7

Regional Characterisation of Constrained Linear Quadratic Optimal Control

7.1 Overview

In Chapter 6 we provided a global characterisation of receding horizon constrained optimal control. This gives practically valuable insights into the form of the control law. Indeed, for many problems it is feasible to compute, store and retrieve the function $u = \mathcal{K}_N(x)$, thus eliminating the need to solve the associated QP on line.

In other cases, this approach may be too complex. Thus, in Chapter 8 we will explore various numerical algorithms aimed at solving the QP on line. The current chapter addresses a question with similar motivation but a different end result. Here we ask the following question: Given that we only ever apply the *first* move from the optimal control sequence of length N, is it possible that the first element of the control law might not change as the horizon increases beyond some modest size at least locally in the state space? We will show, via dynamic programming arguments, that this is indeed the case, at least for special classes of problems. To illustrate the ideas we will consider single input systems with an amplitude input constraint. This class of problems is simple and is intended to motivate the idea of local solutions. In particular, we will consider the simple control law

$$u_k = -\text{sat}_\Delta(Kx_k), \tag{7.1}$$

where $\text{sat}_\Delta(\,\cdot\,)$ is the saturation function defined in (6.10) of Chapter 6, and K is the feedback gain resulting from the *infinite horizon unconstrained* optimal control problem. We will show that there exists a nontrivial region of the state space (which we denote by $\bar{\mathbb{Z}}$) such that (7.1) is the *constrained* optimal control law with arbitrary large horizon. This is a very interesting result which has important practical implications. For example, (7.1) can be thought of as a simple type of anti-windup control law (see Section 7.4) when used in a certainty equivalence form with an appropriate observer for the system state and disturbances. Thus, the result establishes a link between anti-windup and RHC. Also, the result explains the (local) success of this control law in Example 1.2.1 of Chapter 1.

We will see that the characterisation of the region $\bar{\mathbb{Z}}$ is relatively complicated for large horizons. Hence, we will first, in Section 7.2, present the result for horizon $N = 2$. We will then establish the result for general horizons.

7.2 Regional Characterisation for Horizon 2

We consider again single input, linear, discrete time systems in which the magnitude of the control input is constrained to be less than or equal to a positive constant. In particular, let the system be given by

$$x_{k+1} = Ax_k + Bu_k, \tag{7.2}$$

where $x_k \in \mathbb{R}^n$ and $u_k \in \mathbb{R}$. As in Section 6.2 of Chapter 6, we consider the following fixed horizon optimal control problem with horizon 2:

$$\begin{aligned} \mathcal{P}_2(x): \qquad & V_2^{\text{OPT}}(x) \triangleq \min V_2(\{x_k\}, \{u_k\}), \\ & \text{subject to:} \\ & x_{k+1} = Ax_k + Bu_k \quad \text{for } k = 0, 1, \\ & x_0 = x, \\ & u_k \in \mathbb{U} \triangleq [-\Delta, \Delta] \quad \text{for } k = 0, 1, \end{aligned} \tag{7.3}$$

where $\Delta > 0$ is the input constraint level, and the objective function in (7.3) is

$$V_2(\{x_k\}, \{u_k\}) \triangleq \frac{1}{2} x_2^{\mathrm{T}} P x_2 + \frac{1}{2} \sum_{k=0}^{1} \left(x_k^{\mathrm{T}} Q x_k + u_k^{\mathrm{T}} R u_k \right). \tag{7.4}$$

The matrices Q and R in (7.4) are positive definite and P satisfies the algebraic Riccati equation

$$P = A^{\mathrm{T}} P A + Q - K^{\mathrm{T}} \bar{R} K, \tag{7.5}$$

where

$$K \triangleq \bar{R}^{-1} B^{\mathrm{T}} P A, \qquad \bar{R} \triangleq R + B^{\mathrm{T}} P B. \tag{7.6}$$

Let the control sequence that achieves the minimum in (7.3) be

$$\{u_0^{\text{OPT}}, u_1^{\text{OPT}}\}. \tag{7.7}$$

Then the RHC law is given by the first element of (7.7) (which depends on the current state $x = x_0$); that is,

$$\mathcal{K}_2(x) = u_0^{\text{OPT}}. \tag{7.8}$$

For the above special case, we have the following regional characterisation of the fixed horizon optimal control (7.7).

Lemma 7.2.1 (Regional Characterisation of the Fixed Horizon Control (7.7)**)** *Consider the fixed horizon optimal control problem $\mathcal{P}_2$ defined in* (7.3)–(7.6)*. Then for all $x \in \mathbb{Z}$, where*

$$\mathbb{Z} \triangleq \{x : |K(A - BK)x| \leq \Delta\}\,, \tag{7.9}$$

the optimal control sequence (7.7) *that attains the minimum is*

$$u_k^{\text{OPT}} = -\text{sat}_\Delta(Kx_k), \quad k = 0, 1. \tag{7.10}$$

Proof. The result follows from the proof of Theorem 6.2.1 of Chapter 6, in particular, from equations (6.17) and (6.19). □

The above result is quite remarkable in that it shows that the simple policy (7.10) provides a solution to the constrained linear quadratic fixed horizon optimal control problem (7.3)–(7.6) in a region of the state space. Using induction and similar dynamic programming arguments to the proof of Theorem 6.2.1, it is possible to show that a characterisation of the form (7.10) holds for horizon N of *arbitrary length*. We will establish this result in Section 7.3. As we will see, in the case of arbitrary horizon the characterisation is valid in a region $\mathbb{Z}$ of the state space having a more complex description than the one used in Lemma 7.2.1.

In the sequel, we will explore various aspects of the solution provided by Lemma 7.2.1, including a refinement of the set in which the result holds.

7.2.1 Regional Characterisation of RHC

We have seen above that the simple control law (7.10) solves the fixed horizon constrained linear quadratic problem in a special region of the state space. However, before we can use this control law as the solution to the associated RHC problem (see (7.8)), we need to extend the results to the receding horizon formulation of the problem. In particular, in order to guarantee that the RHC mapping (7.8) is regionally given by (7.10), it is essential to know if future states remain in the region in which the result holds or whether they are driven outside this region. Clearly, in the former case, we can implement the RHC algorithm as in (7.10) without further consideration. We thus proceed to examine the conditions under which the state remains in the region $\mathbb{Z}$ where (7.10) applies. We first define the mapping $\phi_{nl} : \mathbb{R}^n \to \mathbb{R}^n$ as

$$\phi_{nl}(x) \triangleq Ax - B\text{sat}_\Delta(Kx), \tag{7.11}$$

so that when the controller (7.10) is employed, the closed loop system satisfies $x_{k+1} = \phi_{nl}(x_k)$. In the sequel we denote by ϕ_{nl}^k the concatenation of ϕ_{nl} with itself k times; for example, $\phi_{nl}^0(x) \triangleq x$, $\phi_{nl}^1(x) = \phi_{nl}(x)$, $\phi_{nl}^2(x) = \phi_{nl}(\phi_{nl}(x))$, and so on.

We also require the following definition.

Definition 7.2.1 *Define the set* $\bar{\mathbb{Z}}$ *as*

$$\bar{\mathbb{Z}} \triangleq \left\{x : \phi_{nl}^k(x) \in \mathbb{Z},\ k = 0, 1, 2, \ldots\right\}. \tag{7.12}$$

◦

From its definition it is clear that $\bar{\mathbb{Z}}$ is the *maximal positively invariant set* contained in $\mathbb{Z}$ for the closed loop system $x_{k+1} = \phi_{nl}(x_k) = Ax_k - B\mathrm{sat}_\Delta(Kx_k)$ (see Definition 4.4.2 in Chapter 4).

We have from Lemma 7.2.1 that, if the initial state $x_0 = x \in \mathbb{Z}$, then $\mathcal{K}_2(x_0) = -\mathrm{sat}_\Delta(Kx_0)$. But, in general, since $\mathbb{Z}$ is not necessarily positively invariant under $\phi_{nl}(\cdot)$, after this control is applied there is no guarantee that the successor state $x_1 = \phi_{nl}(x_0)$ will stay in $\mathbb{Z}$. Hence, in general, $\mathcal{K}_2(x_1) \neq -\mathrm{sat}_\Delta(Kx_1)$. So, in order that the solution (7.10) can be applied to the RHC problem we must ensure that all successor states belong to $\mathbb{Z}$. We can then state:

Theorem 7.2.2 *For all* $x \in \bar{\mathbb{Z}}$ *defined in* (7.12)*, the RHC law* $\mathcal{K}_2$ *in* (7.8) *is given by*

$$\mathcal{K}_2(x) = -\mathrm{sat}_\Delta(Kx). \tag{7.13}$$

Proof. The proof of the theorem follows from the fact that $\bar{\mathbb{Z}} \subseteq \mathbb{Z}$ and that $\bar{\mathbb{Z}}$ is positively invariant for the system (7.2) under the control $\mathcal{K}_2(x) = -\mathrm{sat}_\Delta(Kx)$. Then, for all states in $\bar{\mathbb{Z}}$ the future trajectories of the system will be such that $x \in \bar{\mathbb{Z}} \subseteq \mathbb{Z}$, and from Lemma 7.2.1 we conclude that $\mathcal{K}_2(x) = u_0^{\mathrm{OPT}} = -\mathrm{sat}_\Delta(Kx)$. □

Notice that, if the set $\bar{\mathbb{Z}}$, in which the theorem is valid, were small enough such that the control sequence $\{u_k\} = \{-\mathrm{sat}_\Delta(Kx_k)\}$ stayed unsaturated along the system trajectories, then the result of Theorem 7.2.2 would be trivial, since it would readily follow from the result for the unconstrained case (see, for example, Anderson and Moore 1989). We will show next that $\bar{\mathbb{Z}}$ is not smaller than this trivial case.

Consider the *maximal output admissible set* $\mathcal{O}_\infty$ (introduced in (5.63) of Chapter 5), which in this case is defined as

$$\mathcal{O}_\infty \triangleq \left\{x : |K(A - BK)^i x| \leq \Delta \text{ for } i = 0, 1, \ldots\right\}. \tag{7.14}$$

The following proposition shows that the set $\bar{\mathbb{Z}}$ contains $\mathcal{O}_\infty$.

Proposition 7.2.3 $\mathcal{O}_\infty \subseteq \bar{\mathbb{Z}}$.

Proof. $\bar{\mathbb{Z}}$ is the maximal positively invariant set in $\mathbb{Z}$ for the closed loop system $x_{k+1} = \phi_{nl}(x_k)$. The set $\mathcal{O}_\infty$ is also a positively invariant set for $x_{k+1} = \phi_{nl}(x_k)$ (since $\phi_{nl}(x) = (A - BK)x$ in $\mathcal{O}_\infty$). It suffices, therefore, to establish that $\mathcal{O}_\infty \subseteq \mathbb{Z}$. This is indeed true since, from (7.14), we can write

$$\begin{aligned}\mathcal{O}_\infty &= \{x : |Kx| \le \Delta\} \cap \{x : |K(A-BK)x| \le \Delta\} \cap \\ &\quad \left\{x : |K(A-BK)^2x| \le \Delta\right\} \cap \dots \\ &= \{x : |Kx| \le \Delta\} \cap \mathbb{Z} \cap \left\{x : |K(A-BK)^2x| \le \Delta\right\} \cap \dots\end{aligned}$$

Hence $\mathcal{O}_\infty \subseteq \mathbb{Z}$ and the result then follows. □

We have proved that the set $\bar{\mathbb{Z}}$ contains the maximal positively invariant set in which the control constraints are avoided. Although a complete characterisation of the set $\bar{\mathbb{Z}}$ is not currently known, examples (see Example 7.3.2 at the end of the chapter) show that, in general, the set $\bar{\mathbb{Z}}$ is considerably larger than the set $\mathcal{O}_\infty$. In other words, that the motions of the system $x_{k+1} = \phi_{nl}(x_k)$ involve control sequences $\{u_k\} = \{-\text{sat}_\Delta(Kx_k)\}$ which remain saturated for several steps and, in accordance with Theorem 7.2.2, coincide with the solution provided by the RHC strategy (see the simulation of Example 7.3.2).

7.2.2 An Ellipsoidal Approximation to the Set $\bar{\mathbb{Z}}$

The set $\bar{\mathbb{Z}}$ is, in general, very difficult to characterise explicitly since it involves nonlinear inequalities. Notice however that, for any positively invariant set contained in $\bar{\mathbb{Z}}$, the result of Theorem 7.2.2 is also valid. In principle, a positively invariant inner approximation of the set $\bar{\mathbb{Z}}$ could be obtained by considering a *family* of positively invariant sets, which can be represented with reasonable complexity, and finding the biggest member within this family which is contained in $\mathbb{Z}$. The set $\mathbb{Z}$ is a polyhedral set, which suggests that polyhedral sets could be good candidates for this approximation, having also the advantage of flexibility. However, these sets could be arbitrarily complex (see, for example, Blanchini 1999).

In this section we will consider an alternative mechanism for obtaining positively invariant sets under the control $u = -\text{sat}_\Delta(Kx)$, based on the use of ellipsoidal sets. We will show how to construct an ellipsoidal invariant set $\mathcal{E} \subseteq \bar{\mathbb{Z}}$ based on a quadratic Lyapunov function constructed from the solution P of (7.5)–(7.6). In view of the discussion following Theorem 7.2.2, we will be interested in positively invariant ellipsoidal sets which extend to regions wherein the controls are saturated (that is, such that $|Kx| > \Delta$ for some x in the set). This result is related to the fact that a linear system, with optimal LQR controller, remains closed loop stable when sector-bounded input nonlinearities are introduced (see, for example, Anderson and Moore 1989). This, in turn, translates into bigger positively invariant ellipsoidal sets, under the control $u = -\text{sat}_\Delta(Kx)$, than the case $|Kx| \le \Delta$ for all x.

In the sequel we will need the following result.

Lemma 7.2.4 *Let K be a nonzero row vector, P a symmetric positive definite matrix and ρ a positive constant. Then,*

$$\min\{Kx : x^T Px \le \rho\} = -\sqrt{\rho K P^{-1} K^T}, \tag{7.15}$$

and

$$\max\{Kx : x^T Px \le \rho\} = +\sqrt{\rho K P^{-1} K^T}. \tag{7.16}$$

Proof. We first prove (7.15). The KKT conditions (2.32) in Chapter 2 are, in this case (convex objective function and convex constraint), sufficient for optimality. Now, let $\bar{x}$ be a (the) KKT point and denote by $\mu \ge 0$ its associated Lagrange multiplier. From the dual feasibility condition $K^T + \mu P\bar{x} = 0$ we can see that $\mu > 0$ (since $K^T \neq 0$) and $\bar{x} = -P^{-1}K^T/\mu$. Moreover, from the complementary slackness condition $\mu(\bar{x}^T P\bar{x} - \rho) = 0$ we have that the constraint is active at $\bar{x}$, that is $\bar{x}^T P\bar{x} = \rho$, from where we obtain that $\mu = \sqrt{KP^{-1}K^T/\rho}$. Thus, the minimum value is $K\bar{x} = -KP^{-1}K^T/\mu = -\sqrt{\rho K P^{-1}K^T}$, which proves (7.15). Finally, (7.16) follows from the fact that $\max\{Kx : x^T Px \le \rho\} = -\min\{-Kx : x^T Px \le \rho\}$. □

We define the ellipsoidal set

$$\mathcal{E} \triangleq \{x : x^{\mathrm{T}} P x \le \rho\}.$$

From Lemma 7.2.4 we can see that, if the ellipsoidal radius ρ is computed from $\rho = (1+\bar{\beta})^2\Delta^2/(KP^{-1}K^{\mathrm{T}})$, $\bar{\beta} \ge 0$, then the ellipsoid has the property: $|Kx| \le (1+\bar{\beta})\Delta$ for all $x \in \mathcal{E}$.

Notice that whenever $\bar{\beta}$ is bigger than zero the ellipsoid extends to regions where saturation levels are reached. For this reason, $\bar{\beta}$ is called the *over-saturation index*. We compute the *maximum over-saturation index* $\bar{\beta}_{\max}$ from:

$$\bar{\beta}_{\max} \triangleq \begin{cases} \dfrac{\sqrt{KK^{\mathrm{T}}\left[R(KK^{\mathrm{T}}\bar{R} - q_\varepsilon) + \bar{R}q_\varepsilon\right]} + q_\varepsilon}{KK^{\mathrm{T}}\bar{R} - q_\varepsilon} & \text{if} \quad \dfrac{q_\varepsilon}{KK^{\mathrm{T}}\bar{R}} < 1, \\ M^+ & \text{otherwise,} \end{cases} \tag{7.17}$$

where $q_\varepsilon = (1-\varepsilon)\lambda_{\min}(Q)$, $\varepsilon \in [0,1)$ is an arbitrarily small nonnegative number (introduced to ensure exponential stability; see the result in (7.23) below), $\lambda_{\min}(Q)$ is the minimum eigenvalue of the matrix Q (strictly positive, since Q is assumed positive definite), and M^+ is an arbitrarily large positive number.

Then, the *maximum radius* $\bar{\rho}_{\max}$ is computed from

$$\bar{\rho}_{\max} = \frac{(1+\bar{\beta}_{\max})^2\Delta^2}{KP^{-1}K^{\mathrm{T}}}. \tag{7.18}$$

Theorem 7.2.5 *The ellipsoid $\mathcal{E} = \{x : x^{\mathrm{T}}Px \le \rho\}$, with radius $\rho < \bar{\rho}_{max}$, has the following properties:*

(i) $\mathcal{E}$ is a positively invariant set for system (7.2) under the control $u = -\mathrm{sat}_\Delta(Kx)$.
(ii) The origin is exponentially stable in $\mathcal{E}$ for system (7.2) with control $u = -\mathrm{sat}_\Delta(Kx)$ (and, in particular, $\mathcal{E}$ is contained in the region of attraction of (7.2) for all admissible controls $u \in \mathbb{U} = [-\Delta, \Delta]$).

Proof. Consider the quadratic Lyapunov function: $V(x) = x^{\mathrm{T}} P x$. Let $x = x_k$ and $x^+ = x_{k+1}$. Then, by using the system equation (7.2) with control $u = -\mathrm{sat}_\Delta(Kx)$, and the Riccati equation (7.5), we can express the increment of $V(\cdot)$ along the system trajectory as

$$\begin{aligned}\Delta V(x) &\triangleq V(x^+) - V(x) = [Ax - B\mathrm{sat}_\Delta(Kx)]^{\mathrm{T}} P [Ax - B\mathrm{sat}_\Delta(Kx)] - x^{\mathrm{T}} P x \\ &= -\varepsilon x^{\mathrm{T}} Q x - (1-\varepsilon) x^{\mathrm{T}} Q x \\ &\quad + \bar{R}\left[|Kx|^2 - 2|Kx|\mathrm{sat}_\Delta(|Kx|) + \frac{B^{\mathrm{T}} P B}{\bar{R}} \mathrm{sat}_\Delta(|Kx|)^2\right].\end{aligned} \tag{7.19}$$

Next, define the sequence $\{\bar{\beta}_i\}_{i=1}^{\infty}$ as

$$\bar{\beta}_1 = 0, \quad \ldots, \quad \bar{\beta}_{i+1} = \sqrt{\frac{R}{\bar{R}} + \frac{q_\varepsilon (1+\bar{\beta}_i)^2}{KK^{\mathrm{T}}\bar{R}}}, \quad \ldots, \tag{7.20}$$

where $q_\varepsilon = (1-\varepsilon)\lambda_{\min}(Q)$ and $\varepsilon \in [0,1)$ is an arbitrarily small nonnegative number. It can be shown that the sequence $\{\bar{\beta}_i\}_{i=1}^{\infty}$ grows monotonically, and converges to $\bar{\beta}_{\max}$ defined by (7.17) in the case when $q_\varepsilon / KK^{\mathrm{T}}\bar{R} < 1$, or diverges to $+\infty$ if $q_\varepsilon / KK^{\mathrm{T}}\bar{R} \geq 1$, in which case, for any arbitrarily large positive number M^+, there exists i^+ such that $\bar{\beta}_i > M^+$ for all $i > i^+$.

Consider now the following cases:

Case (a). $|Kx| \leq (1+\bar{\beta}_1)\Delta$: Suppose first that $x \in \mathcal{E}$, $x \neq 0$, is such that

$$|Kx| \leq \Delta = (1+\bar{\beta}_1)\Delta,$$

then $\Delta V(x)$ in (7.19) is equal to

$$\Delta V(x) = -\varepsilon x^{\mathrm{T}} Q x - x^{\mathrm{T}}\left((1-\varepsilon)Q + K^{\mathrm{T}} R K\right) x < -\varepsilon x^{\mathrm{T}} Q x,$$

(from the positive definiteness of Q and R).

Case (b). $(1+\bar{\beta}_1)\Delta < |Kx| \leq (1+\bar{\beta}_2)\Delta$: Suppose next that $x \in \mathcal{E}$, $x \neq 0$, is such that

$$\Delta = (1+\bar{\beta}_1)\Delta < |Kx| \leq (1+\bar{\beta}_2)\Delta,$$

then $\mathrm{sat}_\Delta(|Kx|) = \Delta$, and, by the Cauchy-Schwarz inequality we obtain:

$$|Kx|^2 > (1+\bar{\beta}_1)^2\,\Delta^2 \quad \Rightarrow \quad -x^{\mathrm{T}} x < -(1+\bar{\beta}_1)^2 \frac{\Delta^2}{KK^{\mathrm{T}}}. \tag{7.21}$$

Therefore, an upper bound for $\Delta V(\cdot)$ in (7.19) is

$$\Delta V(x) < -\varepsilon x^{\mathrm{T}} Q x + \bar{R}\left[|Kx|^2 - 2|Kx|\Delta + \left(\frac{B^{\mathrm{T}} P B}{\bar{R}} - \frac{q_\varepsilon (1+\bar{\beta}_1)^2}{KK^{\mathrm{T}}\bar{R}}\right)\Delta^2\right]. \tag{7.22}$$

It is easy to see that the quadratic term in (7.22) is nonpositive if $\Delta < |Kx| \leq (1+\bar{\beta}_2)\Delta$, in which case

$$\Delta V(x) < -\varepsilon x^{\mathrm{T}} Q x.$$

Case (c). $(1+\bar{\beta}_i)\Delta < |Kx| \leq (1+\bar{\beta}_{i+1})\Delta$, $i = 2, 3, \ldots$: Repeating the above argument for $x \in \mathcal{E}$, $x \neq 0$, such that

$$\Delta < (1+\bar{\beta}_i)\Delta < |Kx| \leq (1+\bar{\beta}_{i+1})\Delta,$$

for $i = 2, 3, \ldots$, and, since $\bar{\beta}_i \to \bar{\beta}_{\max}$ (or diverges to $+\infty$, in which case $\bar{\beta}_i$ eventually becomes bigger than $\bar{\beta}_{\max} = M^+$), we can see that

$$\Delta V(x) < -\varepsilon x^{\mathrm{T}} Q x$$

if $|Kx| < (1+\bar{\beta}_{\max})\Delta$, which (from the construction of $\mathcal{E} = \{x : x^{\mathrm{T}} P x \leq \rho\}$ with $\rho < \bar{\rho}_{\max}$) is true for all $x \in \mathcal{E}$.

It follows that,

$$\Delta V(x) < -\varepsilon x^{\mathrm{T}} Q x \quad \text{for all } x \in \mathcal{E}, \tag{7.23}$$

and hence:

(i) The trajectories that start in the ellipsoid $\mathcal{E} \triangleq \{x : x^{\mathrm{T}} P x \leq \rho\}$ will never leave it since $\Delta V(x)$, along the trajectories, is negative definite on the ellipsoid. Therefore the ellipsoid $\mathcal{E}$ is a positively invariant set under the control $u = -\mathrm{sat}_\Delta(Kx)$.

(ii) From Theorem 4.3.3 in Chapter 4, the origin is exponentially stable in $\mathcal{E}$ for system (7.2) with control $u = -\mathrm{sat}_\Delta(Kx)$, with a region of attraction that includes the ellipsoid $\mathcal{E}$. Notice that if we choose $\varepsilon = 0$ we only guarantee asymptotic stability.

□

We have thus found an ellipsoidal set $\mathcal{E}$ that is positively invariant for the system (7.2) with control $u = -\mathrm{sat}_\Delta(Kx)$. Moreover, this control exponentially stabilises (7.2) with a region of attraction that contains $\mathcal{E}$. However, to guarantee that $u = -\mathrm{sat}_\Delta(Kx)$ is also the receding horizon optimal control law in $\mathcal{E}$, we need to further restrict the radius of the ellipsoid so that the trajectories inside $\mathcal{E}$ also remain within the set $\bar{\mathbb{Z}}$ defined in (7.12).

Recall that $\bar{\mathbb{Z}}$ is the maximal positively invariant set for the closed loop system $x_{k+1} = \phi_{nl}(x_k)$ contained in the set $\mathbb{Z}$ given by

$$\mathbb{Z} \triangleq \{x : |K(A - BK)x| \leq \Delta\}. \tag{7.24}$$

We then compute the ellipsoidal radius equal to

$$\bar{\rho} \triangleq \min\left\{ \frac{(1+\bar{\beta})^2 \Delta^2}{K P^{-1} K^{\mathrm{T}}}, \frac{\Delta^2}{(K(A-BK)) P^{-1} (K(A-BK))^{\mathrm{T}}} \right\}, \tag{7.25}$$

where $\bar{\beta} < \bar{\beta}_{\max}$, and $\bar{\beta}_{\max}$ is computed from (7.17) (in practice, one can choose $\bar{\beta}$ arbitrarily close to $\bar{\beta}_{\max}$). Then we have the following corollary of Theorems 7.2.2 and 7.2.5.

Corollary 7.2.6 *Consider the ellipsoidal set* $\mathcal{E} = \{x : x^{\mathrm{T}} P x \leq \bar{\rho}\}$ *where* $\bar{\rho}$ *is computed from* (7.25). *Then:*

(i) The set $\mathcal{E}$ *is a positively invariant set for system* (7.2) *under the control* $u = -\mathrm{sat}_{\Delta}(Kx)$.
(ii) The set $\mathcal{E}$ *is a subset of* $\bar{\mathbb{Z}}$.
(iii) The RHC law (7.8) *is*

$$\mathcal{K}_2(x) = -\mathrm{sat}_{\Delta}(Kx) \quad \text{for all } x \in \mathcal{E}. \tag{7.26}$$

(iv) System (7.2), *with the RHC sequence* (7.26), *is exponentially stable in* $\mathcal{E}$.

Proof.

(i) Notice from (7.18), (7.25), and the fact that $\bar{\beta} < \bar{\beta}_{\max}$, that $\bar{\rho} < \bar{\rho}_{\max}$. Then it follows from Theorem 7.2.5 (i) that $\mathcal{E} = \{x : x^{\mathrm{T}} P x \leq \bar{\rho}\}$ is positively invariant.
(ii) From Lemma 7.2.4 and the definitions (7.24) and (7.25) it follows that $x \in \mathcal{E} \Rightarrow x \in \mathbb{Z}$, and, since $\mathcal{E}$ is positively invariant, this implies that $\phi_{nl}^{k}(x) \in \mathbb{Z},\ k = 0, 1, 2, \ldots$. Clearly then, from the definition of $\bar{\mathbb{Z}}$, $x \in \bar{\mathbb{Z}}$.
(iii) This result follows immediately from (ii) above and Theorem 7.2.2.
(iv) This follows from $\bar{\rho} < \bar{\rho}_{\max}$ and Theorem 7.2.5 (ii).

□

7.3 Regional Characterisation for Arbitrary Horizon

Here we extend the result presented in Section 7.2 to arbitrary horizons. We will build on the special case presented above. This development is somewhat involved and the reader might prefer to postpone reading the remainder of this chapter until a second reading of the book. For clarity of exposition, we present first in Section 7.3.1 some notation and preliminary results. In particular various sets in $\mathbb{R}^n$ are defined. These sets are used in the characterisation of the state space regions in which the solution of the form (7.1) holds.

7.3.1 Preliminaries

We consider the discrete time system

$$x_{k+1} = Ax_k + Bu_k, \tag{7.27}$$

where $x_k \in \mathbb{R}^n$ and $u_k \in \mathbb{R}$. The pair (A, B) is assumed to be stabilisable. We consider the following fixed horizon optimal control problem:

$$\mathcal{P}_N(x): \qquad V_N^{\text{OPT}}(x) \triangleq \min V_N(\{x_k\},\{u_k\}), \tag{7.28}$$

$$\begin{aligned} &\text{subject to:}\\ &x_{k+1} = Ax_k + Bu_k \quad \text{for } k = 0,1,\ldots,N-1,\\ &x_0 = x,\\ &u_k \in \mathbb{U} \triangleq [-\Delta,\Delta] \quad \text{for } k = 0,1,\ldots,N-1, \end{aligned}$$

where $\Delta > 0$ is the input saturation level, and the objective function in (7.28) is

$$V_N(\{x_k\},\{u_k\}) \triangleq \frac{1}{2}x_N^{\mathrm{T}}Px_N + \frac{1}{2}\sum_{k=0}^{N-1}\left(x_k^{\mathrm{T}}Qx_k + u_k^{\mathrm{T}}Ru_k\right). \tag{7.29}$$

We assume that Q and R are positive definite, and P satisfies the algebraic Riccati equation (7.5)–(7.6).

Let the control sequence that achieves the minimum in (7.28) be $\{u_0^{\text{OPT}},\ldots,u_{N-1}^{\text{OPT}}\}$. The associated RHC law, which depends on the current state $x = x_0$, is

$$\mathcal{K}_N(x) = u_0^{\text{OPT}}. \tag{7.30}$$

For each $i = 0,1,2,\ldots,N-1$, the partial value function, is defined by (see similar definitions in (6.8) of Chapter 6)

$$V_{N-i}^{\text{OPT}}(x) \triangleq \min_{u_k\in\mathbb{U}} V_{N-i}(\{x_k\},\{u_k\}), \tag{7.31}$$

where V_{N-i} is the partial objective function

$$V_{N-i}(\{x_k\},\{u_k\}) \triangleq \frac{1}{2}x_N^{\mathrm{T}}Px_N + \frac{1}{2}\sum_{k=i}^{N-1}\left(x_k^{\mathrm{T}}Qx_k + u_k^{\mathrm{T}}Ru_k\right),$$

with x_k, $k = i,\ldots,N$ satisfying (7.27) starting from $x_i = x$. We refer to V_{N-i}^{OPT} as the *partial value function* (or, just the *value function*) "at time i," meaning that the (partial) value function "starts at time i." The partial value function at time N is defined as

$$V_0^{\text{OPT}}(x) \triangleq \frac{1}{2}x^{\mathrm{T}}Px.$$

We also define the functions $\delta_i : \mathbb{R}^n \to \mathbb{R}$ as

$$\delta_i(x) \triangleq Kx - \text{sat}_{\Delta_i}(Kx), \quad i = 1,2,\ldots,N, \tag{7.32}$$

where the saturation bounds Δ_i are defined as

$$\Delta_i \triangleq \left(1 + \sum_{k=0}^{i-2}|KA^kB|\right)\Delta, \quad i = 1,2,\ldots,N. \tag{7.33}$$

In summations, it is to be understood that $\sum_{k=k_1}^{k_2}(\,\cdot\,) = 0$ whenever $k_2 < k_1$, so that, in (7.33) we have

$$\Delta_1 = \Delta, \quad \Delta_2 = \Delta_1 + |KB|\Delta, \quad \ldots, \quad \Delta_{i+1} = \Delta_i + |KA^{i-1}B|\Delta, \ldots.$$

We define, for future use, the sets $X_i \subseteq \mathbb{R}^n$:

$$X_i \triangleq \left\{x : \delta_i\left(A^{i-1}(A - BK)x\right) = 0\right\}, \qquad i = 1, 2, \ldots, N-1. \tag{7.34}$$

Denote

$$\bar{K}_i \triangleq KA^{i-1}(A - BK), \qquad i = 1, 2, \ldots, N-1.$$

Then, the sets X_i are given by the set of linear inequalities:

$$X_i = \left\{x : \bar{K}_i x \le \Delta_i, \ -\bar{K}_i x \le \Delta_i\right\}, \quad i = 1, 2, \ldots, N-1. \tag{7.35}$$

Recall the definition of the nonlinear mapping ϕ_{nl} in (7.11), which we repeat here for convenience:

$$\phi_{nl}(x) \triangleq Ax - B\mathrm{sat}_\Delta(Kx). \tag{7.36}$$

Also, $\phi_{nl}^0(x) = x$ and ϕ_{nl}^k, $k \ge 1$, denotes the concatenation of ϕ_{nl} with itself k times.

We define, for future use, the sets Y_i, $Z_i \subseteq \mathbb{R}^n$:

$$Y_0 \triangleq Y_1 \triangleq \mathbb{R}^n,$$

$$Y_i = \textstyle\bigcap_{j=1}^{i-1} X_j, \quad i = 2, 3, \ldots, N, \tag{7.37}$$

$$Z_0 \triangleq Z_1 \triangleq \mathbb{R}^n,$$

$$Z_i \triangleq \left\{x : \phi_{nl}^k(x) \in Y_{i-k}, \ k = 0, 1, \ldots, i-2\right\}, \ i = 2, 3, \ldots, N, \tag{7.38}$$

so that

$$\begin{aligned} Z_2 &= Y_2, \\ Z_3 &= \{x : x \in Y_3, \ \phi_{nl}(x) \in Y_2\}, \\ Z_4 &= \{x : x \in Y_4, \ \phi_{nl}(x) \in Y_3, \ \phi_{nl}^2(x) \in Y_2\}, \end{aligned}$$

and so on.

We have the following properties of these sets:

Proposition 7.3.1

(i) $Y_{i+1} = Y_i \cap X_i$, $i = 1, 2, \ldots, N-1$.

(ii) The set sequence $\{Z_i : i = 0, 1, \ldots, N\}$ *is monotonically nonincreasing (with respect to inclusion), that is,* $Z_{i+1} \subseteq Z_i$, $i = 0, 1, \ldots, N-1$.

(iii) $Z_{i+1} = Y_{i+1} \cap \{x : \phi_{nl}(x) \in Z_i\}, \quad i = 0, 1, \ldots, N-1$.

Proof.

(i) This follows trivially from (7.37).

(ii) Certainly $Z_{i+1} \subseteq Z_i$ for $i = 0$ and 1. For $i \geq 2$:

$$\begin{aligned}
Z_{i+1} &= \left\{x : \phi_{nl}^k(x) \in \textstyle\bigcap_{j=1}^{i-k} X_j,\ k = 0, 1, \ldots, i-1\right\} \\
&= \left\{x : \phi_{nl}^k(x) \in \textstyle\bigcap_{j=1}^{i-k} X_j,\ k = 0, 1, \ldots, i-2\right\} \cap \left\{x \colon \phi_{nl}^{i-1}(x) \in X_1\right\} \\
&= \left\{x : \phi_{nl}^k(x) \in \textstyle\bigcap_{j=1}^{i-k-1} X_j,\ k = 0, 1, \ldots, i-2\right\} \\
&\quad \cap \left\{x : \phi_{nl}^k(x) \in X_{i-k},\ k = 0, 1, \ldots, i-2\right\} \cap \left\{x \colon \phi_{nl}^{i-1}(x) \in X_1\right\} \\
&= Z_i \cap \left\{x : \phi_{nl}^k(x) \in X_{i-k},\ k = 0, 1, \ldots, i-1\right\}.
\end{aligned}$$

(iii) This is trivial for $i = 0$. For $i \geq 1$:

$$\begin{aligned}
Z_{i+1} &= \left\{x : \phi_{nl}^k(x) \in \textstyle\bigcap_{j=1}^{i-k} X_j,\ k = 0, 1, \ldots, i-1\right\} \\
&= \left\{x : \phi_{nl}^{k+1}(x) \in \textstyle\bigcap_{j=1}^{i-k-1} X_j,\ k = -1, 0, \ldots, i-2\right\} \\
&= \left\{x : x \in \textstyle\bigcap_{j=1}^{i} X_j\right\} \\
&\quad \cap \left\{x : \phi_{nl}^k(\phi_{nl}(x)) \in \textstyle\bigcap_{j=1}^{i-k-1} X_j,\ k = 0, 1, \ldots, i-2\right\} \\
&= Y_{i+1} \cap \left\{x : \phi_{nl}(x) \in Z_i\right\}.
\end{aligned}$$

□

Finally, we require the following key result.

Lemma 7.3.2 *For any $i \in \{1, 2, \ldots, N-1\}$ define the functions $\phi_{nl}(\cdot)$ and $\delta_i(\cdot)$, $\delta_{i+1}(\cdot)$ as in* (7.36) *and* (7.32)*, respectively, and the set X_i as in* (7.34)*. Define, for $i \in \{1, 2, \ldots, N-1\}$ the functions $\mu_1, \mu_2 : \mathbb{R}^n \to [0, +\infty)$ as*

$$\begin{aligned}
\mu_1(x) &\triangleq \delta_i\left(A^{i-1}\phi_{nl}(x)\right)^2, \\
\mu_2(x) &\triangleq \delta_{i+1}\left(A^i x\right)^2.
\end{aligned}$$

Then, $\mu_1(x) = \mu_2(x)$ for all $x \in X_i$.

Proof. The functions $\mu_1, \mu_2 : \mathbb{R}^n \to [0, +\infty)$ can be written as

$$\begin{aligned}
\mu_1(x) &\triangleq \delta_i\left(A^{i-1}\phi_{nl}(x)\right)^2 \\
&= \Big[KA^i x - KA^{i-1}B \operatorname{sat}_\Delta(Kx) \\
&\qquad - \operatorname{sat}_{\Delta_i}\left(KA^i x - KA^{i-1}B \operatorname{sat}_\Delta(Kx)\right)\Big]^2, \\
\mu_2(x) &\triangleq \delta_{i+1}\left(A^i x\right)^2 = \left[KA^i x - \operatorname{sat}_{\Delta_{i+1}}\left(KA^i x\right)\right]^2,
\end{aligned}$$

for $i \in \{1, 2, \ldots, N-1\}$. Notice, from (7.35), that:

$$x \in X_i \Leftrightarrow \left|KA^i x - KA^{i-1}BKx\right| \leq \Delta_i = \left(1 + \sum_{k=0}^{i-2} |KA^k B|\right)\Delta. \qquad (7.39)$$

We will prove that $\mu_1(x) = \mu_2(x)$ for all $x \in X_i$ by considering two separate cases, case (a) where $x \in X_i$ and $|Kx| \leq \Delta$, and case (b) where $x \in X_i$ and $|Kx| > \Delta$.

Case (a). $x \in X_i$ and $|Kx| \leq \Delta$:
Suppose

$$|Kx| \leq \Delta. \tag{7.40}$$

It follows from (7.39) and (7.40) that

$$\begin{aligned}\mu_1(x) &= \big[KA^i x - KA^{i-1}B \,\mathrm{sat}_\Delta(Kx) \\ &\quad - \mathrm{sat}_{\Delta_i}\left(KA^i x - KA^{i-1}B\,\mathrm{sat}_\Delta(Kx)\right)\big]^2 \\ &= \left[KA^i x - KA^{i-1}BKx - \mathrm{sat}_{\Delta_i}\left(KA^i x - KA^{i-1}BKx\right)\right]^2 \\ &= 0.\end{aligned} \tag{7.41}$$

Also, notice from (7.39) and (7.40) that

$$\begin{aligned}\Delta_i &\geq |KA^i x - KA^{i-1}BKx| \geq |KA^i x| - |KA^{i-1}B||Kx| \\ &\geq |KA^i x| - |KA^{i-1}B|\Delta\end{aligned} \tag{7.42}$$

$$\Rightarrow$$

$$|KA^i x| \leq \Delta_i + |KA^{i-1}B|\Delta = \Delta_{i+1}, \tag{7.43}$$

then it follows that

$$\mu_2(x) = \left[KA^i x - \mathrm{sat}_{\Delta_{i+1}}\left(KA^i x\right)\right]^2 = 0, \tag{7.44}$$

and we conclude that, for case (a):

$$\mu_1(x) = \mu_2(x) = 0.$$

Case (b). $x \in X_i$ and $|Kx| > \Delta$:
Suppose:

$$|Kx| > \Delta. \tag{7.45}$$

We will consider two cases for case (b): case (b1) where $x \in X_i$ satisfies (7.45) and $|KA^i x| \leq \Delta_{i+1}$ and case (b2) where $x \in X_i$ satisfies (7.45) and $|KA^i x| > \Delta_{i+1}$.

Case (b1). $x \in X_i$, $|Kx| > \Delta$ and $|KA^i x| \leq \Delta_{i+1}$:
Suppose

$$|KA^i x| \leq \Delta_{i+1} = \Delta_i + |KA^{i-1}B|\Delta. \tag{7.46}$$

Now, suppose also that $KA^{i-1}BKx \leq 0$. Then from (7.39) we have

$$-KA^{i-1}B\mathrm{sat}_\Delta(Kx) \leq -KA^{i-1}BKx \leq -KA^i x + \Delta_i, \tag{7.47}$$

and from (7.45) and (7.46):

$$\begin{aligned}-KA^{i-1}B\text{sat}_\Delta(Kx) &= |KA^{i-1}B|\Delta \\ &\geq |KA^ix| - \Delta_i \geq -KA^ix - \Delta_i.\end{aligned} \tag{7.48}$$

Suppose next $KA^{i-1}BKx > 0$, then it follows from (7.45) and (7.46) that

$$\begin{aligned}-KA^{i-1}B\text{sat}_\Delta(Kx) &= -|KA^{i-1}B|\Delta \\ &\leq -|KA^ix| + \Delta_i \leq -KA^ix + \Delta_i,\end{aligned} \tag{7.47$'$}$$

and from (7.39)

$$-KA^{i-1}B\text{sat}_\Delta(Kx) \geq -KA^{i-1}BKx \geq -KA^ix - \Delta_i. \tag{7.48$'$}$$

We conclude from (7.47) and (7.48) (or, (7.47)$'$ and (7.48)$'$) that

$$|KA^ix - KA^{i-1}B\text{sat}_\Delta(Kx)| \leq \Delta_i, \tag{7.49}$$

and, hence:

$$\begin{aligned}\mu_1(x) = [&KA^ix - KA^{i-1}B\text{sat}_\Delta(Kx) \\ &- \text{sat}_{\Delta_i}\left(KA^ix - KA^{i-1}B\text{sat}_\Delta(Kx)\right)]^2 = 0.\end{aligned} \tag{7.50}$$

Also, it follows immediately from (7.46) that

$$\mu_2(x) = \left[KA^ix - \text{sat}_{\Delta_{i+1}}\left(KA^ix\right)\right]^2 = 0, \tag{7.51}$$

and we conclude that, for case (b1),

$$\mu_1(x) = \mu_2(x) = 0.$$

Case (b2). $x \in X_i$, $|Kx| > \Delta$ and $|KA^ix| > \Delta_{i+1}$:

Suppose

$$|KA^ix| > \Delta_{i+1} = \Delta_i + |KA^{i-1}B|\Delta. \tag{7.52}$$

We will next show that case (b2) is not compatible with

$$\text{sign}(KA^ix) = -\text{sign}(KA^{i-1}BKx). \tag{7.53}$$

To see this, notice that (7.39), (7.45) and (7.53) imply

$$\begin{aligned}\Delta_i &\geq |KA^ix - KA^{i-1}BKx| = |KA^ix| + |KA^{i-1}BKx| \\ &> |KA^ix| + |KA^{i-1}B|\Delta\end{aligned}$$

$$\Rightarrow$$

$$|KA^ix| < \Delta_i - |KA^{i-1}B|\Delta \leq \Delta_i + |KA^{i-1}B|\Delta,$$

which, clearly, contradicts (7.52). We conclude, then, that for case (b2),

$$\mathrm{sign}(KA^i x) = \mathrm{sign}(KA^{i-1}BKx). \tag{7.54}$$

We then have from (7.45) and (7.54), that

$$\begin{aligned}
\mu_1(x) &= \big[KA^i x - KA^{i-1}B\mathrm{sat}_\Delta(Kx) \\
&\quad - \mathrm{sat}_{\Delta_i}\left(KA^i x - KA^{i-1}B\mathrm{sat}_\Delta(Kx)\right)\big]^2, \\
&= [\mathrm{sign}(KA^i x)(|KA^i x| - |KA^{i-1}B|\Delta \\
&\quad - \mathrm{sat}_{\Delta_i}\left(|KA^i x| - |KA^{i-1}B|\Delta\right))]^2, \\
&= \big[|KA^i x| - |KA^{i-1}B|\Delta - \mathrm{sat}_{\Delta_i}\left(|KA^i x| - |KA^{i-1}B|\Delta\right)\big]^2.
\end{aligned} \tag{7.55}$$

Notice, finally, that (7.52) implies

$$|KA^i x| - |KA^{i-1}B|\Delta > \Delta_i, \tag{7.56}$$

which, in turn, implies in (7.55) that

$$\mu_1(x) = \big[|KA^i x| - |KA^{i-1}B|\Delta - \Delta_i\big]^2 = \big[|KA^i x| - \Delta_{i+1}\big]^2. \tag{7.57}$$

It also follows from (7.52) that

$$\mu_2(x) = \big[KA^i x - \mathrm{sat}_{\Delta_{i+1}}\left(KA^i x\right)\big]^2 \tag{7.58}$$

$$= \big[\mathrm{sign}(KA^i x)|KA^i x| - \mathrm{sign}(KA^i x)\Delta_{i+1}\big]^2 \tag{7.59}$$

$$= \big[|KA^i x| - \Delta_{i+1}\big]^2, \tag{7.60}$$

and we conclude that, for case (b2),

$$\mu_1(x) = \mu_2(x) = \big[|KA^i x| - \Delta_{i+1}\big]^2.$$

We can see that for all the cases considered (which cover all the possibilities for $x \in X_i$) the equality $\mu_1(x) = \mu_2(x)$ is satisfied. □

7.3.2 Main Result

The following theorem gives a characterisation of the partial value function (7.31). The proof extends to the case of arbitrary horizon the dynamic programming arguments used in Theorem 6.2.1 of Chapter 6 for horizon $N = 2$.

Theorem 7.3.3 *For all $i \in \{0, 1, \ldots, N\}$, provided $x \in Z_{N-i}$ (see (7.38)), the partial value function (7.31) is given by*

$$V_{N-i}^{\mathrm{OPT}}(x) = \frac{1}{2}x^{\mathrm{T}}Px + \frac{1}{2}\bar{R}\sum_{k=1}^{N-i}\delta_k(A^{k-1}x)^2, \tag{7.61}$$

where δ_k is the function defined in (7.32).

Proof. We prove the theorem by induction. We start from the last value function at $i = N$, and solve the problem backwards in time by using the principle of optimality:

$$V_{N-i}^{\text{OPT}}(x) = \min_{u\in\mathbb{U}} \left\{ \frac{1}{2}x^{\mathrm{T}}Qx + \frac{1}{2}u^{\mathrm{T}}Ru + V_{N-(i+1)}^{\text{OPT}}(Ax+Bu) \right\},$$

where u and x denote, $u = u_i$ and $x = x_i$, respectively.

(i) The value function V_0^{OPT} $(i = N)$:

By definition, the optimal value function at time N is

$$V_0^{\text{OPT}}(x) \triangleq \frac{1}{2}x^{\mathrm{T}}Px \quad \text{for all } x \in Z_0 \equiv \mathbb{R}^n.$$

(ii) The value function V_1^{OPT} $(i = N-1)$:

By the principle of optimality, for all $x \in \mathbb{R}^n$,

$$\begin{aligned} V_1^{\text{OPT}}(x) &= \min_{u\in\mathbb{U}} \left\{ \frac{1}{2}x^{\mathrm{T}}Qx + \frac{1}{2}u^{\mathrm{T}}Ru + V_0^{\text{OPT}}(Ax+Bu) \right\} \\ &= \min_{u\in\mathbb{U}} \left\{ \frac{1}{2}x^{\mathrm{T}}Qx + \frac{1}{2}u^{\mathrm{T}}Ru + \frac{1}{2}(Ax+Bu)^{\mathrm{T}}P(Ax+Bu) \right\} \\ &= \min_{u\in\mathbb{U}} \left\{ \frac{1}{2}x^{\mathrm{T}}Px + \frac{1}{2}\bar{R}(u+Kx)^2 \right\}. \end{aligned} \tag{7.62}$$

In deriving the last line we have made use of the algebraic Riccati equation (7.5)–(7.6). It is clear that the *unconstrained* optimal control is given by $u = -Kx$. From the convexity of the function $\bar{R}(u+Kx)^2$ it then follows that the *constrained* optimal control law is given by

$$u_{N-1}^{\text{OPT}} = \text{sat}_\Delta(-Kx) = -\text{sat}_\Delta(Kx) \quad \text{for all } x \in Z_1 \equiv \mathbb{R}^n, \tag{7.63}$$

and the optimal value function at time $N-1$ is

$$V_1^{\text{OPT}}(x) = \frac{1}{2}x^{\mathrm{T}}Px + \frac{1}{2}\bar{R}\delta_1(x)^2 \quad \text{for all } x \in Z_1 \equiv \mathbb{R}^n.$$

(iii) The value function V_2^{OPT} $(i = N-2)$:

By the principle of optimality, for all $x \in \mathbb{R}^n$,

$$\begin{aligned} V_2^{\text{OPT}}(x) &= \min_{u\in\mathbb{U}} \left\{ \frac{1}{2}x^{\mathrm{T}}Qx + \frac{1}{2}u^{\mathrm{T}}Ru + V_1^{\text{OPT}}(Ax+Bu) \right\} \\ &= \min_{u\in\mathbb{U}} \left\{ \frac{1}{2}x^{\mathrm{T}}Qx + \frac{1}{2}u^{\mathrm{T}}Ru + \frac{1}{2}(Ax+Bu)^{\mathrm{T}}P(Ax+Bu) \right. \\ &\quad \left. + \frac{1}{2}\bar{R}\delta_1(Ax+Bu)^2 \right\} \\ &= \min_{u\in\mathbb{U}} \left\{ \frac{1}{2}x^{\mathrm{T}}Px + \frac{1}{2}\bar{R}(u+Kx)^2 + \frac{1}{2}\bar{R}\delta_1(Ax+Bu)^2 \right\}. \end{aligned}$$

Since $\delta_1(Ax - BKx) = 0$ for $x \in X_1$ (see (7.34)) the *unconstrained* minimum of the right hand side of the above equation occurs at $u = -Kx$ if $x \in X_1$. Because the right hand side is convex in u, the *constrained* minimum occurs at

$$u_{N-2}^{\text{OPT}} = \text{sat}_{\Delta}(-Kx) = -\text{sat}_{\Delta}(Kx) \quad \text{for all } x \in Z_2 \equiv X_1,$$

and the optimal partial value function at time $N-2$ is

$$V_2^{\text{OPT}}(x) = \frac{1}{2}x^{\mathrm{T}}Px + \frac{1}{2}\bar{R}\delta_1(x)^2 + \frac{1}{2}\bar{R}\delta_1(\phi_{nl}(x))^2 \quad \text{for all } x \in Z_2 \equiv X_1.$$

Now we can use the result of Lemma 7.3.2 to express $V_2^{\text{OPT}}(x)$ as

$$\begin{aligned} V_2^{\text{OPT}}(x) &= \frac{1}{2}x^{\mathrm{T}}Px + \frac{1}{2}\bar{R}\delta_1(x)^2 + \frac{1}{2}\bar{R}\delta_2(Ax)^2 \\ &= \frac{1}{2}x^{\mathrm{T}}Px + \frac{1}{2}\bar{R}\sum_{k=1}^{2}\delta_k(A^{k-1}x)^2 \quad \text{for all } x \in Z_2 \equiv X_1. \end{aligned}$$

(iv) The value functions V_{N-i}^{OPT} and $V_{N-(i-1)}^{\text{OPT}}$ $(i \in \{1, 2, \ldots, N-1\})$:

We have established above the theorem for $N-i$, $i = N$, $N-1$ and $N-2$. We will now introduce the *induction hypothesis*. Assume that the value function V_{N-i}^{OPT}, for some $i \in \{1, 2, \ldots, N-1\}$, is given by the general expression (7.61). Based on this assumption, we will now derive the partial value function at time $i-1$.
By the principle of optimality,

$$\begin{aligned} V_{N-(i-1)}^{\text{OPT}} &= \min_{u\in\mathbb{U}}\left\{\frac{1}{2}x^{\mathrm{T}}Qx + \frac{1}{2}u^{\mathrm{T}}Ru + V_{N-i}^{\text{OPT}}(Ax+Bu)\right\} \\ &= \min_{u\in\mathbb{U}}\left\{\frac{1}{2}x^{\mathrm{T}}Qx + \frac{1}{2}u^{\mathrm{T}}Ru + \frac{1}{2}(Ax+Bu)^{\mathrm{T}}P(Ax+Bu)\right. \\ &\quad \left. + \frac{1}{2}\bar{R}\sum_{k=1}^{N-i}\delta_k(A^{k-1}(Ax+Bu))^2\right\} \\ &= \min_{u\in\mathbb{U}}\left\{\frac{1}{2}x^{\mathrm{T}}Px + \frac{1}{2}\bar{R}(u+Kx)^2\right. \\ &\quad \left. + \frac{1}{2}\bar{R}\sum_{k=1}^{N-i}\delta_k(A^{k-1}(Ax+Bu))^2\right\}, \end{aligned} \tag{7.64}$$

for all x such that

$$Ax + Bu_{i-1}^{\text{OPT}} \in Z_{N-i}, \tag{7.65}$$

(since the expression used above for $V_{N-i}^{\text{OPT}}(\cdot)$ is only valid in Z_{N-i}).

Since $\delta_k(A^{k-1}(Ax - BKx)) = 0$ for $k = 1, 2, \ldots, N-i$ if $x \in Y_{N-(i-1)} = X_1 \cap X_2 \cap \cdots \cap X_{N-i}$ (see (7.34)) the *unconstrained* minimum of the right hand side of (7.64) occurs at $u = -Kx$ if $x \in Y_{N-(i-1)}$. Because the right hand side of (7.64) is convex in u, the *constrained* minimum occurs at:

$$u_{i-1}^{\text{OPT}} = \text{sat}_\Delta(-Kx) = -\text{sat}_\Delta(Kx),$$

for all $x \in Y_{N-(i-1)} = \bigcap_{j=1}^{N-i} X_j$ *and*, such that $Ax - B\text{sat}_\Delta(Kx) = \phi_{nl}(x) \in Z_{N-i}$ (see (7.65), that is, for all $x \in Z_{N-(i-1)}$ (Proposition 7.3.1 (iii)).

Therefore the optimal partial value function at time $i-1$ is

$$\begin{aligned} V_{N-(i-1)}^{\text{OPT}}(x) &= \frac{1}{2}x^{\text{T}}Px + \frac{1}{2}\bar{R}\delta_1(x)^2 \\ &\quad + \frac{1}{2}\bar{R}\sum_{k=1}^{N-i}\delta_k(A^{k-1}\phi_{nl}(x))^2 \quad \text{for all } x \in Z_{N-(i-1)}, \end{aligned}$$

and, using the result of Lemma 7.3.2, we can express $V_{N-(i-1)}^{\text{OPT}}(\,\cdot\,)$ as

$$\begin{aligned} V_{N-(i-1)}^{\text{OPT}}(x) &= \frac{1}{2}x^{\text{T}}Px + \frac{1}{2}\bar{R}\delta_1(x)^2 + \frac{1}{2}\bar{R}\sum_{k=1}^{N-i}\delta_{k+1}(A^k x)^2 \\ &= \frac{1}{2}x^{\text{T}}Px + \frac{1}{2}\bar{R}\sum_{k=1}^{N-(i-1)}\delta_k(A^{k-1}x)^2 \quad \text{for all } x \in Z_{N-(i-1)}. \end{aligned}$$

This expression for $V_{N-(i-1)}^{\text{OPT}}(\cdot)$ is of the same form as that of (7.61) for $V_{N-i}^{\text{OPT}}(\cdot)$. The result then follows by induction.

□

The optimal solution of the fixed horizon control problem $\mathcal{P}_N$ easily follows as a corollary of the above result. For a horizon $N \geq 1$, consider the set

$$\begin{aligned} &\mathbb{Z} \triangleq Z_N = \mathbb{R}^n, \quad \text{if } N = 1, \\ &\mathbb{Z} \triangleq Z_N = \left\{x : \phi_{nl}^k(x) \in Y_{N-k},\ k = 0, 1, \dots, N-2\right\}, \text{ if } N \geq 2. \end{aligned} \tag{7.66}$$

Note that, for $N = 2$, (7.66) coincides with (7.9) since $\phi_{nl}^0(x) = x$ and, hence, $\mathbb{Z} = Z_2 = Y_2 = X_1 = \{x : |K(A - BK)x| \leq \Delta\}$ (see (7.37) and (7.35)) in this case.

We then have:

Corollary 7.3.4 *Consider the fixed horizon optimal control problem $\mathcal{P}_N$ defined in* (7.28)–(7.29)*, where x denotes the initial state $x = x_0$ of system* (7.27)*. Then for all $x \in \mathbb{Z}$ the minimum value is*

$$V_N^{\text{OPT}}(x) = \frac{1}{2}x^{\text{T}}Px + \frac{1}{2}\bar{R}\sum_{k=1}^{N}\delta_k(A^{k-1}x)^2, \tag{7.67}$$

and, for all $x \in \mathbb{Z}$ *the minimising sequence* $\{u_0^{\text{OPT}}, \dots, u_{N-1}^{\text{OPT}}\}$ *is*

$$u_k^{\text{OPT}} = \text{sat}_\Delta(-Kx_k) = -\text{sat}_\Delta(Kx_k), \tag{7.68}$$

for $k = 0, 1, \dots, N-1$, *where* $x_k = \phi_{nl}^k(x)$.

Proof. Equation (7.67) follows from (7.61) for $i = 0$. From Proposition 7.3.1 (iii) it follows that $x = x_0 \in \mathbb{Z} = Z_N \Rightarrow x_k = \phi_{nl}^k(x) \in Z_{N-k},\ k = 0, 1, \dots N-1$. Then (7.68) follows from the proof by induction of Theorem 7.3.3. □

The above result extends Lemma 7.2.1 to arbitrary horizons. We next present a simple example for which the solution (7.68) holds globally.

Example 7.3.1 (Scalar System with Cheap Control). Consider a scalar system $x_{k+1} = ax_k + bu_k$, $x_0 = x$, with $b \neq 0$ and weights $Q = 1$, $R = 0$, in the objective function (7.29). Such a design, with no weight on the control input, is a limiting case of the controller considered known as *cheap control*.

For this case, the *unconstrained* optimal control is $u = -Kx$, where K computed from (7.6) is $K = a/b$. Now, notice that, with $K = a/b$, the gain $A - BK$ is zero and, hence, the sets in (7.34)–(7.38) are: $X_i \equiv \mathbb{R}^n$, $Y_i \equiv \mathbb{R}^n$, $Z_i \equiv \mathbb{R}^n$, for all i. It then follows from Corollary 7.3.4 that the optimal control sequence for all $x \in \mathbb{R}$ in this case is

$$u_k^{\text{OPT}} = \text{sat}_\Delta\left(\frac{-ax_k}{b}\right) = -\text{sat}_\Delta\left(\frac{ax_k}{b}\right), \tag{7.69}$$

for $k = 0, 1, \dots, N-1$, where $x_k = \phi_{nl}^k(x)$. Note that here the result (7.69) holds *globally* in the state space. ∘

7.3.3 Regional Characterisation of RHC

As in Section 7.2.1, we turn here to the regional characterisation of the RHC law. That is, we will extend the regional characterisation given in Corollary 7.3.4 for the fixed horizon optimal control problem to its receding horizon formulation. To this end, we define the set

$$\bar{\mathbb{Z}} \triangleq \left\{x : \phi_{nl}^k(x) \in Y_N,\ k = 0, 1, 2, \dots\right\}, \qquad N \geq 2. \tag{7.70}$$

Notice that, from the definitions, $\bar{\mathbb{Z}} \subset \mathbb{Z} \subset Y_N$. The set $\bar{\mathbb{Z}}$ is the maximal positively invariant set contained in $\mathbb{Z}$ and Y_N for the closed loop system $x_{k+1} = \phi_{nl}(x_k) = Ax_k - B\text{sat}_\Delta(Kx_k)$. It is easy to see that (7.70) coincides with the set (7.12) introduced in Definition 7.2.1 for horizon $N = 2$ (since $\mathbb{Z} = Z_2 = Y_2$ in this case, see the discussion after (7.66)).

We then have the following result.

Theorem 7.3.5 *For all* $x \in \bar{\mathbb{Z}}$ *the RHC law* $\mathcal{K}_N$ *in* (7.30) *is given by*

$$\mathcal{K}_N(x) = \text{sat}_\Delta(-Kx) = -\text{sat}_\Delta(Kx). \tag{7.71}$$

Proof. The proof of the theorem follows from the fact that $\bar{\mathbb{Z}} \subseteq \mathbb{Z}$ and that $\bar{\mathbb{Z}}$ is positively invariant under the control $\mathcal{K}_N(x) = -\text{sat}_\Delta(Kx)$. Then, for all states in $\bar{\mathbb{Z}}$ the future trajectories of the system will be such that $x \in \bar{\mathbb{Z}} \subseteq \mathbb{Z}$, and from Corollary 7.3.4 we conclude that $\mathcal{K}_N(x) = u_0^{\text{OPT}} = \text{sat}_\Delta(-Kx) = -\text{sat}_\Delta(Kx)$. □

As we discussed before, if the set $\bar{\mathbb{Z}}$, in which the theorem is valid, were such that the control sequence $\{u_k\} = \{-\text{sat}_\Delta(Kx_k)\}$ stayed unsaturated along the system trajectories, then the result of Theorem 7.3.5 would be trivial. Also, if this set were such that only the first control in the sequence $\{u_k\} = \{-\text{sat}_\Delta(Kx_k)\}$ stayed saturated, then the result would also be trivial (although this is not as evident). This fact can be seen from the proof of Theorem 7.3.3. Assume, for this purpose, that the horizon is $N \geq 2$. Notice that the step $i = N-1$ of the dynamic programming procedure involves the minimisation of the quadratic function in (7.62), whose *constrained minimum* is simply given by $u_{N-1}^{\text{OPT}} = -\text{sat}_\Delta(Kx_{N-1}) \equiv -Kx_{N-1}$ (since we are assuming that *only the first control* saturates; see (7.63)). Following the same argument backwards in time, and assuming that the controls $u_i^{\text{OPT}} = -Kx_i$, $i = N-1, N-2, \ldots, 1$ are not saturated, it can be easily seen—since P satisfies (7.5)—that the same quadratic equation (7.62) will propagate until the initial step $i = 0$, in which case no assumption would be needed for the optimal control to be $u_0^{\text{OPT}} = -\text{sat}_\Delta(Kx_0)$. In fact, $\bar{\mathbb{Z}}$ can be considerably bigger than both of these trivial cases, as we will see later in Example 7.3.2.

Proposition 7.2.3 also extends to the case of horizons of arbitrary length, that is, the set $\bar{\mathbb{Z}}$ defined in (7.70) contains the maximal output admissible set $\mathcal{O}_\infty$, defined in (7.14). We show this below.

Proposition 7.3.6 *$\mathcal{O}_\infty \subseteq \bar{\mathbb{Z}}$.*

Proof. As in the proof of Proposition 7.2.3, since $\bar{\mathbb{Z}}$ is the maximal positively invariant set in Y_N, it suffices to show that $\mathcal{O}_\infty \subseteq Y_N \triangleq \bigcap_{i=1}^{N-1} X_i$ (see (7.37)). Assume, therefore, that $x \in \mathcal{O}_\infty$, so that (see (7.14))

$$|KA_K^j x| \leq \Delta, \quad j = 0, 1, \ldots, \tag{7.72}$$

where $A_K \triangleq A - BK$. For any $i \in \{1, 2, \ldots, N-1\}$,

$$\begin{aligned}
A_K^i &= (A-BK)A_K^{i-1} = AA_K^{i-1} - BKA_K^{i-1} \\
&= A(A-BK)A_K^{i-2} - BKA_K^{i-1} = A^2A_K^{i-2} - ABKA_K^{i-2} - BKA_K^{i-1} \\
&= A^2(A-BK)A_K^{i-3} - ABKA_K^{i-2} - BKA_K^{i-1} \\
&= A^3A_K^{i-3} - A^2BKA_K^{i-3} - ABKA_K^{i-2} - BKA_K^{i-1} \\
&\ \ \vdots \\
&= A^{i-1}A_K - \sum_{j=0}^{i-2} A^jBKA_K^{i-1-j},
\end{aligned}$$

which implies

$$KA^{i-1}A_K x = KA_K^i x + \sum_{j=0}^{i-2} KA^j BKA_K^{i-1-j} x. \tag{7.73}$$

From (7.72) and (7.73), we obtain the inequality

$$|KA^{i-1}A_K x| \le |KA_K^i x| + \sum_{j=0}^{i-2} |KA^j B||KA_K^{i-1-j} x| \tag{7.74}$$

$$\le \left(1 + \sum_{j=0}^{i-2} |KA^j B|\right) \Delta = \Delta_i \quad \text{(see (7.33))}. \tag{7.75}$$

This implies $x \in X_i$ for all $i \in \{1, 2, \ldots, N-1\}$ (see (7.35)), yielding the desired result. □

7.3.4 An Ellipsoidal Approximation to the Set $\bar{\mathbb{Z}}$

We can also construct an ellipsoidal inner approximation to the set $\bar{\mathbb{Z}}$, as was done in Section 7.2.2. To this end, recall that $\bar{\mathbb{Z}}$ is the largest positively invariant set, under the mapping $\phi_{nl}(\cdot)$, contained in the set $Y_N \triangleq \bigcap_{i=1}^{N-1} X_i$. Also, recall from (7.35) that the sets X_i are given by

$$X_i = \left\{x : |\bar{K}_i x| \le \Delta_i \right\}, \quad i = 1, 2, \ldots, N-1.$$

We then compute the ellipsoidal radius from

$$\bar{\rho} = \min \left\{ \frac{(1+\bar{\beta})^2 \Delta^2}{KP^{-1}K^{\mathrm{T}}}, \frac{\Delta_1^2}{\bar{K}_1 P^{-1} \bar{K}_1^{\mathrm{T}}}, \frac{\Delta_2^2}{\bar{K}_2 P^{-1} \bar{K}_2^{\mathrm{T}}}, \ldots, \frac{\Delta_{N-1}^2}{\bar{K}_{N-1} P^{-1} \bar{K}_{N-1}^{\mathrm{T}}} \right\}, \tag{7.76}$$

where $\bar{\beta} < \bar{\beta}_{\max}$, and $\bar{\beta}_{\max}$ is computed from (7.17) (in practice, one can choose $\bar{\beta}$ arbitrarily close to $\bar{\beta}_{\max}$).

Then we have that Corollary 7.2.6 holds for the ellipsoidal set

$$\mathcal{E} = \{x : x^{\mathrm{T}} P x \le \bar{\rho}\},$$

that is,

(i) $\mathcal{E}$ is positively invariant for system (7.27) under the control $u_k = -\mathrm{sat}_\Delta(Kx_k)$.
(ii) $\mathcal{E} \subseteq \bar{\mathbb{Z}}$.
(iii) The RHC law (7.71) holds, and it is optimal, for all $x \in \mathcal{E}$.
(iv) System (7.27), with the RHC sequence (7.71), is exponentially stable in $\mathcal{E}$.

The following example illustrates the regional characterisation of RHC and the different sets used to describe it.

Example 7.3.2. Consider the system $x_{k+1} = Ax_k + Bu_k$ with

$$A = \begin{bmatrix} 1 & 0 \\ 0.4 & 1 \end{bmatrix}, \qquad B = \begin{bmatrix} 0.4 \\ 0.08 \end{bmatrix},$$

which is the zero-order hold discretisation with a sampling period of 0.4 sec of the double integrator

$$\dot{x}^1(t) = u(t), \quad \dot{x}^2(t) = x^1(t).$$

The input constraint level is taken as $\Delta = 1$. The fixed horizon objective function is of the form (7.29) using $N = 10$, $Q = I$ and $R = 0.25$. The matrix P and the gain K were computed from (7.5) and (7.6). The maximum over-saturation index was computed from (7.17) with $\varepsilon = 0$ and is equal to $\bar{\beta}_{\max} = 1.3397$. We then take $\bar{\beta} = 1.3396 < \bar{\beta}_{\max}$ and compute the ellipsoid radius $\bar{\rho}$ from (7.76).

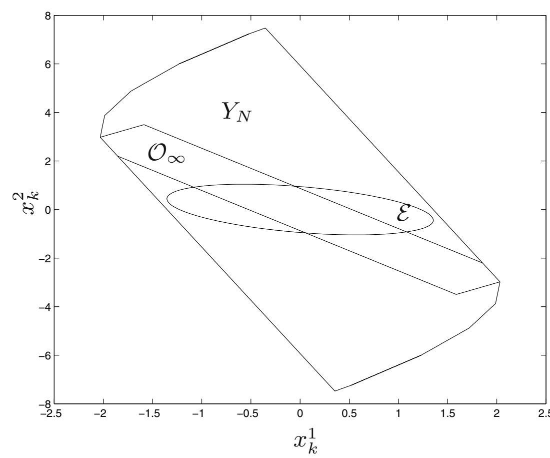
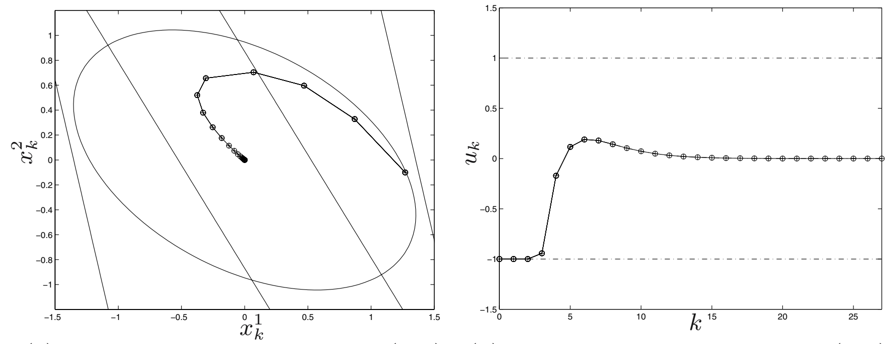


Figure 7.1. Set boundaries for Example 7.3.2.

In Figure 7.1 we show the following sets: $Y_N = \bigcap_{i=1}^{N-1} X_i$ (from (7.37)); the maximal output admissible set $\mathcal{O}_\infty$; and the ellipsoid $\mathcal{E} = \{x : x^{\mathrm{T}}Px \leq \bar{\rho}\}$. In this figure, x_k^1 and x_k^2 denote the components of the state vector x_k in the discrete time model. The sets $\mathcal{O}_\infty$ and $\mathcal{E}$ are positively invariant and are contained in $\bar{\mathbb{Z}}$ (Proposition 7.3.6 and (ii) above), and hence we have that $\mathcal{O}_\infty \cup \mathcal{E} \subseteq \bar{\mathbb{Z}} \subseteq Y_N$, which gives an estimate of the size of $\bar{\mathbb{Z}}$.

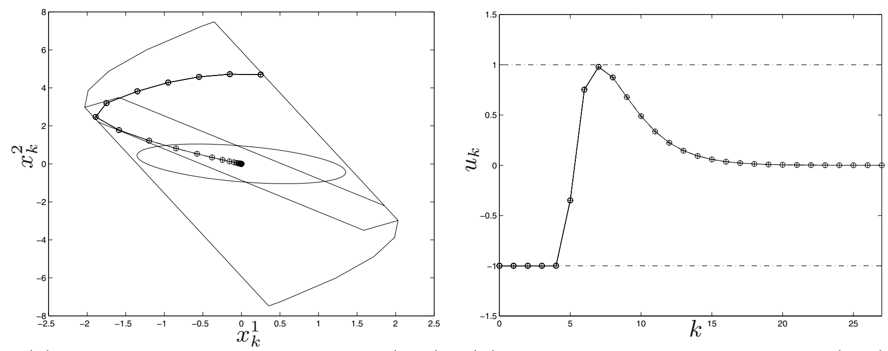
In Figure 7.2 we show the boundaries of the sets discussed above, together with the result of simulating the system with control $u = -\mathrm{sat}_\Delta(Kx)$, and with RHC performed numerically via quadratic programming, for an initial condition contained in the invariant ellipsoid $\mathcal{E} = \{x : x^{\mathrm{T}}Px \leq \bar{\rho}\}$. Notice



(a) State trajectory: with $-\text{sat}_\Delta(Kx)$ (solid-circle) and with (7.30) (dashdot-plus).

(b) Control sequence: $-\text{sat}_\Delta(Kx)$ (solid-circle) and (7.30) (dashdot-plus).

Figure 7.2. State trajectories and control sequence for the initial condition $x_0 = [1.27 \;\; -0.1]^{\text{T}}$. Also shown in the left figure are the set boundaries for Y_N, $\mathcal{O}_\infty$, $\mathcal{E}$.



(a) State trajectory: with $-\text{sat}_\Delta(Kx)$ (solid-circle) and with (7.30) (dashdot-plus).

(b) Control sequence: $-\text{sat}_\Delta(Kx)$ (solid-circle) and (7.30) (dashdot-plus).

Figure 7.3. State trajectories and control sequence for the initial condition $x_0 = [0.25 \; 4.7]^{\text{T}}$. Also shown in the left figure are the set boundaries for Y_N, $\mathcal{O}_\infty$, $\mathcal{E}$.

that both strategies coincide, and that the control remains saturated during the initial three steps.

Figure 7.3 shows a case where the initial condition is not contained in the invariant ellipsoid $\mathcal{E} = \{x : x^{\text{T}}Px \leq \bar{\rho}\}$ but is contained in the set $\bar{\mathbb{Z}}$ (since the trajectory does not leave the set Y_N). Therefore, as established in Theorem 7.3.5, both control sequences coincide and, as Figure 7.3 shows, they stay saturated during the initial five steps.

As can be seen from the simulations, the region in which both strategies coincide is such that the control remains saturated during several steps. Hence, we conclude that this region—the set $\bar{\mathbb{Z}}$—is, in fact, nontrivial (see the discussion following Theorem 7.3.5). ◦

7.4 Further Reading

For complete list of references cited, see References section at the end of book.

General

Further discussion on the regional solution may be found in De Doná (2000), De Doná and Goodwin (2000) and De Doná, Goodwin and Seron (2000).

Link to Anti-Windup Strategies

Another line of attack on the problem of input constraints was developed, beginning from a different perspective to the optimisation approach taken here, grouped under the name of *anti-windup techniques*. The first versions of these techniques can be traced back to PID and integral control, where limitations on the controller's ability to quickly regulate errors to zero, imposed by input saturation, led to unnecessarily high values of the state of the controller integrator. The term "anti-windup" is then used to describe the capability of the technique to prevent the state of the integrator from "winding up" to an excessively high value. Distinctive characteristics of the anti-windup technique are (see, for example, Teel 1999): (i) The original controller is used locally as long as it does not encounter input saturation, and (ii) saturation effects are minimised by modifying the controller structure when the plant input reaches its saturation level. In essence, all the algorithms achieve these goals by letting the controller states "know" about saturation being reached. A unified framework that encompasses many of these algorithms is given in Kothare, Campo, Morari and Nett (1994). They are prime examples of "evolutionary" strategies (see Section 1.2 of Chapter 1).

Consider now that the original controller is the static state feedback $u = -Kx$, that is, that the controller has no dynamics. Particularising the unified framework of Kothare et al. (1994) to this case, one easily obtains that the corresponding anti-windup strategy is equivalent to saturating the control signal, that is, $u = -\mathrm{sat}_\Delta(Kx)$. Comparing with the RHC law (7.71), we conclude that for all $x \in \bar{\mathbb{Z}}$, the RHC and anti-windup strategies have the identical characterisation $u = -\mathrm{sat}_\Delta(Kx)$. That is, the RHC and anti-windup control laws coincide in the region where RHC has the simple finite parameterisation (7.71).