5

Constrained Linear Quadratic Optimal Control

5.1 Overview

Up to this point we have considered rather general nonlinear receding horizon optimal control problems. Whilst we have been able to establish some important properties for these algorithms (for example, conditions for asymptotic stability), the algorithms remain relatively complex. However, remarkable simplifications occur if we specialise to the particular case of linear systems subject to linear inequality constraints. This will be the topic of the current chapter.

We will show how a fixed horizon optimal control problem for linear systems with a quadratic objective function and linear constraints can be set up as a quadratic program. We then discuss some practical aspects of the controller implementation, such as the use of observers to estimate states and disturbances. In particular, we will introduce the *certainty equivalence principle* and address several associated matters including steady state disturbance rejection (that is, provision of integral action) and how one can treat time delays in multivariable plants.

Finally, we show how closed loop stability of the receding horizon control [RHC] implementation can be achieved by specialising the results of Sections 4.4 and 4.5 in Chapter 4.

5.2 Problem Formulation

We consider a system described by the following linear, time-invariant model:

$$x_{k+1} = Ax_k + Bu_k, \tag{5.1}$$

$$y_k = Cx_k + d_k, \tag{5.2}$$

where $x_k \in \mathbb{R}^n$ is the state, $u_k \in \mathbb{R}^m$ is the control input, $y_k \in \mathbb{R}^m$ is the output, and $d_k \in \mathbb{R}^m$ is a time-varying output disturbance.

We assume that (A, B, C) is stabilisable and detectable and, for the moment, that one is not an eigenvalue of A. (See Section 5.4, where we relax the latter assumption.) So as to illustrate the principles involved, we go beyond the set-up described in Chapter 4 to include reference tracking and disturbance rejection.

Thus, we consider the problem where the output y_k in (5.2) is required to track a constant reference y^* in the presence of the disturbance d_k. That is, we wish to regulate, to zero, the output error

$$e_k \triangleq y_k - y^* = Cx_k + d_k - y^*. \tag{5.3}$$

Let $\bar{d}$ denote the steady state value of the output disturbance d_k, that is,

$$\bar{d} \triangleq \lim_{k\to\infty} d_k, \tag{5.4}$$

and denote by u_s, x_s, y_s and e_s, the *setpoints*, or desired steady state values for u_k, x_k, y_k and e_k, respectively. We then have that

$$y_s = y^* = Cx_s + \bar{d}, \tag{5.5}$$

$$e_s = 0, \tag{5.6}$$

and hence

$$u_s = [C(I - A)^{-1}B]^{-1}(y^* - \bar{d}), \tag{5.7}$$

$$x_s = (I - A)^{-1}Bu_s. \tag{5.8}$$

Without loss of generality, we take the current time as zero.

Here we assume knowledge of the disturbance d_k for all $k = 0, \ldots, N-1$, and the current state measurement $x_0 = x$. (In practice, these signals will be obtained from an observer/predictor of some form; see Section 5.5.)

Our aim is to find, for the system (5.1)–(5.3), the M-move control sequence $\{u_0, \ldots, u_{M-1}\}$, and corresponding state sequence $\{x_0, \ldots, x_N\}$ and error sequence $\{e_0, \ldots, e_{N-1}\}$, that minimise the finite horizon objective function:

$$V_{N,M}(\{x_k\}, \{u_k\}\{e_k\}) \triangleq \frac{1}{2}(x_N - x_s)^{\mathrm{T}}P(x_N - x_s) + \frac{1}{2}\sum_{k=0}^{N-1} e_k^{\mathrm{T}}Qe_k + \frac{1}{2}\sum_{k=0}^{M-1}(u_k - u_s)^{\mathrm{T}}R(u_k - u_s), \tag{5.9}$$

where $P \geq 0$, $Q \geq 0$, $R > 0$. In (5.9), N is the *prediction horizon*, $M \leq N$ is the *control horizon*, and u_s, x_s are the input and state setpoints given by (5.7) and (5.8), respectively. The control is set equal to its steady state setpoint after M steps, that is, $u_k = u_s$ for all $k \geq M$.

In the following section, we will show how the minimisation of (5.9) is performed under constraints on the input and output.

The above fixed horizon minimisation problem is solved at each time step for the current state and disturbance values. Then, the first move of the resulting control sequence is used as the current control, and the procedure is repeated at the next time step in a RHC fashion, as described in Chapter 4.

5.3 Quadratic Programming

In the presence of linear constraints on the input and output, the fixed horizon optimisation problem described in Section 5.2 can be transformed into a *quadratic program* [QP] (see Section 2.5.6 in Chapter 2). We show below how this is accomplished.

5.3.1 Objective Function Handling

We will begin by showing how (5.9) can be transformed into an objective function of the form used in QP. We start by writing, from (5.1) with $x_0 = x$, and using the constraint that $u_k = u_s$ for all $k \geq M$, the following set of equations:

$$\begin{aligned}
x_1 &= Ax + Bu_0, \\
x_2 &= A^2x + ABu_0 + Bu_1, \\
&\vdots \\
x_M &= A^M x + A^{M-1}Bu_0 + \cdots + Bu_{M-1}, \\
x_{M+1} &= A^{M+1}x + A^M Bu_0 + \cdots + ABu_{M-1} + Bu_s, \\
&\vdots \\
x_N &= A^N x + A^{N-1}Bu_0 + \cdots + A^{N-M}Bu_{M-1} + \sum_{i=0}^{N-M-1} A^i Bu_s.
\end{aligned} \tag{5.10}$$

Using $x_s = Ax_s + Bu_s$ (from (5.8)) recursively, we can write a similar set of equations for x_s as follows:

$$\begin{aligned}
x_s &= Ax_s + Bu_s, \\
x_s &= A^2x_s + ABu_s + Bu_s, \\
&\vdots \\
x_s &= A^M x_s + A^{M-1}Bu_s + \cdots + Bu_s, \\
x_s &= A^{M+1}x_s + A^M Bu_s + \cdots + ABu_s + Bu_s, \\
&\vdots \\
x_s &= A^N x_s + A^{N-1}Bu_s + \cdots + A^{N-M}Bu_s + \sum_{i=0}^{N-M-1} A^i Bu_s.
\end{aligned} \tag{5.11}$$

We now subtract the set of equations (5.11) from the set (5.10), and rewrite the resulting difference in vector form to obtain

$$\mathbf{x} - \mathbf{x}_s = \Gamma(\mathbf{u} - \mathbf{u}_s) + \Omega(x - x_s), \tag{5.12}$$

where

$$\mathbf{x} \triangleq \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{bmatrix}, \quad \mathbf{x}_s \triangleq \begin{bmatrix} x_s \\ x_s \\ \vdots \\ x_s \end{bmatrix}, \quad \mathbf{u} \triangleq \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{M-1} \end{bmatrix}, \quad \mathbf{u}_s \triangleq \begin{bmatrix} u_s \\ u_s \\ \vdots \\ u_s \end{bmatrix}, \tag{5.13}$$

($\mathbf{x}_s$ is an $nN \times 1$ vector, and $\mathbf{u}_s$ is an $mM \times 1$ vector), and where

$$\Gamma \triangleq \begin{bmatrix} B & 0 & \dots & 0 & 0 \\ AB & B & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ A^{M-1}B & A^{M-2}B & \dots & AB & B \\ A^M B & A^{M-1}B & \dots & A^2B & AB \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ A^{N-1}B & A^{N-2}B & \dots & \dots & A^{N-M}B \end{bmatrix}, \quad \Omega \triangleq \begin{bmatrix} A \\ A^2 \\ \vdots \\ A^N \end{bmatrix}. \tag{5.14}$$

We also define the disturbance vector

$$\mathbf{d} \triangleq \left[(d_1 - \bar{d})^{\mathrm{T}} \; (d_2 - \bar{d})^{\mathrm{T}} \; \cdots \; (d_{N-1} - \bar{d})^{\mathrm{T}} \; 0_{1\times m} \right]^{\mathrm{T}}, \tag{5.15}$$

and the matrices

$$\begin{aligned} \mathbf{Q} &\triangleq \mathrm{diag}\{C^{\mathrm{T}}QC, \dots, C^{\mathrm{T}}QC, P\}, \\ \mathbf{R} &\triangleq \mathrm{diag}\{R, \dots, R\}, \\ \mathbf{Z} &\triangleq \mathrm{diag}\{C^{\mathrm{T}}Q, C^{\mathrm{T}}Q, \dots, C^{\mathrm{T}}Q\}, \end{aligned} \tag{5.16}$$

where $\mathrm{diag}\{A_1, A_2, \dots, A_p\}$ denotes a block diagonal matrix having the matrices A_i as its diagonal blocks. Next, adding $-Cx_s - \bar{d} + y^* = 0$ (from (5.5)) to (5.3), we can express the error as

$$e_k = C(x_k - x_s) + (d_k - \bar{d}). \tag{5.17}$$

We now substitute (5.17) into the objective function (5.9), and rewrite it using the vector notation (5.13), (5.16) and (5.15), as follows:

$$\begin{aligned} V_{N,M} = \frac{1}{2} e_0^{\mathrm{T}} Q e_0 + \frac{1}{2}(\mathbf{x} - \mathbf{x}_s)^{\mathrm{T}} \mathbf{Q} (\mathbf{x} - \mathbf{x}_s) + \frac{1}{2}(\mathbf{u} - \mathbf{u}_s)^{\mathrm{T}} \mathbf{R} (\mathbf{u} - \mathbf{u}_s) \\ + (\mathbf{x} - \mathbf{x}_s)^{\mathrm{T}} \mathbf{Z} \mathbf{d} + \frac{1}{2} \mathbf{d}^{\mathrm{T}} \mathrm{diag}\{Q, Q, \dots, Q\} \mathbf{d}. \end{aligned} \tag{5.18}$$

Next, we substitute (5.12) into (5.18) to yield

$$\begin{aligned} V_{N,M} &= \bar{V} + \frac{1}{2}\mathbf{u}^{\mathrm{T}}(\Gamma^{\mathrm{T}}\mathbf{Q}\Gamma + \mathbf{R})\mathbf{u} + \mathbf{u}^{\mathrm{T}}\Gamma^{\mathrm{T}}\mathbf{Q}\Omega(x - x_s) \\ &\quad - \mathbf{u}^{\mathrm{T}}(\Gamma^{\mathrm{T}}\mathbf{Q}\Gamma + \mathbf{R})\mathbf{u}_s + \mathbf{u}^{\mathrm{T}}\Gamma^{\mathrm{T}}\mathbf{Z}\mathbf{d}, \\ &\triangleq \bar{V} + \frac{1}{2}\mathbf{u}^{\mathrm{T}}H\mathbf{u} + \mathbf{u}^{\mathrm{T}}\big[F(x - x_s) - H\mathbf{u}_s + D\mathbf{d}\big], \end{aligned} \tag{5.19}$$

where $\bar{V}$ is independent of $\mathbf{u}$ and

$$H \triangleq \Gamma^{\mathrm{T}}\mathbf{Q}\Gamma + \mathbf{R}, \qquad F \triangleq \Gamma^{\mathrm{T}}\mathbf{Q}\Omega, \qquad D \triangleq \Gamma^{\mathrm{T}}\mathbf{Z}. \tag{5.20}$$

The last calculation is equivalent to the elimination of the equality constraints given by the state equations (5.1)–(5.3) by substitution into the objective function.

Note that H in (5.20) is positive definite because we have assumed $R > 0$ in the objective function (5.9).

From (5.19) it is clear that, if the problem is *unconstrained*, $V_{N,M}$ is minimised by taking

$$\mathbf{u} = \mathbf{u}_{\mathrm{UC}}^{\mathrm{OPT}} \triangleq -H^{-1}\big[F(x - x_s) - H\mathbf{u}_s + D\mathbf{d}\big]. \tag{5.21}$$

The vector formed by the first m components of (5.21), $u_{0,\mathrm{UC}}^{\mathrm{OPT}}$, has a linear time-invariant feedback structure of the form

$$u_{0,\mathrm{UC}}^{\mathrm{OPT}} = -K(x - x_s) + u_s + K_d\mathbf{d}, \tag{5.22}$$

where K and K_d are defined as the first m rows of the matrices $H^{-1}F$ and $-H^{-1}D$, respectively. By appropriate selection of the weightings in the objective function (5.9), the resulting K is such that the matrix $(A - BK)$ is Hurwitz, that is, all its eigenvalues have moduli smaller than one (see, for example, Bitmead et al. 1990). The control law (5.22) is the control used by the RHC algorithm if the problem is unconstrained. More interestingly, even in the constrained case, the optimal RHC solution has the form (5.22) in a region of the state space that contains the steady state setpoint $x = x_s$. This point will be discussed in detail in Chapters 6 and 7.

5.3.2 Constraint Handling

We now introduce inequality constraints into the problem formulation. Magnitude and rate constraints on the plant *input* and *output* can be expressed as follows:

$$\begin{aligned} u_{\min} \leq u_k &\leq u_{\max}, & k &= 0, \dots, M-1, \\ y_{\min} \leq y_k &\leq y_{\max}, & k &= 1, \dots, N-1, \\ \delta u_{\min} \leq u_k - u_{k-1} &\leq \delta u_{\max}, & k &= 0, \dots, M-1, \end{aligned} \tag{5.23}$$

where u_{-1} is the input used in the previous step of the receding horizon implementation, which has to be stored for use in the current fixed horizon optimisation.

More generally, we may require to impose *state constraints* of the form

$$x_k \in \mathbb{X}_k \quad \text{for } k = 1, \ldots, N, \tag{5.24}$$

where $\mathbb{X}_k$ is a polyhedral set of the form

$$\mathbb{X}_k = \{x \in \mathbb{R}^n : L_k x \leq W_k\}. \tag{5.25}$$

For example, the constraint $x_N \in \mathbb{X}_f$, where $\mathbb{X}_f$ is a set satisfying certain properties, is useful to establish closed loop stability, as discussed in Chapter 4 (see also Section 5.6).

When constraints are present, we require that the setpoint $y_s = y^*$ and the corresponding input and state setpoints u_s and x_s be feasible, that is, that they satisfy the required constraints. For example, in the case of the constraints given in (5.23), we assume that $u_{\min} \leq u_s \leq u_{\max}$, and $y_{\min} \leq y^* \leq y_{\max}$. When the desired setpoint is not feasible in the presence of constraints, then one has to search for a feasible setpoint that is close to the desired setpoint in some sense. A procedure to do this is described in Section 5.4.

The constraints (5.23)–(5.25) can be written as *linear* constraints on $\mathbf{u}$ of the form

$$L\mathbf{u} \leq W, \tag{5.26}$$

where

$$L = \begin{bmatrix} I_{Mm} \\ \Psi \\ E \\ -I_{Mm} \\ -\Psi \\ -E \\ \tilde{L} \end{bmatrix}, \qquad W = \begin{bmatrix} \mathbf{u}_{\max} \\ \mathbf{y}_{\max} \\ \delta\mathbf{u}_{\max} \\ \mathbf{u}_{\min} \\ \mathbf{y}_{\min} \\ \delta\mathbf{u}_{\min} \\ \tilde{W} \end{bmatrix}. \tag{5.27}$$

In (5.27), I_{Mm} is the $Mm \times Mm$ identity matrix (where M is the control horizon and m is the number of inputs). Ψ is the following $(N-1)m \times Mm$ matrix:

$$\Psi = \begin{bmatrix} CB & 0 & \ldots & 0 & 0 \\ CAB & CB & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ CA^{M-1}B & CA^{M-2}B & \ldots & CAB & CB \\ CA^{M}B & CA^{M-1}B & \ldots & CA^2B & CAB \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ CA^{N-2}B & CA^{N-3}B & \ldots & \ldots & CA^{N-M-1}B \end{bmatrix}.$$

E is the following $Mm \times Mm$ matrix:

$$E = \begin{bmatrix} I_m & 0 & \dots & 0 \\ -I_m & I_m & \dots & 0 \\ & \ddots & \ddots & \\ 0 & \dots & -I_m & I_m \end{bmatrix},$$

where I_m is the $m \times m$ identity matrix; and

$$\tilde{L} \triangleq \operatorname{diag}\{L_1, L_2, \dots, L_N\}\Gamma,$$

where $L_1, \dots, L_N$ are the state constraint matrices given in (5.25) and Γ is given in (5.14).

The vectors forming W in (5.27) are as follows

$$\mathbf{u}_{\max} = \begin{bmatrix} u_{\max} \\ \vdots \\ u_{\max} \end{bmatrix}, \qquad \mathbf{u}_{\min} = \begin{bmatrix} -u_{\min} \\ \vdots \\ -u_{\min} \end{bmatrix},$$

$$\delta\mathbf{u}_{\max} = \begin{bmatrix} u_{-1} + \delta u_{\max} \\ \delta u_{\max} \\ \vdots \\ \delta u_{\max} \end{bmatrix}, \qquad \delta\mathbf{u}_{\min} = \begin{bmatrix} -u_{-1} - \delta u_{\min} \\ -\delta u_{\min} \\ \vdots \\ -\delta u_{\min} \end{bmatrix},$$

$$\mathbf{y}_{\max} = \begin{bmatrix} y_{\max} - CAx - d_1 \\ \vdots \\ y_{\max} - CA^M x - d_M \\ y_{\max} - CA^{M+1}x - d_{M+1} - CBu_s \\ \vdots \\ y_{\max} - CA^{N-1}x - d_{N-1} - \sum_{i=0}^{N-M-2} CA^i Bu_s \end{bmatrix},$$

$$\mathbf{y}_{\min} = \begin{bmatrix} -y_{\min} + CAx + d_1 \\ \vdots \\ -y_{\min} + CA^M x + d_M \\ -y_{\min} + CA^{M+1}x + d_{M+1} + CBu_s \\ \vdots \\ -y_{\min} + CA^{N-1}x + d_{N-1} + \sum_{i=0}^{N-M-2} CA^i Bu_s \end{bmatrix},$$

$$\tilde{W} \triangleq -\operatorname{diag}\{L_1, L_2, \dots, L_N\} \begin{bmatrix} Ax \\ \vdots \\ A^M x \\ A^{M+1}x + Bu_s \\ A^{M+2}x + ABu_s \\ \vdots \\ A^N x + \sum_{i=0}^{N-M-1} A^i Bu_s \end{bmatrix} + \begin{bmatrix} W_1 \\ W_2 \\ \vdots \\ W_N \end{bmatrix},$$

where x is the initial state, $u_{\max}$, $u_{\min}$, $\delta u_{\max}$, $\delta u_{\min}$, $y_{\max}$, $y_{\min}$ are the vectors of constraint limits defined in (5.23) and $L_1, \ldots, L_N$, $W_1, \ldots, W_N$ are the matrices and vectors of the state constraint polyhedra (5.25).

5.3.3 The QP Problem

Using the above formalism, we can express the problem of minimising (5.9) subject to the inequality constraints (5.23)–(5.25) as the QP problem of minimising (5.19) subject to (5.26), that is,

$$\begin{aligned}&\min_{\mathbf{u}} \ \frac{1}{2}\mathbf{u}^{\mathrm{T}} H \mathbf{u} + \mathbf{u}^{\mathrm{T}}\big[F(x - x_s) - H\mathbf{u}_s + D\mathbf{d}\big],\\ &\text{subject to:}\\ &L\mathbf{u} \leq W.\end{aligned} \tag{5.28}$$

Note that the term $\bar{V}$ in (5.19) has not been included in (5.28) since it is independent of $\mathbf{u}$.

The optimal solution $\mathbf{u}^{\text{OPT}}(x)$ to (5.28) is then:

$$\mathbf{u}^{\text{OPT}}(x) = \arg\min_{L\mathbf{u} \leq W} \frac{1}{2}\mathbf{u}^{\mathrm{T}} H \mathbf{u} + \mathbf{u}^{\mathrm{T}}\big[F(x - x_s) - H\mathbf{u}_s + D\mathbf{d}\big]. \tag{5.29}$$

The matrix H is called the *Hessian* of the QP. If the Hessian is positive definite, the QP is convex. This is indeed the case for H given (5.20), which, as already mentioned, is positive definite because we have assumed $R > 0$ in the objective function (5.9). In Chapter 11 we will investigate the structure of the Hessian in detail and formulate numerically stable ways to compute it from the problem data.

Standard numerical procedures (called *QP algorithms*) are available to solve the above optimisation problem. In Chapter 8 we will review some of these algorithms.

Once the QP problem (5.29) is solved, the receding horizon algorithm applies, at the current time k, only the first control move, formed by the first m components of the optimal vector $\mathbf{u}^{\text{OPT}}(x)$ in (5.29). This yields a control law of the form

$$u_k = \mathcal{K}(x_k, \bar{d}, y^*, \mathbf{d}), \tag{5.30}$$

where $x_k = x$ is the current state, and where the dependency on $\bar{d}$ and y^* is via u_s, x_s and $\mathbf{d}$ (see (5.7), (5.8), and (5.15)) as data for the optimisation (5.29). Then the whole procedure is repeated at the next time instant, with the optimisation horizon kept constant.

5.4 Embellishments

(i) Systems with integrators. In the above development, we have assumed that one is not an eigenvalue of A. This assumption allowed us to invert the matrix $(I - A)$ in (5.7) and (5.8).

There are several ways to treat the case when one is an eigenvalue of A. For example, in the single input-single output case, when one is an eigenvalue of A, then $u_s = 0$. To calculate x_s, we can write the state space model so that the integrator is shifted to the output, that is,

$$\begin{aligned}\tilde{x}_{k+1} &= \tilde{A}\tilde{x}_k + \tilde{B}u_k,\\ x'_{k+1} &= \tilde{C}\tilde{x}_k + x'_k,\\ y_k &= x'_k + d_k,\end{aligned}$$

where $\tilde{A}$, $\tilde{B}$, and $\tilde{C}$ correspond to the state space model of the reduced-order plant, that is, the plant without the integrator.

With this transformation, $(\tilde{A} - I)$ is nonsingular, and hence the state setpoint is

$$\begin{bmatrix} \tilde{x} \\ \\ x' \end{bmatrix}_s = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ y^* - \bar{d} \end{bmatrix},$$

where y^* is the reference and $\bar{d}$ is the steady state value of the output disturbance.

The optimisation problem is then solved in terms of these transformed state variables.

(ii) Setpoint Calculation for Underactuated Systems. By "underactuation" we mean that the actuators have insufficient authority to cancel the disturbance and reach the desired setpoint in steady state, that is, the desired setpoint is not feasible in the presence of constraints. This is not an uncommon situation. For example, in shape control in rolling mills, the actuators are cooling water sprays across the strip. These sprays change the radius of the rolls and hence influence the cross-directional reduction of the strip. However, these sprays have limited control authority, and thus they are frequently incapable of cancelling certain disturbances. More will be said about this cross-directional control problem in Chapter 15.

Under these conditions, it is clear that, in steady state, we will generally not be able to bring the output to the desired setpoint. Blind application of the receding horizon algorithm will lead to a saturated control, which achieves an optimal (in the quadratic objective function sense) compromise. However, there is an unexpected difficulty that arises from the weighting on the control effort in the objective function. In particular, the term $(u_k - u_s)^{\mathrm{T}}R(u_k - u_s)$ in (5.9), where u_s is the unconstrained steady state input required to achieve the desired setpoint $y_s = y^*$ (see (5.7) and (5.5)) may bias the solution of the optimisation problem.

One way to address this issue is to search for a feasible setpoint that is closest to the desired setpoint in a mean-square sense. In this case, the values of u_s and x_s that are used in the objective function (5.9) may be computed from the following quadratic program (see, for example, Muske and Rawlings 1993):

$$\begin{aligned} &\min_{u_s} [(y^* - \bar{d}) - Cx_s]^{\mathrm{T}}[(y^* - \bar{d}) - Cx_s] \\ &\text{subject to:} \\ &x_s = (I - A)^{-1}Bu_s, \\ &u_{\min} \leq u_s \leq u_{\max}, \\ &y_{\min} \leq Cx_s + \bar{d} \leq y_{\max}. \end{aligned} \tag{5.31}$$

The actual value for the output that will be achieved in steady state is then $Cx_s + \bar{d}$ and u_s has been automatically defined to be consistent so that no bias results.

5.5 Observers and Integral Action

The above development has assumed that the system evolves in a deterministic fashion and that the full state (including disturbances) is measured.

When the state and disturbances are not measured, it is possible to obtain combined state and disturbance estimates via an *observer*. Those estimates can then be used in the control algorithm by means of the *certainty equivalence* [CE] *principle*, which consists of designing the control law assuming knowledge of the states and disturbances (as was done in Sections 5.2 and 5.3), and then using their estimates as if they were the true ones when implementing the controller. (More will be said about the CE principle in Chapter 12.)

In practice, it is also important to ensure that the true system output reaches its desired steady state value, or setpoint, despite the presence of unaccounted constant disturbances and modelling errors. In linear control, this is typically achieved by the inclusion of integrators in the feedback loop; hence, we say that a control algorithm that achieves this property has *integral action*.

In the context of constrained control, there are several alternative ways in which integral action can be included into a control algorithm. For example, using CE, the key idea is to include a model for constant disturbances at the input or output of the system and design an observer for the composite model including system and disturbance models. Then the control is designed to reject the disturbance, assuming knowledge of states and disturbance. Finally, the control is implemented using CE. The resulting observer-based closed loop system has integral action, as we will next show.

We will consider a *model* of the system of the form (5.1)-(5.2) with a constant output disturbance. (One could equally assume a constant input disturbance.) This leads to a composite model of the form

$$\begin{aligned} x_{k+1} &= Ax_k + Bu_k, \\ d_{k+1} &= d_k = \bar{d}, \\ y_k &= Cx_k + d_k. \end{aligned} \tag{5.32}$$

We do not assume that the *model* (5.32) is a correct representation of the *real* system. In fact, we do not assume knowledge of the real system at all, but we assume that we can measure its output, which we denote y_k^{REAL}.

The RHC algorithm described in Sections 5.2 and 5.3 is now applied to the model (5.32) to design a controller for rejection of the constant output disturbance (and tracking of a constant reference) assuming knowledge of the model state and disturbance measurements. At each k the algorithm consists of solving the QP problem (5.29) for the current state $x_k = x$, with u_s and x_s computed from (5.7) and (5.8), and with $\mathbf{d} = 0$ (this follows from (5.15), since $d_k = \bar{d}$ for all k).

To apply the CE principle, we use the model (5.32) and the real system output y_k^{REAL} to construct an observer of the form

$$\begin{aligned} \hat{x}_{k+1} &= A\hat{x}_k + Bu_k + L_1[y_k^{\text{REAL}} - C\hat{x}_k - \hat{d}_k], \\ \hat{d}_{k+1} &= \hat{d}_k + L_2[y_k^{\text{REAL}} - C\hat{x}_k - \hat{d}_k], \end{aligned} \tag{5.33}$$

where L_1 and L_2 are determined via any observer design method (such as the Kalman filter) that ensures that the matrix

$$\begin{bmatrix} A - L_1C & -L_1 \\ -L_2C & I - L_2 \end{bmatrix}$$

is Hurwitz. Then we simply use the estimates $(\hat{x}_k, \hat{d}_k)$ given by (5.33) in the RHC algorithm as if they were the true states, that is, $\hat{x}_k$ replaces x (which is the current state) and $\hat{d}_k$ replaces $\bar{d}$. Specifically, the QP problem (5.29) is solved at each k for $x = \hat{x}_k$, $\mathbf{d} = 0$ and with $u_s = u_{s,k}$ and $x_s = x_{s,k}$ computed as

$$u_{s,k} \triangleq [C(I - A)^{-1}B]^{-1}(y^* - \hat{d}_k), \tag{5.34}$$

$$x_{s,k} \triangleq (I - A)^{-1}Bu_{s,k}. \tag{5.35}$$

Note that now $u_{s,k}$ and $x_{s,k}$ are time-varying variables (compare with (5.7) and (5.8)). Thus, the resulting CE control law has the form (5.30) evaluated at $x_k = \hat{x}_k$, $\bar{d} = \hat{d}_k$, $y^* = y^*$, and $\mathbf{d} = 0$, that is,

$$u_k = \mathcal{K}(\hat{x}_k, \hat{d}_k, y^*, 0). \tag{5.36}$$

We will next show how integral action is achieved. We make the following assumption.

Assumption 5.1 *We assume that the* real system *in closed loop with the (constrained) control law* (5.36) *reaches a steady state in which no constraints are active and where* $\{y_k^{\text{REAL}}\}$ *and* $\{\hat{d}_k\}$ *converge to the constant values* $\bar{y}^{\text{REAL}}$, $\bar{\hat{d}}$. ◦

Note that the assumption of no constraints being active when steady state is achieved implies that the control law, in steady state, must satisfy equation (5.22) (with $\mathbf{d} = 0$) evaluated at the steady state values, that is,

$$\bar{u} = -K(\bar{\hat{x}} - \bar{x}_s) + \bar{u}_s, \tag{5.37}$$

where $\bar{u} \triangleq \lim_{k\to\infty} u_k$, $\bar{\hat{x}} \triangleq \lim_{k\to\infty} \hat{x}_k$, $\bar{x}_s \triangleq \lim_{k\to\infty} x_{s,k}$, and $\bar{u}_s \triangleq \lim_{k\to\infty} u_{s,k}$. In (5.37), $\bar{u}_s$ and $\bar{x}_s$ satisfy, from (5.34) and (5.35),

$$\bar{u}_s = [C(I-A)^{-1}B]^{-1}(y^* - \bar{\hat{d}}), \tag{5.38}$$

$$\bar{x}_s = (I-A)^{-1}B\bar{u}_s, \tag{5.39}$$

where $\bar{\hat{d}} \triangleq \lim_{k\to\infty} \hat{d}_k$.

We make the following assumption on the matrix K in (5.37)

Assumption 5.2 *The matrix $(A-BK)$ is Hurwitz, that is, all its eigenvalues have moduli smaller than one.*

We then have the following result.

Lemma 5.5.1 *Under Assumptions 5.1 and 5.2, the real system output converges to the desired setpoint y^*, that is*

$$\bar{y}^{\text{REAL}} = y^*. \tag{5.40}$$

Proof. From the observer equations (5.33) in steady state we have

$$(I-A)\bar{\hat{x}} = B\bar{u}, \tag{5.41}$$

$$\bar{y}^{\text{REAL}} = C\bar{\hat{x}} + \bar{\hat{d}}. \tag{5.42}$$

Substituting (5.37) in (5.41) and using $(I-A)\bar{x}_s = B\bar{u}_s$ from (5.39), we obtain

$$(I-A)\bar{\hat{x}} = B\bar{u}_s - BK\bar{\hat{x}} + BK\bar{x}_s = (I-A)\bar{x}_s - BK\bar{\hat{x}} + BK\bar{x}_s.$$

Reordering terms in the above equation yields

$$(I-A+BK)\bar{\hat{x}} = (I-A+BK)\bar{x}_s,$$

or

$$\bar{\hat{x}} = \bar{x}_s, \tag{5.43}$$

since $(A-BK)$ is Hurwitz by Assumption 5.2. We then have, from (5.43), (5.39) and (5.38), that

$$C\bar{\hat{x}} = C\bar{x}_s = y^* - \bar{\hat{d}}. \tag{5.44}$$

Thus, the result (5.40) follows upon substitution of (5.44) into (5.42). □

Note that we have not shown (and indeed it will not be true in general) that $\bar{\hat{d}}$ is equal to the true output disturbance. In fact, the disturbance could actually be at the system input. Moreover, since we have not assumed that the model is correct, there need be no connection between $\hat{x}$ and the states of the real system. Lemma 5.5.1 is then an important result since it shows that, subject to the assumption that a steady state is achieved, the *required* output setpoint can be achieved despite uncertainty of different sources.

5.5.1 Observers for Systems with Time Delays

Many systems incorporate time delays between input and output. There is, thus, an issue of how best to deal with this. Naively, one could simply add extra states corresponding to the delays on each input. For example, suppose the delays on inputs 1 to m are τ_1 to τ_m samples, respectively. Let the input vector at time k be

$$u_k \triangleq \left[u_k^1 \; u_k^2 \; \cdots \; u_k^m\right]^{\mathrm{T}}. \tag{5.45}$$

Then we can use the model

$$x_{k+1} = Ax_k + B\xi_k, \tag{5.46}$$

where

$$\xi_k \triangleq \left[\xi_k^1 \; \xi_k^2 \; \cdots \; \xi_k^m\right]^{\mathrm{T}}, \tag{5.47}$$

and where each component ξ_k^i, $i = 1, \ldots, m$, has a model of the form

$$\begin{aligned} \eta_{k+1}^i &= \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \eta_k^i + \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} u_k^i, \\ \xi_k^i &= \left[0 \; 0 \; \cdots \; 0 \; 1\right] \eta_k^i, \end{aligned} \tag{5.48}$$

with η_k^i a vector having τ_i components. That is, ξ_k^i is the input u_k^i delayed τ_i samples. A block diagram illustrating the model (5.46)–(5.48) is shown in Figure 5.1.

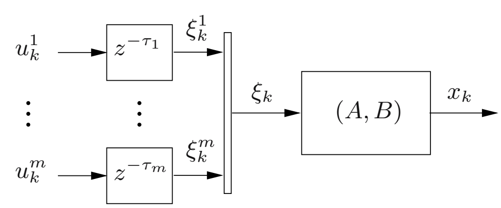

Figure 5.1. Block diagram of the input delayed system modelled by (5.48).

However, there are more parsimonious ways to proceed. As an illustration, consider the case where all input-output transfer functions contain a common delay. (This is typical in cases where measurements are made downstream from a process and all variables suffer the same transport delay. A specific example is the cross-directional control problem of the type discussed in Chapter 15.) Let the common delay be τ samples. Then, since the system model is linear, we can lump this delay at the output (whether it appears there or not). Hence, given output data $\{y_0, \ldots, y_k\}$, we can readily estimate $\hat{x}_{k-\tau}$ using a standard observer without considering any delays, other than delaying the inputs to ensure that we use the correct inputs in the model. By causality, this will utilise inputs up to $u_{k-\tau}$. The reason is that y_k is equivalent to $y'_{k-\tau}$, where y'_k denotes the undelayed output of the system.

For the purpose of the RHC calculations, we need $\hat{x}_k$. Since no measurements are available to compute this value, the best estimate of $\hat{x}_k$ is simply obtained by running the system model in open loop starting from $\hat{x}_{k-\tau}$.

Now we carry out the RHC calculations as usual to evaluate the sequence $\{u_k^{\text{OPT}}, \ldots, u_{k+N}^{\text{OPT}}\}$ and apply the first element u_k^{OPT} to the plant.

The reader will have observed that none of the above calculations have increased complexity resulting from the delay, save for the step of running the model forward from $\hat{x}_{k-\tau}$ to $\hat{x}_k$. Indeed, those readers who are familiar with the Smith predictor of classical control (see, for example, Goodwin et al. 2001) will recognise that the above procedure is a version of the scheme.

Of course, the problem becomes more complicated when there is not a common delay. In this case, we suggest that one should extract the delay of minimum value of all delays and treat that as a bulk delay of τ samples as described above. The residual (interaction) delays can then be dealt with as in (5.45)–(5.48), save that only the difference between the actual input delay and the bulk delay needs to be explicitly modelled.

5.6 Stability

In this section, we study closed loop stability of the receding horizon algorithm described in Sections 5.2 and 5.3. For simplicity, we assume that there are no reference or disturbance signals (that is, $d_k = 0$ for all k, $u_s = 0$ and $x_s = 0$). Also, in the objective function (5.9) we take $M = N$, $R > 0$ and we choose $Q > 0$ as the state (rather than output error) weighting matrix. We thus consider the following optimisation problem:

$$\mathcal{P}_N(x): \qquad V_N^{\text{OPT}}(x) \triangleq \min \left[F(x_N) + \frac{1}{2}\sum_{k=0}^{N-1}(x_k^{\mathrm{T}} Q x_k + u_k^{\mathrm{T}} R u_k) \right], \tag{5.49}$$

subject to:

$$x_{k+1} = Ax_k + Bu_k \quad \text{for } k = 0, \ldots, N-1, \tag{5.50}$$

$$x_0 = x, \tag{5.51}$$

$$u_k \in \mathbb{U} \quad \text{for } k = 0, \ldots, N-1, \tag{5.52}$$

$$x_k \in \mathbb{X} \quad \text{for } k = 0, \ldots, N, \tag{5.53}$$

$$x_N \in \mathbb{X}_f \subset \mathbb{X}, \tag{5.54}$$

as the underlying fixed horizon optimisation problem for the receding horizon algorithm.

Sufficient conditions for stability in the above linear constrained case can be obtained by specialising the results presented in Sections 4.4 and 4.5 of Chapter 4. Note that, with the choices $Q > 0$ and $R > 0$ in (5.49), condition **B1** of Section 4.4 is satisfied with $\gamma(t) = \lambda_{\min}(Q)\ t^2$, where $\lambda_{\min}(Q)$ is the minimum eigenvalue of the matrix Q. In the remainder of this section we will assume that the sets $\mathbb{U}$, $\mathbb{X}$ and $\mathbb{X}_f$ are convex and that the sets $\mathbb{U}$ and $\mathbb{X}_f$ contain the origin of their respective spaces (that is, condition **B5** is satisfied).

The fixed horizon optimal control problem $\mathcal{P}_N(x)$ in (5.49)–(5.54) has an associated set of feasible initial states $\mathbb{S}_N$. We recall from Definition 4.4.1 in Chapter 4 that $\mathbb{S}_N$ is the set of initial states $x \in \mathbb{X}$ for which there exist feasible state and control sequences, that is, sequences $\{x_0, x_1, \dots, x_N\}$, $\{u_0, u_1, \dots, u_{N-1}\}$ satisfying (5.50)–(5.54). The following lemma shows that $\mathbb{S}_N$ is convex if the sets $\mathbb{U}$, $\mathbb{X}$ and $\mathbb{X}_f$ are convex.

Lemma 5.6.1 (Convexity of the Set of Feasible Initial States) *Let the sets $\mathbb{U}$, $\mathbb{X}$ and $\mathbb{X}_f$ in (5.52)–(5.54) be convex. Then the set $\mathbb{S}_N$ of feasible initial states for problem $\mathcal{P}_N(x)$ in (5.49)–(5.54) is convex.*

Proof. Let $x \in \mathbb{S}_N$. Hence there exist feasible state and control sequences $\{x_0, x_1, \dots, x_N\}$, $\{u_0, u_1, \dots, u_{N-1}\}$ satisfying (5.50)–(5.54). Similarly, let $\check{x} \in \mathbb{S}_N$, so that there exist feasible state and control sequences $\{\check{x}_0, \check{x}_1, \dots, \check{x}_N\}$, $\{\check{u}_0, \check{u}_1, \dots, \check{u}_{N-1}\}$ satisfying (5.50)–(5.54).

Let $x^\alpha \triangleq \alpha x + (1-\alpha)\check{x}$, $\alpha \in [0,1]$, and consider the sequences

$$\{x_k^\alpha\} \triangleq \{x_0^\alpha, x_1^\alpha, \dots, x_N^\alpha\}, \tag{5.55}$$

$$\{u_k^\alpha\} \triangleq \{u_0^\alpha, u_1^\alpha, \dots, u_{N-1}^\alpha\}, \tag{5.56}$$

where $x_k^\alpha \triangleq \alpha x_k + (1-\alpha)\check{x}_k$, $k = 0, \dots, N$, and $u_k^\alpha \triangleq \alpha u_k + (1-\alpha)\check{u}_k$, $k = 0, \dots, N-1$. The above sequences are feasible since

$$\begin{aligned}
x_{k+1}^\alpha &= \alpha x_{k+1} + (1-\alpha)\check{x}_{k+1}\\
&= \alpha(Ax_k + Bu_k) + (1-\alpha)(A\check{x}_k + B\check{u}_k)\\
&= Ax_k^\alpha + Bu_k^\alpha \quad \text{for } k = 0, \dots, N-1,\\
x_0^\alpha &= \alpha x_0 + (1-\alpha)\check{x}_0\\
&= \alpha x + (1-\alpha)\check{x}\\
&= x^\alpha,
\end{aligned}$$

and, also,

$$\begin{aligned}
u_k^\alpha &\triangleq \alpha u_k + (1-\alpha)\check{u}_k \in \mathbb{U} \quad \text{for } k = 0, \dots, N-1,\\
x_k^\alpha &\triangleq \alpha x_k + (1-\alpha)\check{x}_k \in \mathbb{X} \quad \text{for } k = 0, \dots, N,\\
x_N^\alpha &\triangleq \alpha x_N + (1-\alpha)\check{x}_N \in \mathbb{X}_f,
\end{aligned}$$

by the convexity of $\mathbb{U}$, $\mathbb{X}$ and $\mathbb{X}_f$. Hence, $x^\alpha \in \mathbb{S}_N$, proving that $\mathbb{S}_N$ is convex. □

With the aid of Lemma 5.6.1, we can show that the value function $V_N^{\text{OPT}}(\,\cdot\,)$ in (5.49) is convex.

Lemma 5.6.2 (Convexity of the Value Function) *Let the sets $\mathbb{U}$, $\mathbb{X}$ and $\mathbb{X}_f$ in* (5.52)–(5.54) *be convex. Suppose that, in* (5.49)*, $Q \geq 0$, $R \geq 0$ and the terminal state weighting is of the form $F(x) = \frac{1}{2}x^{\mathrm{T}}Px$ with $P \geq 0$. Then the value function $V_N^{\text{OPT}}(\,\cdot\,)$ in* (5.49) *is convex.*

Proof. Let $x \in \mathbb{S}_N$, with associated optimal (and hence, feasible) state and control sequences $\{x_k^{\text{OPT}}\} \triangleq \{x_0^{\text{OPT}}, x_1^{\text{OPT}}, \dots, x_N^{\text{OPT}}\}$ and $\{u_k^{\text{OPT}}\} \triangleq \{u_0^{\text{OPT}}, u_1^{\text{OPT}}, \dots, u_{N-1}^{\text{OPT}}\}$, respectively, solution of $\mathcal{P}_N(x)$ in (5.49)–(5.54). Similarly, let $\check{x} \in \mathbb{S}_N$, with associated optimal (and hence, feasible) sequences $\{\check{x}_k^{\text{OPT}}\} \triangleq \{\check{x}_0^{\text{OPT}}, \check{x}_1^{\text{OPT}}, \dots, \check{x}_N^{\text{OPT}}\}$ and $\{\check{u}_k^{\text{OPT}}\} \triangleq \{\check{u}_0^{\text{OPT}}, \check{u}_1^{\text{OPT}}, \dots, \check{u}_{N-1}^{\text{OPT}}\}$.

Let

$$V_N(\{x_k\},\{u_k\}) \triangleq \frac{1}{2}x_N^{\mathrm{T}}Px_N + \frac{1}{2}\sum_{k=0}^{N-1}(x_k^{\mathrm{T}}Qx_k + u_k^{\mathrm{T}}Ru_k).$$

Then, $V_N^{\text{OPT}}(x) = V_N(\{x_k^{\text{OPT}}\},\{u_k^{\text{OPT}}\})$ and $V_N^{\text{OPT}}(\check{x}) = V_N(\{\check{x}_k^{\text{OPT}}\},\{\check{u}_k^{\text{OPT}}\})$.

Now consider $x^\alpha \triangleq \alpha x + (1-\alpha)\check{x}$, $\alpha \in [0,1]$. Similarly to the proof of Lemma 5.6.1, we can show that the sequences (5.56) and (5.55), with $u_k^\alpha \triangleq \alpha u_k^{\text{OPT}} + (1-\alpha)\check{u}_k^{\text{OPT}}$, $k = 0,\dots,N-1$, and $x_k^\alpha \triangleq \alpha x_k^{\text{OPT}} + (1-\alpha)\check{x}_k^{\text{OPT}}$, $k = 0,\dots,N$, are feasible. Hence, by optimality, we have that

$$V_N^{\text{OPT}}(x^\alpha) \leq V_N(\{x_k^\alpha\},\{u_k^\alpha\}). \tag{5.57}$$

Also, by convexity of the quadratic functions $F(x) = \frac{1}{2}x^{\mathrm{T}}Px$ and $L(x,u) = \frac{1}{2}(x^{\mathrm{T}}Qx + u^{\mathrm{T}}Ru)$ we have

$$\begin{aligned}
V_N(\{x_k^\alpha\},\{u_k^\alpha\}) &= F(x_N^\alpha) + \sum_{k=0}^{N-1} L(x_k^\alpha, u_k^\alpha)\\
&\leq \alpha F(x_N^{\text{OPT}}) + (1-\alpha)F(\check{x}_N^{\text{OPT}})\\
&\quad + \sum_{k=0}^{N-1}\Big[\alpha L(x_k^{\text{OPT}}, u_k^{\text{OPT}}) + (1-\alpha)L(\check{x}_k^{\text{OPT}}, \check{u}_k^{\text{OPT}})\Big]\\
&= \alpha V_N(\{x_k^{\text{OPT}}\},\{u_k^{\text{OPT}}\}) + (1-\alpha)V_N(\{\check{x}_k^{\text{OPT}}\},\{\check{u}_k^{\text{OPT}}\})\\
&= \alpha V_N^{\text{OPT}}(x) + (1-\alpha)V_N^{\text{OPT}}(\check{x}).
\end{aligned} \tag{5.58}$$

Combining the inequalities in (5.57) and (5.58), it follows that $V_N^{\text{OPT}}(\,\cdot\,)$ is convex, and the result is then proved. □

Lemmas 5.6.1 and 5.6.2 show that $V_N^{\text{OPT}}(\,\cdot\,)$ is a convex function defined on the convex set $\mathbb{S}_N$. Since we have assumed that the sets $\mathbb{U}$ and $\mathbb{X}_f$ contain the origin of their respective spaces, then $0 \in \mathbb{S}_N$ and hence $\mathbb{S}_N$ is nonempty. From Theorem 2.3.8 in Chapter 2, we conclude that $V_N^{\text{OPT}}(\,\cdot\,)$ is *continuous* on int $\mathbb{S}_N$. This fact will be used below in the proof of asymptotic (exponential) stability.

We present in the following sections three instances of the application of the stability theory for receding horizon control developed in Sections 4.4 and 4.5 of Chapter 4.

5.6.1 Open Loop Stable System with Input Constraints

Assume that the matrix A in (5.50) is Hurwitz, that is, all its eigenvalues have moduli smaller than one. Suppose that there are no state constraints (that is, $\mathbb{X} = \mathbb{X}_f = \mathbb{R}^n$ in (5.53) and (5.54)) and, as a consequence, $\mathbb{S}_N = \mathbb{R}^n$. Then, to apply Theorem 4.4.2 of Chapter 4, we simply choose the terminal state weighting as

$$F(x) = \frac{1}{2}x^{\mathrm{T}}Px, \tag{5.59}$$

where P satisfies the discrete Lyapunov equation

$$P = A^{\mathrm{T}}PA + Q. \tag{5.60}$$

A feasible terminal control is

$$\mathcal{K}_f(x) = 0 \ \text{ for all } x \in \mathbb{X}_f = \mathbb{R}^n. \tag{5.61}$$

Note that, by assumption, the system is open loop stable, hence $F(x)$ is the infinite horizon objective function beginning in state x and using the terminal control (5.61).

Recall that, as discussed before, conditions **B1** and **B5** of Theorem 4.4.2 are satisfied from the assumptions on problem $\mathcal{P}_N(x)$ in (5.49)–(5.54). Clearly conditions **B3** and **B4** hold with the above choices for the terminal triple. Direct calculation yields that $F(x) = x^{\mathrm{T}}Px$ satisfies

$$\begin{aligned} F(f(x,\mathcal{K}_f(x))) - F(x) &= \frac{1}{2}(Ax + B\mathcal{K}_f(x))^{\mathrm{T}}P(Ax + B\mathcal{K}_f(x)) - \frac{1}{2}x^{\mathrm{T}}Px \\ &= \frac{1}{2}x^{\mathrm{T}}(A^{\mathrm{T}}PA - P)x \\ &= -\frac{1}{2}x^{\mathrm{T}}Qx \\ &= -L(x,\mathcal{K}_f(x)) \end{aligned}$$

so that condition **B2** is also satisfied. Thus far, we have verified conditions **B1**–**B5** of Theorem 4.4.2, which establishes global attractivity of the origin. To prove exponential stability, we further need to show that the conditions in part (iv) of the theorem are also fulfilled. Note that $F(x) \leq \lambda_{\max}(P)\|x\|^2$. Also, as shown above, $V_N^{\mathrm{OPT}}(\cdot)$ is continuous. Hence, exponential stability holds in any arbitrarily large compact set of the state space.

5.6.2 General Linear System with Input Constraints

This case is a slight generalisation of the result in Section 5.6.1. Assume that the system has no eigenvalue with modulus equal to one. Suppose that $\mathbb{X} = \mathbb{R}^n$ in (5.53). We factor the system into stable and unstable subsystems as follows:

$$x_{k+1}^s = A_s x_k^s + B_s u_k,$$
$$x_{k+1}^u = A_u x_k^u + B_u u_k,$$

where the eigenvalues of A_s have moduli less than one, and the eigenvalues of A_u have moduli greater than one. Next, we choose $Q > 0$ in (5.49) of the form

$$Q = \begin{bmatrix} Q_s & 0 \\ 0 & Q_u \end{bmatrix}$$

and use as terminal state weighting

$$F(x) = \frac{1}{2}(x^s)^{\mathrm{T}} P_s x^s,$$

where P_s satisfies the discrete Lyapunov equation

$$P_s = A_s^{\mathrm{T}} P_s A_s + Q_s.$$

Finally, we choose $\mathcal{K}_f(x) = 0$ and $\mathbb{X}_f$ in (5.54) as

$$\mathbb{X}_f = \left\{ x = \begin{bmatrix} x^s \\ x^u \end{bmatrix} \in \mathbb{R}^n : x^u = 0 \right\}.$$

It can be easily verified, as done in Section 5.6.1, that the conditions **B1**–**B5** of Theorem 4.4.2 are satisfied with the above choices. Hence, if[1] $0 \in \operatorname{int} \mathbb{S}_N$, asymptotic stability of the origin follows, as proved in part (iii) of Theorem 4.4.2.

Note that the condition $x_N^u = 0$ is not very restrictive because the system $x_{k+1}^u = A_u x_k^u + B_u u_k$ is bounded input-bounded output stable *in reverse time*. Hence the set of initial states x_0^u that are taken by feasible control sequences into *any* terminal set is largely determined by the constraints on the input rather than the values of x_N^u; that is,

$$x_0^u = A_u^{-N} x_N^u - \sum_{k=0}^{N-1} A_u^{-k-1} B_u u_k,$$

and

$$A_u^{-N} \overset{\exp}{\to} 0 \quad \text{as } N \to \infty.$$

(See Sections 11.2 and 11.3 in Chapter 11 for further discussion on solving for unstable modes in reverse time.)

[1] This is the case if $0 \in \operatorname{int} \mathbb{U}$ and N is greater than or equal to the dimension of x^u, since the system is assumed stabilisable.

5.6.3 General Linear Systems with State and Input Constraints

In this case, $F(x)$ in (5.49) is often chosen to be the value function of the infinite horizon, *unconstrained* optimal control problem for the same system (see Scokaert and Rawlings 1998, Sznaier and Damborg 1987). This problem, defined as in (4.32)-(4.33) in Chapter 4, but with no constraints ($\mathbb{U} = \mathbb{R}^m$, $\mathbb{X} = \mathbb{R}^n$), is a standard linear quadratic regulator problem whose value function is $x^{\mathrm{T}}Px$, where P is the positive definite solution of the algebraic Riccati equation

$$P = A^{\mathrm{T}}PA + Q - K^{\mathrm{T}}\bar{R}K,$$

where

$$K \triangleq \bar{R}^{-1}B^{\mathrm{T}}PA, \qquad \bar{R} \triangleq R + B^{\mathrm{T}}PB. \tag{5.62}$$

The terminal state weighting used in this case is then

$$F(x) = \frac{1}{2}x^{\mathrm{T}}Px.$$

The local controller $\mathcal{K}_f(x)$ is chosen to be the optimal linear controller $\mathcal{K}_f(x) = -Kx$, where K is given by (5.62).

The terminal set $\mathbb{X}_f$ is usually taken to be the *maximal output admissible set* $\mathcal{O}_\infty$ (Gilbert and Tan 1991) for the closed loop system using the local controller $\mathcal{K}_f(x)$, defined as

$$\mathcal{O}_\infty \triangleq \{x : K(A-BK)^k x \in \mathbb{U} \text{ and } (A-BK)^k x \in \mathbb{X} \text{ for } k = 0, 1, \ldots\}. \tag{5.63}$$

$\mathcal{O}_\infty$ is the maximal positively invariant set for the system $x_{k+1} = (A-BK)x_k$ (see Definition 4.4.2 in Chapter 4) in which constraints are satisfied.

With the above choice for the terminal triple $(\mathbb{X}_f, \mathcal{K}_f, F)$, conditions **B1**–**B5** of Theorem 4.4.2 are readily established, similarly to Section 5.6.1. This proves attractivity of the origin in $\mathbb{S}_N$. To prove exponential stability, we further need to show that the conditions in part (iv) of the theorem are also fulfilled. Note that $F(x) \leq \lambda_{\max}(P)\|x\|^2$. Also, as shown above, $V_N^{\mathrm{OPT}}(\cdot)$ is continuous on int $\mathbb{S}_N$. Hence, exponential stability holds in any arbitrarily large compact subset contained in the interior of $\mathbb{S}_N$.

An interesting consequence of this choice for the terminal triple is that $V_\infty^{\mathrm{OPT}}(x) = F(x)$ for all x in $\mathbb{X}_f$ and that $V_N^{\mathrm{OPT}}(x) = V_\infty^{\mathrm{OPT}}(x)$ for all $x \in \mathbb{S}_N$. Actually, the horizon N can be chosen large enough for the predicted terminal state x_N^{OPT} (corresponding to the Nth step of the optimal state sequence for initial state x) to belong to $\mathbb{X}_f$ (see Section 5.8 for references to methods to compute lower bounds on such N). If N is so chosen, the terminal constraint may be omitted from the optimisation problem $\mathcal{P}_N(x)$.

5.7 Stability with Observers

A final question raised by the use of observers and the CE principle in RHC is whether or not closed loop stability is retained when the true states are replaced by state estimates in the control law. We will not explicitly address this issue but, instead, refer the reader, in Section 5.8 below, to recent literature dealing with this topic. Again, from a practical perspective, it seems fair to anticipate that, provided the state estimates are reasonably accurate, then stability should not be compromised; recall that we have established, in Sections 5.6.1 and 5.6.3, that exponential stability of the origin holds (under mild conditions) in the case where the states are known.

5.8 Further Reading

For complete list of references cited, see References section at the end of book.

General

The following books give detailed description of receding horizon control in the linear constrained case: Camacho and Bordons (1999), Maciejowski (2002), Borrelli (2003), Rossiter (2003). See also the early paper Muske and Rawlings (1993), as well as the survey paper Bemporad and Morari (1999).

Section 5.6

A method to compute a lower bound on the optimisation horizon N such that the predicted terminal state x_N^{OPT} in the fixed horizon optimal control problem (5.49)–(5.54) belongs to the terminal set $\mathbb{X}_f$ for all initial conditions in a given compact set is presented in Bemporad, Morari, Dua and Pistikopoulos (2002); this method, in turn, uses an algorithm proposed in Chmielewski and Manousiouthakis (1996).

There are various embellishments of the basic idea described in Section 5.6.3. For example, a new terminal triple has been provided for receding horizon control of input constrained linear systems in De Doná, Seron, Goodwin and Mayne (2002). The new triple is an improvement over those previously used in that the terminal constraint set $\mathbb{X}_f$, which we define below, is strictly larger than $\mathcal{O}_\infty$, thus facilitating the solution of the fixed horizon optimal control problem. The improved terminal conditions employ the results of Section 7.3 in Chapter 7 that show that the nonlinear controller

$$\mathcal{K}_f(x) = -\text{sat}(Kx) \tag{5.64}$$

is optimal in a region $\bar{\mathbb{Z}}$, which includes the maximal output admissible set $\mathcal{O}_\infty$. The terminal constraint set $\mathbb{X}_f$ is then selected as the maximal positively invariant set for the system $x_{k+1} = Ax_k - B\text{sat}(Kx)$. We refer the reader to the literature to follow up this and related ideas.

Stability of RHC has been established for neutrally stable systems (that is, systems having nonrepeated roots on the unit circle) using a nonquadratic terminal weighting (see Jadbabaie, Persis and Yoon 2002, Yoon, Kim, Jadbabaie and Persis 2003).

Section 5.7

The stability of the CE implementation of RHC has been addressed for constrained linear systems in, for example, Zheng and Morari (1995), where global asymptotic stability is shown for open loop stable systems, and in Muske, Meadows and Rawlings (1994), where a local stability result is given for general linear systems. Local results for nonlinear systems are reported in, for example, Scokaert et al. (1997), and Magni, De Nicolao and Scattolini (2001). A stability result for nonlinear systems using a moving horizon observer is given in Michalska and Mayne (1995).

See also Findeisen, Imsland, Allgöwer and Foss (2003) for a recent survey and new results on output feedback nonlinear RHC.