7

Imperfect Preventive Maintenance

The maintenance of an operating unit after failure is costly, and sometimes, it requires a long time to repair failed units. It would be an important problem to determine when to maintain preventively the unit before it fails. However, it would be not wise to maintain the unit too often. From this viewpoint, commonly considered maintenance policies are preventive replacement for units with no repair as described in Chapters 3 through 5 and preventive maintenance for units with repair discussed in Chapter 6. It may be wise to maintain units to prevent failures when their failure rates increase with age.

The usual preventive maintenance (PM) of the unit is done before failure at a specified time T after its installation. The mean time to failure (MTTF), the availability, and the expected cost are derived as the reliability measures for maintained units. Optimum PM policies that maximize or minimize these measures have been summarized in Chapter 6. All models have assumed that *the unit after PM becomes as good as new.* Actually, this assumption might not be true. The unit after PM usually might be younger at PM, and occasionally, it might be worse than before PM because of faulty procedures, *e.g.,* wrong adjustments, bad parts, and damage done during PM. Generally, the improvement of the unit by PM would depend on the resources spent for PM.

It was first assumed in [1] that the inspection to detect failures may not be perfect. Similar models such that inspection, test, and detection of failures are uncertain were treated in [2, 3]. The imperfect PM where the unit after PM is not like new with a certain probability was considered, and the optimum PM policies that maximize the availability or minimize the expected cost were discussed in [4–7]. In addition, the PM policies with several reliability levels were presented in [8].

It is imperative to check a computer system and remove as many unit faults, failures, and degradations as possible, by providing fault-tolerant techniques. Imperfect maintenance for a computer system was first treated in [9]. The MTTF and availability were obtained in [10–12] in the case where although the system is usually renewed after PM, it sometimes remains unchanged. The imperfect test of intermittent faults incurred in digital systems was studied in [13].

Two imperfect PM models of the unit were considered [14, 15]: the age becomes x units of time younger at each PM and the failure rate is reduced in proportion to that before PM or to the PM cost. The improvement factor in failure rate after maintenance [16, 17] and the system degradation with time where the PM restores the hazard function to the same shape [18] were introduced. Furthermore, the PM policy that slows the degradation rate was considered in [19].

On the other hand, it was assumed in [20–22] that a failed unit becomes as good as new after repair with a certain probability, and some properties of its failure distribution were investigated. Similar imperfect repair models were generalized by [23–31]. Also, the stochastic properties of imperfect repair models with PM were derived in [32, 33]. Multivariate distributions and their probabilistic quantities of these models were derived in [34–36]. The improvement factors of imperfect PM and repair were statistically estimated in [37–40]. The PM was classified into four terms of its effect [41]: perfect maintenance, minimal maintenance, imperfect maintenance, and worse maintenance. Some chapters [42–44] of recently published books summarized many results of imperfect maintenance.

This chapter summarizes our results of imperfect maintenance models that could be applied to actual systems and would be helpful for further studies in research fields. It is assumed in Section 7.1 that the operating unit is replaced at failure or is maintained preventively at time T. Then, the unit after PM has the same failure rate as before PM with a certain probability. The expected cost rate is obtained and an optimum PM policy that minimizes it is discussed analytically [5]. Section 7.2 considers several imperfect PM models with minimal repair at failures: (1) the unit after PM becomes as good as new with a certain probability; (2) the age becomes younger at each PM; and (3) the age or failure rate after PM reduces in proportion to that before PM. The expected cost rates of four models are obtained and optimum policies for each model are derived [15].

Section 7.3 considers a modified inspection model where the unit after inspection becomes like new with a certain probability. The MTTF, the expected number of inspections, and the total expected cost are obtained [45, 46]. Furthermore, an imperfect inspection model with two human errors is proposed. Section 7.4 considers the imperfect PM of a computer system that is maintained at periodic times [12]. The MTTF and the availability are obtained, and optimum policies that maximize them are discussed. Finally, Section 7.5 suggests a sequential imperfect PM model where the PM is done at successive times and the age or failure rate reduces in proportion to that before PM. The expected cost rates are obtained and optimum policies that minimize them are discussed [47]. It is shown in numerical examples that optimum intervals are uniquely determined when the failure time has a Weibull distribution.

The following notation is used throughout this chapter. A unit begins to operate at time 0, and has the failure distribution $F(t)$ $(t \geq 0)$ with finite mean μ and its density function $f(t) \equiv \mathrm{d}F(t)/\mathrm{d}t$. Furthermore, the failure rate $h(t) \equiv f(t)/\overline{F}(t)$ and the cumulative hazard function $H(t) \equiv \int_0^t h(u)\mathrm{d}u$, where $\overline{\Phi} \equiv 1 - \Phi$.

7.1 Imperfect Maintenance Policy

All models have assumed until now that a unit after any PM becomes as good as new. Actually, this assumption might not be true. It sometimes occurs that a unit after PM is worse than before PM because of faulty procedures, *e.g.,* wrong adjustments, bad parts, and damage done during PM. To include this, it is assumed that the failure rate after PM is the same as before PM with a certain probability, and a unit is not like new. This section derives the expected cost rate of the model with imperfect PM, and discusses an optimum policy that minimizes it.

Consider the imperfect PM policy for a one-unit system that should operate for an infinite time span.

1. The operating unit is repaired at failure or is maintained preventively at time T $(0 < T \leq \infty)$, whichever occurs first, after its installation or previous PM.
2. The unit after repair becomes as good as new.
3. The unit after PM has the same failure rate as it had before PM with probability p $(0 \leq p < 1)$ and becomes as good as new with probability $q \equiv 1 - p$.
4. Cost of each repair is c_1 and cost of each PM is c_2.
5. The repair and PM times are negligible.

Consider one cycle from time $t = 0$ to the time that the unit becomes as good as new by either repair or perfect PM. Then, the expected cost of one cycle is given by the sum of the repair cost and PM cost;

$$\widehat{C}(T;p) = c_1 \Pr\{\text{unit is repaired at failure}\} \\ + c_2 \Pr\{\text{expected number of PMs per one cycle}\}. \tag{7.1}$$

The probability that the unit is repaired at failure is

$$\sum_{j=1}^{\infty} p^{j-1} \int_{(j-1)T}^{jT} \mathrm{d}F(t) = 1 - q \sum_{j=1}^{\infty} p^{j-1} \overline{F}(jT) \tag{7.2}$$

and the expected number of PMs including perfect PM per one cycle is

$$\sum_{j=1}^{\infty} (j-1) p^{j-1} \int_{(j-1)T}^{jT} \mathrm{d}F(t) + q \sum_{j=1}^{\infty} j p^{j-1} \overline{F}(jT) = \sum_{j=1}^{\infty} p^{j-1} \overline{F}(jT). \tag{7.3}$$

Furthermore, the mean time of one cycle is

$$\sum_{j=1}^{\infty} p^{j-1} \int_{(j-1)T}^{jT} t \, \mathrm{d}F(t) + q \sum_{j=1}^{\infty} p^{j-1} (jT) \overline{F}(jT) = \sum_{j=1}^{\infty} p^{j-1} \int_{(j-1)T}^{jT} \overline{F}(t) \, \mathrm{d}t. \tag{7.4}$$

Thus, substituting (7.2) and (7.3) into (7.1), and dividing it by (7.4), the expected cost rate is, from (3.3),

$$C(T;p) = \frac{c_1 \left[1 - q \sum_{j=1}^{\infty} p^{j-1} \overline{F}(jT)\right] + c_2 \sum_{j=1}^{\infty} p^{j-1} \overline{F}(jT)}{\sum_{j=1}^{\infty} p^{j-1} \int_{(j-1)T}^{jT} \overline{F}(t) \, \mathrm{d}t}. \tag{7.5}$$

We clearly have

$$C(0;p) \equiv \lim_{T \to 0} C(T;p) = \infty, \qquad C(\infty;p) \equiv \lim_{T \to \infty} C(T;p) = \frac{c_1}{\mu} \tag{7.6}$$

which is the expected cost for the case where no PM is done and the unit is repaired only at failure.

We seek an optimum PM time T^* that minimizes $C(T;p)$. Let

$$H(t;p) \equiv \frac{\sum_{j=1}^{\infty} p^{j-1} j f(jt)}{\sum_{j=1}^{\infty} p^{j-1} j \overline{F}(jt)}. \tag{7.7}$$

Then, differentiating $C(T;p)$ with respect to T and setting it equal to zero,

$$H(T;p) \sum_{j=1}^{\infty} p^{j-1} \int_{(j-1)T}^{jT} \overline{F}(t) \, \mathrm{d}t - q \sum_{j=1}^{\infty} p^{j-1} F(jT) = \frac{c_2}{c_1 q - c_2}, \tag{7.8}$$

where $c_1 q - c_2 \neq 0$. Denoting the left-hand side of (7.8) by $Q(T;p)$, we easily have that if $H(t;p)$ is strictly increasing then $Q(T;p)$ is also strictly increasing from 0 and

$$Q(\infty;p) \equiv \lim_{T \to \infty} Q(T;p) = \mu H(\infty;p) - 1. \tag{7.9}$$

It is assumed that $H(t;p)$ is strictly increasing in t for any p. Then, we have the following optimum policy.

(i) If $c_1 q > c_2$ and $H(\infty;p) > c_1 q / [\mu(c_1 q - c_2)]$ then there exists a finite and unique T^* that satisfies (7.8), and the resulting cost rate is

$$C(T^*;p) = \left(c_1 - \frac{c_2}{q}\right) H(T^*;p). \tag{7.10}$$

(ii) If $c_1 q > c_2$ and $H(\infty;p) \leq c_1 q / [\mu(c_1 q - c_2)]$, or $c_1 q \leq c_2$ then $T^* = \infty$; *i.e.,* no PM should be done, and the expected cost is given in (7.6).

Table 7.1. Optimum PM time T^* and expected cost rate $C(T^*;p)$ for p when $c_1 = 5$ and $c_2 = 1$

p	T^*	$C(T^*;p)$
0.00	1.31	2.27
0.01	1.32	2.27
0.05	1.36	2.30
0.10	1.43	2.34
0.15	1.52	2.37
0.20	1.64	2.40
0.25	1.80	2.43
0.30	2.02	2.45
0.35	2.33	2.47
0.40	2.79	2.49

When $p = 0$, *i.e.*, the PM is perfect, the model corresponds to a standard age replacement policy, and the above results agree with those of Chapter 3.

Example 7.1. Suppose that $F(t)$ is a gamma distribution with order 2; *i.e.*, $F(t) = 1 - (1+t)\mathrm{e}^{-t}$. Then, $H(t;p)$ in (7.7) is

$$H(t;p) = \frac{t(1+p\mathrm{e}^{-t})}{1 - p\mathrm{e}^{-t} + t(1+p\mathrm{e}^{-t})}$$

which is strictly increasing from 0 to 1. Thus, if $c_1 q > 2c_2$ then there exists a finite and unique T^* that satisfies (7.8), and otherwise, $T^* = \infty$.

Table 7.1 gives the optimum PM time T^* and the expected cost rate $C(T^*;p)$ for $p = 0.0 \sim 0.4$ when $c_1 = 5$ and $c_2 = 1$. Both T^* and $C(T^*;p)$ are increasing when the probability p of imperfect PM is large. The reason is that it is better to repair a failed unit than to perform PM when p is large. ∎

7.2 Preventive Maintenance with Minimal Repair

Earlier results of optimum PM policies have been summarized in Chapter 6. However, almost all models have assumed that a unit becomes as good as new after any PM. In practice, this assumption often might not be true. A unit after PM usually might be younger at PM, and occasionally, it might become worse than before PM because of faulty procedures.

This section considers the following four imperfect PM models with minimal repair at failures.

(1) The unit after PM has the same failure rate as before PM or becomes as good as new with a certain probability q.
(2) The age becomes x units of time younger at each PM.
(3) The age or failure rate after PM reduces to at or $bh(t)$ when it was t or $h(t)$ before PM, respectively.

(4) The age or failure rate is reduced to the original one at the beginning of all PMs in proportion to the PM cost.

For each model, we obtain the expected cost rates and discuss optimum PM policies that minimize them. A numerical example is finally given when the failure time has a Weibull distribution.

(1) Model A – Probability

Consider the periodic PM policy for a one-unit system that should operate for an infinite time span.

1. The operating unit is maintained preventively at times kT $(k = 1, 2, \dots)$, and undergoes only minimal repair at failures between PMs (see Chapter 4).
2. The failure rate $h(t)$ remains undisturbed by minimal repair.
3. The unit after PM has the same failure rate as it had before PM with probability p $(0 \le p < 1)$ and becomes as good as new with probability $q \equiv 1 - p$.
4. Cost of each minimal repair is c_1 and cost of each PM is c_2.
5. The minimal repair and PM times are negligible.
6. The failure rate $h(t)$ is strictly increasing.

Consider one cycle from time $t = 0$ to the time that the unit becomes as good as new by perfect PM. Then, the total expected cost of one cycle is

$$\sum_{j=1}^{\infty} p^{j-1} q \left[c_1 \int_0^{jT} h(t)\,\mathrm{d}t + j c_2 \right] = c_1 q \sum_{j=1}^{\infty} p^{j-1} \int_0^{jT} h(t)\,\mathrm{d}t + \frac{c_2}{q} \tag{7.11}$$

and its mean time is

$$\sum_{j=1}^{\infty} jTp^{j-1} q = \frac{T}{q}. \tag{7.12}$$

Thus, dividing (7.11) by (7.12) and arranging them, the expected cost rate is

$$C_A(T;p) = \frac{1}{T} \left[c_1 q^2 \sum_{j=1}^{\infty} p^{j-1} \int_0^{jT} h(t)\,\mathrm{d}t + c_2 \right]. \tag{7.13}$$

We seek an optimum PM time T^* that minimizes $C_A(T;p)$. Differentiating $C_A(T;p)$ with respect to T and setting it equal to zero,

$$\sum_{j=1}^{\infty} p^{j-1} \int_0^{jT} t\,\mathrm{d}h(t) = \frac{c_2}{c_1 q^2} \tag{7.14}$$

whose left-hand side is strictly increasing from 0 to $\int_0^{\infty} t\mathrm{d}h(t)$, which may be possibly infinity. It is clearly seen that $\int_0^{\infty} t\,\mathrm{d}h(t) \to \infty$ as $h(t) \to \infty$.

Therefore, we have the following optimum policy.

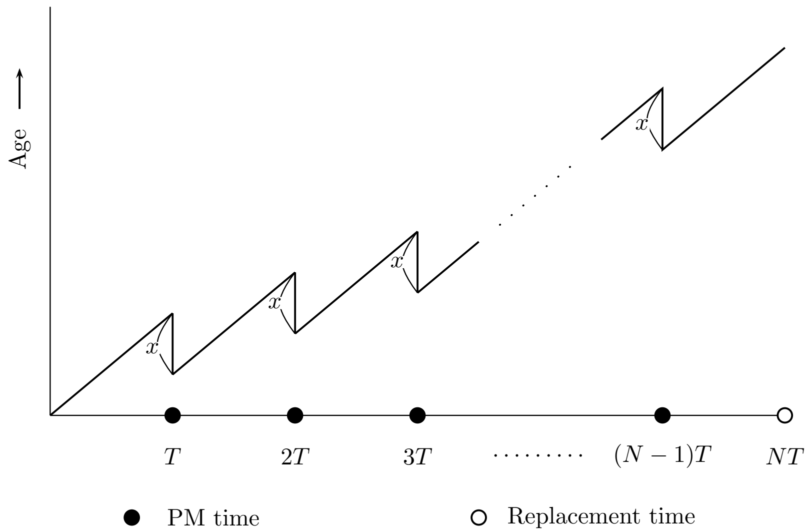

Fig. 7.1. Process of Model B

(i) If $\int_0^\infty t\,\mathrm{d}h(t) > c_2/(c_1q^2)$ then there exists a finite and unique T^* that satisfies (7.14), and the resulting cost rate is

$$C_A(T^*;p) = c_1q^2\sum_{j=1}^{\infty}p^{j-1}jh(jT^*). \tag{7.15}$$

(ii) If $\int_0^\infty t\,\mathrm{d}h(t) \le c_2/(c_1q^2)$ then $T^* = \infty$, and the expected cost rate is

$$C_A(\infty;p) \equiv \lim_{T\to\infty} C_A(T;p) = c_1q^2h(\infty).$$

(2) Model B – Age

The process in Model B is shown in Figure 7.1.

3. The age becomes x units younger at each PM, where x $(0 \le x \le T)$ is constant and previously specified. Furthermore, the unit is replaced if it operates for the time interval NT $(N = 1, 2, \dots, \infty)$.
4. Cost of each minimal repair is c_1, cost of each PM is c_2, and cost of replacement at time NT is c_3 with $c_3 > c_2$.

1, 2, 5, 6. Same as the assumptions of Model A.

The expected cost rate is easily given by

$$C_B(N;T,x) = \frac{1}{NT}\left[c_1\sum_{j=0}^{N-1}\int_{j(T-x)}^{T+j(T-x)}h(t)\,\mathrm{d}t + (N-1)c_2 + c_3\right]$$
$$(N = 1, 2, \dots). \tag{7.16}$$

It is trivial that the expected cost rate is decreasing in x because the failure rate $h(t)$ is increasing.

We seek an optimum replacement number N^* $(1 \le N^* \le \infty)$ that minimizes $C_B(N;T,x)$ for specified $T > 0$ and x. From the inequality $C_B(N+1;T,x) \ge C_B(N;T,x)$, we have

$$L(N;T,x) \ge \frac{(c_3 - c_2)}{c_1} \qquad (N = 1,2,\dots), \tag{7.17}$$

where

$$\begin{aligned} L(N;T,x) &\equiv N \int_{N(T-x)}^{T+N(T-x)} h(t)\,\mathrm{d}t - \sum_{j=0}^{N-1} \int_{j(T-x)}^{T+j(T-x)} h(t)\,\mathrm{d}t \\ &= \sum_{j=0}^{N-1} \int_0^T \{h[t+N(T-x)] - h[t+j(T-x)]\}\,\mathrm{d}t \qquad (N = 1,2,\dots). \end{aligned}$$

In addition, we have

$$\begin{aligned} &L(N+1;T,x) - L(N;T,x) \\ &\quad = (N+1) \int_0^T \{h[t+(N+1)(T-x)] - h[t+N(T-x)]\}\,\mathrm{d}t > 0. \end{aligned}$$

Therefore, we have the following optimum policy.

(i) If $L(\infty;T,x) \equiv \lim_{N\to\infty} L(N;T,x) > (c_3 - c_2)/c_1$ then there exists a finite and unique minimum N^* that satisfies (7.17).

(ii) If $L(\infty;T,x) \le (c_3 - c_2)/c_1$ then $N^* = \infty$, and the expected cost rate is

$$C_B(\infty;T,x) \equiv \lim_{N\to\infty} C_B(N;T,x) = c_1 h(\infty) + \frac{c_2}{T}.$$

We clearly have $N^* < \infty$ if $h(t) \to \infty$ as $t \to \infty$.

(3) Model C – Rate

It is assumed that:

3. The age after PM reduces to at $(0 < a \le 1)$ when it was t before PM; *i.e.*, the age becomes $t(1-a)$ units of time younger at each PM. Furthermore, the unit is replaced if it operates for NT.

1, 2, 4, 5, 6. Same as the assumptions of Model B.

The expected cost rate is

$$C_C(N;T,a) = \frac{1}{NT}\left[c_1 \sum_{j=0}^{N-1} \int_{A_j T}^{(A_j+1)T} h(t)\,\mathrm{d}t + (N-1)c_2 + c_3\right] \qquad (N = 1,2,\dots), \tag{7.18}$$

where $A_j \equiv a + a^2 + \cdots + a^j$ $(j = 1, 2, \dots)$ and $A_0 \equiv 0$.

We can have similar results to Model B. From the inequality $C_C(N+1; T, a) \geq C_C(N; T, a)$,

$$L(N; T, a) \geq \frac{c_3 - c_2}{c_1} \qquad (N = 1, 2, \dots), \tag{7.19}$$

where

$$L(N; T, a) \equiv N \int_{A_N T}^{(A_N+1)T} h(t)\,\mathrm{d}t - \sum_{j=0}^{N-1} \int_{A_j T}^{(A_j+1)T} h(t)\,\mathrm{d}t \qquad (N = 1, 2, \dots)$$

which is strictly increasing in N.

Therefore, we have the following optimum policy.

(i) If $L(\infty; T, a) > (c_3 - c_2)/c_1$ then there exists a finite and unique minimum N^* that satisfies (7.19).
(ii) If $L(\infty; T, a) \leq (c_3 - c_2)/c_1$ then $N^* = \infty$.

If the age after the jth PM reduces to $a_j t$ when it was t before the jth PM, we have the expected cost $C_c(N; T, a_j)$ by denoting that $A_j \equiv a_1 + a_1 a_2 + \cdots + a_1 a_2 \dots a_j$ $(j = 1, 2, \dots)$ and $A_0 \equiv 0$.

Next, it is assumed that:

3. The failure rate after PM reduces to $bh(t)$ $(0 < b \leq 1)$ when it was $h(t)$ before PM.

The expected cost rate is

$$C_C(N; T, b) = \frac{1}{NT} \left[c_1 \sum_{j=0}^{N-1} b^j \int_{jT}^{(j+1)T} h(t)\,\mathrm{d}t + (N-1)c_2 + c_3 \right] \qquad (N = 1, 2, \dots) \tag{7.20}$$

and (7.19) is rewritten as

$$L(N; T, b) \geq \frac{c_3 - c_2}{c_1} \qquad (N = 1, 2, \dots), \tag{7.21}$$

where

$$L(N; T, b) \equiv N b^N \int_{NT}^{(N+1)T} h(t)\,\mathrm{d}t - \sum_{j=0}^{N-1} b^j \int_{jT}^{(j+1)T} h(t)\,\mathrm{d}t \quad (N = 1, 2, \dots)$$

which is strictly increasing in N.

If the failure rate becomes $h_j(t)$ for $jT \leq t < (j+1)T$ between the jth and $(j+1)$th PMs, the expected cost rate in (7.20) is written in the general form

$$C_c(N; T) = \frac{1}{NT} \left[c_1 \sum_{j=0}^{N-1} \int_{jT}^{(j+1)T} h_j(t)\,\mathrm{d}t + (N-1)c_2 + c_3 \right].$$

(4) Model D – Cost

Models B and C have assumed that the age reduced by PM is independent of PM cost. In this model, it is assumed that:

3. The age or failure rate after PM is reduced in proportion to PM cost c_2.
4. Cost of each minimal repair is c_1 and cost of each PM is c_2. Furthermore, the cost c_0 with $c_0 \geq c_2$ is the initial cost of the unit.

1, 2, 5, 6. Same as the assumptions of Model A.

First, suppose that the age after PM reduces to $[1-(c_2/c_0)](x+T)$ at each PM when it was $x+T$ immediately before PM. If the operation of the unit enters into the steady-state then we have the equation

$$\left(1-\frac{c_2}{c_0}\right)(x+T)=x, \qquad i.e., \qquad x=\left(\frac{c_0}{c_2}-1\right)T. \tag{7.22}$$

Thus, the expected cost rate is

$$\begin{aligned} C_D(T;c_0) &= \frac{1}{T}\left[c_1\int_0^T h(t+x)\,\mathrm{d}t+c_2\right] \\ &= \frac{1}{T}\left[c_1\int_{[(c_0/c_2)-1]T}^{(c_0/c_2)T} h(t)\,\mathrm{d}t+c_2\right]. \end{aligned} \tag{7.23}$$

Differentiating $C_D(T;c_0)$ with respect to T and setting it equal to zero,

$$\int_{[(c_0/c_2)-1]T}^{(c_0/c_2)T} t\,\mathrm{d}h(t)=\frac{c_2}{c_1}. \tag{7.24}$$

Next, suppose that the failure rate after PM reduces to $[1-(c_2/c_0)]h(x+T)$ at each PM where it was $h(x+T)$ before PM. In the steady-state, we have

$$\left(1-\frac{c_2}{c_0}\right)h(x+T)=h(x) \tag{7.25}$$

and the expected cost rate is

$$\widetilde{C}_D(T;c_0)=\frac{1}{T}\left[c_1\int_0^T h(t+x)\,\mathrm{d}t+c_2\right]. \tag{7.26}$$

Thus, the age after PM is computed from (7.25), and hence, an optimum PM time T^* is computed by substituting x into (7.26) and changing T to minimize it.

We have considered four imperfect PM models and have obtained the expected cost rates. It is noted that all models are identical and agree with the standard model in Section 4.2 when $p=0$ in Model A, $N=1$ in Models B and C, and $c_0=c_2$ in Model D.

Example 7.2. We finally consider an example when the failure time has a Weibull distribution and show how to determine optimum PM times. When $F(t) = 1 - \exp(-\lambda t^m)$ $(\lambda > 0, m > 1)$, we have the following results for each model.

(1) Model A

The expected cost rate is, from (7.13),

$$C_A(T;p) = \frac{1}{T}\left[c_1 q\lambda T^m g(m) + c_2\right],$$

where $g(m) \equiv q\sum_{j=1}^{\infty} p^{j-1} j^m$ which represents the mth moment of the geometric distribution with parameter p. The optimum PM time is, from (7.14),

$$T^* = \left[\frac{c_2}{c_1 q\lambda(m-1)g(m)}\right]^{1/m}.$$

(2) Model B

The expected cost rate is, from (7.16),

$$C_B(N;T,x) = \frac{1}{NT}\left[c_1\lambda\sum_{j=0}^{N-1}\{[T+j(T-x)]^m - [j(T-x)]^m\} + (N-1)c_2 + c_3\right]$$

and from (7.17),

$$\sum_{j=0}^{N-1}\{[T+N(T-x)]^m - [T+j(T-x)]^m - [N(T-x)]^m + [j(T-x)]^m\} \geq \frac{c_3 - c_2}{\lambda c_1}$$

whose left-hand side is strictly increasing in N to ∞ for $0 \leq x < T$. Thus, there exists a finite and unique minimum N^* $(1 \leq N^* < \infty)$.

(3) Model C

The expected cost rate is, from (7.18),

$$C_C(N;T,a) = \frac{1}{NT}\left[c_1\lambda T^m\sum_{j=0}^{N-1}[(A_j+1)^m - (A_j)^m] + (N-1)c_2 + c_3\right]$$

and from (7.19),

$$T^m \sum_{j=0}^{N-1} [(A_N+1)^m - (A_j+1)^m - (A_N)^m + (A_j)^m] \geq \frac{c_3 - c_2}{\lambda c_1}$$

whose left-hand side is strictly increasing in N to ∞ because $m > 1$. Thus, there exists a finite and unique minimum N^* $(1 \leq N^* < \infty)$. Furthermore, the left-hand side of the above equation is increasing in T for a fixed N and m, and hence, the optimum N^* is a decreasing function of T.

(4) Model D

The expected cost rate is, from (7.23),

$$C_D(T; c_0) = \frac{1}{T}\left\{c_1 \lambda T^m \left[\left(\frac{c_0}{c_2}\right)^m - \left(\frac{c_0}{c_2} - 1\right)^m\right] + c_2\right\}$$

and the optimum PM time is, from (7.24),

$$T^* = \left[\frac{c_2}{c_1 \lambda (m-1)\left\{(c_0/c_2)^m - [(c_0/c_2) - 1]^m\right\}}\right]^{1/m}.$$

Similarly, the expected cost rate in (7.26) is

$$\widetilde{C}_D(T; c_0) = \frac{1}{T}\{c_1 \lambda T^m [D^m - (D-1)^m] + c_2\}$$

and hence, the optimum PM time is

$$T^* = \left\{\frac{c_2}{c_1 \lambda (m-1)[D^m - (D-1)^m]}\right\}^{1/m},$$

where

$$D \equiv \frac{1}{1 - [1 - (c_2/c_0)]^{1/(m-1)}}. \quad \blacksquare$$

7.3 Inspection with Preventive Maintenance

In this section, we check a unit periodically to see whether it is good, and at the same time, provide preventive maintenance. For example, we test a unit, and if needed, we make the overhaul and the repair or replacement of bad parts. This policy could actually be applied to the models of production machines, standby units, and preventive medical checks for diseases [23]. The standard inspection policy is explained in detail in Chapter 8.

We consider a modified inspection model in which the unit after inspection has the same age as before with probability p and becomes as good as new with probability q. Then, we obtain the following reliability quantities: (1) the mean time to failure and (2) the expected number of inspections until failure detection. When the failure rate is increasing, we investigate some properties of these quantities. Furthermore, we derive the total expected cost and the expected cost rate until failure detection. Optimum inspection times that minimize the expected costs are given numerically where the failure time has a Weibull distribution. Moreover, we propose two extended cases where the age becomes younger at each inspection; *i.e.,* the age becomes x units of time younger at each inspection and the age after inspection reduces to at when it was t before inspection. Finally, we consider two types of human error at inspection and obtain the total expected cost.

7.3.1 Imperfect Inspection

Consider the periodic inspection policy with PM for a one-unit system that should operate for an infinite time span.

1. The operating unit is inspected and maintained preventively at times kT $(k = 1, 2, \dots)$ $(0 < T < \infty)$.
2. The failed unit is detected only through inspection.
3. The unit after inspection has the same failure rate as it had before inspection with probability p $(0 \le p \le 1)$ and becomes as good as new with probability $q \equiv 1 - p$.
4. Cost of each inspection is c_1 and cost of time elapsed between a failure and its detection per unit of time is c_2.
5. Inspection and PM times are negligible.

Let $l(T; p)$ be the mean time to failure of a unit. Then, we can form the renewal-type equation:

$$l(T; p) = \sum_{j=1}^{\infty} \left\{ p^{j-1} \int_{(j-1)T}^{jT} t \, \mathrm{d}F(t) + p^{j-1} q \overline{F}(jT)[jT + l(T; p)] \right\}. \qquad (7.27)$$

The first term in the bracket on the right-hand side is the mean time until it fails between $(j-1)$th and jth inspections, and the second term is the mean time until it becomes new at the jth inspection, and after that, it fails. By solving (7.27) and arranging it,

$$l(T; p) = \frac{\sum_{j=0}^{\infty} p^j \int_{jT}^{(j+1)T} \overline{F}(t) \, \mathrm{d}t}{\sum_{j=0}^{\infty} p^j \{\overline{F}(jT) - \overline{F}[(j+1)T]\}}. \qquad (7.28)$$

In particular, when $p = 0$, *i.e.,* the unit always becomes as good as new at each inspection,

$$l(T;0) = \frac{1}{F(T)} \int_0^T \overline{F}(t)\,\mathrm{d}t \tag{7.29}$$

which agrees with (1.6) in Chapter 1. When $p = 1$, *i.e.,* the unit after inspection has the same failure rate as before inspection, $l(T;1) = \mu$ which is the mean failure time of the unit.

Next, let $M(T;p)$ be the expected number of inspections until failure detection. Then, by a similar method to that of obtaining (7.27),

$$M(T;p) = \sum_{j=1}^{\infty} \Big[p^{j-1} j \{\overline{F}[(j-1)T] - \overline{F}(jT)\} + p^{j-1} q \overline{F}(jT)[j + M(T;p)] \Big];$$

i.e.,

$$M(T;p) = \frac{\sum_{j=0}^{\infty} p^j \overline{F}(jT)}{\sum_{j=0}^{\infty} p^j \left\{\overline{F}(jT) - \overline{F}[(j+1)T]\right\}}. \tag{7.30}$$

In particular,

$$M(T;0) = \frac{1}{F(T)}, \qquad M(T;1) = \sum_{j=0}^{\infty} \overline{F}(jT). \tag{7.31}$$

It is easy to see that

$$T\overline{F}[(j+1)T] \le \int_{jT}^{(j+1)T} \overline{F}(t)\,\mathrm{d}t \le T\overline{F}(jT)$$

because $\overline{F}(t)$ is a nonincreasing function of t. Thus, from (7.28) and (7.30),

$$T[M(T;p) - 1] \le l(T;p) \le TM(T;p). \tag{7.32}$$

Furthermore, it has been proved in [16] that if the failure rate is increasing then both $l(T;p)$ and $M(T;p)$ are decreasing functions of p for a fixed T. From this result, we have the inequalities

$$\mu \le l(T;p) \le \frac{1}{F(T)} \int_0^T \overline{F}(t)\,\mathrm{d}t \tag{7.33}$$

$$\sum_{j=0}^{\infty} \overline{F}(jT) \le M(T;p) \le \frac{1}{F(T)}, \tag{7.34}$$

where all equalities hold when F is exponential.

The total expected cost until failure detection is (see Equation (8.1) in Chapter 8),

Table 7.2. Optimum inspection time T^* for p and m when $c_1 = 10$ and $c_2 = 1$

p	m				
	1.0	1.5	2.0	2.5	3.0
0.00	97	171	236	289	330
0.01	97	170	234	286	328
0.05	97	168	228	275	314
0.10	97	164	219	262	295
0.20	97	158	204	237	260
0.30	97	151	189	214	231
0.40	97	144	175	195	207

$$C(T;p) = \sum_{j=1}^{\infty} \left\{ p^{j-1} \int_{(j-1)T}^{jT} [c_1 j + c_2(jT - t)]\,\mathrm{d}F(t) + p^{j-1} q \overline{F}(jT)[c_1 j + C(T;p)] \right\}.$$

Solving the above renewal equation with respect to $C(T;p)$, we have

$$C(T;p) = \frac{(c_1 + c_2 T)\sum_{j=0}^{\infty} p^j \overline{F}(jT) - c_2 \sum_{j=0}^{\infty} p^j \int_{jT}^{(j+1)T} \overline{F}(t)\,\mathrm{d}t}{\sum_{j=0}^{\infty} p^j \{\overline{F}(jT) - \overline{F}[(j+1)T]\}}$$
$$= (c_1 + c_2 T) M(T;p) - c_2 l(T;p). \tag{7.35}$$

It is easy to see that $\lim_{T \to 0} C(T;p) = \lim_{T \to \infty} C(T;p) = \infty$. Thus, there exists a finite and positive T^* that minimizes the expected cost $C(T;p)$. Also, from the relation of (7.32), we have

$$\frac{c_1 l(T;p)}{T} \leq C(T;p) \leq c_1 M(T;p) + c_2 T. \tag{7.36}$$

Example 7.3. We give a numerical example when the failure time has a Weibull distribution with shape parameter m ($m \geq 1$). Suppose that $\overline{F}(t) = \exp[-(\lambda t)^m]$, $1/\lambda = 500$, $c_1 = 10$, and $c_2 = 1$. Table 7.2 presents the optimum inspection time T^* that minimizes the expected cost $C(T;p)$ for several values of p and m. It is noted that optimum times T^* are independent of p for the particular case of $m = 1$. Except for $m = 1$, they are small when p is large. The reason is that when the failure rate increases with age, it is better to inspect early for large p. ∎

7.3.2 Other Inspection Models

Consider two inspection models with PM where the age becomes younger at each inspection. It is assumed that the age becomes x ($0 \leq x \leq T$) units of time younger at each inspection. Then, the probability that the unit does not fail until time t is

$$\overline{S}(t;T,x) = \overline{\lambda}[k(T-x);t-kT]\prod_{j=0}^{k-1}\overline{\lambda}[j(T-x);T] \quad \text{for } kT \le t < (k+1)T, \tag{7.37}$$

where $\lambda(t;x) \equiv [F(t+x)-F(t)]/\overline{F}(t)$ is the probability that the unit with age t fails during $(t, t+x]$. Thus, the mean time to failure is

$$\begin{aligned} l(T;x) &= \sum_{k=0}^{\infty}\int_{kT}^{(k+1)T}\overline{S}(t;T,x)\,\mathrm{d}t \\ &= \sum_{k=0}^{\infty}\left\{\prod_{j=0}^{k-1}\overline{\lambda}[j(T-x);T]\right\}\frac{\int_{k(T-x)}^{k(T-x)+T}\overline{F}(t)\,\mathrm{d}t}{\overline{F}[k(T-x)]}, \end{aligned} \tag{7.38}$$

where $\prod_0^{-1} \equiv 1$, and the expected number of inspections until failure detection is

$$\begin{aligned} M(T;x) &= \sum_{k=0}^{\infty} k\lambda[k(T-x);T]\prod_{j=0}^{k-1}\overline{\lambda}[j(T-x);T] \\ &= \sum_{k=0}^{\infty}\prod_{j=0}^{k}\overline{\lambda}[j(T-x);T]. \end{aligned} \tag{7.39}$$

Next, it is assumed that the age after inspection reduces to at $(0 \le a \le 1)$ where it was t before inspection. Then, in similar ways to those of obtaining (7.38) and (7.39),

$$l(T;a) = \sum_{k=0}^{\infty}\left\{\prod_{j=0}^{k-1}\overline{\lambda}[A_jT;T]\right\}\frac{\int_{A_kT}^{(A_k+1)T}\overline{F}(t)\,\mathrm{d}t}{\overline{F}(A_kT)} \tag{7.40}$$

$$M(T;a) = \sum_{k=0}^{\infty}\prod_{j=0}^{k}\overline{\lambda}[A_jT;T], \tag{7.41}$$

where $A_j \equiv a + a^2 + \cdots + a^j$ $(j = 1, 2, \dots)$ and $A_0 \equiv 0$.

Note that the mean times $l(T;\cdot)$ and the expected numbers $M(T;\cdot)$ of three models are equal in both cases of $p = a = 0$ and $x = T$ (*i.e.,* the unit becomes as good as new by perfect inspection), and $p = a = 1$ and $x = 0$ (*i.e.,* the unit has the same age by imperfect inspection). Furthermore, substituting (7.38), (7.39) and (7.40), (7.41) into (7.35), respectively, we obtain two expected costs until failure detection.

7.3.3 Imperfect Inspection with Human Error

It is well known that a high percentage of failures in most systems is directly due to human error [48]. There are the following types of human error when we inspect a standby unit at periodic times kT $(k = 1, 2, \dots)$ [2, 49–51]:

1. Type A human error: The unit in a good state, *i.e.,* in a normal condition, is judged to be bad and is repaired.
2. Type B human error: The unit in a bad state, *i.e.,* in a failed state, is judged to be good.

It is assumed that the probabilities of type A error and type B error are α and β, respectively, where $0 \le \alpha + \beta < 1$. Then, the expected number of inspections until a failed unit is detected is

$$\sum_{j=0}^{\infty} j\beta^{j-1}(1-\beta) = \frac{1}{1-\beta}.$$

Consider one cycle from time $t = 0$ to the time when a failed unit is detected by perfect inspection or a good unit is repaired by type A error, whichever occurs first. Then, the total expected cost of one cycle is given by

$$\begin{aligned} C(T;\alpha,\beta) &= \sum_{j=0}^{\infty}(1-\alpha)^j \left[\int_{jT}^{(j+1)T} c_1\left(j + \frac{1}{1-\beta}\right) \mathrm{d}F(t) \right. \\ &\quad \left. + \alpha c_1 (j+1)\overline{F}((j+1)T) + \int_{jT}^{(j+1)T} c_2\left(jT + \frac{T}{1-\beta} - t\right) \mathrm{d}F(t) \right] \\ &= (c_1 + c_2 T)\left\{ \frac{1}{1-\beta} \sum_{j=0}^{\infty}(1-\alpha)^j[\overline{F}(jT) - \overline{F}((j+1)T)] \right. \\ &\quad \left. + \sum_{j=0}^{\infty}(1-\alpha)^j \overline{F}((j+1)T) \right\} - c_2 \sum_{j=0}^{\infty}(1-\alpha)^j \int_{jT}^{(j+1)T} \overline{F}(t)\,\mathrm{d}t. \end{aligned} \tag{7.42}$$

When $\alpha = \beta = 0$, *i.e.,* the inspection is perfect, Equation (7.42) is equal to that of a standard periodic inspection policy (see Section 8.1).

In particular, when $F(t) = 1 - \mathrm{e}^{-\lambda t}$, the expected cost is rewritten as

$$C(T;\alpha,\beta) = (c_1 + c_2 T)\frac{(1-\mathrm{e}^{-\lambda T})/(1-\beta) + \mathrm{e}^{-\lambda T}}{1-(1-\alpha)\mathrm{e}^{-\lambda T}} - \frac{c_2}{\lambda}\frac{1-\mathrm{e}^{-\lambda T}}{1-(1-\alpha)\mathrm{e}^{-\lambda T}}. \tag{7.43}$$

Differentiating $C(T;\alpha,\beta)$ with respect to T and setting it equal to zero,

$$\frac{\mathrm{e}^{\lambda T}-1}{\lambda}[1 - \beta(1-\alpha)\mathrm{e}^{-\lambda T}] - (1-\alpha-\beta)T = \frac{c_1}{c_2}(1-\alpha-\beta). \tag{7.44}$$

Note that the left-hand side of (7.44) is strictly increasing from 0 to ∞. Therefore, there exists a finite and unique T^* that satisfies (7.44).

7.4 Computer System with Imperfect Maintenance

Periodic maintenance of a computer system is imperative in order to inspect and remove as many component faults, failures, and degradations as possible. In most cases, it has been assumed that the system becomes like new and operates normally after maintenance. However, the system occasionally becomes worse for one or more of the following reasons:

(1) Hidden faults and failures that are not detected during maintenance;
(2) Human errors such as wrong adjustments and further damage done during maintenance; or
(3) Replacement with faulty parts.

It is useful to develop an imperfect maintenance strategy for a computer system.

This section considers a system that is maintained at periodic times kT $(k = 1, 2, \dots)$. Due to imperfect PM, one of the following results occurs: the system is not changed, is renewed, or is put in a failed state and needs repair. The MTTF and availability of the system are derived by the usual probability calculations. Furthermore, we calculate an optimum PM time T^* that maximizes the availability, and show that T^* is determined by a unique solution of an equation under certain conditions. A numerical example is given for a triple redundant system that fails when two or more units have failed.

A computer system begins to operate at time 0 and should operate for an infinite time span.

1. The system is maintained preventively at periodic times kT $(k = 1, 2, \dots)$ $(0 < T \le \infty)$.
2. The failed system is repaired immediately when it fails, and becomes as good as new after repair.
3. One of the following cases after PM results.
   (a) The system is not changed with probability p_1; *viz*, PM is imperfect.
   (b) The system becomes as good as new with probability p_2; *viz*, PM is perfect.
   (c) The system fails with probability p_3; *viz*, PM fails, where $p_1+p_2+p_3 = 1$ and $p_2 > 0$.
4. The mean times to repair actual failure in case 2 and maintenance failure in (c) are β_1 and β_2 with $\beta_1 \ge \beta_2$, respectively.
5. The PM time is negligible.

The probability that the system is renewed by repair upon actual failure is

$$\sum_{j=1}^{\infty} p_1^{j-1} \int_{(j-1)T}^{jT} \mathrm{d}F(t) = (1-p_1) \sum_{j=1}^{\infty} p_1^{j-1} F(jT), \tag{7.45}$$

the probability that the system is renewed by perfect maintenance is

$$p_2 \sum_{j=1}^{\infty} p_1^{j-1} \overline{F}(jT), \tag{7.46}$$

and the probability that the system is renewed by repair after maintenance failure is

$$p_3 \sum_{j=1}^{\infty} p_1^{j-1} \overline{F}(jT), \tag{7.47}$$

where (7.45) + (7.46) + (7.47) = 1.

Furthermore, the mean time of one cycle from time $t = 0$ to the time when the system is renewed by either repair or perfect maintenance is

$$\sum_{j=1}^{\infty} p_1^{j-1} \int_{(j-1)T}^{jT} t \, \mathrm{d}F(t) + (p_2 + p_3) \sum_{j=1}^{\infty} jT p_1^{j-1} \overline{F}(jT)$$
$$= (1 - p_1) \sum_{j=1}^{\infty} p_1^{j-1} \int_0^{jT} \overline{F}(t) \, \mathrm{d}t. \tag{7.48}$$

Therefore, the mean time to failure is

$$l(T; p_1, p_2, p_3) = \sum_{j=1}^{\infty} \Bigg\{ p_1^{j-1} \int_{(j-1)T}^{jT} t \, \mathrm{d}F(t)$$
$$+ p_1^{j-1} \overline{F}(jT) \left[p_2(jT + l(T; p_1, p_2, p_3)) + p_3 jT \right] \Bigg\};$$

i.e.,

$$l(T; p_1, p_2, p_3) = \frac{(1 - p_1) \sum_{j=1}^{\infty} p_1^{j-1} \int_0^{jT} \overline{F}(t) \, \mathrm{d}t}{1 - p_2 \sum_{j=1}^{\infty} p_1^{j-1} \overline{F}(jT)} \tag{7.49}$$

which agrees with (5) of [11] when $p_3 = 0$, and (9) of [13].

The availability is, from (6.10) in Chapter 6,

$$A(T; p_1, p_2, p_3) = \frac{(1 - p_1) \sum_{j=1}^{\infty} p_1^{j-1} \int_0^{jT} \overline{F}(t) \, \mathrm{d}t}{\left[\begin{array}{c} (1 - p_1) \sum_{j=1}^{\infty} p_1^{j-1} \int_0^{jT} \overline{F}(t) \, \mathrm{d}t + \beta_2 p_3 \sum_{j=1}^{\infty} p_1^{j-1} \overline{F}(jT) \\ + \beta_1 (1 - p_1) \sum_{j=1}^{\infty} p_1^{j-1} F(jT) \end{array} \right]} \tag{7.50}$$

which agrees with (10) of [11] when $p_3 = 0$.

First, we seek an optimum PM time T_1^* that maximizes MTTF $l(T; p_1, p_2, p_3)$ in (7.49). It is evident that

$$l(0;p_1,p_2,p_3) \equiv \lim_{T\to 0} l(T;p_1,p_2,p_3) = 0$$
$$l(\infty;p_1,p_2,p_3) \equiv \lim_{T\to \infty} l(T;p_1,p_2,p_3) = \mu. \tag{7.51}$$

Thus, there exists some positive T_1^* $(0 < T_1^* \leq \infty)$ that maximizes $l(T;p_1,p_2,p_3)$. Differentiating $l(T;p_1,p_2,p_3)$ with respect to T and setting it equal to zero, we have

$$H(T;p_1)\sum_{j=1}^{\infty} p_1^{j-1}\int_0^{jT}\overline{F}(t)\,\mathrm{d}t + \sum_{j=1}^{\infty} p_1^{j-1}\overline{F}(jT) = \frac{1}{p_2}, \tag{7.52}$$

where

$$H(T;p_1) \equiv \frac{\sum_{j=1}^{\infty} p_1^{j-1} jf(jT)}{\sum_{j=1}^{\infty} p_1^{j-1} j\overline{F}(jT)}.$$

It can be shown that the left-hand side of (7.52) is strictly increasing from $1/(1-p_1)$ to $\mu H(\infty;p_1)/(1-p_1)$ when $H(t;p_1)$ is strictly increasing. Thus, the optimum policy is:

(i) If $H(T;p_1)$ is strictly increasing and $H(\infty;p_1) > (1-p_1)/(\mu p_2)$ then there exists a finite and unique T_1^* that satisfies (7.52), and the resulting MTTF is

$$l(T_1^*;p_1,p_2,p_3) = \frac{1-p_1}{p_2 H(T_1^*;p_1)}. \tag{7.53}$$

(ii) If $H(T;p_1)$ is nonincreasing, or $H(T;p_1)$ is strictly increasing and $H(\infty;p_1) \leq (1-p_1)/(\mu p_2)$, then $T_1^* = \infty$; *viz*, no PM should be done, and the MTTF is given in (7.51).

Next, we seek an optimum PM time T_2^* that maximizes the availability $A(T;p_1,p_2,p_3)$ in (7.50). Differentiating $A(T;p_1,p_2,p_3)$ with respect to T and setting it equal to zero imply

$$H(T;p_1)\sum_{j=1}^{\infty} p_1^{j-1}\int_0^{jT}\overline{F}(t)\,\mathrm{d}t + \sum_{j=1}^{\infty} p_1^{j-1}\overline{F}(jT) = \frac{\beta_1}{\beta_1(1-p_1)-\beta_2 p_3}. \tag{7.54}$$

Note that $\beta_1(1-p_1) > \beta_2 p_3$ because $\beta_1 \geq \beta_2$.

Thus, we have a similar optimum policy to the previous case. Also, it is of interest that $T_1^* \geq T_2^*$ because $\beta_1/[\beta_1(1-p_1)-\beta_2 p_3] \leq 1/p_2$.

Example 7.4. Consider a triple redundant system that consists of three units, and fails when two or more units have failed. This system is a 2-out-of-3 system and is applied to the design of a fail-safe system. The failure distribution of the system is $\overline{F}(t) = 3\mathrm{e}^{-2t} - 2\mathrm{e}^{-3t}$, and the mean time to failure is $\mu = 5/6$. In addition, we have

$$H(t;p_1) = \frac{6\sum_{j=1}^{\infty} p_1^{j-1} j(\mathrm{e}^{-2jt} - \mathrm{e}^{-3jt})}{\sum_{j=1}^{\infty} p_1^{j-1} j(3\mathrm{e}^{-2jt} - 2\mathrm{e}^{-3jt})}$$

$$H(0;p_1) = 0, \qquad H(\infty;p_1) = 2$$

$$\begin{aligned}\frac{\mathrm{d}H(t;p_1)}{6\,\mathrm{d}t} &= \frac{1}{D}\Bigg[6\sum_{j=1}^{\infty} p_1^{j-1} j^2(\mathrm{e}^{-2jt} - \mathrm{e}^{-3jt})\sum_{j=1}^{\infty} p_1^{j-1} j(\mathrm{e}^{-2jt} - \mathrm{e}^{-3jt}) \\ &\quad - \sum_{j=1}^{\infty} p_1^{j-1} j^2(2\mathrm{e}^{-2jt} - 3\mathrm{e}^{-3jt})\sum_{j=1}^{\infty} p_1^{j-1} j(3\mathrm{e}^{-2jt} - 2\mathrm{e}^{-3jt})\Bigg] \\ &= \frac{1}{D}\left[\sum_{j=1}^{\infty}\sum_{i=1}^{\infty} p_1^{i+j-2}(i^2 j)(3\mathrm{e}^{-it} - 2\mathrm{e}^{-jt})\mathrm{e}^{-2(i+j)t}\right] > 0,\end{aligned}$$

where

$$D \equiv \left[\sum_{j=1}^{\infty} p_1^{j-1} j(3\mathrm{e}^{-2jt} - 2\mathrm{e}^{-3jt})\right]^2.$$

Thus, $H(t;p_1)$ is strictly increasing from 0 to 2.

Therefore, if $1 - p_1 > (5/2)(\beta_2/\beta_1)p_3$ then there exists a finite and unique T_2^* that satisfies

$$\frac{H(T;p_1)}{6}\left\{\sum_{j=1}^{\infty} p_1^{j-1}[9(1 - \mathrm{e}^{-2jT}) - 4(1 - \mathrm{e}^{-3jT})]\right\} + \sum_{j=1}^{\infty} p_1^{j-1}(3\mathrm{e}^{-2jT} - 2\mathrm{e}^{-3jT}) = \frac{\beta_1}{\beta_1(1 - p_1) - \beta_2 p_3}$$

and otherwise, $T_2^* = \infty$.

Table 7.3 shows the optimum PM time T_2^* ($\times 10^2$) for $p_1 = 10^{-3}$, 10^{-2}, 10^{-1}, $p_3 = 10^{-4}$, 10^{-3}, 10^{-2}, 10^{-1}, and $\beta_2/\beta_1 = 0.1$, 1.0. For example, when $p_1 = 0.1$, $p_3 = 0.01$, and $\beta_2/\beta_1 = 0.1$, $T_2^* = 1.72 \times 10^{-2}$. If the MTTF of each unit is 10^4 hours then $T_2^* = 172$ hours. These results indicate that the system should be maintained about once a week. Furthermore, it is of great interest that the optimum T_2^* depends considerably on the product of β_2/β_1 and p_3, but depends little on p_1. When $(\beta_2/\beta_1)p_3 = 10^{-4}$, 10^{-3}, 10^{-2}, 10^{-1}, the approximate optimum times T_2^* are $0.005, 0.018, 0.06, 0.28$, respectively. ∎

7.5 Sequential Imperfect Preventive Maintenance

We consider the following two PM policies, by introducing improvement factors [15,52] in failure rate and age for a sequential PM policy [53,54]: the PM

Table 7.3. Optimum PM time T_2^* $(\times 10^2)$ to maximize availability $A(T; p_1, p_2, p_3)$ for p_1, p_2 and β_2/β_1

p_3	$\beta_2/\beta_1 = 0.1$			$\beta_2/\beta_1 = 1.0$		
	p_1					
	10^{-3}	10^{-2}	10^{-1}	10^{-3}	10^{-2}	10^{-1}
10^{-4}	0.183	0.181	0.166	0.582	0.578	0.529
10^{-3}	0.582	0.578	0.529	1.88	1.87	1.72
10^{-2}	1.88	1.87	1.72	6.42	6.37	5.98
10^{-1}	6.42	6.37	5.98	28.1	28.1	27.6

is done at fixed intervals T_k $(k = 1, 2, \dots, N-1)$ and is replaced at the Nth PM; if the system fails between PMs, it undergoes only minimal repair. The PM is imperfect as follows.

(1) The age after the kth PM reduces to $a_k t$ when it was t before PM.
(2) The failure rate after the kth PM becomes $b_k h(t)$ when it was $h(t)$ in the period of the kth PM.

The imperfect PM model that combines two policies was considered in [55].

The expected cost rates of two models are obtained and optimum sequences $\{T_k^*\}$ are derived. When the failure time has a Weibull distribution, optimum policies are computed explicitly.

(1) Model A – Age

Consider the sequential PM policy for a one-unit system for an infinite time span. It is assumed that (see Figure 7.2):

1. The PM is done at fixed intervals T_k $(k = 1, 2, \dots, N-1)$ and is replaced at the Nth PM; *i.e.,* the unit is maintained preventively at successive times $T_1 < T_1 + T_2 < \cdots < T_1 + T_2 + \cdots + T_{N-1}$ and is replaced at time $T_1 + T_2 + \cdots + T_N$, where $T_0 \equiv 0$.
2. The unit undergoes only minimal repair at failures between replacements and becomes as good as new at replacement.
3. The age after the kth PM reduces to $a_k t$ when it was t before PM; *i.e.,* the unit with age t becomes $t(1 - a_k)$ units of time younger at the kth PM, where $0 = a_0 < a_1 \le a_2 \le \cdots \le a_N < 1$.
4. Cost of each minimal repair is c_1, cost of each PM is c_2, and cost of replacement at the Nth PM is c_3.
5. The times for PM, repair, and replacement are negligible.

The unit is aged from $a_{k-1}(T_{k-1} + a_{k-2}T_{k-2} + \cdots + a_{k-2}a_{k-3} \dots a_2 a_1 T_1)$ after the $(k-1)$th PM to $T_k + a_{k-1}(T_{k-1} + a_{k-2}T_{k-2} + \cdots + a_{k-2}a_{k-3} \dots a_2 a_1 T_1)$ before the kth PM, *i.e.,* from $a_{k-1}Y_{k-1}$ to Y_k, where $Y_k \equiv T_k + a_{k-1}T_k + \cdots + a_{k-1}a_{k-2} + \cdots + a_2 a_1 T_1$ $(k = 1, 2, \dots)$, which is the age immediately before the kth PM. Thus, the expected cost rate is

$$C_A(Y_1, Y_2, \dots, Y_N) = \frac{c_1 \sum_{k=1}^{N} \int_{a_{k-1}Y_{k-1}}^{Y_k} h(t)\,\mathrm{d}t + (N-1)c_2 + c_3}{\sum_{k=1}^{N-1}(1-a_k)Y_k + Y_N} \quad (N = 1, 2, \dots) \tag{7.55}$$

because $T_k = Y_k - a_{k-1}Y_{k-1}$ and $\sum_{k=1}^{N} T_k = \sum_{k=1}^{N-1}(1-a_k)Y_k + Y_N$.

To find an optimum sequence $\{Y_k\}$ that minimizes $C_A(Y_1, Y_2, \dots, Y_N)$, differentiating $C_A(Y_1, Y_2, \dots, Y_N)$ with respect to Y_k and setting it equal to zero,

$$\frac{h(Y_k) - a_k h(a_k Y_k)}{1 - a_k} = h(Y_N) \qquad (k = 1, 2, \dots, N-1) \tag{7.56}$$

$$c_1 h(Y_N) = C_A(Y_1, Y_2, \dots, Y_N). \tag{7.57}$$

Suppose that Y_N $(0 < Y_N < \infty)$ is fixed. If $h(t)$ is strictly increasing then there exists some Y_k $(0 < Y_k < Y_N)$ that satisfies (7.56), because

$$\frac{h(0) - a_k h(0)}{1 - a_k} < h(Y_N), \qquad \frac{h(Y_N) - a_k h(a_k Y_N)}{1 - a_k} > h(Y_N).$$

Furthermore, if $\mathrm{d}h(t)/\mathrm{d}t$ is also strictly increasing then a solution to (7.56) is unique.

Thus, substituting each Y_k into (7.57), its equation becomes a function of Y_N only which is

$$h(Y_N)\left[\sum_{k=1}^{N-1}(1-a_k)Y_k + Y_N\right] - \sum_{k=1}^{N}\int_{a_{k-1}Y_{k-1}}^{Y_k} h(t)\,\mathrm{d}t = \frac{(N-1)c_2 + c_3}{c_1}, \tag{7.58}$$

where each Y_k $(k = 1, 2, \dots, N-1)$ is given by some function of Y_N. If there exists a solution Y_N to (7.58) then a sequence $\{Y_k\}$ minimizes the expected cost $C_A(Y_1, Y_2, \dots, Y_N)$.

Finally, suppose that $Y_1, Y_2, \dots, Y_N$ are determined from (7.56) and (7.58). Then, from (7.57), the resulting cost rate is $c_1 h(Y_N)$, which is a function of N. To complete an optimum PM schedule, we may seek an optimum number N^* that minimizes $h(Y_N)$.

From the above discussion, we can specify the computing procedure for obtaining the optimum PM schedule.

1. Solve (7.56) and express Y_k $(k = 1, 2, \dots, N-1)$ by a function of Y_N.
2. Substitute Y_k into (7.58) and solve it with respect to Y_N.
3. Determine N^* that minimizes $h(Y_N)$.
4. Compute T_k^* $(k = 1, 2, \dots, N^*)$ from $T_k = Y_k - a_{k-1}Y_{k-1}$.

(2) Model B – Failure rate

3. The failure rate after the kth PM becomes $b_k h(t)$ when it was $h(t)$ before PM; *i.e.*, the unit has the failure rate $B_k h(t)$ in the kth PM period, where $1 = b_0 < b_1 \le b_2 \le \cdots \le b_{N-1}$, $B_k \equiv \prod_{j=0}^{k-1} b_j$ $(k = 1, 2, \ldots, N)$ and $1 = B_1 < B_2 < \cdots < B_N$.

1, 2, 4, 5. Same as the assumptions of Model A.

The expected cost rate is

$$C_B(T_1, T_2, \ldots, T_N) = \frac{c_1 \sum_{k=1}^{N} B_k \int_0^{T_k} h(t)\,\mathrm{d}t + (N-1)c_2 + c_3}{T_1 + T_2 + \cdots + T_N} \qquad (N = 1, 2, \ldots). \tag{7.59}$$

Differentiating $C_B(T_1, T_2, \ldots, T_N)$ with respect to T_k and setting it equal to zero, we have

$$B_1 h(T_1) = B_2 h(T_2) = \cdots = B_N h(T_N) \tag{7.60}$$

$$c_1 B_k h(T_k) = C_B(T_1, T_2, \ldots, T_N) \qquad (k = 1, 2, \ldots, N). \tag{7.61}$$

When the failure rate is strictly increasing to infinity, we can specify the computing procedure for obtaining an optimum schedule.

1. Solve $B_k h(T_k) = D$ and express T_k $(k = 1, 2, \ldots, N)$ by a function of D.
2. Substitute T_k into (7.60) and solve it with respect to D.
3. Determine N^* that minimizes D.

Example 7.5. Suppose that the failure time has a Weibull distribution; *i.e.*, $h(t) = mt^{m-1}$ for $m > 1$. From the computing procedure of Model A, by solving (7.56), we have

$$Y_k = \left(\frac{1 - a_k}{1 - a_k^m}\right)^{1/(m-1)} Y_N \qquad (k = 1, 2, \ldots, N-1). \tag{7.62}$$

Substituting Y_k into (7.58) and arranging it,

$$Y_N = \left[\frac{(N-1)c_2 + c_3}{(m-1)c_1 \sum_{k=0}^{N-1} d_k}\right]^{1/m}, \tag{7.63}$$

where

$$d_k \equiv (1 - a_k)\left(\frac{1 - a_k}{1 - a_k^m}\right)^{1/(m-1)} \qquad (k = 0, 1, 2, \ldots, N-1).$$

Next, we consider the problem that minimizes

$$C_A(N) \equiv \frac{(N-1)c_2 + c_3}{\sum_{k=0}^{N-1} d_k} \qquad (N = 1, 2, \dots) \tag{7.64}$$

which is the same problem as minimizing $h(Y_N)$, *i.e.*, $C_A(Y_1, Y_2, \dots, Y_N)$. From the inequality $C_A(N+1) \geq C_A(N)$, we have

$$L_A(N) \geq \frac{c_3}{c_2} \qquad (N = 1, 2, \dots), \tag{7.65}$$

where

$$L_A(N) \equiv \sum_{k=0}^{N-1} \frac{d_k}{d_N} - (N-1) \qquad (N = 1, 2, \dots). \tag{7.66}$$

If d_k is decreasing in k then $L_A(N)$ is increasing in N. Thus, there exists a finite and unique minimum N^* that satisfies (7.65) if $L_A(\infty) > c_3/c_2$.

We show that d_k is decreasing in k from the assumption that $a_k < a_{k+1}$. Let $g(x) \equiv (1-x)^m/(1-x^m)$ $(0 < x < 1)$ for $m > 1$. Then, $g(x)$ is decreasing from 1 to 0, and hence,

$$\frac{(1-a_k)^m}{1-a_k^m} > \frac{(1-a_{k+1})^m}{1-a_{k+1}^m}$$

which follows that $d_k > d_{k+1}$. Furthermore, if $a_k \to 1$ as $k \to \infty$ then

$$\lim_{k\to\infty} d_k = \lim_{x\to 1} [g(x)]^{1/(m-1)} = 0;$$

i.e., $L_A(N) \to \infty$ as $N \to \infty$, and a finite N^* exists uniquely.

Therefore, if $a_k \to 1$ as $k \to \infty$ then an N^* is a finite and unique minimum that satisfies (7.65), and the optimum intervals are $T_k^* = Y_k - a_{k-1}Y_{k-1}$ $(k = 1, 2, \dots, N^*)$, where Y_k and Y_N are given in (7.62) and (7.63).

For Model B, by solving $B_k h(T_k) = D$, we have

$$T_k = \left(\frac{D}{mB_k}\right)^{1/(m-1)} \qquad (k = 1, 2, \dots, N). \tag{7.67}$$

Substituting T_k into (7.61) and arranging it,

$$D = \left[\frac{(N-1)c_2 + c_3}{c_1\left(1 - \frac{1}{m}\right)\sum_{k=1}^{N}\left[(1/mB_k)\right]^{1/(m-1)}}\right]^{(m-1)/m} \tag{7.68}$$

which is a function of N. Let us denote D by $D(N)$. Then, from the inequality $D(N+1) \geq D(N)$, an N^* to minimize D is given by a unique minimum that satisfies

$$L_B(N) \geq \frac{c_3}{c_2} \qquad (N = 1, 2, \dots), \tag{7.69}$$

Table 7.4. Optimum N^* and PM intervals of Model A when $c_1/c_2 = 3$

c_3/c_2	2	5	10	20	40
N^*	1	2	4	7	11
T_1	0.54	0.82	1.07	1.40	1.84
T_2		0.82	0.43	0.56	0.74
T_3			0.28	0.36	0.48
T_4			0.92	0.27	0.35
T_5				0.21	0.28
T_6				0.18	0.23
T_7				1.13	0.20
T_8					0.17
T_9					0.15
T_{10}					0.14
T_{11}					1.45

Table 7.5. Optimum N^* and PM intervals of Model B when $c_1/c_2 = 3$

c_3/c_2	2	5	10	20	40
N^*	2	3	4	5	6
T_1	0.77	1.06	1.37	1.82	2.45
T_2	0.52	0.71	0.92	1.21	1.64
T_3		0.43	0.55	0.73	0.98
T_4			0.31	0.42	0.56
T_5				0.23	0.31
T_6					0.17

where

$$L_B(N) \equiv \sum_{k=1}^{N} \left(\frac{B_{N+1}}{B_k}\right)^{1/(m-1)} - (N-1) \qquad (N = 1, 2, \dots)$$

which is increasing in N because B_k is increasing in k. Also, if $B_k \to \infty$ as $k \to \infty$ then $L_B(N) \to \infty$ as $N \to \infty$, and hence, a finite N^* exists uniquely in (7.69), and the optimum intervals T_k^* $(k = 1, 2, \dots, N^*)$ are given in (7.67) and (7.68).

Tables 7.4 and 7.5 present the optimum number N^* and the PM intervals T_1^*, T_2^*, ..., T_N^* for $c_3/c_2 = 2$, 5, 10, 20, 40, where $c_1/c_2 = 3$, $m = 2$, and $a_k = k/(k+1)$, $b_k = 1 + k/(k+1)$ $(k = 0, 1, 2, \dots)$. These examples show that $T_1^* > T_2^* > \cdots > T_N^*$ for Model B, but $T_1^* > T_N^* > T_2^*$ for $c_3/c_2 = 10$, 20, 40 of Model A. This indicates that it would be reasonable to do frequent PM with age, but it would be better to do the last PM as late as possible because the system should be replaced at the next PM. Figure 7.2 shows the graph of Model A for time and age when $c_3/c_2 = 10$. ∎

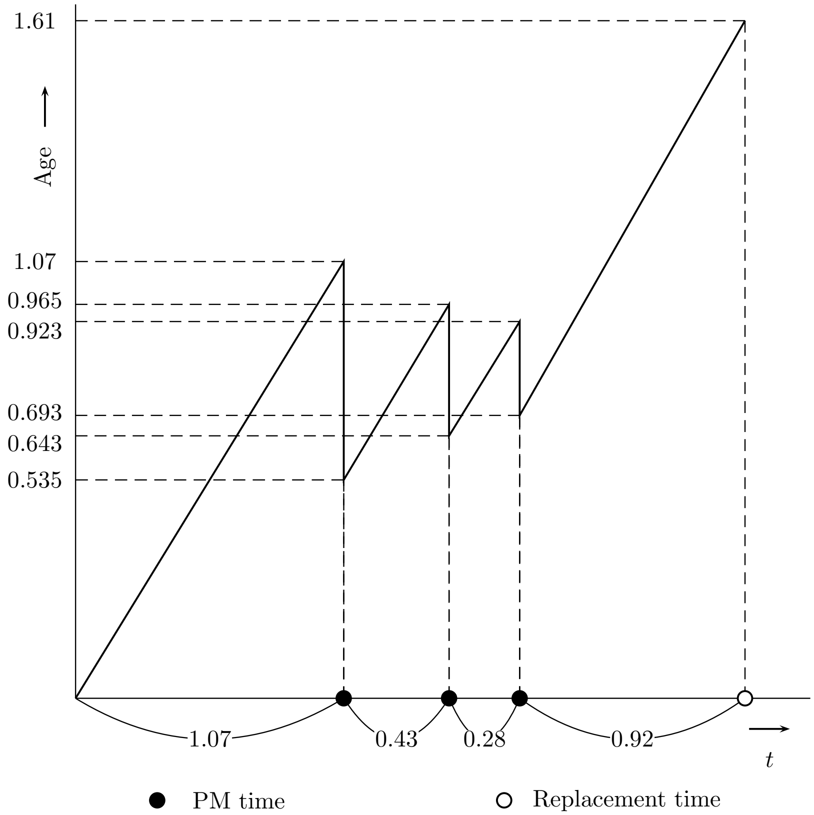

Fig. 7.2. Graph of Model A when $c_3/c_2 = 10$

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