4

Periodic Replacement

When we consider large and complex systems that consist of many kinds of units, we should make only minimal repair at each failure, and make the planned replacement or do preventive maintenance at periodic times. We consider the following replacement policy which is called *periodic replacement with minimal repair at failures* [1]. A unit is replaced periodically at planned times kT $(k = 1, 2, \dots)$. Only minimal repair after each failure is made so that the failure rate remains undisturbed by any repair of failures between successive replacements.

This policy is commonly used with complex systems such as computers and airplanes. A practical procedure for applying the policy to large motors and small electrical parts was given in [2]. More general cost structures and several modified models were provided in [3–11]. On the other hand, the policy regarding the version that a unit is replaced at the Nth failure and $(N-1)$th previous failures are corrected with minimal repair proposed in [12]. The stochastic models to describe the failure pattern of repairable units subject to minimal maintenance are dealt with [13].

This chapter summarizes the periodic replacement with minimal repair based on our original work with reference to the book [1]. In Section 4.1, we make clear the theoretical definition of minimal repair, and give some useful theorems that can be applied to the analysis of optimum policies [14]. In Section 4.2, we consider the periodic replacement policy in which a unit is replaced at planned time T and any failed units undergo minimal repair between replacements. We obtain the expected cost rate as an objective function and analytically derive an optimum replacement time T^* that minimizes it [15]. In Section 4.3, we propose the extended replacement policy in which a unit is replaced at time T or at the Nth failure, whichever occurs first. Using the results in Section 4.1, we derive an optimum number N^* that minimizes the expected cost rate for a specified T [16–18]. Furthermore, in Section 4.4, we show five models of replacement with discounting and replacement in discrete time [15], replacement of a used unit [15], replacement with random and wearout failures, and replacement with threshold level [19]. Finally, in Section 4.5, we introduce periodic replacements with two types of failures [16] and with two types of units [20].

4.1 Definition of Minimal Repair

Suppose that a unit begins to operate at time 0. If a unit fails then it undergoes minimal repair and begins to operate again. It is assumed that the time for repair is negligible. Let us denote by $0 \equiv Y_0 \le Y_1 \le \cdots \le Y_n \le \cdots$ the successive failure times of a unit. The times between failures $X_n \equiv Y_n - Y_{n-1}$ $(n = 1, 2, \dots)$ are nonnegative random variables.

We define *to make minimal repair at failure* as follows.

Definition 4.1. Let $F(t) \equiv \Pr\{X_1 \le t\}$ for $t \ge 0$. A unit undergoes minimal repair at failures if and only if

$$\Pr\{X_n \le x | X_1 + X_2 + \cdots + X_{n-1} = t\} = \frac{F(t+x) - F(t)}{\overline{F}(t)} \quad (n = 2, 3, \dots) \tag{4.1}$$

for $x > 0$, $t \ge 0$ such that $F(t) < 1$, where $\overline{F} \equiv 1 - F$.

The function $[F(t+x) - F(t)]/\overline{F}(t)$ is called the failure rate and represents the probability that a unit with age t fails in a finite interval $(t, t+x]$. The definition means that the failure rate remains undisturbed by any minimal repair of failures; *i.e.,* a unit after each minimal repair has the same failure rate as before failure.

Assume that $F(t)$ has a density function $f(t)$ and $h(t) \equiv f(t)/\overline{F}(t)$, which is continuous. The function $h(t)$ is also called the instantaneous failure rate or simply the failure rate and has the same monotone property as $[F(t+x) - F(t)]/\overline{F}(t)$ as shown in Section 1.1. Moreover, $H(t) \equiv \int_0^t h(u)\mathrm{d}u$ is called the *cumulative hazard function* and satisfies a relation $\overline{F}(t) = \mathrm{e}^{-H(t)}$.

Theorem 4.1. Let $G_n(x) \equiv \Pr\{Y_n \le x\}$ and $F_n(x) \equiv \Pr\{X_n \le x\}$ $(n = 1, 2, \dots)$. Then,

$$G_n(x) = 1 - \sum_{j=0}^{n-1} \frac{[H(x)]^j}{j!} \mathrm{e}^{-H(x)} \qquad (n = 1, 2, \dots) \tag{4.2}$$

$$F_n(x) = 1 - \int_0^\infty \overline{F}(t+x) \frac{[H(t)]^{n-2}}{(n-2)!} h(t)\,\mathrm{d}t \qquad (n = 2, 3, \dots). \tag{4.3}$$

Proof. By mathematical induction, we have

$$\begin{aligned}
G_1(x) &= F_1(x) = F(x) \\
G_{n+1}(x) &= \int_0^\infty \Pr\{X_{n+1} \le x - t | Y_n = t\}\, \mathrm{d}G_n(t) \\
&= \int_0^x \frac{F(x) - F(t)}{\overline{F}(t)} \frac{[H(t)]^{n-1}}{(n-1)!} f(t)\, \mathrm{d}t \\
&= 1 - \sum_{j=0}^{n-1} \frac{[H(x)]^j}{j!} \mathrm{e}^{-H(x)} - \mathrm{e}^{-H(x)} \int_0^x \frac{[H(t)]^{n-1}}{(n-1)!} h(t)\, \mathrm{d}t \\
&= 1 - \sum_{j=0}^{n} \frac{[H(x)]^j}{j!} \mathrm{e}^{-H(x)} \qquad (n = 1, 2, \dots).
\end{aligned}$$

Similarly,

$$\begin{aligned}
F_{n+1}(x) &= \int_0^\infty \Pr\{X_{n+1} \le x | Y_n = t\}\, \mathrm{d}G_n(t) \\
&= \int_0^\infty \frac{F(t+x) - F(t)}{\overline{F}(t)} \frac{[H(t)]^{n-1}}{(n-1)!} f(t)\, \mathrm{d}t \\
&= 1 - \int_0^\infty \overline{F}(t+x) \frac{[H(t)]^{n-1}}{(n-1)!} h(t)\, \mathrm{d}t \qquad (n = 1, 2, \dots). \quad \blacksquare
\end{aligned}$$

It easily follows from Theorem 4.1 that

$$E\{Y_n\} \equiv \int_0^\infty \overline{G}_n(x)\mathrm{d}x = \sum_{j=0}^{n-1} \int_0^\infty \frac{[H(x)]^j}{j!} \mathrm{e}^{-H(x)}\, \mathrm{d}x \quad (n = 1, 2, \dots) \tag{4.4}$$

$$E\{X_n\} = E\{Y_n\} - E\{Y_{n-1}\} = \int_0^\infty \frac{[H(x)]^{n-1}}{(n-1)!} \mathrm{e}^{-H(x)}\, \mathrm{d}x \quad (n = 1, 2, \dots). \tag{4.5}$$

In particular, when $F(t) = 1 - \mathrm{e}^{-\lambda t}$, *i.e.*, $H(t) = \lambda t$,

$$F_n(x) = 1 - \mathrm{e}^{-\lambda x}, \qquad G_n(x) = 1 - \sum_{j=0}^{n-1} \frac{(\lambda x)^j}{j!} \mathrm{e}^{-\lambda x} \quad (n = 1, 2, \dots)$$

$$E\{X_n\} = \frac{1}{\lambda}, \qquad E\{Y_n\} = \frac{n}{\lambda}.$$

Let $N(t)$ be the number of failures of a unit during $[0, t]$; *i.e.*, $N(t) \equiv \max_n\{Y_n \le t\}$. Clearly,

$$\begin{aligned}
p_n(t) \equiv \Pr\{N(t) = n\} &= \Pr\{Y_n \le t < Y_{n+1}\} = G_n(t) - G_{n+1}(t) \\
&= \frac{[H(t)]^n}{n!} \mathrm{e}^{-H(t)} \quad (n = 0, 1, 2, \dots)
\end{aligned} \tag{4.6}$$

and moreover,

$$E\{N(t)\} = V\{N(t)\} = H(t); \tag{4.7}$$

that is, failures occur at a non-homogeneous Poisson process with mean-value function $H(t)$ in Section 1.3 [21].

Next, assume that the failure rate $[F(t+x)-F(t)]/\overline{F}(t)$ or $h(t)$ is increasing in t for $x > 0$, $t \geq 0$. Then, there exists $\lim_{t\to\infty} h(t) \equiv h(\infty)$, which may possibly be infinity.

Theorem 4.2. If the failure rate is increasing then $E\{X_n\}$ is decreasing in n, and converges to $1/h(\infty)$ as $n \to \infty$, where $1/h(\infty) = 0$ whenever $h(\infty) = \infty$.

Proof. Let

$$\gamma(t) \equiv \int_0^\infty \left[1 - \frac{F(t+x) - F(t)}{\overline{F}(t)}\right] \mathrm{d}x$$

which represents the mean residual lifetime of a unit with age t. Then, $\gamma(t)$ is decreasing in t from the assumption that $[F(t+x)-F(t)]/\overline{F}(t)$ is increasing, and

$$\lim_{t\to\infty} \gamma(t) = \lim_{t\to\infty} \frac{1}{\overline{F}(t)} \int_t^\infty \overline{F}(x)\mathrm{d}x = \frac{1}{h(\infty)}.$$

Furthermore, noting from (4.1) that

$$E\{X_{n+1}\} = E\{\gamma(Y_n)\}$$

and using the relation $Y_{n+1} \geq Y_n$, we have the inequality

$$E\{X_{n+1}\} = E\{\gamma(Y_n)\} \leq E\{\gamma(Y_{n-1})\} = E\{X_n\} \quad (n = 1, 2, \dots).$$

Therefore, because $Y_n \to \infty$ as $n \to \infty$, we have, by monotone convergence,

$$\lim_{n\to\infty} E\{\gamma(Y_n)\} = \frac{1}{h(\infty)}$$

which completes the proof. ∎

Theorem 4.3. If failure rate $h(t)$ is increasing then

$$\frac{\int_0^T \{[H(t)]^n/n!\} f(t)\, \mathrm{d}t}{\int_0^T \{[H(t)]^n/n!\} \overline{F}(t)\, \mathrm{d}t} \tag{4.8}$$

is increasing in n and converges to $h(T)$ as $n \to \infty$ for any $T > 0$.

Proof. Letting

$$q(T) \equiv \int_0^T [H(t)]^{n+1} f(t)\,\mathrm{d}t \int_0^T [H(t)]^n \overline{F}(t)\,\mathrm{d}t - \int_0^T [H(t)]^n f(t)\,\mathrm{d}t \int_0^T [H(t)]^{n+1} \overline{F}(t)\,\mathrm{d}t$$

we obviously have that $\lim_{T\to 0} q(T) = 0$, and because $h(t)$ is increasing,

$$\frac{\mathrm{d}q(T)}{\mathrm{d}T} = [H(T)]^n \overline{F}(T) \int_0^T [H(t)]^n \overline{F}(t)[H(T) - H(t)][h(T) - h(t)]\,\mathrm{d}t \geq 0.$$

Thus, $q(T)$ is increasing in T from 0, and hence, $q(T) \geq 0$ for all $T > 0$, which implies that the function (4.8) is increasing in n.

Next, to prove that the function (4.8) converges to $h(T)$ as $n \to \infty$, we introduce the following result. If $\phi(t)$ and $\psi(t)$ are continuous, $\phi(b) \neq 0$ and $\psi(b) \neq 0$, then for $0 \leq a < b$,

$$\lim_{n\to\infty} \frac{\int_a^b t^n \phi(t)\,\mathrm{d}t}{\int_a^b t^n \psi(t)\,\mathrm{d}t} = \frac{\phi(b)}{\psi(b)}. \tag{4.9}$$

For, putting $t = bx$, $c = a/b$, $\phi(bx) = f(x)$, and $\psi(bx) = g(x)$, Equation (4.9) is rewritten as

$$\lim_{n\to\infty} \frac{\int_c^1 x^n f(x)\,\mathrm{d}x}{\int_c^1 x^n g(x)\,\mathrm{d}x} = \frac{f(1)}{g(1)}.$$

This is easily shown from the fact that

$$\lim_{n\to\infty} (n+1) \int_c^1 x^n f(x)\,\mathrm{d}x = f(1)$$

for any c $(0 \leq c < 1)$. Thus, letting $H(t) = x$ in (4.8) and using (4.9), it follows that

$$\lim_{n\to\infty} \frac{\int_0^T \{[H(t)]^n/n!\} f(t)\,\mathrm{d}t}{\int_0^T \{[H(t)]^n/n!\} \overline{F}(t)\,\mathrm{d}t} = \lim_{n\to\infty} \frac{\int_0^{H(T)} x^n \mathrm{e}^{-x}\,\mathrm{d}x}{\int_0^{H(T)} x^n \mathrm{e}^{-x}/h(H^{-1}(x))\,\mathrm{d}x} = h(T),$$

where $H^{-1}(x)$ is the inverse function of $x = H(t)$. ∎

In particular, when $F(t) = 1 - \mathrm{e}^{-\lambda t}$,

$$\frac{\int_0^T \{[H(t)]^n/n!\} f(t)\,\mathrm{d}t}{\int_0^T \{[H(t)]^n/n!\} \overline{F}(t)\,\mathrm{d}t} = \lambda \qquad (n = 0, 1, 2, \dots).$$

Let $G(t)$ represent any distribution with failure rate $r(t) \equiv g(t)/\overline{G}(t)$ and finite mean, where $g(t)$ is a density function of $G(t)$ and $\overline{G} \equiv 1 - G$.

Theorem 4.4. If both $h(t)$ and $r(t)$ are continuous and increasing then

$$\frac{\int_0^\infty \{[H(t)]^{n-1}/(n-1)!\}\overline{G}(t)f(t)\,\mathrm{d}t}{\int_0^\infty \{[H(t)]^n/n!\}\overline{G}(t)\overline{F}(t)\,\mathrm{d}t} \tag{4.10}$$

is increasing in n and converges to $h(\infty)+r(\infty)$ as $n\to\infty$.

Proof. Integrating by parts, we have

$$\int_0^\infty \frac{[H(t)]^{n-1}}{(n-1)!}\overline{G}(t)f(t)\,\mathrm{d}t = \int_0^\infty \frac{[H(t)]^n}{n!}\overline{G}(t)f(t)\,\mathrm{d}t + \int_0^\infty \frac{[H(t)]^n}{n!}\overline{F}(t)g(t)\,\mathrm{d}t.$$

First, we show

$$\frac{\int_0^\infty [H(t)]^n\overline{G}(t)f(t)\,\mathrm{d}t}{\int_0^\infty [H(t)]^n\overline{G}(t)\overline{F}(t)\,\mathrm{d}t} \tag{4.11}$$

is increasing in n when $h(t)$ is increasing. By a similar method to that of proving Theorem 4.3, letting

$$\begin{aligned} q(T) \equiv& \int_0^T [H(t)]^{n+1}\overline{G}(t)f(t)\,\mathrm{d}t \int_0^T [H(t)]^n\overline{G}(t)\overline{F}(t)\,\mathrm{d}t \\ &- \int_0^T [H(t)]^n\overline{G}(t)f(t)\,\mathrm{d}t \int_0^T [H(t)]^{n+1}\overline{G}(t)\overline{F}(t)\,\mathrm{d}t \end{aligned}$$

for any $T>0$, we have $\lim_{T\to 0} q(T)=0$ and $\mathrm{d}q(T)/\mathrm{d}T \geq 0$. Thus, $q(T)\geq 0$ for all $T>0$, and hence, the function (4.11) is increasing in n. Similarly,

$$\frac{\int_0^\infty [H(t)]^n\overline{F}(t)g(t)\,\mathrm{d}t}{\int_0^\infty [H(t)]^n\overline{F}(t)\overline{G}(t)\,\mathrm{d}t} \tag{4.12}$$

is also increasing in n. Therefore, from (4.11) and (4.12), the function (4.10) is also increasing in n.

Next, we show that

$$\lim_{n\to\infty} \frac{\int_0^\infty [H(t)]^n\overline{G}(t)f(t)\,\mathrm{d}t}{\int_0^\infty [H(t)]^n\overline{G}(t)\overline{F}(t)\,\mathrm{d}t} = h(\infty). \tag{4.13}$$

Clearly,

$$\frac{\int_0^\infty [H(t)]^n\overline{G}(t)f(t)\,\mathrm{d}t}{\int_0^\infty [H(t)]^n\overline{G}(t)\overline{F}(t)\,\mathrm{d}t} \leq h(\infty).$$

On the other hand, we have, for any $T>0$,

$$\begin{aligned} \frac{\int_0^\infty [H(t)]^n\overline{G}(t)f(t)\,\mathrm{d}t}{\int_0^\infty [H(t)]^n\overline{G}(t)\overline{F}(t)\,\mathrm{d}t} &= \frac{\int_0^T [H(t)]^n\overline{G}(t)f(t)\,\mathrm{d}t + \int_T^\infty [H(t)]^n\overline{G}(t)f(t)\,\mathrm{d}t}{\int_0^T [H(t)]^n\overline{G}(t)\overline{F}(t)\,\mathrm{d}t + \int_T^\infty [H(t)]^n\overline{G}(t)\overline{F}(t)\,\mathrm{d}t} \\ &\geq \frac{h(T)\int_T^\infty [H(t)]^n\overline{G}(t)\overline{F}(t)\,\mathrm{d}t}{\int_0^T [H(t)]^n\overline{G}(t)\overline{F}(t)\,\mathrm{d}t + \int_T^\infty [H(t)]^n\overline{G}(t)\overline{F}(t)\,\mathrm{d}t} \\ &= \frac{h(T)}{1+\{\int_0^T [H(t)]^n\overline{G}(t)\overline{F}(t)\,\mathrm{d}t \,/ \int_T^\infty [H(t)]^n\overline{G}(t)\overline{F}(t)\,\mathrm{d}t\}}. \end{aligned}$$

Furthermore, the bracket of the denominator is, for $T < T_1$,

$$\frac{\int_0^T [H(t)]^n \overline{G}(t)\overline{F}(t)\,\mathrm{d}t}{\int_T^\infty [H(t)]^n \overline{G}(t)\overline{F}(t)\,\mathrm{d}t} \le \frac{[H(T)]^n \int_0^T \overline{G}(t)\overline{F}(t)\,\mathrm{d}t}{\int_{T_1}^\infty [H(t)]^n \overline{G}(t)\overline{F}(t)\,\mathrm{d}t}$$

$$\le \frac{[H(T)]^n}{[H(T_1)]^n} \frac{\int_0^T \overline{G}(t)\overline{F}(t)\,\mathrm{d}t}{\int_{T_1}^\infty \overline{G}(t)\overline{F}(t)\,\mathrm{d}t} \to 0 \quad \text{as } n \to \infty.$$

Thus, we have

$$h(\infty) \ge \lim_{n\to\infty} \frac{\int_0^\infty [H(t)]^n \overline{G}(t) f(t)\,\mathrm{d}t}{\int_0^\infty [H(t)]^n \overline{G}(t)\overline{F}(t)\,\mathrm{d}t} \ge h(T)$$

which implies (4.13) because T is arbitrary. Similarly,

$$\lim_{n\to\infty} \frac{\int_0^\infty [H(t)]^n \overline{F}(t) g(t)\,\mathrm{d}t}{\int_0^\infty [H(t)]^n \overline{F}(t)\overline{G}(t)\,\mathrm{d}t} = r(\infty). \tag{4.14}$$

Therefore, combining (4.13) and (4.14), we complete the proof. ∎

From Theorems 4.3 and 4.4, we easily have that for any function $\phi(t)$ that is continuous and $\phi(t) \neq 0$ for any $t > 0$, if the failure rate $h(t)$ is increasing then

$$\frac{\int_0^T \{[H(t)]^n/n!\}\phi(t) f(t)\,\mathrm{d}t}{\int_0^T \{[H(t)]^n/n!\}\phi(t)\overline{F}(t)\,\mathrm{d}t} \tag{4.15}$$

is increasing in n and converges to $h(T)$ as $n \to \infty$ for any $T > 0$.

In all results of Theorems 4.2 through 4.4 it can easily be seen that if the failure rates are strictly increasing then $E\{X_n\}$, the functions (4.8), (4.10), and (4.15) are also strictly increasing.

4.2 Periodic Replacement with Minimal Repair

A new unit begins to operate at time $t = 0$, and when it fails, only minimal repair is made. Also, a unit is replaced at periodic times kT $(k = 1, 2, \dots)$ independent of its age, and any unit becomes as good as new after replacement (Figure 4.1). It is assumed that the repair and replacement times are negligible. Suppose that the failure times of a unit have a density function $f(t)$ and a distribution $F(t)$ with finite mean $\mu \equiv \int_0^\infty \overline{F}(t)\mathrm{d}t < \infty$ and its failure rate $h(t) \equiv f(t)/\overline{F}(t)$.

Consider one cycle with constant time T $(0 < T \le \infty)$ from the planned replacement to the next one. Let c_1 be the cost of minimal repair and c_2 be the cost of the planned replacement. Then, the expected cost of one cycle is, from (3.2),

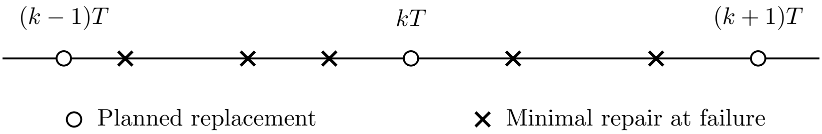

Fig. 4.1. Process of periodic replacement with minimal repair

$$c_1 E\{N_1(T)\} + c_2 E\{N_2(T)\} = c_1 H(T) + c_2$$

because the expected number of failures during one cycle is $E\{N_1(T)\} = \int_0^T h(t)\mathrm{d}t \equiv H(T)$ from (4.7). Therefore, from (3.3), the expected cost rate is [1, p. 99],

$$C(T) = \frac{1}{T}[c_1 H(T) + c_2]. \tag{4.16}$$

If a unit is never replaced (*i.e.*, $T = \infty$) then $\lim_{T\to\infty} H(T)/T = h(\infty)$ if it exists, which may possibly be infinite, and $C(\infty) \equiv \lim_{T\to\infty} C(T) = c_1 h(\infty)$.

Furthermore, suppose that a unit is replaced when the total operating time is T. Then, the availability is given by

$$A(T) = \frac{T}{T + \beta_1 H(T) + \beta_2}, \tag{4.17}$$

where β_1 = time of minimal repair and β_2 = time of replacement. Thus, the policy maximizing $A(T)$ is the same as minimizing the expected cost rate $C(T)$ in (4.16) by replacing β_i with c_i.

We seek an optimum planned time T^* that minimizes the expected cost rate $C(T)$ in (4.16). Differentiating $C(T)$ with respect to T and setting it equal to zero, we have

$$Th(T) - H(T) = \frac{c_2}{c_1} \qquad \text{or} \qquad \int_0^T t\,\mathrm{d}h(t) = \frac{c_2}{c_1}. \tag{4.18}$$

Suppose that the failure rate $h(t)$ is continuous and strictly increasing. Then, the left-hand side of (4.18) is also strictly increasing because

$$\begin{aligned}
&(T + \Delta T)h(T + \Delta T) - H(T + \Delta T) - Th(T) + H(T)\\
&\quad = T[h(T + \Delta T) - h(T)] + \int_T^{T+\Delta T} [h(T + \Delta T) - h(t)]\,\mathrm{d}t > 0
\end{aligned}$$

for any $\Delta T > 0$. Thus, if a solution T^* to (4.18) exists then it is unique, and the resulting cost rate is

$$C(T^*) = c_1 h(T^*). \tag{4.19}$$

In addition, if $\int_0^\infty t dh(t) > c_2/c_1$ then there exists a finite solution to (4.18). Also, from (4.18),

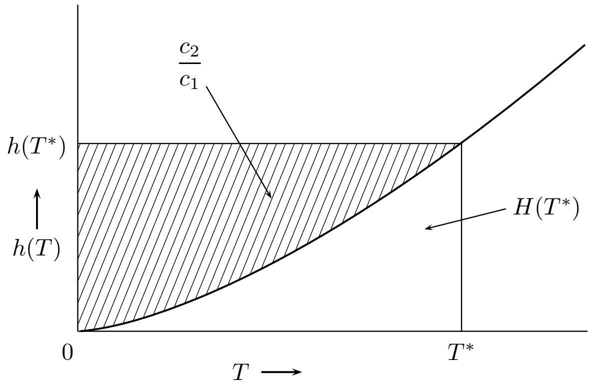

Fig. 4.2. Optimum T^* for failure rate $h(T)$

$$Th(T) - H(T) > T_1 h(T) - H(T_1)$$

for any $T > T_1$. Thus, if $h(t)$ is strictly increasing to infinity then there exists a finite and unique T^* that satisfies (4.18).

When $h(t)$ is strictly increasing, we have, from Theorem 3.3,

$$Th(T) - \int_0^T h(t)\,\mathrm{d}t \geq h(T)\int_0^T \overline{F}(t)\,\mathrm{d}t - F(T)$$

whose right-hand side agrees with (3.9). That is, an optimum T^* is not greater than that of an age replacement in Section 3.1. Thus, from Theorem 3.2, if $h(\infty) > (c_1 + c_2)/(\mu c_1)$ then a finite solution to (4.18) exists.

Figure 4.2 shows graphically an optimum time T^* given in (4.18) for the failure rate $h(T)$. If $h(T)$ were roughly drawn then T^* could be given by the time when the area covered with slash lines becomes equal to the ratio of c_2/c_1. So that, when $h(T)$ is a concave function, $H(T^*) > c_2/c_1$, and when $h(T)$ is a convex function, $H(T^*) < c_2/c_1$. For example, when the failure distribution is Weibull, *i.e.*, $F(t) = 1 - \exp(-t^m)$ $(m > 1)$, $H(T^*) > c_2/c_1$ for $1 < m < 2$, $= c_2/c_1$ for $m = 2$ and $< c_2/c_1$ for $m > 2$. If the cumulative hazard function $H(t)$ were statistically estimated, the replacement time that satisfies $H(T) = c_2/c_1$ could be utilized as one indicator of replacement time [22] (see Example 3.1 in Chapter 3).

If the cost of minimal repair depends on the age t of a unit and is given by $c_1(t)$, the expected cost rate is

$$C(T) = \frac{1}{T}\left[\int_0^T c_1(t)h(t)\,\mathrm{d}t + c_2\right]. \tag{4.20}$$

Finally, we consider a system consisting of n identical units that operate independently of each other. It is assumed that all are replaced together at times kT $(k = 1, 2, \dots)$ and each failed unit between replacements undergoes minimal repair. Then, the expected cost rate is

$$C(T;n) = \frac{1}{T}[nc_1 H(T) + c_2], \tag{4.21}$$

where c_1 = cost of minimal repair for one failed unit, and c_2 = cost of planned replacement for all units at time T.

4.3 Periodic Replacement with *N*th Failure

A unit is replaced at time T or at the Nth $(N = 1, 2, \dots)$ failure after its installation, whichever occurs first, where T is a positive constant and previously specified. A unit undergoes only minimal repair at failures between replacements. This policy is called *Policy IV* [12].

From Theorem 4.1, the mean time to replacement is

$$\begin{aligned} T\Pr\{Y_N > T\} + \int_0^T t\,\mathrm{d}\Pr\{Y_N \le t\} &= \int_0^T \Pr\{Y_N > t\}\,\mathrm{d}t \\ &= \sum_{j=0}^{N-1} \int_0^T p_j(t)\,\mathrm{d}t, \end{aligned}$$

where $p_j(t)$ is given in (4.6), and the expected number of failures before replacement is

$$\begin{aligned} &\sum_{j=0}^{N-1} j\Pr\{N(T) = j\} + (N-1)\Pr\{Y_N \le T\} \\ &\qquad = N - 1 - \sum_{j=0}^{N-1} (N-1-j)p_j(T). \end{aligned}$$

Therefore, from (3.3), the expected cost rate is

$$C(N;T) = \frac{c_1\left[N - 1 - \sum_{j=0}^{N-1}(N-1-j)p_j(T)\right] + c_2}{\sum_{j=0}^{N-1}\int_0^T p_j(t)\,\mathrm{d}t} \quad (N = 1, 2, \dots), \tag{4.22}$$

where c_1 = cost of minimal repair and c_2 = cost of planned replacement at time T or at number N. It is evident that

$$C(\infty;T) \equiv \lim_{N\to\infty} C(N;T) = \frac{1}{T}[c_1 H(T) + c_2]$$

which agrees with (4.16) for the periodic replacement with planned time T.

Let T^* be the optimum time that minimizes $C(\infty;T)$ and is given by a unique solution to (4.18) if it exists, or $T^* = \infty$ if it does not. We seek an optimum number N^* such that $C(N^*;T) = \min_N C(N;T)$ for a fixed $0 < T \leq \infty$, when the failure rate $h(t)$ is continuous and strictly increasing.

Theorem 4.5. Suppose that $0 < T^* \leq \infty$.

(i) If $T > T^*$ then there exists a finite and unique minimum N^* that satisfies

$$L(N;T) \geq \frac{c_2}{c_1} \quad (N = 1, 2, \dots), \tag{4.23}$$

where

$$\begin{aligned} L(N;T) \equiv & \frac{\sum_{j=N}^{\infty} p_j(T) \sum_{j=0}^{N-1} \int_0^T p_j(t)\,\mathrm{d}t}{\int_0^T p_N(t)\,\mathrm{d}t} \\ & - \left[N - 1 - \sum_{j=0}^{N-1} (N-1-j) p_j(T)\right] \quad (N = 1, 2, \dots). \end{aligned}$$

(ii) If $T \leq T^*$ or $T^* = \infty$ then no N^* satisfying (4.23) exists.

Proof. For simplicity of computation, we put $C(0;T) = \infty$. To find an N^* that minimizes $C(N;T)$ for a fixed T, we form the inequality $C(N+1;T) \geq C(N;T)$, and have (4.23). Hence, we may seek a minimum N^* that satisfies (4.23).

Using the relation

$$\sum_{j=N+1}^{\infty} \frac{[H(T)]^j}{j!} \mathrm{e}^{-H(T)} = \int_0^T \frac{[H(t)]^N}{N!}\,\mathrm{d}F(t) \quad (N = 0, 1, 2, \dots)$$

we have, from Theorem 4.3,

$$\begin{aligned} & L(N+1;T) - L(N;T) \\ & \quad = \sum_{j=0}^{N} \int_0^T p_j(t)\,\mathrm{d}t \left[\frac{\sum_{j=N+1}^{\infty} p_j(T)}{\int_0^T p_{N+1}(t)\,\mathrm{d}t} - \frac{\sum_{j=N}^{\infty} p_j(T)}{\int_0^T p_N(t)\,\mathrm{d}t}\right] > 0 \end{aligned}$$

and

$$L(\infty;T) \equiv \lim_{N\to\infty} L(N;T) = Th(T) - H(T)$$

which is equal to the left-hand side of (4.18) and is strictly increasing in T.

Suppose that $0 < T^* < \infty$. If $L(\infty;T) > c_2/c_1$, *i.e.*, $T > T^*$, then there exists a finite and unique minimum N^* that satisfies (4.23). On the other hand, if $L(\infty;T) \leq c_2/c_1$, *i.e.*, $T \leq T^*$, then $C(N;T)$ is decreasing in N, and no solution satisfying (4.23) exists. Finally, if $T^* = \infty$ then no solution to (4.23) exists inasmuch as $L(\infty;T) < c_2/c_1$ for all T. ∎

This theorem describes that when a unit is planned to be replaced at time $T > T^*$ for some reason, it also should be replaced at the N^*th failure before time T.

If c_2 is the cost of planned replacement at the Nth failure and c_3 is the cost at time T, then the expected cost rate in (4.22) is rewritten as

$$C(N;T) = \frac{c_1\left[N-1-\sum_{j=0}^{N-1}(N-1-j)p_j(T)\right] + c_2\sum_{j=N}^{\infty}p_j(T) + c_3\sum_{j=0}^{N-1}p_j(T)}{\sum_{j=0}^{N-1}\int_0^T p_j(t)\,\mathrm{d}t}. \tag{4.24}$$

Similar replacement policies were discussed in [23–33].

Next, suppose that a unit is replaced only at the Nth failure. Then, the expected cost rate is, from (4.22),

$$C(N) \equiv \lim_{T\to\infty} C(N;T) = \frac{c_1(N-1)+c_2}{\sum_{j=0}^{N-1}\int_0^\infty p_j(t)\,\mathrm{d}t} \quad (N=1,2,\dots). \tag{4.25}$$

In a similar way to that of obtaining Theorem 4.5, we derive an optimum number N^* that minimizes $C(N)$.

Theorem 4.6. If $h(\infty) > c_2/(\mu c_1)$ then there exists a finite and unique minimum N^* that satisfies

$$L(N) \geq \frac{c_2}{c_1} \quad (N=1,2,\dots) \tag{4.26}$$

and the resulting cost rate is

$$\frac{c_1}{\int_0^\infty p_{N^*-1}(t)\,\mathrm{d}t} < C(N^*) \leq \frac{c_1}{\int_0^\infty p_{N^*}(t)\,\mathrm{d}t}, \tag{4.27}$$

where

$$L(N) \equiv \lim_{T\to\infty} L(N;T) = \frac{\sum_{j=0}^{N-1}\int_0^\infty p_j(t)\,\mathrm{d}t}{\int_0^\infty p_N(t)\,\mathrm{d}t} - (N-1) \quad (N=1,2,\dots).$$

Proof. The inequality $C(N+1) \geq C(N)$ implies (4.26). It is easily seen that $L(N+1) - L(N) > 0$ from Theorem 4.2. Thus, if a solution to (4.26) exists then it is unique.

Furthermore, we have the inequality

$$L(N) \geq \frac{\mu}{\int_0^\infty p_N(t)\,\mathrm{d}t} \tag{4.28}$$

because $\int_0^\infty p_N(t)\mathrm{d}t$ is decreasing in N from Theorem 4.2. Therefore, if

$$\lim_{N\to\infty}\frac{\mu}{\int_0^\infty p_N(t)\,\mathrm{d}t} > \frac{c_2}{c_1},$$

i.e., if $h(\infty) > c_2/(\mu c_1)$, then a solution to (4.26) exists, and it is unique. Also, we easily have (4.27) from the inequalities $L(N^*-1) < c_2/c_1$ and $L(N^*) \geq c_2/c_1$. ∎

Suppose that $h(\infty) > c_2/(\mu c_1)$. Then, from (4.28), there exists a finite and unique minimum $\overline{N}$ that satisfies

$$\int_0^\infty p_N(t)\,\mathrm{d}t \leq \frac{\mu c_1}{c_2} \quad (N = 1, 2, \dots) \tag{4.29}$$

and $N^* \leq \overline{N}$.

Example 4.1. Suppose that the failure time of a unit has a Weibull distribution; *i.e.,* $\overline{F}(t) = \exp(-t^m)$ for $m > 1$. Then, $h(t)$ is strictly increasing from 0 to infinity, and

$$\int_0^\infty \frac{[H(t)]^N}{N!}\mathrm{e}^{-H(t)}\,\mathrm{d}t = \frac{1}{m}\frac{\Gamma(N+1/m)}{\Gamma(N+1)}$$
$$\sum_{j=0}^{N-1}\int_0^\infty \frac{[H(t)]^j}{j!}\mathrm{e}^{-H(t)}\,\mathrm{d}t = \frac{\Gamma(N+1/m)}{\Gamma(N)}.$$

Thus, there exists a finite and unique minimum that satisfies (4.26), which is given by

$$N^* = \left[\frac{c_2 - c_1}{(m-1)c_1}\right] + 1,$$

where $[x]$ denotes the greatest integer contained in x. ∎

4.4 Modified Replacement Models

We show the following modified models of periodic replacement with minimal repair at failures: (1) replacement with discounting, (2) replacement in discrete time, (3) replacement of a used unit, (4) replacement with random and wearout failures, and (5) replacement with threshold level. The detailed derivations are omitted and optimum policies for each model are directly given.

(1) Replacement with Discounting

Suppose that all costs are discounted with rate α $(0 < \alpha < \infty)$. In a similar way to that for obtaining (3.14) in **(1)** of Section 3.2, the total expected cost for an infinite time span is

$$C(T;\alpha) = \frac{c_1\int_0^T \mathrm{e}^{-\alpha t}h(t)\,\mathrm{d}t + c_2\mathrm{e}^{-\alpha T}}{1-\mathrm{e}^{-\alpha T}}. \tag{4.30}$$

Differentiating $C(T;\alpha)$ with respect to T and setting it equal to zero

$$\frac{1-\mathrm{e}^{-\alpha T}}{\alpha}h(T) - \int_0^T \mathrm{e}^{-\alpha t}h(t)\,\mathrm{d}t = \frac{c_2}{c_1} \tag{4.31}$$

and the resulting cost rate is

$$C(T^*;\alpha) = \frac{c_1}{\alpha}h(T^*) - c_2. \tag{4.32}$$

Note that $\lim_{\alpha\to 0}\alpha C(T;\alpha) = C(T)$ in (4.16), and (4.31) agrees with (4.18) as $\alpha \to 0$.

(2) Replacement in Discrete Time

A unit is replaced at cycles kN $(k = 1,2,\dots)$ and a failed unit between planned replacements undergoes only minimal repair. Then, using the same notation and methods in **(2)** of Section 3.2, the expected cost rate is

$$C(N) = \frac{1}{N}\left[c_1\sum_{j=1}^{N} h_j + c_2\right] \quad (N = 1,2,\dots) \tag{4.33}$$

and an optimum number N^* is given by a minimum solution that satisfies

$$Nh_{N+1} - \sum_{j=1}^{N} h_j \geq \frac{c_2}{c_1} \quad (N = 1,2,\dots). \tag{4.34}$$

(3) Replacement of a Used Unit

Consider the periodic replacement with minimal repair at failures for a used unit. A unit is replaced at times kT $(k = 1,2,\dots)$ by the same used unit with age x, where x $(0 \leq x < \infty)$ is previously specified. Then, the expected cost rate is, from (4.16),

$$C(T;x) = \frac{1}{T}\left[c_1\int_x^{T+x} h(t)\,\mathrm{d}t + c_2(x)\right], \tag{4.35}$$

where c_1 = cost of minimal repair and $c_2(x)$ = acquisition cost of a used unit with age x which may be decreasing in x. In this case, (4.18) and (4.19) are rewritten as

$$Th(T+x) - \int_x^{T+x} h(t)\,\mathrm{d}t = \frac{c_2(x)}{c_1} \tag{4.36}$$

$$C(T^*;x) = c_1 h(T^*+x). \tag{4.37}$$

Next, consider the problem that it is most economical to use a unit of a certain age. Suppose that x is a variable, and inversely, T is constant and $c_2(x)$

is differentiable. Then, differentiating $C(T;x)$ with respect to x and setting it equal to zero imply

$$h(T+x) - h(x) = -\frac{c_2'(x)}{c_1} \tag{4.38}$$

which is a necessary condition that a finite x minimizes $C(T;x)$ for a fixed T.

(4) Replacement with Random and Wearout Failures

We consider a modified replacement policy for a unit with random and wearout failure periods, where an operating unit enters a wearout failure period at a fixed time T_0, after it has operated continuously in a random failure period. It is assumed that a unit is replaced at planned time $T+T_0$, where T_0 is constant and previously given, and it undergoes only minimal repair at failures between replacements [34, 35].

Suppose that a unit has a constant failure rate λ for $0 < t \le T_0$ in a random failure period and $\lambda + h(t-T_0)$ for $t > T_0$ in a wearout failure period. Then, the expected cost rate is

$$C(T;T_0) = c_1\lambda + \frac{c_1 H(T) + c_2}{T+T_0}. \tag{4.39}$$

Thus, if $h(t)$ is strictly increasing and there exists a solution T^* that satisfies

$$(T+T_0)h(T) - H(T) = \frac{c_2}{c_1} \tag{4.40}$$

then it is unique and the resulting cost rate is

$$C(T^*;T_0) = c_1[\lambda + h(T^*)]. \tag{4.41}$$

Furthermore, it is easy to see that T^* is a decreasing function of T_0 because the left-hand side of (4.40) is increasing in T_0 for a fixed T. Thus, an optimum time T^* is less than the optimum one given in (4.18) as we have expected.

(5) Replacement with Threshold Level

Suppose that if more failures have occurred between periodic replacements then the total cost would be higher than expected. For example, if more than K failures have occurred and the number of K parts is needed for providing against $K-1$ spares during a planned interval, an extra cost would result from the downtime, the ordering and delivery of spares, and repair. Let $N(T)$ be the total number of failures during $(0,T]$ and K be its threshold number. Then, from (4.16) and (4.6), the expected cost rate is

$$C(T;K) = \frac{1}{T}\left[c_1 H(T) + c_2 + c_3 \Pr\{N(T) \geq K\}\right]$$
$$= \frac{1}{T}\left[c_1 H(T) + c_2 + c_3 \sum_{j=K}^{\infty} p_j(T)\right], \tag{4.42}$$

where c_3 = additional cost when the number of failures has exceeded a threshold level K.

4.5 Replacements with Two Different Types

Periodic replacement with minimal repair is modified and extended in several ways. We show typical models of periodic replacement with (1) two types of failures and (2) two types of units.

(1) Two Types of Failures

We may generally classify failure into failure modes: partial and total failures, slight and serious failures, minor and major failures, or simply faults and failures. Generalized replacement models of two types of failures were proposed in [36–40].

Consider a unit with two types of failures. When a unit fails, type 1 failure occurs with probability p $(0 \leq p \leq 1)$ and is removed by minimal repair, and type 2 failure occurs with probability $1-p$ and is removed by replacement. Type 1 failure is a minor failure that is easily restored to the same operating state by minimal repair, and type 2 failure incurs a total breakdown and needs replacement or repair.

A unit is replaced at the time of type 2 failure or Nth type 1 failure, whichever occurs first. Then, the expected number of minimal repairs, *i.e.*, type 1 failures before replacement, is

$$(N-1)p^N + \sum_{j=1}^{N}(j-1)p^{j-1}(1-p) = \begin{cases} \dfrac{p-p^N}{1-p} & \text{for } 0 \leq p < 1 \\ N-1 & \text{for } p = 1. \end{cases}$$

Thus, the expected cost rate is, from (4.25),

$$C(N;p) = \frac{c_1[(p-p^N)/(1-p)] + c_2}{\sum_{j=0}^{N-1} p^j \int_0^\infty p_j(t)\,\mathrm{d}t} \quad (N = 1, 2, \dots) \tag{4.43}$$

for $0 \leq p < 1$, where c_1 = cost of minimal repair for type 1 failure and c_2 = cost of replacement at the Nth type 1 or type 2 failure. When $p \to 1$, $C(N;1) \equiv \lim_{p\to 1} C(N;p)$ is equal to (4.25) and the optimum policy is given in Theorem 4.6. When $p = 0$, $C(N;0) = c_2/\mu$, which is constant for all N, and a unit is replaced only at type 2 failure. Therefore, we need only discuss an optimum policy in the case of $0 < p < 1$ when the failure rate $h(t)$ is strictly increasing. To simplify equations, we denote $\mu_p \equiv \int_0^\infty [\overline{F}(t)]^p \, \mathrm{d}t = \int_0^\infty \mathrm{e}^{-pH(t)} \, \mathrm{d}t$. When $p = 1$, $\mu_1 = \mu$ which is the mean time to failure of a unit.

Theorem 4.7. (i) If $h(\infty) > [c_1 p + c_2(1-p)]/[c_1(1-p)\mu_{1-p}]$ then there exists a finite and unique minimum $N^*(p)$ that satisfies

$$L(N;p) \geq \frac{c_2}{c_1} \quad (N = 1, 2, \dots), \tag{4.44}$$

where

$$L(N;p) \equiv \frac{\sum_{j=0}^{N-1} p^j \int_0^\infty p_j(t) \, \mathrm{d}t}{\int_0^\infty p_N(t) \, \mathrm{d}t} - \frac{p - p^N}{1 - p} \quad (N = 1, 2, \dots).$$

(ii) If $h(\infty) \leq [c_1 p + c_2(1-p)]/[c_1(1-p)\mu_{1-p}]$ then $N^*(p) = \infty$, and the resulting cost rate is

$$C(\infty;p) \equiv \lim_{N\to\infty} C(N;p) = \frac{c_1[p/(1-p)] + c_2}{\mu_{1-p}}. \tag{4.45}$$

Proof. The inequality $C(N+1;p) \geq C(N;p)$ implies (4.44). Furthermore, it is easily seen from Theorem 4.2 that $L(N;p)$ is an increasing function of N, and hence, $\lim_{N\to\infty} L(N;p) = \mu_{1-p} h(\infty) - [p/(1-p)]$. Thus, in a similar way to that of obtaining Theorem 4.6, if $h(\infty) > [c_1 p + c_2(1-p)]/[c_1(1-p)\mu_{1-p}]$ then there exists a finite and unique minimum $N^*(p)$ that satisfies (4.44). On the other hand, if $h(\infty) \leq [c_1 p + c_2(1-p)]/[c_1(1-p)\mu_{1-p}]$ then $L(N;p) < c_2/c_1$ for all N, and hence, $N^*(p) = \infty$, and we have (4.45). ∎

It is easily noted that $\partial L(N;p)/\partial p > 0$ for all N. Thus, if $h(\infty) > [c_1 p + c_2(1-p)]/[c_1(1-p)\mu_{1-p}]$ for $0 < p < 1$ then $N^*(p)$ is decreasing in p, and $\overline{N} \geq N^*(p) \geq N^*$, where both N^* and $\overline{N}$ exist and are given in (4.26) and (4.29), respectively.

Until now, it has been assumed that the replacement costs for both the Nth type 1 failure and type 2 failure are the same. In reality, they may be different from each other. It is supposed that c_2 is the replacement cost of the Nth type 1 failure and c_3 is the replacement cost of the type 2 failure. Then, the expected cost rate in (4.43) is rewritten as

$$C(N;p) = \frac{c_1[(p - p^N)/(1-p)] + c_2 p^N + c_3(1 - p^N)}{\sum_{j=0}^{N-1} p^j \int_0^\infty p_j(t) \, \mathrm{d}t} \quad (N = 1, 2, \dots). \tag{4.46}$$

Example 4.2. We compute an optimum number $N^*(p)$ that minimizes the expected cost rate $C(N;p)$ in (4.46) when $\overline{F}(t) = \exp(-t^m)$ for $m > 1$. When

$c_2 = c_3$, it is shown from Theorem 4.7 that $N^*(p)$ exists uniquely and is decreasing in p for $0 < p < 1$. Furthermore, when $p = 1$, $N^*(p)$ is given in Example 4.1. If $c_1 + (c_3 - c_2)(1 - p) > 0$ then $N^*(p)$ is given by a minimum value such that

$$\frac{(1-p)\Gamma(N+1)}{\Gamma(N+1/m)}\sum_{j=0}^{N-1}\frac{p^j\Gamma(j+1/m)}{\Gamma(j+1)} + p^N \geq \frac{c_1 p + c_3(1-p)}{c_1 + (c_3 - c_2)(1-p)}.$$

It is easily seen that $N^*(p)$ is small when c_1/c_2 or c_3/c_2 for $c_2 > c_1$ is large. Conversely, if $c_1 + (c_3 - c_2)(1 - p) \leq 0$ then $N^*(p) = \infty$.

Table 4.1. Variation in the optimum number $N^*(p)$ for probability p of type 1 failure and ratio of c_3 to c_2 when $m = 2$ and $c_1/c_2 = 0.1$

p	c_3/c_2						
	0.8	0.9	1.0	1.2	1.5	2.0	3.0
0.1	∞	∞	30	6	2	1	1
0.2	∞	∞	27	6	3	1	1
0.3	∞	220	24	6	3	2	1
0.4	∞	112	22	7	3	2	1
0.6	288	39	17	7	4	2	1
0.7	64	25	15	8	5	3	2
0.8	26	17	13	8	6	4	2
0.9	14	12	11	9	7	5	4
1.0	10	10	10	10	10	10	10

Table 4.1 gives the optimum number $N^*(p)$ for probability p of type 1 failure and the ratio of cost c_3 to cost c_2 when $m = 2$ and $c_1/c_2 = 0.1$. It is of great interest that $N^*(p)$ is increasing in p for $c_3 > c_2$, however, it is decreasing for $c_3 \leq c_2$. We can explain the reason why $N^*(p)$ is increasing in p for c_3/c_2. If $c_3 > c_2$ then the replacement cost for type 1 failure is cheaper than that for type 2 failure and the number of its failures increases with p, and so, $N^*(p)$ is large when p is large. This situation reflects a real situation. On the other hand, if $c_3 \leq c_2$ then it is not useful to replace the unit frequently before type 2 failure, however, the total cost of minimal repairs for type 1 increases as the number of its failures does with p. Thus, it may be better to replace the unit preventively at some number N when p is large. Evidently, $N^*(p)$ is rapidly increasing when c_1 is small enough. ∎

(2) Two Types of Units

Most systems consist of vital and nonvital parts or essential and nonessential units. If vital parts fail then a system becomes dangerous or incurs high cost. It would be wise to make replacements or overhauls before failure at periodic times. The optimum replacement policies for systems with two units were derived in [42–48]. Furthermore, the optimum inspection schedule of a production system [49] and a storage system [50] with two types of units was studied.

Consider a system with two types of units that operate statistically independently. When unit 1 fails, it undergoes minimal repair instantaneously and begins to operate again. When unit 2 fails, the system is replaced without repairing unit 2. Unit 1 has a failure distribution $F_1(t)$, the failure rate $h_1(t)$ and $H_1(t) \equiv \int_0^t h_1(u)\mathrm{d}u$, which have the same assumptions as those in Section 4.2, whereas unit 2 has a failure distribution $F_2(t)$ with finite mean μ_2 and the failure rate $h_2(t)$, where $\overline{F}_i \equiv 1 - F_i$ $(i = 1, 2)$.

Suppose that the system is replaced at the time of unit 2 failure or Nth unit 1 failure, whichever occurs first. Then, the mean time to replacement is

$$\sum_{j=0}^{N-1} \int_0^\infty t p_j(t)\,\mathrm{d}F_2(t) + \int_0^\infty t\overline{F}_2(t) p_{N-1}(t) h_1(t)\,\mathrm{d}t = \sum_{j=0}^{N-1} \int_0^\infty \overline{F}_2(t) p_j(t)\,\mathrm{d}t,$$

where $p_j(t) = \{[H_1(t)]^j/j!\}\mathrm{e}^{-H_1(t)}$ $(j = 0, 1, 2, \dots)$, and the expected number of minimal repairs before replacement is

$$\sum_{j=0}^{N-1} j \int_0^\infty p_j(t)\,\mathrm{d}F_2(t) + (N-1)\int_0^\infty \overline{F}_2(t) p_{N-1}(t) h_1(t)\,\mathrm{d}t$$
$$= \sum_{j=0}^{N-2} \int_0^\infty \overline{F}_2(t) p_j(t) h_1(t)\,\mathrm{d}t,$$

where $\sum_{j=0}^{-1} \equiv 0$. Thus, the expected cost rate is

$$C(N) = \frac{c_1 \sum_{j=0}^{N-2} \int_0^\infty \overline{F}_2(t) p_j(t) h_1(t)\,\mathrm{d}t + c_2}{\sum_{j=0}^{N-1} \int_0^\infty \overline{F}_2(t) p_j(t)\,\mathrm{d}t} \quad (N = 1, 2, \dots). \tag{4.47}$$

When $\overline{F}_2(t) \equiv 1$ for $t \geq 0$, $C(N)$ is equal to (4.25), and when $\overline{F}_2(t) \equiv 1$ for $t \leq T$ and 0 for $t > T$, this is equal to (4.22).

We have the following optimum number N^* that minimizes $C(N)$.

Theorem 4.8. Suppose that $h_1(t)$ is continuous and increasing. If there exists a minimum N^* that satisfies

$$L(N) \geq \frac{c_2}{c_1} \quad (N = 1, 2, \dots) \tag{4.48}$$

then it is unique and it minimizes $C(N)$, where

$$L(N) \equiv \frac{\int_0^\infty \overline{F}_2(t)p_{N-1}(t)h_1(t)\,\mathrm{d}t}{\int_0^\infty \overline{F}_2(t)p_N(t)\,\mathrm{d}t} \sum_{j=0}^{N-1} \int_0^\infty \overline{F}_2(t)p_j(t)\,\mathrm{d}t - \sum_{j=0}^{N-2} \int_0^\infty \overline{F}_2(t)p_j(t)h_1(t)\,\mathrm{d}t \quad (N = 1, 2, \dots).$$

Proof. The inequality $C(N+1) \geq C(N)$ implies (4.48). In addition,

$$L(N+1) - L(N) = \sum_{j=0}^{N} \int_0^\infty \overline{F}_2(t)p_j(t)\,\mathrm{d}t \times \left[\frac{\int_0^\infty \overline{F}_2(t)p_N(t)h_1(t)\,\mathrm{d}t}{\int_0^\infty \overline{F}_2(t)p_{N+1}(t)\,\mathrm{d}t} - \frac{\int_0^\infty \overline{F}_2(t)p_{N-1}(t)h_1(t)\,\mathrm{d}t}{\int_0^\infty \overline{F}_2(t)p_N(t)\,\mathrm{d}t} \right] \geq 0$$

because $\int_0^\infty \overline{F}_2(t)p_N(t)h_1(t)\mathrm{d}t / \int_0^\infty \overline{F}_2(t)p_{N+1}(t)\mathrm{d}t$ is increasing in N from Theorem 4.4, when $h_1(t)$ is increasing. Thus, if a minimum solution to (4.48) exists then it is unique. ∎

Furthermore, we also have, from Theorem 4.4,

$$L(\infty) \equiv \lim_{N\to\infty} L(N) = \mu_2[h_1(\infty) + h_2(\infty)] - \int_0^\infty \overline{F}_2(t)h_1(t)\,\mathrm{d}t.$$

Thus, if $h_1(t)+h_2(t)$ is strictly increasing and $h_1(\infty)+h_2(\infty) > (1/\mu_2)[(c_2/c_1) + \int_0^\infty \overline{F}_2(t)h_1(t)\,\mathrm{d}t]$ then there exists a finite and unique minimum N^* that satisfies (4.48). For example, suppose that $h_2(t)$ is strictly increasing and $h_1(t)$ is increasing. Then, because $L(\infty) \geq \mu_2 h_2(\infty)$, if $h_2(\infty) > c_2/(\mu_2 c_1)$ then a finite minimum to (4.48) exists uniquely.

If c_2 is the replacement cost of the Nth failure of unit 1 and c_3 is the replacement cost of unit 2 failure, then the expected cost rate $C(N)$ in (4.47) is rewritten as

$$C(N) = \frac{c_1 \sum_{j=0}^{N-2} \int_0^\infty \overline{F}_2(t)p_j(t)h_1(t)\,\mathrm{d}t + c_2 \int_0^\infty \overline{F}_2(t)p_{N-1}(t)h_1(t)\,\mathrm{d}t + c_3\left[1 - \int_0^\infty \overline{F}_2(t)p_{N-1}(t)h_1(t)\,\mathrm{d}t\right]}{\sum_{j=0}^{N-1} \int_0^\infty \overline{F}_2(t)p_j(t)\,\mathrm{d}t} \quad (N = 1, 2, \dots). \tag{4.49}$$

References

1. Barlow RE and Proschan F (1965) Mathematical Theory of Reliability. J Wiley & Sons, New York.
2. Holland CW, McLean RA (1975) Applications of replacement theory. AIIE Trans 7:42–47.

3. Tilquin C, Cléroux R (1975) Periodic replacement with minimal repair at failure and adjustment costs. Nav Res Logist Q 22:243–254.
4. Boland PJ (1982) Periodic replacement when minimal repair costs vary with time. Nav Res Logist Q 29:541–546.
5. Boland PJ, Proschan F (1982) Periodic replacement with increasing minimal repair costs at failure. Oper Res 30:1183–1189.
6. Chen M, Feldman RM (1997) Optimal replacement policies with minimal repair and age-dependent costs. Eur J Oper Res 98:75–84.
7. Aven T (1983) Optimal replacement under a minimal repair strategy–A general failure model. Adv Appl Prob 15:198–211.
8. Makis V, Jardine AKS (1991) Optimal replacement in the proportional hazard model. INFOR 30:172–183.
9. Makis V, Jardine AKS (1992) Optimal replacement policy for a general model with imperfect repair. J Oper Res Soc 43:111–120.
10. Makis V, Jardine AKS (1993) A note on optimal replacement policy under general repair. Eur J Oper Res 69:75–82.
11. Bagai I, Jain K (1994) Improvement, deterioration, and optimal replacement under age-replacement with minimal repair. IEEE Trans Reliab 43:156–162.
12. Morimura H (1970) On some preventive maintenance policies for IFR. J Oper Res Soc Jpn 12:94–124.
13. Pulcini G (2003) Mechanical reliability and maintenance models. In: Pham H (ed) Handbook of Reliability Engineering. Springer, London:317–348.
14. Nakagawa T, Kowada M (1983) Analysis of a system with minimal repair and its application to replacement policy. Eur J Oper Res 12:176–182.
15. Nakagawa T (1981) A summary of periodic replacement with minimal repair at failure. J Oper Res Soc Jpn 24:213–227.
16. Nakagawa T (1981) Generalized models for determining optimal number of minimal repairs before replacement. J Oper Res Soc Jpn 24:325–337.
17. Nakagawa T (1983) Optimal number of failures before replacement time. IEEE Trans Reliab R-32:115–116.
18. Nakagawa T (1984) Optimal policy of continuous and discrete replacement with minimal repair at failure. Nav Res Logist Q 31:543–550.
19. Nakagawa T, Yasui K (1991) Periodic-replacement models with threshold levels. IEEE Trans Reliab 40:395–397.
20. Nakagawa T (1987) Optimum replacement policies for systems with two types of units. In: Osaki S, Cao JH (eds) Reliability Theory and Applications Proceedings of the China-Japan Reliability Symposium, Shanghai, China.
21. Murthy DNP (1991) A note on minimal repair. IEEE Trans Reliab 40:245–246.
22. Kumar D (2000) Maintenance scheduling using monitored parameter values. In: Ben-Daya M, Duffuaa SO, Raouf A (eds) Maintenance, Modeling and Optimization. Kluwer Academic, Boston:345–374.
23. Park KS (1979) Optimal number of minimal repairs before replacement. IEEE Trans Reliab R-28:137–140.
24. Tapiero CS, Ritcken P (1985) Note on the (N, T) replacement rule. IEEE Trans Reliab R-34:374–376.
25. Stadje W, Zuckerman D (1990) Optimal strategies for some repair replacement models. Adv Appl Prob 22:641–656.
26. Lam Y (1988) A note on the optimal replacement problem. Adv Appl Prob 20:479–482.

27. Lam Y (1990) A repair replacement model. Adv Appl Prob 22:494–497.
28. Lam Y (1991) An optimal repairable replacement model for deteriorating system. J Appl Prob 28:843–851.
29. Lam Y (1999) An optimal maintenance model for a combination of secondhand-new or outdated-updated system. Eur J Oper Res 119:739–752.
30. Lam Y, Zhang YL (2003) A geometric-process maintenance model for a deteriorating system under a random environment. IEEE Trans Reliab 52:83–89.
31. Zhang YL (2002) A geometric-process repair-model with good-as-new preventive repair. IEEE Trans Reliab 51:223–228.
32. Ritchken P, Wilson JG (1990) (m, T) group maintenance policies. Manage Sci 36:632–639.
33. Sheu SH (1993) A generalized model for determining optimal number of minimal repairs before replacement. Eur J Oper Res 69:38–49.
34. Mine H, Kawai H (1974) Preventive replacement of a 1-unit system with a wearout state. IEEE Trans Reliab R23:24–29.
35. Maillart LM, Pollock SM (2002) Cost-optimal condition-monitoring for predictive maintenance of 2-phase systems. IEEE Trans Reliab 51:322–330.
36. Beichelt F, Fischer K (1980) General failure model applied to preventive maintenance policies. IEEE Trans Reliab R-29:39–41.
37. Beichelt F (1981) A generalized block-replacement policy. IEEE Trans Reliab R-30:171–172.
38. Murthy DNP, Maxwell MR (1981) Optimal age replacement policies for items from a mixture. IEEE Trans Reliab R-30:169–170.
39. Block HW, Borges WS, Savitz TH (1985) Age-dependent minimal repair. J Appl Prob 22:370–385.
40. Block HW, Borges WS, Savitz TH (1988) A general age replacement model with minimal repair. Nav Res Logist Q 35:365–372.
41. Berg M (1995) The marginal cost analysis and its application to repair and replacement policies. Eur J Oper 82:214–224.
42. Scheaffer RL (1975) Optimum age replacement in the bivariate exponential case. IEEE Trans Reliab R-24: 214–215.
43. Berg M (1976) Optimal replacement policies for two-unit machines with increasing running costs I. Stoch Process Appl 4:80–106.
44. Berg M (1978) General trigger-off replacement procedures for two-unit systems. Nav Res Logist Q 25:15–29.
45. Yamada S, Osaki S (1981) Optimum replacement policies for a system composed of components. IEEE Trans Reliab R-30:278–283.
46. Bai DS, Jang JS, Kwon YI (1983) Generalized preventive maintenance policies for a system subject to deterioration. IEEE Trans Reliab R-32:512–514.
47. Murthy DNP, Nguyen DG (1985) Study of two-component system with failure interaction. Nav Res Logist Q 32:239–247.
48. Pullen KW, Thomas MU (1986) Evaluation of an opportunistic replacement policy for a 2-unit system. IEEE Trans Reliab R-35:320–324.
49. Makis V, Jiang X (2001) Optimal control policy for a general EMQ model with random machine failure. In: Rahim MA, Ben-Daya M (eds) Integrated Models in Production, Planning, Inventory, Quality, and Maintenance. Kluwer Academic, Boston:67–78.
50. Ito K, Nakagawa T (2000) Optimal inspection policies for a storage system with degradation at periodic tests. Math Comput Model 31:191–195.