2

Repair Maintenance

The most basic maintenance policy for units is to do some maintenance of failed units which is called *corrective maintenance*; *i.e.,* when units fail, they may undergo repair or may be scrapped and replaced. After the repair completion, units can operate again. A system with several units forms semi-Markov processes and Markov renewal processes in stochastic processes. Such reliability models are called *repairman problems* [1], and some useful expressions of reliability measures of many redundant systems were summarized in [2, 3]. Early results of two-unit systems and their maintenance (see Section 6.2) were surveyed in [4]. Furthermore, imperfect repair models that do not always become like new after repair were proposed in [5, 6] (see Chapter 7).

In this chapter, we are concerned only with reliability characteristics of repairable systems such as mean time to system failure, availability, and expected number of system failures. Such reliability measures are obtained by using the techniques of stochastic processes as described in Section 1.3.

In Section 2.1, we consider the most fundamental one-unit system and survey its reliability quantities such as transition probabilities, downtime distribution, and availabilities. Another point of interest is the *repair limit policy* where the repair of a failed unit is stopped if it is not completed within a planned time T [7]. It is shown that there exists an optimum repair limit time T^* that minimizes the expected cost rate when the repair cost is proportional to time. In Section 2.2, we consider a system with a main unit supported by n spare units, and obtain the mean time to system failure and the expected number of failed spare units [8]. Using these results, we propose several optimization problems. Finally, in Section 2.3, we consider $(n + 1)$-unit standby and parallel systems, and derive transition probabilities and first-passage time distributions.

2.1 One-Unit System

An operating unit is repaired or replaced when it fails. When the failed unit undergoes repair, it takes a certain time which may not be negligible. When the repair is completed, the unit begins to operate again. If the failed unit cannot be repaired and spare units are not on hand, it takes a replacement time which may not be negligible.

We consider one operating unit that is repaired immediately when it fails. The failed unit is returned to the operating state when its repair is completed and becomes as good as new. It is assumed that the switchover time from the operating state to the repair state and from the repair state to the operating state are instantaneous. The successive operating times between failures are independently and identically distributed. The successive repair times are also independently, identically distributed and independent of the operating times. Of course, we can consider the repair time as the time required to make a replacement. In this case, the failed unit is replaced with a new one, and its unit operates as same as the failed one.

This system is the most fundamental system that repeats up and down states alternately. The process of such a system can be described by a Markov renewal process with two states, *i.e.,* an alternating renewal process given in Section 1.3 [9]. Many of the known results were summarized in [1, 10].

This section surveys the reliability quantities of a one-unit system and considers a repair limit policy in which the unit under repair is replaced with a new one when the repair is not completed by a fixed time.

2.1.1 Reliability Quantities

(1) Renewal Functions and Transition Probabilities

In the analysis of stochastic models, we are interested in the expected number of system failures during $(0, t]$ and the probability that the system is operating at time t. We obtain the stochastic behavior of a one-unit system by using the techniques in Markov renewal processes.

Assume that the failure time of an operating unit has a general distribution $F(t)$ with finite mean μ and the repair time of failed units has a general distribution $G(t)$ with finite mean β, where $\overline{\Phi} \equiv 1 - \Phi$ for any function Φ, where, in general, μ and β are referred to as *mean time to failure* (MTTF) and *mean time to repair* (MTTR), respectively. To analyze the system, we define the following states.

State 0: Unit is operating.
State 1: Unit is under repair.

Suppose that the unit begins to operate at time 0. The system forms a Markov renewal or semi-Markov process with two states of up and down as shown in Figure 1.4 of Section 1.3.2.

Define the mass function $Q_{ij}(t)$ from state i to state j by the probability that after making a transition into state i, the system next makes a transition into state j $(i, j = 0, 1)$, in an amount of time less than or equal to time t. Then, from a Markov renewal process, we can easily have

$$Q_{01}(t) = F(t), \qquad Q_{10}(t) = G(t).$$

Let $M_{ij}(t)$ denote the expected number of occurrences of state j during $(0, t]$ when the system goes into state i at time 0, where the first visit to state j is not counted when $i = j$. Then, from Section 1.3, we have the following renewal equations:

$$M_{01}(t) = Q_{01}(t) * [1 + M_{11}(t)], \quad M_{10}(t) = Q_{10}(t) * [1 + M_{00}(t)],$$

and $M_{11}(t) = Q_{10}(t) * M_{01}(t)$, $M_{00}(t) = Q_{01}(t) * M_{10}(t)$, where the asterisk denotes the pairwise Stieltjes convolution; i.e., $a(t) * b(t) \equiv \int_0^t a(t - u) \mathrm{d}b(u)$. Thus, forming the Laplace–Stieltjes (LS) transforms of both sides of these equations and solving them, we have

$$M_{01}^*(s) = \frac{Q_{01}^*(s)}{1 - Q_{01}^*(s)Q_{10}^*(s)} = \frac{F^*(s)}{1 - F^*(s)G^*(s)} \tag{2.1}$$

$$M_{10}^*(s) = \frac{Q_{10}^*(s)}{1 - Q_{01}^*(s)Q_{10}^*(s)} = \frac{G^*(s)}{1 - F^*(s)G^*(s)} \tag{2.2}$$

and $M_{11}^*(s) = G^*(s)M_{01}^*(s) = M_{00}^*(s) = F^*(s)M_{10}^*(s)$, where the asterisk of the function denotes the LS transform with itself; i.e., $\Phi^*(s) \equiv \int_0^\infty \mathrm{e}^{-st}\mathrm{d}\Phi(t)$ for any function $\Phi(t)$.

Furthermore, let $P_{ij}(t)$ denote the probability that the system is in state j at time t if it starts in state i at time 0. Then, from Section 1.3,

$$P_{00}(t) = 1 - Q_{01}(t) + Q_{01}(t) * P_{10}(t)$$
$$P_{11}(t) = 1 - Q_{10}(t) + Q_{10}(t) * P_{01}(t)$$

and $P_{10}(t) = Q_{10}(t) * P_{00}(t)$, $P_{01}(t) = Q_{01}(t) * P_{11}(t)$. Thus, again forming the LS transforms,

$$P_{00}^*(s) = \frac{1 - Q_{01}^*(s)}{1 - Q_{01}^*(s)Q_{10}^*(s)} = \frac{1 - F^*(s)}{1 - F^*(s)G^*(s)} \tag{2.3}$$

$$P_{11}^*(s) = \frac{1 - Q_{10}^*(s)}{1 - Q_{01}^*(s)Q_{10}^*(s)} = \frac{1 - G^*(s)}{1 - F^*(s)G^*(s)} \tag{2.4}$$

and $P_{10}^*(s) = G^*(s)P_{00}^*(s)$, $P_{01}^*(s) = F^*(s)P_{11}^*(s)$. Thus, from (2.1) to (2.4), we have the following relations.

$$P_{01}(t) = M_{01}(t) - M_{00}(t), \qquad P_{10}(t) = M_{10}(t) - M_{11}(t).$$

Moreover, we have

$$P_{01}^*(s) = \frac{F^*(s)[1 - G^*(s)]}{1 - F^*(s)G^*(s)} = \int_0^\infty e^{-st}\overline{G}(t - u)\,dM_{01}(u)$$

$$P_{10}^*(s) = \frac{G^*(s)[1 - F^*(s)]}{1 - F^*(s)G^*(s)} = \int_0^\infty e^{-st}\overline{F}(t - u)\,dM_{10}(u);$$

i.e.,

$$P_{01}(t) = \int_0^t \overline{G}(t - u)\,dM_{01}(u), \qquad P_{10}(t) = \int_0^t \overline{F}(t - u)\,dM_{10}(u).$$

These relations with renewal functions and transition probabilities would be useful for the analysis of more complex systems.

Next, let $h(t)$ and $r(t)$ be the failure rate and the repair rate of the unit, respectively; *i.e.*, $h(t) \equiv f(t)/\overline{F}(t)$ and $r(t) \equiv g(t)/\overline{G}(t)$, where f and g are the respective density functions of F and G. Then, from (2.1) to (2.4), we also have

$$\min_{x \leq t} h(x) \int_0^t P_{00}(u)\,du \leq M_{01}(t) \leq \max_{x \leq t} h(x) \int_0^t P_{00}(u)\,du$$

$$\min_{x \leq t} r(x) \int_0^t P_{11}(u)\,du \leq M_{10}(t) \leq \max_{x \leq t} r(x) \int_0^t P_{11}(u)\,du.$$

All inequalities equal when both F and G are exponential, which is shown in Example 2.1.

There exist $P_j \equiv \lim_{t\to\infty} P_{ij}(t)$ and $M_j \equiv \lim_{t\to\infty} M_{ij}(t)/t$, independent of an initial state i, because the system forms a Markov renewal process with one positive recurrent. Thus, from (1.63) we have

$$M_0 = \lim_{s\to 0} sM_{00}^*(s) = \frac{1}{\mu + \beta} = M_1 \tag{2.5}$$

$$P_0 = \lim_{s\to 0} P_{00}^*(s) = \frac{\mu}{\mu + \beta} = 1 - P_1. \tag{2.6}$$

In general, it is often impossible to invert explicitly the LS transforms of $M_{ij}^*(s)$ and $P_{ij}^*(s)$ in (2.1) to (2.4), and it is very difficult even to invert them numerically [11,12]. However, we can state the following asymptotic describing behaviors for small t and large t.

First, we consider the approximation calculation for small t. Reliability calculations for small t are needed in considering the near-term future security of an operating bulk power system [13]. We can rewrite (2.3) as

$$P_{00}^*(s) = 1 - F^*(s) + F^*(s)G^*(s) - [F^*(s)]^2 G^*(s) + \cdots.$$

Because the probability that the process makes more than two transitions in a short time is very small, by dropping the terms with higher degrees than $F^*(s)G^*(s)$, we have

$$P_{00}^*(s) \approx 1 - F^*(s) + F^*(s)G^*(s);$$

i.e.,

$$P_{00}(t) \approx \overline{F}(t) + \int_0^t G(t-u)\,\mathrm{d}F(u). \tag{2.7}$$

Similarly,

$$P_{01}(t) \approx \int_0^t \overline{G}(t-u)\,\mathrm{d}F(u) \tag{2.8}$$

$$M_{00}(t) \approx \int_0^t G(t-u)\,\mathrm{d}F(u), \qquad M_{01}(t) \approx F(t). \tag{2.9}$$

Next, we obtain the asymptotic forms for large t [9]. By expanding e^{-st} in a Taylor series on the LS transforms of $F^*(s)$ and $G^*(s)$ as $s \to 0$, it follows that

$$F^*(s) = 1 - \mu s + \frac{1}{2}(\mu^2 + \sigma_\mu^2)s^2 + o(s^2)$$

$$G^*(s) = 1 - \beta s + \frac{1}{2}(\beta^2 + \sigma_\beta^2)s^2 + o(s^2),$$

where σ_μ^2 and σ_β^2 are the variances of F and G, respectively, and $o(s)$ is an infinite decimal higher than s. Thus, substituting these equations into (2.1), we have

$$M_{01}^*(s) = \frac{1}{\mu+\beta}\frac{1}{s} - \frac{\mu}{\mu+\beta} + \frac{1}{2} + \frac{\sigma_\mu^2 + \sigma_\beta^2}{2(\mu+\beta)^2} + o(1).$$

Formal inversion of $M_{01}^*(s)$ gives that for large t,

$$M_{01}(t) = \frac{t}{\mu+\beta} - \frac{\mu}{\mu+\beta} + \frac{1}{2} + \frac{\sigma_\mu^2 + \sigma_\beta^2}{2(\mu+\beta)^2} + o(1). \tag{2.10}$$

Similarly,

$$M_{00}(t) = \frac{t}{\mu+\beta} - \frac{1}{2} + \frac{\sigma_\mu^2 + \sigma_\beta^2}{2(\mu+\sigma)^2} + o(1) \tag{2.11}$$

$$P_{00}(t) = \frac{\mu}{\mu+\beta} + o(1), \qquad P_{01}(t) = \frac{\beta}{\mu+\beta} + o(1). \tag{2.12}$$

Example 2.1. Suppose that $F(t) = 1 - \mathrm{e}^{-\lambda t}$ and $G(t) = 1 - \mathrm{e}^{-\theta t}$ $(\theta \neq \lambda)$. Then, it is easy to invert the LS transforms of $P_{01}^*(s)$ and $M_{01}^*(s)$,

$$P_{01}(t) = \frac{\lambda}{\lambda+\theta}[1 - \mathrm{e}^{-(\lambda+\theta)t}]$$

$$M_{01}(t) = \frac{\lambda\theta t}{\lambda+\theta} + \left(\frac{\lambda}{\lambda+\theta}\right)^2[1 - \mathrm{e}^{-(\lambda+\theta)t}].$$

Furthermore, for small t,

$$P_{01}(t) \approx \frac{\lambda}{\theta-\lambda}(\mathrm{e}^{-\lambda t} - \mathrm{e}^{-\theta t}), \qquad M_{01}(t) \approx 1 - \mathrm{e}^{-\lambda t}$$
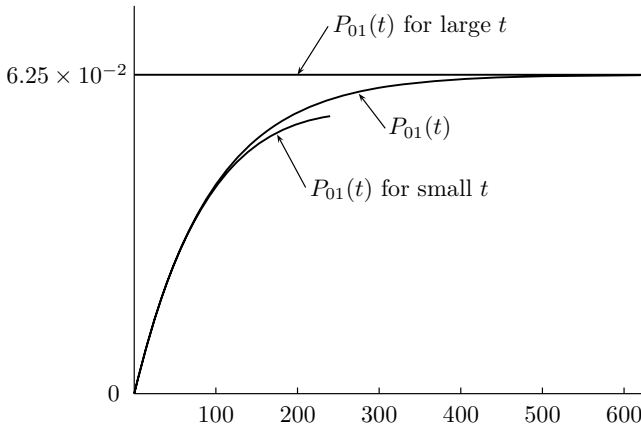
Fig. 2.1. Comparisons of $P_{01}(t)$

and for large t,

$$P_{01}(t) \approx \frac{\lambda}{\lambda + \theta}, \qquad M_{01}(t) \approx \frac{\lambda \theta t}{\lambda + \theta} + \left(\frac{\lambda}{\lambda + \theta}\right)^2.$$

Figure 2.1 shows the value of $P_{01}(t)$ and the approximate values of $P_{01}(t)$ for small t and large t when $1/\lambda = 1500$ hours and $1/\theta = 100$ hours. In this case, we can use these approximate values for about fewer than 100 hours and more than 500 hours. This indicates that these approximations are comparatively fitted for a long interval of time t. ∎

Example 2.2.   When $F(t) = 1 - (1 + \lambda t)e^{-\lambda t}$ and the time for repair is constant β,

$$P_{01}^*(s) = \frac{\lambda^2(1 - e^{-\beta s})}{(s + \lambda)^2 - \lambda^2 e^{-\beta s}}.$$

Furthermore, for small t,

$$P_{01}(t) \approx 1 - (1 + \lambda t)e^{-\lambda t} - \begin{cases} 0 & \text{for } t < \beta \\ 1 - [1 + \lambda(t - \beta)]e^{-\lambda(t-\beta)} & \text{for } t \geq \beta \end{cases}$$

and for large t,

$$P_{01}(t) \approx \frac{\beta}{2/\lambda + \beta}. \quad ∎$$

(2) Downtime Distribution

A quantity of most interest is the behavior of system down or system failure. It is of great importance to know how long and how many times the system is down during $(0, t]$, because the system down is sometimes costly and/or dangerous. It was shown in [10] that the downtime distribution of a one-unit system is given from the result of a stochastic process [14]. The excess time which is the time spent in t due to failures was proposed and its stochastic properties were reviewed in [15, 16]. Furthermore, the downtime distribution was derived in the case where failure and repair times are dependent [17].

We have already derived in **(1)**: the probability $P_{01}(t)$ that the system is down at time t, the mean downtime $\int_0^t P_{01}(u)\mathrm{d}u$ during $(0, t]$, and the expected number $M_{01}(t)$ of system down during $(0, t]$. Of other interest is to show (i) the downtime distribution, (ii) the mean time that the total downtime during $(0, t]$ exceeds a specified level $\delta > 0$ for the first time, and (iii) the first time that an amount of downtime exceeds a specified level c.

Suppose that the unit begins to operate at time 0. Let $D(t)$ denote the total amount of downtime during $(0, t]$. Then, the distribution of downtime $D(t)$ is, from (1.35) in Section 1.3,

$$\Omega(t, x) \equiv \Pr\{D(t) \le x\}$$
$$= \begin{cases} \sum_{n=0}^{\infty} G^{(n)}(x)[F^{(n)}(t - x) - F^{(n+1)}(t - x)] & \text{for } t > x \\ 1 & \text{for } t \le x, \end{cases} \tag{2.13}$$

where $F^{(n)}(t)$ $(G^{(n)}(t))$ denotes the n-fold Stieltjes convolution of F (G) with itself, and $F^{(0)}(t) = G^{(0)}(t) \equiv 1$ for $t \ge 0$ and 0 for $t < 0$. Equation (2.13) can also be written as

$$\Omega(t + x, x) = \Pr\{D(t + x) \le x\}$$
$$= \sum_{n=0}^{\infty} G^{(n)}(x)[F^{(n)}(t) - F^{(n+1)}(t)] \tag{2.14}$$

which is called *excess time* [15]. Furthermore, the survival distribution of downtime is

$$1 - \Omega(t, x) = \Pr\{D(t) > x\}$$
$$= \begin{cases} \sum_{n=0}^{\infty} [G^{(n)}(x) - G^{(n+1)}(x)]F^{(n+1)}(t - x) & \text{for } t > x \\ 0 & \text{for } t \le x. \end{cases} \tag{2.15}$$

Takács also proved the following important theorem.

Theorem 2.1.   Suppose that μ, β and σ_μ^2, σ_β^2 are the means and variances of distributions $F(t)$ and $G(t)$, respectively. If $\sigma_\mu^2 < \infty$ and $\sigma_\beta^2 < \infty$ then

$$\lim_{t\to\infty} \Pr\left\{ \frac{D(t) - \beta t/(\mu+\beta)}{\sqrt{[(\beta\sigma_\mu)^2 + (\mu\sigma_\beta)^2]t/(\mu+\beta)^3}} \le x \right\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-u^2/2} du. \quad (2.16)$$

That is, if the means and variances of F and G are statistically estimated then the probability of the amount of $D(t)$ is approximately obtained for large t, by using a standard normal distribution.

Next, let $T_\delta \equiv \min_t\{D(t) > \delta\}$ be the first time that the total downtime exceeds a specified level $\delta > 0$. Then, from (2.15),

$$J_\delta(t) \equiv \Pr\{T_\delta \le t\} = \Pr\{D(t) > \delta\}$$

$$= \sum_{n=0}^{\infty} [G^{(n)}(\delta) - G^{(n+1)}(\delta)]F^{(n+1)}(t-\delta) \qquad \text{for } t > \delta. \quad (2.17)$$

The mean time that the total time first exceeds δ is

$$l_\delta \equiv \int_0^\infty \overline{J}_\delta(t)\,dt = \delta + \mu \sum_{n=0}^{\infty} G^{(n)}(\delta). \quad (2.18)$$

Example 2.3.   Suppose that $F(t) = 1 - e^{-\lambda t}$ and the time for repair is constant β [1, pp. 78–79]. Then, the downtime distribution is

$$\Omega(t,x) = \sum_{n=0}^{[x/\beta]} \frac{[\lambda(t-x)]^n}{n!} e^{-\lambda(t-x)} \qquad \text{for } t > x$$

and

$$l_\delta = \delta + \frac{1}{\lambda}\left\{ \left[\frac{\delta}{\beta}\right] + 1 \right\},$$

where $[x]$ denotes the greatest integer contained in x. In addition, the expected number of systems down during $(0,t]$ is

$$M_{01}(t) = \left[\frac{t}{\beta}\right] + 1 - \sum_{j=0}^{[t/\beta]} \sum_{k=0}^{j} \frac{\lambda^k(t-\beta j)^k e^{-\lambda(t-\beta j)}}{k!}$$

and the probability that the system is down at time t is

$$P_{01}(t) = 1 - \sum_{j=0}^{[t/\beta]} \frac{\lambda^j(t-\beta j)^j e^{-\lambda(t-\beta j)}}{j!}. \quad \blacksquare$$

Finally, we consider the first time that an amount of a single downtime exceeds a fixed time $c > 0$, where c is considered to be a critically allowed time for repair [18]. For example, we can give a fuel charge and discharge system for a nuclear reactor that shuts down spontaneously when the system

has failed more than time c [19]. The distribution $L(t)$ of the first time that an amount of downtime first exceeds time c is given by applying a terminating renewal process. Then, from (1.39) and (1.40), the LS transform of $L(t)$ and its mean time l are, respectively,

$$L^*(s) = \frac{F^*(s)\mathrm{e}^{-sc}\overline{G}(c)}{1 - F^*(s)\int_0^c \mathrm{e}^{-st}\,\mathrm{d}G(t)}, \qquad l = \frac{\mu + \int_0^c \overline{G}(t)\,\mathrm{d}t}{\overline{G}(c)}. \tag{2.19}$$

(3) Availability

We derive the exact expressions of availabilities for a one-unit system with repair introduced in Section 1.1. Suppose that the unit begins to operate at time 0.

(i) *Pointwise availability*: From (2.3),

$$A(t) = P_{00}(t) = \overline{F}(t)*[1+F(t)*G(t)+F(t)*G(t)*F(t)*G(t)+\cdots]; \tag{2.20}$$

i.e.,

$$A(t) = \overline{F}(t) + \int_0^t \overline{F}(t-u)\,\mathrm{d}M_{00}(u)$$

and its LS transform is

$$A^*(s) = \frac{1 - F^*(s)}{1 - F^*(s)G^*(s)}. \tag{2.21}$$

Furthermore, when $m_{01}(t) \equiv \mathrm{d}M_{01}(t)/\mathrm{d}t$ exists, from the results **(1)** of Section 2.1.1, we have

$$\min_{x\leq t} h(x)A(t) \leq m_{01}(t) \leq \max_{x\leq t} h(x)A(t)$$

$$\overline{A}(t) \equiv 1 - A(t) = \int_0^t \overline{G}(t-u)m_{01}(u)\,\mathrm{d}u.$$

Thus, we have the inequality [20, p. 107]

$$\overline{A}(t) \leq \max_{x\leq t} h(x) \int_0^t \overline{G}(t-u)A(u)\,\mathrm{d}u$$

$$\leq \max_{x\leq t} h(x) \int_0^t \overline{G}(u)\,\mathrm{d}u \leq \max_{x\leq t} h(x)\beta \tag{2.22}$$

which give the upper bounds of the unavailability at time t.

(ii) *Interval availability*:

$$\frac{1}{t}\int_0^t A(u)\,\mathrm{d}u = \frac{1}{t}\int_0^t P_{00}(u)\,\mathrm{d}u. \tag{2.23}$$

(iii) *Limiting interval availability*:

$$A = \lim_{t\to\infty} P_{00}(t) = \frac{\mu}{\mu + \beta} = \frac{\text{MTTF}}{\text{MTTF} + \text{MTTR}} \qquad (2.24)$$

which is sometimes called simply *availability*.

(iv) *Multiple cycle availability*:

$$A(n) = \int_0^\infty \int_0^\infty \frac{x}{x + y}\, dF^{(n)}(x)\, dG^{(n)}(y) \qquad (n = 1, 2, \ldots). \qquad (2.25)$$

(v) *Multiple cycle availability with probability*: Because

$$\Pr\left\{ a\sum_{i=1}^n X_i \geq \sum_{i=1}^n Y_i \right\} = \int_0^\infty G^{(n)}(ax)\, dF^{(n)}(x) \quad \text{for } a > 0$$

$$\Pr\left\{ \frac{\sum_{i=1}^n X_i}{\sum_{i=1}^n (X_i + Y_i)} \geq y \right\} = \int_0^\infty G^{(n)}\left(\frac{x}{y} - x\right) dF^{(n)}(x) \quad \text{for } 0 < y \leq 1.$$

Thus, putting $y = A_\nu(n)$ in the above equation,

$$\int_0^\infty G^{(n)}\left(\frac{x}{A_\nu(n)} - x\right) dF^{(n)}(x) = \nu \quad (n = 1, 2, \ldots). \qquad (2.26)$$

(vi) *Interval availability with probability*: Let $U(t)$ denote the total amount of uptime during $(0, t]$; *i.e.*, $U(t) \equiv t - D(t)$. Then, from the downtime distribution in (2.13),

$$\Pr\left\{ \frac{U(t)}{t} \geq y \right\} = \Pr\{D(t) \leq t - ty\}$$

$$= \sum_{n=0}^\infty G^{(n)}(t - ty)[F^{(n)}(ty) - F^{(n+1)}(ty)] \quad \text{for } 0 < y \leq 1.$$

Thus, it is given by solving

$$\sum_{n=0}^\infty G^{(n)}(t - tA_\nu(t))[F^{(n)}(tA_\nu(t)) - F^{(n+1)}(tA_\nu(t))] = \nu. \qquad (2.27)$$

Furthermore, the interval reliability is, from (1.14),

$$R(x; t) = \overline{F}(t + x) + \int_0^t \overline{F}(t + x - u)\, dM_{00}(u) \qquad (2.28)$$

and its Laplace transform is

$$R^*(x; s) = \int_0^\infty e^{-st} R(x; t)\, dt = \frac{e^{sx} \int_x^\infty e^{-st} \overline{F}(t)\, dt}{1 - F^*(s)G^*(s)}. \qquad (2.29)$$

Thus, the limiting interval reliability is [21, 22]

$$R(x) \equiv \lim_{t\to\infty} R(x; t) = \lim_{s\to 0} sR^*(x; s) = \frac{\int_x^\infty \overline{F}(t)\, dt}{\mu + \beta}. \qquad (2.30)$$

We give the exact expressions of the above availabilities for two particular cases [10, 23–26].

Example 2.4.   When $F(t) = 1 - e^{-\lambda t}$ and $G(t) = 1 - e^{-\theta t}$, the availabilities are given as follows.

(i) $A(t) = \dfrac{\theta}{\lambda + \theta} + \dfrac{\lambda}{\lambda + \theta} e^{-(\lambda + \theta)t}$

$\overline{A}(t) = \dfrac{\lambda}{\lambda + \theta}(1 - e^{-(\lambda + \theta)t}) \leq \dfrac{\lambda}{\lambda + \theta} < \dfrac{\lambda}{\theta}.$

(ii) $\dfrac{1}{t}\displaystyle\int_0^t A(u)\,\mathrm{d}u = \dfrac{\theta}{\lambda + \theta} + \dfrac{\lambda}{(\lambda + \theta)^2 t}(1 - e^{-(\lambda + \theta)t})$

$\dfrac{1}{t}\displaystyle\int_0^t \overline{A}(u)\,\mathrm{d}u = \dfrac{\lambda}{\lambda + \theta} - \dfrac{\lambda}{(\lambda + \theta)^2 t}(1 - e^{-(\lambda + \theta)t}) \leq \dfrac{\lambda t}{2}.$

(iii) $A = \dfrac{\theta}{\lambda + \theta}.$

(iv) $A(n) = \displaystyle\int_0^\infty \dfrac{n(\lambda\theta)^n}{(n-1)!} y^{2n-1}\Gamma(-n, \lambda y)e^{(\lambda - \theta)y}\,\mathrm{d}y,$

where $\Gamma(\alpha, x) \equiv \int_x^\infty u^{\alpha - 1}e^{-u}\,\mathrm{d}u$. In particular,

$$A(1) = \begin{cases} \dfrac{\theta}{\theta - \lambda} + \dfrac{\lambda\theta}{(\theta - \lambda)^2}\log\dfrac{\lambda}{\theta} & \text{for } \lambda \neq \theta \\[2ex] \dfrac{1}{2} & \text{for } \lambda = \theta. \end{cases}$$

(v) $A_\nu(n)$ is given by solving

$$\sum_{j=0}^{n-1}\binom{n+j-1}{j}\left\{\dfrac{\theta[(1/A_\nu(n))-1]}{\lambda+\theta[(1/A_\nu(n))-1]}\right\}^j\left\{\dfrac{\lambda}{\lambda+\theta[1/(A_\nu(n))-1]}\right\}^n = 1 - \nu.$$

In particular,

$$A_\nu(1) = \dfrac{(1-\nu)\theta}{\lambda\nu + (1-\nu)\theta}.$$

(vi) $A_\nu(t)$ is given by solving

$$e^{-\lambda t A_\nu(t)}\left[1 + \sqrt{\lambda\theta t A_\nu(t)}\int_0^{t(1-A_\nu(t))} e^{-\theta y}y^{-1/2}I_1(2\sqrt{\lambda\theta y A_\nu(t)})\,\mathrm{d}y\right] = \nu,$$

where $I_1(x) \equiv \sum_{j=0}^\infty (x/2)^{2j+1}/[j!(j+1)!]$.

The interval reliability is

$$R(x; t) = \left[\dfrac{\theta}{\lambda + \theta} + \dfrac{\lambda}{\lambda + \theta}e^{-(\lambda+\theta)t}\right]e^{-\lambda x} = A(t)\overline{F}(x)$$

and its limiting interval reliability is

$$R(x) = \frac{\theta}{\lambda + \theta} e^{-\lambda x} = A\overline{F}(x). \quad \blacksquare$$

Example 2.5. Suppose that $F(t) = 1 - e^{-\lambda t}$ and the time for repair is constant β.

(i) $A(t) = \sum_{j=0}^{[t/\beta]} \frac{\lambda^j (t - \beta j)^j}{j!} e^{-\lambda(t-\beta j)}$.

(ii) $\frac{1}{t} \int_0^t A(u)\,\mathrm{d}u = \frac{1}{\lambda t}\left\{\left[\frac{t}{\beta}\right] + 1 - \sum_{j=0}^{[t/\beta]}\sum_{k=0}^j \frac{\lambda^k (t - \beta j)^k}{k!} e^{-\lambda(t-\beta j)}\right\}$.

(iii) $A = \dfrac{1/\lambda}{1/\lambda + \beta}$.

(iv) $A(n) = n(n\lambda\beta)^n e^{n\lambda\beta} \Gamma(-n, n\lambda\beta)$.
   In particular,
$$A(1) = 1 - \lambda\beta e^{\lambda\beta} \int_{\lambda\beta}^{\infty} u^{-1} e^{-u}\,\mathrm{d}u.$$

(v) $A_\nu(n)$ is given by solving

$$\sum_{j=0}^{n-1} \frac{[n\lambda\beta A_\nu(n)/(1 - A_\nu(n))]^j}{j!} \exp[-n\lambda\beta A_\nu(n)/(1 - A_\nu(n))] = \nu.$$

   In particular,
$$A_\nu(1) = \frac{\log(1/\nu)}{\lambda\beta + \log(1/\nu)}.$$

(vi) $A_\nu(t)$ is given by solving

$$\sum_{j=0}^{[t(1-A_\nu(t))/\beta]} \frac{[\lambda t A_\nu(t)]^j}{j!} \exp[-\lambda t A_\nu(t)] = \nu. \quad \blacksquare$$

Finally, we give the example of asymptotic behavior shown in $[1, 26]$.

Example 2.6. We wish to compute the time T when the probability that the system is down more than T in $t = 10{,}000$ hours of operation is given by 0.90, and the availability $A_\nu(t)$ when $\nu = 0.90$. The failure and repair distributions are unknown, but from the sample data, the estimates of means and variances are:

$$\mu = 1{,}000, \qquad \sigma_\mu^2 = 100{,}000, \qquad \beta = 100, \qquad \sigma_\beta^2 = 400.$$

Then, from Theorem 2.1, when $t = 10{,}000$,

$$\frac{D(t) - \beta t/(\mu + \beta)}{\sqrt{[(\beta\sigma_\mu)^2 + (\mu\sigma_\beta)^2]t/(\mu + \beta)^3}} = \frac{D(10{,}000) - 909.09}{102.56}$$

is approximately normally distributed with mean 0 and variance 1. Thus,

$$\Pr\{D(t) > T\} = \Pr\left\{\frac{D(10,000) - 909.09}{102.56} > \frac{T - 909.09}{102.56}\right\}$$

$$\approx \frac{1}{\sqrt{2\pi}} \int_{(T-909.09)/102.56}^{\infty} e^{-u^2/2}\, du = 0.90.$$

Because $u_0 = -1.28$ such that $(1/\sqrt{2\pi}) \int_{u_0}^{\infty} e^{-u^2/2} du = 0.90$, we have

$$T = 909.09 - 102.56 \times 1.28 = 777.81.$$

Moreover, from the relation $U(t) = t - D(t)$, we have

$$\Pr\left\{\frac{U(t)/t - \mu/(\mu+\beta)}{\sqrt{[(\beta\sigma_\mu)^2 + (\mu\sigma_\beta)^2]/[t(\mu+\beta)^3]}} > -y\right\} \approx \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{y} e^{-u^2/2}\, du$$

$$= 0.90.$$

Thus, we have approximately

$$A_\nu(t) = \frac{\mu}{\mu+\beta} + u_0\sqrt{\frac{(\beta\sigma_\mu)^2 + (\mu\sigma_\beta)^2}{t(\mu+\beta)^3}} = 0.896.$$

In this case, it can be said that with probability 0.90 the system will operate for at least 89.6 percent of the time interval $10,000$ hours. ∎

2.1.2 Repair Limit Policy

Until now, we have analyzed a one-unit system which is repaired upon failure and then returns to operation without having any preventive maintenance (PM). The first PM policy for an operating unit, in which it is repaired at failure or at time T, whichever occurs first, was defined in [27]. The optimum PM policy that maximizes the availability was derived in [10]. We discuss some PM policies in Chapters 6 and 7.

An alternative considered here is to repair a failed unit if the repair time is short or to replace it if the repair time is long. This is achieved by stopping the repair if it is not completed within a repair limit time, and the unit is replaced. This policy is optimum over both deterministic and random repair limit time policies [28]. We discuss optimum repair limit policies that minimize the expected cost rates for an infinite time span. An optimum repair limit time is analytically obtained in the case where the repair cost is proportional to time.

Similar repair limit problems can be applied to army vehicles [29–33]. When a unit requires repair, it is first inspected and its repair cost is estimated. If the estimated cost exceeds a certain amount, the unit is not repaired but

is replaced. The authors further derived the repair limiting value, in which the expected future cost per vehicle-year when the failed vehicle is repaired is equal to the cost when the failed vehicle is scrapped and a new one is substituted. They used three methods of optimizing the repair limit policies such as simulation, hill-climbing, and dynamic programming. More general forms of repair costs were given in [34]. Using the nonparametric and graphical methods, several problems were solved in [35, 36].

Consider a one-unit system that is repaired or replaced if it fails. Let μ denote the finite mean failure time of the unit and $G(t)$ denote the repair distribution of the failed unit with finite mean β. It is assumed that a failure of the unit is immediately detected, and it is repaired or replaced and becomes as good as new upon repair or replacement.

When the unit fails, its repair is started immediately, and when the repair is not completed within time T $(0 \le T \le \infty)$, which is called the *repair limit time*, it is replaced with a new one. Let c_1 be the replacement cost of a failed unit that includes all costs caused by failure and replacement. Let $c_r(t)$ be the expected repair cost during $(0, t]$, which also includes all costs incurred due to repair and downtime during $(0, t]$, and be bounded on a finite interval.

Consider one cycle from the beginning of an operative unit to the repair or replacement completion. Each cycle is independently and identically distributed, and hence, a sequence of cycles forms a renewal process. Then, the expected cost of one cycle is

$$[c_1 + c_r(T)]\overline{G}(T) + \int_0^T c_r(t)\,dG(t) = c_1\overline{G}(T) + \int_0^T \overline{G}(t)\,dc_r(t)$$

and the mean time of one cycle is

$$\mu + T\overline{G}(T) + \int_0^T t\,dG(t) = \mu + \int_0^T \overline{G}(t)\,dt.$$

Thus, from Theorem 1.6, the expected cost rate for an infinite span (see (3.3) in Chapter 3) is

$$C(T) = \frac{c_1\overline{G}(T) + \int_0^T \overline{G}(t)\,dc_r(t)}{\mu + \int_0^T \overline{G}(t)\,dt}. \tag{2.31}$$

It is evident that

$$C(0) \equiv \lim_{T \to 0} C(T) = \frac{c_1}{\mu} \tag{2.32}$$

$$C(\infty) \equiv \lim_{T \to \infty} C(T) = \frac{\int_0^\infty \overline{G}(t)\,dc_r(t)}{\mu + \beta} \tag{2.33}$$

which represent the expected cost rates with only replacement and only repair maintenance, respectively.

Consider the special case where the repair cost is proportional to time; i.e., $c_r(t) = at^b$ for $a > 0$ and $b \ge 0$. The repair cost would be dependent on

downtime and repairpersons, both of which are approximately proportional to time. In this case, the expected cost rate is

$$C(T) = \frac{c_1 \overline{G}(T) + ab \int_0^T t^{b-1} \overline{G}(t) \, dt}{\mu + \int_0^T \overline{G}(t) \, dt}. \qquad (2.34)$$

If $\int_0^\infty t^b dG(t) \equiv \beta_b < \infty$ then

$$C(\infty) = \frac{a\beta_b}{\mu + \beta}. \qquad (2.35)$$

We find an optimum repair limit time T^* that minimizes $C(T)$. It is assumed that there exists a density function $g(t)$ of $G(t)$ and let $r(t) \equiv g(t)/\overline{G}(t)$ be the repair rate. Then, differentiating $C(T)$ with respect to T and setting it equal to zero yield

$$r(T) \left[\mu + \int_0^T \overline{G}(t) \, dt \right] + \overline{G}(T)$$

$$= \frac{ab}{c_1} \left\{ T^{b-1} \left[\mu + \int_0^T \overline{G}(t) \, dt \right] - \int_0^T t^{b-1} \overline{G}(t) \, dt \right\}. \qquad (2.36)$$

If there exists a finite and positive T^* that minimizes $C(T)$, it has to satisfy (2.36). Otherwise, an optimum repair limit time is $T^* = 0$ or $T^* = \infty$.

Consider the particular case of $b = 1$; *i.e.*, $c_r(t) = at$. Let

$$k \equiv \frac{a\mu - c_1}{c_1 \mu}, \qquad K \equiv \frac{a\mu}{c_1(\mu + \beta)},$$

where k might be negative. Substituting $b = 1$ into (2.36),

$$r(T) \left[\mu + \int_0^T \overline{G}(t) \, dt \right] + \overline{G}(T) = \frac{a\mu}{c_1}. \qquad (2.37)$$

Letting $Q(T)$ be the left-hand side of (2.37), we have

$$Q(0) \equiv \mu r(0) + 1, \qquad Q(\infty) = (\mu + \beta) r(\infty)$$

and furthermore, $Q(T)$ and $r(T)$ are monotonic together. Hence, if $r(t)$ is strictly decreasing and $Q(0) > a\mu/c_1 > Q(\infty)$; *i.e.*, $r(0) > k$ and $r(\infty) < K$, there exists uniquely a finite and positive T^* that minimizes $C(T)$, and

$$C(T^*) = a - c_1 r(T^*). \qquad (2.38)$$

If $r(0) \le k$ then $Q(T) < a\mu/c_1$ and $dC(T)/dT > 0$ for any $T > 0$. Thus, the optimum time is $T^* = 0$; *i.e.*, no repair should be made. If $r(\infty) \ge K$ then $Q(T) > a\mu/c_1$ and $dC(T)/dT < 0$ for any $T < \infty$. Thus, the optimum time is $T^* = \infty$; *i.e.*, no replacement should be made.

From the above discussions, we have the following optimum policy when $r(t)$ is continuous and strictly decreasing.

Table 2.1. Optimum repair limit time T^* and expected cost rate $C(T^*)$ when $a = 3$, $\mu = 10$, and $c_1 = 10$

θ	T^*	$C(T^*)$
0.1	0.062	0.989
0.2	0.239	0.953
0.3	0.510	0.900
0.4	0.854	0.836
0.5	1.252	0.766
0.6	1.693	0.694
0.7	2.170	0.624
0.8	2.682	0.557
0.9	3.229	0.496
1.0	3.813	0.439

(i) If $r(0) > k$ and $r(\infty) < K$ then there exists a finite and unique T^* $(0 < T^* < \infty)$ that satisfies (2.37), and the resulting cost rate is given in (2.38).
(ii) If $r(0) \leq k$ then $T^* = 0$ and the expected cost rate is given in (2.32).
(iii) If $r(\infty) \geq K$ then $T^* = \infty$ and the expected cost rate is given in (2.35).

It is evident in the above result that if $r(t)$ is not decreasing then $T^* = 0$ or $T^* = \infty$. In this case, if $a/c_1 > 1/\mu + 1/\beta$ then $T^* = 0$, and conversely, if $a/c_1 < 1/\mu + 1/\beta$ then $T^* = \infty$. In other cases of $b \neq 1$, it is, in general, difficult to discuss an optimum repair limit policy. However, it could compute an optimum time T^* that satisfies (2.36) if the parameters a, b, and $G(t)$ are specified.

Example 2.7. Suppose that $c_r(t) = at$ and $G(t) = 1 - e^{-\theta\sqrt{t}}$. Then, $r(t) = \theta/(2\sqrt{t})$ which is strictly decreasing from infinity to zero. Then, from (2.37), there exists a unique solution T^* that satisfies

$$\frac{a\mu}{c_1}\sqrt{T} - \frac{1}{\theta}(1 - e^{-\theta\sqrt{T}}) = \frac{\theta\mu}{2}$$

and from (2.38), the expected cost rate is $C(T^*) = a - c_1\theta/(2\sqrt{T^*})$. Table 2.1 shows a numerical example of the optimum repair limit time T^* and the resulting cost rate $C(T^*)$ for $\theta = 0.1 \sim 1.0$ when $a = 3$, $\mu = 10$, and $c_1 = 10$. ∎

Example 2.8. Suppose that $c_r(t) = at^2$ and $G(t) = 1 - e^{-\theta t}$. Then, from (2.36), there exists a unique solution T^* that satisfies

$$T - \frac{1 - e^{-\theta T}}{\theta(\mu\theta + 1)} = \frac{c_1\theta}{2a}$$

because the left-hand side is strictly increasing from 0 to ∞, and from (2.34), the expected cost rate is $C(T^*) = 2aT^* - c_1\theta$. Table 2.2 shows a numerical example of T^* and $C(T^*)$ for θ when $a = 3$, $\mu = 10$, and $c_1 = 10$. ∎

Table 2.2. Optimum repair limit time T^* and expected cost rate $C(T^*)$ when $a = 3$, $\mu = 10$, and $c_1 = 10$

θ	T^*	$C(T^*)$
0.1	0.330	0.981
0.2	0.489	0.931
0.3	0.647	0.883
0.4	0.804	0.826
0.5	0.961	0.763
0.6	1.116	0.697
0.7	1.272	0.632
0.8	1.428	0.568
0.9	1.584	0.507
1.0	1.742	0.450

Until now, we have discussed the case where the repair cost is not estimated when an operating unit fails. However, if the repair cost can be previously estimated when an operating unit fails and the decision can be made as to whether the failed unit should be repaired or replaced, the expected cost rate is easily given by

$$C(T) = \frac{c_1 \overline{G}(T) + \int_0^T c_r(t)\, \mathrm{d}G(t)}{\mu + \int_0^T t\, \mathrm{d}G(t)}. \tag{2.39}$$

Finally, we introduce the following earnings in specifying the repair limit policy. Let e_0 be a net earning per unit of time made by the production of an operating unit, e_1 be an earning gained for replacing a failed unit, and e_2 be an earning rate per unit of time while the unit is under repair, where both e_1 and e_2 would usually be negative. Then, by the similar method to that of obtaining (2.31), the expected earning rate is

$$C(T) = \frac{e_0\mu + e_1 \overline{G}(T) + e_2 \int_0^T \overline{G}(t)\, \mathrm{d}t}{\mu + \int_0^T \overline{G}(t)\, \mathrm{d}t}. \tag{2.40}$$

Checking up on these models with actual systems, modifying, and extending them, we could get an optimum repair limit policy.

2.2 Standby System with Spare Units

Most standby systems with spare units have been discussed only for the case where any failed units are repaired and become as good as new upon the repair completion. In the real world, it may be worthwhile to scrap some failed units without repairing, depending on the nature of the failed units. For instance, we have scrapped failed units according to the repair limit policy proposed in Section 2.1.2.

Consider the system with a main unit and n spare subunits that are statistically not identical to each other, but any spare ones have the same function as the main unit if they take over operation. The system functions as follows. When the main unit fails, it undergoes repair immediately and one of the spare units replaces it. As soon as the repair of the main unit is completed, it begins to operate and the operating spare unit is available for further use. Any failed spare units are scrapped. The system functions until the nth spare unit fails; *i.e.,* system failure occurs when the last spare unit fails while the main unit is under repair. This model often occurs when something is broken or lost, and we temporarily use a substitute until it is repaired or replaced. We believe that this could be applicable to other practical fields.

We are interested in the following operating characteristics of the system.

(i) The distribution and the mean time to first system failure, given that n spare units are provided at time 0.
(ii) The probability that the number of failed spare units is exactly equal to n and its expected number during $(0, t]$.

These quantities are derived by forming renewal equations, and using them, two optimization problems to determine an initial number of spares to stock are considered.

We adopt the expected cost per unit of time for an infinite time span; *i.e.,* the expected cost rate (see Section 3.1) as an appropriate objective function. First, we compare two systems with (1) both main and spare units and (2) only unrepairable spare units. Secondly, we do the preventive maintenance (PM) of the main unit. When the main unit works for a specified time T ($0 \le T \le \infty$) without failure, its operation is stopped and one of the spare units takes over operation. The main unit is serviced on failure or its age T, whichever occurs first. The costs incurred for each failed unit and each PM are introduced. Then, we derive an optimum PM policy that minimizes the expected cost rate under suitable conditions.

2.2.1 Reliability Quantities

Suppose that the failure time of the main unit has a general distribution $F(t)$ with finite mean μ and its repair time has a general distribution $G(t)$ with finite mean β, where $\overline{\Phi} \equiv 1 - \Phi$ for any function. The failure time of each spare unit also has a general distribution $F_s(t)$ with finite mean μ_s, even if it has been used before; *i.e.,* the life of spare units is not affected by past operation. It is assumed that all random variables considered here are independent, and all units are good at time 0. Furthermore, any failures are instantly detected and repaired or scrapped, and each switchover is perfect and its time is instantaneous.

Let $L_j(t)$ ($j = 1, 2, \ldots, n$) denote the first-passage time distribution to system failure when j spares are provided at time 0. Then, we have the following renewal equation.

$$L_n(t) = F(t) * \left\{ \int_0^t \overline{G}(u) \, dF_s^{(n)}(u) \right.$$

$$\left. + \sum_{j=0}^{n-1} L_{n-j}(t) * \int_0^t [F_s^{(j)}(u) - F_s^{(j+1)}(u)] \, dG(u) \right\} \quad (n = 1, 2, \ldots), \quad (2.41)$$

where the asterisk represents the Stieltjes convolution, and $F_s^{(j)}(t)$ $(j = 1, 2, \ldots)$ represents the j-fold Stieltjes convolution of $F_s(t)$ with itself and $F_s^{(0)}(t) \equiv 1$ for $t \geq 0$. The first term of the bracket on the right-hand side is the time distribution that all of n spares have failed before the first repair completion of the failed main unit, and the second term is the time distribution that j $(j = 0, 1, \ldots, n - 1)$ spares fail exactly before the first repair completion, and then, the main unit with $n - j$ spares operates again.

The first-passage time distribution $L_n(t)$ to system failure can be calculated recursively and determined from (2.41). To obtain $L_n(t)$ explicitly, we introduce the notation of the generating function of LS transforms;

$$L^*(z, s) \equiv \sum_{j=1}^{\infty} z^j \int_0^{\infty} e^{-st} \, dL_j(t) \quad \text{for } |z| < 1.$$

Then, taking the LS transform on both sides of (2.41) and using the generating function $L^*(z, s)$, we have

$$L^*(z, s) = \frac{F^*(s) \sum_{j=1}^{\infty} z^j \int_0^{\infty} e^{-st} \overline{G}(t) \, dF_s^{(j)}(t)}{1 - F^*(s) \sum_{j=0}^{\infty} z^j \int_0^{\infty} e^{-st} [F_s^{(j)}(t) - F_s^{(j+1)}(t)] \, dG(t)}, \quad (2.42)$$

where $F^*(s) \equiv \int_0^{\infty} e^{-st} dF(t)$.

Moreover, let l_n denote the mean first-passage time to system failure. Then, by a similar method to that of (2.41), we easily have

$$l_n = \mu + \int_0^{\infty} [1 - F_s^{(n)}(t)] \overline{G}(t) \, dt + \sum_{j=0}^{n-1} l_{n-j} \int_0^{\infty} [F_s^{(j)}(t) - F_s^{(j+1)}(t)] \, dG(t)$$

$$(n = 1, 2, \ldots) \quad (2.43)$$

and hence, the generating function is

$$l^*(z) \equiv \sum_{j=1}^{\infty} z^j l_j = \frac{\mu[z/(1-z)] + \sum_{j=1}^{\infty} z^j \int_0^{\infty} [1 - F_s^{(j)}(t)] \overline{G}(t) \, dt}{1 - \sum_{j=0}^{\infty} z^j \int_0^{\infty} [F_s^{(j)}(t) - F_s^{(j+1)}(t)] \, dG(t)}. \quad (2.44)$$

In a similar way, we obtain the expected number of failed spares during $(0, t]$. Let $p_n(t)$ be the probability that the total number of failed spares during $(0, t]$ is exactly n. Then, we have

$$p_0(t) = \overline{F}(t) + F(t) * \left[\overline{F}_s(t)\overline{G}(t) + p_0(t) * \int_0^t \overline{F}_s(u)\,\mathrm{d}G(u)\right] \tag{2.45}$$

$$p_n(t) = F(t) * \left\{\left[F_s^{(n)}(t) - F_s^{(n+1)}(t)\right]\overline{G}(t)\right.$$

$$\left. + \sum_{j=0}^n p_{n-j}(t) * \int_0^t \left[F_s^{(j)}(u) - F_s^{(j+1)}(u)\right]\mathrm{d}G(u)\right\} \quad (n = 1, 2, \dots).$$
$$\tag{2.46}$$

Introducing the notation

$$p^*(z, s) \equiv \sum_{n=0}^\infty z^n \int_0^\infty \mathrm{e}^{-st}\,\mathrm{d}p_n(t) \quad \text{for } |z| < 1$$

we have, from (2.45) and (2.46),

$$p^*(z, s) = \frac{1 - F^*(s)\left[1 - \sum_{j=0}^\infty z^j \int_0^\infty \mathrm{e}^{-st}\,\mathrm{d}\{[F_s^{(j)}(t) - F_s^{(j+1)}(t)]\overline{G}(t)\}\right]}{1 - F^*(s)\sum_{j=0}^\infty z^j \int_0^\infty \mathrm{e}^{-st}[F_s^{(j)}(t) - F_s^{(j+1)}(t)]\,\mathrm{d}G(t)},$$
$$\tag{2.47}$$

where note that $p^*(1, s) \equiv \lim_{z\to 1} p^*(z, s) = 1$. Thus, the LS transform of the expected number $M(t) \equiv \sum_{n=1}^\infty np_n(t)$ of failed spares during $(0, t]$ is

$$M^*(s) \equiv \sum_{n=1}^\infty \int_0^\infty \mathrm{e}^{-st}\,\mathrm{d}M(t) = \lim_{z\to 1} \frac{\partial p^*(z, s)}{\partial z}$$

$$= \frac{F^*(s)\int_0^\infty \mathrm{e}^{-st}\overline{G}(t)\,\mathrm{d}M_s(t)}{1 - F^*(s)G^*(s)}, \tag{2.48}$$

where $M_s(t) \equiv \sum_{j=1}^\infty F_s^{(j)}(t)$ is the renewal function of $F_s(t)$. Furthermore, the limit of the expected number of failed spares per unit of time is

$$M \equiv \lim_{t\to\infty} \frac{M(t)}{t} = \lim_{s\to 0} sM^*(s) = \frac{\int_0^\infty M_s(t)\,\mathrm{d}G(t)}{\mu + \beta}. \tag{2.49}$$

The result of M can be intuitively derived because the numerator represents the total expected number of failed spares during the repair time of the main unit and the denominator represents the mean time from the operation to the repair completion of the main unit.

Example 2.9. Suppose that $G(t) = 1 - \mathrm{e}^{-\theta t}$. In this case, from (2.44), when n spares are provided at time 0, the mean time to system failure is

$$l_n = \mu + n\left(\mu + \frac{1}{\theta}\right)\frac{1 - F_s^*(\theta)}{F_s^*(\theta)}.$$

Note that by adding one spare unit to the system, the mean time increases a constant $\alpha \equiv (\mu + 1/\theta)[1 - F_s^*(\theta)]/F_s^*(\theta)$. Furthermore, the LS transform of the expected number of failed spares during $(0, t]$ is

$$M^*(s) = \frac{F^*(s)F_s^*(s + \theta)}{\{1 - [\theta/(s + \theta)]F^*(s)\}[1 - F_s^*(s + \theta)]}$$

and its limit per unit of time is

$$M = \frac{F_s^*(\theta)}{(\mu + 1/\theta)[1 - F_s^*(\theta)]}$$

which is equal to $1/\alpha$; *i.e.*, $l_n = \mu + n/M$. ∎

2.2.2 Optimization Problems

First, we obtain the expected cost rate, by introducing costs incurred for each failed main unit and spare unit. This expected cost rate is easily deduced from the expected number of failed units. We compare two expected costs of the system with both main unit and spares and the system with only spares, and determine which of the systems is more economical.

Cost c_1 is incurred for each failed main unit, which includes all costs resulting from its failure and repair, and cost c_s is incurred for each failed spare, which includes all costs resulting from its failure, replacement, and cost of itself. Let $C(t)$ be the total expected cost during $(0, t]$. Then, the expected cost rate is, from Theorems 1.2 and 1.6 in Section 1.3,

$$C \equiv \lim_{t \to \infty} \frac{C(t)}{t} = c_1 M_1 + c_s M, \tag{2.50}$$

where M_1 is the expected number of the failed main unit per unit of time, and from (2.5), $M_1 = 1/(\mu + \beta)$.

Thus, from (2.49) the expected cost rate is

$$C = \frac{c_1 + c_s \int_0^\infty M_s(t)\,dG(t)}{\mu + \beta} \tag{2.51}$$

which is also equal to the expected cost per one cycle from the beginning of the operation to the repair completion of the main unit. If only spare units are allowed then the expected cost rate is

$$C_s \equiv \frac{c_s}{\mu_s}. \tag{2.52}$$

Therefore, comparing (2.51) and (2.52), we have $C \leq C_s$ if and only if

$$c_1 \leq c_s \left[\frac{\mu + \beta}{\mu_s} - \int_0^\infty M_s(t)\,dG(t) \right]$$

and *vice versa.*

In general, it is hard to compute the above costs directly. However, simple results that would be useful in practical fields can be obtained in the following particular cases. Because $M_s(t) \geq t/\mu_s - 1$ [1, p. 53], if $c_1 > c_s(\mu/\mu_s + 1)$ then $C > C_s$. In the case of Example 2.9, we have the relation $C \leq C_s$ if and only if

$$c_1 \leq c_s \left[\frac{\mu + 1/\theta}{\mu_s} - \frac{F_s^*(\theta)}{1 - F_s^*(\theta)} \right]$$

and *vice versa.*

Next, consider the PM policy where the operating main unit is preventively maintained at time T $(0 \leq T \leq \infty)$ after its installation or is repaired at failure, whichever occurs first. The several PM policies are discussed fully in Chapter 6. In this model, spare units work temporarily during the interval of repair or PM time of the main unit. It is assumed that the PM time has the same distribution $G(t)$ with finite mean β as the repair time. The main unit becomes as good as new upon repair or PM, and begins to operate immediately. The costs incurred for each failed main unit and each failed spare are the same as c_1 and c_s, respectively, as those in the previous model. The PM cost c_2 with $c_2 < c_1$ incurs for each nonfailed main unit that is preventively maintained.

The total expected cost of one cycle from the operation to the repair or PM completion of the main unit is

$$F(T) \left[c_1 + c_s \int_0^\infty M_s(t) \, dG(t) \right] + \overline{F}(T) \left[c_2 + c_s \int_0^\infty M_s(t) \, dG(t) \right]$$

and the mean time of one cycle is

$$\int_0^T (t + \beta) \, dF(t) + \overline{F}(T)(T + \beta) = \int_0^T \overline{F}(t) \, dt + \beta.$$

Thus, in a similar way to that of obtaining (2.51), the expected cost rate is

$$C(T) = \frac{\tilde{c}_1 F(T) + \tilde{c}_2 \overline{F}(T)}{\int_0^T \overline{F}(t) \, dt + \beta}, \tag{2.53}$$

where $\tilde{c}_1 \equiv c_1 + c_s \int_0^\infty M_s(t) \, dG(t)$ and $\tilde{c}_2 \equiv c_2 + c_s \int_0^\infty M_s(t) \, dG(t)$, and $\tilde{c}_1 > \tilde{c}_2$ from $c_1 > c_2$.

We find an optimum PM time T^* that minimizes $C(T)$. Clearly, $C(0) = \tilde{c}_2/\beta$ is the expected cost in the case where the main unit is always under PM, and $C(\infty)$ is the expected cost of the main unit with no PM and is given in (2.51). Let $h(t) \equiv f(t)/\overline{F}(t)$ be the failure rate of $F(t)$ with $h(0) \equiv \lim_{t \to 0} h(t)$ and $h(\infty) \equiv \lim_{t \to \infty} h(t)$, and $k \equiv \tilde{c}_2/[\beta(\tilde{c}_1 - \tilde{c}_2)]$ and $K \equiv \tilde{c}_1/[(\mu + \beta)(\tilde{c}_1 - \tilde{c}_2)]$. Then, we have the following optimum policy.

Theorem 2.2. Suppose that the failure rate $h(t)$ is continuous and strictly increasing.

(i) If $h(0) < k$ and $h(\infty) > K$ then there exists a finite and unique T^* $(0 < T^* < \infty)$ that satisfies

$$h(T) \left[\int_0^T \overline{F}(t)\, \mathrm{d}t + \beta \right] - F(T) = \frac{\tilde{c}_2}{\tilde{c}_1 - \tilde{c}_2} \tag{2.54}$$

and the resulting expected cost rate is

$$C(T^*) = (c_1 - c_2)h(T^*). \tag{2.55}$$

(ii) If $h(0) \geq k$ then $T^* = 0$.

(iii) If $h(\infty) \leq K$ then $T^* = \infty$.

Proof.   Differentiating $C(T)$ in (2.53) with respect to T and putting it equal to zero, we have (2.54). Letting $Q(T)$ be the left-hand side of (2.54), it is easily proved that $Q(0) = \beta h(0)$, $Q(\infty) = (\mu + \beta)h(\infty) - 1$, and $Q(T)$ is strictly increasing because $h(t)$ is strictly increasing. Thus, if $h(0) < k$ and $h(\infty) > K$ then $Q(0) < \tilde{c}_2/(\tilde{c}_1 - \tilde{c}_2) < Q(\infty)$, and hence, there exists a finite and unique T^* that satisfies (2.54) and minimizes $C(T)$. Furthermore, from (2.54), we have (2.55).

    If $h(0) \geq k$ then $Q(0) \geq \tilde{c}_2/(\tilde{c}_1 - \tilde{c}_2)$. Thus, $C(T)$ is strictly increasing, and hence, $T^* = 0$. Finally, if $h(\infty) \leq K$ then $Q(\infty) \leq \tilde{c}_2/(\tilde{c}_1 - \tilde{c}_2)$. Thus, $C(T)$ is strictly decreasing, and $T^* = \infty$. ∎

    It is easily noted in Theorem 2.2 that if the failure rate $h(t)$ is non-increasing then $T^* = 0$ or $T^* = \infty$. Similar theorems are derived in Section 3.1.

    Until now, it has been assumed that the time to the PM completion has the same repair distribution $G(t)$. In reality, the PM time might be smaller than the repair time. So that, suppose that the repair time is $G_1(t)$ with mean β_1 and the PM time is $G_2(t)$ with mean β_2. Then, the expected cost rate is similarly given by

$$C(T) = \frac{\left[c_1 + c_s \int_0^\infty M_s(t)\, \mathrm{d}G_1(t) \right] F(T) + \left[c_2 + c_s \int_0^\infty M_s(t)\, \mathrm{d}G_2(t) \right] \overline{F}(T)}{\int_0^T \overline{F}(t)\, \mathrm{d}t + \beta_1 F(T) + \beta_2 \overline{F}(T)}. \tag{2.56}$$

Example 2.10.   Consider the optimization problem of ensuring that sufficient numbers of spares are initially provided to protect against shortage. If the probability α of occurrences of no shortage during $(0, t]$ is given *a priori*, we can find a minimum number of spares to maintain this level of confidence. One solution of this problem can be shown by computing a minimum n such that $\sum_{i=0}^n p_i(t) \geq \alpha$. If we need a minimum number of initial stocks during $(0, t]$ *on the average* without probabilistic guarantee, we might compute a minimum n such that $l_n \geq t$, or $M(t) \leq n$.

Table 2.3. Optimum PM time T^*, its cost rates $C(T^*)$, and C when $1/\lambda_s = 1$, $1/\theta = 5$, $c_1 = 10$, $c_2 = 1$, and $c_s = 2$

$2/\lambda$	T^*	$C(T^*)$	C
1	0.06	2.18	3.33
2	0.31	2.13	2.86
3	0.78	2.06	2.50
4	1.54	1.94	2.22
5	2.63	1.84	2.00
6	4.08	1.72	1.82
7	5.91	1.61	1.67
8	8.14	1.50	1.54
9	10.78	1.41	1.43
10	13.88	1.32	1.33

Next, compare two systems with main and spare units, and with only spares, when $F(t) = 1 - (1 + \lambda t)\mathrm{e}^{-\lambda t}$, $F_s(t) = 1 - \exp(-\lambda_s t)$ and $G(t) = 1 - \mathrm{e}^{-\theta t}$. Then, from (2.51) and (2.52), the expected cost rates are

$$C = \frac{c_1 + c_s(\lambda_s/\theta)}{2/\lambda + 1/\theta}, \qquad C_s = \lambda_s c_s.$$

Thus, $C \le C_s$ if and only if $c_1 \le c_s\ (2\lambda_s/\lambda)$ and *vice versa*.

Furthermore, when $F(t) = 1 - (1 + \lambda t)\mathrm{e}^{-\lambda t}$, the failure rate is $h(t) = \lambda^2 t/(1 + \lambda t)$ which is strictly increasing from 0 to λ. Thus, from (i) of Theorem 2.2, if $\lambda(\tilde{c}_1 - \tilde{c}_2) > \theta(2\tilde{c}_2 - \tilde{c}_1)$ then there exists a finite and unique T^* $(0 < T^* < \infty)$ that satisfies

$$\frac{1}{1 + \lambda T}\left[\frac{\lambda^2 T}{\theta} + \lambda T - (1 - \mathrm{e}^{-\lambda T})\right] = \frac{\tilde{c}_2}{\tilde{c}_1 - \tilde{c}_2}$$

and the expected cost rate is

$$C(T^*) = \frac{\lambda^2 T^*}{1 + \lambda T^*}(c_1 - c_2).$$

Table 2.3 gives the optimum PM time T^*, its cost rates $C(T^*)$, and C with no PM for $2/\lambda$ when $1/\lambda_s = 1$, $1/\theta = 5$, $c_1 = 10$, $c_2 = 1$, and $c_s = 2$. This indicates that when the mean failure time $2/\lambda$ is small, the PM time T^* is small and it is very effective. In this case, because $C_s = 2$, we have that $C \ge C_s$ for $2/\lambda \le 5$ and $C(T^*) > C_s$ for $2/\lambda \le 3$. ∎

2.3 Other Redundant Systems

In this section, we briefly mention redundant systems with repair maintenance without detailed derivations [37–40]. For the analysis of redundant systems,

it is of great importance to know the behavior of system failure; *i.e.,* the probability that the system will be in system failure, the mean time to system failure, and the expected number of system failures. For instance, if the system failure is catastrophic, we have to make the time to system failure as long as possible, by doing the PM and providing standby units.

2.3.1 Standby Redundant System

Consider an $(n+1)$-unit standby redundant system with $n+1$ repairpersons and one operating unit supported by n identical spares (refer to [40] for s ($1 \leq s \leq n+1$) repairpersons). Each unit fails according to a general distribution $F(t)$ with finite mean μ and undergoes repair immediately. When the repair is completed, the unit rejoins the spares. It is also assumed that the repair time of each failed unit is an independent random variable with an exponential distribution $(1 - e^{-\theta t})$ for $0 < \theta < \infty$. Let $\xi(t)$ denote the number of units under repair at time t. The system is said to be in state k at time t if $\xi(t) = k$. In particular, it is also said that system failure occurs when the system is in state $n + 1$. Furthermore, let $0 \equiv t_0 < t_1 < \cdots < t_m \ldots$ be the failure times of an operating unit. If we define $\xi_m \equiv \xi(t_m - 0)$ ($m = 0, 1, \ldots$) then ξ_m represents the number of units under repair immediately before the mth failure occurs. Then, we present only the results of transition probabilities and first-passage time distributions.

The Laplace transform of the binomial moment of transition probabilities $p_{ik}(t) \equiv \Pr\{\xi(t) = k | \xi_0 = i\}$ ($i = 0, 1, \ldots, n; k = 0, 1, \ldots, n+1$) is

$$
\begin{aligned}
\Psi_{ir}(s) &\equiv \sum_{k=r}^{n+1} \binom{k}{r} \int_0^\infty e^{-st} p_{ik}(t)\, dt \\
&= \frac{B_{r-1}(s)}{s + r\theta} \left\{ \sum_{j=0}^r \binom{i+1}{j} \frac{1}{B_{j-1}(s)} - \sum_{j=0}^{i+1} \binom{i+1}{j} \frac{1}{B_{j-1}(s)} \right. \\
&\quad \left. \times \frac{\sum_{j=0}^{r-1} \binom{n+1}{j}(s+j\theta)/B_{j-1}(s)}{\sum_{j=0}^{n+1} \binom{n+1}{j}(s+j\theta)/B_{j-1}(s)} \right\} \qquad (r = 0, 1, \ldots, n+1)
\end{aligned}
$$

and the limiting probability $p_k \equiv \lim_{t \to \infty} p_{ik}$ ($k = 0, 1, \ldots, n+1$) is

$$
\begin{aligned}
\Psi_r &\equiv \sum_{k=r}^{n+1} \binom{k}{r} p_k \\
&= \frac{(n+1)B_{r-1}(0)}{r} \frac{\sum_{j=r-1}^n \binom{n}{j}/B_j(0)}{1 + (n+1)(\mu\theta) \sum_{j=0}^n \binom{n}{j}/B_j(0)} \qquad (r = 1, 2, \ldots, n+1)
\end{aligned}
$$

and $\Psi_0 \equiv 1$, where $\sum_{j=0}^{-1} \equiv 0$, $B_{-1}(s) = B_0(0) \equiv 1$ and

$$B_r(s) \equiv \prod_{j=0}^{r} \frac{F^*(s+j\theta)}{1-F^*(s+j\theta)} \qquad (r = 0, 1, 2, \dots)$$

$$B_r(0) \equiv \prod_{j=1}^{r} \frac{F^*(j\theta)}{1-F^*(j\theta)} \qquad (r = 1, 2, \dots).$$

Thus, by the inversion formula of binomial moments,

$$p_{ik}^*(s) \equiv \int_0^\infty e^{-st} p_{ik}(t)\, \mathrm{d}t = \sum_{r=k}^{n+1} (-1)^{r-k} \binom{r}{k} \Psi_{ir}(s)$$

$$(i = 0, 1, \dots, n; k = 0, 1, \dots, n+1) \tag{2.57}$$

$$p_k = \sum_{r=k}^{n+1} (-1)^{r-k} \binom{r}{k} \Psi_r \quad (k = 0, 1, \dots, n+1). \tag{2.58}$$

It was shown in [41] that there exists the limiting probability p_k for $\mu < \infty$.

Next, the LS transform of the first-passage time distribution $F_{ik}(t) \equiv \sum_{m=1}^{\infty} \Pr\{\xi_m = k, \xi_j \neq k \text{ for } j = 1, 2, \dots, m-1, t_m \leq t \mid \xi_0 = i\}$ is, for $i < k$,

$$F_{ik}^*(s) \equiv \int_0^\infty e^{-st}\, \mathrm{d}F_{ik}(t) = \frac{\sum_{j=0}^{i+1} \binom{i+1}{j}/B_{j-1}(s)}{\sum_{j=0}^{k+1} \binom{k+1}{j}/B_{j-1}(s)} \quad (k = 0, 1, \dots, n) \tag{2.59}$$

and its mean time is

$$l_{ik} \equiv \int_0^\infty t\, \mathrm{d}F_{ik}(t) = \mu \sum_{j=1}^{k+1} \left[\binom{k+1}{j} - \binom{i+1}{j} \right] \frac{1}{B_{j-1}(0)}$$

$$(k = 0, 1, \dots, n), \tag{2.60}$$

where $\binom{i}{j} \equiv 0$ for $j > i$. The mean time l_{ik} when $i = -1$ and $k = n$ agrees with the result of [37], where state -1 means the initial condition that one unit begins to operate and n units are on standby at time 0.

The expected number M_k $(k = 0, 1, \dots, n-1)$ of visits to state k before system failure is

$$M_k = \sum_{r=k}^{n} (-1)^{r-k} \binom{r}{k} B_r(0) \sum_{j=r+1}^{n+1} \binom{n+1}{j} \frac{1}{B_{j-1}(0)} \quad (k = 0, 1, \dots, n-1). \tag{2.61}$$

Thus, the total expected number M of unit failures before system failure from state 0 is

$$M \equiv 1 + \sum_{k=0}^{n-1} M_k = \sum_{j=1}^{n+1} \binom{n+1}{j} \frac{1}{B_{j-1}(0)} \tag{2.62}$$

and the expected number of repairs before system failure is $M - (n+1)$. It is noted that μM is also the mean time to system failure $l_{-1\,n}$ in (2.60).

In the case of one repairperson, the first-passage time from state i to state k for $i < k$ coincides with that of queue $G/M/1$. Thus, for $i < k$ [42],

$$F_{ik}^*(s) = \frac{1 + [1 - F^*(s)][A_{i+1}(s) - \delta_{i+1\,0}]}{1 + [1 - F^*(s)]A_{k+1}(s)} \quad (k = 0, 1, \ldots, n), \qquad (2.63)$$

where $\delta_{ik} = 1$ for $i = k$ and 0 for $i \neq k$,

$$\sum_{j=0}^{\infty} A_j(s)z^j \equiv z^2 \Big[(1 - z)\{F^*[s + \theta(1 - z)] - z\}\Big] \quad \text{for } |z| < 1.$$

From the relation of transition probability and first-passage time distribution, we easily have

$$p_{ik}(t) = \int_0^t p_{k-1\,k}(t - u)\,\mathrm{d}F_{i\,k-1}(u)$$

$$p_{n\,n+1}(t) = \mathrm{e}^{-\theta t} + \int_0^t p_{n\,n+1}(t - u)\,\mathrm{d}F_{nn}(u)$$

$$F_{n\,n}(t) = \int_0^t F_{n-1\,n}(t - u)\theta\mathrm{e}^{-\theta u}\,\mathrm{d}u.$$

Thus, forming the Laplace transforms of the above equations and using the result of $F_{ik}^*(s)$,

$$p_{i\,n+1}^*(s) = \frac{1 + [1 - F^*(s)][A_{i+1}(s) - \delta_{i+1\,0}]}{s + [1 - F^*(s)]\{sA_{n+1}(s) + \theta[A_{n+1}(s) + \delta_{n\,0} - A_n(s)]\}} \qquad (2.64)$$

$$p_{n+1} = \frac{1}{1 + (\mu\theta)[A_{n+1}(0) + \delta_{n\,0} - A_n(0)]}. \qquad (2.65)$$

2.3.2 Parallel Redundant System

Consider an $(n + 1)$-unit parallel redundant system with one repairperson. Then, it can be easily seen that this system is equivalent to a standby system with $n+1$ repairpersons as described in Section 2.3.1 wherein the notations of failure and repair change one another. For instance, the transition probability p_{ik} in (2.57) becomes the transition probability for the number of units under operation. The LS transform of the busy period of a repairperson is

$$F_{n-1\,n}^*(s) = \frac{\sum_{j=0}^n \binom{n}{j}/B_{j-1}(s)}{\sum_{j=0}^{n+1} \binom{n+1}{j}/B_{j-1}(s)} \qquad (2.66)$$

and its mean time is

$$l_{n-1\,n} = \mu \sum_{j=0}^n \binom{n}{j}\frac{1}{B_j(0)}. \qquad (2.67)$$

In addition, when a system has $n + 1$ repairpersons (*i.e.,* there are as many repairpersons as the number of units), we may consider only n one-unit systems [1, p. 145]. In this model, we have

$$p_{ik}(t) = \sum_{j_1} \sum_{j_2} \binom{i}{j_1} \binom{n-i}{j_2} [P_{11}(t)]^{j_1} [P_{10}(t)]^{i-j_1} [P_{01}(t)]^{j_2} [P_{00}(t)]^{n-i-j_2},$$

(2.68)

where the summation takes over $j_1 + j_2 = k$, $j_1 \le i$, and $j_2 \le n - i$, and $P_{ij}(t)$ $(i, j = 0, 1)$ are given in (2.3) and (2.4).

Finally, consider n parallel units in which system failure occurs where k $(1 \le k \le n)$ out of n units are down simultaneously. The LS transform of the distribution of time to system failure and its mean time were obtained in [43], by applying a birth and death process, and 2-out-of-n systems were discussed in [4].

References

1. Barlow RE and Proschan F (1965) Mathematical Theory of Reliability. J Wiley & Sons, New York.
2. Ushakov IA (1994) Handbook of Reliability Engineering. J Wiley & Sons, New York.
3. Birolini A (1999) Reliability Engineering Theory and Practice. Springer, New York.
4. Nakagawa T (2002) Two-unit redundant models. In: Osaki S (ed) Stochastic Models in Reliability and Maintenance. Springer, New York:165–185.
5. Brown M, Proschan F (1983) Imperfect repair. J Appl Prob 20:851–859.
6. Fontenot RA, Proschan F (1984) Some imperfect maintenance models. In: Abdel-Hameed MS, Çinlar E, Quinn J (eds) Reliability Theory and Models. Academic, Orlando, FL:83–101.
7. Nakagawa T, Osaki S (1974) The optimum repair limit replacement policies. Oper Res Q 25:311–317.
8. Nakagawa T, Osaki S (1976) Reliability analysis of a one-unit system with unrepairable spare units and its optimization applications. Oper Res Q 27:101–110.
9. Cox DR (1962) Renewal Theory. Methuen, London.
10. Barlow RE, Hunter LC (1961) Reliability analysis of a one-unit system. Oper Res 9:200–208.
11. Bellman R, Kalaba RE, Lockett J (1966) Numerical Inversion of the Laplace Transform. American Elsevier, New York.
12. Abate J, Choudury G, Whitt W (1999) An introduction to numerical transform inversion and its application to probability models. In: Grassmann WK (ed) Commutational Probability. Kluwer Academic, The Netherlands:257–323.
13. Patton AD (1972) A probability method for bulk power system security assessment, I-basic concepts. IEEE Trans Power Apparatus Syst PAS-91:54–61.
14. Takács L (1957) On certain sojourn time problems in the theory of stochastic processes. Acta Math Acad Sci Hungary 8:169–191.
15. Muth EJ (1968) A method for predicting system downtime. IEEE Trans Reliab R-17:97–102.

16. Muth EJ (1970) Excess time, a measure of system repairability. IEEE Trans Reliab R-19:16–19.
17. Suyona, Van der Weide JAM (2003) A method for computing total downtime distributions in repairable systems. J Appl Prob 40:643–653.
18. Calabro SR (1962) Reliability Principles and Practices. McGraw-Hill, New York.
19. Buzacott JA (1973) Reliability analysis of a nuclear reactor fuel charging system. IEEE Trans Reliab R-22:88–91.
20. Aven T, Jensen U (1999) Stochastic Models in Reliability. Springer, New York.
21. Baxter LA (1981) Availability measures for a two-state system. J Appl Prob 18:227–235.
22. Mi J (1998) Some comparison results of system availability. Nav Res Logist 45:205–218.
23. Welker EL (1966) System effectiveness. In: Ireson WG (ed) Reliability Handbook Section 1. McGraw-Hill, New York.
24. Kabak IW (1969) System availability and some design implications. Oper Res 17:827–837.
25. Martz Jr HF (1971) One single-cycle availability. IEEE Trans Reliab R-20:21–23.
26. Nakagawa T and Goel AL (1973) A note on availability for a finite interval. IEEE Trans Reliab R-22:271–272.
27. Morse PM (1958) Queues, Inventories, and Maintenance. J Wiley & Sons, New York.
28. Nguyen DG, Murthy DNP (1980) A note on the repair limit replacement policy. J Oper Res Soc 31:103–104.
29. Drinkwater RW, Hastings NAJ (1967) An economic replacement model. Oper Res Q 18:121–138.
30. Hastings NAJ (1968) Some notes on dynamic programming and replacement. Oper Res Q 19:453–464.
31. Hastings NAJ (1969) The repair limit replacement method. Oper Res Q 20:337–349.
32. Love CE, Rodger R, Blazenko G (1982) Repair limit policies for vehicle replacement. INFOR 20:226–237.
33. Love CE, Guo R (1996) Utilizing Weibull failure rates in repair limit analysis for equipment replacement/preventive maintenance decision. J Oper Res Soc 47:1366–1376.
34. Choi CH, Yun WY (1998) A note on pseudodynamic cost limit replacement model. Int J Reliab Qual Saf Eng 5:287–292.
35. Dohi T, Matsushima N, Kaio N, Osaki S (1996) Nonparametric repair limit replacement policies with imperfect repair. Eur J Oper Res 96:260–273.
36. Dohi T, Takeshita K, Osaki S (2000) Graphical methods for determining/estimating optimal repair-limit replacement policies. Int J Reliab Qual Saf Eng 7:43–60.
37. Srinivasan VS (1968) First emptiness in the spare parts problem for repairable components. Oper Res 16:407–415.
38. Natarajan R (1968) A reliability problem with spares and multiple repair facilities. Oper Res 16:1041–1057.
39. Bhat UN (1973) Reliability of an independent component, s-spare system with exponential life times and general repair times. Technometrics 15:529–539.

40. Nakagawa T (1974) The expected number of visits to state k before a total system failure of a complex system with repair maintenance. Oper Res 22:108–116.
41. Takács, L (1962) Introduction to the Theory of Queues. Oxford University Press, New York.
42. Cohen JW (1969) The Single Server Queue. North-Holland, Amsterdam.
43. Downton F (1966) The reliability of multiplex systems with repair. J Roy Stat Soc B 28:459–476.