4. Universal Generating Function in Analysis of Series-Parallel Multi-state Systems

4.1 Reliability Block Diagram Method

Having a generic model of an MSS in the form of Equations (3.3) and (3.4) we can obtain the measures of MSS reliability by applying the following steps:

1. Represent the p.m.f. of the random performance of each system element j, Equations (3.1) and (3.2), in the form of the u-function

$$u_j(z) = \sum_{i=0}^{k_j-1} p_{ji} z^{g_{ji}}, \quad 1 \le j \le n \tag{4.1}$$

2. Obtain the u-function of the entire system (representing the p.m.f. of the random variable G) applying the composition operator that uses the system structure function.
3. Obtain the u-functions representing the random functions F, $\tilde{G}$ and D using operators (3.8)-(3.10).
4. Obtain the system reliability measures by calculating the values of the derivatives of the corresponding u-functions at $z = 1$ and applying Equations (3.11)-(3.14).

While steps 1, 3 and 4 are rather trivial, step 2 may involve complicated computations. Indeed, the derivation of a system structure function for various types of system is usually a difficult task.

As shown in Chapter 1, representing the functions in the recursive form is beneficial from both the derivation clarity and computation simplicity viewpoints. In many cases, the structure function of the entire MSS can be represented as the composition of the structure functions corresponding to some subsets of the system elements (MSS subsystems). The u-functions of the subsystems can be obtained separately and the subsystems can be further treated as single equivalent elements with the performance p.m.f. represented by these u-functions.

The method for distinguishing recurrent subsystems and replacing them with single equivalent elements is based on a graphical representation of the system structure and is referred to as the reliability block diagram method. This approach is usually applied to systems with a complex series-parallel configuration.

While the structure function of a binary series-parallel system is unambiguously determined by its configuration (represented by the reliability block diagram), the structure function of a series-parallel MSS also depends on the physical meaning of the system and of the elements' performance and on the nature of the interaction among the elements.

4.1.1 Series Systems

In the flow transmission MSS, where performance is defined as capacity or productivity, the total capacity of a subsystem containing n independent elements connected in series is equal to the capacity of a bottleneck element (the element with least performance). Therefore, the structure function for such a subsystem takes the form

$$\phi_{\text{ser}}(G_1,...,G_n) = \min\{G_1,...,G_n\} \tag{4.2}$$

In the task processing MSS, where the performance is defined as the processing speed (or operation time), each system element has its own operation time and the system's total task completion time is restricted. The entire system typically has a time resource that is larger than the time needed to perform the system's total task. But unavailability or deteriorated performance of the system elements may cause time delays, which in turn would cause the system's total task performance time to be unsatisfactory. The definition of the structure function for task processing systems depends on the discipline of the elements' interaction in the system.

When the system operation is associated with consecutive discrete actions performed by the ordered line of elements, each element starts its operation after the previous one has completed its operation. Assume that the random performances G_j of each element j is characterized by its processing speed. The random processing time T_j of any system element j is defined as $T_j = 1/G_j$. The total time of task completion for the entire system is

$$T = \sum_{j=1}^{n} T_j = \sum_{j=1}^{n} G_j^{-1} \tag{4.3}$$

The entire system processing speed is therefore

$$G = 1/T = (\sum_{j=1}^{n} G_j^{-1})^{-1} \tag{4.4}$$

Note that if for any j $G_j = 0$ the equation cannot be used, but it is obvious that in this case $G = 0$. Therefore, one can define the structure function for the series task processing system as

$$\phi_{\text{ser}}(G_1,\ldots,G_n) = \times(G_1,\ldots,G_n) = \begin{cases} 1/\sum_{j=1}^{n} G_j^{-1} & \text{if } \prod_{j=1}^{n} G_j \neq 0 \\ 0 & \text{if } \prod_{j=1}^{n} G_j = 0 \end{cases} \tag{4.5}$$

One can see that the structure functions presented above are associative and commutative (*i.e.* meet conditions (1.26) and (1.28)). Therefore, the u-functions for any series system of described types can be obtained recursively by consecutively determining the u-functions of arbitrary subsets of the elements. For example the u-function of a system consisting of four elements connected in a series can be determined in the following ways:

$$\begin{aligned} &[(u_1(z) \underset{\phi_{\text{ser}}}{\otimes} u_2(z)) \underset{\phi_{\text{ser}}}{\otimes} u_3(z)] \underset{\phi_{\text{ser}}}{\otimes} u_4(z) \\ &= (u_1(z) \underset{\phi_{\text{ser}}}{\otimes} u_2(z)) \underset{\phi_{\text{ser}}}{\otimes} (u_3(z) \underset{\phi_{\text{ser}}}{\otimes} u_4(z)) \end{aligned} \tag{4.6}$$

and by any permutation of the elements' u-functions in this expression.

Example 4.1

Consider a system consisting of n elements with the total failures connected in series. Each element j has only two states: operational with a nominal performance of g_{j1} and failure with a performance of zero. The probability of the operational state is p_{j1}. The u-function of such an element is presented by the following expression:

$$u_j(z) = (1 - p_{j1})z^0 + p_{j1}z^{g_{j1}}, \quad j = 1, \ldots, n$$

In order to find the u-function for the entire MSS, the corresponding $\otimes_{\phi_{\text{ser}}}$ operators should be applied. For the MSS with the structure function (4.2) the system u-function takes the form

$$U(z) = \underset{\min}{\otimes}(u_1(z),\ldots,u_n(z)) = (1 - \prod_{j=1}^{n} p_{j1})z^0 + \prod_{j=1}^{n} p_{j1} z^{\min\{g_{11},\ldots,g_{n1}\}}$$

For the MSS with the structure function (4.5) the system u-function takes the form

$$U(z) = \underset{\times}{\otimes}\{u_1(z),\ldots,u_n(z)\} = (1 - \prod_{j=1}^{n} p_{j1})z^0 + \prod_{j=1}^{n} p_{j1} z^{(\sum_{j=1}^{n} g_{j1}^{-1})^{-1}}$$

Since the failure of each single element causes the failure of the entire system, the MSS can have only two states: one with the performance level of zero (failure of at least one element) and one with the performance level $\hat{g} = \min\{g_{11},..., g_{n1}\}$ for the flow transmission MSS and $\hat{g} = 1/\sum_{j=1}^{n} g_{j1}^{-1}$ for the task processing MSS.

The measures of the system performance $A(w) = \Pr\{G \ge w\}$, $\Delta^{-}(w) = E(\max(w-G,0))$ and $\varepsilon = E(G)$ are presented in the Table 4.1.

Table 4.1. Measures of MSS performance

w	$A(w)$	$\Delta^{-}(w)$	ε
$w > \hat{g}$	0	$w(1-\prod_{j=1}^{n} p_{j1}) + (w-\hat{g})\prod_{j=1}^{n} p_{j1} = w - \hat{g}\prod_{j=1}^{n} p_{j1}$	$\hat{g}\prod_{j=1}^{n} p_{j1}$
$0 < w \le \hat{g}$	$\prod_{j=1}^{n} p_{j1}$	$w(1-\prod_{j=1}^{n} p_{j1})$	

The u-function of a subsystem containing n identical elements ($p_{j1}=p$, $g_{j1}=g$ for any j) takes the form

$$(1-p^n)z^0 + p^n z^g \tag{4.7}$$

for the system with the structure function (4.2) and takes the form

$$(1-p^n)z^0 + p^n z^{g/n} \tag{4.8}$$

for the system with the structure function (4.5).

4.1.2 Parallel Systems

In the flow transmission MSS, in which the flow can be dispersed and transferred by parallel channels simultaneously (which provides the work sharing), the total capacity of a subsystem containing n independent elements connected in parallel is equal to the sum of the capacities of the individual elements. Therefore, the structure function for such a subsystem takes the form

$$\phi_{\text{par}}(G_1,...,G_n) = +(G_1,...,G_n) = \sum_{j=1}^{n} G_j \tag{4.9}$$

In some cases only one channel out of n can be chosen for the flow transmission (no flow dispersion is allowed). This happens when the transmission is associated with the consumption of certain limited resources that does not allow simultaneous use of more than one channel. The most effective way for such a system to function is by choosing the channel with the greatest transmission capacity from the set of available channels. In this case, the structure function takes the form

$$\phi_{\text{par}}(G_1,...,G_n) = \max\{G_1,...,G_n\}. \tag{4.10}$$

In the task processing MSS, the definition of the structure function depends on the nature of the elements' interaction within the system.

First consider a system without work sharing in which the parallel elements act in a competitive manner. If the system contains n parallel elements, then all the elements begin to execute the same task simultaneously. The task is assumed to be completed by the system when it is completed by at least one of its elements. The entire system processing time is defined by the minimum element processing time and the entire system processing speed is defined by the maximum element processing speed. Therefore, the system structure function coincides with (4.10).

Now consider a system of n parallel elements with work sharing for which the following assumptions are made:

1. The work x to be performed can be divided among the system elements in any proportion.
2. The time required to make a decision about the optimal work sharing is negligible, the decision is made before the task execution and is based on the information about the elements state during the instant the demand for the task executing arrives.
3. The probability of the elements failure during any task execution is negligible.

The elements start performing the work simultaneously, sharing its total amount x in such a manner that element j has to perform x_j portion of the work and $x = \sum_{j=1}^{n} x_j$. The time of the work processing by element j is x_j/G_j. The system processing time is defined as the time during which the last portion of work is completed: $T = \max_{1 \le j \le n}\{x_j / G_j\}$. The minimal time of the entire work completion can be achieved if the elements share the work in proportion to their processing speed G_j: $x_j = xG_j / \sum_{k=1}^{n} G_k$. The system processing time T in this case is equal to $x / \sum_{k=1}^{n} G_k$ and its total processing speed G is equal to the sum of the processing speeds of its elements. Therefore, the structure function of such a system coincides with the structure function (4.9).

One can see that the structure functions presented also meet the conditions (1.26) and (1.28). Therefore, the u-functions for any parallel system of described types can be obtained recursively by the consecutive determination of u-functions of arbitrary subsets of the elements.

Example 4.2

Consider a system consisting of two elements with total failures connected in parallel. The elements have nominal performance g_{11} and g_{21} ($g_{11}<g_{21}$) and the

probability of operational state p_{11} and p_{21} respectively. The performances in the failed states are $g_{10} = g_{20} = 0$. The u-function for the entire MSS is

$$U(z) = u_1(z) \underset{\phi_{\text{par}}}{\otimes} u_2(z)$$

$$= [(1-p_{11})z^0 + p_{11}z^{g_{11}}] \underset{\phi_{\text{par}}}{\otimes} [(1-p_{21})z^0 + p_{21}z^{g_{21}}]$$

which for structure function (4.9) takes the form

$$U(z) = (1-p_{11})(1-p_{21})z^0 + p_{11}(1-p_{21})z^{g_{11}}$$
$$+ p_{21}(1-p_{11})z^{g_{21}} + p_{11}p_{21}z^{g_{11}+g_{21}}$$

and for structure function (4.10) takes the form

$$U(z) = (1-p_{11})(1-p_{21})z^0 + p_{11}(1-p_{21})z^{g_{11}} + p_{21}(1-p_{11})z^{g_{21}}$$
$$+ p_{11}p_{21}z^{\max(g_{11},g_{21})} = (1-p_{11})(1-p_{21})z^0 + p_{11}(1-p_{21})z^{g_{11}} + p_{21}z^{g_{21}}$$

The measures of the system output performance for MSSs of both types are presented in Tables 4.2 and 4.3.

Table 4.2. Measures of MSS performance for system with structure function (4.9)

w	$A(w)$	$\Delta^-(w)$	ε
$w>g_{11}+g_{21}$	0	$w-p_{11}g_{11}-p_{21}g_{21}$	
$g_{21}<w\le g_{11}+g_{21}$	$p_{11}p_{21}$	$g_{11}p_{11}(p_{21}-1)+g_{21}p_{21}(p_{11}-1)+w(1-p_{11}p_{21})$	
$g_{11}<w\le g_{21}$	p_{21}	$(1-p_{21})(w-g_{11}p_{11})$	$p_{11}g_{11}+p_{21}g_{21}$
$0<w\le g_{11}$	$p_{11}+p_{21}-p_{11}p_{21}$	$(1-p_{11})(1-p_{21})w$	

Table 4.3. Measures of MSS performance for system with structure function (4.10)

w	$A(w)$	$\Delta^-(w)$	ε
$w>g_{21}$	0	$w-p_{11}g_{11}-p_{21}g_{21}+p_{11}p_{21}g_{11}$	
$g_{11}<w\le g_{21}$	p_{21}	$(1-p_{21})(w-g_{11}p_{11})$	$p_{11}(1-p_{21})g_{11}+p_{21}g_{21}$
$0<w\le g_{11}$	$p_{11}+p_{21}-p_{11}p_{21}$	$(1-p_{11})(1-p_{21})w$	

The u-function of a subsystem containing n identical parallel elements ($p_{j1} = p$, $g_{j1} = g$ for any j) can be obtained by applying the operator $\otimes_{\phi_{\text{par}}}(u(z),....,u(z))$ over n identical u-functions $u(z)$ of an individual element. The u-function of this subsystem takes the form

$$\sum_{k=0}^{n} \frac{n!}{k!(n-k)!} p^k (1-p)^{n-k} z^{kg} \tag{4.11}$$

for the structure function (4.9) and

$$(1-p)^n z^0 + (1-(1-p)^n) z^g \tag{4.12}$$

for the structure function (4.10).

4.1.3 Series-Parallel Systems

The structure functions of complex series-parallel systems can always be represented as compositions of the structure functions of statistically independent subsystems containing only elements connected in a series or in parallel. Therefore, in order to obtain the *u*-function of a series-parallel system one has to apply the composition operators recursively in order to obtain *u*-functions of the intermediate pure series or pure parallel structures.

The following algorithm realizes this approach:

1. Find the pure parallel and pure series subsystems in the MSS.
2. Obtain *u*-functions of these subsystems using the corresponding $\otimes_{\phi_{\text{ser}}}$ and $\otimes_{\phi_{\text{par}}}$ operators.
3. Replace the subsystems with single elements having the *u*-function obtained for the given subsystem.
4. If the MSS contains more then one element return to step 1.

The resulting *u*-function represents the performance distribution of the entire system.

The choice of the structure functions used for series and parallel subsystems depends on the type of system. Table 4.4 presents the possible combinations of structure functions corresponding to the different types of MSS.

Table 4.4. Structure functions for a purely series and for purely parallel subsystems

No of MSS type	Description of MSS	Structure function for series elements (ϕ_{ser})	Structure function for parallel elements (ϕ_{par})
1	Flow transmission MSS with flow dispersion	(4.2)	(4.9)
2	Flow transmission MSS without flow dispersion	(4.2)	(4.10)
3	Task processing MSS with work sharing	(4.5)	(4.9)
4	Task processing MSS without work sharing	(4.5)	(4.10)

Example 4.3

In order to illustrate the recursive approach (the reliability block diagram method) consider the series-parallel system presented in Figure 4.1A.

First, one can find only one pure series subsystem consisting of elements with the u-functions $u_2(z)$, $u_3(z)$ and $u_4(z)$. By calculating the u-function $U_1(z)= u_2(z) \underset{\phi_{\text{ser}}}{\otimes} u_3(z) \underset{\phi_{\text{ser}}}{\otimes} u_4(z)$ and replacing the three elements with a single element with the u-function $U_1(z)$ one obtains a system with the structure presented in Figure 4.1B. This system contains a purely parallel subsystem consisting of elements with the u-functions $U_1(z)$ and $u_5(z)$, which in their turn can be replaced by a single element with the u-function $U_2(z) = U_1(z) \underset{\phi_{\text{par}}}{\otimes} u_5(z)$ (Figure 4.1C). The structure obtained has three elements connected in a series that can be replaced with a single element having the u-function $U_3(z) = u_1(z) \underset{\phi_{\text{ser}}}{\otimes} U_2(z) \underset{\phi_{\text{ser}}}{\otimes} u_6(z)$ (Figure 4.1D). The resulting structure contains two elements connected in parallel. The u-function of this structure representing the p.m.f. of the entire MSS performance is obtained as $U(z) = U_3(z) \underset{\phi_{\text{par}}}{\otimes} u_7(z)$.

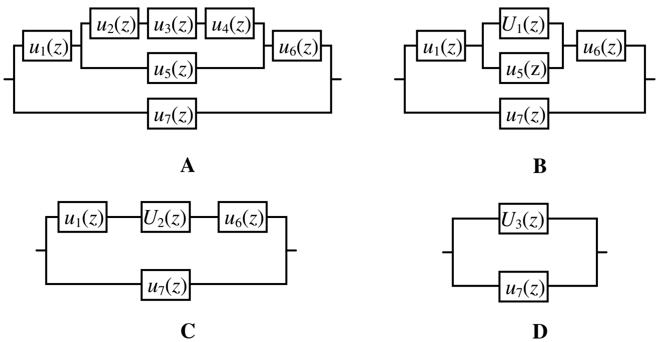


Figure 4.1. Example of recursive determination of the MSS u-function

Assume that in the series-parallel system presented in Figure 4.1A all of the system elements can have two states (elements with total failure) and have the parameters presented in Table 4.5. Each element j has a nominal performance rate g_{j1} in working state and performance rate of zero when it fails. The system is repairable and the steady-state probability that element j is in working state (element availability) is p_{j1}.

Table 4.5. Parameters of elements of series-parallel system

j	1	2	3	4	5	6	7
g_{j1}	5	3	5	4	2	6	3
p_{j1}	0.9	0.8	0.9	0.7	0.6	0.8	0.8

The process of calculating $U(z)$ for the flow transmission system with flow dispersion (for which ϕ_{ser} and ϕ_{par} functions are defined by Equations (4.2) and (4.9) respectively) is as follows:

$$u_2(z) \underset{\min}{\otimes} u_3(z) = (0.8z^3+0.2z^0) \underset{\min}{\otimes} (0.9z^5+0.1z^0) = 0.72z^3+0.28z^0$$

$$U_1(z) = (u_2(z) \underset{\min}{\otimes} u_3(z)) \underset{\min}{\otimes} u_4(z)$$

$$= (0.72z^3+0.28z^0) \underset{\min}{\otimes} (0.7z^4+0.3z^0) = 0.504z^3+0.496z^0$$

$$U_2(z) = U_1(z) \underset{+}{\otimes} u_5(z) = (0.504z^3+0.496z^0) \underset{+}{\otimes} (0.6z^3+0.4z^0)$$

$$= 0.3024z^6+0.4992z^3+0.1984z^0$$

$$u_1(z) \underset{\min}{\otimes} U_2(z) = (0.9z^5+0.1z^0) \underset{\min}{\otimes} (0.3024z^6+0.4992z^3+0.1984z^0)$$

$$= 0.27216z^5+0.44928z^3+0.27856z^0;$$

$$U_3(z) = (u_1(z) \underset{\min}{\otimes} U_2(z)) \underset{\min}{\otimes} u_6(z) = (0.27216z^5+0.44928z^3$$

$$+0.27856z^0) \underset{\min}{\otimes} (0.8z^6+0.2z^0) = 0.217728z^5+0.359424z^3+0.422848z^0$$

$$U(z) = U_3(z) \underset{+}{\otimes} u_7(z)$$

$$= (0.217728z^5+0.359424z^3+0.422848z^0) \underset{+}{\otimes} (0.8z^3+0.2z^0)$$

$$= 0.1741824z^8+0.2875392z^6+0.0435456z^5+0.4101632z^3+0.0845696z^0$$

Having the system u-function that represents its performance distribution one can easily obtain the system mean performance $\varepsilon = U'(1) = 4.567$. The system availability for different demand levels can be obtained by applying the operator δ_w (3.15) over the u-function $U(z)$:

$A(w) = 0.91543$ for $0<w\le 3$

$A(w) = 0.50527$ for $3<w\le 5$

$A(w) = 0.461722$ for $5<w\le 6$

$A(w) = 0.174182$ for $6<w\le 8$

$A(w) = 0$ for $w>8$

The process of calculating $U(z)$ for the task processing system without work sharing (for which ϕ_{ser} and ϕ_{par} functions are defined by Equations (4.5) and (4.10) respectively) is as follows:

$$u_2(z) \underset{\times}{\otimes} u_3(z) = (0.8z^3+0.2z^0) \underset{\times}{\otimes} (0.9z^5+0.1z^0) = 0.72z^{1.875}+0.28z^0;$$

$$U_1(z) = (u_2(z) \underset{\times}{\otimes} u_3(z)) \underset{\times}{\otimes} u_4(z)$$

$$= (0.72z^{1.875}+0.28z^0) \underset{\times}{\otimes} (0.7z^4+0.3z^0) = 0.504z^{1.277}+0.496z^0$$

$$U_2(z) = U_1(z) \underset{\max}{\otimes} u_5(z)) = (0.504z^{1.277}+0.496z^0) \underset{\max}{\otimes} (0.6z^2+0.4z^0)$$

$$= 0.6z^2+0.2016z^{1.277}+0.1984z^0$$

$$u_1(z) \underset{\times}{\otimes} U_2(z) = (0.9z^5+0.1z^0) \underset{\times}{\otimes} (0.6z^2+0.2016z^{1.277}+0.1984z^0)$$

$$= 0.54z^{1.429}+0.18144z^{1.017}+0.27856z^0$$

$$U_3(z) = (u_1(z) \underset{\times}{\otimes} U_2(z)) \underset{\times}{\otimes} u_6(z) = (0.54z^{1.429}+0.18144z^{1.017}$$

$$+0.27856z^0) \underset{\times}{\otimes} (0.8z^6+0.2z^0) = 0.432z^{1.154}+0.145152z^{0.87}+0.422848z^0$$

$$U(z) = U_3(z) \underset{\max}{\otimes} u_7(z) = (0.432z^{1.154}+0.145152z^{0.87}+0.422848z^0)$$

$$\underset{\max}{\otimes} (0.8z^3+0.2z^0) = 0.8z^3+0.0864z^{1.154}+0.0290304z^{0.87}$$

$$+0.08445696z^0$$

The main performance measures of this system are:

$$\varepsilon = U'(1) = 2.549$$

$$A(w) = 0.91543 \text{ for } 0<w\le 0.87,\ A(w) = 0.8864 \text{ for } 0.87<w\le 1.429$$

$$A(w) = 0.8 \text{ for } 1.429<w\le 3,\ A(w) = 0 \text{ for } w>3$$

The procedure described above obtains recursively the same MSS u-function that can be obtained directly by operator $\underset{\phi}{\otimes}(u_1(z), u_2(z), u_3(z), u_4(z), u_5(z))$ using the following structure function:

$$\phi(G_1, G_2, G_3, G_4, G_5, G_6, G_7)$$

$$= \phi_{par}(\phi_{ser}(G_1, \phi_{par}(\phi_{ser}(G_2, G_3, G_4), G_5), G_6), G_7)$$

The recursive procedure for obtaining the MSS u-function is not only more convenient than the direct one, but, and much more important, it allows one to reduce the computational burden of the algorithm considerably. Indeed, using the direct procedure corresponding to Equation (1.20) one has to evaluate the system structure function for each combination of values of random variables G_1, ..., G_7 ($\prod_{j=1}^{7} k_j$ times, where k_j is the number of states of element j). Using the recursive algorithm one can take advantage of the fact that some subsystems have the same performance rates in different states, which makes these states indistinguishable and reduces the total number of terms in the corresponding u-functions.

In Example 4.3 the number of evaluations of the system structure function using the direct Equation (1.20) for the system with two-state elements is $2^7 = 128$. Each evaluation requires calculating a function of seven arguments. Using the reliability block diagram method one obtains the system u-function just by 30 procedures of structure function evaluation (each procedure requires calculating simple functions of just two arguments). This is possible because of the reduction in the lengths of intermediate u-functions by like terms collection. For example, it can be easily seen that in the subsystem of elements 2, 3 and 4 all eight possible combinations of the elements' states produce just two different values of the subsystem performance: 0 and $\min(g_{21}, g_{31}, g_{41})$ in the case of the flow transmission system, or 0 and $g_{21}g_{31}g_{41}/(g_{21}g_{31}+g_{21}g_{41}+g_{31}g_{41})$ in the case of the task processing system. After obtaining the u-function $U_1(z)$ for this subsystem and collecting like terms one gets a two-term equivalent u-function that is used further in the recursive algorithm. Such simplification is impossible when the entire expression (1.20) is used.

Example 4.4

Assume that in the series-parallel system presented in Figure 4.1A all of the system elements can have two states (elements with total failure). The system is unrepairable and the reliability of each element is defined by the Weibull hazard function

$$h(t) = \lambda^{\gamma}\gamma t^{\gamma-1}$$

The accumulated hazard function takes the form

$$H(t) = (\lambda t)^{\gamma}$$

The elements' nominal performance rates g_{j1}, the hazard function scale parameters λ_j and the shape parameters γ_j are presented in Table 4.6. One can see that some elements have increasing failure rates ($\gamma > 1$) that correspond to their aging and some elements have constant failure rates ($\gamma = 1$).

Since the MSS reliability varies with the time, in order to obtain the performance measures of the system the reliability of its elements $\Pr\{G_j = g_{j1}\} = \exp(-H_j(t))$ should be calculated for each time instant. Then the entire system characteristics can be evaluated for the given demand w. Figures 4.2, 4.3 and 4.4

present ε, $R(w)$ and $\Delta^-(w)$ as functions of time for different types of system (numbered according to Table 4.4).

Table 4.6. Parameters of system elements

No of element	Nominal performance rate	Hazard function parameters	
	g	λ	γ
1	5	0.018	1.0
2	3	0.010	1.2
3	5	0.015	1.0
4	4	0.022	1.0
5	2	0.034	1.0
6	6	0.012	2.2
7	3	0.025	1.8

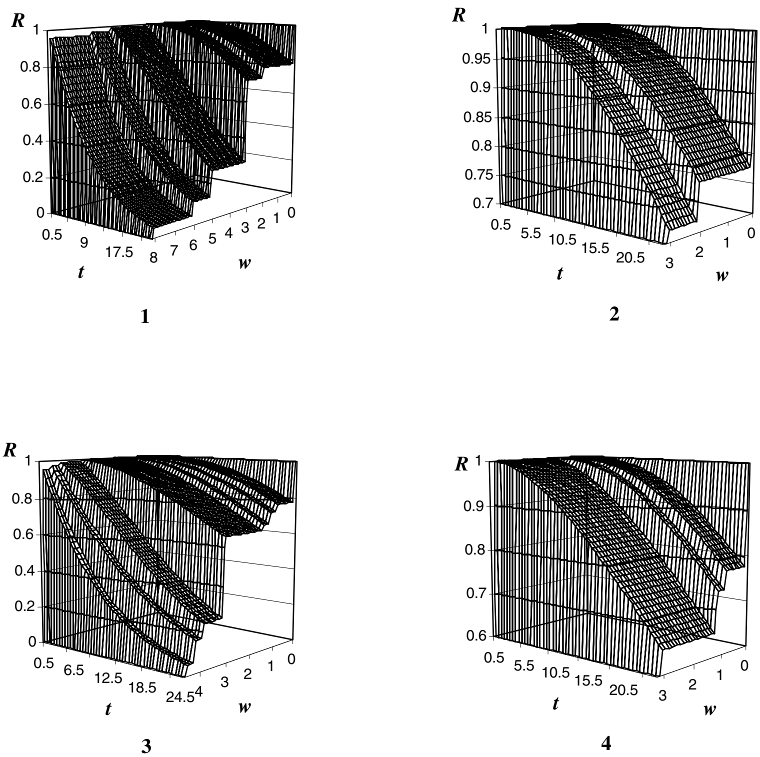


Figure 4.2. System reliability function for different types of MSS

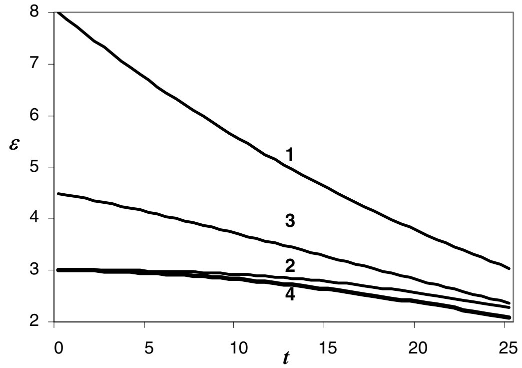


Figure 4.3. System mean performance for different types of MSS

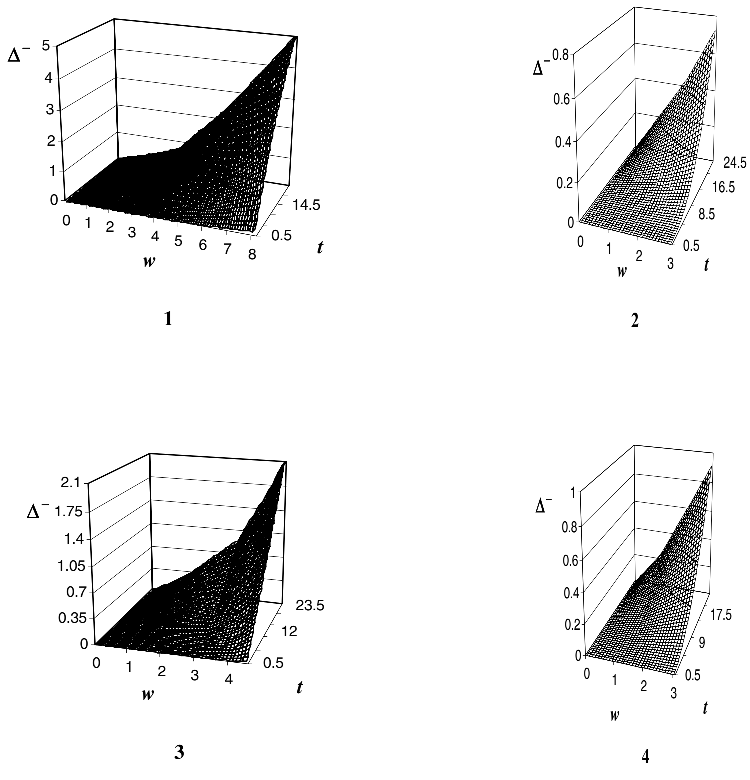


Figure 4.4. System performance deficiency for different types of MSS

4.1.4 Series-Parallel Multi-state Systems Reducible to Binary Systems

In some special cases the reliability (availability) of the entire system can be obtained without derivation of its *u*-function. In the final stage of reliability evaluation, such systems can be treated as binary systems.

Consider, for example, a flow transmission system consisting of n independent multi-state components connected in a series (each component in its turn can be a series-parallel subsystem). Let G_j be the random performance of component j. The structure function of the series flow transmission system is $G=\phi(G_1,...,G_n)=\min\{G_1,...,G_n\}$.

Assume that the system should meet a constant demand w. Therefore, the system acceptability function takes the form $F(G,w)=1(G\geq w)$. It can be seen that in this special case

$$F(G,w)=1(\min\{G_1,...,G_n\}\geq w)=\prod_{j=1}^{n}1(G_j\geq w) \tag{4.13}$$

The system's reliability is defined as the probability that G is no less than w and takes the form

$$\begin{aligned}R(w) &= \Pr\{F(G,w)=1\}=\Pr\{\prod_{j=1}^{n}1(G_j\geq w)=1\}\\ &=\prod_{j=1}^{n}\Pr\{1(G_j\geq w)=1\}=\prod_{j=1}^{n}\Pr\{F(G_j,w)=1\}\end{aligned} \tag{4.14}$$

This means that the system's reliability is equal to the product of the reliabilities of its components.

Each component j can be considered to be a binary element with the state variable $X_j=F(G_j,w)$ and the entire system becomes the binary series system with the state variable X and the binary structure function φ:

$$F(G,w)=X=\varphi(X_1,...,X_n)=\prod_{j=1}^{n}X_j \tag{4.15}$$

The algorithm for evaluating the system reliability can now be simplified. It consists of the following steps:

1. Obtain the *u*-functions $U_j(z)$ of all of the series components.
2. Obtain the reliability of each component j as $R_j(w)=\delta_w(U_j(z))$.

3. Calculate the entire system reliability as $R = \prod_{j=1}^{n} R_j(w) = \prod_{j=1}^{n} \delta_w(U_j(z))$.

It can easily be seen that for the discrete random demand with p.m.f. $\boldsymbol{w} = \{w_1, \ldots, w_M\}$, $\boldsymbol{q} = \{q_1, \ldots, q_M\}$ the system reliability takes the form

$$R = \sum_{m=1}^{M} q_m R(w_m) = \sum_{m=1}^{M} q_m \prod_{j=1}^{n} R_j(w_m) = \sum_{m=1}^{M} q_m \prod_{j=1}^{n} \delta_{w_m}(U_j(z)) \tag{4.16}$$

Another example is a flow transmission system without flow dispersion consisting of n independent multi-state components connected in parallel. The structure function of such a system is $G = \phi(G_1, \ldots, G_n) = \max\{G_1, \ldots, G_n\}$. If the system should meet a constant demand w, its acceptability function also takes the form $F(G, w) = 1(G \geq w)$. The probability of the system's failure is

$$\begin{aligned}
&\Pr\{F(G,w)=0\} = \Pr\{G < w\} = \Pr\{\max\{G_1,\ldots,G_n\} < w\} \\
&= \Pr\{\prod_{j=1}^{n} 1(G_j < w)\} = \prod_{j=1}^{n} \Pr\{1(G_j < w)\} = \prod_{j=1}^{n} (1 - \Pr\{1(G_j \geq w)\}) \\
&= \prod_{j=1}^{n} (1 - \Pr\{F(G_j, w) = 1\})
\end{aligned} \tag{4.17}$$

The entire system reliability can now be determined as

$$\begin{aligned}
&\Pr\{F(G,w)=1\} = 1 - \Pr\{F(G,w)=0\} \\
&= 1 - \prod_{j=1}^{n} (1 - \Pr\{F(G_j, w) = 1\})
\end{aligned} \tag{4.18}$$

This means that each component j can be considered to be a binary element with the state variable $X_j = F(G_j, w)$ and the entire system becomes the binary parallel system with the state variable X and the binary structure function σ

$$F(G,w) = X = \sigma(X_1, \ldots, X_n) = 1 - \prod_{j=1}^{n} (1 - X_j) \tag{4.19}$$

After obtaining the u-functions $U_j(z)$ of all of the parallel components one can calculate the system reliability as

$$R(w) = 1 - \prod_{j=1}^{n} (1 - R_j(w)) = 1 - \prod_{j=1}^{n} (1 - \delta_w(U_j(z))) \tag{4.20}$$

for the constant demand and as

$$R=\sum_{m=1}^{M}q_m\{1-\prod_{j=1}^{n}[1-R_j(w_m)]\}=\sum_{m=1}^{M}q_m\{1-\prod_{j=1}^{n}[1-\delta_{w_m}(U_j(z))]\} \quad (4.21)$$

for the discrete random demand.

Example 4.5

Consider the flow transmission series-parallel system presented in Figure 4.5. The system consists of three components connected in a series. The first component consists of two different elements and constitutes a subsystem without flow dispersion. The second and third components are subsystems with a flow dispersion consisting of two and three identical elements respectively. Each element j can have only two states: total failure (corresponding to a performance of zero) and operating with the nominal performance g_{j1}. The availability of element j is p_{j1}.

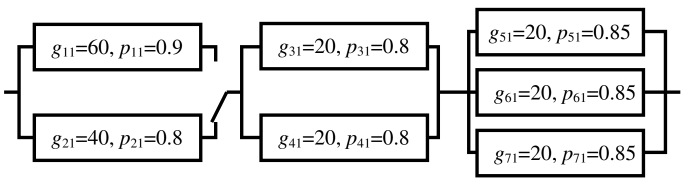


Figure 4.5. Example of a series-parallel system reducible to a binary system

The u-functions of the individual elements are:

$$u_1(z)=0.9z^{60}+0.1z^{0},\ u_2(z)=0.8z^{40}+0.2z^{0}$$

$$u_3(z)=u_4(z)=0.8z^{20}+0.2z^{0}$$

$$u_5(z)=u_6(z)=u_7(z)=0.85z^{20}+0.15z^{0}$$

The u-functions of the components are obtained using the corresponding $\underset{\phi_{\text{par}}}{\otimes}$ operators:

$$U_1(z)=(0.9z^{60}+0.1z^{0})\underset{\max}{\otimes}(0.8z^{40}+0.2z^{0})=0.9z^{60}+0.08z^{40}+0.02z^{0}$$

$$U_2(z)=(0.8z^{20}+0.2z^{0})\underset{+}{\otimes}(0.8z^{20}+0.2z^{0})=0.64z^{40}+0.32z^{20}+0.04z^{0}$$

$$U_3(z)=(0.85z^{20}+0.15z^{0})\underset{+}{\otimes}(0.85z^{20}+0.15z^{0})\underset{+}{\otimes}(0.85z^{20}+0.15z^{0})$$

$$=(0.85z^{20}+0.15z^{0})^3=0.6141z^{60}+0.3251z^{40}+0.0574z^{20}+0.0034z^{0}$$

For demand $w = 20$ we obtain

$$\delta_{20}(U_1(z)) = \delta_{20}(0.9z^{60} + 0.08z^{40} + 0.02z^{0}) = 0.98$$

$$\delta_{20}(U_2(z)) = \delta_{20}(0.64z^{40} + 0.32z^{20} + 0.04z^{0}) = 0.96$$

$$\delta_{20}(U_3(z)) = \delta_{20}(0.6141z^{60} + 0.3251z^{40} + 0.0574z^{20} + 0.0034z^{0}) = 0.9966$$

The entire system availability is

$$A(20) = \delta_{20}(U_1(z))\delta_{20}(U_2(z))\delta_{20}(U_3(z)) = 0.98 \times 0.96 \times 0.9966 = 0.9376$$

For demand $w = 40$ we obtain

$$\delta_{40}(U_1(z)) = \delta_{40}(0.9z^{60} + 0.08z^{20} + 0.02z^{0}) = 0.9$$

$$\delta_{40}(U_2(z)) = \delta_{40}(0.64z^{40} + 0.32z^{20} + 0.04z^{0}) = 0.64$$

$$\delta_{40}(U_3(z)) = \delta_{40}(0.6141z^{60} + 0.3251z^{40} + 0.0574z^{20} + 0.0034z^{0}) = 0.9392$$

The entire system availability is

$$A(40) = \delta_{40}(U_1(z))\delta_{40}(U_2(z))\delta_{40}(U_3(z)) = 0.9 \times 0.64 \times 0.9392 = 0.541$$

If the demand is the random variable W with p.m.f. $\boldsymbol{w} = \{20, 40\}$, $\boldsymbol{q} = \{0.7, 0.3\}$, the system availability is

$$A = 0.7A(20) + 0.3A(40) = 0.7 \times 0.9376 + 0.3 \times 0.541 = 0.8186$$

It should be noted that only the reliability (availability) of series-parallel systems can be evaluated using the MSS reduction to the binary system. The evaluation of the mean performance and the performance deviation measures still require the derivation of the u-function of the entire system.

4.2 Controllable Series-Parallel Multi-state Systems

Some series-parallel systems can change their configuration following certain rules aimed at achieving maximal system efficiency. Such systems belong to the class of controllable systems. If the rules that determine the system configuration depend on external factors then the system reliability measures should be determined for each possible configuration. If the rules are based on the states of the system elements then they can be incorporated into algorithms evaluating the system reliability measures. The application of simple operators $\otimes_{\phi_{\text{ser}}}$ and $\otimes_{\phi_{\text{par}}}$ over u-functions of the system elements is usually not enough in order to obtain the u-function of the entire system since its structure function is affected by the configuration control rules.

Examples of systems with controllable configuration are systems that contain elements with fixed resource consumption [105]. Many technical devices (processes) can work only if the available amount of some of the resources that they consume is not lower than the specified limits. If this requirement is not met, then the device (process) fails to work. An example of such a situation is a control system that stops the controlled process if a decrease in its computational resources does not allow the necessary information to be processed within the required cycle time. Another example is a metalworking machine that cannot perform its task if the flow of coolant supplied is less than required.

For a resource-consuming system that consists of several units, the amount of resource necessary to provide the normal operation of a given composition of the main producing units (controlled processes or machines) is fixed. Any deficit of the resource makes it impossible for all of the units from the composition to operate together (in parallel), because no unit can reduce the amount of resource it consumes. Therefore, any resource deficit leads to turning off some of the producing units.

Consider a system consisting of H resource-generating subsystems (RGSs) that supply different (not interchangeable) resources to the main producing system (MPS). RGSs can have an arbitrary series-parallel configuration, while the MPS consists of n elements connected in parallel (Figure 4.6). Each element of the MPS is an element with total failure and can perform in its working state only by consuming a fixed amount of resources. The MPS is the flow transmission system with flow dispersion. If, following failures, in any RGS there are not enough resources to allow all of the available producing elements to work, some of these elements should be turned off. We assume that the choice of the working MPS elements is made by a control system in such a way as to maximize the total performance rate of the MPS under the given resource constraints.

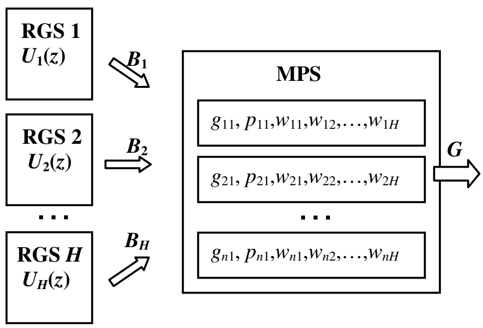


Figure 4.6. Structure of controllable system with fixed resource consumption

Assume that the RGS h produces a random amount B_h of the resource. The p.m.f. of B_h is represented by the u-function $U_h(z)=\sum_{i=0}^{k_h-1} q_{hi} z^{\beta_{hi}}$, where β_{hi} is the performance rate of RGS h in state i and q_{hi} = Pr{$B_h = \beta_{hi}$}. Each element j of the MPS has a nominal performance g_{j1} and availability p_{j1} and requires the amount w_{jh} of each resource h ($1 \le h \le H$) for normal functioning (if different MPS elements consume different subsets of the set of H resources, this can be represented by assigning zero to w_{jh} for any resource h that is not required by element j). The p.m.f. of the random performance G_j of element j is represented by the u-function $u_j(z) = p_{j1} z^{g_{j1}} + (1-p_{j1}) z^0$. The distribution of the available performance of the entire MPS G_{MPS} can be obtained as $U_{\text{MPS}}(z) = \otimes_{+}(u_1(z),...,u_n(z))$. Observe that the performance G_{MPS} represents the potential performance ability of the MPS. It does not always coincide with the output performance of the entire system G. $U_{\text{MPS}}(z)$ represents the conditional distribution of G corresponding to a situation when the resources are supplied without limitations. In order to take into account the possible deficiency of the resources supplied we have to incorporate the MPS control rule (the rule of turning the MPS elements off and on) into the derivation of the system u-function $U(z)$ representing the p.m.f. of G.

4.2.1 Systems with Identical Elements in the Main Producing System

If an MPS contains only identical elements with $g_{j1} = g$, $p_{j1} = p$ and $w_{jh} = w_h>0$ for any j and h, the number of elements that can work in parallel when the available amount of resource h is β_{hi} is $\lfloor \beta_{hi}/w_h \rfloor$, which corresponds to the total system performance $\gamma_{hi} = g \lfloor \beta_{hi}/w_h \rfloor$ (the remainder of the MPS elements must be turned off). It must be noted that γ_{hi} represents the total theoretical performance, which can be achieved by using the available resource h by an unlimited number of producing elements. In terms of the entire system output performance, the u-function of the RGS h can be obtained in the following form:

$$U_h^{\gamma}(z) = \sum_{i=0}^{k_h-1} q_{hi} z^{\gamma_{hi}} = \sum_{i=0}^{k_h-1} q_{hi} z^{g\lfloor \beta_{hi}/w_h \rfloor} \tag{4.22}$$

The RGS, which can provide the work of a minimal number of producing units, becomes the system's bottleneck. This RGS limits the total system performance. Therefore, the u-function for a system containing H different RGS in terms of system output performance can be obtained as

$$U_{\text{RGS}}(z) = \underset{\min}{\otimes}(U_1^{\gamma}(z),...,U_H^{\gamma}(z)) \tag{4.23}$$

Function $U_{\mathrm{RGS}}(z)$ represents the entire system performance distribution in the case of an unlimited number of available elements in the MPS.

The entire system performance is equal to the minimum of the total theoretical performance, which can be achieved using available resources and the total performance of the available MPS elements. To obtain the *u*-function $U(z)$ of the entire system representing the p.m.f. of its performance *G,* the same operator $\underset{\min}{\otimes}$ should be applied over the *u*-functions $U_{\mathrm{RGS}}(z)$ and $U_{\mathrm{MPS}}(z)$:

$$U(z) = U_{\mathrm{RGS}}(z) \underset{\min}{\otimes} U_{\mathrm{MPS}}(z) = \underset{\min}{\otimes}(U_1^{\gamma}(z),...,U_H^{\gamma}(z),U_{\mathrm{MPS}}(z))$$
$$= \underset{\min}{\otimes}(U_1^{\gamma}(z),...,U_H^{\gamma}(z),\underset{+}{\otimes}(u_1(z),...,u_n(z))) \tag{4.24}$$

4.2.2 System with Different Elements in the Main Producing System

If the MPS consists of *n* different elements, then it can be in one of 2^n possible states corresponding to the different combinations of the available elements. Let *S* be a random set of numbers of available MPS elements and S_k be a realization of *S* in state *k* $(1 \le k \le 2^n)$. The probability of state *k* can be evaluated as follows:

$$\tilde{P}_k = \prod_{j=1}^{n} p_{j1}^{1(j\in S_k)}(1-p_{j1})^{1(i\notin S_k)} \tag{4.25}$$

The maximal possible performance of the MPS and the corresponding maximal resources consumption in state *k* are

$$g_k^{\max} = \sum_{j=1}^{n} (g_{j1})^{1(j\in S_k)} \tag{4.26}$$

and

$$w_{hk}^{\max} = \sum_{j=1}^{n}(w_{jh})^{1(j\in S_k)} \; (1 \le h \le H) \tag{4.27}$$

respectively.

Let us define a *u*-function representing the distribution of the random set of available elements. For a single element *j* this *u*-function takes the form

$$u_j(z) = p_{j1}z^{\{j\}} + (1-p_{j1})z^{\varnothing} \tag{4.28}$$

Using the union procedure $\cup$ in the composition operator $\underset{\cup}{\otimes}$ we can obtain the distribution of the random set of available elements in the system consisting of several elements. For example, if the MPS consists of two elements

$$u_1(z) = p_{11}z^{\{1\}} + (1 - p_{11})z^{\varnothing},\ u_2(z) = p_{21}z^{\{2\}} + (1 - p_{21})z^{\varnothing} \tag{4.29}$$

and the distribution of the set of available elements takes the form

$$\begin{aligned}
U_{\mathrm{MPS}}(z) &= u_1(z)\underset{\cup}{\otimes}u_2(z) \\
&= [p_{11}z^{\{1\}} + (1 - p_{11})z^{\varnothing}]\underset{\cup}{\otimes}[p_{21}z^{\{2\}} + (1 - p_{21})z^{\varnothing}] \\
&= p_{11}p_{21}z^{\{1\}\cup\{2\}} + p_{11}(1 - p_{21})z^{\{1\}\cup\varnothing} + (1 - p_{11})p_{21}z^{\varnothing\cup\{2\}} \\
&+ (1 - p_{11})(1 - p_{21})z^{\varnothing\cup\varnothing} = p_{11}p_{21}z^{\{1,2\}} + p_{11}(1 - p_{21})z^{\{1\}} \\
&+ (1 - p_{11})p_{21}z^{\{2\}} + (1 - p_{11})(1 - p_{21})z^{\varnothing}
\end{aligned} \tag{4.30}$$

For an MPS consisting of n elements the u-function representing the distribution of a random set of available elements takes the form

$$U_{MPS}(z) = \underset{\cup}{\otimes}(u_1(z),\ldots,u_n(z)) = \sum_{k=1}^{2^n}\tilde{P}_k z^{S_k} \tag{4.31}$$

When each RGS h is in state i_h the amount β_{hi_h} of the resource generated by this RGS can be not enough to provide the maximal performance of the MPS at state k. In order to provide the maximum possible performance G of the MPS under the resource constraints one has to solve the following linear programming problem for any combination of states $i_1,\ldots,i_H$ of H RDSs and state k of the MPS:

$$\begin{aligned}
&\mathrm{opt}(\beta_{1i_1},\beta_{2i_2},\ldots,\beta_{Hi_H},S_k) = \max \sum_{j\in S_k} g_{j1}x_j \\
&\text{subject to} \\
&\sum_{j\in S_k} w_{jh}x_j \le \beta_{hi_h},\quad \text{for } 1\le h\le H \\
&x_j \in \{0,1\}
\end{aligned} \tag{4.32}$$

where $x_j = 1$ if the available element j is turned on (works providing performance rate g_{j1} and consuming w_{jh} of each resource $1 \le h \le H$) and $x_j = 0$ if the element is turned off.

The performance distribution of the entire system can be obtained by considering all of the possible combinations of the available resources generated by the RGS and the states of the MPS. For each combination, a solution of the above formulated optimization problem defines the system's performance. The u-function representing the p.m.f. of the entire system performance G can be defined as follows:

$$U(z) = \underset{\text{opt}}{\otimes}(U_1(z),...,U_H(z),U_{\text{MPS}}(z))$$
$$= \sum_{i_1=0}^{k_1-1} \sum_{i_2=0}^{k_2-1} ... \sum_{i_H=0}^{k_H-1} \left\{ (\prod_{h=1}^{H} q_{hi_h}) \sum_{k=1}^{2^n} \tilde{P}_k z^{\text{opt}(\beta_{1i_1},\beta_{2i_2},...,\beta_{Hi_H},S_k)} \right\} \tag{4.33}$$

To obtain the system u-function, its optimal performance should be determined for each unique combination of available resources and for each unique state of the MPS. In general, the total number of linear programs to be solved in order to obtain $U(z)$ is $2^n \prod_{h=1}^{H} k_h$. In practice, the number of programs to be solved can be reduced drastically using the following rules:

1. If for the given vector $\beta_{1i_1},...,\beta_{Hi_H}$ and for each element j from the given set of MPS elements S_k there exists h for which $\beta_{hi_h} < w_{jh}$, then the system performance is equal to zero. In this case the system performance is equal to zero also for all combinations $(\beta_{1j_1},...,\beta_{Hj_H},S_m)$ such that $\beta_{1j_1} \le \beta_{1i_1},...,\beta_{Hj_H} \le \beta_{Hi_H}$ and $S_m \subseteq S_k$.

2. If element $j \in S_k$ exists for which $\beta_{hi_h} < w_{jh}$ for some h, this means that in the program (4.32) x_j must be zeroed. In this case, the integer program dimension can be reduced by removing all such elements from S_k.

3. If for the given vector $\beta_{1i_1},\beta_{2i_2}...,\beta_{Hi_H}$ and for the given set S_k the solution of the integer program (4.32) determines subset $\hat{S}_k$ of turned-on MPS elements ($j \in \hat{S}_k$ if x_j=1), then the same solution must be optimal for the MPS states characterized by any set S_m: $\hat{S}_k \subset S_m \subset S_k$. This allows one to avoid solving many integer programs by assigning the value of $\text{opt}(\beta_{1i_1},\beta_{2i_2}...,\beta_{Hi_H},S_k)$ to all the $\text{opt}(\beta_{1i_1},\beta_{2i_2}...,\beta_{Hi_H},S_m)$.

Example 4.6

Three different metalworking units (Figure 4.7) have the respective productivities and availabilities g_{11}=10, p_{11}=0.8, g_{21}=15, p_{21}=0.9 and g_{31}=20, p_{31}=0.85. The system productivity should be no less than a constant demand w. Each unit consumes two resources: electrical power and coolant.

The constant power consumption of the units is w_{11} = 5, w_{21} = 2, w_{31} = 3. The power is supplied by the system consisting of two transformers that work without load sharing (only one of the two transformers can work at any moment). The power of the transformers is 10 and 6. The availability of the transformers is 0.9 and 0.8 respectively. The constant coolant flow consumed by the units is w_{12} = 4,

$w_{22} = 5$ $w_{32} = 7$. Two identical pumps working in parallel supply the coolant (both pumps can work simultaneously). The nominal coolant flow provided by each pump is 9. The availability of each pump is 0.8.

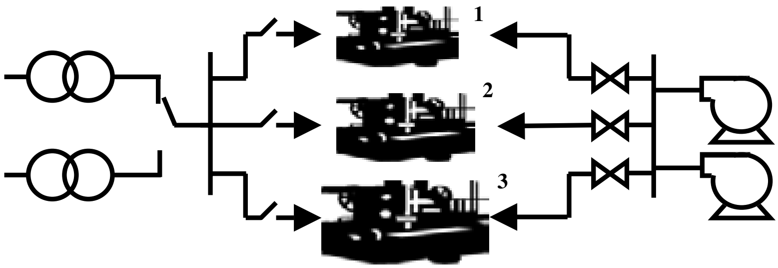


Figure 4.7. Example of controllable series-parallel system

The u-function representing the distribution of available power takes the form

$$U_1(z) = (0.9z^{10} + 0.1z^0) \underset{\max}{\otimes} (0.8z^6 + 0.2z^0) = 0.9z^{10} + 0.08z^6 + 0.02z^0$$

and the u-function representing the distribution of the available coolant takes the form

$$U_2(z) = (0.8z^9 + 0.2z^0) \underset{+}{\otimes} (0.8z^9 + 0.2z^0) = 0.64z^{18} + 0.32z^9 + 0.04z^0$$

The u-function representing the distribution of the set of available metalworking units takes the form

$$U_{\text{MPS}}(z) = (0.8z^{\{1\}} + 0.02z^{\varnothing}) \underset{\cup}{\otimes} (0.9z^{\{2\}} + 0.1z^{\varnothing}) \underset{\cup}{\otimes} (0.85z^{\{3\}}$$

$$+ 0.15z^{\varnothing}) = 0.003z^{\varnothing} + 0.012z^{\{1\}} + 0.027z^{\{2\}} + 0.017z^{\{3\}}$$
$$+ 0.108z^{\{1,2\}} + 0.068z^{\{1,3\}} + 0.153z^{\{2,3\}} + 0.612z^{\{1,2,3\}}$$

The values of the opt function obtained for all of the possible combinations of available metalworking units (realizations S_k of the random set S) and available resources (realizations of B_1 and B_2) are presented in Table 4.7. The table contains the maximal possible productivity of the MPS $g_k^{\max}$ and the corresponding maximal required resources $w_{hk}^{\max}$ for any set S_k that is not empty. It also contains the optimal system productivity G (values of the opt function) and the corresponding sets of turned-on elements $\hat{S}_k$.

Table 4.7. Solutions of a linear program for a system with different elements in an MPS

				B_1=6, B_2=9		B_1=6, B_2=18		B_1=10, B_2=9		B_1=10, B_2=18	
S_k	g_k^{max}	w_{1k}^{max}	w_{2k}^{max}	$\hat{S}_k$	G	$\hat{S}_k$	G	$\hat{S}_k$	G	$\hat{S}_k$	G
{1}	10	5	4	{1}	10	{1}	10	{1}	10	{1}	10
{2}	15	2	5	{2}	15	{2}	15	{2}	15	{2}	15
{3}	20	3	7	**{3}**	**20**	{3}	20	{3}	20	{3}	20
{1,2}	25	7	9	{2}	15	{2}	15	**{1,2}**	**25**	{1,2}	25
{1,3}	30	8	11	**{3}**	**20**	{3}	20	{3}	20	{1,3}	30
{2,3}	35	5	12	**{3}**	**20**	**{2,3}**	**35**	{3}	20	{2,3}	35
{1,2,3}	45	10	16	*{3}*	*20*	*{2,3}*	*35*	*{1,2}*	*25*	{1,2,3}	45

It is obvious that if $S_k = \varnothing$ then the entire system performance is equal to zero. If $B_1 = 0$ or $B_2 = 0$ then the entire system performance is also equal to zero according to rule 1 (these solutions are not included in the table). Note that the solutions marked in bold are obtained without solving the linear program (they were obtained using rule 3 from the solutions marked in italic).

The u-function of the entire system obtained in accordance with Table 4.7 after collecting the like terms takes the following form:

$$U(z) = \underset{\text{opt}}{\otimes}[(0.9z^{10} + 0.08z^{6} + 0.02z^{0}), (0.64z^{18} + 0.32z^{9} + 0.04z^{0}),$$
$$(0.003z^{\varnothing} + 0.012z^{\{1\}} + 0.027z^{\{2\}} + 0.017z^{\{3\}} + 0.108z^{\{1,2\}} + 0.068z^{\{1,3\}}$$
$$+ 0.153z^{\{2,3\}} + 0.612z^{\{1,2,3\}})] = 0.062z^{0} + 0.0113z^{10} + 0.0337z^{15}$$
$$+ 0.104z^{20} + 0.270z^{25} + 0.039z^{30} + 0.127z^{35} + 0.353z^{45}$$

Having the system u-function we can easily obtain its mean performance

$$\varepsilon = U'(1) = 0.0113 \times 10 + 0.0337 \times 15 + 0.104 \times 20 + 0.270 \times 25$$
$$+ 0.039 \times 30 + 0.127 \times 35 + 0.353 \times 45 = 30.94$$

and availability. For example, for system demand w = 20:

$$A(20) = \delta_{20}(U(z)) = \delta_{20}(0.062z^{0} + 0.0113z^{10} + 0.0337z^{15}$$
$$+ 0.104z^{20} + 0.270z^{25} + 0.039z^{30} + 0.127z^{35} + 0.353z^{45})$$
$$= 0.104 + 0.270 + 0.039 + 0.127 + 0.353 = 0.893$$

The system availability as a function of demand is presented in Figure 4.8.

Now consider the same system in which the MPS consists of three identical units with parameters g_{j1} = 20, p_{j1} = 0.85, w_{j1} = 3 and w_{j2} = 7. The reliability measures of such a system can be obtained in an easier manner by using the algorithm presented in Section 4.2.1.

From the u-functions of the RGSs $U_1(z)$ and $U_2(z)$ by applying Equation (4.22) we obtain for the first RGS:

$$U_1^{\gamma}(z) = 0.9z^{20\lfloor 10/3 \rfloor} + 0.08z^{20\lfloor 6/3 \rfloor} + 0.02z^{20\lfloor 0/5 \rfloor}$$
$$= 0.9z^{60} + 0.08z^{40} + 0.02z^{0}$$

and for the second RGS:

$$U_2^{\gamma}(z) = 0.64z^{20\lfloor 18/7 \rfloor} + 0.32z^{20\lfloor 9/7 \rfloor} + 0.04z^{20\lfloor 0/7 \rfloor}$$
$$= 0.64z^{40} + 0.32z^{20} + 0.04z^{0}$$

The u-function of the MPS is

$$U_{\text{MPS}}(z) = (0.85z^{20} + 0.15z^{0}) \underset{+}{\otimes} (0.85z^{20}$$

$$+ 0.15z^{0}) \underset{+}{\otimes} (0.85z^{20} + 0.15z^{0}) = (0.85z^{20} + 0.15z^{0})^{3}$$

$$= 0.6141z^{60} + 0.3251z^{40} + 0.0574z^{20} + 0.0034z^{0}$$

The u-function of the entire system after collecting the like terms takes the form:

$$U(z) = U_1^{\gamma}(z) \underset{\min}{\otimes} U_2^{\gamma}(z) \underset{\min}{\otimes} U_{\text{MPS}}(z)$$

$$= (0.9z^{60} + 0.08z^{40} + 0.02z^{0}) \underset{\min}{\otimes} (0.64z^{40} + 0.32z^{20} + 0.04z^{0})$$

$$\underset{\min}{\otimes} (0.6141z^{60} + 0.3251z^{40} + 0.0574z^{20} + 0.0034z^{0})$$

$$= 0.589z^{40} + 0.3486z^{20} + 0.0624z^{0}$$

From the system u-function we can obtain its mean performance

$$\varepsilon = U'(1) = 0.589 \times 40 + 0.3486 \times 20 = 30.532$$

and its availability. For example, for $w = 20$:

$$A(20) = \delta_{20}(U(z)) = \delta_{20}(0.589z^{40} + 0.3486z^{20} + 0.0624z^{0})$$
$$= 0.589 + 0.3486 = 0.9375$$

The system availability as a function of demand is presented in Figure 4.8.

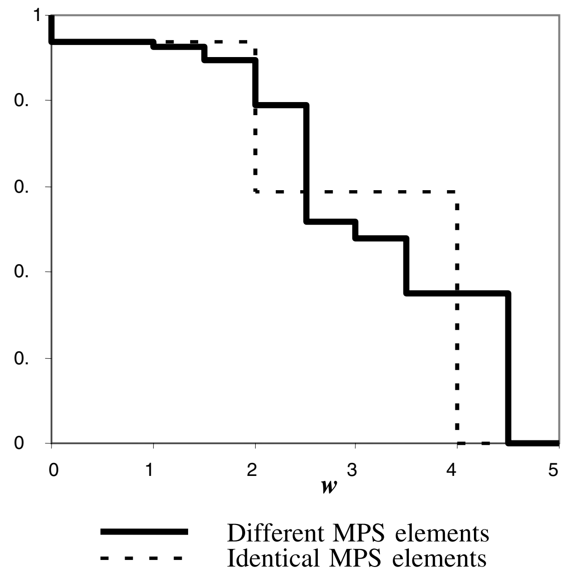


Figure 4.8. Availability of controllable series-parallel system as a function of demand

Since the system consists of three subsystems connected in a series and can be considered as a flow transmission system, its availability for any given demand can be obtained without derivation of the entire system u-function $U(z)$ using the simplified technique described in Section 4.1.4. The availability of the system for $w = 20$ is calculated using this simplified technique in Example 4.5.

The RGS-MPS model considered can easily be expanded to systems with a multilevel hierarchy. When analyzing multilevel systems, the entire RGS-MPS system (with its performance distribution represented by its u-function) may be considered in its turn as one of the RGSs for a higher level MPS (Figure 4.9).

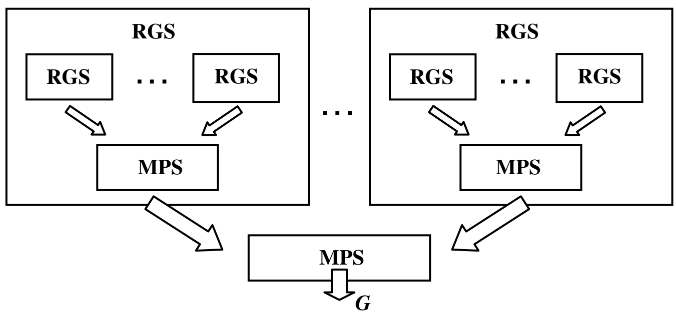


Figure 4.9. RGS-MPS system with hierarchical structure

4.3 Multi-state Systems with Dependent Elements

One of the main assumptions made in the previous sections is statistical independency of system elements. This assumption is not true for many technical systems. Fortunately, the UGF approach can be extended to cases when the performance distributions of some system elements are influenced by the states of other elements or subsystems [106].

4.3.1 *u*-functions of Dependent Elements

Consider a subsystem consisting of a pair of multi-state elements i and j in which the performance distribution of element j (p.m.f. of random performance G_j) depends on the state of element i. Since the states of the elements are distinguished by their corresponding performance rates, we can assume that the performance distribution of element j is determined by the performance rate of element i. Let $\boldsymbol{g}_i=\{g_{ih}: 1\le h\le k_i\}$ be the set of possible performance rates of element i. In general, this set can be separated into M mutually disjoint subsets $\boldsymbol{g}_i^m$ $(1\le m\le M)$:

$$\bigcup_{m=1}^{M}\boldsymbol{g}_i^m=\boldsymbol{g}_i,\quad \boldsymbol{g}_i^m\bigcap\boldsymbol{g}_i^l=\varnothing,\ \text{ if } m\neq l \tag{4.34}$$

such that when element i has the performance rate $g_{ih}\in\boldsymbol{g}_i^m$ the PD of element j is defined by the ordered sets $\boldsymbol{g}_{j|m}=\{g_{jc|m}, 1\le c\le C_{j|m}\}$ and $\boldsymbol{q}_{j|m}=\{q_{jc|m}, 1\le c\le C_{j|m}\}$, where

$$q_{jc|m}=\Pr\{G_j=g_{jc|m}\mid G_i=g_{ih}\in\boldsymbol{g}_i^m\} \tag{4.35}$$

If each performance rate of element i corresponds to a different PD of element j, then we have $M=k_i$ and $\boldsymbol{g}_i^m=\{g_{im}\}$.

We can define the set of all of the possible values of the performance rate of element j as $\boldsymbol{g}_j=\bigcup_{m=1}^{M}\boldsymbol{g}_{j|m}$ and redefine the conditional PD of element j when element i has the performance rate $g_{ih}\in\boldsymbol{g}_i^m$ using two ordered sets $\boldsymbol{g}_j=\{g_{jc}, 1\le c\le C_j\}$ and $\boldsymbol{p}_{j|m}=\{p_{jc|m}, 1\le c\le C_j\}$, where:

$$p_{jc|m}=\begin{cases}0, & g_{jc}\notin\boldsymbol{g}_{j|m}\\ q_{jc|m}, & g_{jc}\in\boldsymbol{g}_{j|m}\end{cases} \tag{4.36}$$

According to this definition

$$p_{jc|m} = \Pr\{G_j = g_{jc} \mid G_i = g_{ih} \in \boldsymbol{g}_i^m\} \tag{4.37}$$

for any possible realization of G_j and any possible realization of $G_i \in \boldsymbol{g}_i^m$.

Since the sets $\boldsymbol{g}_i^m$ ($1 \le m \le M$) are mutually disjoint, the unconditional probability that $G_j = g_{jc}$ can be obtained as

$$\begin{aligned} p_{jc} &= \sum_{m=1}^{M} \Pr\{G_j = g_{jc} \mid G_i \in \boldsymbol{g}_i^m\} \Pr\{G_i \in \boldsymbol{g}_i^m\} \\ &= \sum_{m=1}^{M} p_{jc|m} \sum_{h=1}^{k_i} p_{ih} 1(p_{ih} \in \boldsymbol{g}_i^m) \end{aligned} \tag{4.38}$$

In the case when $\boldsymbol{g}_i^m = \{g_{im}\}$

$$p_{jc} = \sum_{m=1}^{k_i} p_{im} p_{jc|m} \tag{4.39}$$

The unconditional probability of the combination $G_i = g_{ih}$, $G_j = g_{jc}$ is equal to $p_{ih} p_{jc|\mu(h)}$, where $\mu(h)$ is the number of the set to which g_{ih} belongs: $g_{ih} \in \boldsymbol{g}_i^{\mu(h)}$.

Example 4.7

Assume that element 1 has the PD $\boldsymbol{g}_1 = \{0, 1, 2, 3\}$, $\boldsymbol{p}_1 = \{0.1, 0.2, 0.4, 0.3\}$ and the PD of element 2 depends on the performance rate of element 1 such that when $G_1 \le 2$ ($G_1 \in \boldsymbol{g}_1^1 = \{0,1,2\}$) element 2 has the PD $\boldsymbol{g}_{2|1} = \{0, 10\}$, $\boldsymbol{q}_{2|1} = \{0.3, 0.7\}$ while when $G_1 > 2$ ($G_1 \in \boldsymbol{g}_1^2 = \{3\}$) element 2 has the PD $\boldsymbol{g}_{2|2} = \{0, 5\}$, $\boldsymbol{q}_{2|2} = \{0.1, 0.9\}$. The conditional PDs of element 2 can be represented by the sets $\boldsymbol{g}_2 = \{0,5,10\}$ and $\boldsymbol{p}_{2|1} = \{0.3, 0, 0.7\}$, $\boldsymbol{p}_{2|2} = \{0.1, 0.9, 0\}$.

The unconditional probabilities p_{2c} are:

$$\begin{aligned} p_{21} &= \Pr\{G_2 = 0\} = \Pr\{G_2 = 0 \mid G_1 \in \boldsymbol{g}_1^1\} \Pr\{G_1 \in \boldsymbol{g}_1^1\} \\ &+ \Pr\{G_2 = 0 \mid G_1 \in \boldsymbol{g}_1^2\} \Pr\{G_1 \in \boldsymbol{g}_1^2\} = p_{21|1}(p_{11} + p_{12} + p_{13}) + p_{21|2}(p_{14}) \\ &= 0.3(0.1+0.2+0.4)+0.1(0.3) = 0.24 \\ p_{22} &= \Pr\{G_2 = 5\} = \Pr\{G_2 = 5 \mid G_1 \in \boldsymbol{g}_1^1\} \Pr\{G_1 \in \boldsymbol{g}_1^1\} \\ &+ \Pr\{G_2 = 5 \mid G_1 \in \boldsymbol{g}_1^2\} \Pr\{G_1 \in \boldsymbol{g}_1^2\} = p_{22|1}(p_{11} + p_{12} + p_{13}) + p_{22|2}(p_{14}) \\ &= 0(0.1+0.2+0.4)+0.9(0.3) = 0.27 \\ p_{23} &= \Pr\{G_2 = 10\} = \Pr\{G_2 = 10 \mid G_1 \in \boldsymbol{g}_1^1\} \Pr\{G_1 \in \boldsymbol{g}_1^1\} \\ &+ \Pr\{G_2 = 10 \mid G_1 \in \boldsymbol{g}_1^2\} \Pr\{G_1 \in \boldsymbol{g}_1^2\} = p_{23|1}(p_{11} + p_{12} + p_{13}) + p_{23|2}(p_{14}) \\ &= 0.7(0.1+0.2+0.4)+0(0.3)=0.49 \end{aligned}$$

The probability of the combination $G_1 = 2$, $G_2 = 10$ is

$$p_{13}p_{23|\mu(3)} = p_{13}p_{23|1} = 0.4\times 0.7 = 0.28.$$

The probability of the combination $G_1 = 3$, $G_2 = 10$ is

$$p_{14}p_{23|\mu(4)} = p_{14}p_{23|2} = 0.3\times 0 = 0.$$

The sets $\boldsymbol{g}_j$ and $\boldsymbol{p}_{j|m}$ $1\le m\le M$ define the conditional PDs of element j. They can be represented in the form of the u-function with vector coefficients:

$$\bar{u}_j(z) = \sum_{c=1}^{C_j} \bar{\boldsymbol{p}}_{jc} z^{g_{jc}} \tag{4.40}$$

where

$$\bar{\boldsymbol{p}}_{jc} = (p_{jc|1}, p_{jc|2}, \ldots, p_{jc|M}) \tag{4.41}$$

Since each combination of the performance rates of the two elements $G_i = g_{ih}$, $G_j = g_{jc}$ corresponds to the subsystem performance rate $\phi(g_{ih}, g_{jc})$ and the probability of the combination is $p_{ih}p_{jc|\mu(h)}$, we can obtain the u-function of the subsystem as follows:

$$\begin{aligned} u_i(z) \underset{\phi}{\overset{\Rightarrow}{\otimes}} \bar{u}_j(z) &= \sum_{h=1}^{k_i} p_{ih} z^{g_{ih}} \underset{\phi}{\overset{\Rightarrow}{\otimes}} \sum_{c=1}^{C_j} \bar{\boldsymbol{p}}_{jc} z^{g_{jc}} \\ &= \sum_{h=1}^{k_i} p_{ih} \sum_{c=1}^{C_j} p_{jc|\mu(h)} z^{\phi(g_{ih}, g_{jc})} \end{aligned} \tag{4.42}$$

The function $\phi(g_{ih}, g_{jc})$ should be substituted by $\phi_{\text{par}}(g_{ih}, g_{jc})$ or $\phi_{\text{ser}}(g_{ih}, g_{jc})$ in accordance with the type of connection between the elements. If the elements are not connected in the reliability block diagram sense (the performance of element i does not directly affect the performance of the subsystem, but affects the PD of element j) the last equation takes the form

$$u_i(z) \overset{\Rightarrow}{\otimes} \bar{u}_j(z) = \sum_{h=1}^{k_i} p_{ih} z^{g_{ih}} \overset{\Rightarrow}{\otimes} \sum_{c=1}^{C_j} \bar{\boldsymbol{p}}_{jc} z^{g_{jc}} = \sum_{h=1}^{k_i} p_{ih} \sum_{c=1}^{C_j} p_{jc|\mu(h)} z^{g_{jc}} \tag{4.43}$$

Example 4.8

Consider two dependent elements from Example 4.7 and assume that these elements are connected in parallel in a flow transmission system (with flow dispersion). Having the sets $\boldsymbol{g}_1=\{0, 1, 2, 3\}$, $\boldsymbol{p}_1=\{0.1, 0.2, 0.4, 0.3\}$ and $\boldsymbol{g}_2=\{0,5,10\}$, $\boldsymbol{p}_{2|1}=\{0.3, 0, 0.7\}$, $\boldsymbol{p}_{2|2}=\{0.1, 0.9, 0\}$ we define the u-functions of the elements as

$$u_1(z) = 0.1z^0 + 0.2z^1 + 0.4z^2 + 0.3z^3$$
$$\bar{u}_2(z) = (0.3,0.1)z^0 + (0,0.9)z^5 + (0.7,0)z^{10}$$

The u-function representing the cumulative performance of the two elements is obtained according to (4.42):

$$u_1(z) \underset{+}{\overset{\Rightarrow}{\otimes}} \bar{u}_2(z) = \sum_{h=1}^{4} p_{1h} \sum_{c=1}^{3} p_{2c|\mu(h)} z^{g_{1h}+g_{2c}}$$
$$= 0.1(0.3z^{0+0} + 0z^{0+5} + 0.7z^{0+10}) + 0.2(0.3z^{1+0}$$
$$+ 0z^{1+5} + 0.7z^{1+10}) + 0.4(0.3z^{2+0} + 0z^{2+5} + 0.7z^{2+10})$$
$$+ 0.3(0.1z^{3+0} + 0.9z^{3+5} + 0z^{3+10}) = 0.03z^0 + 0.06z^1$$
$$+ 0.12z^2 + 0.03z^3 + 0.27z^8 + 0.07z^{10} + 0.14z^{11} + 0.28z^{12}$$

Now assume that the system performance is determined only by the output performance of the second element. The PD of the second element depends on the state of the first element (as in the previous example). According to (4.43) we obtain the u-function representing the performance of the second element:

$$u_1(z) \underset{+}{\overset{\Rightarrow}{\otimes}} \bar{u}_2(z) = \sum_{h=1}^{4} p_{1h} \sum_{c=1}^{3} p_{2c|\mu(h)} z^{g_{2c}}$$
$$= 0.1(0.3z^0 + 0z^5 + 0.7z^{10}) + 0.2(0.3z^0 + 0z^5 + 0.7z^{10})$$
$$+ 0.4(0.3z^0 + 0z^5 + 0.7z^{10}) + 0.3(0.1z^0 + 0.9z^5 + 0z^{10})$$
$$= 0.24z^0 + 0.27z^5 + 0.49z^{10}$$

4.3.2 *u*-functions of a Group of Dependent Elements

Consider a pair of elements e and j. Assume that both of these elements depend on the same element i and are mutually independent given the element i is in a certain state h. This means that the elements e and j are conditionally independent given the state of element i. For any state h of the element i ($g_{ih} \in \boldsymbol{g}_i^{\mu(h)}$) the PDs of the elements e and j are defined by the pairs of vectors $\boldsymbol{g}_e$, $\boldsymbol{p}_{e|\mu(h)}$ and $\boldsymbol{g}_j$, $\boldsymbol{p}_{j|\mu(h)}$, where $\boldsymbol{p}_{e|\mu(h)} = \{p_{ec|\mu(h)} \mid 1 \le c \le C_e\}$. Having these distributions, one can obtain the u-function corresponding to the conditional PD of the subsystem consisting of elements e and j by applying the operators

$$\sum_{c=1}^{C_e} p_{ec|\mu(h)} z^{g_{ec}} \underset{\phi}{\otimes} \sum_{s=1}^{C_j} p_{js|\mu(h)} z^{g_{js}}$$
$$= \sum_{c=1}^{C_e} \sum_{s=1}^{C_j} p_{ec|\mu(h)} p_{js|\mu(h)} z^{\phi(g_{ec}, g_{js})} \tag{4.44}$$

where the function $\phi(g_{ec}, g_{js})$ is substituted by $\phi_{\text{par}}(g_{ec}, g_{js})$ or $\phi_{\text{ser}}(g_{ec}, g_{js})$ in accordance with the type of connection between the elements. Applying the Equation (4.44) for any subset $\boldsymbol{g}_i^m$ ($1 \le m \le M$) we can obtain the u-function representing all of the subsystem's conditional PDs consisting of elements e and j using the following operator over the u-functions $\overline{u}_e(z)$ and $\overline{u}_j(z)$:

$$\overline{u}_e(z) \overset{\circ}{\underset{\phi}{\otimes}} \overline{u}_j(z) = \sum_{c=1}^{C_e} \overline{\boldsymbol{p}}_{ec} z^{g_{ec}} \overset{\circ}{\underset{\phi}{\otimes}} \sum_{s=1}^{C_j} \overline{\boldsymbol{p}}_{js} z^{g_{js}}$$
$$= \sum_{c=1}^{C_e} \sum_{s=1}^{C_j} \overline{\boldsymbol{p}}_{ec} \circ \overline{\boldsymbol{p}}_{js} z^{\phi(g_{ec}, g_{js})} \tag{4.45}$$

where

$$\overline{\boldsymbol{p}}_{ec} \circ \overline{\boldsymbol{p}}_{js} = (p_{ec|1} p_{js|1}, p_{ec|2} p_{js|2}, \ldots, \; p_{ec|M} p_{js|M}) \tag{4.46}$$

Example 4.9

A flow transmission system (with flow dispersion) consists of three elements connected in parallel. Assume that element 1 has the PD $\boldsymbol{g}_1 = \{0, 1, 3\}$, $\boldsymbol{p}_1 = \{0.2, 0.5, 0.3\}$.

The PD of element 2 depends on the performance rate of element 1 such that when $G_1 \le 1$ ($G_1 \in \{0, 1\}$) element 2 has the PD $\boldsymbol{g}_2 = \{0,3\}$, $\boldsymbol{q}_2 = \{0.3, 0.7\}$ while when $G_1 > 1$ ($G_1 \in \{3\}$) element 2 has the PD $\boldsymbol{g}_2 = \{0, 5\}$, $\boldsymbol{q}_2 = \{0.1, 0.9\}$.

The PD of element 3 depends on the performance rate of element 1 such that when $G_1 = 0$ ($G_1 \in \{0\}$) element 3 has the PD $\boldsymbol{g}_3 = \{0, 2\}$, $\boldsymbol{q}_3 = \{0.8, 0.2\}$ while when $G_1 > 0$ ($G_1 \in \{1, 3\}$) element 3 has the PD $\boldsymbol{g}_3 = \{0, 3\}$, $\boldsymbol{q}_3 = \{0.2, 0.8\}$.

The set $\boldsymbol{g}_1$ should be divided into three subsets corresponding to different PDs of dependent elements such that

for $G_1 \in \boldsymbol{g}_1^1 = \{0\}$ $\boldsymbol{g}_{2|1} = \{0, 3\}$, $\boldsymbol{q}_{2|1} = \{0.3, 0.7\}$ and $\boldsymbol{g}_{3|1} = \{0, 2\}$, $\boldsymbol{q}_{3|1} = \{0.8, 0.2\}$

for $G_1 \in \boldsymbol{g}_1^2 = \{1\}$ $\boldsymbol{g}_{2|2} = \{0, 3\}$, $\boldsymbol{q}_{2|2} = \{0.3, 0.7\}$ and $\boldsymbol{g}_{3|2} = \{0, 3\}$, $\boldsymbol{q}_{3|2} = \{0.2, 0.8\}$

for $G_1 \in \boldsymbol{g}_1^3 = \{3\}$ $\boldsymbol{g}_{2|3} = \{0, 5\}$, $\boldsymbol{q}_{2|3} = \{0.1, 0.9\}$ and $\boldsymbol{g}_{3|3} = \{0, 3\}$, $\boldsymbol{q}_{3|3} = \{0.2, 0.8\}$

The conditional PDs of elements 2 and 3 can be represented in the following form:

$$\boldsymbol{g}_2 = \{0,3,5\}, \boldsymbol{p}_{2|1} = \boldsymbol{p}_{2|2} = \{0.3, 0.7, 0\}, \; \boldsymbol{p}_{2|3} = \{0.1, 0, 0.9\}$$

$$\boldsymbol{g}_3 = \{0,2,3\}, \boldsymbol{p}_{3|1} = \{0.8, 0.2, 0\}, \boldsymbol{p}_{3|2} = \boldsymbol{p}_{3|3} = \{0.2, 0, 0.8\}$$

The u-functions $\bar{u}_1(z)$ and $\bar{u}_2(z)$ take the form

$$\bar{u}_2(z) = (0.3,0.3,0.1)z^0 + (0.7,0.7,0)z^3 + (0,0,0.9)z^5$$

$$\bar{u}_3(z) = (0.8,0.2,0.2)z^0 + (0.2,0,0)z^2 + (0,0.8,0.8)z^3$$

The u-function of the subsystem consisting of elements 2 and 3 according to (4.45) is

$$\begin{aligned}
\bar{U}_4(z) &= \bar{u}_2(z) \underset{+}{\overset{\circ}{\otimes}} \bar{u}_3(z) = [(0.3,0.3,0.1)z^0 + (0.7,0.7,0)z^3 \\
&+ (0,0,0.9)z^5] \underset{+}{\overset{\circ}{\otimes}} [(0.8,0.2,0.2)z^0 + (0.2,0,0)z^2 + (0,0.8,0.8)z^3] \\
&= (0.24,0.06,0.02)z^0 + (0.06,0,0)z^2 + (0,0.24,0.08)z^3 \\
&+ (0.56,0.14,0)z^3 + (0.14,0,0)z^5 + (0,0.56,0)z^6 \\
&+ (0,0,0.18)z^5 + (0,0,0)z^7 + (0,0,0.72)z^8 \\
&= (0.24,0.06,0.02)z^0 + (0.06,0,0)z^2 + (0.56,0.38,0.08)z^3 \\
&+ (0.14,0,0.18)z^5 + (0,0.56,0)z^6 + (0,0,0.72)z^8
\end{aligned}$$

Now we can replace elements 2 and 3 by a single equivalent element with the u-function $\bar{U}_4(z)$ and consider the system as consisting of two elements with u-functions $u_1(z)$ and $\bar{U}_4(z)$. The u-function of the entire system according to (4.42) is:

$$\begin{aligned}
U(z) &= u_1(z) \underset{+}{\overset{\Rightarrow}{\otimes}} \bar{U}_4(z) = \sum_{h=1}^{3} p_{1h} \sum_{c=1}^{6} p_{4c|\mu(h)} z^{g_{1h}+g_{4c}} \\
&= 0.2(0.24z^{0+0} + 0.06z^{0+2} + 0.56z^{0+3} + 0.14z^{0+5}) + 0.5(0.06z^{1+0} \\
&+ 0.38z^{1+3} + 0.56z^{1+6}) + 0.3(0.02z^{3+0} + 0.08z^{3+3} + 0.18z^{3+5} \\
&+ 0.72z^{3+8}) = 0.048z^0 + 0.03z^1 + 0.012z^2 + 0.118z^3 + 0.19z^4 \\
&+ 0.028z^5 + 0.024z^6 + 0.28z^7 + 0.054z^8 + 0.216z^{11}
\end{aligned}$$

Note that the conditional independence of two elements e and j does not imply their unconditional independence. The two elements are conditionally independent if for any states c, s and h

$$\begin{aligned}
&\Pr\{G_e = g_{ec}, G_j = g_{js} \mid G_i = g_{ih}\} \\
&= \Pr\{G_e = g_{ec} \mid G_i = g_{ih}\}\Pr\{G_j = g_{js} \mid G_i = g_{ih}\}
\end{aligned}$$

The condition of independence of elements e and j

$$\Pr\{G_e = g_{ec}, G_j = g_{js}\} = \Pr\{G_e = g_{ec}\}\Pr\{G_j = g_{js}\}$$

does not follow from the previous equation. In our example we have

$$\Pr\{G_2 = 3\} = p_{22|1}p_{11} + p_{22|2}p_{12} + p_{22|3}p_{13}$$

$$= 0.7\times0.2 + 0.7\times0.5 + 0\times0.3 = 0.49$$

$$\Pr\{G_3 = 3\} = p_{33|1}p_{11} + p_{33|2}p_{12} + p_{33|3}p_{13}$$

$$= 0\times0.2 + 0.8\times0.5 + 0.8\times0.3 = 0.64$$

Hence

$$\Pr\{G_2 = 3\}\Pr\{G_3 = 3\} = 0.49\times0.64 = 0.3136$$

while

$$\Pr\{G_2 = 3, G_3 = 3\} = p_{22|1}p_{33|1}p_{11} + p_{22|2}p_{33|2}p_{12} + p_{22|3}p_{33|3}p_{13}$$

$$= 0.7\times0\times0.2 + 0.7\times0.8\times0.5 + 0\times0.8\times0.3 = 0.28$$

4.3.3 *u*-functions of Multi-state Systems with Dependent Elements

Consecutively applying the operators $\otimes_{\varphi}$, $\overset{\Rightarrow}{\otimes}_{\varphi}$ and $\overset{\Rightarrow}{\otimes}$ and replacing pairs of elements by auxiliary equivalent elements, one can obtain the u-function representing the performance distribution of the entire system. The following recursive algorithm obtains the system u-function:

1. Define the u-functions for all of the independent elements.
2. Define the u-functions for all of the dependent elements in the form (4.40) and (4.41).
3. If the system contains a pair of mutually independent elements connected in parallel or in a series, replace this pair with an equivalent element with the u-function obtained by $\otimes_{\varphi_{\text{par}}}$ or $\otimes_{\varphi_{\text{ser}}}$ operator respectively (if both elements depend on the same external element, i.e. they are conditionally independent, operators $\overset{\circ}{\otimes}_{\varphi_{\text{par}}}$ or $\overset{\circ}{\otimes}_{\varphi_{\text{ser}}}$ (4.45) should be applied instead of $\otimes_{\varphi_{\text{par}}}$ or $\otimes_{\varphi_{\text{ser}}}$ respectively).
4. If the system contains a pair of dependent elements, replace this pair with an equivalent element with the u-function obtained by $\overset{\Rightarrow}{\otimes}_{\varphi_{\text{par}}}$, $\overset{\Rightarrow}{\otimes}_{\varphi_{\text{ser}}}$ or $\overset{\Rightarrow}{\otimes}$ operator.
5. If the system contains more than one element, return to step 3.

The performance distribution of the entire system is represented by the *u*-function of the remaining single equivalent element.

Example 4.10

Consider an information processing system consisting of three independent computing blocks (Figure 4.10). Each block consists of a high-priority processing unit and a low-priority processing unit that share access to a database. When the high-priority unit operates with the database, the low-priority unit waits for access. Therefore, the processing speed of the low-priority unit depends on the load (processing speed) of the high-priority unit. The processing speed distributions of the high-priority units (elements 1, 3 and 5) are presented in Table 4.8.

Table 4.8. Unconditional PDs of system elements 1, 3 and 5

g_1	50	40	30	20	10	0
p_1	0.2	0.5	0.1	0.1	0.05	0.05
g_3	60	20	0			
p_3	0.2	0.7	0.1			
g_5	100	80	0			
p_5	0.7	0.2	0.1			

The conditional distributions of the processing speed of the low-priority units (elements 2, 4 and 6) are presented in Table 4.9. The high- and low-priority units share their work in proportion to their processing speed.

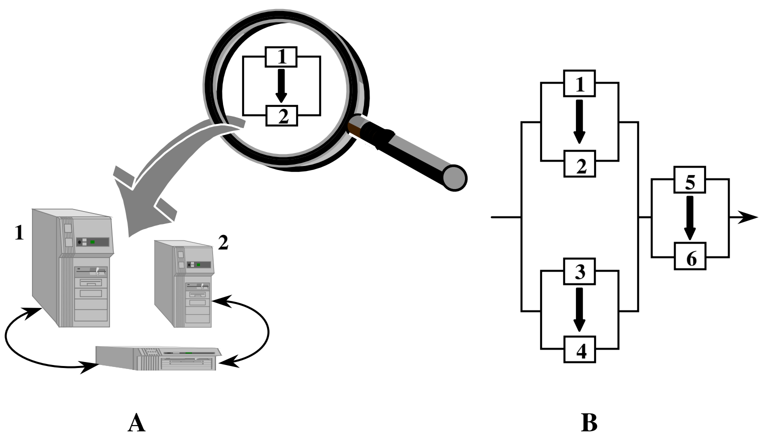


Figure 4.10. Information processing system
(A: structure of computing block; B: system logic diagram)

Table 4.9. Conditional PDs of system elements 2, 4 and 6

Condition for element 2	g_2:	30	15	0
$0\le G_1<15$	$p_{2\|m}$:	0.8	0.15	0.05
$15\le G_1<35$		0.4	0.55	0.05
$35\le G_1<70$		0	0.9	0.1
Condition for element 4	g_4:	30	15	0
$0\le G_3<15$	$p_{4\|m}$:	0.8	0.15	0.05
$15\le G_3<35$		0.6	0.35	0.05
$35\le G_3<70$		0	0.95	0.05
Condition for element 6	g_6:	50	30	0
$0\le G_5<30$	$p_{6\|m}$:	0.8	0.15	0.05
$30\le G_5<90$		0.5	0.4	0.1
$90\le G_5<150$		0.3	0.6	0.1

The first two computing blocks also share the computational load in proportion to their processing speed. The third block obtains the output of the first two blocks and starts processing when these blocks complete their work. The system fails if its processing speed is lower than the demand *w*.

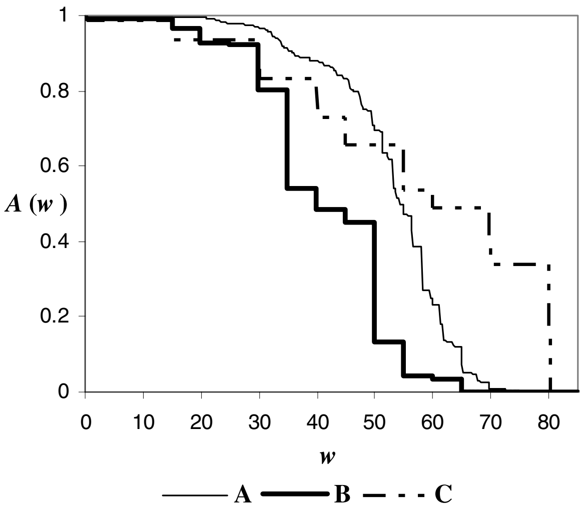


Figure 4.11. System availability as a function of demand *w*

The system belongs to the task processing type. In order to obtain the UGF representing the system PD, we first define the *u*-functions $u_1(z)$, $u_3(z)$, $u_5(z)$ from the unconditional PDs of the corresponding elements and the *u*-functions $\overline{u}_2(z), \overline{u}_4(z), \overline{u}_6(z)$ in accordance with (4.40) and (4.41):

$u_1(z) = 0.2z^{50}+0.5z^{40}+0.1z^{30}+0.1z^{20}+0.05z^{10}+0.05z^0$

$u_3(z) = 0.2z^{60}+0.7z^{20}+0.1z^0$, $u_5(z) = 0.7z^{100}+0.2z^{80}+0.1z^0$

$\bar{u}_2(z) = (0.8, 0.4, 0)z^{30}+(0.15, 0.55, 0.9)z^{15}+(0.05, 0.05, 0.1)z^0$

$\bar{u}_4(z) = (0.8, 0.6, 0)z^{30}+(0.15, 0.35, 0.95)z^{15}+(0.05, 0.05, 0.05)z^0$

$\bar{u}_6(z) = (0.8, 0.5, 0.3)z^{50}+(0.15, 0.4, 0.6)z^{30}+(0.05, 0.1, 0.1)z^0$

Then we apply the following operators producing the u-functions of the auxiliary equivalent elements:

$$U_7(z) = u_1(z) \underset{+}{\overset{\Rightarrow}{\otimes}} \bar{u}_2(z), \; U_8(z) = u_3(z) \underset{+}{\overset{\Rightarrow}{\otimes}} \bar{u}_4(z)$$

$$U_9(z) = u_5(z) \underset{+}{\overset{\Rightarrow}{\otimes}} \bar{u}_6(z)$$

The obtained u-functions represent the PD of the three computing blocks. The PD of the subsystem consisting of two parallel blocks (equivalent element 10) is represented by

$$U_{10}(z) = U_7(z) \underset{+}{\otimes} U_8(z)$$

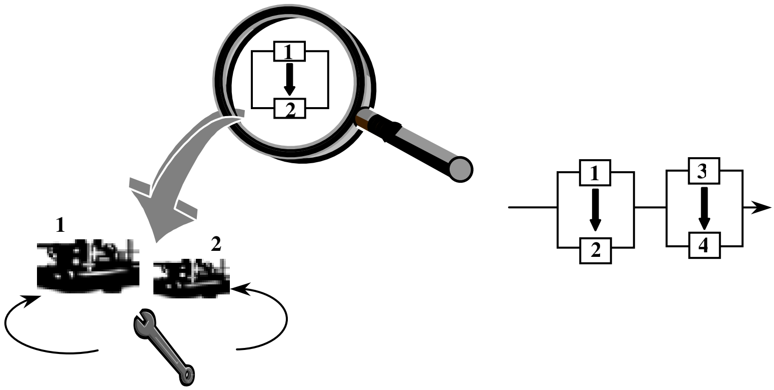
The entire system can be represented as two elements with u-functions $u_{10}(z)$ and $u_9(z)$ connected in series. Since the system belongs to the task processing type, its u-function is obtained by the operator (4.5)

$$U(z) = U_{10}(z) \underset{\times}{\otimes} U_9(z)$$

The system availability can now be obtained by applying the operator δ_w over $U(z)$: $A(w)=\delta_w(U(z))$. The system availability, as a function of demand w, is presented in Figure 4.11 (curve A).

Example 4.11

A continuous production system (Figure 4.12) consists of two consecutive production blocks. Each block consists of a main production unit and an auxiliary production unit that share some preventive maintenance resources (cleaning, lubrication, *etc.*). When the main production unit is intensively loaded, the lack of resources prevents the auxiliary unit from being intensively loaded with high availability.



A B

Figure 4.12. Continuous production system
(A: structure of production block; B: system logic diagram)

The productivity distributions of the main production units (elements 1 and 3) are presented in Table 4.8. The conditional distributions of the auxiliary units' productivities (elements 2 and 4) are presented in Table 4.9. The system fails if it does not meet the demand *w*.

The system belongs to the flow transmission type. In order to obtain the UGF representing the system PD, first we define the *u*-functions $u_1(z)$ and $u_3(z)$ from the unconditional PDs of the corresponding elements and the *u*-functions $\overline{u}_2(z)$ and $\overline{u}_4(z)$ in accordance with (4.40) and (4.41) (as in the previous example).

Then we apply the following operators producing the *u*-functions of auxiliary equivalent elements corresponding to the production blocks:

$$U_5(z) = u_1(z) \overset{\Rightarrow}{\underset{+}{\otimes}} \overline{u}_2(z)\,,\ U_6(z) = u_3(z) \overset{\Rightarrow}{\underset{+}{\otimes}} \overline{u}_4(z)$$

The entire system can be represented as two elements with *u*-functions $U_5(z)$ and $U_6(z)$ connected in a series. Since the system belongs to the flow transmission type, its u-function takes the form:

$$U(z) = U_5(z) \underset{\min}{\otimes} U_6(z)$$

The system availability is obtained as $A(w) = \delta_w(U(z))$. The system availability as a function of demand *w* is presented in Figure 4.11 (curve B).

Example 4.12

Consider a system with indirect influence of part of the elements on its performance. A chemical reactor contains six heating elements and two identical mixers (Figure 4.13). Two heating elements have nominal heating power 8 and availability 0.9, four heating elements have heating power 5 and availability 0.85. The heating elements are powered by two independent power sources with nominal power 25 and availability 0.95 for each one. The heating power of the elements cannot exceed the total power of the available sources.

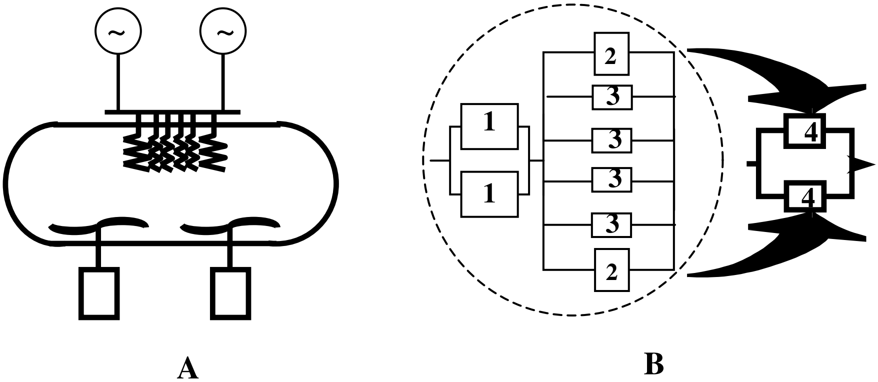


Figure 4.13. Chemical reactor (A: structure of reactor; B: system logic diagram)

The productivity distribution of each mixer depends on the cumulative power of the heaters. The greater the heating effect, the greater the productivity and availability of the mixers. The mixers are conditionally independent given the state of the heating subsystem. The conditional distributions of the mixers' productivities (element 4) are presented in Table 4.10. The total productivity of the reactor is equal to the cumulated productivity of the two mixers. The system fails if it does not meet the demand *w*.

Table 4.10. Conditional performance distributions of the mixers

Condition	g_4:	40	30	15	0
$0\le G_h<10$		0	0	0.2	0.8
$10\le G_h<20$	$p_{4\|m}$:	0	0	0.8	0.2
$20\le G_h<25$		0	0.2	0.6	0.2
$25\le G_h<30$		0.3	0.4	0.2	0.1
$30\le G_h<40$		0.7	0.1	0.1	0.1

The heating subsystem is the series-parallel system of flow transmission type. In order to obtain the UGF representing the subsystem PD, first we define the u-functions $u_1(z)$, $u_2(z)$, $u_3(z)$ as

$$u_1(z) = 0.95z^{25}+0.05z^0,\ u_2(z) = 0.9z^8+0.1z^0,\ u_3(z)=0.85z^5+0.15z^0$$

and then obtain the u-function representing the PD of the subsystem by consecutively applying the composition operators. The u-function of the power supply system is

$$U_5(z) = u_1(z) \underset{+}{\otimes} u_1(z)$$

The u-function of the heaters is obtained as follows:

$$U_6(z) = u_2(z) \underset{+}{\otimes} u_2(z),\ U_7(z) = U_6(z) \underset{+}{\otimes} u_3(z),\ U_8(z) = U_7(z) \underset{+}{\otimes} u_3(z)$$

$$U_9(z) = U_8(z) \underset{+}{\otimes} u_3(z),\ U_{10}(z) = U_9(z) \underset{+}{\otimes} u_3(z)$$

Observe that this u-function can be obtained in a simpler manner by defining an auxiliary element with the u-function $U_7(z)$ equivalent to the u-function of two parallel elements 3:

$$U_6(z) = u_2(z) \underset{+}{\otimes} u_2(z), U_7(z) = u_3(z) \underset{+}{\otimes} u_3(z)$$

$$U_8(z) = U_7(z) \underset{+}{\otimes} U_7(z),\ U_{10}(z) = U_6(z) \underset{+}{\otimes} U_8(z)$$

The u-function of the entire heating system (power sources and heaters) is

$$U_h(z) = U_5(z) \underset{\min}{\otimes} U_{10}(z)$$

The mechanical system consists of two parallel mixers and belongs to the flow transmission type. Having the u-function $\overline{u}_4(z)$ of a single mixer defined in accordance with (4.40) and (4.41) as

$$\overline{u}_4(z) = (0, 0, 0, 0.3, 0.7)z^{40} + (0, 0, 0.2, 0.4, 0.1)z^{30}$$

$$+ (0.2, 0.8, 0.6, 0.2, 0.1)z^{15} + (0.8, 0.2, 0.2, 0.1, 0.1)z^{0}$$

we obtain the u-function representing the conditional PDs of the system:

$$\overline{U}_{11}(z) = \overline{u}_4(z) \underset{+}{\overset{\circ}{\otimes}} \overline{u}_4(z)$$

Since the heating system affects the reactor's productivity only by influencing the PD of the mixers, we apply the $\overset{\Rightarrow}{\otimes}$ operator:

$$U(z) = U_h(z) \overset{\Rightarrow}{\otimes} \overline{U}_{11}(z)$$

The system availability can now be obtained as $A(w)=\delta_w(U(z))$. The system availability as a function of demand w is presented in Figure 4.11 (curve C).

4.4 Common Cause Failures in Multi-state Systems

Common cause (CC) failures (CCFs) are the failures of multiple elements due to a common cause (single occurrence or condition). The origin of CC events can be outside the system elements they affect (lightning or seismic events, sudden changes in the environment, a wide range of human interventions from maintenance errors, to intended enemy attacks), or they can originate from the elements themselves, causing other elements to fail (examples of such events are voltage surges caused by inappropriate switching in power systems leading to failure propagation, and pipe-whip events in high-pressure systems). The condition of a CCF occurring exists when some coupling factors affect a group of elements. These include the elements being:

- involved in the same process or procedure
- sharing a common resource
- having similar design or interface
- having the same manufacturer
- having the same or close location, *etc*.

CCFs increase joint-failure probabilities, thereby reducing the reliability of the technical systems.

It is assumed that all of the elements that can fail due to a certain CC belong to a corresponding CC group (CCG). There can be several CCGs in a system, since several factors can affect the functioning of its elements. Within each CCG, several failure processes can exist that cause the simultaneous failure of different subgroups of this CCG. In order to estimate the system's reliability, the characteristics of these failure processes should be included in the system model. The description of the methods for estimating the effect of CCFs on the reliability of the binary systems can be found in [107, 108].

4.4.1 Incorporating Common Cause Failures into Multi-state Systems Reliability Analysis

An algorithm presented in this section for incorporating the CCFs into the MSS reliability analysis is based on an implicit method suggested by Vaurio [109]. This implicit method uses formulas (derived by Chae and Clark [110]) for probabilities that specific elements subject to the same CCF remain in a working condition during a given time.

Consider an MSS consisting of two-state elements (elements with total failures). The elements are mutually independent (except for the elements belonging to the same CCG).

The system contains J CCGs such that each CCG j is defined by the set C_j of numbers of MSS elements belonging to this group.

Each element can belong to a single CCG (the CCGs are disjoint): $C_i \cap C_j = \varnothing$ if $i \neq j$. Each CCG j consists of L_j elements.

All of the elements subject to the same CC (belonging to the same CCG) have the same statistical characteristics (are statistically identical).

All elements belonging to the same CCG are subject to CCF by a number of different failure events. Each failure event ϑ_{jk} is independent and constitutes the simultaneous failures of a specific subset of k elements of CCG j. The probability of each failure event depends on the number of elements that fail, but it does not depend on the particular elements involved. Each particular element cannot individually affect the probability of the failure event it is involved in.

The implicit method for incorporating CCFs into the system reliability analysis suggested in [109] consists of the following three steps:

1. Assign the unique reliability p_j to all the basic system elements j.
2. Determine the expression for the system reliability in terms of the reliabilities of the basic elements without considering any CCF. This expression is in the form of an algebraic sum of the products (terms) of the basic element reliabilities.
3. In any term containing a product of k element reliabilities (*i.e.* $p_1p_2...p_k$) belonging to the same CCG j $(1 \le k \le L_j)$, replace that product with the probability $R^{(k)}_{j,L_j}$ that these specific k elements (which are subject to failure events $\vartheta_{j1},...,\vartheta_{jL_j}$) all remain in a working state.

This probability can be obtained recursively as follows [110]:

$$R^{(k)}_{j,n} = \prod_{i=n-k+1}^{n} R^{(1)}_{j,i} \tag{4.47}$$

$$R^{(1)}_{j,n} = \prod_{k=1}^{n} [\widetilde{P}_{jk}]^{\binom{n-1}{k-1}} \tag{4.48}$$

where $R^{(k)}_{j,n}$ is the probability that specific k elements belonging to CCG j, which contains a total of the n elements, all remain in working condition $(R^{(0)}_{j,n} = 1$ for any j and n by definition) and $\widetilde{P}_{jk}$ is the probability of the non-occurrence of the failed state caused by the event ϑ_{jk}.

The implicit method can be easily applied to an MSS if the final expression for its reliability is obtained in an explicit analytical form. Obtaining the analytical expressions for complex MSSs using the UGF method is an extremely time-consuming task. In contrast, the method provides simple numerical algorithms for computing the system's reliability for arbitrary time and demand without obtaining analytical expressions. To adapt the implicit method to the numerical algorithms, the modified u-function technique has been suggested [111].

In the u-function of the MSS subsystem e

$$U_e(z)=\sum_{i=1}^{k_e}\alpha_{ei}z^{g_{ei}} \tag{4.49}$$

the coefficients α_{ei} are products of the reliabilities of the individual elements. In order to keep track of the occurrence of different reliability functions in these coefficients, the u-function is modified as follows:

$$\widetilde{U}_e(z)=\sum_{i=1}^{k_e}\overset{*}{\alpha}_{ei}z^{g_{ei},\boldsymbol{s}_{ei}} \tag{4.50}$$

To obtain the system u-function in the form (4.50) from u-function (4.49), one has to perform the following steps for each term $\alpha_{ei}z^{g_{ei}}$:

1. Assign **0** to the vector $\boldsymbol{s}_{ei}$ that consists of J integer numbers.
2. Obtain coefficient $\overset{*}{\alpha}_{ei}$ by replacing in the product $\alpha_{ei}=p_1p_2...p_k$ all of the reliabilities of the individual elements belonging to any CCG with 1.
3. When replacing reliability p_h of element h belonging to CCG j, increment by 1 the corresponding element $s_{ei}(j)$ of the vector-indicator $\boldsymbol{s}_{ei}$. Finally each element $s_{ei}(j)$ of the vector-indicator $\boldsymbol{s}_{ei}$ contains a number of replaced reliabilities of elements belonging to CCG j.

Based on these steps one can obtain the u-function $\tilde{u}_i(z)$ of a single two-state MSS element i not belonging to any CCG as

$$\tilde{u}_i(z)=p_{i1}z^{g_{i1},\mathbf{0}}+(1-p_{i1})z^{f,\mathbf{0}}=p_{i1}z^{g_{i1},\mathbf{0}}+z^{f,\mathbf{0}}-p_{i1}z^{f,\mathbf{0}} \tag{4.51}$$

where g_{i1} and p_{i1} are the nominal performance and reliability of the element respectively, f is a performance rate in the failed state.

The u-function $\tilde{u}_l(z)$ of the MSS element belonging to CCG j takes the form

$$\tilde{u}_l(z)=z^{g_{l1},\mathbf{s}_l}+z^{f,\mathbf{0}}-z^{f,\mathbf{s}_l} \tag{4.52}$$

where $s_l(k)=1(k=j)$ for $1\le k\le J$.

The composition operators over u-functions (4.50) are the same as regular composition operators $\otimes_{\varphi}$ except for the rule that defines the treatment of vector-indicators:

$$\begin{aligned}\underset{\phi,+}{\otimes}(\widetilde{U}_1(z),\widetilde{U}_2(z))&=\underset{\phi,+}{\otimes}\Big(\sum_{i=1}^{k_1}\overset{*}{\alpha}_{1i}z^{g_{1i},\mathbf{s}_{1i}},\sum_{j=1}^{k_2}\overset{*}{\alpha}_{2j}z^{g_{2j},\mathbf{s}_{2j}}\Big)\\&=\sum_{i=1}^{k_1}\sum_{j=1}^{k_2}\overset{*}{\alpha}_{1i}\overset{*}{\alpha}_{2j}z^{\phi(g_{1i},g_{2j}),\,\mathbf{s}_{1i}+\mathbf{s}_{2j}}\end{aligned} \tag{4.53}$$

The vector-indicators are always summed independently on function ϕ chosen for a specific operator.

Consequently, applying the composition operators (4.53) in accordance with the reliability block diagram method (described in Section 4.1.3), one can obtain the u-function of the entire MSS in the form (4.50). In each term i of this sum, α_{ei}^{*} is a product of the reliabilities of the basic elements not belonging to any CCG, and g_{ei} is the total MSS output performance in state i of the system. Each element $s_{ei}(j)$ of vector-indicator $\boldsymbol{s}_{ei}$ contains a number of elements belonging to CCG j that should also be taken into account when calculating the probability of the corresponding MSS state. Multiplying the α_{ei}^{*} coefficients by the probabilities that specific $s_{ei}(j)$ elements of each CCG j do not fail, one can obtain the probability of state i which should be the coefficient of the ith term of the u-function of the MSS calculated with respect to the CCF.

Thus, the u-function of an MSS can be obtained by applying the following Ω operator over the u-function of the MSS:

$$
\begin{aligned}
U(z) &= \Omega(\tilde{U}_e(z)) = \Omega[\sum_{i=1}^{k_e} \alpha_{ei}^{*} z^{g_{ei},(s_{ei}(1),\ldots,s_{ei}(J))}] \\
&= \sum_{i=1}^{k_e} \alpha_{ei}^{*} \prod_{j=1}^{J} R_{j,L_j}^{(s_{ei}(j))} z^{g_{ei}}
\end{aligned}
\tag{4.54}
$$

The numerical algorithm for the evaluation of the entire MSS u-function with respect to CCF is as follows:

1. Determine the reliabilities of the individual system elements p_i and $R_{j,L_j}^{(k)}$ $(0 \le k \le L_j)$ values for each CCG j $(1 \le j \le J)$ using (4.47) and (4.48).
2. Determine the u-functions of the individual MSS elements using definitions (4.51) and (4.52).
3. For a given MSS topology, obtain the entire system u-function $\tilde{U}(z)$ by applying the composition operators (4.53) over the u-functions of the individual system elements (the ϕ functions should be chosen in accordance with the system type and connection between the elements).
4. Obtain the u-function of the MSS using the Ω operator (4.54) over $\tilde{U}(z)$.

Example 4.13

Consider a series-parallel task processing MSS (with work sharing) containing two subsystems (components) connected in a series (Figure 4.14A).

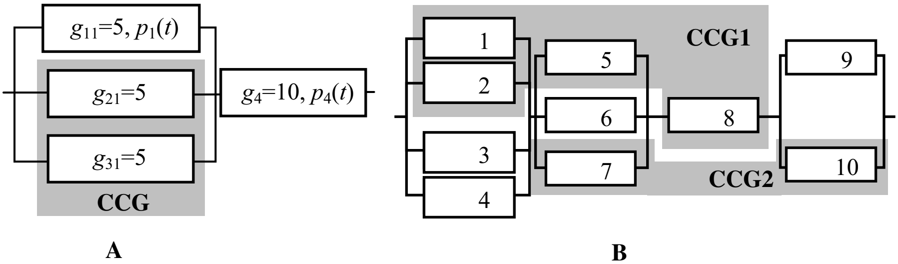


Figure 4.14. Examples of series-parallel MSS with CCF

The first component has three parallel elements with the same nominal performance rate: $g_{11} = g_{21} = g_{31} = 5$. The reliability of the first element is p_1. Two other elements of the first component compose a CCG, which is characterized by the probabilities $\tilde{P}_{11}$ and $\tilde{P}_{12}$. For this CCG, $R_{1,2}^{(1)} = \tilde{P}_{11}\tilde{P}_{12}$ and $R_{1,2}^{(2)} = \tilde{P}_{11}^2\tilde{P}_{12}$. The second component has a single element with a nominal performance rate of g_{41}=10 and the reliability p_4. All of the elements have the performance f=0 when they fail.

Following (4.51) and (4.52), we obtain the u-function for the first element as

$$u_1(z) = p_1 z^{5,(0)} + z^{0,(0)} - p_1 z^{0,(0)}$$

for elements belonging to the CCG as

$$u_2(z) = u_3(z) = z^{5,(1)} + z^{0,(0)} - z^{0,(1)}$$

and for element of the second component as

$$u_4(z) = p_4 z^{10,(0)} + (1-p_4) z^{0,(0)}$$

The u-function of the first component is obtained using the $\underset{+}{\otimes}$ operator:

$$U_1(z) = \underset{+}{\otimes}(u_1(z), u_2(z), u_3(z))$$

$$= (p_1 z^{5,(0)} + z^{0,(0)} - p_1 z^{0,(0)}) \underset{+}{\otimes} (z^{5,(1)} + z^{0,(0)} - z^{0,(1)}) \underset{+}{\otimes} (z^{5,(1)} + z^{0,(0)} - z^{0,(1)})$$

$$= p_1 z^{15,(2)} + (1-3p_1) z^{10,(2)} + 2p_1 z^{10,(1)} + (3p_1-2) z^{5,(2)}$$
$$+ 2(1-2p_1) z^{5,(1)} + p_1 z^{5,(0)} + (1-p_1) z^{0,(2)} + 2(p_1-1) z^{0,(1)} + (1-p_1) z^{0,(0)}$$

The u-function of the second component is equal to the u-function of its single element $U_2(z)=u_4(z)$.

To obtain the u-function $\tilde{U}(z)$ corresponding to the entire system we use the $\underset{\times}{\otimes}$ operator:

$$\tilde{U}(z) = \underset{\times}{\otimes}(U_1(z), U_2(z)) = p_1 p_4 z^{6,(2)} + p_4(1-3p_1) z^{5,(2)} + 2p_1 p_4 z^{5,(1)}$$

$$+p_4(3p_1-2)z^{3.3,(2)}+2p_4(1-2p_1)z^{3.3,(1)}+p_1p_4z^{3.3,(0)}+p_4(1-p_1)z^{0,(2)}$$

$$+p_4(p_1-1)z^{0,(1)}+(1-p_1p_4)z^{0,(0)}$$

Now, using operator Ω, we obtain the u-function for the entire system with respect to CCF:

$$U(z)=\Omega\,(\tilde{U}(z))=q_1z^6+q_2z^5+q_3z^{3.3}+q_4z^0$$

where

$$q_1=R_{1,2}^{(2)}\,p_1p_4=\tilde{P}_{11}^2\tilde{P}_{12}\,p_1p_4$$

$$q_2=R_{1,2}^{(2)}\,p_4(1-3p_1)+2\,R_{1,2}^{(1)}\,p_1p_4=\tilde{P}_{11}^2\tilde{P}_{12}\,p_4(1-3p_1)+2\,\tilde{P}_{11}\tilde{P}_{12}\,p_1p_4$$

$$q_3=R_{1,2}^{(2)}\,p_4(3p_1-2)+2\,R_{1,2}^{(1)}\,p_4(1-2p_1)+p_1p_4$$

$$=\tilde{P}_{11}^2\tilde{P}_{12}\,p_4(3p_1-2)+2\,\tilde{P}_{11}\tilde{P}_{12}\,p_4(1-2p_1)+p_1p_4$$

$$q_4=R_{1,2}^{(2)}\,p_4(1-p_1)+2\,R_{1,2}^{(1)}\,p_4(p_1-1)+(1-p_1p_4)$$

$$=\tilde{P}_{11}^2\tilde{P}_{12}\,p_4(1-p_1)+2\,\tilde{P}_{11}\tilde{P}_{12}\,p_4(p_1-1)+(1-p_1p_4)$$

The system performance distribution is determined by the vectors

$$\boldsymbol{g}=\{6,5,3.3,0\},\ \boldsymbol{q}=\{q_1,q_2,q_3,q_4\}$$

Using the operators δ_w we can obtain the system reliability for any demand w:

$$R(w)=\delta_w(U(z))=\begin{cases}0, & w>6\\ q_1, & 5<w\le 6\\ q_1+q_2, & 3.3<w\le 5\\ q_1+q_2+q_3, & 0<w\le 3.3\\ q_1+q_2+q_3+q_4=1, & w=0\end{cases}$$

Example 4.14

The non-repairable series-parallel MSS (Figure 4.14B) consists of four components connected in a series. All of the MSS elements have Weibull cumulative hazard functions $H(t) = (\alpha t)^{\beta}$. Parameters of the elements are presented in Table 4.11. The performance of any element in a failed state is f=0.

Table 4.11. Parameters of MSS elements

No of element i	No of component	Nominal performance g_{i1}	Parameters of individual element cumulative hazard function $H(t)=(\alpha t)^{\beta}$		No of CCG
			α	β	
1	1	0.20	-	-	1
2	1	0.20	-	-	1
3	1	0.20	0.004	1.0	-
4	1	0.50	0.001	0.5	-
5	2	0.60	-	-	1
6	2	0.30	0.008	1.0	-
7	2	0.20	-	-	2
8	3	1.30	-	-	1
9	4	0.85	0.0012	1.0	-
10	4	0.25	-	-	2

There are two CCGs in the given MSS: C_1={1, 2, 5, 8}, C_2={7, 10}. The failure processes ϑ_{jk} in these CCGs that govern simultaneous failures of a specific set of k elements are characterized by the cumulative hazard functions $H_{jk}(t)$. The probability of the non-occurrence of the failure event governed by the process ϑ_{jk} in time interval [0, t] is $\widetilde{P}_{jk}(t) = \exp(-H_{jk}(t))$.

For CCG 1:

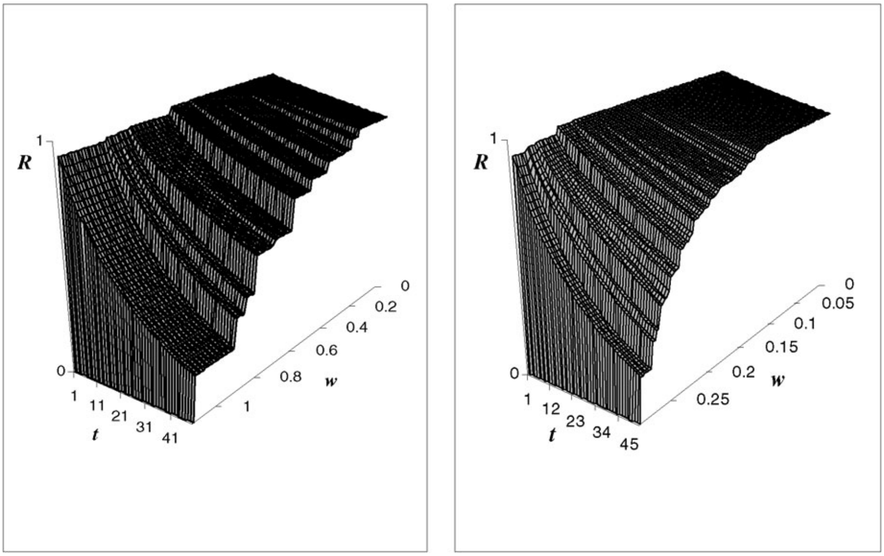
$$H_{11}(t)=(0.001t)^{0.8}, \quad H_{12}(t)=0.08H_{11}(t)$$

$$H_{13}(t)=0.02H_{11}(t), \quad H_{14}(t)=0.007H_{11}(t)$$

For CCG 2:

$$H_{21}(t)=0.003t, \quad H_{22}(t)=0.2H_{21}(t)$$

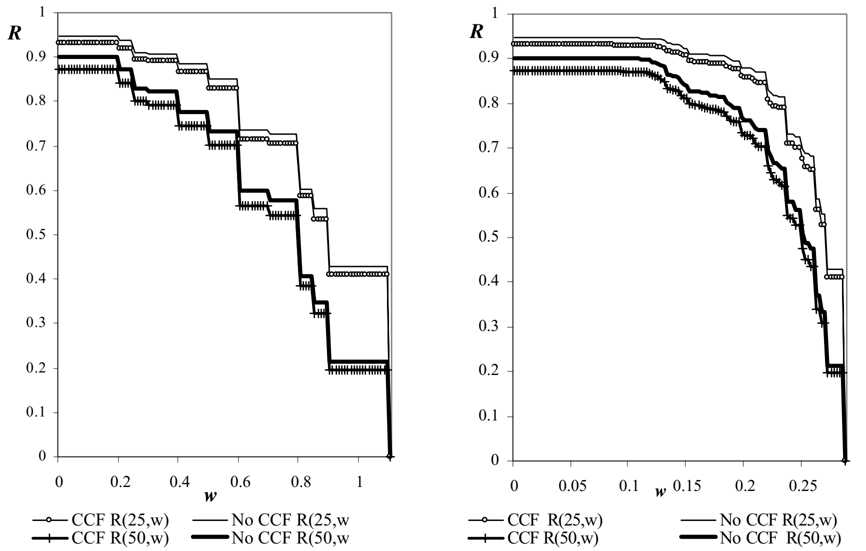
The structure presented is interpreted as flow transmission MSS with flow dispersion and task processing MSS with work sharing. The reliability functions $R(t,w)$ for both MSSs obtained using the numerical algorithm described above are presented in Figure 4.15. One can see that the task processing MSS has more different levels of PD than the flow transmission MSS. This is due to the nature of the operator $\otimes_{\min}$, which, as distinct from the operator $\otimes_{\times}$, reduces the diversity of the possible performance levels.



A **B**

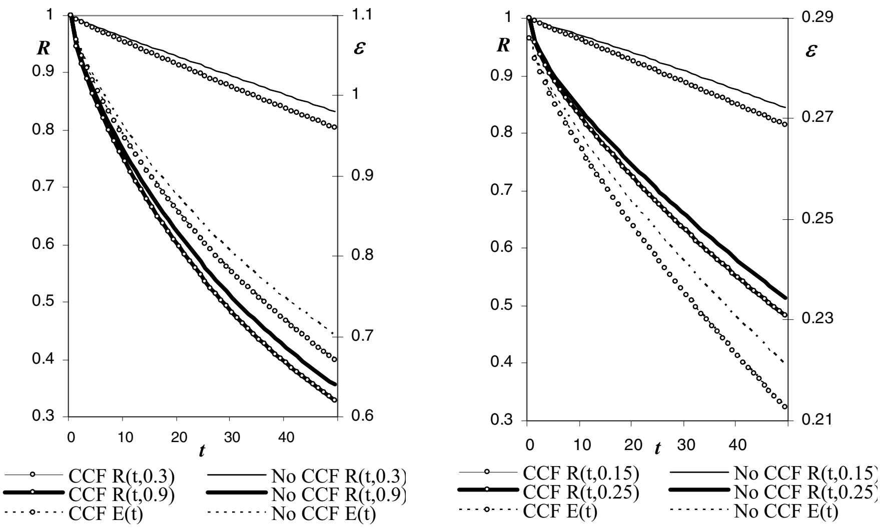
Figure 4.15. Reliability functions $R\,(t, w)$ for MSSs with CCF (A: flow transmission system; B: task processing system)

To estimate the influence of CCF on MSS reliability we compare two systems of each type: an MSS without CCF in which elements belonging to the CCG j have their individual reliability functions equal to $\widetilde{P}_{j1}(t)$, and the same MSS with CCF. Since it is difficult visually to distinguish the differences between the three-dimensional representations of reliability functions for the MSSs with and without CCF, we present them for fixed values of t (Figure 4.16) as $R(w)$ and for fixed values of w (Figure 4.17) as $R(t)$. One can see the effect of CCF in decreasing MSS reliability. In addition, the expected MSS performances $\varepsilon(t)$ are presented for MSSs with and without CCF (Figure 4.17).



A B

Figure 4.16. Reliability functions *R* (25, *w*) and *R* (50, *w*) for MSSs with and without CCF (A: flow transmission system; B: task processing system)



A B

Figure 4.17. Functions *R* (*t*, 0.3), *R* (*t*, 0.9) and $\varepsilon(t)$ for MSSs with and without CCF (A: flow transmission system; B: task processing system)

4.4.2 Multi-state Systems with Total Common Cause Failures

In some cases CCFs lead to the total outage of all of the elements belonging to the corresponding CCG. Usually, such total failures occur when a group of elements share the same resource (energy source, space, protection, *etc.*) that has limited availability. Examples of such situations include an electrical supply failure that causes an outage of all production units supplied from the same source or the failure of a waterproof casing that causes water penetration into the hermetic compartment and destruction of all the equipment located there. The algorithm for incorporating the total CCF in reliability analysis of MSSs is simpler than the general algorithm considered in the previous section. This algorithm can be easily extended to MSS with multi-state elements [112].

Consider a subsystem consisting of several elements that compose a series-parallel structure. Assume that the elements are subject to a total CCF occurring with probability v. The total CCF leads to outage of all of the subsystem elements. The entire subsystem can have different performance rates, depending on the internal states of its elements. However, when the CCF occurs, the performance rate of the subsystem is f, which corresponds to its total failure.

The total or partial failures of subsystem elements and the entire subsystem failure due to common cause are independent events. Probabilities of all the states of the subsystem itself now should be treated as conditional probabilities, given the CCF does not occur. The only possible subsystem state, when the CCF occurs, is the state with the performance equal to f. If the u-function of a combination of elements composing the subsystem is $U_j(z)$, then the u-function of the subsystem which takes into account the CCF can be determined using the following operator ξ:

$$\xi(\mathrm{U_j(z)}) = (1-v)U_j(z) + vz^f \tag{4.55}$$

One can model the subsystem with CCF as a series connection of the subsystem itself and an element representing the CCF, which has PD

$$\Pr(G = x_1) = 1-v,\ \ \Pr(G = x_2) = v \tag{4.56}$$

where x_1 corresponds to the state when CCF does not occur and x_2 corresponds to the state when CCF occurs. Such a model should reflect the fact that the subsystem performance rate will be changed to f with probability υ and will not be changed with probability $1-v$. In order to provide this property, one has to define the values of x_1 and x_2 such that for any G

$$\phi_{\mathrm{ser}}(G, x_1) = G \quad \text{and} \quad \phi_{\mathrm{ser}}(G, x_2) = f. \tag{4.57}$$

For any type of series-parallel systems described in Section 4.1, where f corresponds to the performance rate 0, x_1=∞ and x_2=0 meet the requirement (4.57).

Using the $\underset{\phi_{ser}}{\otimes}$ operator over $U_j(z)$ and the u-function representing the PD (4.56) one obtains

$$\xi(U_j(z)) = \underset{\phi_{ser}}{\otimes}(U_j(z), (1-v)z^{\infty} + vz^0) = (1-v)U_j(z) + vz^0 \tag{4.58}$$

Replacing any CCG with the u-function $U_j(z)$ by an equivalent element with the u-function $\xi(U_j(z))$ one can use the reliability block diagram method for obtaining the reliability of series-parallel systems with total CCF.

Example 4.15

Consider the series-parallel flow transmission MSS with flow dispersion presented in Figure 4.18.

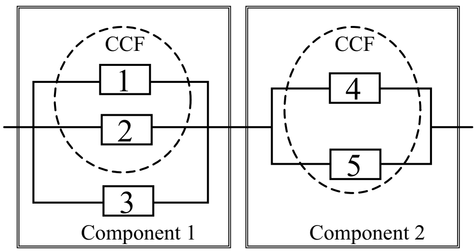


Figure 4.18. Series-parallel MSS with total CCF

The system consists of two components connected in a series. The first component contains three parallel elements. The first and second elements are subject to CCF, which has probability v_1 = 0.1. The second component contains two parallel elements that are subject to CCF with probability v_2 = 0.2. Each element j can have two states: total failure with performance rate zero and normal functioning with nominal performance g_{j1}. The availability p_{j1} and nominal performance of the elements are presented in Table 4.12. The system should meet the constant demand w = 2.

First, we determine the u-functions for the individual elements as follows:

$$u_1(z) = 0.9z^1+0.1z^0,\ u_2(z) = 0.8z^2+0.2z^0,\ u_3(z) = 0.8z^2+0.2z^0$$

$$u_4(z) = 0.9z^2+0.1z^0,\ u_5(z) = 0.8z^3+0.2z^0$$

Using the operator $\underset{+}{\otimes}$, we determine the u-functions for the subsystem consisting of two parallel elements, 1 and 2:

$$u_1(z)\underset{+}{\otimes} u_2(z) = (0.9z^1+0.1z^0)(0.8z^2+0.2z^0)=0.72z^3+0.08z^2+0.18z^1+0.02z^0$$

and for the subsystem consisting of two parallel elements 4 and 5:

$$u_4(z) \underset{+}{\otimes} u_5(z) = (0.9z^2+0.1z^0)(0.8z^3+0.2z^0) = 0.72z^5+0.08z^3+0.18z^2+0.02z^0$$

Table 4.12. Parameters of MSS elements

No of element j	Availability p_{j1}	Nominal performance rate g_{j1}
1	0.90	1.0
2	0.80	2.0
3	0.72	2.0
4	0.90	2.0
5	0.80	3.0

To incorporate the total CCF into u-functions of the subsystems, we use the operator ξ (4.58):

$$\xi(u_1(z) \underset{+}{\otimes} u_2(z)) = (1-v_1)(u_1(z) \underset{+}{\otimes} u_2(z))+v_1z^0 = 0.9(0.72z^3+0.08z^2$$

$$+ 0.18z^1+0.02z^0)+0.1z^0 = 0.648z^3+0.072z^2+0.162z^1+0.118z^0$$

$$\xi(u_4(z) \underset{+}{\otimes} u_5(z)) = (1-v_2)(u_4(z) \underset{+}{\otimes} u_5(z))+v_2z^0 = 0.8(0.72z^5+0.08z^3$$

$$+ 0.18z^2+0.02z^0)+0.2z^0 = 0.567z^5+0.064z^3+0.144z^2+0.216z^0$$

To obtain u-functions $U_1(z)$ for the entire first component, we consider it as a parallel connection of subsystem that has u-function $\xi(u_1(z) \underset{+}{\otimes} u_2(z))$ and the element 3 with u-function $u_3(z)$:

$$U_1(z) = \xi(u_1(z) \underset{+}{\otimes} u_2(z)) \underset{+}{\otimes} u_3(z)$$

$$= (0.648z^3+0.072z^2+0.162z^1+0.118z^0)(0.72z^2+0.28z^0)$$

$$= 0.4666z^5+0.0518z^4+0.298z^3+0.1051z^2+0.0454z^1+0.033z^0$$

The u-function of the second component, consisting of elements 4 and 5, is $U_2(z) = \xi(u_4(z) \underset{+}{\otimes} u_5(z))$. In order to obtain the u-function for the entire system consisting of two components connected in a series, we use the operator $\underset{\min}{\otimes}$ over u-functions $U_1(z)$ and $U_2(z)$:

$$U(z) = U_1(z) \underset{\min}{\otimes} U_2(z) = (0.4666z^5+0.0518z^4+0.298z^3+0.1051z^2$$

$$+0.0454z^1+0.033z^0) \underset{\min}{\otimes} (0.567z^5+0.064z^3+0.144z^2+0.216z^0)$$

$$= 0.269z^5+0.03z^4+0.224z^3+0.2z^2+0.035z^1+0.242z^0$$

This u-function represents the performance distribution of the entire MSS. Using the $\delta_2(U(z))$ operator we obtain the system availability as

$$A(2) = 0.269+0.03+0.224+0.2 = 0.723$$

4.4.3 Multi-state Systems with Nested Common Cause Groups

In the previous sections we assumed that the CCFs affecting different CCGs are independent. In many cases this model is not relevant because statistical dependence between the different CCFs exists. The typical examples of such a situation are systems with a multilevel protection. Such systems are used in many applications (nuclear, military, underwater, airspace systems, *etc.*) and are designed according to the so-called defence-in-depth methodology [113].

The multilevel protection means that a subsystem and its inner level protection are in turn protected by the protection of the outer level. This double-protected subsystem has its outer protection, and so forth. In such systems, the protected subsystems can be destroyed only if all of the levels of their protection are destroyed. Each level of protection can be destroyed only if all of the outer levels of protection are destroyed. This creates statistical dependence among the destruction events of the different protection levels (different CCFs). The systems with multilevel protection can be considered as systems with nested CCGs in which the CCF in any group can occur only if the CCFs in all CCGs containing this group have occurred.

In this section we consider series-parallel MSSs with nested CCGs and total CCFs and make the following assumptions:

- The elements belonging to any CCG compose a series-parallel structure (Figure 4.19A).
- Any CCG can belong to another CCG. For any pair of CCGs A and B $A\cap B\neq\varnothing$ means that $A\subseteq B$ or $B\subseteq A$, *i.e.* part of any CCG cannot belong to another CCG (Figure 4.19B).
- CCF in any group m cannot occur if this group belongs to another group and the CCF in the outer group has not occurred. If the CCFs in all of the outer CCGs that include the CCG m have occurred, the CCF in CCG m can occur with the probability v_m.
- Any element fails with probability 1 if CCFs in all of the CCGs that this element belongs to have occurred.
- The performance of any failed element is equal to f.
- The element failure caused by the CCFs and the transitions of this element in the space of states caused by its individual failures and repairs are independent events.

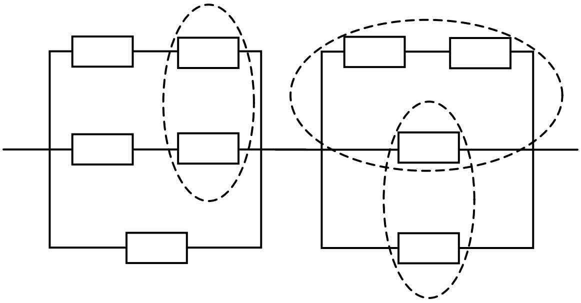


Figure 4.19. Impossible CCGs. (A: elements of CCG do not compose a series-parallel structure; B: two CCGs have common elements)

The probability of each state of an element (or subsystem) belonging to some CCG depends on the CC event. Therefore, each subsystem belonging to a CCG is characterized by two conditional performance distributions: the first corresponds to the case when the CCF in this group occurs and the second corresponds to the case when the CCF in the group does not occur. In order to represent the performance distributions of a subsystem m belonging to some CCG, we introduce the following double u-function (d-function) $d_m(z)=<U_m(z), \tilde{U}_m(z)>$, where $U_m(z)$ and $\tilde{U}_m(z)$ represent performance distributions for the first and second cases respectively.

If CCF in a group consisting of a single basic element occurs, then this element fails with probability 1 and has the performance rate f. Therefore, for a basic single element j that has a performance distribution represented by the u-function $u_j(z)$

$$d_j(z)=<z^f, u_j(z)> \tag{4.59}$$

It can easily be seen that any pair of elements with d-functions $d_j(z)$ and $d_i(z)$ belonging to the same CCG can be replaced by the equivalent element (Figure 4.20) with the d-function

$$\begin{aligned} d_j(z) \underset{\phi}{\otimes} d_i(z) &=< U_j(z), \tilde{U}_j(z) > \underset{\phi}{\otimes} < U_i(z), \tilde{U}_i(z) > \\ &=< U_j(z) \underset{\phi}{\otimes} U_i(z), \tilde{U}_j(z) \underset{\phi}{\otimes} \tilde{U}_i(z) > \end{aligned} \tag{4.60}$$

where ϕ should be substituted by ϕ_{ser} or ϕ_{par} in accordance with the type of connection between the elements.

Assume that the d-function of a series-parallel subsystem that constitutes CCG m obtained without respect to CCF in this group is $d_m(z)=<U_m(z), \tilde{U}_m(z)>$. Assume also that the group m belongs to an outer CCG h. If the CCF in group h occurs, then the CCF in group m can occur with probability v_m. If this CCF occurs, then the subsystem has its performance distribution represented by the u-function $U_m(z)$; if the CCF does not occur (with probability $1-v_m$), then the subsystem has its performance distribution represented by the u-function $\tilde{U}_m(z)$. Therefore, the conditional performance distribution of the group m given CCF in group h has occurred can be represented by the u-function

$$v_m U_m(z) + (1-v_m)\tilde{U}_m(z) \tag{4.61}$$

In the case when the CCF in CCG h have not occurred, CCF in the group m also cannot occur and its conditional performance distribution is represented by the u-function $\tilde{U}_m(z)$.

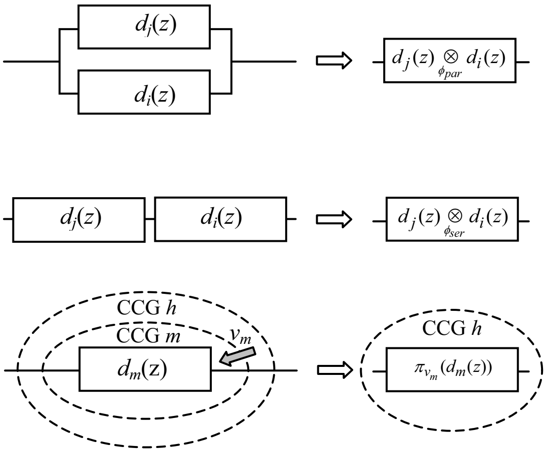


Figure 4.20. Basic equivalent transformations of system elements

These considerations allow one to incorporate the CCF that occurs in the CCG m with probability v_m into the d-function of this group by replacing the group with an equivalent element (Figure 4.20) with the d-function obtained by applying the following operator π_{v_m} over $d_m(z)$:

$$\begin{aligned} \pi_{v_m}(d_m(z)) &= \pi_{v_m} < U_m(z), \tilde{U}_m(z) > \\ &=< v_m U_m(z) + (1-v_m)\tilde{U}_m(z), \tilde{U}_m(z) > \end{aligned} \tag{4.62}$$

It can be seen that when $v_m = 1$ the operator π_{v_m} does not change the d-function. Indeed, the totally vulnerable protection (which is equivalent to absence of any protection) cannot affect the performance distribution of the subsystem it protects.

Consecutively applying the operators (4.60) and (4.62) and replacing the subsystems and the CCGs with equivalent elements, one can obtain the d-function representing the performance distribution of the entire system. The algorithm for obtaining the d-function is based on the assumption that any system element belongs to at least one CCG. In order to make this algorithm universal we can always assume that the entire system belongs to an outer CCG (is protected by an outer protection). If such protection does not exist, then the outer protection with vulnerability $v = 1$ can be added without changing the system performance distribution. The following recursive algorithm obtains the system d-function:

1. Obtain the d-functions of all of the system elements using Equation (4.59).
2. If the system contains a pair of elements connected in parallel or in a series and belonging to the same CCG, replace this pair with an equivalent element with the d-function obtained by the $\underset{\phi_{\text{par}}}{\otimes}$ or $\underset{\phi_{\text{ser}}}{\otimes}$ operator.
3. If the system contains a CCG consisting of a single element, replace this CCG with a single equivalent element with the d-function obtained using the π_{v_m} operator.
4. If the system contains more than one element or a CCG not replaced by a single element, return to step 2.
5. Determine the d-function of the entire series-parallel system as the d-function of the remaining single equivalent element $d(z)=<U(z),\tilde{U}(z)>$.

According to the definition of the d-function, the u-function $\tilde{U}(z)$ corresponds to the case when the CCFs in the system do not occur while the u-function $U(z)$ represents the entire system performance distribution in which all probabilities of the CCFs that can occur in the system are incorporated. The system reliability (or any other performance measure) can now be obtained by applying the corresponding operators over the u-function $U(z)$.

Example 4.16

Consider the system with multiple protection presented in Figure 4.21A. In this system, each CCG corresponds to a subsystem that has its own protection. Each CCG can contain other CCGs (protected subsystems). The CCF in any CCG corresponds to the destruction of the corresponding protection. If the protection of the CCG is destroyed, all unprotected elements in this CCG fail (the performance of a failed element is zero). The protection cannot be destroyed if an outer protection is not destroyed.

Assume that the performance distribution of each individual element j is represented by the u-function $u_j(z)$. The destruction probability v_m of each protection m is assumed to be known. The d-functions of the individual elements are

$$d_1(z) = <z^0, u_1(z)>, d_2(z) = <z^0, u_2(z)>, d_3(z) = <z^0, u_3(z)>$$

$$d_4(z) = <z^0, u_4(z)>, d_5(z) = <z^0, u_5(z)>$$

According to the recursive algorithm, in order to obtain the system's availability one has to perform the following steps:

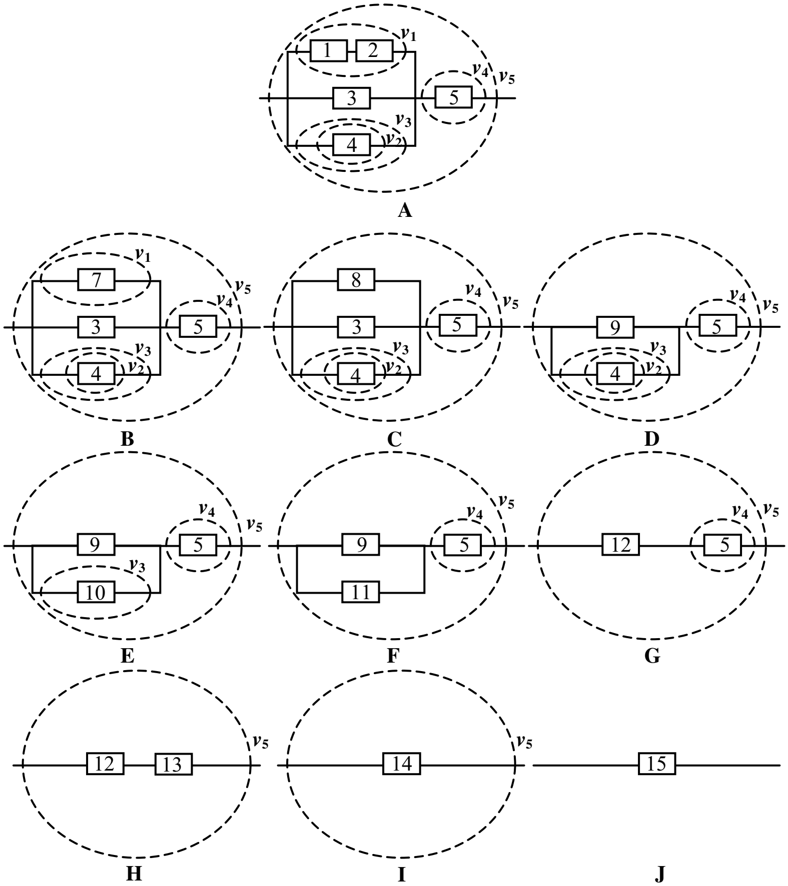


Figure 4.21. Example of recursive algorithm

Replace elements 1 and 2 connected in series by a single equivalent element 7 with the *d*-function

$$d_7(z) = d_1(z) \underset{\phi_{\text{ser}}}{\otimes} d_2(z) = <z^0, u_1(z)> \underset{\phi_{\text{ser}}}{\otimes} <z^0, u_2(z)> = <z^0, u_1(z) \underset{\phi_{\text{ser}}}{\otimes} u_2(z)>$$

(see Figure 4.21B).

Replace element 7 with its protection by an equivalent element 8 with the *d*-function

$$d_8(z) = \pi_{v_1}(d_7(z)) = \pi_{v_1} <z^0, u_1(z) \underset{\phi_{\text{ser}}}{\otimes} u_2(z)>$$

$$= <v_1 z^0 + (1-v_1)u_1(z) \underset{\phi_{\text{ser}}}{\otimes} u_2(z), u_1(z) \underset{\phi_{\text{ser}}}{\otimes} u_2(z)>$$

(see Figure 4.21C).

Replace elements 8 and 3 connected in parallel by a single equivalent element 9 with the d-function (taking into account that $u(z) \underset{\phi_{\text{par}}}{\otimes} z^0 = u(z)$ for any $u(z)$)

$$d_9(z) = d_8(z) \underset{\phi_{\text{par}}}{\otimes} d_3(z)$$

$$= \langle v_1 z^0 + (1-v_1)u_1(z) \underset{\phi_{\text{ser}}}{\otimes} u_2(z), u_1(\mathrm{z}) \underset{\phi_{\text{ser}}}{\otimes} u_2(z)\rangle \underset{\phi_{\text{par}}}{\otimes} \langle z^0, u_3(z)\rangle$$

$$= \langle v_1 z^0 + (1-v_1)u_1(z) \underset{\phi_{\text{ser}}}{\otimes} u_2(z), \;\; (u_1(z) \underset{\phi_{\text{ser}}}{\otimes} u_2(z)) \underset{\phi_{\text{par}}}{\otimes} u_3(z)\rangle$$

(see Figure 4.21D).

Replace element 4 with its inner protection by an equivalent element 10 with the d-function

$$d_{10}(z) = \pi_{v_2}(d_4(z)) = \pi_{v_2} \langle z^0, u_4(z)\rangle = \langle v_2 z^0 + (1-v_2)u_4(z), u_4(z)\rangle$$

see (Figure 4.21E).

Replace element 10 with its protection by an equivalent element 11 with the d-function

$$d_{11}(z) = \pi_{v_3}(d_{10}(z)) = \pi_{v_3} \langle v_2 z^0 + (1-v_2)u_4(z), u_4(z)\rangle$$

$$= \langle v_3 v_2 z^0 + v_3(1-v_2)u_4(z) + (1-v_3)u_4(z), \;\; u_4(z)\rangle$$

$$= \langle v_2 v_3 z^0 + (1-v_3 v_2)u_4(z), u_4(z)\rangle$$

(see Figure 4.21F).

Replace elements 9 and 11 connected in parallel by a single equivalent element 12 with the d-function

$$d_{12}(z) = d_9(z) \underset{\phi_{\text{par}}}{\otimes} d_{11}(z) = \langle v_1 z^0 + (1-v_1)u_1(z)$$

$$\underset{\phi_{\text{ser}}}{\otimes} u_2(z), (u_1(z) \underset{\phi_{\text{ser}}}{\otimes} u_2(z)) \underset{\phi_{\text{par}}}{\otimes} u_3(z)\rangle \underset{\phi_{\text{par}}}{\otimes} \langle v_2 v_3 z^0 + (1-v_2 v_3)u_4(z), u_4(z)\rangle$$

$$= \langle v_1 v_2 v_3 z^0 + (1-v_1)v_2 v_3 u_1(z) \underset{\phi_{\text{ser}}}{\otimes} u_2(z) + v_1(1-v_2 v_3)u_4(z)$$

$$+(1-v_1)(1-v_2 v_3)(u_1(z) \underset{\phi_{\text{ser}}}{\otimes} u_2(z)) \underset{\phi_{\text{par}}}{\otimes} u_4(z), (u_1(z) \underset{\phi_{\text{ser}}}{\otimes} u_2(z)) \underset{\phi_{\text{par}}}{\otimes} u_3(z) \underset{\phi_{\text{par}}}{\otimes} u_4(z)\rangle$$

(see Figure 4.21G).

Replace element 5 with its protection by an equivalent element 13 with the d-function

$$d_{13}(z) = \pi_{v_4}(d_5(z)) = \pi_{v_4} \langle z^0, u_5(z)\rangle = \langle v_4 z^0 + (1-v_4)u_5(z), u_5(z)\rangle$$

(see Figure 4.21H).

Replace elements 12 and 13 connected in series by a single equivalent element 14 with the *d*-function

$$d_{14}(z) = d_{12}(z) \underset{\phi_{\mathrm{ser}}}{\otimes} d_{13}(z) = <v_1v_2v_3z^0 + (1-v_1)v_2v_3u_1(z) \underset{\phi_{\mathrm{ser}}}{\otimes} u_2(z)$$

$$+ v_1(1-v_2v_3)u_4(z) + (1-v_1)(1-v_2v_3)(u_1(z) \underset{\phi_{\mathrm{ser}}}{\otimes} u_2(z)) \underset{\phi_{\mathrm{par}}}{\otimes} u_4(z), (u_1(z) \underset{\phi_{\mathrm{ser}}}{\otimes} u_2(z))$$

$$\underset{\phi_{\mathrm{par}}}{\otimes} u_3(z) \underset{\phi_{\mathrm{par}}}{\otimes} u_4(z)> \underset{\phi_{\mathrm{ser}}}{\otimes} <v_4z^0 + (1-v_4)u_5(z), u_5(z)>$$

$$= <v_4z^0+v_1v_2v_3(1-v_4)z^0 + (1-v_1)v_2v_3(1-v_4)u_1(z) \underset{\phi_{\mathrm{ser}}}{\otimes} u_2(z) \underset{\phi_{\mathrm{ser}}}{\otimes} u_5(z)$$

$$+ v_1(1-v_2v_3)(1-v_4)u_4(z) \underset{\phi_{\mathrm{ser}}}{\otimes} u_5(z)$$

$$+ (1-v_1)(1-v_2v_3)(1-v_4)((u_1(z) \underset{\phi_{\mathrm{ser}}}{\otimes} u_2(z)) \underset{\phi_{\mathrm{par}}}{\otimes} u_4(z))$$

$$\underset{\phi_{\mathrm{ser}}}{\otimes} u_5(z), ((u_1(z) \underset{\phi_{\mathrm{ser}}}{\otimes} u_2(z)) \underset{\phi_{\mathrm{par}}}{\otimes} u_3(z) \underset{\phi_{\mathrm{par}}}{\otimes} u_4(z)) \underset{\phi_{\mathrm{ser}}}{\otimes} u_5(z)>$$

(see Figure 4.21I).

Finally, replace element 14 with its protection by an equivalent element 15 with the *d*-function

$$d_{15}(z) = \pi_{v_5}(d_{14}(z)) = \pi_{v_5} <v_4z^0 + v_1v_2v_3(1\text{-}v_4)z^0$$

$$+ (1-v_1)v_2v_3(1-v_4)u_1(z) \underset{\phi_{\mathrm{ser}}}{\otimes} u_2(z) \underset{\phi_{\mathrm{ser}}}{\otimes} u_5(z) + v_1(1-v_2v_3)(1-v_4)u_4(z) \underset{\phi_{\mathrm{ser}}}{\otimes} u_5(z)$$

$$+ (1-v_1)(1-v_2v_3)(1-v_4)((u_1(z) \underset{\phi_{\mathrm{ser}}}{\otimes} u_2(z)) \underset{\phi_{\mathrm{par}}}{\otimes} u_4(z)) \underset{\phi_{\mathrm{ser}}}{\otimes} u_5(z), ((u_1(z) \underset{\phi_{\mathrm{ser}}}{\otimes} u_2(z))$$

$$\underset{\phi_{\mathrm{par}}}{\otimes} u_3(z) \underset{\phi_{\mathrm{par}}}{\otimes} u_4(z)) \underset{\phi_{\mathrm{ser}}}{\otimes} u_5(z))>$$

$$= < v_4v_5z^0 + v_1v_2v_3(1-v_4)v_5z^0 + (1-v_1)v_2v_3(1-v_4)v_5u_1(z) \underset{\phi_{\mathrm{ser}}}{\otimes} u_2(z) \underset{\phi_{\mathrm{ser}}}{\otimes} u_5(z)$$

$$+ v_1(1-v_2v_3)(1-v_4)v_5u_4(z) \underset{\phi_{\mathrm{ser}}}{\otimes} u_5(z)$$

$$+ (1-v_1)(1-v_2v_3)(1-v_4)v_5((u_1(z) \underset{\phi_{\mathrm{ser}}}{\otimes} u_2(z)) \underset{\phi_{\mathrm{par}}}{\otimes} u_4(z)) \underset{\phi_{\mathrm{par}}}{\otimes} u_5(z)$$

$$+ (1-v_5)((u_1(z) \underset{\phi_{\mathrm{ser}}}{\otimes} u_2(z)) \underset{\phi_{\mathrm{par}}}{\otimes} u_3(z) \underset{\phi_{\mathrm{par}}}{\otimes} u_4(z)) \underset{\phi_{\mathrm{par}}}{\otimes} u_5(z), ((u_1(z) \underset{\phi_{\mathrm{ser}}}{\otimes} u_2(z))$$

$$\underset{\phi_{\mathrm{par}}}{\otimes} u_3(z) \underset{\phi_{\mathrm{par}}}{\otimes} u_4(z)) \underset{\phi_{\mathrm{ser}}}{\otimes} u_5(z) >$$

(see Figure 4.21J).

The entire system performance distribution is represented by the first u-function of $d_{15}(z)$

$$U(z) = v_4v_5z^0 + v_1v_2v_3(1-v_4)v_5z^0 + (1-v_1)v_2v_3(1-v_4)v_5u_1(z) \underset{\phi_{ser}}{\otimes} u_2(z) \underset{\phi_{ser}}{\otimes} u_5(z)$$

$$+ v_1(1-v_2v_3)(1-v_4)v_5u_4(z) \underset{\phi_{ser}}{\otimes} u_5(z)$$

$$+ (1-v_1)(1-v_2v_3)(1-v_4)v_5((u_1(z) \underset{\phi_{ser}}{\otimes} u_2(z)) \underset{\phi_{par}}{\otimes} u_4(z)) \underset{\phi_{par}}{\otimes} u_5(z)$$

$$+ (1-v_5)((u_1(z) \underset{\phi_{ser}}{\otimes} u_2(z)) \underset{\phi_{par}}{\otimes} u_3(z) \underset{\phi_{par}}{\otimes} u_4(z))\otimes u_5(z)$$

Example 4.17

Consider a series-parallel MSS (power substation) that consists of three basic subsystems (Figure 4.22A):

1. blocks of commutation equipment (elements 1-5);
2. power transformers (elements 6-8);
3. output medium voltage line sections (elements 9-12).

All of the elements of this flow transmission system (with flow dispersion) are two-state units with nominal performance rates (the power that the elements can transform/transmit) g_{j1} and the availabilities p_{j1} presented in Table 4.13. The failed elements have performance zero.

Table 4.13. Parameters of elements of power substation

j	1	2	3	4	5	6	7	8	9	10	11	12
g_{j1}	2	6	6	3	5	5	4	5	4	3	4	5
p_{j1}	0.92	0.90	0.95	0.88	0.95	0.97	0.97	0.97	0.93	0.96	0.90	0.94

The d-function of two-state element j takes the form

$$d_j(z) = <z^0, p_{j1}z^{g_{j1}} + (1-p_{j1})z^0>$$

In order to increase the system survivability (the probability that the system meets demand w) in the case of an external attack, the system can be divided into four spatially separated groups represented by the following sets of elements: {1,2,3}, {4}, {6,7,9,10,11} and {5,8,12}. The probability of impact in the case of attack is v_1 = 0.3. Since the groups are separated, no more than one group can be affected by a single impact. Four subsystems belonging to the separated groups can be protected (located indoors within concrete constructions). These subsystems include elements 2 and 3, element 6, elements 9 and 10, elements 5, 8 and 12. The probability of protection destruction in the case of impact is v_2 = 0.6, while the probability of destruction of the unprotected elements in the case of impact is 1 (unprotected elements do not survive the impact).

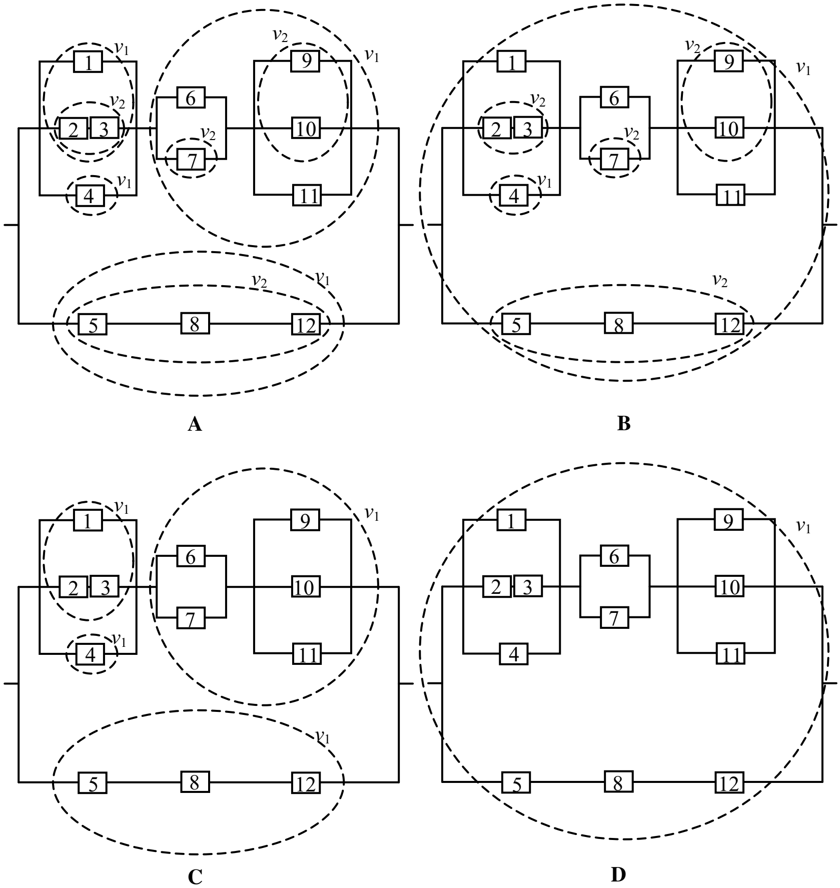


Figure 4.22. Series-parallel power substation (system with multilevel protection)

In order to evaluate the influence of each type of protection, four different configurations are compared:

A. Both separation and indoor allocation are applied (Figure 4.22A).

B. All of the elements are gathered in the same place (no separation). Indoor allocation is applied (Figure 4.22B).

C. The groups of elements are separated, but all of the elements are located outdoors (Figure 4.22C).

D. All of the elements are gathered in the same place and located outdoors (Figure 4.22D).

In Figure 4.23, one can see the system survivability (obtained using the method presented in this section) as a function of the demand for cases A, B, C, and D.

Observe that the protection of parts of the system is not effective when the system tolerates only a very small decrease of its performance below its maximal possible performance. In our case the indoor allocation of some system elements can increase the system survivability only when $w \le 9$ (compare curves B and D).

Indeed, in the case of impact, even if all of the elements located indoors survive, they cannot provide the system's performance greater than 9.

The separation is also effective only when the demand is considerably smaller than the maximal possible system performance. Moreover, the separation can decrease the system's survivability when the demand is close to its maximal performance. Indeed, by separating the system elements one creates additional vulnerable CCGs, which contribute to an additional overall system exposure to the impact. When the demand is relatively small, the separation increases the system's survivability because the smaller parts can be destroyed by a single impact. In our case, the separation is effective for $w \le 5$. When $w > 5$ the separation decreases the system's survivability (compare curves C and D).

The total survivability improvement achieved by separation and protection of its elements for $w \le 5$ is greater than 23%.

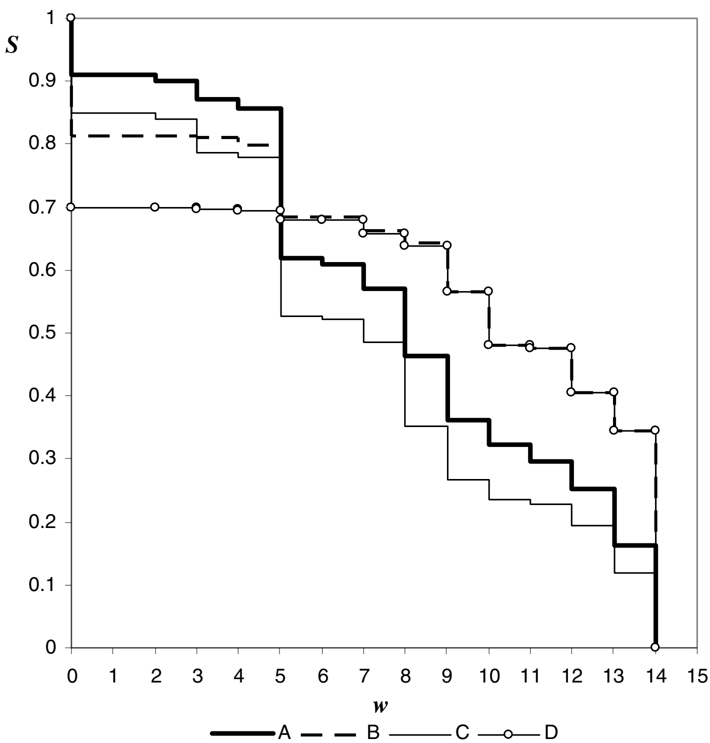


Figure 4.23. Survivability of power substation as a function of demand

4.5 Importance Analysis in Multi-state Systems

Information about the importance of the elements that constitute a system with respect to its safety, reliability, availability, and performance, is of great practical aid to system designers and managers. Indeed, the identification of which elements most influence the overall system performance allows one to trace technical bottlenecks and provides guidelines for effective actions of system improvement. In this sense, importance measures (IMs) are used to quantify the contribution of individual elements to the system's performance measures (*e.g.* reliability, availability, mean performance, expected performance deficiency).

IMs were first introduced by Birnbaum [114]. The Birnbaum importance measure gives the contributions to the system's reliability due to the reliability of the various system elements. Elements for which a variation in reliability results in the largest variation of the entire system's reliability have the highest importance. Fussell and Vesely later proposed a measure based on the cut-sets importance [115]. According to the Fussell-Vesely measure, the importance of an element depends on the number and the order of the cut-sets in which it appears. Other concepts of importance measures have been proposed and used based on different views of the elements' influence on the system's performance. Structural IMs account for the topographic importance of the logic position of the element in the system [116, 117]. Criticality IMs consider the conditional probability of the failure of an element, given that the system has failed [118, 119]. Joint IMs account for the introduction of the elements' interactions in their contribution to the system's reliability [120, 121].

IMs are being widely used in risk-informed applications of the nuclear industry to characterize the importance of basic events, *i.e.* element failures, human errors, common cause failures, etc., with respect to the risk associated to the system [122-125]. In this framework, the risk importance measures are based on two other IMs: the performance reduction worth and the performance achievement worth [122]. The former is a measure of the 'worth' of the basic failure event in achieving the present level of system performance and, when applied to elements, it highlights the importance of maintaining the current level of element reliability (with respect to the basic failure event). The latter, the performance achievement worth, is associated to the variation of the system's performance consequent to an improvement of the element reliability.

In a general context, the IMs reflect the changes in distribution of the performance of the entire system caused by constraints imposed n the performance of one of its elements. Once the system PD is determined, one can focus on specific system performance measures, *e.g.* system availability, for the definition of the relevant measures of element importance.

4.5.1 Reliability Importance of Two-state Elements in Multi-state Systems

Consider a system consisting of two-state elements. Each element j has performance g_{j1} in the state of perfect functioning and performance g_{i0} in the state of total failure, which corresponds to its u-function $u_j(z) = p_{j1}z^{g_{j1}} + (1-p_{j1})z^{g_{j0}}$.

Let O be a system output performance measure ($O \equiv A$ for availability or reliability; $O \equiv \varepsilon$ for mean system performance, $O \equiv \Delta^{-}$ for expected performance deficiency). The system performance measure (PM) O can be expressed for the given demand distribution as a function of parameters of system elements

$$O(p_{11}, g_{11}, g_{10}, \ldots, p_{j11}, g_{j1}, g_{j0}, \ldots, p_{n1}, g_{n1}, g_{n0}) \tag{4.63}$$

In order to obtain this index, one has to determine the u-functions of individual elements $u_j(z)$ for $1 \le i \le n$, to obtain the u-function of the PMs of interest (see Section 3.3) using the corresponding operators and to calculate the derivatives of these u-functions at $z = 1$.

Let O_{j0} and O_{j1} be the system PM when element j is fixed in its faulty and functioning state respectively, while the remainder of the elements are free to randomly change their states. The PMs O_{j0} and O_{j1} according to their definition are

$$O_{j0} = O(p_{11}, g_{11}, g_{10}, \ldots, 0, g_{j1}, g_{j0}, \ldots, p_{n1}, g_{n1}, g_{n0}) \tag{4.64}$$

$$O_{j1} = O(p_{11}, g_{11}, g_{10}, \ldots, 1, g_{j1}, g_{j0}, \ldots, p_{n1}, g_{n1}, g_{n0}) \tag{4.65}$$

O_{j0} corresponds to the system PM when the element j is in the state of total failure with probability $p_{j0} = 1$ (which can be represented by the u-function $u_j^{-}(z) = z^{g_{j0}}$). O_{j1} corresponds to the system PM when the element j is in the state of perfect functioning with probability $p_{j1} = 1$ (which can be represented by the u-function $u_j^{+}(z) = z^{g_{j1}}$). Therefore, O_{j0} and O_{j1} can be obtained by substituting $u_j(z)$ by $u_j^{-}(z)$ and $u_j^{+}(z)$ respectively before using the procedure of system PM determination.

The system output performance measure O can be expressed as

$$O = O_{j0}p_{j0} + O_{j1}p_{j1} = O_{j0}(1-p_{j1}) + O_{j1}p_j \tag{4.66}$$

Definitions of four of the most frequently used IMs with reference to PM O and element j are as follows

The *performance reduction worth* is the ratio of the actual system PM to the valueofthePMwhenelementjisconsideredas always failed:

$$I_{Oj} = O/O_{j0} \tag{4.67}$$

This index measures the potential damage to the system's performance caused by the total unavailability of element j.

The *performance achievement worth* is the ratio of the system PM obtained when element j is always in the operable state to the actual value of the system's PM (when all of the elements including element j are left free to change their states randomly in accordance with their PD):

$$I_O a_j = O_{j1}/O \tag{4.68}$$

This index measures the contribution of element j to enhancing the system's performance by considering the maximum improvement on the system's PM achievable by making the element fully available.

The *Fussell-Vesely measure* represents the relative PM reduction due to the total failure of element j:

$$I_O f_j = (O - O_{j0})/O = 1 - 1/I_O r_j \tag{4.69}$$

Similarly, one can define the relative PM achievement when element j is always in the operable state:

$$I_O v_j = (O_{j1} - O)/O = I_O a_j - 1 \tag{4.70}$$

The *Birnbaum importance measure* represents the variation of the system PM when element j switches from the condition of perfect functioning to the condition of total failure. It is a differential measure of the importance of element j, since it is equal to the rate at which the system PM changes with respect to changes in the reliability of element j:

$$I_O b_j = \partial O / \partial p_{j1} = \partial (p_{j1} O_{j1} + (1 - p_{j1}) O_{j0}) / \partial p_{j1} = O_{j1} - O_{j0} \tag{4.71}$$

Note that for the Fussel-Vesely and Birnbaum IMs, depending on the system's PM, an improvement in the system's performance can correspond either to an increase of the considered PM (*e.g.* the availability or mean performance) or to a decrease (*e.g.* the expected performance deficiency). In the latter case, the absolute values of Iv_j, If_j and Ib_j are taken as the importance values.

The IMs for each MSS element depend strongly on that element's place in the system, its nominal performance level, and the system's demand. The notion of element relevancy is closely connected to the element's importance. The element is relevant if some changes in its state that take place without changes in the states of the reminder of the elements cause changes in the PM of the entire system. According to this definition, if the element j is irrelevant then $O_{j0} = O_{j1} = O$. Therefore, for the irrelevant element

$$I_O r_j = I_O a_j = 1 \tag{4.72}$$

while

$$I_O f_j = I_O v_j = I_O b_j = 0 \tag{4.73}$$

Example 4.18

Consider a system consisting of n elements with total failures connected in a series described in Example 4.1. For any element j $g_{j0} = 0$. The reliability measures of this system are presented in Table 4.1. The corresponding analytically obtained IMs are presented in Tables 4.14. - 4.18. In these tables $\pi = \sum_{i=1}^{n} p_{i1}$.

The element with the minimal availability has the greatest impact on MSS availability ("a chain fails at its weakest link"). The importance indices associated with the system's availability do not depend on the elements' performance rates or on demand. IMs associated with the system's mean performance and performance deficiency also do not depend on the performance rate of the individual element j; however, the performance rate g_{j1} can influence these indices if it affects the entire system performance $\hat{g}$.

Table 4.14. Performance reduction worth IMs for series MSS

w	$I_A r_j$	$I_{\Delta^-} r_j$	$I_\varepsilon r_j$
$w > \hat{g}$	not defined	$1 - \hat{g}\pi / w$	not defined
$0 \le w \le \hat{g}$	not defined	$1 - \pi$	

Table 4.15. Performance achievement worth IMs for series MSS

w	$I_A a_j$	$I_{\Delta^-} a_j$	$I_\varepsilon a_j$
$w > \hat{g}$	not defined	$\dfrac{w p_{j1} - \hat{g}\pi}{p_{j1}(w - \hat{g}\pi)}$	$1 / p_{j1}$
$0 \le w \le \hat{g}$	$1 / p_{j1}$	$\dfrac{p_{j1} - \pi}{p_{j1}(1 - \pi)}$	

Table 4.16. Relative performance reduction IMs for series MSS

w	$I_A f_j$	$I_{\Delta^-} f_j$	$I_\varepsilon f_j$
$w > \hat{g}$	not defined	$\dfrac{\hat{g}\pi}{w - \hat{g}\pi}$	1
$0 \le w \le \hat{g}$	1	$\dfrac{\pi}{1 - \pi}$	

Table 4.17. Relative performance achievement IMs for series MSS

w	$I_A v_j$	$I_{\Delta^-} v_j$	$I_\varepsilon v_j$
$w > \hat{g}$	not defined	$\frac{(1-p_{j1})\hat{g}\pi}{p_{j1}(w-\hat{g}\pi)}$	$1/p_{j1}-1$
$0 \le w \le \hat{g}$	$1/p_{j1}-1$	$\frac{(1-p_{j1})\pi}{p_{j1}(1-\pi)}$	

Table 4.18. Birnbaum importance IMs for series MSS

w	$I_A b_j$	$I_{\Delta^-} b_j$	$I_\varepsilon b_j$
$w > \hat{g}$	0	$\hat{g}\pi/p_{j1}$	$\hat{g}\pi/p_{j1}$
$0 \le w \le \hat{g}$	π/p_{j1}	$w\pi/p_{j1}$	

Example 4.19

Consider a task processing system without work sharing presented in Example 4.2. The system consists of two elements with total failures ($g_{10} = g_{20} = 0$) connected in parallel. The analytically obtained system reliability measures are presented in Table 4.3. The importance measures can also be obtained analytically. The measures $I_O r_j$, $I_O a_j$ and $I_O b_j$ are presented in Tables 4.19-4.21.

Table 4.19. Performance reduction worth IMs for parallel MSS

w	$I_A r_1$	$I_{\Delta^-} r_1$	$I_\varepsilon r_1$
$w>g_{21}$	not defined	$\frac{w-p_{11}g_{11}-p_{21}g_{21}+p_{11}p_{21}g_{11}}{w-p_{21}g_{21}}$	$1+\frac{p_{11}(1-p_{21})g_{11}}{p_{21}g_{21}}$
$g_{11}<w\le g_{21}$	1	$\frac{w-p_{11}g_{11}}{w}$	
$0<w\le g_{11}$	$1-p_{11}+p_{11}/p_{21}$	$1-p_{11}$	
	$I_A r_2$	$I_{\Delta^-} r_2$	$I_\varepsilon r_2$
$w>g_{21}$	not defined	$\frac{w-p_{11}g_{11}-p_{21}g_{21}+p_{11}p_{21}g_{11}}{w-p_{11}g_{11}}$	$1-p_{21}+\frac{p_{21}g_{11}}{p_{11}g_{11}}$
$g_{11}<w\le g_{21}$	not defined	$1-p_{21}$	
$0<w\le g_{11}$	$1-p_{21}+p_{21}/p_{11}$	$1-p_{21}$	

Table 4.20. Performance achievement worth IMs for parallel MSS

w	$I_A a_1$	$I_{\Delta^-} a_1$	$I_\varepsilon a_1$
$w>g_{21}$	not defined	$\dfrac{w-g_{11}-p_{21}(g_{21}-g_{11})}{w-p_{11}g_{11}-p_{21}g_{21}+p_{11}p_{21}g_{11}}$	
$g_{11}<w\le g_{21}$	1	$\dfrac{w-g_{11}}{w-p_{11}g_{11}}$	$\dfrac{(1-p_{21})g_{11}+p_{21}g_{21}}{(1-p_{21})p_{11}g_{11}+p_{21}g_{21}}$
$0<w\le g_{11}$	$1/(p_{11}+p_{21}-p_{11}p_{21})$	0	
	$I_A a_2$	$I_{\Delta^-} a_2$	$I_\varepsilon a_2$
$w>g_{21}$	not defined	$\dfrac{w-g_{21}}{w-p_{11}g_{11}-p_{21}g_{21}+p_{11}p_{21}g_{11}}$	
$g_{11}<w\le g_{21}$	$1/p_{21}$	0	$\dfrac{g_{21}}{(1-p_{21})p_{11}g_{11}+p_{21}g_{21}}$
$0<w\le g_{11}$	$1/(p_{11}+p_{21}-p_{11}p_{21})$	0	

Table 4.21. Birnbaum IMs for parallel MSS

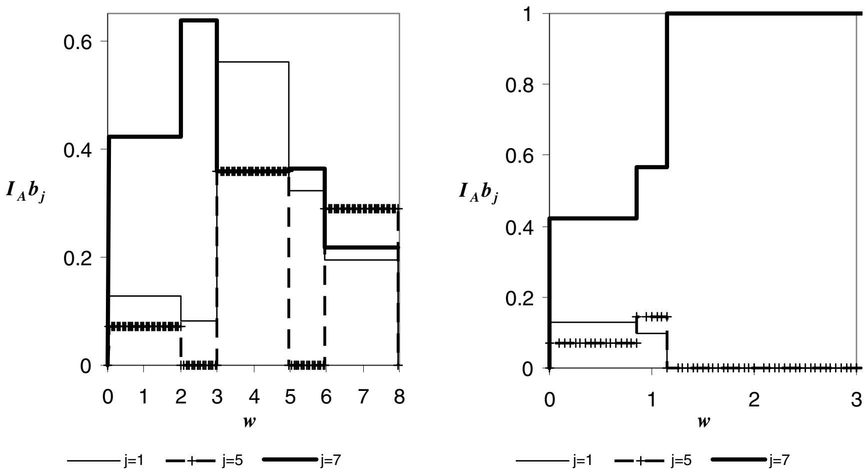
w	$I_A b_1$	$I_{\Delta^-} b_1$	$I_\varepsilon b_1$
$w>g_{21}$	0	$(1-p_{21})g_{11}$	
$g_{11}<w\le g_{21}$	0	$(1-p_{21})g_{11}$	$g_{11}(1-p_{21})$
$0<w\le g_{11}$	$1-p_{21}$	$(1-p_{21})w$	
	$I_A b_2$	$I_{\Delta^-} b_2$	$I_\varepsilon b_2$
$w>g_{21}$	0	$g_{21}-p_{11}g_{11}$	
$g_{11}<w\le g_{21}$	1	$w-p_{11}g_{11}$	$g_{21}-p_{11}g_{11}$
$0<w\le g_{11}$	$1-p_{11}$	$(1-p_{11})w$	

Example 4.20

Consider the series-parallel system from Example 4.3 (Figure 4.1A). The IMs I_Ab_j of elements 1, 5, and 7 as functions of system demand w are presented in Figure 4.24A and B for the system interpreted as a flow transmission MSS with flow dispersion and task processing MSS without work sharing. Observe that $I_Ab_j(w)$ are step nonmonotonic functions.

One can see that the values of w exist for which the importance of some elements is equal to zero. This means that these elements are irrelevant (have no influence on the system's entire availability). For example, in the case of the task processing system, the subsystem consisting of elements from 1 to 6 cannot have a performance that is greater then 1.154. Therefore, when $1.154<w\le 3$, the system satisfies the demand only when element 7 is available. In this case, the entire system availability is equal to the availability of element 7, which is reflected by the element's importance index: $I_Ab_7(w) = 1$. The remainder of the elements are irrelevant for demands greater than 1.154: $I_Ab_j(w) = 0$ for $1\le j\le 6$. Note that, although for the task processing system element 7 has the greatest importance, the importance of this element for the flow transmission system can be lower than the importance of some other elements at certain intervals of demand variance. For

example, for 3<w<5 the importance of element 1 is greater than the importance of element 7.



A **B**

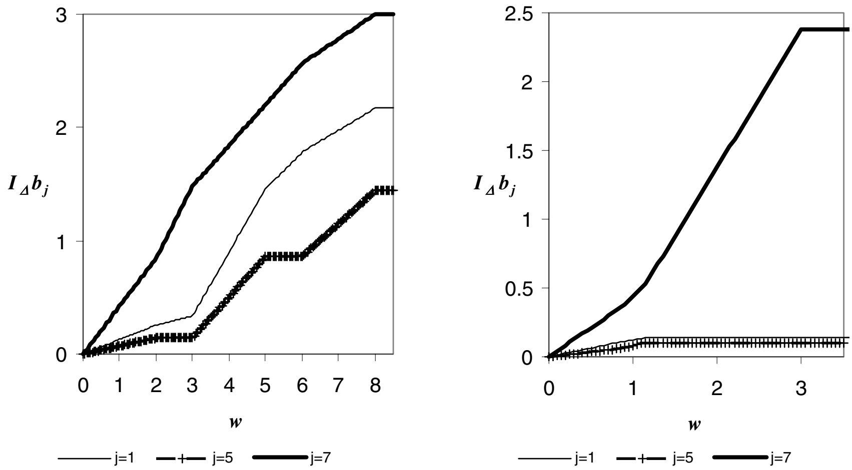
Figure 4.24. IM $I_A b_j(w)$ of system elements in flow transmission MSS (A) and task processing MSS (B)

Unlike the IM associated with the system availability $I_A b_j$, the IM associated with the system mean performance $I_\varepsilon b_j$ for element 7 is the greatest for both types of system. The values of $I_\varepsilon b_j$ for j = 1, …, 7 are presented in Table 4.22.

Table 4.22. The IMs $I_\varepsilon b_j$ for elements of series-parallel system

No of element	1	2	3	4	5	6	7
Flow transmission MSS	2.170	1.361	1.210	1.555	1.440	2.441	3.000
Task processing MSS	0.139	0.032	0.028	0.036	0.103	0.156	2.375

The IMs $I_{\Delta^-} b_j$ as functions of system demand w are presented in Figure 4.25 for j = 1, 5 and 7. Observe that $I_{\Delta^-} b_j(w)$ are piecewise linear functions. The demand intervals when the function $I_{\Delta^-} b_j(w)$ is constant always correspond to the irrelevancy of system element j.



A **B**

Figure 4.25. IM $I_{\Delta}\text{-}b_j(w)$ of system elements in flow transmission MSS (A) and task processing MSS (B)

4.5.2 Importance of Common Cause Group Reliability

In systems that contain CCGs with total CCF, the reliabilities of the groups (the probabilities that the groups do not fail) affect the reliability of the entire system. If a system consists of nonidentical elements and has a complex structure with nested CCGs, reliabilities of different groups play different roles in providing for the system's reliability. The evaluation of the relative influence of the group's reliability on the reliability of the entire system provides useful information about the importance of these groups.

For example, in systems with complex multilevel protection, the protection survivability (the ability to tolerate destructive external impacts) can depend on the type and location of the protection. The importance of each protection depends not only on its survivability but also on characteristics of the subsystem it protects.

Importance evaluation is an essential point in tracing bottlenecks in protected systems and in identifying the most important protections. The protection survivability importance analysis can also help the analyst to find the irrelevant protections, *i.e.* protections that have no impact on the entire system's reliability. Elimination of irrelevant protections simplifies the system and reduces its cost. In the complex multi-state systems with multilevel protection, finding the irrelevant protections is not a trivial task.

In order to evaluate the CCG reliability importance we use the MSS model with nested CCGs. The algorithm presented in Section 4.4.3 allows one to evaluate the system's performance measures O as a function of the probabilities of total CCFs in its CCGs.

Assume that the system has M CCGs. For the given system structure and the fixed performance distributions of the system elements, the system PM O is a function of the CCF probabilities in these CCGs: $O(v_1,\ldots,v_m,\ldots,v_M)$. Since the reliability of CCG m s_m is defined as the probability of non-occurrence of CCF in this group ($s_m = 1-v_m$) we can express the system PM as a function of CCG reliabilities $O(s_1,\ldots,s_m,\ldots,s_M)$ and define in accordance with (4.64) and (4.65):

$$O_{m0} = O(s_1, \ldots, 0, \ldots, s_M) \text{ and } O_{m1} = O(s_1, \ldots, 1, \ldots, s_M) \tag{4.74}$$

where O_{m0} corresponds to the system PM when the failure in CCG m has occurred (in accordance with Equation (4.62), this can be represented by the d-function $< U_m(z), \tilde{U}_m(z) >$ of this CCG) and O_{m1} corresponds to the system PM when the failure in CCG m has not occurred (which can be represented by the d-function $< \tilde{U}_m(z), \tilde{U}_m(z) >$). Therefore, O_{m0} and O_{m1} can be obtained by substituting $d_m(z)$ by $< U_m(z), \tilde{U}_m(z) >$ and $< \tilde{U}_m(z), \tilde{U}_m(z) >$ respectively in the procedure of determining the system's PM. The corresponding IMs can be obtained using Equation (4.67)-(4.71).

Example 4.21

Consider the simplest binary systems with multiple protections. In order to evaluate the protections' survivability importance we use the Birnbaum IM $I_A b_m$.

The system consists of identical binary elements with availability a. The d-function of each element can be represented as:

$$d_m(z) = < z^0, az^1 + (1-a)z^0 >$$

where performance 1 corresponds to its normal state and performance 0 corresponds to failure. The entire system succeeds (survives) if its performance is $G = 1$. Consider the following cases.

Case 1: n-level (concentric) protection of a single element (Figure 4.26A). The system's availability and the survivability importance of mth protection are respectively:

$$A = a[1 - \prod_{i=1}^{n}(1-s_i)] \text{ and } I_A b_m = a\frac{\prod_{i=1}^{n}(1-s_i)}{1-s_m}$$

This means that the protection with the greatest survivability has the greatest importance. The increase of protection survivability lowers the importance of the rest of the protections.

Case 2: n identical protected elements connected in a series (Figure 4.26B). The system's availability and the importance of mth protection are respectively

$$A = a^n \prod_{i=1}^{n} s_i \text{ and } I_A b_m = a^n \frac{\prod_{i=1}^{n} s_i}{s_m}$$

This means that the protection with the lowest survivability has the greatest importance. The increase of protection survivability increases the importance of the remainder of the protections. It can be easily seen that the absence of protection in at least one of the elements makes all of the protections irrelevant (if for any i $s_i = 0$ then $A = 0$ and $I_A b_j = 0$ for all of $j \neq i$). This means that the protection of the elements connected in a series has no sense if at least one element remains unprotected (see protection 1 in Figure 4.27).

Case 3: n identical protected elements connected in parallel (Figure 4.26C). The system's availability and the importance of the mth protection are respectively

$$A = 1 - \prod_{i=1}^{n} (1 - as_i) \text{ and } I_A b_m = a \frac{\prod_{i=1}^{n} (1 - as_i)}{1 - as_m}$$

As in the case of a single element with multiple protections, the protection with the greatest survivability has the greatest importance and the increase of protection survivability lowers the importance of the remainder of the protections.

While in complex systems composed of different multi-state elements, the relations between the elements' survivability and importance are more complicated, the general dependencies are the same as in the cases considered.

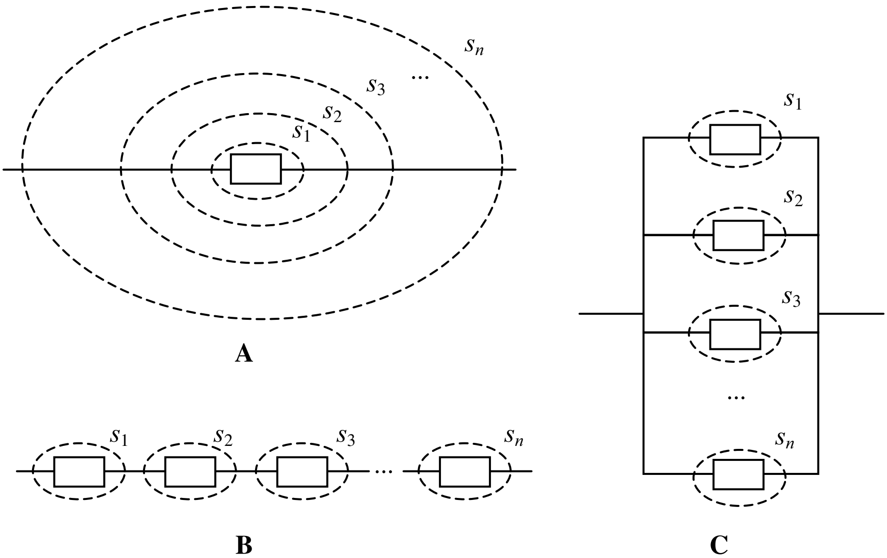


Figure 4.26. Simplest binary systems with multiple protections

Example 4.22

Consider the multi-state flow transmission series-parallel system (with flow dispersion) presented in Figure 4.27. The system consists of seven elements (with performance distributions as presented in Table 4.23) and six protection groups. The survivability of any protection is 0.8. The survivability importance of the protections as functions of demand w are presented in Figure 4.28.

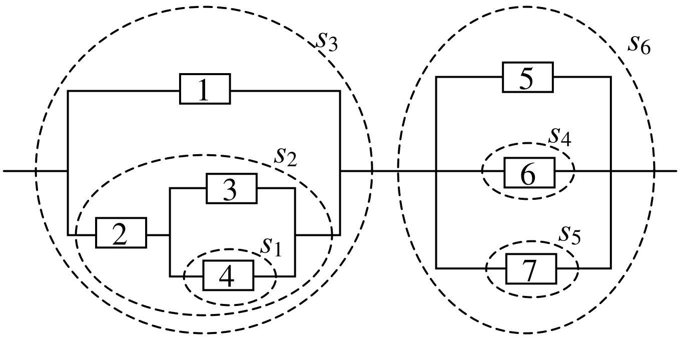


Figure 4.27. Structure of series-parallel MSS with multiple protections

Table 4.23. Performance distributions of multi-state elements

State (h)	No of element (j)													
	1		2		3		4		5		6		7	
	p_{jh}	g_{jh}	p_{jh}	g_{jh}	p_{jh}	g_{jh}	g_{jh}	g_{jh}	p_{jh}	g_{jh}	p_{jh}	g_{jh}	p_{jh}	g_{jh}
0	0.05	0	0.10	0	0.10	0	0.10	0	0.10	0	0.05	0	0.25	0
1	0.05	3	0.05	2	0.10	1	0.30	3	0.20	2	0.95	5	0.75	6
2	0.15	5	0.85	8	0.80	4	0.60	4	0.70	4	-	-	-	-
3	0.75	7	-	-	-	-	-	-	-	-	-	-	-	-

First observe that protection 1 is irrelevant for any w ($I_A b_1(w) = 0$). Indeed, when protection 2 is not destroyed, protection 1 does not affect the system's survivability. When protection 2 is destroyed then element 2 is always destroyed and the subsystem consisting of elements 2, 3 and 4 has a performance rate of 0 independent of the state of protection 1.

Some protections can be irrelevant only for certain intervals of w. For example, protection 2 affects the system's survivability only when protection 3 is destroyed. In this case, element 1 is always destroyed, which prevents the system from having a performance rate greater than 8. Therefore $I_A b_2(w) = 0$ for $w>8$.

Protection 4 affects the system's survivability only when protection 6 is destroyed. In this case, element 5 is always destroyed. If element 7 is in a normal state, then the performance rate of the subsystem remaining after the destruction of protection 6 (elements 6 and 7) is not less than 6. If element 7 does not perform its task, then the performance of the subsystem is no greater than 5 (maximal performance of element 6). This does not depend on the state of protection 4. Therefore, $I_A b_4(w) = 0$ for $5<w\le 6$.

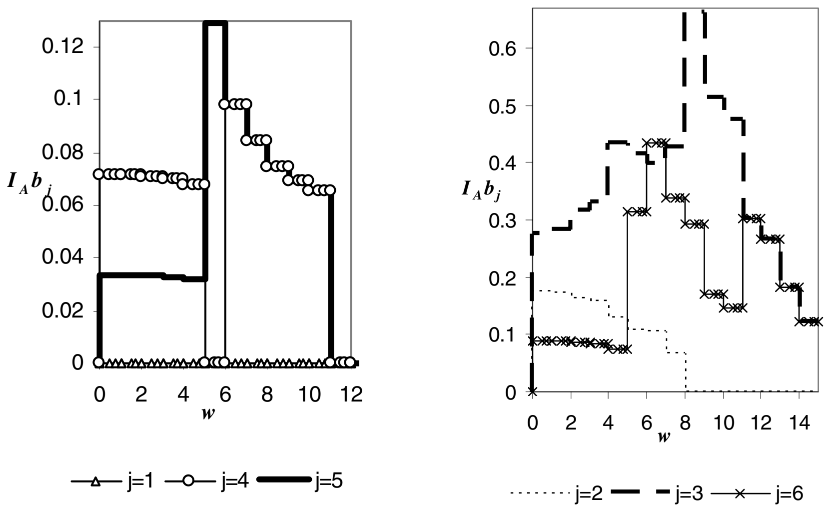


Figure 4.28. Survivability importance of protections as functions of demand

For $w>11$ $I_A b_3(w) = I_A b_6(w)$. Indeed, the system can provide a performance greater than 11 only if both protections 3 and 6 survive. It is the same for protections 4 and 5: when protection 6 is destroyed, the system can provide a performance greater than 6 only if both protections 4 and 5 survive. Therefore, for $w>6$ $I_A b_4(w) = I_A b_5(w)$.

Note also that the greater the availability of the two-state element, the greater the importance of its individual protection. For example, when $w\le 5$ both elements 6 and 7 can meet the demand, but $I_A b_4(w)>I_A b_5(w)$.

In general, the outer-level protections are more important than the inner-level ones, since they protect more elements. In our case, protections 3 and 6 have the greatest importance for any w.

In order to estimate the effect of survivability of protections on their importance, consider Figure 4.29 representing the functions $I_A b_j(s_m)$ for different j and m when the system should meet the demand $w = 5$. Observe that although the relations among the different protections in complex MSSs are much more complicated than in the simple binary systems considered above, the general tendencies are the same. Observe, for example, that the mutual influence of the protections in pairs 2 and 3, 4 and 6, 5 and 7 resembles the mutual influence of protections in Case 1 of Example 4.21, since these pairs of protections are partly concentric (both protect the same subsystems). The greater the survivability of one of the protections in the pair the lower the importance of the other one. When the outer protection becomes invulnerable, the inner protection becomes irrelevant ($I_A b_2 = 0$ when $s_3 = 1$ and $I_A b_4 = I_A b_5 = 0$ when $s_6 = 1$).

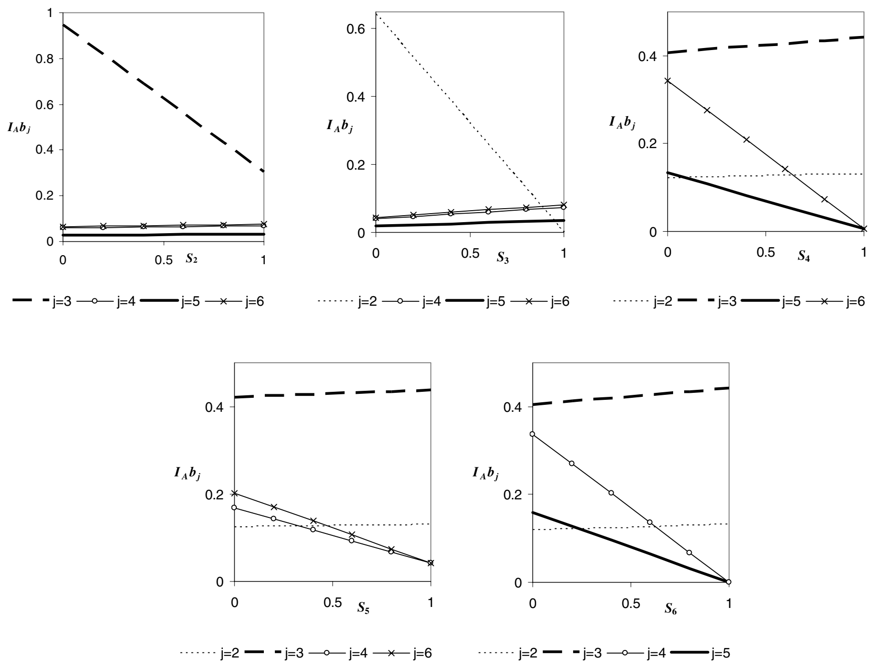


Figure 4.29. Survivability importance of protections as functions of protection survivability

The mutual influence of protections in pairs 2 and 4, 3 and 4, 2 and 6, and 3 and 6 resembles the mutual influence of the protections in Case 2 of Example 4.21, since these pairs of protections protect subsystems connected in the series. In this case the greater the survivability of one of the protections in the pair, the greater the importance of another one.

The mutual influence of protections 4 and 5 resembles the mutual influence of protections in Case 3 of Example 4.21, since this pair of protections protects parallel elements. In this case, the greater the survivability of one of the protection in the pair the lower the importance of another one.

4.5.3 Reliability Importance of Multi-state Elements in Multi-state Systems

Early progress towards the extension of IMs to the case of MSSs can be found in [126, 127], where the measures related to the occupancy of a given state by an element have been proposed. These measures characterize the importance of a given element being in a certain state or moving to the neighbouring state with respect to the system's performance. The IM of a given element is, therefore, represented by a vector of values, one for each state of the element. Such representation may be difficult for the practical reliability analyst to interpret. In the following sections we consider integrated IMs based on element performance restriction.

4.5.3.1 Extension of Importance Measures to Multi-state Elements

Assume that the states of each element j are ordered in such a manner that $g_{j0} \le g_{j1} \le .... \le g_{jk_j-1}$. One can introduce a performance threshold α and divide this set into two ordered subsets corresponding respectively to the element performance above and below the level α. Let element j be constrained to a performance rate not greater than α, while the remainder of the elements of the MSS are not constrained: we denote by $O_j^{\le\alpha|\mathrm{M}}$ the system PM obtained in this situation. Similarly, we denote by $O_j^{>\alpha|\mathrm{M}}$ the system PM resulting from the dual situation in which element j is constrained to performances above α. The MSS performance measures so introduced rely on a restriction of the achievable performance of the MSS elements. Different modelling assumptions in the enforcement of this restriction will lead to different performance values. The letter M in the definitions of $O_j^{\le\alpha|\mathrm{M}}$ and $O_j^{>\alpha|\mathrm{M}}$ is used to code the modelling approach to the restriction of element behaviour. Substituting the measures $O_j^{\le\alpha|\mathrm{M}}$ and $O_j^{>\alpha|\mathrm{M}}$ to the binary equivalents O_{j0} and O_{j1}, we can define importance measures for multi-state elements:

performance reduction worth

$$I_O r_j^{\alpha|\mathrm{M}} = O / O_j^{\le\alpha|\mathrm{M}} \tag{4.75}$$

performance achievement worth

$$I_O a_j^{\alpha|\mathrm{M}} = O_j^{>\alpha|\mathrm{M}} / O \tag{4.76}$$

relative performance reduction (Fussell-Vesely)

$$I_O f_j^{\alpha|\mathrm{M}} = (O - O_j^{\le\alpha|\mathrm{M}}) / O \tag{4.77}$$

relative performance achievement

$$I_O v_j^{\alpha|\mathrm{M}} = (O_j^{>\alpha|\mathrm{M}} - O) / O \tag{4.78}$$

Birnbaum importance

$$I_O b_j^{\alpha|\mathrm{M}} = O_j^{>\alpha|\mathrm{M}} - O_j^{\leq\alpha|\mathrm{M}} \tag{4.79}$$

This latter IM extends the concept of the IM introduced in [126]. Combining the different definitions of importance measures with different types of the system PM and different model assumptions M relative to the types of element restriction, one can obtain many different importance measures for MSS, each one bearing specific physical information. The choice of the proper IM to use depends on the system's mission and the type of improvement actions that one is aiming at in the system design or operation.

In the following section we consider two models of element performance restriction and discuss their application with respect to the importance measures $I_O f_j^{\alpha|\mathrm{M}}$ and $I_O b_j^{\alpha|\mathrm{M}}$.

4.5.3.2 State-space Reachability Restriction Approach

Let O_{jh} be the PM of the MSS when element j is in a fixed state h while the rest of the elements evolve stochastically among their corresponding states with performance distributions $\{g_{ih}, p_{ih}\}$, $1 \leq i \leq n$, $i \neq j$, $0 \leq h < k_i$. Using pivotal decomposition, we obtain the overall expected system performance

$$O = \sum_{h=0}^{k_j-1} p_{jh} O_{jh} \tag{4.80}$$

We denote by $h_{j\alpha}$ the state in the ordered set of states of element j whose performance $g_{jh_{j\alpha}}$ is equal to or immediately below α, *i.e.* $g_{jh_{j\alpha}} \leq \alpha < g_{jh_{j\alpha}+1}$. The conditional probability $\breve{p}_{jh}$ that element j is in a state h characterized by a performance g_{jh} not greater than a prespecified level threshold α $(h \leq h_{j\alpha})$ can be obtained as

$$\begin{aligned} \breve{p}_{jh} &= \Pr\{G_j = g_{jh} \mid G_j \leq \alpha\} = p_{jh} / \Pr\{G_j \leq \alpha\} \\ &= p_{jh} / \sum_{m=0}^{h_{j\alpha}} p_{jm} = p_{jh} / p_j^{\leq\alpha} \end{aligned} \tag{4.81}$$

Similarly, the conditional probability $\hat{p}_{jh}$ of element j being in a state h when it is known that $h > h_{j\alpha}$ is

$$\begin{aligned} \hat{p}_{jh} &= \Pr\{G_j = g_{jh} \mid G_j > \alpha\} = p_{jh} / \Pr\{G_j > \alpha\} \\ &= p_{jh} / \sum_{m=h_{j\alpha}+1}^{k_j-1} p_{jm} = p_{jh} / p_j^{>\alpha} \end{aligned} \tag{4.82}$$

In Equation (4.81) and (4.82), $p_j^{\le\alpha}$ and $p_j^{>\alpha}$ are probabilities that element j is in states with performance not greater than α and greater than α respectively.

The state-space reachability restriction model (coded with the letter s: $\mathrm{M} \equiv s$) is based on the restrictive condition on the states reachable by element j. In this model we define as $O_j^{\le\alpha|s}$ the system PM obtained when element j is forced to visit only states with performance not greater than α:

$$O_j^{\le\alpha|s} = \sum_{m=0}^{h_{j\alpha}} \breve{p}_{jm} O_{jm} = \sum_{m=0}^{h_{j\alpha}} p_{jm} O_{jm} / p_j^{\le\alpha} \tag{4.83}$$

Similarly, we define as $O_j^{>\alpha|s}$ the system performance measure obtained under the condition that the element j stays in states with performance greater than α:

$$O_j^{>\alpha|s} = \sum_{m=h_{j\alpha}+1}^{k_j-1} \hat{p}_{jm} O_{jm} = \sum_{m=h_{j\alpha}+1}^{k_j-1} p_{jm} O_{jm} / p_j^{>\alpha} \tag{4.84}$$

According to these definitions

$$O = O_j^{\le\alpha|s} p_j^{\le\alpha} + O_j^{>\alpha|s} p_j^{>\alpha} \tag{4.85}$$

Using the definition of the performance measures O, $O_j^{\le\alpha|s}$ and $O_j^{>\alpha|s}$ we can specify the IMs. For example, the Birnbaum importance takes the form

$$I_O b_j^{\alpha|s} = O_j^{>\alpha|s} - O_j^{\le\alpha|s} \tag{4.86}$$

From (4.86) and since $p_j^{\le\alpha} + p_j^{>\alpha} = 1$

$$O = O_j^{\le\alpha|s} + I_O b_j^{\alpha|s} p_j^{>\alpha} = O_j^{>\alpha|s} - I_O b_j^{\alpha|s} p_j^{\le\alpha} \tag{4.87}$$

And thus

$$I_O b_j^{\alpha|s} = (O - O_j^{\le\alpha|s}) / p_j^{>\alpha} = (O_j^{>\alpha|s} - O) / p_j^{\le\alpha} \tag{4.88}$$

From Equation (4.66) and (4.71) we can see that for two-state elements:

$$O = O_{j0} + I_O b_j p_{j1} = O_{j1} - I_O b_j p_{j0} \tag{4.89}$$

Comparing (4.87) and (4.89) we see that the $I_O b_j^{\alpha|s}$ measure for MSSs is really an extension of the definition of the Birnbaum importance for two-state elements, for which $k_j = 2$ and $\alpha = 0$. As such, it measures the rate of improvement of the system PM deriving from improvements on the probability $p_j^{>\alpha}$ of element j occupying states characterized by performance higher than α.

The Fussel-Vesely importance measure (relative performance reduction) takes the form

$$I_O f_j^{\alpha|s} = 1 - O_j^{\le\alpha|s} / O \tag{4.90}$$

It can be easily seen that

$$I_O b_j^{\alpha|s} = I_O f_j^{\alpha|s} O / p_j^{>\alpha} \tag{4.91}$$

The element IMs based on the state-space reachability restriction approach quantify the effect on the system performance of element j remaining confined in the dual subspaces of states corresponding to performances greater or not greater than α.

4.5.3.3 Performance Level Limitation Approach

We consider again a threshold α on the performance of element j. However, we assume that the space of reachable states of element j is not restricted, *i.e.* element j can visit any of its states independently on whether the associated performance is below or above α and it can do so with the original state probability distribution. Limitations, however, are imposed on the performance rate of element j: we consider a deteriorated version of the element that is not capable of providing a performance greater than α, in spite of the possibility of reaching any state, and an enhanced version of element j that provides performances always not less than α, even when residing in states below $h_{j\alpha}$. The limitation on the performance is such that, when in states $h > h_{j\alpha}$, the deteriorated element j is not capable of providing the design performance corresponding to its state; in these cases it is assumed that it provides the performance α. On the other hand, when the enhanced element is working in states $h < h_{j\alpha}$, it is assumed that it provides the performance α. We code this modelling approach by the letter w: $\mathrm{M} \equiv w$.

The output performance measures $O_j^{\le\alpha|w}$ and $O_j^{>\alpha|w}$ in this model take the form

$$O_j^{\le\alpha|w} = \sum_{m=0}^{h_{j\alpha}} p_{jm} O_{jm} + O_j^{\alpha} \sum_{m=h_{j\alpha}+1}^{k_j-1} p_{jm}$$
$$= \sum_{m=0}^{h_{j\alpha}} p_{jm} O_{jm} + O_j^{\alpha} p_j^{>\alpha} \tag{4.92}$$

and

$$O_j^{>\alpha|w} = O_j^{\alpha} \sum_{m=0}^{h_{j\alpha}} p_{jm} + \sum_{m=h_{j\alpha}+1}^{k_j-1} p_{jm} O_{jm}$$
$$= O_j^{\alpha} p_j^{\le\alpha} + \sum_{m=h_{j\alpha}+1}^{k_j-1} p_{jm} O_{jm} \tag{4.93}$$

where O_j^{α} is the system PM when element j remains fixed operating with performance α while the remainder of the system elements visit their states in accordance with their performance distributions. It can be seen that

$$O_j^{\le\alpha|w} + O_j^{>\alpha|w} = O + O_j^{\alpha} \tag{4.94}$$

In this case, the Birnbaum importance takes the form

$$\begin{aligned} I_O b_j^{\alpha|w} &= O_j^{>\alpha|w} - O_j^{\le\alpha|w} \\ &= O_j^{\alpha} \sum_{m=0}^{h_{j\alpha}} p_{jm} + \sum_{m=h_{j\alpha}+1}^{k_j-1} p_{jm} O_{jm} - \sum_{m=0}^{h_{j\alpha}} p_{jm} O_{jm} - O_j^{\alpha} \sum_{m=h_{j\alpha}+1}^{k_j-1} p_{jm} \\ &= \sum_{m=0}^{h_{j\alpha}} p_{jm} (O_j^{\alpha} - O_{jm}) + \sum_{m=h_{j\alpha}+1}^{k_j-1} p_{jm} (O_{jm} - O_j^{\alpha}) \\ &= \sum_{m=0}^{k_j-1} p_{jm} \,| O_j^{\alpha} - O_{jm} | \end{aligned} \tag{4.95}$$

Hence, in the performance level limitation model the Birnbaum IM is equal to the expected value of the absolute deviation of the system PM from its value when element j has performance α.

The Fussel-Vesely IM (relative performance reduction) takes the form

$$I_O f_j^{\alpha|w} = 1 - O_j^{\le\alpha|w} / O = (O_j^{>\alpha|w} - O_j^{\alpha}) / O \tag{4.96}$$

Birnbaum and Fussel-Vesely IMs are related as follows:

$$I_O b_j^{\alpha|w} = I_O f_j^{\alpha|w} \cdot O + (O_j^{\alpha} - O_j^{\le\alpha|w}) \tag{4.97}$$

The element IMs based on a limitation of the achievable performance level give information on which level α of element performance is the most beneficial from the point of view of the entire system PM.

Observe that according to the definitions (4.83), (4.84) and (4.92), (4.93)

$$O_j^{\le\alpha|w} = O_j^{\le\alpha|s} p_j^{\le\alpha} + O_j^{\alpha} p_j^{>\alpha} \tag{4.98}$$

and

$$O_j^{>\alpha|w} = O_j^{>\alpha|s} p_j^{>\alpha} + O_j^{\alpha} p_j^{\le\alpha} \tag{4.99}$$

From these equations one can obtain relations between Birnbaum and Fussel-Vesely IMs as defined according to the two approaches

$$I_O b_j^{\alpha|w} = I_O b_j^{\alpha|s} p_j^{>\alpha} + (O_j^{\alpha} - O_j^{\le\alpha|s})(2p_j^{\le\alpha} - 1) \tag{4.100}$$

and

$$I_O f_j^{\alpha|w} = I_O f_j^{\alpha|s} - (O_j^{\alpha} - O_j^{\le\alpha|s}) p_j^{>\alpha} / O \tag{4.101}$$

4.5.3.4 Evaluating System Performance Measures

In order to evaluate the system PM O when all of its elements are not restricted, one has to apply the reliability block diagram technique over u-functions of the individual elements representing their performance distributions in the form:

$$u_j(z) = \sum_{h=0}^{k_j - 1} p_{jh} z^{g_{jh}} \tag{4.102}$$

In order to obtain the IMs in accordance with the state-space reachability restriction approach, one has to modify the u-function of element j as follows:

$$u_j(z) = \sum_{h=0}^{h_{j\alpha}} (p_{jh} / p_j^{\le\alpha}) z^{g_{jh}} \tag{4.103}$$

when evaluating $O_j^{\le\alpha|s}$ and

$$u_j(z) = \sum_{h=h_{j\alpha}+1}^{k_j-1} (p_{jh} / p_j^{>\alpha}) z^{g_{jh}} \tag{4.104}$$

when evaluating $O_j^{>\alpha|s}$ and then apply the reliability block diagram technique.

In order to obtain the PMs in accordance with the performance level limitation approach one has to modify the u-function of element j as follows:

$$u_j(z) = \sum_{h=0}^{h_{j\alpha}} p_{jh} z^{g_{jh}} + p_j^{>\alpha} z^{\alpha} \tag{4.105}$$

when evaluating $O_j^{\le\alpha|w}$ and

$$u_j(z) = p_j^{\le\alpha} z^{\alpha} + \sum_{h=h_{j\alpha}+1}^{k_j-1} p_{jh} z^{g_{jh}} \tag{4.106}$$

when evaluating $O_j^{>\alpha|w}$ and then apply the reliability block diagram technique.

Note that the PM O_j^{α} can also be easily obtained by using the u-function of element j in the form $u_j(z) = z^{\alpha}$.

Example 4.23

Consider the series-parallel flow transmission system (with flow dispersion) presented in Figure 4.30 with elements having performance distributions given in Table 4.24.

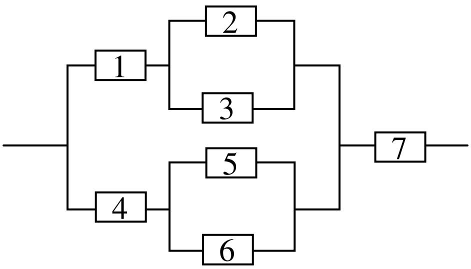


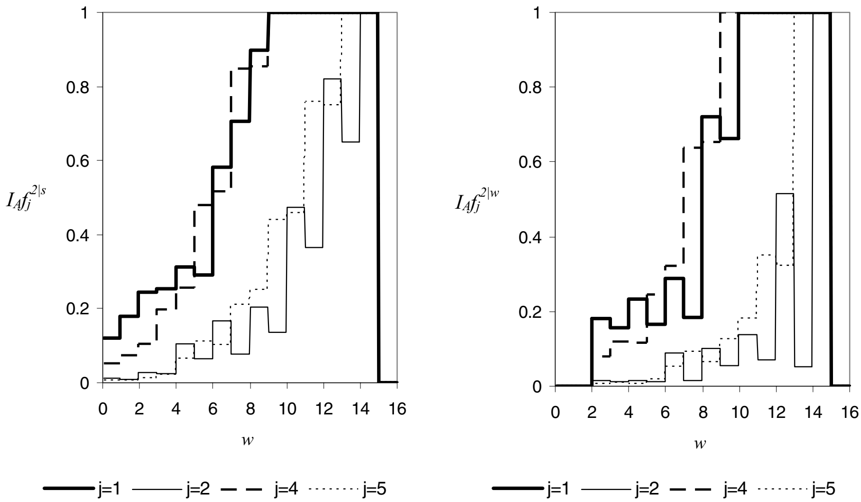
Figure 4.30. Structure of series-parallel MSS with multi-state elements

Elements 2, 3, 5, and 6 are identical. However, the pairs of elements 2, 3 and 5, 6 have different influences on the system's entire performance, since they are connected in a series with different elements (1 and 4 respectively). Therefore, while we expect elements 2 and 3 have the same importance (as well as elements 5 and 6), the importance of element 2 (or 3) differs from the importance of element 5 (or 6). The demand w is assumed to be constant in time, but different values of the constant will be considered.

Table 4.24. Performance distributions of multi-state elements

State (h)	No of element (j)													
	1		2		3		4		5		6		7	
	p_{jh}	g_{jh}	p_{jh}	g_{jh}	p_{jh}	g_{jh}	g_{jh}	g_{jh}	p_{jh}	g_{jh}	p_{jh}	g_{jh}	p_{jh}	g_{jh}
0	0.10	0	0.10	0	0.10	0	0.20	0	0.10	0	0.10	0	0.15	0
1	0.05	1	0.05	2	0.05	2	0.10	2	0.05	2	0.05	2	0.15	6
2	0.15	3	0.85	4	0.85	4	0.45	6	0.85	4	0.85	4	0.05	10
3	0.35	5	-	-	-	-	0.25	8	-	-	-	-	0.45	14
4	0.35	7	-	-	-	-	-	-	-	-	-	-	0.20	18

In this example we perform the importance analysis based on the Fussel-Vesely IM (relative performance reduction). In Figure 4.31 the $I_A f_j^{2|s}(w)$ and $I_A f_j^{2|w}(w)$ measures are presented for elements 1, 2 (identical to 3) and 4 and 5 (identical to 6) for different time-constant system demands w. The first measure shows how critical it is for the MSS availability that the element visits only states with performance below or equal to $\alpha = 2$. The second measure shows how critical for the MSS availability it is to limit the performance of the element below the threshold value $\alpha = 2$.



A B

Figure 4.31. Behaviour of the elements' IMs. A: $I_A f_j^{2|s}(w)$; B: $I_A f_j^{2|w}(w)$

The functions $I_A f_j^{2|s}(w)$ and $I_A f_j^{2|w}(w)$ differ significantly. While $I_A f_j^{2|w}(w) = 0$ for $w \le 2$, since $O_j^{\le\alpha|w} = O$ for these demands, $I_A f_j^{2|s}(w) > 0$, since the reduction of the state-space for obtaining $O_j^{\le\alpha|s}$ changes the probabilities of being in the states with $g_{jh} \le 2$, and, therefore, $O_j^{\le\alpha|s} \ne$O.

Recall also that from the definitions, $I_A f_j^{2|s} = 1$ or $I_A f_j^{2|w} = 1$ means that, when the element j has a performance restricted below α, the entire system fails. The importance measure $I_A f_j^{2|s}$ for elements $j = 1$ and $j = 4$ becomes 1 when $w = 9$. Indeed, the greatest performance of the subsystem of unrestricted elements 4, 5, and 6 is 8 while the greatest performance of the subsystem of elements 1, 2, and 3 is 1 when element 1 is allowed to visit only states with a performance not greater than $\alpha = 2$ (*i.e.* $g_{10} = 0$ or $g_{11} = 1$). Therefore, the MSS cannot have a performance greater than 8+1 = 9. Similarly, the greatest performance of the subsystem of unrestricted elements 1, 2, and 3 is 7, while the greatest performance of the subsystem of elements 4, 5, and 6 is 2 when element 4 is allowed to visit only states with a performance not greater than $\alpha = 2$ ($g_{40} = 0$ or $g_{41} = 2$). Therefore, in this case the MSS cannot have a performance greater than 7+2 = 9.

On the contrary, the importance $I_A f_j^{2|w}$ for elements $j = 1$ and $j = 4$ becomes 1 for different values of w. When the performance of element 1 is restricted by $\alpha = 2$, the MSS cannot have a performance greater than 8+2 = 10; when the performance of element 4 is restricted by $\alpha = 2$, the MSS cannot have a performance greater than 7+2 = 9. Therefore, $I_A f_1^{2|w} = 1$ for $w > 10$ while $I_A f_4^{2|w} = 1$ for $w > 9$.

Figure 4.31 also shows that an element which is the most important with respect to a value of the demand w can be less important for a different value. This is a typical situation in MSSs. For example, when $5 < w < 6$ element 4 is the most important one among elements 1-6 when their ability to perform above $\alpha = 2$ is considered, while for $w \le 5$ it becomes less important than element 1.

The $I_\Delta f_j^{2|s}(w)$ and $I_\Delta f_j^{2|w}(w)$ functions are presented in Figure 4.32. Analogously to $I_A f_j^{2|w}(w)$, the function $I_\Delta f_j^{2|w}(w)$ is equal to zero when $w \le 2$, since in this case $\Delta_j^{\le 2|w} = \Delta$. For increasing demand values, the difference between $\Delta_j^{\le\alpha|w}$ and Δ (system performance deficiency when element j is not constrained) increases from zero to a constant level. Therefore, the ratio $I_\Delta f_j^{2|w}(w) = (\Delta_j^{\le 2|w} - \Delta)/\Delta$ first increases and then begins to decrease.

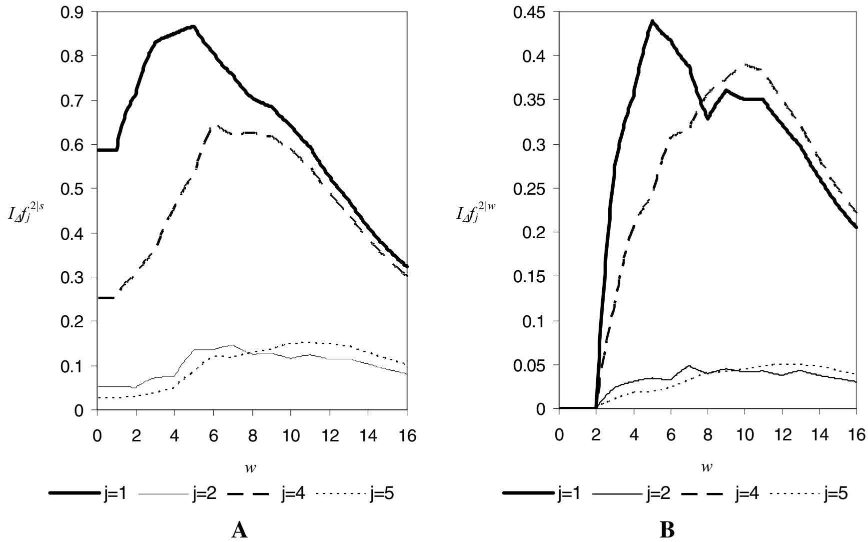


Figure 4.32. Behaviour of the elements' IMs. A: $I_\Delta f_j^{2|s}(w)$; B: $I_\Delta f_j^{2|w}(w)$

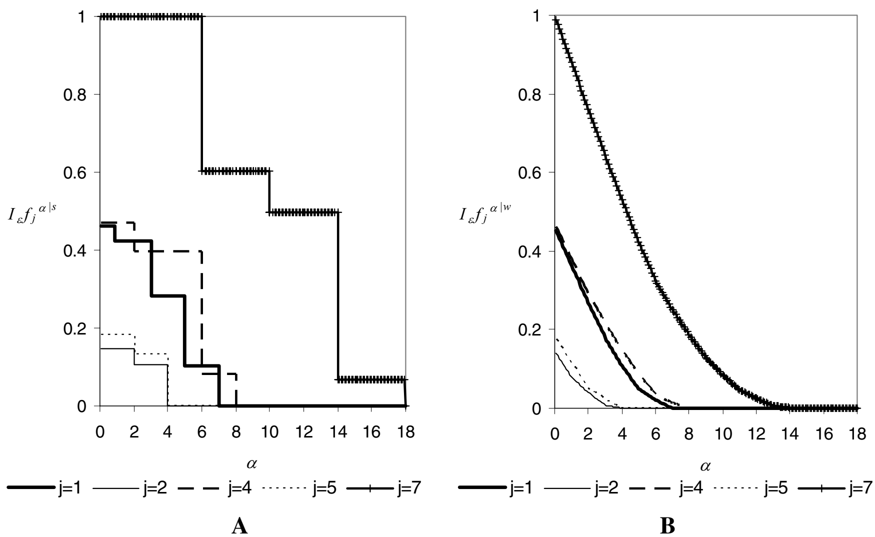
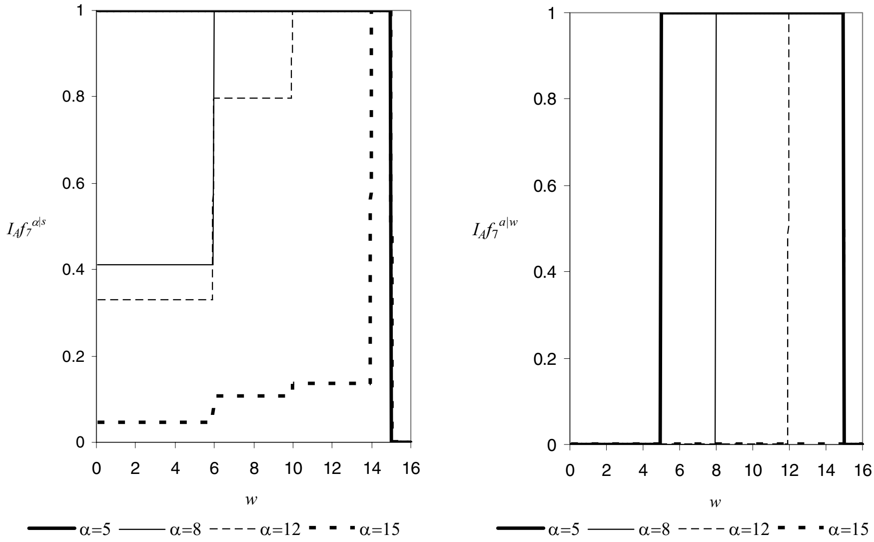
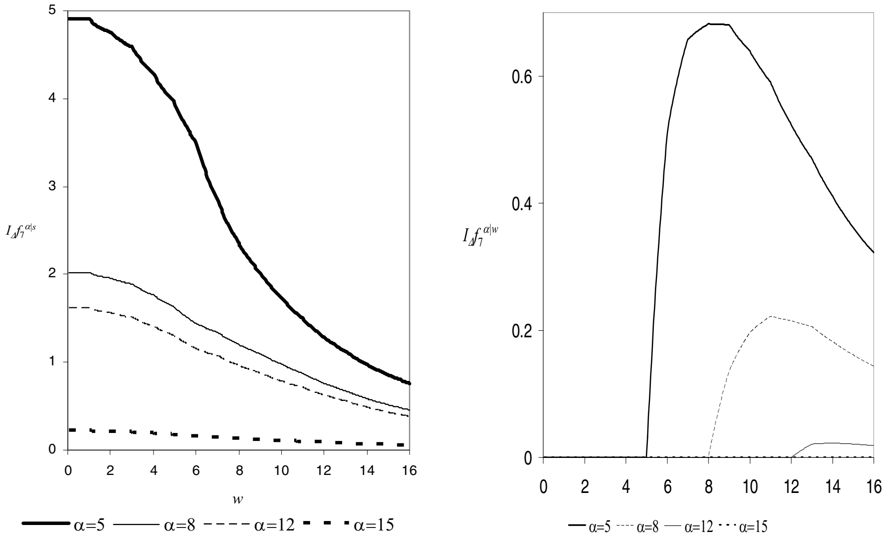


Figure 4.33. Behaviour of the elements' IMs. A: $I_\varepsilon f_j^{\alpha|s}(\alpha)$; B: $I_\varepsilon f_j^{\alpha|w}(\alpha)$



A **B**

Figure 4.34. Behaviour of the elements' IMs. A: $I_A f_7^{\alpha|s}(w)$; B: $I_A f_7^{\alpha|w}(w)$



A **B**

Figure 4.35. Behaviour of the elements' IMs. A: $I_\Delta f_7^{\alpha|s}(w)$; B: $I_\Delta f_7^{\alpha|w}(w)$

A similar behaviour is shown by $I_\Delta f_j^{2|s}(w)$. It can be seen that values of demand w exist for which the increase of the element performance above the threshold α causes the greatest relative reduction of the system performance deficiency (maxima of the curves $I_\Delta f_j^{2|w}(w)$ and $I_\Delta f_j^{2|s}(w)$ in Figure 4.32). It is also confirmed that the relative importance of different elements depends on the value of the demand (for example, element 2 is more important than element 5 for w<8 and less important for w>8).

The mean system performance ε does not depend on the demand. Figure 4.33 reports the indices $I_\varepsilon f_j^{\alpha|s}$ and $I_\varepsilon f_j^{\alpha|w}$ as functions of α. Note that while $I_\varepsilon f_j^{\alpha|w}(\alpha)$ are continuous functions, $I_\varepsilon f_j^{\alpha|s}(\alpha)$ are stepwise functions since $I_\varepsilon f_j^{\alpha|s}(\alpha_1) = I_\varepsilon f_j^{\alpha|s}(\alpha_2)$ for any α_1 and α_2 such that $g_{jh_{j\alpha}} \le \alpha_1 < \alpha_2 < g_{jh_{j\alpha}+1}$. Both functions are decreasing, which means that the higher levels of performance threshold α cause a less relative increase of the system's performance.

Note that both $I_\varepsilon f_j^{\alpha|s}$ and $I_\varepsilon f_j^{\alpha|w}$ take a value of zero (*i.e.* $O_j^{\le\alpha|w} = O_j^{\le\alpha|s} = O$) when the α level is above or equal to the maximum performance achievable by element j, g_{jk_j}.

Improvement of the performance of a certain element above a given threshold α may be achieved, either by increasing the probability of residing in states with performances larger than α (as indicated by the $I_O f_j^{\alpha|s}$ measures) or by increasing the performances of some states (as indicated by $I_O f_j^{\alpha|w}$ measures). Consider, for example, element 7, whose IMs for different threshold values α as functions of the demand w are given in Figures 4.34 and 4.35. Observe that $I_A f_7^{\alpha|w}(w) = 0$ when $w\le\alpha$ and $I_A f_7^{\alpha|w}(w) = 1$ when $w>\alpha$, since the logic of the system is such that its performance is not affected by limitations on the performance of element 7 if its threshold α is set to a value greater than the demand w, whereas the system fails completely if element 7 has a performance below the system's demand. Also, the $I_A f_7^{\alpha|s}(w)$ function does not depend on α, when α varies within the performance intervals 1-6, 6-10, 10-14, 14-18. The jumps in the step-functions $I_A f_7^{\alpha|s}(w)$ occur at values $w = g_{7h}$ and correspond to the restrictions to state h with $w > g_{7h}$. Functions $I_\Delta f_7^{\alpha|s}(w)$ and $I_\Delta f_7^{\alpha|w}(w)$ are continuous. When α increases, the relative reduction of the system's performance deficiency becomes smaller (because a smaller number of states are subject to restriction). Note that the demand w for which the greatest relative reduction of system performance deficiency is achieved (maximum of the function $I_\Delta f_7^{\alpha|w}(w)$) increases with the increase of α.

4.6 Universal Generating Function in Analysis of Continuum-state Systems

Some systems and elements exhibit continuous performance variation (for example, when their performance degrades due to gradual failures). In these cases, one can discern a continuum of different states. The structure functions $\phi(G_1,...,G_n)$ representing such continuous-state systems are mappings $[g_{1\min}, g_{1\max}] \times [g_{2\min}, g_{2\max}] \times ... \times [g_{n\min}, g_{n\max}] \rightarrow [g_{\min}, g_{\max}]$, where $[g_{j\min}, g_{j\max}]$ is the closed interval of performance variation of element j and $[g_{\min}, g_{\max}]$ is the closed interval of performance variation of the entire system. Such functions were introduced in [128-130] and are called the continuum structure functions.

The stochastic behaviour of continuous-state systems and elements may be specified through the complemented distribution functions [131]:

$$C_j(x) = \Pr\{G_j \geq x\}, \quad C(x) = \Pr\{\phi(G_1,...,G_n) \geq x\} \tag{4.107}$$

An example of such a function (cumulated curve) is presented in Figure 4.36.

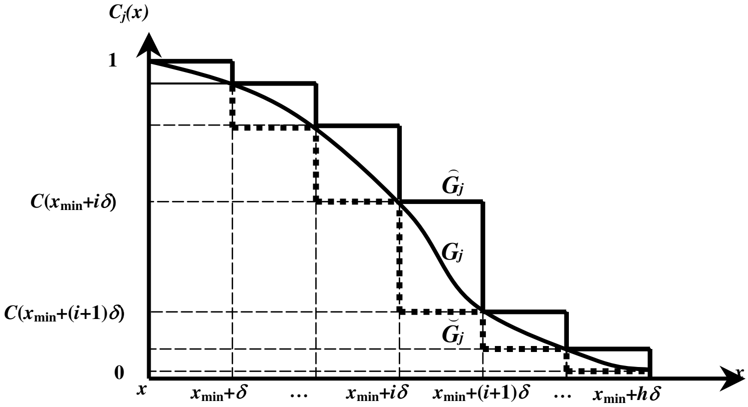


Figure 4.36. Complemented distribution functions for continuous and discrete variables

The method for estimating the boundary points for performance measures of continuum-state systems suggested by Lisnianski [132] uses the approximation of continuous performance distributions by discrete performance distributions. This method is based on the assumptions that the continuum structure functions are monotonic, *i.e.*

$$\phi(G_1,...,G_n) \leq \phi(\widehat{G}_1,...,\widehat{G}_n) \text{ if } G_j \leq \widehat{G}_j \text{ for } 1 \leq j \leq n \tag{4.108}$$

or

$$\phi(G_1,...,G_n) \geq \phi(\hat{G}_1,...,\hat{G}_n) \text{ if } G_j \leq \hat{G}_j \text{ for } 1 \leq j \leq n \tag{4.109}$$

and that the functions $C_j(x)$ for all of the elements are continuous (with possible jumps at the end points). These assumptions are relevant for many types of practical system.

In order to obtain the discrete approximation of the continuous performance distribution of element j, we divide the interval $[g_{j\min}, g_{j\max}]$ into h equal subintervals. The length of each subinterval is

$$\partial_j = \frac{g_{j\max} - g_{j\min}}{h} \tag{4.110}$$

In order to obtain the lower and upper bound approximations of distribution of performance G_j, we introduce discrete random variables $\breve{G}_j$ and $\hat{G}_j$ such that

$$\begin{aligned}&\Pr\{\breve{G}_j \geq g_{j\min} + i\partial_j\} = \Pr\{\hat{G}_j \geq g_{j\min} + i\partial_j\}\\&= \Pr\{G_j \geq g_{j\min} + i\partial_j\},\ \ 0 \leq i \leq h\end{aligned} \tag{4.111}$$

and

$$\begin{aligned}&\Pr\{\breve{G}_j \geq g_{j\min} + i\partial_j + x\} = \Pr\{G_j \geq g_{j\min} + (i+1)\partial_j\}\\&< \Pr\{G_j \geq g_{j\min} + i\partial_j + x\},\ \ 0 \leq i < h,\ \ 0 \leq x < \partial_j\end{aligned} \tag{4.112}$$

$$\begin{aligned}&\Pr\{\hat{G}_j \geq g_{j\min} + i\partial_j + x\} = \Pr\{G_j \geq g_{j\min} + i\partial_j\}\\&> \Pr\{G_j \geq g_{j\min} + i\partial_j + x\},\ \ 0 \leq i < h,\ \ 0 \leq x < \partial_j\end{aligned} \tag{4.113}$$

The complemented distribution functions of $\breve{G}_j$ and $\hat{G}_j$ are presented in Figure 4.36. Since for any variable X with a complemented distribution function $C(x)$ $\Pr\{x_1 \leq X < x_2\} = C(x_1) - C(x_2)$, we can obtain that for $\hat{G}_j$

$$\begin{aligned}&\Pr\{\hat{G}_j = g_{j\min}\} = 0\\&\Pr\{\hat{G}_j = g_{j\min} + i\partial_j\}\\&= C_j(g_{j\min} + (i-1)\partial_j) - C_j(g_{j\min} + i\partial_j),\ \ 1 \leq i < h\\&\Pr\{\hat{G}_j = g_{j\min} + h\partial_j\} = \Pr\{\hat{G}_j = g_{j\max}\}\\&= C_j(g_{j\min} + (h-1)\partial_j) = C_j(g_{j\max} - \partial_j)\\&\Pr\{\hat{G}_j = g_{j\min} + i\partial_j + x\} = 0,\ \ 0 \leq i < h,\ \ 0 < x < \partial_j\end{aligned} \tag{4.114}$$

and for $\breve{G}_j$

$$\Pr\{\breve{G}_j = g_{j\min} + i\partial_j\}$$
$$= C_j(g_{j\min} + i\partial_j) - C_j(g_{j\min} + (i+1)\partial_j),\ \ 0 \le i < h$$
$$\Pr\{\breve{G}_j = g_{j\min} + h\partial_j\} = \Pr\{\breve{G}_j = g_{j\max}\} = C_j(g_{j\max})$$
$$\Pr\{\breve{G}_j = g_{j\min} + i\partial_j + x\} = 0,\ \ 0 \le i < h,\ \ 0 < x < \partial_j \tag{4.115}$$

These expressions define the p.m.f. of the discrete variables $\breve{G}_j$ and $\hat{G}_j$.

Observe that the inequalities (4.112) and (4.113) guarantee that for any j $E(\breve{G}_j) \le E(G_j)$ and $E(\hat{G}_j) \ge E(G_j)$. Therefore, for any increasing monotonic function f:

$$E(f(\breve{G}_1,...,\breve{G}_n)) \le E(f(G_1,...,G_n)) \le E(f(\hat{G}_1,...,\hat{G}_n)) \tag{4.116}$$

and for any decreasing monotonic function v:

$$E(v(\hat{G}_1,...,\hat{G}_n)) \le E(v(G_1,...,G_n)) \le E(v(\breve{G}_1,...,\breve{G}_n)) \tag{4.117}$$

Since the system performance measures are defined as expected values of functions of performances of individual elements (see Section 3.3), the upper and lower bounds for these measures can be obtained by replacing the continuous-state elements with multi-state elements having discrete performances distributed as defined by Equations (4.114) and (4.115).

Having the complemented distribution functions $C_j(x)$ of system elements, one can determine the u-functions of the corresponding multi-state elements with discrete performance as

$$\hat{u}_j(z) = \sum_{i=1}^{h-1}\{C_j[g_{j\min} + (i-1)\partial_j] - C_j(g_{j\min} + i\partial_j)\}z^{g_{j\min}+i\partial_j}$$
$$+ C_j(g_{j\max} - \partial)z^{g_{j\max}} \tag{4.118}$$

$$\breve{u}_j(z) = \sum_{i=0}^{h-1}\{C_j(g_{j\min} + i\partial_j) - C_j[g_{j\min} + (i+1)\partial_j]\}z^{g_{j\min}+i\partial_j}$$
$$+ C_j(g_{j\max})z^{g_{j\max}} \tag{4.119}$$

Applying the reliability block diagram technique over u-functions $\hat{u}_j(z)$ one obtains the u-function $\hat{U}(z)$ representing the p.m.f. of the entire system consisting of elements with discrete performance distributions with p.m.f. (4.114). Applying this technique over u-functions $\breve{u}_j(z)$ one obtains the u-function $\breve{U}(z)$ representing the p.m.f. of the entire system consisting of elements with discrete performance distributions with p.m.f. (4.115). Having the u-functions $\hat{U}(z)$ and $\breve{U}(z)$ one can obtain the boundary points for the system performance measures as described in Section 3.3.

Example 4.24

Consider the series-parallel continuum-state system presented in Figure 4.37. Each element of the system can be either available or totally unavailable due to a catastrophic failure. If the element is available, then its performance rate varies continuously depending on the state of the element's operating environment. The performance rate of the unavailable element is zero.

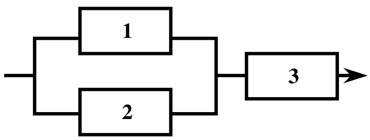


Figure 4.37. Structure of continuum-state system

Observe that if the element's availability is a_j and its complemented distribution function given that the element is available is $C_j^*(x)$, then the performance distribution of this element is defined by the complemented distribution function $C_j(x)$ that takes the form

$$C_j(x) = \begin{cases} 1, & x \le 0 \\ a_2, & 0 < x \le g_{\min} \\ a_2 C_j^*(x), & x > g_{\min} \end{cases}$$

The first element has the availability a_1 = 0.8 and exponentially distributed performance with mean $\mu_1 = 40$ and $g_{1\min}$ = 0 (the probability that $G_1 > g_{1\max}$ = 1000 is neglected). The second element has availability a_2 = 0.7 and uniformly distributed performance with $g_{2\min}$ = 30 and $g_{2\max}$ = 60. The third element has availability a_3 = 0.95 and normally distributed performance with mean $\mu_3 = 70$ and standard deviation $\sigma_3 = 10$ (the probabilities that $G_3 < g_{3\min}$ = 0 and that $G_3 > g_{3\max}$ = 1000 are neglected). In the state of failure, all the elements have performance zero. The system fails if its performance is less that the constant demand w = 20.

The complemented distribution functions of the element performances taking into account the element availabilities are

$$C_1(x) = \begin{cases} 1, & x \le 0 \\ a_1 e^{-x/\mu_1}, & x > 0 \end{cases}$$

$$C_2(x) = \begin{cases} 1, & x \le 0 \\ a_2, & 0 < x \le g_{\min} \\ a_2 (g_{\max} - x)/(g_{\max} - g_{\min}), & g_{\min} \le x \le g_{\max} \\ 0, & x > g_{\max} \end{cases}$$

$$C_3(x) = \begin{cases} 1, & x \le 0 \\ a_3 (1 - \frac{1}{\sigma_3 \sqrt{2\pi}} \int\limits_{-\infty}^{x} e^{-(t-\mu_3)^2/(2\sigma_3^2)} dt), & x > 0 \end{cases}$$

Considering the complemented distribution functions in the interval [0, 1000] and assigning $\partial = 1$, Lisnianski [132] has obtained for the system interpreted as a flow transmission MSS with flow dispersion $\widehat{\varepsilon} = \widehat{U}'(1) = 47.21$ and $\breve{\Delta}^- = 3.07$ when the element performance distributions are represented by u-functions $\widehat{u}_j(z)$, and $\breve{\varepsilon} = \breve{U}'(1) = 46.23$ and $\widehat{\Delta}^- = 3.19$ when the element performance distributions are represented by u-functions $\breve{u}_j(z)$. Using these boundary points, one can estimate the performance measures with maximal relative errors

100×(47.21–46.23)/46.23=2.1%

for mean performance and

100×(3.19–3.07)/3.07=3.9%

for expected performance deficiency.

For the system interpreted as a task processing MSS without work sharing, $\widehat{\varepsilon} = \widehat{U}'(1) = 24.23$ and $\breve{\Delta}^- = 3.32$ when the element performance distributions are represented by u-functions $\widehat{u}_j(z)$, and $\breve{\varepsilon} = \breve{U}'(1) = 23.83$ and $\widehat{\Delta}^- = 3.45$ when the element performance distributions are represented by u-functions $\breve{u}_j(z)$. This gives the estimations of the performance measures with maximal relative errors

$$100\times(24.23-23.83)/46.23=1.7\%$$

for mean performance and

$$100\times(3.45-3.32)/3.32=3.9\%$$

for expected performance deficiency.

The upper and lower boundary points for mean performance and expected unsupplied demand are presented in Figure 4.38 as functions of step ∂ for both types of systems. The decrease of step ∂ provides improvement in the accuracy of boundary points estimation. However, it considerably increases the computational burden, since the number of terms in the u-functions $\breve{u}_j(z)$ and $\hat{u}_j(z)$ is proportional to $1/\partial$.

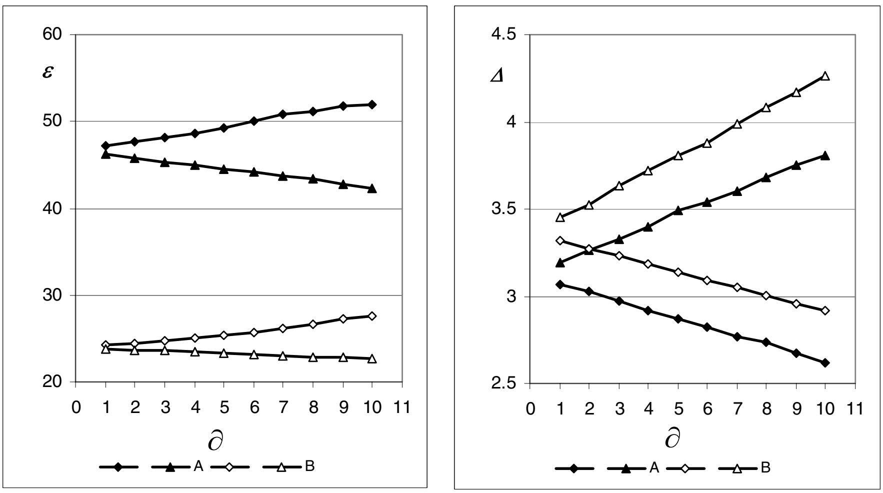


Figure 4.38. Boundary points for expected performance deficiency and mean performance (A: flow transmission system; B: task processing system)