4

Receding Horizon Optimal Control with Constraints

4.1 Overview

The goal of this chapter is to introduce the principle of receding horizon optimal control. The idea is to start with a fixed optimisation horizon, of length N say, using the current state of the plant as the initial state. We then optimise the objective function over this fixed interval *accounting for constraints*, obtain an optimal sequence of N control moves, and apply only the first control move to the plant. Time then advances one step and the same N-step optimisation problem is considered using the new state of the plant as the initial state. Thus one continuously revises the *current control action* based on the *current state* and accounting for the constraints over an optimisation horizon of length N. This chapter will expand on this intuitively reasonable idea.

4.2 The Receding Horizon Optimisation Principle

Fixed horizon optimisation leads to a control sequence $\{u_i, \dots, u_{i+N-1}\}$, which begins at the current time i and ends at some future time $i + N - 1$. This fixed horizon solution suffers from two potential drawbacks:

(i) Something unexpected may happen to the system at some time over the future interval $[i, i + N - 1]$ that was not predicted by (or included in) the model. This would render the fixed control choices $\{u_i, \dots, u_{i+N-1}\}$ obsolete.
(ii) As one approaches the final time $i+N-1$, the control law typically "gives up trying" since there is too little time to go to achieve anything useful in terms of objective function reduction. Of course, there do exist problems where time does indeed "run out" because the problem is simply such that no further time is available. This is typical of so-called, batch control problems. However, in other cases, the use of a fixed optimisation horizon

is principally dictated by computational needs rather than the absolute requirement that everything must be "wrapped up" at some fixed future time $i + N - 1$.

The above two problems are addressed by the idea of *receding horizon optimisation*. As foreshadowed in Section 4.1, this idea can be summarised as follows:

(i) At time i and for the current state x_i, solve an optimal control problem over a fixed future interval, say $[i, i + N - 1]$, taking into account the *current* and *future* constraints.
(ii) Apply only the first step in the resulting optimal control sequence.
(iii) Measure the state reached at time $i + 1$.
(iv) Repeat the fixed horizon optimisation at time $i+1$ over the future interval $[i + 1, i + N]$, starting from the (now) current state x_{i+1}.

Of course, in the absence of disturbances, the state measured at step (iii) will be the same as that predicted by the model. Nonetheless, it seems prudent to use the *measured state* rather than the predicted state just to be sure. The above description assumes that the state is indeed measured at time $i + 1$. In practice, the available measurements would probably cover only a subset of the full state vector. In this case, it seems reasonable that one should use some form of observer to estimate x_{i+1} based on the available data. More will be said about the use of observers in Section 5.5 of Chapter 5, and on the general topic of *output feedback* in Chapter 12. For the moment, we will assume that the full state vector is indeed measured and we will ignore the impact of disturbances.

If the model and objective function are time invariant, then it is clear that the same input u_i will result whenever the state takes the same value. That is, the receding horizon optimisation strategy is really an "alibi" for generating a particular time-invariant feedback control law. In particular, we can set $i = 0$ in the formulation of the open loop control problem without loss of generality. Then at the current time, and for the current state x, we solve:

$$\mathcal{P}_N(x): \qquad V_N^{\text{OPT}}(x) \triangleq \min V_N(\{x_k\}, \{u_k\}), \tag{4.1}$$

subject to:

$$x_{k+1} = f(x_k, u_k) \quad \text{for } k = 0, \ldots, N-1, \tag{4.2}$$

$$x_0 = x, \tag{4.3}$$

$$u_k \in \mathbb{U} \quad \text{for } k = 0, \ldots, N-1, \tag{4.4}$$

$$x_k \in \mathbb{X} \quad \text{for } k = 0, \ldots, N, \tag{4.5}$$

$$x_N \in \mathbb{X}_f \subset \mathbb{X}, \tag{4.6}$$

where

$$V_N(\{x_k\},\{u_k\}) \triangleq F(x_N) + \sum_{k=0}^{N-1} L(x_k,u_k), \tag{4.7}$$

and where $\{x_k\}$, $x_k \in \mathbb{R}^n$, $\{u_k\}$, $u_k \in \mathbb{R}^m$, denote the state and control sequences $\{x_0,\dots,x_N\}$ and $\{u_0,\dots,u_{N-1}\}$, respectively, and $\mathbb{U} \subset \mathbb{R}^m$, $\mathbb{X} \subset \mathbb{R}^n$, and $\mathbb{X}_f \subset \mathbb{R}^n$ are constraint sets. All sequences $\{u_0,\dots,u_{N-1}\}$ and $\{x_0,\dots,x_N\}$ satisfying the constraints (4.2)–(4.6) are called *feasible* sequences. A pair of feasible sequences $\{u_0,\dots,u_{N-1}\}$ and $\{x_0,\dots,x_N\}$ constitute a *feasible solution* of (4.1)–(4.7). The functions F and L in the objective function (4.7) are the *terminal state weighting* and the *per-stage weighting*, respectively.

In the sequel we make the following assumptions:

- f, F and L are continuous functions of their arguments;
- $\mathbb{U} \subset \mathbb{R}^m$ is a compact set, $\mathbb{X} \subset \mathbb{R}^n$ and $\mathbb{X}_f \subset \mathbb{R}^n$ are closed sets;
- there exists a feasible solution to the optimisation problem (4.1)–(4.7).

Because N is finite, these assumptions are sufficient to ensure the existence of a minimum by Weierstrass' theorem (see Theorem 2.2.2 of Chapter 2). Typical choices for the weighting functions F and L are quadratic functions of the form $F(x) = x^{\mathrm{T}}Px$ and $L(x,u) = x^{\mathrm{T}}Qx + u^{\mathrm{T}}Ru$, where $P = P^{\mathrm{T}} \geq 0$, $Q = Q^{\mathrm{T}} \geq 0$ and $R = R^{\mathrm{T}} > 0$. More generally, one could use functions of the form $F(x) = \|Px\|_p$ and $L(x,u) = \|Qx\|_p + \|Ru\|_p$, where $\|y\|_p$ with $p = 1,2,\dots,\infty$, is the p-norm of the vector y.

Denote the minimising control sequence, which is a function of the current state x_i, by

$$\mathscr{U}_{x_i}^{\mathrm{OPT}} \triangleq \{u_0^{\mathrm{OPT}}, u_1^{\mathrm{OPT}}, \dots, u_{N-1}^{\mathrm{OPT}}\}\,; \tag{4.8}$$

then the control applied to the plant at time i is the first element of this sequence, that is,

$$u_i = u_0^{\mathrm{OPT}}\,. \tag{4.9}$$

Time is then stepped forward one instant, and the above procedure is repeated for another N-step-ahead optimisation horizon. The first element of the new N-step input sequence is then applied. The above procedure is repeated endlessly. The idea is illustrated in Figure 4.1 for a horizon $N = 5$. In this figure, each plot shows the minimising control sequence $\mathscr{U}_{x_i}^{\mathrm{OPT}}$ given in (4.8), computed at time $i = 0,1,2$. Note that only the shaded inputs are actually applied to the system. We can see that we are continually looking ahead to judge the impact of current and future decisions on the future response before we "lock in" the current input by applying it to the plant.

The above receding horizon procedure *implicitly* defines a time-invariant control policy $\mathcal{K}_N : \mathbb{X} \to \mathbb{U}$ of the form

$$\mathcal{K}_N(x) = u_0^{\mathrm{OPT}}\,. \tag{4.10}$$

Note that the strict definition of the function $\mathcal{K}_N(\cdot)$ requires the minimiser to be unique. Most of the problems treated in this book are convex and hence

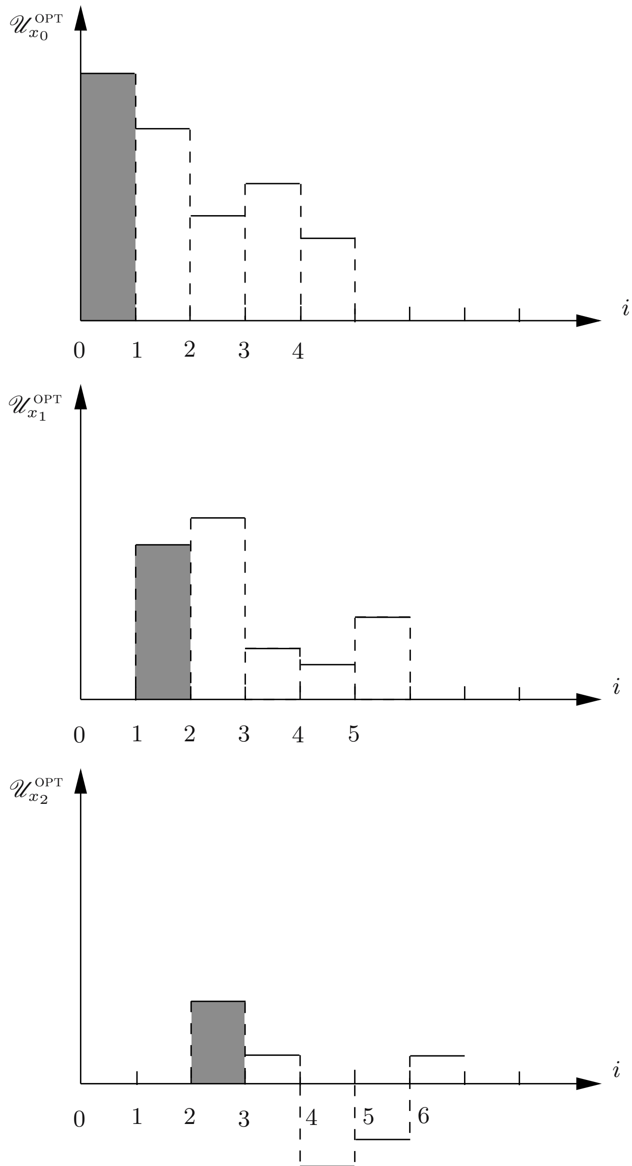

Figure 4.1. Receding horizon optimisation principle. The shaded rectangles indicate the inputs actually applied to the plant.

satisfy this condition. One exception is the "finite alphabet" optimisation case of Chapter 13, where the minimiser is not necessarily unique. However, in such cases, one can adopt a rule to select one of the minimisers (see, for example, the discussion following Definition 13.3.1 in Chapter 13).

It is common in receding horizon control applications to compute *numerically*, at time i, and for the current state $x_i = x$, the optimal control move $\mathcal{K}_N(x)$. In this case, we call it an *implicit receding horizon optimal policy.* In some cases, we can explicitly evaluate the *control law* $\mathcal{K}_N(\cdot)$. In this case, we say that we have an *explicit receding horizon optimal policy.* We will expand on the above skeleton description of receding horizon optimal constrained control as the book evolves. For example, we will treat linear constrained problems in subsequent chapters. When the system model is linear, the objective function quadratic and the constraint sets polyhedral, the fixed horizon optimal control problem $\mathcal{P}_N(\cdot)$ is a quadratic programme of the type discussed in Section 2.5.6 of Chapter 2. In Chapters 5 to 8 we will study the solution of this quadratic program in some detail. If, on the other hand, the system model is nonlinear, $\mathcal{P}_N(\cdot)$ is, in the general case, nonconvex, so that only local solutions are available.

The remainder of the present chapter is devoted to the analysis of the stability properties of receding horizon optimal control. However, before we embark on these issues, we pause to review concepts from stability theory. As for the results on optimisation presented in Chapter 2, the results on stability presented below in Section 4.3 find widespread application beyond constrained control and estimation.

4.3 Background on Stability Theory

4.3.1 Notions of Stability

We will utilise the following notions of stability:

Definition 4.3.1 (Stability Properties) *Let $\mathbb{S}$ be a set in $\mathbb{R}^n$ that contains the origin. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be such that $f(\mathbb{S}) \subset \mathbb{S}$. Suppose that the system*

$$x_{i+1} = f(x_i), \tag{4.11}$$

with $x_i \in \mathbb{R}^n$, has an equilibrium point *at the origin $x = 0$, that is, $f(0) = 0$. Let $x_0 \in \mathbb{S}$ and let $\{x_i\} \subset \mathbb{S}$, $i \geq 0$, be the resulting sequence satisfying* (4.11).

We say that the equilibrium point is:

(i) (Lyapunov) stable in $\mathbb{S}$: *if for any $\varepsilon > 0$, there exists $\delta > 0$ such that*

$$x_0 \in \mathbb{S} \text{ and } \|x_0\| < \delta \implies \|x_i\| < \varepsilon \text{ for all } i \geq 0\,; \tag{4.12}$$

(ii) attractive in $\mathbb{S}$: *if there exists $\eta > 0$ such that*

$$x_0 \in \mathbb{S} \text{ and } \|x_0\| < \eta \implies \lim_{i\to\infty} x_i = 0\,;$$

(iii) globally attractive in $\mathbb{S}$: *if*

$$x_0 \in \mathbb{S} \implies \lim_{i\to\infty} x_i = 0\,;$$

(iv) asymptotically stable in $\mathbb{S}$: *if it is both stable in* $\mathbb{S}$ *and attractive in* $\mathbb{S}$*;*
(v) exponentially stable in $\mathbb{S}$: *if there exist constants* $\theta > 0$ *and* $\rho \in (0,1)$ *such that*

$$x_0 \in \mathbb{S} \implies \|x_i\| \leq \theta \|x_0\| \rho^i \text{ for all } i \geq 0\,; \tag{4.13}$$

In cases (iii), and (v) above, we say that the set $\mathbb{S}$ *is contained in the* region of attraction[1] *of the equilibrium point.* ○

4.3.2 Tests for Stability

Testing for stability properties is facilitated if one can find a function $V : \mathbb{S} \to [0,\infty)$ (called a Lyapunov function) satisfying certain conditions. The following results use this fact.

Theorem 4.3.1 (Attractivity in $\mathbb{S}$) *Let* $\mathbb{S}$ *be a nonempty set in* $\mathbb{R}^n$*. Let* $f : \mathbb{R}^n \to \mathbb{R}^n$ *be such that* $f(0) = 0$ *and* $f(\mathbb{S}) \subset \mathbb{S}$*. Assume that there exists a (Lyapunov) function* $V : \mathbb{S} \to [0,\infty)$ *satisfying the following properties:*[2]

(i) $V(\cdot)$ *decreases along the trajectories of* (4.11) *that start in* $\mathbb{S}$ *in the following way: there exists a continuous function* $\gamma : [0,\infty) \to [0,\infty)$*,* $\gamma(t) > 0$ *for all* $t > 0$*, such that*

$$V(f(x)) - V(x) \leq -\gamma(\|x\|) \quad \text{for all } x \in \mathbb{S}\,. \tag{4.14}$$

(ii) for every unbounded sequence $\{y_i\} \subset \mathbb{S}$ *there is some* j *such that* [3]

$$\limsup_{i\to\infty} V(y_i) > V(y_j)\,.$$

Then:

(a) $0 \in \operatorname{cl}\mathbb{S}$*, and*
(b) For all $x_0 \in \mathbb{S}$*, the resulting sequence* $\{x_i\}$*,* $i \geq 0$*, satisfying* (4.11) *is such that* $\lim_{i\to\infty} x_i = 0$*, that is, if* $0 \in \mathbb{S}$*, the origin is globally attractive in* $\mathbb{S}$*.*

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[1] The region of attraction of an equilibrium point of (4.11) is the set of all initial states $x_0 \in \mathbb{R}^n$ that originate state trajectories $\{x_i\}$, $i \geq 0$, solution of (4.11), which converge to the equilibrium point as $i \to \infty$.
[2] Property (ii) can be omitted if $\mathbb{S}$ is bounded.
[3] We recall that, if $\{a_i\}$ is a sequence in $[-\infty,\infty]$, and $b_k = \sup\{a_k, a_{k+1}, a_{k+2}, \dots\}$, $k = 1, 2, 3, \dots$, then the *upper limit* of $\{a_i\}$, denoted by $\beta = \limsup_{i\to\infty} a_i$, is defined as $\beta \triangleq \inf\{b_1, b_2, b_3, \dots\}$.

Proof. Let $x_0 \in \mathbb{S}$ and let $\{x_i\}$, $i \geq 0$, be the resulting sequence satisfying (4.11). The associated sequence of Lyapunov function values $\{V(x_i)\} \subset [0, \infty)$ is nonincreasing, since, from (4.14),

$$V(x_{i+1}) = V(f(x_i)) \leq V(x_i) - \gamma(\|x_i\|) \leq V(x_i)\,.$$

Hence, $c = \lim_{i\to\infty} V(x_i) \geq 0$, exists.

The sequence $\{x_i\}$ is bounded; otherwise, from property (ii) above, there would exist j such that $c > V(x_j)$, but $c \leq V(x_i)$ for all i. Thus, there exists $R > 0$ such that $\|x_i\| \leq R$ for all $i \geq 0$.

Now assume that there exists μ, $0 < \mu < R$, such that $\|x_i\| \geq \mu$ for infinitely many i. Let

$$\alpha = \min_{\mu \leq t \leq R} \gamma(t)\,.$$

Note that α exists by Weierstrass' theorem (Theorem 2.2.2 in Chapter 2) and that $\alpha > 0$ since $\gamma(t) > 0$ for all $t > 0$. From

$$V(x_k) = V(x_0) + \sum_{j=0}^{k-1} V(x_{j+1}) - V(x_j),$$

it follows that

$$\begin{aligned} c &= V(x_0) + \sum_{j=0}^{\infty} V(x_{j+1}) - V(x_j) \\ &\leq V(x_0) - \sum_{j=0}^{\infty} \gamma(\|x_j\|) \\ &= -\infty, \end{aligned}$$

since $\gamma(\|x_j\|) \geq \alpha > 0$ for infinitely many j and $\gamma(t) \geq 0$ for all $t \geq 0$. The above is a contradiction since $c \geq 0$. It follows that x_i converges to 0 as i tends to infinity, showing that $0 \in \mathrm{cl}\,\mathbb{S}$ and that, if $0 \in \mathbb{S}$, the origin is attractive in $\mathbb{S}$ for (4.11). The theorem is then proved. □

Remark 4.3.1. If the Lyapunov function $V : \mathbb{S} \to [0, \infty)$ is continuous, and $f : \mathbb{S} \to \mathbb{S}$ in (4.11) is continuous, $\mathbb{S}$ is closed, and $V(0) = 0$, then inequality (4.14) in Theorem 4.3.1 can be replaced by

$$V(f(x)) - V(x) < 0 \quad \text{for all } x \in \mathbb{S},\ x \neq 0\,.$$

◦

Theorem 4.3.2 (Stability) *Let $\mathbb{S}$ be a set in $\mathbb{R}^n$ that contains an open neighbourhood of the origin $N_\eta(0) \triangleq \{x \in \mathbb{R}^n : \|x\| < \eta\}$. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be such that $f(0) = 0$ and $f(\mathbb{S}) \subset \mathbb{S}$. Assume that there exists a (Lyapunov) function $V : \mathbb{S} \to [0, \infty)$, $V(0) = 0$, satisfying the following properties:*

(i) $V(\cdot)$ *is continuous on* $N_\eta(0)$*;*
(ii) if $\{y_k\} \subset \mathbb{S}$ *is such that* $\lim_{k\to\infty} V(y_k) = 0$ *then* $\lim_{k\to\infty} y_k = 0$*;*
(iii) $V(f(x)) - V(x) \leq 0$ *for all* $x \in N_\eta(0)$*.*

Then the origin is a stable equilibrium point for (4.11) *in* $\mathbb{S}$*.*

Proof. Let $\varepsilon \in (0, \eta)$ and $N_\varepsilon(0) \triangleq \{x \in \mathbb{R}^n : \|x\| < \varepsilon\}$. We first show that there exists $\beta > 0$ such that $V^{-1}[0, \beta] \triangleq \{x \in \mathbb{S} : V(x) \in [0, \beta]\} \subset N_\varepsilon(0)$. Suppose no such β exists. Then for every $k = 1, 2, \ldots$, there exists $y_k \in V^{-1}[0, \frac{1}{k}]$ such that $\|y_k\| > \varepsilon$. But, from property (ii), we have that $\lim_{k\to\infty} y_k = 0$, which is a contradiction. Thus,

$$V^{-1}[0, \beta] \subset N_\varepsilon(0)\,. \tag{4.15}$$

Since $V(\cdot)$ is continuous on $N_\eta(0)$ and $V(0) = 0$, there exists $\delta \in (0, \varepsilon)$ such that $\|x\| < \delta \implies V(x) < \beta$. Then, combining with (4.15), we have

$$\|x\| < \delta \implies V(x) < \beta \implies x \in V^{-1}[0, \beta] \implies \|x\| < \varepsilon\,.$$

Now let $\|x_0\| < \delta$. We show by induction that $x_i \in V^{-1}[0, \beta]$ for all $i \geq 0$. It clearly holds for $i = 0$. Suppose $x_i \in V^{-1}[0, \beta]$. Note that, from (4.15), $\|x_i\| < \varepsilon$, so that $x_i \in N_\eta(0)$. Then, using property (iii), we have

$$V(x_{i+1}) = V(f(x_i)) \leq V(x_i) \leq \beta \implies x_{i+1} \in V^{-1}[0, \beta]\,.$$

Hence, $\|x_0\| < \delta \implies x_0 \in V^{-1}[0, \beta] \implies x_i \in V^{-1}[0, \beta]$ for all $i \geq 0 \implies \|x_i\| < \varepsilon$ for all $i \geq 0$. We have thus shown that given $\varepsilon \in (0, \eta)$ there exists $\delta > 0$ such that (4.12) holds. The result then follows. □

The following theorem gives a sufficient condition for exponential stability.

Theorem 4.3.3 (Exponential Stability) *Let* $\mathbb{S}$ *be a set in* $\mathbb{R}^n$ *containing a nonzero element. Let* $f : \mathbb{R}^n \to \mathbb{R}^n$ *be such that* $f(0) = 0$ *and* $f(\mathbb{S}) \subset \mathbb{S}$*. Assume that there exists a (Lyapunov) function* $V : \mathbb{S} \to \mathbb{R}$*, and positive constants* a*,* b*,* c *and* σ *satisfying*

(i) $a\|x\|^\sigma \leq V(x) \leq b\|x\|^\sigma$ *for all* $x \in \mathbb{S}$*,*
(ii) $V(f(x)) - V(x) \leq -c\|x\|^\sigma$ *for all* $x \in \mathbb{S}$*.*

Then, if $0 \in \mathbb{S}$*, the origin is exponentially stable in* $\mathbb{S}$ *for the system* (4.11)*.*

Proof. Let $f^0(x) \triangleq x$, $f^1(x) \triangleq f(x)$, $\ldots$, $f^{i+1}(x) \triangleq f^i(f(x))$. We first show that

$$V(f^i(x)) \leq \left(1 - \frac{c}{b}\right)^i V(x), \quad \text{for all } x \in \mathbb{S}, \tag{4.16}$$

for all $i \geq 0$. Clearly, (4.16) holds for $i = 0$. Moreover, from the assumptions on $V(x)$, we have

$$V(f(x)) - V(x) \leq -c\|x\|^\sigma \leq -\frac{c}{b}V(x)\,.$$

Thus,

$$V(f(x)) \leq \left(1 - \frac{c}{b}\right) V(x) \text{ for all } x \in \mathbb{S}\,.$$

Choose $0 \neq y \in \mathbb{S}$. Then $V(y) \geq a\|y\|^\sigma > 0$. Thus, $1 - \frac{c}{b} \geq 0$, and therefore $0 \leq 1 - \frac{c}{b} < 1$.

Now assume that (4.16) holds for some $i \geq 1$. Then,

$$V(f^{i+1}(x)) = V(f^i(f(x))) \leq \left(1 - \frac{c}{b}\right)^i V(f(x)) \leq \left(1 - \frac{c}{b}\right)^{i+1} V(x)\,.$$

Hence, by induction, (4.16) holds for all $i \geq 0$. Finally, for all $x \in \mathbb{S}$ and all $i \geq 0$, we have that

$$\left\|f^i(x)\right\|^\sigma \leq \frac{1}{a} V(f^i(x)) \leq \frac{1}{a}\left(1 - \frac{c}{b}\right)^i V(x) \leq \frac{b}{a}\left(1 - \frac{c}{b}\right)^i \|x\|^\sigma,$$

from which (4.13) follows with $\theta = \left(\frac{b}{a}\right)^{1/\sigma} > 0$ and $\rho = \left(1 - \frac{c}{b}\right)^{1/\sigma} \in (0, 1)$. □

4.4 Stability of Receding Horizon Optimal Control

4.4.1 Ingredients

We now return to receding horizon control as described in Section 4.2. Although the receding horizon control idea seems intuitively reasonable, it is important that one be able to establish concrete results about its associated properties. Here we examine the question of closed loop stability which is a minimal performance goal.

Unfortunately, proving/guaranteeing that an optimisation scheme (such as receding horizon optimal control) leads to a stable closed loop system is a nontrivial task. One may well ask what possible tool could be used. After all, the only thing we know is that the *fixed horizon control sequence* is optimal. Luckily, *optimality* can be turned into a notion of stability by utilising the value function (that is, the function $V_N^{\text{OPT}}(x)$ in (4.1), which is a function of the initial state x only) as a *Lyapunov function.*

However, another difficulty soon arises. Namely, the optimisation problems that we are solving are only defined over a *finite* future horizon, yet stability is a property that must hold over an *infinite* future horizon. A trick that is frequently utilised to resolve this conflict is to add an appropriate weighting on the terminal state in the finite horizon problem so as to account for the impact of events that lie beyond the end of the fixed horizon. This effectively turns the fixed horizon problem into an infinite horizon one.

Following this line of reasoning, we will define a terminal control law and an associated terminal state weighting in the objective function that captures the impact of using the terminal control law over *infinite time*. Usually, the chosen terminal control laws are relatively simple and only "feasible" in a restricted (local) region. This implies that one must be able to steer the system into this restricted terminal region over the finite time period available in the optimisation window. (More will be said about this crucial point later.) It is also important to ensure that the terminal region is invariant under the terminal control law, that is, once the state reaches the terminal set, it remains inside the set if the terminal control law is used. Thus, in summary, the ingredients typically employed to provide *sufficient* (though by no means necessary) conditions for stability are captured by the following *terminal triple*:

Ingredients for Stability: The Terminal Triple $(\mathbb{X}_f, \mathcal{K}_f, F)$

(i) a *terminal constraint set* $\mathbb{X}_f$ in the state space which is invariant under the terminal control law;
(ii) a feasible *terminal control law* $\mathcal{K}_f$ that holds in the terminal constraint set;
(iii) a *terminal state weighting* F on the finite horizon optimisation problem, which usually corresponds to the objective function value generated by the use of the terminal control law over infinite time.

We will show below how, based on these "ingredients," Lyapunov-like tests, such as those described in Section 4.3.2, can be used to establish stability of receding horizon control.

4.4.2 Stability Results for Receding Horizon Control

As mentioned above, we will employ the value function $V_N^{\text{OPT}}(x)$ of the fixed horizon optimal control problem (4.1)–(4.7) as a Lyapunov function to establish asymptotic stability of the receding horizon implementation. We will first establish stability under simplifying assumptions. A more general stability analysis will be given later; however, this will follow essentially the same lines as the simplified "prototype" proof given below.

Let us define the set $\mathbb{S}_N$ of *feasible initial states.*

Definition 4.4.1 *The set* $\mathbb{S}_N$ *of* feasible initial states *is the set of initial states* $x \in \mathbb{X}$ *for which there exist feasible state and control sequences for the fixed horizon optimal control problem* $\mathcal{P}_N(x)$ *in* (4.1)–(4.7). ◦

We also require the following definition.

Definition 4.4.2 *The set* $\mathbb{S} \subset \mathbb{R}^n$ *is said to be* positively invariant *for the system* $x_{i+1} = f(x_i, u_i)$ *under the control* $u_i = \mathcal{K}(x_i)$ *(or* positively invariant for the closed loop system $x_{i+1} = f(x_i, \mathcal{K}(x_i))$*) if* $f(x, \mathcal{K}(x)) \in \mathbb{S}$ *for all* $x \in \mathbb{S}$. ◦

We make the following assumptions on the data of problem $\mathcal{P}_N(x)$ in (4.1)–(4.7).

A1 The terminal constraint set in (4.6) is the origin, that is, $\mathbb{X}_f = \{0\}$.
A2 The control constraint set in (4.4) contains the origin, that is, $0 \in \mathbb{U}$.
A3 $L(x, u)$ in (4.7) satisfies $L(0, 0) = 0$ and $L(x, u) \geq \gamma(\|x\|)$ for all $x \in \mathbb{S}_N, u \in \mathbb{U}$, where $\gamma : [0, \infty) \to [0, \infty)$ is continuous, $\gamma(t) > 0$ for all $t > 0$, and $\lim_{t\to\infty} \gamma(t) = \infty$.
A4 There is no terminal state weighting in the objective function, that is, $F(x) \equiv 0$ in (4.7).

Under these conditions, we have the following stability result:

Theorem 4.4.1 *Consider the system*

$$x_{i+1} = f(x_i, u_i) \quad \textit{for } i \geq 0, \qquad f(0, 0) = 0, \tag{4.17}$$

controlled by the receding horizon algorithm (4.1)–(4.9) *and subject to Assumptions* **A1**–**A4** *above. Then:*

(i) The set $\mathbb{S}_N$ of feasible initial states is positively invariant for the closed loop system.
(ii) The origin is globally attractive in $\mathbb{S}_N$ for the closed loop system.
(iii) If, in addition to **A1**–**A4**, $0 \in \operatorname{int} \mathbb{S}_N$ *and the value function $V_N^{\text{OPT}}(x)$ in* (4.1) *is continuous on some neighbourhood of the origin, then the origin is asymptotically stable in $\mathbb{S}_N$ for the closed loop system.*

Proof. (i) *Positive invariance of $\mathbb{S}_N$.*

Let $x_i = x \in \mathbb{S}_N$. At step i, and for the current state $x_i = x$, the receding horizon algorithm solves the optimisation problem $\mathcal{P}_N(x)$ in (4.1)–(4.7) to obtain the optimal control and state sequences

$$\mathscr{U}_x^{\text{OPT}} \triangleq \{u_0^{\text{OPT}}, u_1^{\text{OPT}}, \ldots, u_{N-1}^{\text{OPT}}\}, \tag{4.18}$$

$$\mathscr{X}_x^{\text{OPT}} \triangleq \{x_0^{\text{OPT}}, x_1^{\text{OPT}}, \ldots, x_{N-1}^{\text{OPT}}, x_N^{\text{OPT}}\}. \tag{4.19}$$

Then the actual control applied to (4.17) at time i is the first element of (4.18), that is,

$$u_i = \mathcal{K}_N(x) = u_0^{\text{OPT}}. \tag{4.20}$$

Note that, in the optimal state sequence (4.19), we have, from Assumption **A1**, that

$$x_N^{\text{OPT}} = 0. \tag{4.21}$$

Let $x^+ \triangleq x_{i+1} = f(x, \mathcal{K}_N(x)) = f(x, u_0^{\text{OPT}})$ be the successor state. A *feasible* (but not necessarily optimal) control sequence, and corresponding feasible state sequence for the next step $i + 1$ in the receding horizon computation $\mathcal{P}_N(x^+)$ are then

$$\tilde{\mathscr{U}} = \{u_1^{\text{OPT}}, \dots, u_{N-1}^{\text{OPT}}, 0\}, \tag{4.22}$$

$$\tilde{\mathscr{X}} = \{x_1^{\text{OPT}}, \dots, x_{N-1}^{\text{OPT}}, 0, 0\}, \tag{4.23}$$

where the last two zeros in (4.23) follow from (4.21) and $f(0,0) = 0$. Thus, there exist feasible sequences (4.22) and (4.23) for the successor state $x^+ = f(x, \mathcal{K}_N(x))$ and hence $x^+ \in \mathbb{S}_N$. This shows that $\mathbb{S}_N$ is positively invariant for the closed loop system $x^+ = f(x, \mathcal{K}_N(x))$.

(ii) *Attractivity.*

Note first that, since $L(0,0) = 0$, $F(0) = 0$, $0 \in \mathbb{U}$ and $0 \in \mathbb{X}_f$, then the optimal sequences in (4.1)–(4.7) corresponding to $x = 0$ have all their elements equal to zero. Thus, $\mathcal{K}_N(0) = 0$. Since, in addition, $f(0,0) = 0$, then the origin is an equilibrium point for the closed loop system $x^+ = f(x, \mathcal{K}_N(x))$.

We will next use the value function $V_N^{\text{OPT}}(\cdot)$ in (4.1) as a Lyapunov function. We first show that $V_N^{\text{OPT}}(\cdot)$ satisfies property (i) in Theorem 4.3.1. Let $x \in \mathbb{S}_N$. The increment of the Lyapunov function, upon using the true optimal input (4.20) and moving from x to $x^+ = f(x, \mathcal{K}_N(x))$, satisfies

$$V_N^{\text{OPT}}(x^+) - V_N^{\text{OPT}}(x) = V_N(\mathscr{X}_{x^+}^{\text{OPT}}, \mathscr{U}_{x^+}^{\text{OPT}}) - V_N(\mathscr{X}_x^{\text{OPT}}, \mathscr{U}_x^{\text{OPT}}). \tag{4.24}$$

However, by optimality we know that

$$V_N(\mathscr{X}_{x^+}^{\text{OPT}}, \mathscr{U}_{x^+}^{\text{OPT}}) \leq V_N(\tilde{\mathscr{X}}, \tilde{\mathscr{U}}), \tag{4.25}$$

where $\tilde{\mathscr{U}}$ and $\tilde{\mathscr{X}}$ are the feasible sequences defined in (4.22)–(4.23). Combining (4.24) and (4.25) yields

$$V_N^{\text{OPT}}(x^+) - V_N^{\text{OPT}}(x) \leq V_N(\tilde{\mathscr{X}}, \tilde{\mathscr{U}}) - V_N(\mathscr{X}_x^{\text{OPT}}, \mathscr{U}_x^{\text{OPT}}). \tag{4.26}$$

Substituting (4.18), (4.19), (4.22) and (4.23) in the objective function expression (4.7), and using the fact that the optimal and feasible sequences share common terms, we obtain that the right hand side of (4.26) is equal to $-L(x, \mathcal{K}_N(x))$. It then follows that

$$\begin{aligned} V_N^{\text{OPT}}(x^+) - V_N^{\text{OPT}}(x) &\leq -L(x, \mathcal{K}_N(x)) \\ &\leq -\gamma(\|x\|), \end{aligned}$$

where, in the last inequality, we have used Assumption **A3**. Thus, $V_N^{\text{OPT}}(\cdot)$ satisfies property (i) in Theorem 4.3.1.

In addition, from Assumptions **A3** and **A4**, $V_N^{\text{OPT}}(\cdot)$ satisfies

$$V_N^{\text{OPT}}(x) \geq L(x, u_0^{\text{OPT}}) \geq \gamma(\|x\|) \text{ for all } x \in \mathbb{S}_N. \tag{4.27}$$

Hence, from the assumption on γ, $V_N^{\text{OPT}}(x) \to \infty$ when $\|x\| \to \infty$, and therefore $V_N^{\text{OPT}}(\cdot)$ satisfies property (ii) in Theorem 4.3.1. It then follows from Theorem 4.3.1 that the origin is globally attractive in $\mathbb{S}_N$ for the closed loop system.

(iii) *Asymptotic stability.*

To show asymptotic stability of the origin, note first that $V_N^{\text{OPT}}(0) = 0$ (since, as shown before, the optimal sequences in (4.1)–(4.7) corresponding to $x = 0$ have all their elements equal to zero). Next, note from (4.27) and the properties of γ that $V_N^{\text{OPT}}(\cdot)$ satisfies property (ii) in Theorem 4.3.2 with $\mathbb{S} = \mathbb{S}_N$. If, in addition, $0 \in \operatorname{int} \mathbb{S}_N$ and $V_N^{\text{OPT}}(\cdot)$ is continuous on some neighbourhood of the origin, then Theorem 4.3.2 shows that the origin is a stable equilibrium point for the closed loop system, and hence it is asymptotically stable in $\mathbb{S}_N$ (that is, both stable and attractive in $\mathbb{S}_N$).

□

Assumptions **A1** to **A4** were made to keep the proof of Theorem 4.4.1 simple in order to introduce the reader to the core idea of the stability proof. The assumptions can be relaxed. (For example, Assumption **A1** can be replaced by the assumption that x_N enters a terminal set in which "nice properties" hold. Similarly, Assumption **A3** can be relaxed to requiring that the system be "detectable" in the objective function.)

We next modify the assumptions given above to provide a more comprehensive result by specifying some more general terminal conditions.

Conditions for Stability:

B1 The per-stage weighting $L(x, u)$ in (4.7) satisfies $L(0, 0) = 0$ and $L(x, u) \geq \gamma(\|x\|)$ for all $x \in \mathbb{S}_N, u \in \mathbb{U}$, where $\gamma : [0, \infty) \to [0, \infty)$ is continuous, $\gamma(t) > 0$ for all $t > 0$, and $\lim_{t\to\infty} \gamma(t) = \infty$.

B2 The terminal state weighting $F(x)$ in (4.7) satisfies $F(0) = 0$, $F(x) \geq 0$ for all $x \in \mathbb{X}_f$, and the following property: there exists a terminal control law $\mathcal{K}_f : \mathbb{X}_f \to \mathbb{U}$ such that $F(f(x, \mathcal{K}_f(x))) - F(x) \leq -L(x, \mathcal{K}_f(x))$ for all $x \in \mathbb{X}_f$.

B3 The set $\mathbb{X}_f$ is positively invariant for the system (4.17) under $\mathcal{K}_f(x)$, that is, $f(x, \mathcal{K}_f(x)) \in \mathbb{X}_f$ for all $x \in \mathbb{X}_f$.

B4 The terminal control $\mathcal{K}_f(x)$ satisfies the control constraints in $\mathbb{X}_f$, that is, $\mathcal{K}_f(x) \in \mathbb{U}$ for all $x \in \mathbb{X}_f$.

B5 The sets $\mathbb{U}$ and $\mathbb{X}_f$ contain the origin of their respective spaces.

Using the above conditions, which include more general conditions on the terminal triple $(\mathbb{X}_f, \mathcal{K}_f, F)$, we obtain the following more general theorem.

Theorem 4.4.2 (Stability of Receding Horizon Control) *Consider the closed loop system formed by system* (4.17)*, controlled by the receding* *horizon algorithm* (4.1)–(4.9), *and suppose that Conditions* **B1** to **B5** *are satisfied. Then:*

(i) The set $\mathbb{S}_N$ of feasible initial states is positively invariant for the closed loop system.
(ii) The origin is globally attractive in $\mathbb{S}_N$ for the closed loop system.
(iii) If, in addition to **B1**–**B5**, $0 \in \operatorname{int} \mathbb{S}_N$ *and the value function $V_N^{\text{OPT}}(\cdot)$ in* (4.1) *is continuous on some neighbourhood of the origin, then the origin is asymptotically stable in $\mathbb{S}_N$ for the closed loop system.*
(iv) If, in addition to **B1**–**B5**, $0 \in \operatorname{int} \mathbb{X}_f$, *$\mathbb{S}_N$ is compact, $\gamma(t) \geq a t^{\sigma}$ in* **B1**, *$F(x) \leq b\|x\|^{\sigma}$ for all $x \in \mathbb{X}_f$ in* **B2**, *where $a > 0$, $b > 0$ and $\sigma > 0$ are some real constants, and the value function $V_N^{\text{OPT}}(\cdot)$ in* (4.1) *is continuous on $\mathbb{S}_N$, then the origin is exponentially stable in $\mathbb{S}_N$ for the closed loop system.*

Proof. (i) *Positive invariance of $\mathbb{S}_N$.*

We will use the optimal sequences (4.18), (4.19) for the initial state $x \in \mathbb{S}_N$, and the following feasible sequences for the successor state $x^+ = f(x, \mathcal{K}_N(x)$:

$$\tilde{\mathscr{U}} = \{u_1^{\text{OPT}}, \dots, u_{N-1}^{\text{OPT}}, \mathcal{K}_f(x_N^{\text{OPT}})\}, \tag{4.28}$$

$$\tilde{\mathscr{X}} = \{x_1^{\text{OPT}}, \dots, x_{N-1}^{\text{OPT}}, x_N^{\text{OPT}}, f(x_N^{\text{OPT}}, \mathcal{K}_f(x_N^{\text{OPT}}))\}\,. \tag{4.29}$$

Indeed, the first $N-1$ elements of (4.28) lie in $\mathbb{U}$ (see the control constraint (4.4)) since they are elements of (4.18); also, by **B4**, the last element of (4.28) lies in $\mathbb{U}$ since $x_N^{\text{OPT}} \in \mathbb{X}_f$. Finally, by **B3**, the terminal state $f(x_N^{\text{OPT}}, \mathcal{K}_f(x_N^{\text{OPT}}))$ in (4.29) also lies in $\mathbb{X}_f$. Thus, there exist feasible sequences (4.28) and (4.29) for the successor state $x^+ = f(x, \mathcal{K}_N(x))$ and hence $x^+ \in \mathbb{S}_N$. This shows the result (i) that $\mathbb{S}_N$ is positively invariant for the closed loop system $x^+ = f(x, \mathcal{K}_N(x))$.

(ii) *Attractivity.*

As in Theorem 4.4.1, we can show that the origin is an equilibrium point for the closed loop system $x^+ = f(x, \mathcal{K}_N(x))$.

We next show that the value function $V_N^{\text{OPT}}(\cdot)$ satisfies property (i) in Theorem 4.3.1. The increment of $V_N^{\text{OPT}}(\cdot)$, upon using the receding horizon optimal input (4.20) and moving from $x \in \mathbb{S}_N$ to $x^+ = f(x, \mathcal{K}_N(x))$ satisfies (4.24), and, by optimality, (4.25) also holds for the feasible sequences (4.28), (4.29). We thus have, in a fashion similar to the proof of Theorem 4.4.1,

$$\begin{aligned} V_N^{\text{OPT}}(x^+) - V_N^{\text{OPT}}(x) &\leq V_N(\tilde{\mathscr{X}}, \tilde{\mathscr{U}}) - V_N(\mathscr{X}_x^{\text{OPT}}, \mathscr{U}_x^{\text{OPT}}) \\ &= -L(x, \mathcal{K}_N(x)) + L(x_N^{\text{OPT}}, \mathcal{K}_f(x_N^{\text{OPT}})) \\ &\quad + F(f(x_N^{\text{OPT}}, \mathcal{K}_f(x_N^{\text{OPT}}))) - F(x_N^{\text{OPT}})\,. \end{aligned}$$

From **B2**, and since $x_N^{\text{OPT}} \in \mathbb{X}_f$, the sum of the last three terms on the right hand side of the above inequality is less than or equal to zero. Thus,

$$V_N^{\text{OPT}}(x^+) - V_N^{\text{OPT}}(x) \le -L(x, \mathcal{K}_N(x)) \le -\gamma(\|x\|) \text{ for all } x \in \mathbb{S}_N, \quad (4.30)$$

where, in the last inequality, we have used the bound in Condition **B1**. Thus $V_N^{\text{OPT}}(\cdot)$ satisfies property (i) of Theorem 4.3.1. In a fashion similar to the proof of Theorem 4.4.1, we can show that

$$V_N^{\text{OPT}}(x) \ge \gamma(\|x\|) \text{ for all } x \in \mathbb{S}_N, \quad (4.31)$$

and hence $V_N^{\text{OPT}}(\cdot)$ also satisfies property (ii) of Theorem 4.3.1. We then conclude using Theorem 4.3.1 that the origin is globally attractive in $\mathbb{S}_N$ for the closed loop system. The result (ii) is then proved.

(iii) *Asymptotic stability.*

As in Theorem 4.4.1, we can show that the value function $V_N^{\text{OPT}}(\cdot)$ satisfies property (ii) in Theorem 4.3.2 with $\mathbb{S} = \mathbb{S}_N$, and that $V_N^{\text{OPT}}(0) = 0$. If, in addition, the origin is in the interior of $\mathbb{S}_N$ and $V_N^{\text{OPT}}(\cdot)$ is continuous on a neighbourhood of the origin, then Theorem 4.3.2 shows that the origin is a stable equilibrium point for the closed loop system, and hence, combined with attractivity in $\mathbb{S}_N$, it is asymptotically stable in $\mathbb{S}_N$. This shows the result (iii).

(iv) *Exponential stability.*

By assumption, $F(x) \le b\|x\|^\sigma$ for all $x \in \mathbb{X}_f$, for some constants $b > 0$ and $\sigma > 0$. It is easily shown that $V_N^{\text{OPT}}(x) \le F(x)$ for all $x \in \mathbb{X}_f$. To see this, let x be an arbitrary point in $\mathbb{X}_f$ and denote by $\{x_k^f(x) : k = 0, 1, 2, \ldots\}$, $x_0^f(x) \triangleq x$, the state sequence resulting from initial state x and controller $\mathcal{K}_f(x)$ in (4.17). Then, by **B2**,

$$F(x) \ge \sum_{k=0}^{N-1} L(x_k^f(x), \mathcal{K}_f(x_k^f(x))) + F(x_N^f(x)),$$

where, by **B3**, $x_k^f(x) \in \mathbb{X}_f$ for all $k = 0, 1, \ldots, N$ and, by **B4**, $\mathcal{K}_f(x_k^f(x)) \in \mathbb{U}$ for all $k = 0, 1, \ldots, N-1$. Note that the above state and control sequences are feasible since $x_N^f(x) \in \mathbb{X}_f$. Hence, by optimality,

$$V_N^{\text{OPT}}(x) \le \sum_{k=0}^{N-1} L(x_k^f(x), \mathcal{K}_f(x_k^f(x))) + F(x_N^f(x)).$$

Thus, $V_N^{\text{OPT}}(x) \le F(x) \le b\|x\|^\sigma$ for all $x \in \mathbb{X}_f$. We now show that there exists a constant $\bar{b} > 0$ such that $V_N^{\text{OPT}}(x) \le \bar{b}\|x\|^\sigma$ for all $x \in \mathbb{S}_N$. Consider the function $h : \mathbb{S}_N \to \mathbb{R}$ defined as

$$h(x) \triangleq \begin{cases} \frac{V_N^{\text{OPT}}(x)}{\|x\|^\sigma} & \text{if } x \neq 0, \\ b & \text{if } x = 0\,. \end{cases}$$

Then $h(x)$ is continuous on the compact set $\text{cl}\,(\mathbb{S}_N \setminus \mathbb{X}_f)$, since $V_N^{\text{OPT}}(x)$ is continuous on $\mathbb{S}_N$ and $\mathbb{X}_f$ contains a neighbourhood of the origin. Hence, $h(x)$ is bounded in $\text{cl}\,(\mathbb{S}_N \setminus \mathbb{X}_f)$, say $h(x) \leq M$. It then follows that

$$V_N^{\text{OPT}}(x) \leq \bar{b}\|x\|^\sigma \quad \text{for all } x \in \mathbb{S}_N,$$

where $\bar{b} \geq \max\{b, M\}$. Combining the above inequality with (4.30) and (4.31), and using the assumption that $\gamma(t) \geq at^\sigma$ for some constant $a > 0$, it follows from Theorem 4.3.3 that the closed loop system has an exponentially stable equilibrium point at the origin. This shows (iv) and concludes the proof of the theorem. □

4.5 Terminal Conditions for Stability

In this section, we consider possible choices for the terminal triple $(\mathbb{X}_f, \mathcal{K}_f, F)$ that satisfy conditions **B1**–**B5** of Theorem 4.4.2.

One choice for the terminal state weighting $F(x)$ is the value function $V_\infty^{\text{OPT}}(x)$ for the associated infinite horizon constrained optimal control problem, defined as follows:

$$\mathcal{P}_\infty(x): \qquad V_\infty^{\text{OPT}}(x) \triangleq \min V_\infty(\{x_k\}, \{u_k\}), \tag{4.32}$$

$$\begin{aligned} &\text{subject to:} \\ &x_{k+1} = f(x_k, u_k) \quad \text{for } k = 0, 1, \ldots, \\ &x_0 = x, \\ &u_k \in \mathbb{U} \quad \text{for } k = 0, 1, \ldots, \\ &x_k \in \mathbb{X} \quad \text{for } k = 0, 1, \ldots, \end{aligned}$$

where $\{x_k\}$ and $\{u_k\}$ are now infinite sequences, and

$$V_\infty(\{x_k\}, \{u_k\}) \triangleq \sum_{k=0}^{\infty} L(x_k, u_k)\,. \tag{4.33}$$

Note that $\mathcal{P}_\infty(x)$ does not have either a terminal state weighting nor a terminal state constraint; both are irrelevant since, if a solution to the problem exists, the state must converge to zero as $k \to \infty$ (since L is assumed to satisfy condition **B1**). In this case, it follows from the principle of optimality (see Section 3.4 of Chapter 3) that the finite horizon value function for problem $\mathcal{P}_N(x)$ in (4.1) is $V_N^{\text{OPT}}(x) = V_\infty^{\text{OPT}}(x)$. With this choice, on-line optimisation is unnecessary, and the advantages of an infinite horizon problem automatically accrue. However, constraints generally render this approach impossible.

Usually, then, $\mathbb{X}_f$ is chosen to be an appropriate neighbourhood of the origin in which $V_\infty^{\text{OPT}}(x)$ is exactly (or approximately) known, and $F(x)$ is set equal to $V_\infty^{\text{OPT}}(x)$ or its approximation.

In the rather general framework discussed so far, it is hard to visualise how Theorem 4.4.2 might be utilised in practice. However, when we specialise to *linear constrained* control problems it turns out that it is rather easy to satisfy the required conditions. This will be taken up in the next chapter.

4.6 Further Reading

For complete list of references cited, see References section at the end of book.

General

General treatment of nonlinear receding horizon control can be found in the book Allgöwer and Zhen (2000), and in, for example, the papers Keerthi and Gilbert (1988), Mayne and Michalska (1990), Alamir and Bornard (1994), Jadbabaie, Yu and Hauser (2001). See also the recent special issue Magni (2003).

An overview of industrial applications of receding horizon control is given in Qin and Badgwell (1997).

For a more detailed treatment of stability for general discrete time systems see Vidyasagar (2002), Kalman and Bertram (1960), Scokaert, Rawlings and Meadows (1997).

Stability for continuous-time nonlinear systems is thoroughly covered in several recent books, including Khalil (1996), Sastry (1999) and Vidyasagar (2002).

Section 4.4

The idea of using terminal state weighting to turn the finite horizon optimisation problem into an infinite horizon problem can be traced back to Kleinman (1970), Thomas (1975), and Kwon and Pearson (1977). More recent work appears in Chen and Shaw (1982), Kwon, Bruckstein and Kailath (1983), Garcia, Prett and Morari (1989), Bitmead et al. (1990). Related results for input-output systems appear in Mosca, Lemos and Zhang (1990), Clarke and Scattolini (1981) and Mosca and Zhang (1992).

The three main "ingredients" for stability used in Section 4.4 are implicit (in various combinations) in early literature dealing with constrained receding horizon control, see Sznaier and Damborg (1987), (1990), Keerthi and Gilbert (1988), Mayne and Michalska (1990), Rawlings and Muske (1993), Bemporad, Chisci and Mosca (1995), Chmielewski and Manousiouthakis (1996), De Nicolao, Magni and Scattolini (1996), Scokaert and Rawlings (1998). The form in which we have presented them is based on the clear and elegant synthesis provided by Mayne, Rawlings, Rao and Scokaert (2000).