1{}_1 = 1.6, -\tau\psi^0{}_1 = -1.2, (1-\tau)\psi^1{}_2 = 0.4, -\tau\psi^0{}_2 = -1.8$$

For the given weights of the second unit:

$$V^0{}_2 = \psi^0{}_2 = 3, V^1{}_2 = \psi^1{}_2 = 1$$

$$-\tau V^0{}_2 = -1.8, (1-\tau)V^1{}_2 = 0.4$$

The u-functions of the units are

$$u_{01}(z) = 10^{-2}(2z^{1.6} + \mathbf{97z^{-1.2}} + z^0), \quad u_{11}(z) = 10^{-2}(\mathbf{2z^{-1.2}} + 97z^{1.6} + z^0)$$

$$u_{02}(z) = 10^{-2}(2z^{0.4} + 96z^{-1.8} + 2z^0), \quad u_{12}(z) = 10^{-2}(5z^{-1.8} + 93z^{0.4} + 2z^0)$$

There are no 1-terms in $u_{01}(z)$ and $u_{11}(z)$. The 0-terms are marked in bold.
First, assign

$$Q_{00} = Q_{10} = 0$$

$$\tilde{U}_{01}(z) = u_{01}(z), \tilde{U}_{11}(z) = u_{11}(z)$$

After the 0-terms removal we obtain

$$Q_{00} = 0.97, Q_{10} = 0.02$$

$$\tilde{U}_{01}(z) = 0.02z^{1.6} + 0.01z^0, \tilde{U}_{11}(z) = 0.97z^{1.6} + 0.01z^0$$

The u-functions for two voting units are

$$\tilde{U}_{02}(z) = \tilde{U}_{01}(z) \underset{+}{\otimes} u_{02}(z) = 10^{-4}(2z^{1.6}+z^{0})(2z^{0.4}+96z^{-1.8}+2z^{0})$$

$$= 10^{-4}(4z^{2}+\mathbf{192z^{-0.2}}+4z^{1.6}+2z^{0.4}+\mathbf{96z^{-1.8}}+2z^{0}),$$

$$\tilde{U}_{12}(z) = \tilde{U}_{11}(z) \underset{+}{\otimes} u_{12}(z) = 10^{-4}(97z^{1.6}+1z^{0})(5z^{-1.8}+93z^{0.4}+2z^{0})$$

$$= 10^{-4}(\mathbf{485z^{-0.2}}+9021z^{2}+194z^{1.6}\mathbf{+5z^{-1.8}}+93z^{0.4}+2z^{0})$$

The terms with the negative exponents are marked in bold. Finally we obtain:

$$Q_{00} = 0.97+0.0192+0.0096 = 0.9988$$

$$Q_{10} = 0.02+0.0485+0.0005 = 0.0690$$

$$Q_{1x} = q_{1x}^{(1)} q_{1x}^{(2)} = 0.01 \times 0.02 = 0.0002$$

$$R = P_0 Q_{00} + P_1(1 - Q_{10} - Q_{1x})$$

$$= 0.5 \times 0.9988 + 0.5 \times (1 - 0.069 - 0.0002) = 0.9648$$

6.3.5.2 Optimization of Asymmetric Weighted Voting Systems

The parameter optimization problem for asymmetric WVSs can be formulated as follows:

$$R(\psi^{0}_{1}, \psi^{1}_{1}, \ldots, \psi^{0}_{n}, \psi^{1}_{n}, \tau) \rightarrow \max \tag{6.94}$$

The natural representation of a WVS weight distribution is by a $2n$-length integer string $\boldsymbol{a}$ in which the values in a_{2j-1} and a_{2j} correspond to the weights ψ^{0}_{j} and ψ^{1}_{j} respectively. The unit weights can be normalized in such a way that the total weight is always equal to some constant c. The normalized weights from arbitrary integer string $\boldsymbol{a} = (a_1, \ldots, a_{2n})$ are obtained as follows:

$$\psi_j^0 = a_{2j-1} c \Big/ \sum_{i=1}^{2n} a_i, \qquad \psi_j^1 = a_{2j} c \Big/ \sum_{i=1}^{2n} a_i \tag{6.95}$$

where c is a constant.

The solution decoding procedure determines the value of τ maximizing R for given unit weights. The obtained maximal $R(\boldsymbol{a})$ is considered as a solution fitness, which is used to compare different solutions.

Example 6.19

Consider a WVS consisting of five voting units with reliability indices presented in Table 6.24 [180]. The optimal weights of units obtained for a symmetric WVS ($\psi_j = \psi^0{}_j = \psi^1{}_j$) and for an asymmetric WVS when $P_0 = P_1 = 0.5$ are also presented in this table. The optimal values of the threshold, the decision probabilities, and the reliability of symmetric and asymmetric WVS obtained are presented in Table 6.25.

Table 6.24. Parameters of voting units

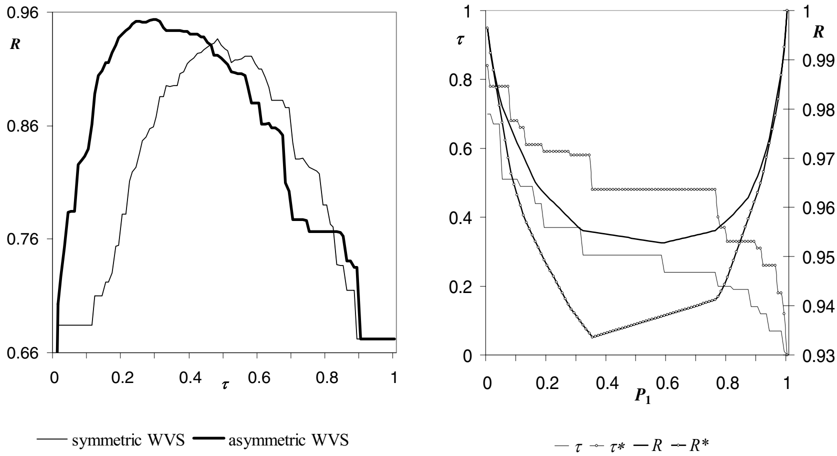
No. of unit	1	2	3	4	5
q_{01}	0.224	0.243	0.208	0.000	0.204
q_{0x}	0.209	0.077	0.073	0.249	0.168
q_{10}	0.287	0.219	0.103	0.197	0.133
q_{1x}	0.025	0.106	0.197	0.014	0.067
ψ (symmetric WVS)	1.155	1.763	1.915	3.040	2.128
ψ^1	0.402	1.796	2.848	0.526	0.557
ψ^0	0.372	0.124	0.031	2.693	0.650

Table 6.25. Parameters of optimal WVSs

	Symmetric WVS	Asymmetric WVS
τ	0.48	0.30
Q_{00}	0.927	0.958
Q_{01}	0.073	0.042
Q_{0x}	4.9E-05	4.9E-05
Q_{10}	0.054	0.051
Q_{11}	0.946	0.949
Q_{1x}	4.9E-07	4.9E-07
R	0.936	0.954

Observe that the asymmetric WVS is more reliable than the symmetric one. The system reliability as a function of the threshold value is presented in Figure 6.25A for the both WVSs with weights from Table 6.24 (note that for $\tau = 0$ $R = P_1(1-\prod_{j=1}^{n} q_{1x}^{(j)})$ and for $\tau = 1$, $R = P_0(1-\prod_{j=1}^{n} q_{01}^{(j)} - \prod_{j=1}^{n} q_{0x}^{(j)}) + P_1\prod_{j=1}^{n} q_{11}^{(j)})$

For an existing WVS with given weights, the "turning" problem can arise in which just the threshold value maximizing the system reliability should be found subject to changing conditions. For example, based on information about the probability distribution P_i between propositions that should be accepted or rejected, one can modify the threshold value to achieve the greatest reliability. To estimate the effect of the input probability distribution on WVS reliability, the optimal threshold values, and the corresponding system reliabilities were obtained for the two WVSs as functions of P_1. These functions are presented in Figure 6.25B, where $\tau(P_1)$ and $R(P_1)$ correspond to the asymmetric WVS and $\tau^*(P_1)$ and $R^*(P_1)$ correspond to the symmetric WVS (weights of the both WVSs are optimal for $P_1 = 0.5$).



A **B**

Figure 6.25. WVS reliability as a function of threshold and input reliability distribution

6.3.6 Weighted Voting System Decision-making Time

This section addresses the aspect of the WVS decision-making time. In many technical systems the time when the output (decision) of each voting unit is available is predetermined. For example, the decision time of a chemical analyzer is determined by the time of a chemical reaction. The decision time of a target detection radar system is determined by the time of the radio signal return and by the time of the signal processing by the electronic subsystem. In both these cases the variation of the decision times for a single voting unit is usually negligibly small.

On the contrary, the decision time of the entire WVS composed of voting units with different constant decision times can vary because in some cases the decisions of the slow voting units do not affect the decision of the entire system since this decision becomes evident after the faster units have voted. This happens when the total weight of the units voting for the proposition acceptance or rejection is enough to guarantee the system's decision independently of the decisions of the units that have not yet voted. In such situations, the voting process can be terminated without waiting for the slow units' decisions and the WVS decision can be made in a shorter time.

6.3.6.1 Determination of Weighted Voting System Decision Time Distribution

Assume that each voting unit j needs a fixed time t_j to produce its decision and all the WVS units are arranged in order of the decision time increase: $t_j<t_{j+1}$. In this case, u-functions $\tilde{U}_{im}(z)$ represent the distribution of score $\tilde{G}_{im}$ obtained by the voting of m fastest units. As was shown in Section 6.3.1.2, the 1-terms and 0-terms in the u-function $\tilde{U}_{im}(z)$ correspond to combinations of decisions of the first m

units that guarantee the entire WVS decision (proposition acceptance and rejection respectively) independent of the decisions of the rest of the units. The sum of the coefficients of these terms in $\tilde{U}_{im}(z)$ is equal to π_{im}, the conditional probability that the WVS decision can be made at time t_m given the system input is i. By determining π_{im} as the sum of the coefficients of the removed 0-terms and 1-terms for each u-function $\tilde{U}_{im}(z)$ in step 3 of the algorithm presented in Section 6.3.1.2, we obtain the probabilities that the WVS decision time is equal to t_m.

Having the WVS decision time distribution represented by values of π_{im} and t_m for $m = 1, \ldots, n$ we obtain the expected WVS decision-making time as

$$\varepsilon = P_0 \sum_{m=1}^{n} \pi_{0m} t_m + P_1 \sum_{m=1}^{n} \pi_{1m} t_m \tag{6.96}$$

Example 6.20

Consider a WVS with parameters

$$n = 3,\ P_0 = P_1 = 0.5,\ \tau = 0.6$$

The parameters of the three voting units are:

$$q_{01}^{(1)} = 0.02,\ q_{10}^{(1)} = 0.02,\ q_{0\mathrm{x}}^{(1)} = q_{1x}^{(1)} = 0.01,\ \psi^0{}_1 = 3,\ \psi^1{}_1 = 5,\ t_1=1$$

$$q_{01}^{(2)} = 0.02,\ q_{10}^{(2)} = 0.05,\ q_{0x}^{(2)} = q_{1x}^{(2)} = 0.02,\ \psi^0{}_2 = 4,\ \psi^1{}_2 = 3,\ t_2=2$$

$$q_{01}^{(3)} = 0.01,\ q_{10}^{(3)} = 0.03,\ q_{0x}^{(3)} = q_{1x}^{(3)} = 0.0,\ \psi^0{}_3 = 3,\ \psi^1{}_3 = 2,\ t_3=4$$

For the given parameters we have

$$(1-\tau)\psi^1{}_1 = 2,\ (1-\tau)\psi^1{}_2 = 1.2,\ (1-\tau)\psi^1{}_3 = 0.8$$

$$-\tau\psi^0{}_1 = -1.8,\ -\tau\psi^0{}_2 = -2.4,\ -\tau\psi^0{}_3 = -1.8$$

and

$$\tau V^0{}_2 = \tau(\psi^0{}_2+\psi^0{}_3) = 0.6\times 7 = 4.2,\ \tau V^0{}_3 = \tau\psi^0{}_3 = 0.6\times 3=1.8,\ \tau V^0{}_4 = 0$$

$$(\tau-1)V^1{}_2 = (\tau-1)(\psi^1{}_2+\psi^1{}_3) = -0.4\times 8 = -3.2$$

$$(\tau-1)V^1{}_3 = (\tau-1)\psi^1{}_3 = -0.4\times 2 = -0.8,\ (\tau-1)V^1{}_3 = 0$$

The u-functions for the individual voting units are

$$u_{01}(z) = 10^{-2}(2z^{2}+z^{0}+97z^{-1.8}),\ \ u_{11}(z) = 10^{-2}(2z^{-1.8}+z^{0}+97z^{2})$$

$$u_{02}(z) = 10^{-2}(2z^{1.2}+2z^0+96z^{-2.4}),\ \ u_{12}(z) = 10^{-2}(5z^{-2.4}+2z^0+93z^{1.2})$$

$$u_{03}(z) = 10^{-2}(1z^{0.8}+99z^{-1.8}),\ \ u_{13}(z) = 10^{-2}(3z^{-1.8}+97z^{0.8})$$

First, we assign

$$Q_{00} = Q_{11} = 0$$

$$\tilde{U}_{01}(z) = u_{01}(z),\ \tilde{U}_{11}(z) = u_{11}(z)$$

The u-functions $u_{01}(z)$ and $u_{11}(z)$ contain neither 1-terms nor 0-terms. This means that the WVS cannot make any decision based on voting of the first unit and

$$\pi_{01} = \pi_{11} = 0$$

We also can add nothing to Q_{00} and Q_{11}. The u-functions for the subsystem consisting of two units are

$$\tilde{U}_{02}(z) = \tilde{U}_{01}(z) \underset{+}{\otimes} u_{02}(z) = 10^{-4}(2z^2+z^0+97z^{-1.8})(2z^{1.2}+2z^0+96z^{-2.4})$$

$$= 10^{-4}(\underline{4z^{3.2}}+\underline{4z^2}+192z^{-0.4}+2z^{1.2}+2z^0+\mathbf{96z^{-2.4}}+194z^{-0.6}+\mathbf{194z^{-1.8}}+\mathbf{9312z^{-4.2}})$$

$$\tilde{U}_{12}(z) = \tilde{U}_{11}(z) \underset{+}{\otimes} u_{12}(z) = 10^{-4}(2z^{-1.8}+z^0+97z^2)(5z^{-2.4}+2z^0+93z^{1.2})$$

$$=10^{-4}(\mathbf{10z^{-4.2}}+\mathbf{4z^{-1.8}}+186z^{-0.6}+\mathbf{5z^{-2.4}}+2z^0+93z^{1.2}+485z^{-0.4}+\underline{194z^2}+\underline{9021z^{3.2}})$$

In these u-functions, the 1-terms are underlined and the 0-terms are marked in bold.

The sums of the coefficients of all of the marked terms are

$$\pi_{02} = 10^{-4}(4+4+96+194+9312) = 0.961$$

$$\pi_{12} = 10^{-4}(10+4+5+194+9021) = 0.9234$$

The sums of coefficients of 0-terms are

$$Q_{00} = 10^{-4}(96+194+9312) = 0.9602$$

$$Q_{10} = 10^{-4}(10+4+5) = 0.0019$$

After removing the marked terms, the u-functions take the form

$$\tilde{U}_{02}(z) = 10^{-4}(192z^{-0.4}+2z^{1.2}+2z^0+194z^{-0.6})$$

$$\tilde{U}_{12}(z) = 10^{-4}(186z^{-0.6}+2z^{0}+93z^{1.2}+485z^{-0.4})$$

The UGF for a subsystem consisting of three units is

$$\tilde{U}_{03}(z) = \tilde{U}_{02}(z) \underset{+}{\otimes} u_{03}(z)$$

$$= 10^{-6}(192z^{-0.4}+2z^{1.2}+2z^{0}+194z^{-0.6})(1z^{0.8}+99z^{-1.8}) = 10^{-6}(\underline{192z^{0.4}}$$

$$+\underline{2z^{2.0}}+\underline{2z^{0.8}}+\underline{194z^{0.2}}+\mathbf{19008}z^{\mathbf{-2.2}}+\mathbf{198}z^{\mathbf{-0.6}}+\mathbf{198}z^{\mathbf{-1.8}}+\mathbf{19206}z^{\mathbf{-2.4}})$$

$$\tilde{U}_{13}(z) = \tilde{U}_{12}(z) \underset{+}{\otimes} u_{13}(z)$$

$$= 10^{-6}(186z^{-0.6}+2z^{0}+93z^{1.2}+485z^{-0.4})(3z^{-1.8}+97z^{0.8}) = 10^{-6}(\mathbf{558}z^{\mathbf{-2.4}}$$

$$\mathbf{+6}z^{\mathbf{-1.8}}\mathbf{+279}z^{\mathbf{-0.6}}\mathbf{+1455}z^{\mathbf{-2.2}}+\underline{18042z^{0.2}}+\underline{194z^{0.8}}+\underline{9021z^{3.0}}+\underline{47045z^{0.4}})$$

In the final u-function, all of the terms are either 1-terms or 0-terms. Summing the coefficients of the terms we obtain

$$\pi_{03} = 10^{-6}(192+2+2+194+19008+198+198+19206) = 0.039$$

$$\pi_{13} = 10^{-6}(558+6+279+1455+18042+194+9021+47045) = 0.0766$$

and adding the coefficients of 0-terms to Q_{I0} we obtain

$$Q_{00} = 0.9602+10^{-6}(19008+198+198+19206) = 0.99881$$

$$Q_{10} = 0.0019+10^{-6}(558+6+279+1455) = 0.004198$$

Since $Q_{1x} = q_{1x}^{(1)}\, q_{1x}^{(2)}\, q_{1x}^{(3)} = 0$

$$Q_{11} = 1 - Q_{10} = 0.995802$$

The WVS reliability is

$$R = P_0Q_{00} + P_1Q_{11} = 0.5\times0.99881+0.5\times0.995802 = 0.997306$$

The expected decision time is

$$\varepsilon = 0.5(0.961\times2 + 0.039\times4) + 0.5(0.9234\times2 + 0.0766\times4) = 2.1156$$

6.3.6.2 Weighted Voting System Reliability Optimization Subject to Decision-time Constraint

The number of combinations of unit decisions that allow the entire system's decision to be obtained before the outputs of all of the units become available depends on the unit weights distribution and on the threshold value. By increasing the weights of the fastest units one makes the WVS more decisive in the initial stage of voting and, therefore, reduces the mean system decision time by the price of making it less reliable.

In applications where the WVS should make many decisions in a limited time, the expected system decision time is considered to be a measure of its performance. Since the units' weights and threshold affect both the WVS's reliability and its expected decision time, the problem of the optimal system turning can be formulated as follows: find the voting units' weights and the threshold that maximize the system reliability R while providing the expected decision time ε not greater than a prespecified value ε^*:

$$R(\psi^0{}_1, \psi^1{}_1, \ldots, \psi^0{}_n, \psi^1{}_n, \tau) \rightarrow \max$$

$$\text{subject to } \varepsilon(\psi^0{}_1, \psi^1{}_1, \ldots, \psi^0{}_n, \psi^1{}_n, \tau) \leq \varepsilon^* \tag{6.97}$$

The solution encoding for solving this problem by the GA is the same as in Section 6.3.5.2. The only difference is in the solution fitness formulation. In the constrained problem, the fitness of a solution defined by the integer string $\boldsymbol{a}$ is determined as $R(\boldsymbol{a}) - \pi \max(\varepsilon - \varepsilon^*, 0)$, where π is a penalty coefficient.

Example 6.21

A WVS consists of six voting units with the voting times and fault probabilities presented in Table 6.26. The optimal voting unit weights and thresholds and the parameters of the optimal WVS obtained for $\varepsilon^* = 35$ (when P_0=0.7, P_0=0.5, P_0=0.3) are presented in Tables 6.27 and 6.28.

Table 6.26. Parameters of voting units

No. of unit j	t_j	$q_{01}^{(j)}$	$q_{0x}^{(j)}$	$q_{10}^{(j)}$	$q_{1x}^{(j)}$
1	10	0.22	0.31	0.29	0.12
2	12	0.35	0.07	0.103	0.30
3	38	0.24	0.08	0.22	0.15
4	48	0.10	0.05	0.2	0.01
5	55	0.08	0.10	0.15	0.07
6	70	0.08	0.01	0.10	0.05

The system abstention probabilities do not depend on its weights and threshold. For any solution, $Q_{0x} = 0.868\times10^{-7}$ and $Q_{1x} = 1.89\times10^{-7}$.

It can be seen that for $P_0 \neq 0.5$ the WVS takes advantage of the knowledge about statistical asymmetry of the input and provides greater reliability than in the case where $P_0 = 0.5$. Observe that when $P_0>0.5$ the WVS provides Q_{00} greater than Q_{11}, and *vice versa* when $P_0<0.5$ $Q_{00}<Q_{11}$.

The R vs. ε trade-off curves for the WVS are presented in Figure 6.26. These curves are obtained by solving the optimization problem (6.97) for different values of time constraint ε*.

Table 6.27. Optimal unit weights for ε^* = 35

	No. of unit j	1	2	3	4	5	6
p_0=0.7	ψ^1_j	0.018	0.240	0.564	0.300	0.388	0.476
	ψ^0_j	1.958	0.018	0.370	2.487	1.005	0.176
p_0=0.5	ψ^1_j	0.017	2.367	0.497	0.017	0.635	1.475
	ψ^0_j	2.281	0.360	0.189	1.561	0.566	0.034
p_0=0.3	ψ^1_j	0.019	2.597	1.243	0.019	0.019	0.742
	ψ^0_j	1.688	0.334	0.204	1.967	0.909	0.260

Table 6.28. Parameters of WVS optimal for ε^* = 35

	P_0 = 0.7	P_0 = 0.5	P_0 = 0.3
τ	0.76	0.50	0.45
Q_{00}	0.9798	0.9005	0.8477
Q_{01}	0.0202	0.0995	0.1523
Q_{10}	0.1611	0.0719	0.0283
Q_{11}	0.8389	0.9281	0.97166
R	0.9375	0.9143	0.9345
ε	34.994	34.987	34.994

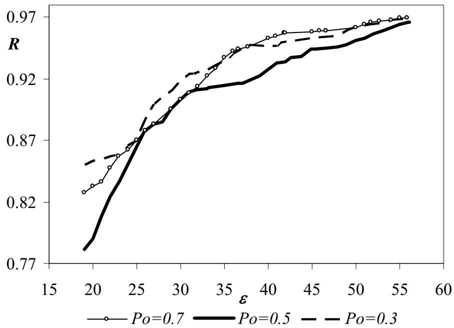


Figure 6.26. Reliability vs. expected decision time for P_0 = 0.7, P_0 = 0.5, P_0 = 0.3

6.3.7 Weighted Voting Classifiers

The weighted voting classifier (WVC) should classify objects belonging to a set of H classes. When an object belonging to some class I ($1\le I\le H$) is presented to the system, its classification decision $D(I)$ is based on the classification decisions made by a set of n independent voting units. Each unit j while identifying objects from class I generates its individual classification decision $d_j(I) \in \{1, \ldots, H\}$. This decision can be correct $d_j(I) = I$ or incorrect $d_j(I) \neq I$. The unit can also abstain from voting $d_j(I) = 0$ (note that the abstention is always considered to be the wrong decision because $I \neq 0$). Each unit j has its individual weight ψ_j depending on the importance of its decision to the entire system.

Given the outputs of the individual units, the WVC can calculate for each classification decision h ($0 \le h \le H$) the sum of the weights of the units supporting this decision:

$$\Psi_I^h = \sum_{d_j(I)=h} \psi_j \tag{6.98}$$

The decision h' that obtained the greatest sum of the weights is determined as

$$\Psi_I^{h'} \ge \Psi_I^h \text{ for } 1 \le h \le H \tag{6.99}$$

(if there are several such decisions any of them can be chosen at random). The second best decision h'' can be determined as

$$\Psi_I^{h''} \ge \Psi_I^h \text{ for any } h \neq h' \tag{6.100}$$

There exist two different ways of making the entire WVC decision. The first one is based on a plurality voting rule. Using this rule, the entire WVC output is calculated as follows:

$$D(I) = \begin{cases} h', & \Psi_I^{h'} > \Psi_I^{h''} \\ 0, & \Psi_I^{h'} = \Psi_I^{h''} \end{cases} \tag{6.101}$$

which means that the WVC is able to classify the input if there exists an ultimate majority of weighted votes corresponding to some output h'. One can see that the system abstains from making a decision in two cases:

- all the units abstain from making a decision;
- more than one decision has the same support while the remaining decisions are supported less.

The second manner of decision making is based on a threshold voting rule. Using this rule, the WVC output is calculated as follows:

$$D(I) = \begin{cases} h', & \Psi_I^{h'} > \tau V_1 \\ 0, & \text{otherwise} \end{cases} \tag{6.102}$$

where, according to (6.62), V_1 is the sum of the weights of all of the voting units. If $\tau \geq 0.5$, then no more than one decision can satisfy the condition $\Psi_i^h > \tau V_1$. The greater τ, the less decisive the WVC. Indeed, the system is inclined to abstain when τ grows, since a lower number of combinations of voters outputs produces the winning decisions.

The entire WVC reliability can be defined as the probability that it makes the correct decisions: $R = \Pr\{D(I) = I\}$.

Each system unit has the probabilities of incorrect classification and abstention. It is natural that the probability of incorrect output depends on the class of the input object for each unit (for example, in target detecting systems some targets can be unrecognizable by speed detectors while highly recognizable by heat radiation detectors and *vice versa*). The same is true for the unit abstention probability $\Pr\{d_j(I) = 0\}$. Therefore, to define the probabilistic behavior of units one has to determine their fault probabilities $q_{ih}^{(j)}$ for $1 \leq i \leq H$, $0 \leq h \leq H$ ($h \neq i$), $1 \leq j \leq n$, where $q_{ih}^{(j)} = \Pr\{d_j(I) = h \mid I = i\}$.

Conditional unit success (correct classification) probability, given the system input is $I = i$, can be determined, therefore, as

$$q_{ii}^{(j)} = \Pr\{d_j(I) = i \mid I = i\} = 1 - \sum_{0 \leq h \leq H, h \neq i} q_{ih}^{(j)} \tag{6.103}$$

The probability of correct classification of an object belonging to class i by the entire WVC $r_i = \Pr\{D(I) = I \mid I = i\}$ depends on the probabilities $q_{ih}^{(j)}$. Since the correct identifications of the objects belonging to different classes are mutually exclusive events, one can obtain the entire system reliability as

$$R = \sum_{i=1}^{H} P_i r_i \tag{6.104}$$

where the probabilities $P_i = \Pr\{I = i\}$ for $1 \leq i \leq H$ define the input probability distribution. (In the most common special case of evenly distributed input $R = \frac{1}{H} \sum_{i=1}^{H} r_i$).

The different states of WVC can be distinguished by the unit output distribution (UOD). WVC consisting of n voting units can have $(H+1)^n$ different states corresponding to different combinations of unit outputs (each unit can produce $H+1$ different outputs). Each WVC state can be characterized by a distribution of the weights of the units supporting different classification decisions named voting weight distribution (VWD).

Note that some different UODs can result in the same VWD. (For example, in a WVC with $n = 3$, $\psi_1 = \psi_2 = 1$ and $\psi_3 = 2$, UOD $d_1(I) = 1$, $d_2(I) = 1$ and $d_3(I) = 0$ results in the same VWD $\Psi_I^0 = \Psi_I^1 = 2$ as that of UOD $d_1(I) = 0$, $d_2(I) = 0$, $d_3(I) = 1$). From the entire WVC output point of view, these different UODs are indistinguishable and, therefore, can be treated as the same state.

To define the VWD of a classifier in state k one can use vector $\boldsymbol{g}_k=\{g_k(h)\}$, $1\le h\le H$, in which $g_k(h)$ is equal to Ψ_I^h in state k. Using the UGF approach one can describe distributions of random VWD $\boldsymbol{G}_j$ of an individual unit j as

$$u_{ij}(z)=\sum_{k=0}^{H} q_{ik}^{(j)} z^{\boldsymbol{g}_{jk}} \tag{6.105}$$

In this u-function each state k has the probability $q_{ik}^{(j)}$ and corresponds to the unit output $d_j(I)=k$ (when $I=i$) and, therefore, to VWD $\boldsymbol{g}_{jk}$ in which

$$g_{jk}(i)=\begin{cases}\psi_j, & i=k\\ 0, & i\neq k\end{cases} \tag{6.106}$$

(for $k=0$, corresponding to element abstention, the vector contains only zeros).

Since the total weight of votes supporting any classification decision in the WVC is equal to the sum of weights of individual units supporting this decision, the resulting system VWD $\boldsymbol{G}$ can be obtained by summing the random VWDs $\boldsymbol{G}_j$ ($1\le j\le n$) of individual voters. Therefore, the distribution $\boldsymbol{G}$ can be represented by the u-function

$$U_i(z)=\underset{+}{\otimes}(u_{i1}(z),\ldots,u_{in}(z))=\sum_k p_{ik} z^{\boldsymbol{g}_k} \tag{6.107}$$

(note that in this operator the exponents are obtained as sums of vectors, not scalar variables). Since the procedure of vector summation possesses commutative and associative properties, the u-function of the entire WVC can be obtained recursively by the consecutive determination of u-functions of arbitrary subsets of elements. For example, it can be obtained by the recursive procedure (6.57).

By following the decision rule (6.101) one can obtain the entire WVC output in each state k (for each term of $U_i(z)$) as

$$D_k(I)=\begin{cases}x, & g_k(x)>\max\limits_{1\le h\le H, h\neq x} g_k(h)\\ 0, & \text{otherwise}\end{cases} \tag{6.108}$$

By following the decision rule (6.102) one can obtain the entire WVC output in each state k (for each term of $U_i(z)$) as

$$D_k(I) = \begin{cases} x, & g_k(x) > \tau V_1 \\ 0, & \text{otherwise} \end{cases} \quad (6.109)$$

The correct classification corresponds to states in which $D_k(I) = i$ when $I = i$ and, therefore, to those u-function terms (further referred to as cc-terms) for which $g_k(i) > \max_{1 \le h \le H, h \ne i} g_k(h)$ for plurality voting or $g_k(i) > \tau V_1$ for threshold voting.

Using the acceptability function

$$F_i(\boldsymbol{G}) = \begin{cases} 1, & D(I) = i \\ 0, & D(I) \ne i \end{cases} \quad (6.110)$$

over $U_i(z)$ representing all the possible WVC classification results, one can obtain the probability of successfully identifying the object of class i for plurality voting as follows:

$$r_i = E(F_i(\boldsymbol{G})) = \sum_k p_{ik} 1(g_k(i) > \max_{1 \le h \le H, h \ne i} g_k(h)) \quad (6.111)$$

and for threshold voting as follows:

$$r_i = E(F_i(\boldsymbol{G})) = \sum_k p_{ik} 1(g_k(i) > \tau V_1) \quad (6.112)$$

Example 6.22

Consider a WVC consisting of two units ($n = 2$) that classifies objects belonging to three different classes ($H = 3$). The probabilities of wrong classification for each type of input object are presented in Table 2.29, as well as the probabilities of correct classification calculated in accordance with (6.103).

Table 6.29. Parameters of WVC units

		Unit 1			Unit 2		
		$i=1$	$i=2$	$i=3$	$i=1$	$i=2$	$i=3$
q_{ih}	h=1	0.94	0.01	0.06	0.68	0.3	0.01
	h=2	0.02	0.95	0.05	0.3	0.63	0.01
	h=3	0.04	0.02	0.85	0.01	0.05	0.97
	h=0	0.0	0.02	0.04	0.01	0.02	0.01

Note that unit 2 can scarcely distinguish objects of class 1 and 2 whereas specializing in the identification of objects of class 3. On the contrary, unit 1 specializes in recognizing objects of classes 1 and 2. Weights of units are $\psi_1 = 2$ and $\psi_2 = 1$. Input probability distribution is $P_1 = P_2 = P_3 = 1/3$. The threshold value is $\tau = 0.5$ ($\tau V_1 = 1.5$).

The u-functions of individual units are as follows:

$$u_{11}(z) = 10^{-2}(94z^{(200)}+2z^{(020)}+4z^{(002)})$$

$$u_{12}(z) = 10^{-2}(68z^{(100)}+30z^{(010)}+z^{(001)}+z^{(000)})$$

for objects of class 1;

$$u_{21}(z) = 10^{-2}(z^{(200)}+95z^{(020)}+2z^{(002)}+2z^{(000)})$$

$$u_{22}(z) = 10^{-2}(30z^{(100)}+63z^{(010)}+5z^{(001)}+2z^{(000)})$$

for objects of class 2;

$$u_{31}(z) = 10^{-2}(6z^{(200)}+5z^{(020)}+85z^{(002)}+4z^{(000)})$$

$$u_{32}(z) = 10^{-2}(z^{(100)}+z^{(010)}+97z^{(001)}+z^{(000)})$$

for objects of class 3.

The u-functions for the entire WVC are as follows:

$$U_1(z)= u_{11}(z) \underset{+}{\otimes} u_{12}(z) = 10^{-4}(\underline{\mathbf{6392z^{(300)}}}+136z^{(120)}+272z^{(102)}+\underline{\mathbf{2820z^{(210)}}}$$

$$+60z^{(030)}+120z^{(012)}+\underline{\mathbf{94z^{(201)}}}+2z^{(021)}+4z^{(003)}+\underline{\mathbf{94z^{(200)}}}+2z^{(020)}+4z^{(002)})$$

for objects of class 1;

$$U_2(z) = u_{21}(z) \underset{+}{\otimes} u_{22}(z) = 10^{-4}(30z^{(300)}+\underline{\mathbf{2850z^{(120)}}}+60z^{(102)}+\underline{\underline{60z^{(100)}}}$$

$$+63z^{(210)}+\underline{\mathbf{5985z^{(030)}}}+126z^{(012)}+\underline{\underline{\mathbf{126z^{(010)}}}}+5z^{(201)}+\underline{\mathbf{475z^{(021)}}}+10z^{(003)}$$

$$+\underline{\underline{10z^{(001)}}}+2z^{(200)}+\underline{\mathbf{190z^{(020)}}}+4z^{(002)}+\underline{\underline{4z^{(000)}}})$$

for objects of class 2;

$$U_3(z) = u_{31}(z) \underset{+}{\otimes} u_{32}(z) = 10^{-4}(6z^{(300)}+ 5z^{(120)}+\underline{\mathbf{85z^{(102)}}}+\underline{\underline{4z^{(100)}}} +6z^{(210)}$$

$$+5z^{(030)}+\underline{\mathbf{85z^{(012)}}}+\underline{\underline{4z^{(010)}}}+582z^{(201)}+485z^{(021)}+\underline{\mathbf{8245z^{(003)}}}+\underline{\underline{\mathbf{388z^{(001)}}}}$$

$$+6z^{(200)}+5z^{(020)}+\underline{\mathbf{85z^{(002)}}}+\underline{\underline{4z^{(000)}}})$$

for objects of class 3.

The cc-terms in these u-functions are marked in bold for the plurality voting rule and are underlined for the threshold voting rule. The terms corresponding to

WVC abstention are marked in italics for the plurality voting and are double underlined for the threshold voting. The probability of correct classification for each class can now be obtained as the sum of the coefficients of cc-terms for the corresponding u-function:

$$r_1 = 10^{-4}(6392+2820+94+94) = 0.94$$

$$r_2 = 10^{-4}(2850+5985+126+475+190) = 0.9626$$

$$r_3 = 10^{-4}(85+85+8245+388+85) = 0.8888$$

for plurality voting and

$$r_1 = 10^{-4}(6392+2820+94+94) = 0.94$$

$$r_2 = 10^{-4}(2850+5985+475+190) = 0.95$$

$$r_3 = 10^{-4}(85+85+8245+85) = 0.85$$

for threshold voting.

The entire WVC reliability is

$$R = (0.94+0.9626+0.8888)/3 = 0.9305$$

for plurality voting and

$$R = (0.94+0.95+0.85)/3 = 0.9133$$

for threshold voting. Note that the probability of recognizing objects of class 3 is much lower than the same for classes 1 and 2.

The probabilities of WVS abstaining are

$$\Pr\{D(I) = 0 \mid I = 1\} = 0.0$$

$$\Pr\{D(I) = 0 \mid I = 2\} = \Pr\{D(I) = 0 \mid I = 3\} = 0.0004$$

for plurality voting and

$$\Pr\{D(I) = 0 \mid I = 2\} = 10^{-4}(60+126+10+4) = 0.02$$

$$\Pr\{\mathrm{D}(I) = 0 \mid I = 3\} = 10^{-4}(4+4+388+4) = 0.04$$

for threshold voting. One can obtain the probability of wrong classifications for both types of system as

$$\Pr\{D(I) \neq I\} = \sum_{i=1}^{3} P_i(1 - r_i - \Pr\{D(I) = 0 \mid I = i\})$$

For the plurality voting WVC this index is equal to 0.069 and for the threshold voting it is equal to 0.067. One can see that the plurality voting WVC is more "decisive". It provides both correct and incorrect decisions with greater probability than the threshold voting classifier (even with the minimal possible threshold factor) and has a smaller abstention probability.

The description of the simplification technique used in the WVC reliability evaluation algorithm, as well as algorithms for solving the WVC optimization problem, can be found in [59, 181, 182].

6.4 Sliding Window Systems

The linear multi-state sliding window system (SWS) consists of n linearly ordered, statistically independent, multi-state elements (MEs). Each ME j has the random performance G_j and can be in one of k_j different states. Each state $i \in \{0, 1, \ldots, k_j-1\}$ of ME j is characterized by its fixed performance rate g_{ji} and probability $p_{jk} = \Pr\{G_j = g_{ji}\}$ (where $\sum_{i=0}^{k_j-1} p_{ji} = 1$). The SWS fails if the performance rates of any r consecutive MEs do not satisfy some condition. In terms of acceptability function, the failure criteria can be expressed as

$$F(G_1,\ldots,G_n) = \prod_{h=1}^{n-r+1} f(G_h,\ldots,G_{h+r-1}) = 0 \tag{6.113}$$

where F is the acceptability function for the entire SWS and f is the acceptability function for any group of r consecutive MEs. For example, if the sum of the performance rates of any r consecutive MEs should be not lower than the demand w, then Equation (6.113) takes the form

$$F(G_1,\ldots,G_n) = \prod_{h=1}^{n-r+1} 1\Big(\sum_{m=h}^{h+r-1} G_m \geq w\Big) = 0 \tag{6.114}$$

The special case of SWS where all of the n MEs are identical and have two states with performance rates of 0 and 1, $w = r-k+1$ and the acceptability function takes the form (6.114) is a k-out-of-r-from-n:F system.

6.4.1. Evaluating the Reliability of the Sliding Window Systems

The algorithm for evaluating the reliability of the sliding window system is very similar to that described in Section 2.4 for a k-out-of-r-from-n:F system.

6.4.1.1 Implementing the Universal Generating Function

The u-function representing p.m.f. of the random performance rate of ME j G_j takes the form:

$$u_j(z) = \sum_{i=0}^{k_j-1} p_{ji} z^{g_{j,i}} \tag{6.115}$$

The performance of a group consisting of r MEs numbered from h to $h+r-1$ is represented by the random vector $\boldsymbol{G}_h = (G_h, ..., G_{h+r-1})$ consisting of random performance values corresponding to all of the MEs belonging to the group.

Having the p.m.f. of independent random variables G_h, …, G_{h+r-1} one can obtain the p.m.f. of the random vector $\boldsymbol{G}_h$ by evaluating the probabilities of each combination of realizations of these values. Doing so by a recursive procedure, one can first obtain the p.m.f. of the r-length vector (0, …, 0, G_h) (corresponding to a single ME), then obtain the p.m.f. of r-length vector (0, …, 0, G_h, G_{h+1}) (corresponding to a pair of MEs), and so on until obtaining the p.m.f. of the vector (G_h, …, G_{h+r-1}).

Let the u-function $U_{-r+h}(z)$ represent the p.m.f. of a vector consisting of $r-h+1$ zeros and random values from G_1 to G_{h-1}. This u-function represents the PDs of MEs from 1 to $h-1$. In order to obtain the PD of a group of MEs from 1 to h, one has to evaluate all possible combinations of the realizations of a random vector (0,…,0, G_1, …, G_{h-1}) and a random variable G_h. Therefore, the u-function $U_1(z)$, representing the p.m.f. of the random vector $\boldsymbol{G}_1 = (G_1, ..., G_r)$, can be obtained by assigning

$$U_{1-r}(z) = z^{\boldsymbol{g}_0} \tag{6.116}$$

where the vector $\boldsymbol{g}_0$ consists of r zeros and the consecutive application of the shift operator $\underset{\leftarrow}{\otimes}$ (2.63):

$$U_{-r+h+1}(z) = U_{-r+h}(z) \underset{\leftarrow}{\otimes} u_h(z) \text{ for } h = 1, \dots, r \tag{6.117}$$

where the procedure $\boldsymbol{x} \leftarrow y$ over arbitrary r-length vector $\boldsymbol{x}$ and value y shifts all of the vector elements one position to the left, $x(s-1) = x(s)$ for $s = 2, \dots, r$ in sequence, and adds the value y to the right position, $x(r) = y$. (The first element of vector $\boldsymbol{x}$ disappears after applying the operator).

Having the PD of the first r MEs, one can obtain the PD of the next group of MEs (from 2 to r+1) by estimating all of the possible combinations of random vector values represented by $U_1(z)$, and random variable G_{r+1} represented by $u_{r+1}(z)$. Note that the performance G_1 does not influence the PD of this group. Therefore, in order to obtain the vector $\boldsymbol{G}_2$ one has to remove G_1 from vector $\boldsymbol{G}_1$ and replace it with G_{r+1}. By replacing the first element of the random vector with the new element corresponding to the following ME, one obtains vectors corresponding to the next groups of MEs.

By applying the shift operator $\underset{\leftarrow}{\otimes}$ further for $h = r+1, \ldots, n$ one obtains the u-functions for all of the possible groups of r consecutive MEs: $U_2(z), \ldots, U_{n-r+1}(z)$. The SWS contains exactly $n-r+1$ groups of r consecutive MEs, with each ME belonging to no more than r groups.

Let u-function $U_h(z)=\sum_{i=0}^{E_h-1} q_{hi} z^{\boldsymbol{g}_{hi}}$ for $1 \le h \le n-r+1$ represent the p.m.f. of vector $\boldsymbol{G}_h$. By summing the probabilities of all of the realizations $\boldsymbol{g}_{hi}$ of vector $\boldsymbol{G}_h$ producing zero values of the acceptability function $f(\boldsymbol{G}_h) = f(G_h, \ldots, G_{h+r-1})$, one can obtain the probability of failure Q_h of the hth group of r consecutive MEs:

$$Q_h = E(1-f(G_h)) = \theta_f(U_h(z)) = \sum_{i=0}^{E_{h-1}} q_{hi}(1-f(\boldsymbol{g}_{hi})) \tag{6.118}$$

Consider the u-function $U_h(z)$. For each combination of values of $G_{h+1}, \ldots, G_{h+r-1}$, it contains exactly k_h different terms corresponding to different values of G_h, which takes all of the possible values of the performance rate of ME h. After applying the operator $\otimes_{\leftarrow}$, G_h disappears from the vector $\boldsymbol{G}_{h+1}$ and is replaced with G_{h+1}. This produces k_h terms in $U_{h+1}(z)$, corresponding to the same value of vector $\boldsymbol{G}_{h+1}$. Collecting these like terms, one obtains a single term for each vector $\boldsymbol{G}_{h+1}$. Therefore, the number of different terms in each u-function $U_h(z)$ is equal to $E_h = \prod_{i=h}^{h+r-1} k_i$.

By applying the operator θ_f (6.118) over $U_h(z)$ one can obtain the probability Q_h that the group consisting of MEs $h, \ldots, h+r-1$ fails. If for some combination of MEs' states the group fails, the entire SWS fails independently of the states of the MEs that do not belong to this group. Therefore, the terms corresponding to the group failure can be removed from $U_h(z)$, since they should not participate in determining further state combinations that cause system failures. This consideration lies at the base of the following algorithm for SWS availability evaluation:

1. Assign: $x = 0$; $U_{-r+1}(z) = z^{\boldsymbol{g}_0}$. Determine the u-functions of the individual MEs using (6.115).

2. Main loop. Repeat the following for $h = 1, \ldots, n$:

2.1. Obtain $U_{-r+h+1}(z) = U_{-r+h}(z) \underset{\leftarrow}{\otimes} u_h(z)$.

2.2. If $h \geq r$ add value $\theta_f(U_{h+1-r}(z))$ to x and remove all of the terms with the exponents producing the zero acceptability function from $U_{h+1-r}(z)$.

3. Obtain the SWS availability as $R = 1-x$.

Example 6.23

Consider an SWS with five MEs (n = 5) in which the sum of the performance rates of any three (r = 3) adjacent MEs should not be less than four. Each ME has two states: total failure (corresponding to a performance rate of zero) and functioning with a nominal performance rate. The nominal performance rates of the MEs from 1 to 5 are 1, 2, 3, 1 and 1 respectively.

The u-functions of the individual MEs are:

$$u_1(z) = p_{10}z^0+p_{11}z^1,\ u_2(z) = p_{20}z^0+p_{21}z^2,\ u_3(z)=p_{30}z^0+p_{31}z^3$$

$$u_4(z) = p_{40}z^0+p_{41}z^1,\ u_5(z) = p_{50}z^0+p_{51}z^1$$

First, we assign

$$x = 0,\ U_{-2}(z) = z^{(0,0,0)}$$

Following step 2 of the algorithm, we obtain

$$U_{-1}(z) = U_{-2}(z) \underset{\leftarrow}{\otimes} u_1(z)) = z^{(0,0,0)} \underset{\leftarrow}{\otimes} (p_{10}z^0+p_{11}z^1) = p_{10}z^{(0,0,0)}+p_{11}z^{(0,0,1)}$$

$$U_0(z) = U_{-1}(z) \underset{\leftarrow}{\otimes} u_2(z) = (p_{10}z^{(0,0,0)}+p_{11}z^{(0,0,1)}) \underset{\leftarrow}{\otimes} (p_{20}z^0+p_{21}z^2)$$

$$= p_{10}p_{20}z^{(0,0,0)}+p_{11}p_{20}z^{(0,1,0)}+p_{10}p_{21}z^{(0,0,2)}+p_{11}p_{21}z^{(0,1,2)}$$

$$U_1(z) = U_0(z) \underset{\leftarrow}{\otimes} u_3(z)$$

$$= (p_{10}p_{20}z^{(0,0,0)}+p_{11}p_{20}z^{(0,1,0)}+p_{10}p_{21}z^{(0,0,2)}+p_{11}p_{21}z^{(0,1,2)}) \underset{\leftarrow}{\otimes} (p_{30}z^0+p_{31}z^3)$$

$$= \boldsymbol{p_{10}p_{20}p_{30}z^{(0,0,0)}+p_{11}p_{20}p_{30}z^{(1,0,0)}+p_{10}p_{21}p_{30}z^{(0,2,0)}+p_{11}p_{21}p_{30}z^{(1,2,0)}}$$

$$\boldsymbol{+p_{10}p_{20}p_{31}z^{(0,0,3)}}+p_{11}p_{20}p_{31}z^{(1,0,3)}+p_{10}p_{21}p_{31}z^{(0,2,3)}+p_{11}p_{21}p_{31}z^{(1,2,3)}$$

The terms of $U_1(z)$ with exponents in which sums of elements are less than 4 are marked in bold. Following step 2.2 of the algorithm, we obtain

$$x = p_{10}p_{20}p_{30}+p_{11}p_{20}p_{30}+p_{10}p_{21}p_{30}+p_{11}p_{21}p_{30}+p_{10}p_{20}p_{31}$$

After removing the marked terms, $U_1(z)$ takes the form

$$U_1(z) = p_{11}p_{20}p_{31}z^{(1,0,3)}+p_{10}p_{21}p_{31}z^{(0,2,3)}+p_{11}p_{21}p_{31}z^{(1,2,3)}$$

Applying further the $\underset{\leftarrow}{\otimes}$ operator, we obtain

$$U_2(z)= U_1(z) \underset{\leftarrow}{\otimes} u_4(z)$$

$$= (p_{11}\, p_{20}\, p_{31}\, z^{(1,0,3)}+p_{10}\, p_{21}\, p_{31}\, z^{(0,2,3)}+p_{11}\, p_{21}\, p_{31}\, z^{(1,2,3)}) \underset{\leftarrow}{\otimes} (p_{40}z^{0}+p_{41}z^{1})$$

$$= \boldsymbol{p_{11}p_{20}p_{31}p_{40}z^{(0,3,0)}}+p_{10}p_{21}p_{31}p_{40}z^{(2,3,0)}+p_{11}p_{21}p_{31}p_{40}\, z^{(2,3,0)}$$

$$+p_{11}\, p_{20}\, p_{31}\, p_{41}\, z^{(0,3,1)}+p_{10}p_{21}\, p_{31}\, p_{41}\, z^{(2,3,1)}+p_{11}\, p_{21}\, p_{31}\, p_{41}z^{(2,3,1)}$$

Following step 2.2 of the algorithm, we modify x as follows:

$$x = p_{10}p_{20}p_{30}+p_{11}p_{20}p_{30}+p_{10}p_{21}p_{30}+p_{11}p_{21}p_{30}+p_{10}p_{20}p_{31}+p_{11}\, p_{20}p_{31}\, p_{40}$$

After removing the marked term and collecting like terms, $U_2(z)$ takes the form:

$$U_2(z) = p_{21}p_{31}p_{40}z^{(2,3,0)}+p_{11}p_{20}p_{31}p_{41}z^{(0,3,1)}+p_{21}p_{31}p_{41}z^{(2,3,1)}$$

Following steps 2.1 and 2.2 of the algorithm we obtain

$$U_3(z) = U_2(z) \underset{\leftarrow}{\otimes} u_5(z)$$

$$= (p_{21}p_{31}p_{40}z^{(2,3,0)}+p_{11}p_{20}p_{31}p_{41}z^{(0,3,1)}+p_{21}p_{31}p_{41}z^{(2,3,1)}) \underset{\leftarrow}{\otimes} (p_{50}\mathrm{z}^{0}+p_{51}\mathrm{z}^{1})$$

$$= \boldsymbol{p_{21}p_{31}p_{40}p_{50}z^{(3,0,0)}}+p_{11}p_{20}p_{31}p_{41}p_{50}z^{(3,1,0)}+p_{21}p_{31}p_{41}p_{50}z^{(3,1,0)}$$

$$+p_{21}p_{31}p_{40}p_{51}z^{(3,0,1)}+p_{11}p_{20}p_{31}p_{41}p_{51}z^{(3,1,1)}+p_{21}p_{31}p_{41}p_{51}z^{(3,1,1)}$$

After adding the coefficient of the marked term to x we have

$$x = p_{10}p_{20}p_{30}+p_{11}p_{20}p_{30}+p_{10}p_{21}p_{30}+p_{11}p_{21}p_{30}+p_{10}p_{20}p_{31}+p_{11}\, p_{20}p_{31}\, p_{40}$$

$$+p_{21}p_{31}\, p_{40}\, p_{50}$$

Finally:

$$R=1-x=1-p_{30}-p_{31}[p_{10}p_{20}+(p_{11}p_{20}+p_{21}p_{50})p_{40}]$$

6.4.1.2 Simplification Technique

Note that the first elements of vectors $\boldsymbol{g}_{hi}$ in the u-function $U_h(z)$ do not participate in determining $U_{h+1}(z)$ (according to the definition of the procedure $\boldsymbol{x}\leftarrow y$), which leads to producing k_h like terms in $U_{h+1}(z)$. In order to avoid excessive term multiplication procedures in operator $\otimes_{\leftarrow}$, one can perform a like term collection in $U_h(z)$. To do this, one can, after step 2.2 of the algorithm, replace the first elements in all vectors of $U_h(z)$ with zeros and collect like terms.

The algorithm can be further simplified if the SWS acceptability function takes the form (6.114). Consider the sth term $q_{hs}z^{\boldsymbol{g}_{hs}}$ of a u-function $U_h(z)$ after replacing the first element $g_{hs}(1)$ of vector $\boldsymbol{g}_{hs}$ with zero. If $\tilde{g}_{h+r}$ is the greatest possible value of the performance rate of the $(h+r)$th ME and

$$\sum_{i=2}^{r} g_{hs}(i) < w - \tilde{g}_{h+r} \tag{6.119}$$

any combination of the term $q_{hs}z^{\boldsymbol{g}_{hs}}$ with terms of $u_{h+r}(z)$ produces terms corresponding to SWS failure. This means that, in the u-function $U_{h+1}(z)$, all of the terms with coefficients $q_{hs}p_{h+r,i}$ should be removed and the sum of the corresponding coefficients should be added to x. Since $\sum_{i=0}^{k_{h+r}-1} p_{h+r,i} = 1$, the sum of these coefficients is equal to q_{hs}. In order to avoid k_{h+r} redundant term multiplication procedures, one can remove the term $q_{hs}z^{\boldsymbol{g}_{hs}}$ meeting condition (6.119) from $U_h(z)$ and add its coefficient to x.

In order to reduce the algorithm computation complexity considerably using the considerations described above, one has to apply to any newly obtained u-function $U_m(z)$ (in step 2 of the algorithm) for $m = 0, \ldots, n-r$ the following operator φ, which:

- replaces all of the first elements of vectors $\boldsymbol{g}_{hs}$ with zeros;
- collects like terms in the u-function;
- removes the terms meeting (6.119) and adds the coefficients of the replaced terms to x.

Example 6.24

Consider the Example 6.23 and apply to it the simplification technique. First, we obtain

$$w-\tilde{g}_3 = 4-3 = 1,\ w-\tilde{g}_4 = w-\tilde{g}_5 = 4-1 = 3$$

The operator φ applied to

$$U_0(z) = \boldsymbol{p_{10}p_{20}z^{(0,0,0)}} + p_{11}p_{20}z^{(0,1,0)} + p_{10}p_{21}z^{(0,0,2)} + p_{11}p_{21}z^{(0,1,2)}$$

removes the term $p_{10}p_{20}z^{(0,0,0)}$, since $0+0+0<w-\tilde{g}_3 = 1$, and adds $p_{10}p_{20}$ to x. After applying the operator, $U_0(z)$ takes the form

$$\varphi(U_0(z)) = p_{11}p_{20}z^{(0,1,0)}+p_{10}p_{21}z^{(0,0,2)}+p_{11}p_{21}z^{(0,1,2)}$$

Following steps 2.1 and 2.2 of the algorithm, we obtain

$$U_1(z) = (\varphi(U_0(z)) \underset{\leftarrow}{\otimes} u_3(z))$$

$$= (p_{11}p_{20}z^{(0,1,0)}+p_{10}p_{21}z^{(0,0,2)}+p_{11}p_{21}z^{(0,1,2)} \underset{\leftarrow}{\otimes} (p_{30}z^{0}+p_{31}z^{3})$$

$$= \boldsymbol{p_{11}p_{20}p_{30}z^{(1,0,0)}+p_{10}p_{21}p_{30}z^{(0,2,0)} +p_{11}p_{21}p_{30}z^{(1,2,0)}}+p_{11}p_{20}p_{31}z^{(1,0,3)}$$

$$+p_{10}p_{21}p_{31}z^{(0,2,3)}+p_{11}p_{21}p_{31}z^{(1,2,3)}$$

Applying the operator φ that removes the terms meeting condition (6.119) (marked in bold) one obtains

$$x = p_{10}p_{20}+p_{11}p_{20}p_{30}+p_{10}p_{21}p_{30}+p_{11}p_{21}p_{30},$$

$$\varphi(U_1(z)) = p_{11}p_{20}p_{31}z^{(0,0,3)}+p_{21}p_{31}z^{(0,2,3)}$$

Further:

$$U_2(z) = (\varphi(U_1(z)) \underset{\leftarrow}{\otimes} u_4(z)) = p_{11}p_{20}p_{31}z^{(0,0,3)}+p_{21}p_{31}z^{(0,2,3)}) \underset{\leftarrow}{\otimes} (p_{40}z^{0}+p_{41}z^{1})$$

$$= \boldsymbol{p_{11}p_{20}p_{31}p_{40}z^{(0,3,0)}}+p_{21}p_{31}p_{40}z^{(2,3,0)}+p_{11}p_{20}p_{31}p_{41}z^{(0,3,1)}+p_{21}p_{31}p_{41}z^{(2,3,1)}$$

Applying the operator φ over $U_2(z)$ one obtains

$$x = p_{10}p_{20}+p_{11}p_{20}p_{30}+p_{10}p_{21}p_{30}+p_{11}p_{21}p_{30}+p_{11}p_{20}p_{31}p_{40}$$

$$\varphi(U_2(z)) = p_{21}p_{31}p_{40}z^{(0,3,0)}+(p_{11}p_{20}+p_{21})p_{31}\mathrm{p}_{41}z^{(0,3,1)}$$

For the last group of MEs:

$$U_3(z) = (\varphi(U_2(z)) \underset{\leftarrow}{\otimes} u_5(z))$$

$$= (p_{21}p_{31}p_{40}z^{(0,3,0)}+(p_{11}p_{20}+p_{21})p_{31}\mathrm{p}_{41}z^{(0,3,1)}) \underset{\leftarrow}{\otimes} (p_{50}z^{0}+p_{51}z^{1})$$

$$= \boldsymbol{p_{21}p_{31}p_{40}p_{50}z^{(3,0,0)}}+(p_{11}p_{20}+p_{21})p_{31}p_{41}p_{50}z^{(3,1,0)}$$

$$+p_{21}p_{31}p_{40}p_{50}z^{(3,0,1)}+(p_{11}p_{20}+p_{21})p_{31}p_{41}p_{50}z^{(3,1,1)}$$

$$x = p_{10}p_{20}+p_{11}p_{20}p_{30}+p_{10}p_{21}p_{30}+p_{11}p_{21}p_{30}+p_{11}p_{20}p_{31}p_{40}+p_{21}p_{31}p_{40}p_{50}$$

Finally:

$$R = 1 - x$$

$$= 1 - p_{10}p_{20} - p_{11}p_{20}p_{30} - p_{10}p_{21}p_{30} - p_{11}p_{21}p_{30} - p_{11}p_{20}p_{31}p_{40} - p_{21}p_{31}p_{40}p_{50}$$

Taking into account that $p_{10}p_{20} = p_{10}p_{20}p_{30} + p_{10}p_{20}p_{31}$, one obtains the same result:

$$R = 1 - p_{30} - p_{31}[p_{10}p_{20} + (p_{11}p_{20} + p_{21}p_{50})p_{40}]$$

Using the SWS reliability evaluation procedure described, one can analyze the effect of demand variation on the overall system reliability.

Example 6.25

Consider the two following SWSs [183]. The first one consists of 10 identical three-state MEs. The probabilities of the MEs' states are $p_{j0} = 0.1$, $p_{j1} = 0.3$, $p_{j2} = 0.6$. The corresponding performance rates are $g_{j0} = 0$, $g_{j1} = 1$, $g_{j2} = 3$.

The SWS reliability, as a function of constant demand w, is presented in Figure 6.27A for different r ($2 \le r \le 10$). Note that, because the cumulative performance of groups of MEs takes a finite number of discrete values, the $R(w)$ is a step function. One can see that the greater the r, the greater the SWS reliability for the same w. This is natural, because the growth of r provides growing redundancy in each group.

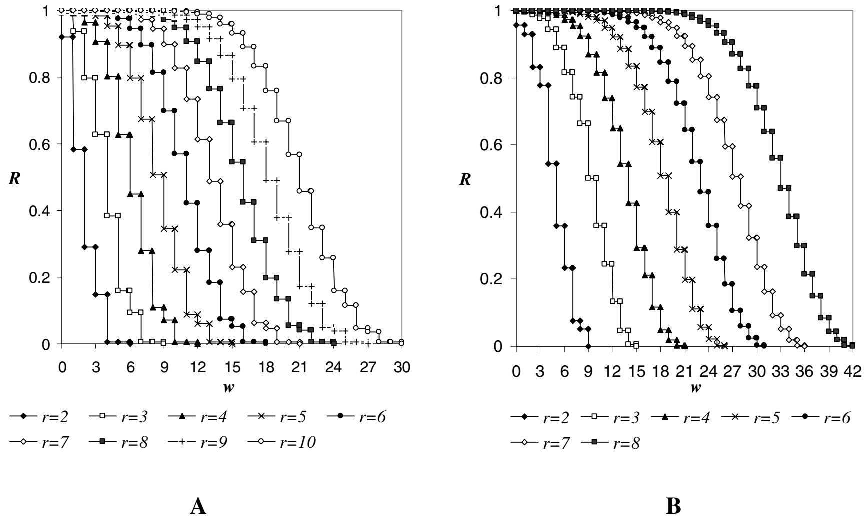


Figure 6.27. Reliability of SWS as a function of w and r
(A: SWS with 10 identical MEs. B: SWS with eight different MEs)

The second SWS consists of eight different MEs. The number of states of these MEs varies from two to five. The performance distributions of the MEs are presented in Table 6.30. The SWS reliability as a function of constant demand w is presented in Figure 6.27B for different r ($2 \le r \le 8$). Observe that the functions $R(w)$ for the second SWS have more steps than the functions $R(w)$ for the first one. Indeed, different MEs produce a greater variety of levels of cumulative group performance rates than the identical ones.

Table 6.30. Parameters of SWS elements

No. of ME	1		2		3		4		5		6		7		8	
No. of state	*p*	*G*	*p*	*g*	*p*	*g*	*p*	*g*	*p*	*g*	*p*	*g*	*p*	*g*	*p*	*g*
0	0.03	0	0.10	0	0.17	0	0.05	0	0.08	0	0.01	0	0.20	0	0.05	0
1	0.22	2	0.10	1	0.83	6	0.25	3	0.20	1	0.22	4	0.10	3	0.25	4
2	0.75	5	0.40	2	-	-	0.40	5	0.15	2	0.77	5	0.10	4	0.70	6
3	-	-	0.40	4	-	-	0.30	6	0.45	4	-	-	0.60	5	-	-
4	-	-	-	-	-	-	-	-	0.12	5	-	-	-	-	-	-

6.4.2 Multiple Sliding Window Systems

The existence of multiple failure criteria is a common situation for complex systems, especially for consecutive-type systems. In this section we consider an extension of the linear SWS model to a multi-criteria case. In this multiple sliding window system (MSWS) a vector $\boldsymbol{r} = (r_i: 1 \le i \le Y)$ is defined such that $r_i < r_{i+1}$, and $1 \le r_i \le n$ for any i. The system fails if for any i ($1 \le i \le Y$) at least one of the functions f_i over the performance rates of any r_i consecutive MEs is equal to zero. The entire MSWS acceptability function takes the form

$$F(G_1,...,G_n) = \prod_{i=1}^{Y} \prod_{h=1}^{n-r_i+1} f_i(G_h,...,G_{h+r_i-1}) = 0 \tag{6.120}$$

The introduction of the linear MSWS model is motivated by the following examples.

Example 6.26

Consider a sequence of service stations in which each station should process the same sequence of n different tasks. Each station i can process r_i incoming tasks simultaneously according to the first-in-first-out rule using a limited resource w_i. Each incoming task can have different states and the amount of the resource needed to process the task is different for each state of each task. The total resource needed to process r_i consecutive tasks should not exceed the available amount of the resource w_i. The system fails if in at least one of the stations there is no available resource to process r_i tasks simultaneously.

The simplest example of such a model is a transportation system in which n randomly ordered containers are carried by consecutive conveyors characterized by a different length and allowable load. The number of containers r_i that are loaded onto each conveyor i is defined by its length. The transportation system fails if the total load of any one of the conveyors is greater than its maximal allowed load w_i.

An example of the transportation system is presented in Figure 6.28. The system consists of Y = 3 conveyors and transports n = 14 randomly ordered containers of four types (each type m is characterized by its weight g_m). The first conveyor can simultaneously carry r_1 = 2 containers, the second and third conveyors can carry r_2 = 6 and r_3 = 3 containers respectively. The maximal allowable loads of conveyors 1, 2 and 3 are w_1, w_2 and w_3 respectively. The system fails if the total weight of any two adjacent containers is greater than w_1, or if the total weight of any six adjacent containers is greater than w_2, or if the total weight of any three adjacent containers is greater than w_3. The weight of the jth container in the line can be represented by a random value G_j: $G_j \in \{g_1, g_2, g_3, g_4\}$. The acceptability function for each conveyor i can be determined as

$$f_i(G_h,...,G_{h+r_i-1}) = 1(\sum_{j=h}^{h+r_i-1} G_j \le w_i),\ \ 1 \le i \le 3$$

for any group of r_i adjacent containers starting with hth one (r_1 = 2, r_2 = 6, r_3 = 3).

The system reliability (defined as its expected acceptability) takes the form

$$R = E(\prod_{i=1}^{3}\prod_{h=1}^{15-r_i} 1(\sum_{j=h}^{h+r_i-1} G_j \le w_i)),\ \text{where}\ r_1{=}2,\ r_2{=}6,\ r_3{=}3$$

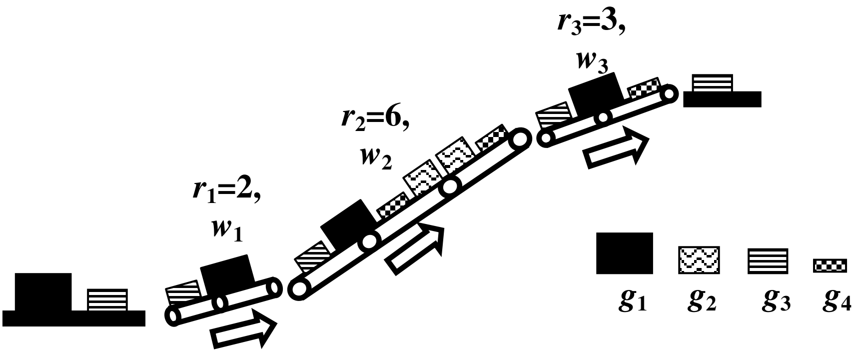


Figure 6.28. Example of transportation MSWS

Example 6.27

Consider a heating system that should provide a certain temperature along several lines with moving parts placed at different distances from the heaters. The temperature at each point of the line i is determined by a cumulative effect of r_i closest heaters. Each heater consists of several electrical heating elements. The heating effect of each heater depends on the availability of its heating elements and, therefore, can vary discretely (if the heaters are different, then the number of different levels of heat radiation and the intensity of the radiation at each level are specific to each heater). In order to provide the temperature, which is not less than some specified value at each point of line i, any r_i adjacent heaters should be in states where the sum of their radiation intensity is greater than the minimum allowed level w_i. The system fails if any group of r_i adjacent heaters provides the cumulative radiation intensity lower than w_i.

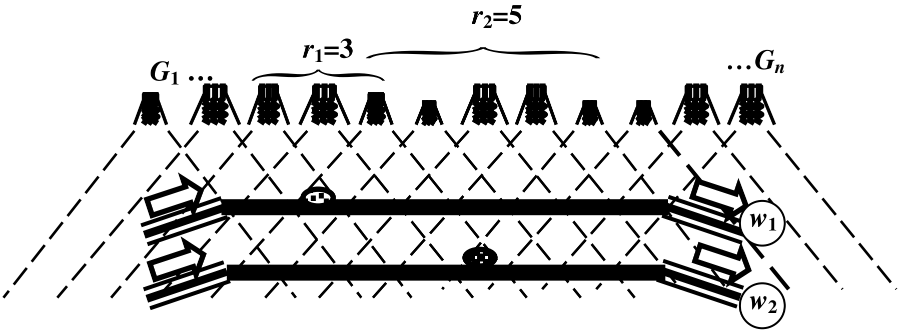


Figure 6.29. Example of manufacturing MSWS

In the example presented in Figure 6.29 there are 12 heaters providing random radiation intensity G_j ($1 \le j \le 12$). The parts located at any point of the close conveyor are heated by three adjacent heaters. The cumulative heating intensity along this conveyor should not be lower than w_1. The parts located at any point of the remote conveyor are heated by five adjacent heaters. The cumulative heating intensity along this conveyor should not be lower than w_2. The system fails if any three adjacent heaters fail to provide the desired heating intensity w_1 or if any five adjacent heaters fail to provide the desired heating intensity w_2. The acceptability function for each conveyor i can be defined as

$$f_i(G_h,...,G_{h+r_i-1}) = 1(\sum_{j=h}^{h+r_i-1} G_j \ge w_i),\ \ 1 \le i \le 2$$

which corresponds to any group of r_i adjacent heaters starting with the hth one ($r_1 = 3$, $r_2 = 5$).

The system reliability takes the form

$$R = E(\prod_{i=1}^{2} \prod_{h=1}^{13-r_i} 1(\sum_{j=h}^{h+r_i-1} G_j \ge w_i))\text{, where } r_1 = 3,\ r_2 = 5$$

A variety of other systems also fit the model: quality control systems that detect deviations from given values of parameters in product samples, combat systems that should provide certain fire density along a defence line, *etc.*

6.4.2.1 Evaluating the Multiple Sliding Window System Reliability

Let the *u*-function $U_{-r_Y+1}(z)$ take the form (6.116) where the vector $\boldsymbol{g}_0$ consists of r_Y zeros. According to the algorithm presented in Section 6.4.1.1, by applying the operator (6.117) for $h = 1, \ldots, n$ one obtains distributions for all of the possible random vectors of performance rates of r_Y consecutive MEs.

Note that the vectors of length r_Y considered also contain all of the possible vectors of the smaller length. For any $r_i<r_Y$ the last r_i elements of vectors in the exponents of *u*-functions $U_{-r_Y+1+r_i}(z)$, $U_{-r_Y+2+r_i}(z) \ldots$, $U_{-r_Y+1+n}(z)$ represent all of the possible vectors of performance rates of r_i consecutive MEs. Therefore, in each *u*-function $U_{-r_Y+1+h}(z)$ obtained by the recursive operator (6.117) for $r_1 \le h \le n$, one can obtain the failure probability of groups of r_i consecutive MEs for any $r_1 \le h \le r_Y$ satisfying the condition $r_i \le h$ by applying operators $\theta_{f_i}(U_{-r_Y+1+h}(z))$ in which acceptability functions f_i take as arguments r_i the last elements of vectors from the exponents of the *u*-function. These considerations lead to the following algorithm for MSWS reliability evaluation:

1. Assign: $x = 0$; $U_{-r_Y+1}(z) = z^{\boldsymbol{g}_0}$. Determine the *u*-functions of the individual MEs using (6.115).
2. Main loop. Repeat the following for $h = 1, \ldots, n$:
    2.1. Obtain $U_{-r_Y+h+1}(z) = U_{-r_Y+h}(z) \underset{\leftarrow}{\otimes} u_h(z)$.
    2.2. For $i = 1, \ldots, Y$: if $h \ge r_i$ add value $\theta_{f_i}(U_{-r_Y+h+1}(z))$ to x and remove from $U_{-r_Y+h+1}(z)$ terms with the exponents in which the last r_i elements produce zero acceptability function f_i.
3. Obtain the SWS availability as $R = 1-x$.

Alternatively, the system reliability can be obtained as the sum of the coefficients of the last *u*-function $U_{-r_Y+n+1}(z)$.

Example 6.28

Consider an MSWS with $n = 5$, $Y = 2$, $r_1 = 3$, $r_2 = 4$, $f_1(x_1, x_2, x_3) = 1(\sum_{j=1}^{3} x_j \ge 5)$, $f_2(x_1, x_2, x_3, x_4) = 1(\sum_{j=1}^{4} x_j \ge 6)$. Each ME has two states: total failure (corresponding to a performance rate of zero) and functioning with a nominal performance rate. The nominal performance rates of the MEs are 2, 2, 3, 1, and 2.

In the initial step of the algorithm, a value of zero is assigned to x. The *u*-functions of the individual MEs are

$$u_1(z) = p_{10}z^0+p_{11}z^2,\ u_2(z) = p_{20}z^0+p_{21}z^2,\ u_3(z) = p_{30}z^0+p_{31}z^3$$

$$u_4(z) = p_{40}z^0+p_{41}z^1,\ u_5(z) = p_{50}z^0+p_{51}z^2$$

Since, in the MSWS considered, $r_Y = r_2 = 4$, the initial u-function takes the form

$$U_{-3}(z) = z^{(0,0,0,0)}$$

Following step 2 of the algorithm we obtain:
for $h = 1$

$$U_{-2}(z) = U_{-3}(z) \underset{\leftarrow}{\otimes} u_1(z) = z^{(0,0,0,0)} \underset{\leftarrow}{\otimes} (p_{10}z^0+p_{11}z^2) = p_{10}z^{(0,0,0,0)}+p_{11}z^{(0,0,0,2)}$$

for $h = 2$

$$U_{-1}(z) = U_{-2}(z) \underset{\leftarrow}{\otimes} u_2(z)) = (p_{10}z^{(0,0,0,0)}+p_{11}z^{(0,0,0,2)}) \underset{\leftarrow}{\otimes} (p_{20}z^0+p_{21}z^2)$$

$$= p_{10}p_{20}z^{(0,0,0,0)}+p_{11}p_{20}z^{(0,0,2,0)}+p_{10}p_{21}z^{(0,0,0,2)}+p_{11}p_{21}z^{(0,0,2,2)}$$

for $h = 3$

$$U_0(z) = U_{-1}(z) \underset{\leftarrow}{\otimes} u_3(z)$$

$$= (p_{10}p_{20}z^{(0,0,0,0)}+p_{11}p_{20}z^{(0,0,2,0)}+p_{10}p_{21}z^{(0,0,0,2)}+p_{11}p_{21}z^{(0,0,2,2)}) \underset{\leftarrow}{\otimes} (p_{30}z^0+p_{31}z^3)$$

$$= \mathbf{p_{10}p_{20}p_{30}z^{(0,0,0,0)}+p_{11}p_{20}p_{30}z^{(0,2,0,0)}+p_{10}p_{21}p_{30}z^{(0,0,2,0)}+p_{11}p_{21}p_{30}z^{(0,2,2,0)}}$$

$$\mathbf{+p_{10}p_{20}p_{31}z^{(0,0,0,3)}}+p_{11}p_{20}p_{31}z^{(0,2,0,3)}+p_{10}p_{21}p_{31}z^{0,0,2,3)}+p_{11}p_{21}p_{31}z^{(0,2,2,3)}$$

In this step, operator θ_{f_1} should be applied to $U_0(z)$. The terms of $U_0(z)$ with $f_1 = 0$ are marked in bold. The value of $\theta_{f_1}(U_0(z))$ is added to x:

$$x = p_{10}p_{20}p_{30}+p_{11}p_{20}p_{30}+p_{10}p_{21}p_{30}+p_{11}p_{21}p_{30}+p_{10}p_{20}p_{31}$$

After removing the marked terms, $U_0(z)$ takes the form

$$U_0(z) = p_{11}p_{20}p_{31}z^{(0,2,0,3)}+p_{10}p_{21}p_{31}z^{0,0,2,3)}+p_{11}p_{21}p_{31}z^{(0,2,2,3)}$$

Proceeding for $h = 4$ we obtain

$$U_1(z) = U_1(z) \underset{\leftarrow}{\otimes} u_4(z) = (p_{11}p_{20}p_{31}z^{(0,2,0,3)}+p_{10}p_{21}p_{31}z^{(0,0,2,3)}$$

$$+p_{11}p_{21}p_{31}z^{(0,2,2,3)}) \underset{\leftarrow}{\otimes} (p_{40}z^0+p_{41}z^1) = \underline{\mathbf{p_{11}p_{20}p_{31}p_{40}z^{(2,0,3,0)}}}$$

$$+\underline{p_{10}p_{21}p_{31}p_{40}z^{(0,2,3,0)}}+p_{11}p_{21}p_{31}p_{40}z^{(2,2,3,0)}\mathbf{+p_{11}p_{20}p_{31}p_{41}z^{(2,0,3,1)}}$$

$$+p_{10}p_{21}p_{31}p_{41}z^{(0,2,3,1)}+p_{11}p_{21}p_{31}p_{41}z^{(2,2,3,1)}$$

Both operators θ_{f_1} and θ_{f_2} should be applied to $U_1(z)$. The terms of $U_1(z)$ with $f_1 = 0$ are marked in bold; the terms with $f_2 = 0$ are underlined. One can see that in the first term both $f_1 = 0$ and $f_2 = 0$, in the second term only $f_2 = 0$, and in the fourth term only $f_1 = 0$. First, the value of

$$\theta_{f_1}(U_1(z)) = p_{11}p_{20}p_{31}p_{40}+p_{11}p_{20}p_{31}p_{41}$$

is added to x and the terms with $f_1 = 0$ are removed. Then, in the remaining u-function $U_1(z)$, the value of

$$\theta_{f_2}(U_1(z)) = p_{10}p_{21}p_{31}p_{40}$$

is added to x and the terms with $f_2 = 0$ are removed.

After removing all of the marked terms, $U_1(z)$ takes the form

$$U_1(z) = p_{11}p_{21}p_{31}p_{40}z^{(2,2,3,0)}+p_{10}p_{21}p_{31}p_{41}z^{(0,2,3,1)}+p_{11}p_{21}p_{31}p_{41}z^{(2,2,3,1)}$$

Finally, for $h = 5$

$$U_2(z) = U_1(z) \underset{\leftarrow}{\otimes} u_5(z)) = (p_{11}p_{21}p_{31}p_{40}z^{(2,2,3,0)}+p_{10}p_{21}p_{31}p_{41}z^{(0,2,3,1)}$$

$$+p_{11}p_{21}p_{31}p_{41}z^{(2,2,3,1)}) \underset{\leftarrow}{\otimes} (p_{50}z^{0}+p_{51}z^{2}) = \underline{\mathbf{p_{11}p_{21}p_{31}p_{40}p_{50}z^{(2,3,0,0)}}}$$

$$+\mathbf{(p_{10}+p_{11})p_{21}p_{31}p_{41}p_{50}z^{(2,3,1,0)}}+p_{11}p_{21}p_{31}p_{40}p_{51}z^{(2,3,0,2)}$$

$$+(p_{10}+p_{11})p_{21}p_{31}p_{41}p_{51}z^{(2,3,1,2)}$$

The terms of $U_2(z)$ with $f_1 = 0$ are marked in bold and the terms with $f_2 = 0$ are underlined. After adding the value of

$$\theta_{f_1}(U_2(z)) = p_{11}p_{21}p_{31}p_{40}p_{50}+(p_{10}+p_{11})p_{21}p_{31}p_{41}p_{50}$$

to x and removing the corresponding terms from $U_2(z)$, this u-function does not contain terms with $f_2 = 0$. Now x is equal to the system unreliability and $R = 1-x$.

The final u-function U_2(z) takes the form

$$U_2(z) = p_{11}p_{21}p_{31}p_{40}p_{51}z^{(2,3,0,2)}+(p_{10}+p_{11})p_{21}p_{31}p_{41}p_{51}z^{(2,3,1,2)}$$

$$= p_{11}p_{21}p_{31}p_{40}p_{51}z^{(2,3,0,2)}+p_{21}p_{31}p_{41}p_{51}z^{(2,3,1,2)}$$

The system reliability can also be obtained as the sum of the coefficients of the resulting *u*-function:

$$R = p_{11}p_{21}p_{31}p_{40}p_{51}+p_{21}p_{31}p_{41}p_{51} = p_{21}p_{31}p_{51}(1-p_{10}p_{40})$$

Example 6.29

An MSWS with n = 10, Y = 3, r_1 = 3, r_2 = 5, r_3 = 7 consists of identical two-state elements. Total failure of the elements corresponds to a performance rate of 0, and a normal state corresponds to performance rate of 1. The reliability of each element j is p_{j1} = 0.8. The system fails if the total performance of any r_i adjacent elements is less then w_i. The graphs of the MSWS reliability as a function of the demands w_1, w_2 and w_3 are presented in Figure 6.30.

When $w_i = r_i$ the system becomes a series one and its reliability is equal to 0.8^{10} = 0.1074. Observe that the variation of demands w_i do not necessarily influence the system's reliability because of failure criteria superposition. For example, satisfying one of the system success conditions

$$\prod_{h=1}^{11-r} 1(\sum_{m=h}^{h+r_i-1} G_m \geq w_i) = 1$$

for r_1 = 3, w_1 = 2 guarantees satisfying this condition for r_2 = 5, w_2 = 3. Therefore, the reliability of the MSWS with w_1 = 2 does not depend on w_2 if $w_2 \leq 3$.

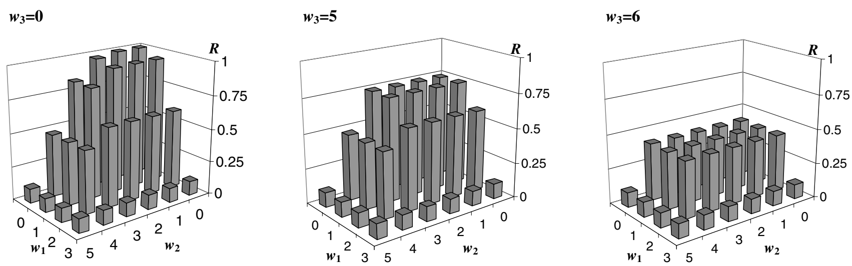


Figure 6.30. Reliability of MSWS as a function of demands

Satisfying the system success condition for r_2 = 5, w_2 = 4 guarantees satisfying this condition for r_1 = 3, w_1 = 2. Therefore, the reliability of the MSWS with w_2 = 4 does not depend on w_1 if $w_1 \leq 2$.

Satisfying the system success condition for r_3 = 7, w_3 = 6 guarantees satisfying this condition for both r_1 = 3, w_1 = 2 and r_2 = 5, w_2 = 4. Therefore, the reliability of the MSWS with w_3 = 6 does not depend on w_1 if $w_1 \leq 2$ and does not depend on w_2 if $w_2 \leq 4$.

6.4.3 Element Reliability Importance in Sliding Window System

The elements' importance measures and the methods of their evaluation for SWSs are the same as for the series-parallel systems. In order to evaluate the importance measures one has to apply the technique described Section 4.5 using the algorithm for SWS reliability evaluation instead of the series-parallel block diagram method.

For SWSs consisting of identical elements it may be important to know how the improvement of all of the elements' reliability influences the entire system's reliability. In order to obtain this importance measure one has to calculate the values of the system reliability for the different values of element reliability, simultaneously changing parameters of the *u*-functions of all of the elements.

Example 6.30

Consider an MSWS with $n = 10$, $Y = 2$, $r_1 = 3$, $r_2 = 5$. The MSWS consists of identical two-state elements. Total failure of the elements corresponds to a performance rate of 0, and anormal state corresponds to a performance rate of 1. The system fails if the total performance of any r_i adjacent elements is less than w_i.

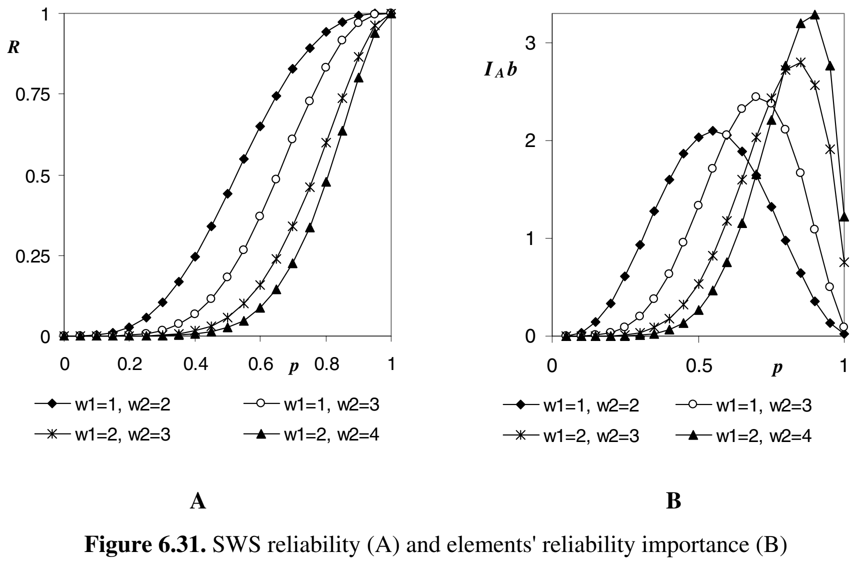


Figure 6.31. SWS reliability (A) and elements' reliability importance (B) as functions of the elements' reliability

In Figure 6.31A one can see the reliability of the MSWS considered as a function of p for different combinations of w_1 and w_2. For the same p, the system reliability decreases with the growth of w_1 and w_2. The elements' Birnbaum reliability importance indices $I_A b = \mathrm{d}R/\mathrm{d}p$ as functions of p are presented in Figure 6.31B. One can see that, until a certain level of p corresponding to a maximal $I_A b$, the more reliable the elements the greater the entire system benefits from further improvement of the elements' reliability. After achieving the maximal value of $I_A b$, the influence of the element's reliability improvement on the system's reliability is drastically reduced (this means that further improvement of the elements' reliability is less justified).

With the growth of w_1 and w_2, the element's reliability corresponding to the maximal reliability importance moves toward the greater values.

In SWSs consisting of nonidentical elements, different elements play different roles in providing for the system's reliability. Evaluating the relative influence of the element's reliability on the reliability of the entire system provides useful information for tracing system bottlenecks.

Example 6.31

Consider an SWS with n = 10 and r = 3 [184]. The parameters of the two-state system elements are presented in Table 6.31. Total failure of any element j corresponds to a performance rate of 0, and a normal state corresponds to a performance rate of g_{j1}. The element's reliability is p_{j1}. The system fails if the cumulative performance of any three adjacent elements is less than the demand w. The system's reliability as a function of demand w is presented in Figure 6.32A.

Table 6.31. Parameters of SWS elements

No. of element j	1	2	3	4	5	6	7	8	9	10
p_{j1}	0.87	0.90	0.83	0.95	0.92	0.89	0.80	0.85	0.82	0.95
g_{j1}	200	200	400	300	100	400	100	200	300	200

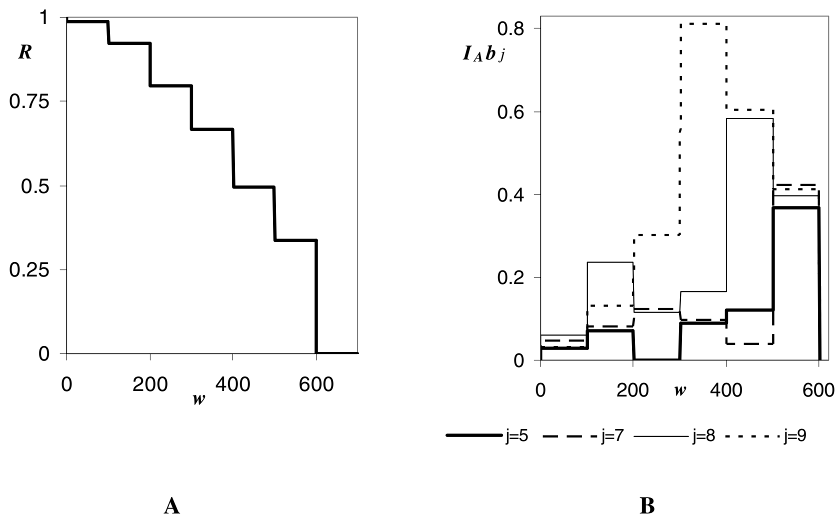


Figure 6.32. SWS reliability (A) and the elements' reliability importance (B) as functions of demand

The reliability importance indices for several elements as functions of the system's demand are presented in Figure 6.32B. Observe that the relative importance of the elements changes with the demand variation. For example, when

$100<w<200$, element 8 is the most important one, whereas when $200<w<300$ this element becomes less important than element 9. This means that in making a decision about the system's reliability enhancement one has to take into account the range of the possible demand levels.

One can see that for some w the importance of the elements can be equal to zero. This means that these elements have no influence on the entire SWS availability and can be removed. Indeed, consider element 5 when $200<w<300$. This element belongs to three triplets with the following nominal performance rates $\{g_{31} = 400, g_{41} = 300, g_{51} = 100\}$, $\{g_{41} = 300, g_{51} = 100, g_{61} = 400\}$, $\{g_{51} = 100, g_{61} = 400, g_{71} = 100\}$. The cumulative performance rate of the first triplet is greater than w if at least one of elements 3 and 4 works and is less than w if both of these elements fail. The cumulative performance rate of the second triplet is greater than w if at least one of elements 4 and 6 works and is less than w if both of these elements fail. The cumulative performance rate of the third triplet is greater than w if element 6 works and is less than w if this element fails. The state of element 5 does not affect the value of the acceptability function for any one of the three triplets.

6.4.4 Optimal Element Sequencing in Sliding Window Systems

Having a given set of MEs, one can achieve considerable reliability improvement of the linear SWS by choosing the elements' proper arrangement along a line. Indeed, it can be easily seen that the order of the tasks' arrivals to the service system (Example 6.26) or allocation of heaters along a line (Example 6.27) can strongly affect the system's entire reliability. For the set of MEs with a given performance rate distribution, the only factor affecting the entire SWS reliability (for fixed r and w) is the sequence of MEs. Papastavridis and Sfakianakis [185] first considered the optimal element arrangement problem for SWSs with binary elements having a different reliability. In this section, the optimal element arrangement problem is considered for the general SWS model. This problem is formulated as follows: find the sequence of MEs in the SWS that maximizes the system reliability.

6.4.4.1 Implementing the Genetic Algorithm

In order to represent the sequence of n MEs in the SWS in the GA one can consider a line with n consequent positions and use a string $\boldsymbol{a} = (a_1, \ldots, a_n)$ in which a_j is equal to the number of the position occupied by ME j (see Section 1.3.2.4). One can see that the total number of different arrangement solutions (number of different possible vectors $\boldsymbol{a}$) is equal to $n!$ (number of possible permutations in a string of n different numbers).

The solution decoding procedure should apply the algorithm for SWS reliability determination for the given sequence of MEs represented by string $\boldsymbol{a}$. The solution's fitness is equal to the value of system reliability $R(\boldsymbol{a}, r, w)$ obtained.

Example 6.32

Consider an SWS with n = 10 [186]. The parameters of the system MEs are presented in Table 6.32. Three element sequencing solutions were obtained by the GA for the SWS with r = 3 (for w = 6, w = 8 and w = 10) and three solutions were obtained for the same SWS with r = 5 (for w = 10, w = 15, w = 20). These solutions are presented in Table 6.33. The system's reliability as a function of demand w is presented in Figure 6.33 for the ME sequences obtained. One can see that the greater r, the greater the SWS reliability for the same w. This is natural, because the growth of r provides a growing redundancy in each group.

Table 6.32. Performance distributions of SWS elements

No. of ME	1		2		3		4		5		6		7		8		9		10	
State	*p*	*g*	*p*	*g*	*p*	*g*	*p*	*g*	*p*	*g*	*p*	*g*	*p*	*g*	*p*	*g*	*p*	*g*	*p*	*g*
0	0.03	0	0.10	0	0.17	0	0.05	0	0.08	0	0.01	0	0.20	0	0.05	0	0.20	0	0.05	0
1	0.22	2	0.10	1	0.83	6	0.25	3	0.20	1	0.22	4	0.10	3	0.25	4	0.10	3	0.25	2
2	0.75	5	0.40	2	-	-	0.40	5	0.15	2	0.77	5	0.10	4	0.70	6	0.15	4	0.70	6
3	-	-	0.40	4	-	-	0.30	6	0.45	4	-	-	0.60	5	-	-	0.55	5	-	-
4	-	-	-	-	-	-	-	-	0.12	5	-	-	-	-	-	-	-	-	-	-

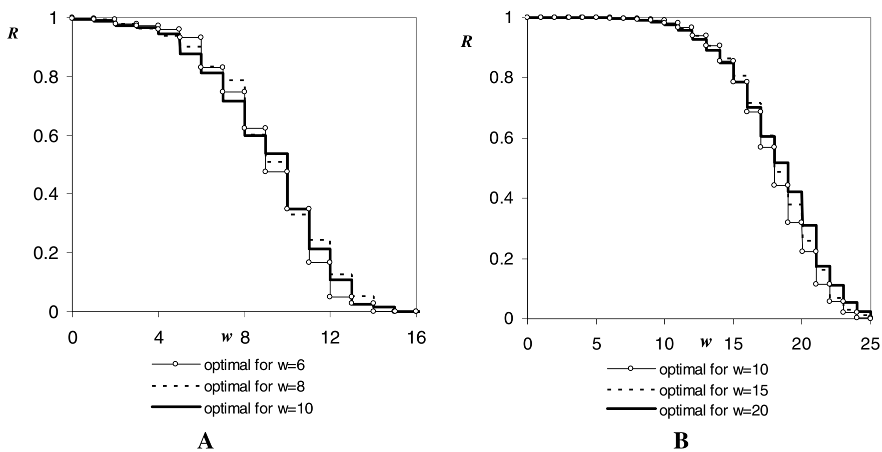


Figure 6.33. Reliability of SWS with the optimal element arrangements as function of demand. A: for r = 3; B: for r = 5

Table 6.33. Parameters of the solutions obtained

r	*w*	*R*	Sequence of SWS elements									
3	6	0.931	2	1	6	5	4	8	7	10	3	9
	8	0.788	5	1	8	9	6	4	7	3	10	2
	10	0.536	5	9	3	1	4	7	10	8	6	2
5	10	0.990	2	5	1	4	6	8	10	3	7	9
	15	0.866	9	7	3	10	1	6	8	4	5	2
	20	0.420	2	5	4	8	3	6	10	7	1	9

Note that, for solutions which provide the greatest SWS reliability for a certain w, the reliability for the rest of the values of w is less than for the solutions optimal for those values of w. Indeed, the optimal allocation provides the greatest system probability of meeting just the specified demand by the price of reducing the probability of meeting greater demands.

6.4.5 Optimal Uneven Element Allocation in Sliding Window Systems

While the problem of the optimal ordering of tasks' arrivals to the service system (Example 6.26) presumes the arrival of one task at a time (only one task can be in each position in the service line), in the problem of the optimal arrangement of heaters (Example 6.27) we can assume that n positions are distributed along a line and the heaters may be allocated unevenly at these positions (several heaters can be gathered at the same position while some positions remain empty).

In many cases such uneven allocation of the MEs in an SWS results in greater system reliability than the even allocation.

Example 6.33

Consider a simple case in which four MEs should be allocated within an SWS with four positions. Each ME j has two states: a failure state with a performance of 0 and a normal state with a performance of 1. The probability of a normal state is p_j, the probability of a failure is $q_j = 1-p_j$. For $r = 3$ and $w = 2$, the system succeeds if each three consecutive positions contain at least two elements in a normal state. Consider two possible allocations of the MEs within the SWS (Figure 6.34):

A. MEs are evenly distributed among the positions.

B. Two MEs are allocated at second position and two MEs are allocated at third position.

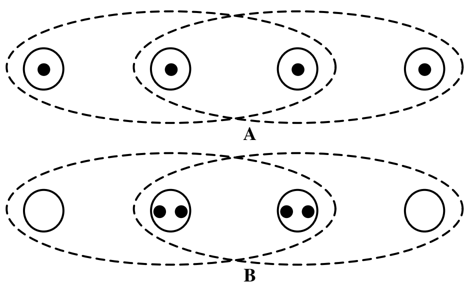


Figure 6.34. Two possible allocations of MEs in an SWS

In case A, the SWS succeeds either if no more than one ME fails or if MEs in the first and fourth positions fail and MEs in the second and third positions are in a normal state. Therefore, the system reliability is

$$R_A = p_1p_2p_3p_4+q_1p_2p_3p_4+p_1q_2p_3p_4+p_1p_2q_3p_4+p_1p_2p_3q_4+q_1p_2p_3q_4$$

For identical MEs with $p_j = p$

$$R_A = p^4+4qp^3+q^2p^2$$

In case B, the SWS succeeds if at least two MEs are in a normal state. The system reliability in this case is

$$R_B = p_1p_2p_3p_4+q_1p_2p_3p_4+p_1q_2p_3p_4+p_1p_2q_3p_4+p_1p_2p_3q_4$$
$$+q_1q_2p_3p_4+q_1p_2q_3p_4+q_1p_2p_3q_4+p_1q_2q_3p_4+p_1q_2p_3q_4+p_1p_2q_3q_4$$

For identical MEs

$$R_B = p^4+4qp^3+6q^2p^2$$

One can see that the uneven allocation B is more reliable:

$$R_B - R_A=5q^2p^2=5(1-p)^2p^2$$

Consider now the same system when w=3. In case A the system succeeds only if it does not contain any failed ME:

$$R_A = p_1p_2p_3p_4$$

In case B it succeeds if it contains no more than one failed element:

$$R_B = p_1p_2p_3p_4+q_1p_2p_3p_4+p_1q_2p_3p_4+p_1p_2q_3p_4+p_1p_2p_3q_4$$

For identical MEs:

$$R_A = p^4,\ R_B = p^4+4qp^3 \text{ and } R_B-R_A = p^4+4qp^3-p^4 = 4(1-p)p^3$$

Observe that, even for $w = 4$, when in case A the system is unable to meet the demand ($R_A = 0$) because $w>r$, in case B it still succeeds with probability $R_B = p_1p_2p_3p_4$.

In this section we consider a general optimal allocation problem in which the number of MEs m is not necessarily equal to the number of positions n ($m \neq n$) and an arbitrary number of elements can be allocated at each position (some positions may be empty):

The SWS consists of n consecutively ordered positions. At each position any group of MEs can be allocated. The allocation problem can be considered as a problem of partitioning a set of m items into a collection of n mutually disjoint subsets. This partition can be represented by the integer string $\boldsymbol{a} = (a_j\text{: } 1\le j\le m)$,

$1 \le a_j \le n$, where a_j is the number of the position at which ME j locates. It is assumed that the SWS has the acceptability function (6.114). The total performance of the group of the MEs located at the same position is equal to the sum of the performances of these MEs. The empty position can be represented by an element with the constant performance of zero.

For any given integer string $\boldsymbol{a}$, the GA determines the solution fitness (equal to the SWS reliability) using the following procedure:

1. Assign $\tilde{u}_i(z) = z^0$ for each $i = 1, \dots, n$, corresponding to SWS positions.

Determine u-functions $u_j(z)$ for each individual ME j ($1 \le j \le m$) in the form (6.115) in accordance with their performance distributions.

2. According to the given string $\boldsymbol{a}$ for each $j = 1, \dots, m$ modify $\tilde{u}_{a_j}(z)$ as follows:

$$\tilde{u}_{a_j}(z) = \tilde{u}_{a_j}(z) \underset{+}{\otimes} u_j(z) \tag{6.121}$$

3. Apply the algorithm for SWS reliability evaluation described in Section 6.4.1 over n u-functions $\tilde{u}_i(z)$.

Example 6.34

Consider an SWS with $n = 10$ positions in which $m = 10$ identical binary MEs are to be allocated [187]. The performance distribution of each ME j is $p_{j1} = \Pr\{G_j = 1\} = 0.9$, $p_{j0} = \Pr\{G_j = 0\} = 0.1$.

Table 6.34 presents allocation solutions obtained for different r and w (number of identical elements in each position). The reliability of the SWS corresponding to the allocations obtained is compared with its reliability corresponding to the case when the MEs are evenly distributed among the positions. One can see that the reliability improvement achieved by the free allocation increases with the increase of r and w. On the contrary, the number of occupied positions in the best solutions obtained decreases when r and w grow. Figure 6.35 presents the SWS reliability as a function of demand w for $r = 2$, $r = 3$ and $r = 4$ for even ME allocation and for unconstrained allocation obtained by the GA.

Table 6.34. Solutions of ME allocation problem (SWS with identical MEs)

Position	r=2, w=1	r=3, w=2	r=4, w=3
1			
2	2	1	
3		3	
4	2		5
5		3	
6	2		
7			5
8	2	3	
9			
10	2		
	Reliability		
Free allocation	0.951	0.941	0.983
Even allocation	0.920	0.866	0.828
Improvement	3.4%	8.7%	18.7%

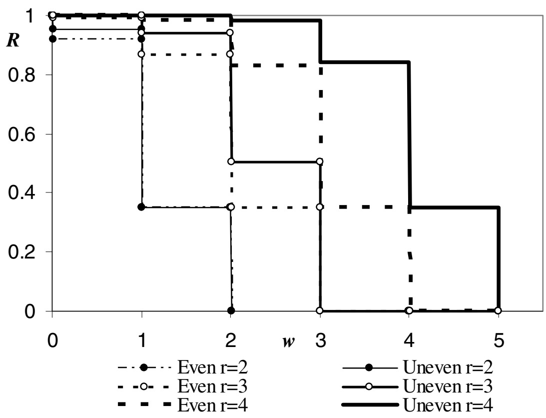


Figure 6.35. Reliability of SWS with identical MEs for different r and ME allocations

Example 6.35

Consider the SWS allocation problem from Example 6.32, in which $n = m = 10$, $r = 3$ and $w = 10$. The best ME allocation solutions obtained by the GA are presented in Table 6.35 (list of elements located at each position).

Table 6.35. Solutions of ME allocation problem (SWS with different MEs)

Position	$m = 10$		$m = 9$	$m = 8$	$m = 7$
	Even allocation	Uneven allocation			
1	5				
2	9				1
3	3	6, 7, 10	2, 5, 8, 9	3, 6	7
4	1				
5	4	2, 5	7		3
6	7	1, 4	1, 4	5, 7, 8	6
7	10				
8	8	3, 8, 9	3	4	4, 5
9	6		6	1, 2	2
10	2				
Reliability	0.536	0.765	0.653	0.509	0.213

The best even allocation solution obtained in Example 6.32 improves considerably when the even allocation constraint is removed. One can see that the best unconstrained allocation solution obtained by the GA in which only 4 out of 10 positions are occupied by the MEs provides a 42% reliability increase over even allocation. The system reliability as a function of demand for the even and unconstrained allocations obtained is presented in Figure 6.36.

Table 6.35 also presents the best allocations of the first m MEs from Table 6.32 (for $m = 9$, $m = 8$ and $m = 7$). Observe that uneven allocation of nine MEs in the SWS still provides greater reliability than does even allocation of 10 MEs.

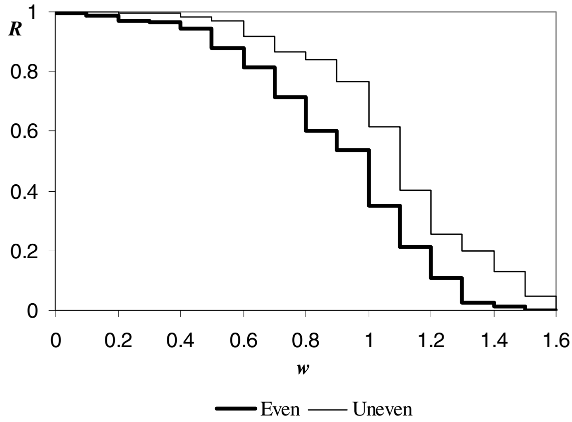


Figure 6.36. Reliability of SWS with different MEs

6.4.6 Sliding Window System Reliability Optimization in the Presence of Common Cause Failures

In many cases, when SWS elements are subject to CCFs, the system can be considered as consisting of mutually disjoint CCGs with total CCFs. The origin of CCFs can be outside the system's elements they affect (external impact), or they can originate from the elements themselves, causing other elements to fail. Usually, the CCFs occur when a group of elements share the same resource (energy source, space, *etc*.)

Example 6.36

Consider the manufacturing heating system from Example 6.27 and assume that the power to the heaters is supplied by B independent power sources (Figure 6.37). Each heater is connected to one of these sources. The heaters supplied from the same source compose the CCG. Each source has a certain failure probability. When the source fails, all of the heaters connected to this source (belonging to the corresponding CCG) are in a state of total failure. Therefore, the failure of any power source causes the CCF in the heating system.

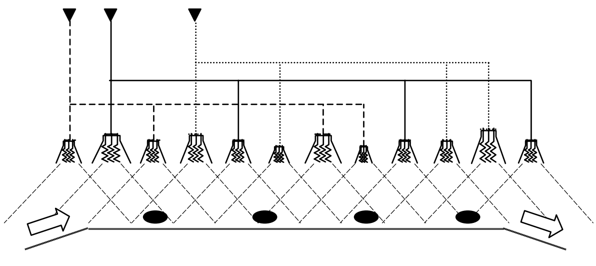


Figure 6.37. Example of SWS with several CCGs

6.4.6.1 Evaluating Sliding Window System Reliability in the Presence of Common Cause Failures

The u-function $u_j(z)$ representing the p.m.f. of the random performance rate of ME j G_j takes the form (6.115) only when the element is not the subject of CCF.

We assume that the performance rate of element j when it is a subject of CCF is equal to zero. The u-function corresponding to this case takes the form $u_j^0(z)=z^0$. The u-functions $u_j(z)$ and $u^0_j(z)$ represent, therefore, conditional performance distributions of the performance rate of element j.

Let the entire SWS consists of a set Λ of n ordered MEs and has B independent CCGs. Each CCG can be in two states (normal state and failure). The failure probability of CCG i is f_i. It can be seen that the total number of the combinations of CCG states is 2^B. Each CCG i can be defined by a subset $\lambda_i \subseteq \Lambda$ such that

$$\bigcup_{i=1}^{B}\lambda_i=\Lambda,\ \lambda_i\cap\lambda_e=\varnothing \text{ for } i\neq e \tag{6.122}$$

Let binary variable s_i define the state of CCG i such that $s_i = 1$ corresponds to the normal state of the group and $s_i = 0$ corresponds to failure of the group. When $s_i = 1$ the performance of each ME j belonging to λ_i is a random value having the distribution determined by its u-function $u_j(z)$. When $s_i = 0$ the performance of each ME belonging to λ_i is equal to zero, which corresponds to the u-function $u^0_j(z)$. One can connect the state s_i of each individual CCG with a number of state combination h in the following way:

$$s_i(h)=\text{mod}_2\lfloor h/2^{i-1}\rfloor \tag{6.123}$$

When h varies from 0 to 2^B-1 one obtains all the possible combinations of states of CCGs using Equation (6.123) for $1\le i\le B$.

The probability of each CCG state combination h is

$$q_h=\prod_{i=1}^{B}(f_i)^{1-s_i(h)}(1-f_i)^{s_i(h)} \tag{6.124}$$

If one defines the u-function of each ME $1\le j\le n$ as

$$\tilde{u}_j(z)=\begin{cases}u_j^0(z), & \text{if} \quad j\in\lambda_i, \quad s_i=0\\ u_j(z), & \text{otherwise}\end{cases} \tag{6.125}$$

and applies the algorithm for SWS reliability evaluation (described in Section 6.4.1) over u-functions $\tilde{u}_j(z)$, then one obtains the conditional probability of the SWS success r_h when the CCG state combination is h. Since all of the 2^B

combinations of CCG states are mutually exclusive, in order to calculate the unconditional probability of the SWS success (SWS reliability) one can apply the following equation:

$$R = \sum_{h=0}^{2^B-1} q_h r_h \tag{6.126}$$

Now we can evaluate the SWS reliability using the following algorithm:

1. Assign $R = 0$. For each j ($1 \le j \le n$) determine two u-functions $u_j(z)$ (in accordance with Equation (6.115) and $u^0_j(z) = z^0$.

2. Repeat the following for $h = 0, \ldots, 2^B-1$:

2.1. For $i = 1, \ldots, B$ determine $s_i(h)$ using Equation (6.123).

2.2. Determine q_h (the probability of CCG state combination h) using Equation (6.124).

2.3. For each $i = 1, \ldots, B$ determine the numbers of elements belonging to the CCG i and define the u-functions of these elements $\tilde{u}_j(z)$ in accordance with Equation (6.125).

2.4. Determine r_k (the conditional SWS reliability for CCG state combination h) applying the procedure described in Section 6.4.1 over u-functions $\tilde{u}_j(z)$ ($1 \le j \le n$) for a given demand w.

2.5. Add the product $q_h r_h$ to R.

6.4.6.2 Optimal Distribution of Multi-state Elements among Common Cause Groups

The way the MEs are distributed among CCGs strongly affects the SWS reliability. Consider a simple example in which SWS with $r = 2$ consists of four MEs composing two CCGs. Each ME has two states with performance rates of 0 and 1. The system demand is 1. When $\lambda_1 = \{1, 2\}$ and $\lambda_2 = \{3, 4\}$ each CCF causes the system's failure. When $\lambda_1 = \{1, 3\}$ and $\lambda_2 = \{2, 4\}$ the SWS can succeed in the case of a single CCF if both MEs not belonging to the failed CCG are in the operational state.

The elements' distribution problem can be considered as a problem of partitioning a set Λ of n MEs into a collection of B mutually disjoint subsets λ_i ($1 \le i \le B$). Each set can contain from 0 to n elements. The partition of set Λ can be represented in the GA by the integer string $\boldsymbol{a} = \{a_j: 1 \le j \le n\}$, $0 \le a_j \le B$, where a_j is the number of the subset to which ME j belongs: $j \in \lambda_{a_j}$.

Example 6.36

Consider the SWS with $n = 10$ from Example 6.32 [188]. The parameters of the system's MEs are presented in Table 6.32. It is assumed that the failure probability f_i of each CCG i is equal to 0.2.

First, distribution solutions were obtained by the GA for a SWS with fixed B. The solutions that provide the greatest SWS reliability for a certain demand w

provide a reliability for the remainder of the values of w that is less than the reliability of the optimal solutions for those values of w. Indeed, the optimal allocation provides the greatest system probability of meeting just the specified demand by the price of reducing the probability of meeting other demands. Therefore, the optimal distribution solution depends on system demand. The solutions obtained for different demands for SWS with $r = 3$ and $r = 5$ when $B = 2$ are presented in Table 6.36.

When $r = 3$, solution A is optimal for demand $0<w<3$ and solution B is optimal for demand $3 \le w \le 5$. When $r = 5$, solution D is optimal for demand $7 \le w \le 11$ and solution E is optimal for demand $12 \le w \le 16$. The system reliabilities, as functions of demand for the solutions obtained, are presented in Figure 6.38. The solutions C (for $r = 3$) and F (for $r = 5$), in which adjacent elements belong to different CCGs, are presented for comparison. These solutions provide lower SWS reliability than the optimal ones.

Table 6.36. Solutions of ME grouping problem obtained for $B = 2$

r		Distribution of MEs
3	A	{1, 4, 6, 7, 9, 10} {2, 3, 5, 8}
	B	{1, 2, 4, 5, 7, 8, 10} {3, 6, 9}
	C	{1, 3, 5, 7, 9} {2, 4, 6, 8, 10}
5	D	{1, 2, 5, 6, 7, 10} {3, 4, 8, 9}
	E	{1, 2, 3, 6, 7, 8} {4, 5, 9, 10}
	F	{1, 3, 5, 7, 9} {2, 4, 6, 8, 10}

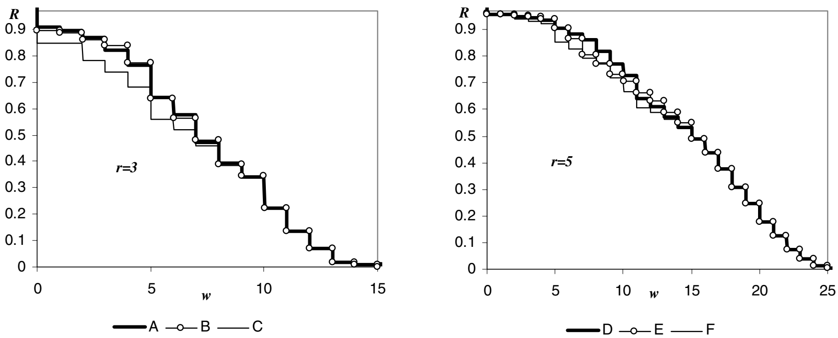


Figure 6.38. SWS reliability for the best ME distribution solutions obtained for $B = 2$

Observe that with the growth of w the difference of system reliability provided by the different distribution solutions becomes negligible. This is because, when w is great, the system becomes intolerant to any common supply failure without regard to the structure of the failed CCG.

The problem of choosing the optimal number of CCGs is also of interest. The increase in the number of CCGs reduces the damage caused by a single CCG to the system, but, on the other hand, it increases the probability that at least one CCG has failed. Therefore, when the system does not tolerate the loss of even a small portion of its elements (which happens when w is great), the increase of B decreases the system's overall reliability. The smaller the system demand w, the greater the benefit from the elements' distribution among different CCGs. Table 6.37 presents the solutions obtained for the given SWS for different B. For each $B>1$ the solution that provides the system's reliability greater than the reliability of a system with a single CCG for the greatest w is chosen (this means that for the given B, the solution presented is better than the solution with $B = 1$ for a given demand w^*, but, when the demand is greater than w^*, no distribution solution with B CCGs exists that outperforms the single CCG solution).

Table 6.37. Solutions of ME grouping problem obtained for different *B*

r	*B*	Distribution of MEs
3	1	{1, 2, 3, 4, 5, 6, 7, 8, 9, 10}
	2	{1, 2, 3, 6, 7, 8} {4, 5, 9, 10}
	3	{1, 4, 7, 10} {2, 5, 8} {3, 6, 9}
5	1	{1, 4, 6, 7, 9, 10} {2, 3, 5, 8}
	2	{1, 4, 5, 6, 9, 10} {3, 4, 7, 8}
	3	{1, 5, 6, 10} {2, 4, 7, 9} {3, 8}
	4	{1, 5, 6, 10} {2, 7} {3, 8} {4, 9}
	5	{1, 6} {2, 7} {3, 8} {4, 9} {5, 10}

The SWS reliabilities as functions of demand for the solutions obtained are presented in Figure 6.39. The single CCG solution is the worst for the small demands and the best for the great demands. On the contrary, the solutions with $B = r$, in which any r adjacent elements belong to different CCGs, provide the greatest system reliability for small demands and provide the lowest system reliability for great demands. The solutions with $1<B<r$ provide intermediate values of system reliability. Therefore, when the number of CCGs is not fixed, the greatest reliability solution is either with $B = r$ for low demands or with $B = 1$ for great demands.

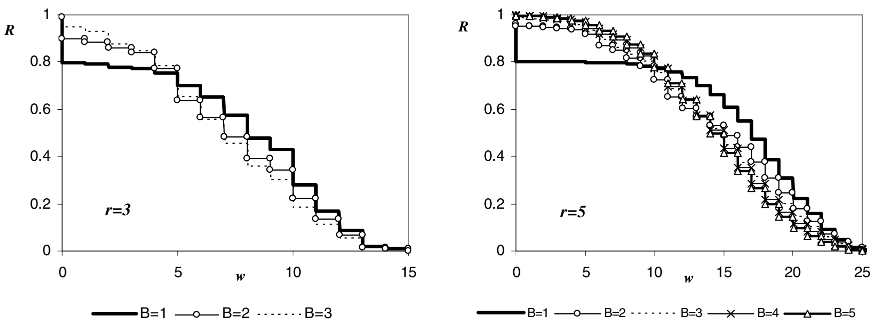


Figure 6.39. SWS reliability for the best ME distribution solutions obtained for different *B*