2

Overview of Optimisation Theory

2.1 Overview

As foreshadowed in Chapter 1, the core idea underlying the approach described in this book to constrained control and estimation will be optimisation theory. This will be the topic of the current chapter. Optimisation theory has huge areas of potential application which extend well beyond the boundaries of control and estimation. However, control and estimation do present an ideal framework within which the basic elements of optimisation theory can be presented.

Key ideas that we present in this chapter include convexity, the Karush–Kuhn–Tucker optimality conditions and Lagrangian duality. These ideas will be drawn upon in following chapters when we apply them to the specific topics of constrained control and estimation. The material for this chapter has been extracted mainly from Bazaraa, Sherali and Shetty (1993). We refer the reader to this reference, as well as to the others mentioned in Section 2.8, for a more complete treatment of optimisation theory and a number of illustrative examples.

2.2 Preliminary Concepts

In this section we review some basic topological properties of sets that will be used throughout the book. Also, we review the definition of differentiability of real-valued functions defined on a subset S of $\mathbb{R}^n$.

2.2.1 Sets and Sequences

Given a point $x \in \mathbb{R}^n$, an *ε-neighbourhood* around x is defined as the set $N_\varepsilon(x) = \{y \in \mathbb{R}^n : ||y - x|| < \varepsilon\}$, for $\varepsilon > 0$, where $||\cdot||$ denotes the Euclidean norm of a vector in $\mathbb{R}^n$.

Let S be an arbitrary set in $\mathbb{R}^n$. A point x is said to be in the *closure* of S, denoted by cl S, if $S \cap N_\varepsilon(x) \neq \emptyset$ for every $\varepsilon > 0$. In other words, the closure of a set S is the set of all points that are arbitrarily close to S. If $S = \mathrm{cl}\, S$, then S is called *closed*. A point $x \in S$ is in the *interior* of S, denoted by int S, if $N_\varepsilon(x) \subset S$ for some $\varepsilon > 0$. If $S = \mathrm{int}\, S$, then S is called *open*.

A point x is in the *boundary* of S, denoted by ∂S, if $N_\varepsilon(x)$ contains at least one point in S and one point not in S for every $\varepsilon > 0$. Hence, a set S is closed if and only if it contains all its boundary points. Moreover, cl $S \equiv S \cup \partial S$ is the smallest closed set containing S. Similarly, a set S is open if and only if it does not contain any of its boundary points. Clearly, a set may be neither open nor closed, and the only sets in $\mathbb{R}^n$ that are both open and closed are the empty set and $\mathbb{R}^n$ itself. Also, note that any point $x \in S$ must be either an interior or a boundary point of S. However, in general, $S \neq \mathrm{int}\, S \cup \partial S$, since S need not contain its boundary points. On the other hand, since int $S \subseteq S$, we have, int $S = S - \partial S$, whilst, in general, $\partial S \neq S - \mathrm{int}\, S$.

A *sequence* of points, or vectors, $\{x_1, x_2, x_3, \ldots\}$, is said to *converge* to the *limit point* $\bar{x}$ if $||x_k - \bar{x}|| \to 0$ as $k \to \infty$; that is, if for any given $\varepsilon > 0$, there is a positive integer N such that $||x_k - \bar{x}|| < \varepsilon$ for all $k \geq N$. The sequence will be denoted by $\{x_k\}$, and the limit point $\bar{x}$ is represented by $x_k \to \bar{x}$ as $k \to \infty$. Any converging sequence has a unique limit point. By deleting certain elements of a sequence $\{x_k\}$, we obtain a *subsequence*, denoted by $\{x_k\}_K$, where K is a subset of all positive integers. To illustrate, let K be the set of all even positive integers, then $\{x_k\}_K$ denotes the subsequence $\{x_2, x_4, x_6, \ldots\}$.

An equivalent definition of closed sets, that is useful when demonstrating that a set is closed, is based on sequences of points contained in S. A set S is closed if and only if, for any convergent sequence of points $\{x_k\}$ contained in S with limit point $\bar{x}$, we also have $\bar{x} \in S$.

A set is *bounded* if it can be contained in a neighbourhood of sufficiently large but bounded radius. A *compact* set is one that is both closed and bounded. For every sequence $\{x_k\}$ in a compact set S, there is a convergent subsequence with a limit in S.

2.2.2 Differentiable Functions

We next investigate differentiability of a real-valued function f defined on a subset S of $\mathbb{R}^n$.

Definition 2.2.1 (Differentiable Function) *Let S be a set in $\mathbb{R}^n$ with a nonempty interior, and let $f : S \to \mathbb{R}$. Then, f is said to be* differentiable *at $\bar{x} \in \mathrm{int}\, S$ if there exists a vector $\nabla f(\bar{x})^{\mathrm{T}} \in \mathbb{R}^n$, called the* gradient vector,[1] *and a function $\alpha : \mathbb{R}^n \to \mathbb{R}$, such that*

[1] Although nonstandard, here we will consider the gradient vector ∇f a row vector to be consistent with notation used in the remainder of the book.

$$f(x) = f(\bar{x}) + \nabla f(\bar{x})(x - \bar{x}) + \|x - \bar{x}\|\alpha(\bar{x}, x - \bar{x}) \quad \text{for all } x \in S, \tag{2.1}$$

where $\lim_{x \to \bar{x}} \alpha(\bar{x}, x - \bar{x}) = 0$. *The function* f *is said to be* differentiable on the open set $S' \subseteq S$ *if it is differentiable at each point in* S'. *The above representation of* f *is called a* first-order (Taylor series) expansion *of* f *at* $\bar{x}$. ◦

Note that if f is differentiable at $\bar{x}$, then there can be only one gradient vector, and this vector consists of the partial derivatives, that is,

$$\nabla f(\bar{x}) = \left(\frac{\partial f(\bar{x})}{\partial x_1}, \frac{\partial f(\bar{x})}{\partial x_2}, \ldots, \frac{\partial f(\bar{x})}{\partial x_n} \right).$$

Definition 2.2.2 (Twice-Differentiable Function) *Let* S *be a set in* $\mathbb{R}^n$ *with a nonempty interior, and let* $f : S \to \mathbb{R}$. *Then,* f *is said to be* twice-differentiable *at* $\bar{x} \in \text{int } S$ *if there exists a vector* $\nabla f(\bar{x})^{\mathrm{T}} \in \mathbb{R}^n$, *and an* $n \times n$ *symmetric matrix* $H(\bar{x})$, *called the* Hessian matrix, *and a function* $\alpha : \mathbb{R}^n \to \mathbb{R}$, *such that*

$$f(x) = f(\bar{x}) + \nabla f(\bar{x})(x - \bar{x}) + \frac{1}{2}(x - \bar{x})^{\mathrm{T}} H(\bar{x})(x - \bar{x}) + \|x - \bar{x}\|^2 \alpha(\bar{x}, x - \bar{x}) \quad \text{for all } x \in S,$$

where $\lim_{x \to \bar{x}} \alpha(\bar{x}, x - \bar{x}) = 0$. *The function* f *is said to be* twice-differentiable on the open set $S' \subseteq S$ *if it is twice-differentiable at each point in* S'. *The above representation of* f *is called a* second-order (Taylor series) expansion *of* f *at* $\bar{x}$. ◦

For a twice-differentiable function, the Hessian matrix $H(\bar{x})$ comprises the second-order partial derivatives, that is, the element in row i and column j of the Hessian matrix is the second partial derivative $\partial^2 f(\bar{x})/\partial x_i \partial x_j$.

A useful theorem, which applies to differentiable functions defined on a convex set, is the *mean value theorem*, stated below. (Convex sets are formally defined in the next section.)

Theorem 2.2.1 (Mean Value Theorem) *Let* S *be a nonempty open convex set in* $\mathbb{R}^n$, *and let* $f : S \longrightarrow \mathbb{R}$ *be differentiable. Then, for every* x_1 *and* x_2 *in* S, *we must have*

$$f(x_2) = f(x_1) + \nabla f(x)(x_2 - x_1),$$

where $x = \lambda x_1 + (1 - \lambda)x_2$ *for some* $\lambda \in (0, 1)$. ◦

2.2.3 Weierstrass' Theorem

The following result, based on the foregoing concepts, relates to the existence of a minimising solution for an optimisation problem. We shall say that $\min\{f(x) : x \in S\}$ exists if there exists a minimising solution $\bar{x} \in S$ such that $f(\bar{x}) \leq f(x)$ for all $x \in S$. On the other hand, we say that $\alpha = \inf\{f(x) : x \in S\}$ if α is the greatest lower bound of f on S. We now prove that if S is nonempty, closed and bounded, and if f is continuous on S, then a minimum exists.

Theorem 2.2.2 (Weierstrass' Theorem: Existence of a Solution) *Let $S \subset \mathbb{R}^n$ be a nonempty, compact set, and let $f : S \longrightarrow \mathbb{R}$ be continuous on S. Then $f(x)$ attains its minimum in S, that is, there exists a minimising solution to the problem* $\min\{f(x) : x \in S\}$.

Proof. Since f is continuous on S, and S is both closed and bounded, f is bounded below on S. Consequently, since $S \neq \emptyset$, there exists a greatest lower bound $\alpha = \inf\{f(x) : x \in S\}$. Now, let $0 < \varepsilon < 1$, and consider the set $S_k = \{x \in S : \alpha \leq f(x) \leq \alpha + \varepsilon^k\}$ for $k = 1, 2, \ldots$. By the definition of an infimum, $S_k \neq \emptyset$ for each k, and so we can construct a sequence of points $\{x_k\} \subseteq S$ by selecting a point $x_k \in S_k$ for each $k = 1, 2, \ldots$. Since S is bounded, there exists a convergent subsequence $\{x_k\}_K \to \bar{x}$, indexed by the set K. By the closedness of S, we have $\bar{x} \in S$; and by the continuity of f, since $\alpha \leq f(x_k) \leq \alpha + \varepsilon^k$ for all k, we have $\alpha = \lim_{k\to\infty, k\in K} f(x_k) = f(\bar{x})$. Hence, we have shown that there exists a solution $\bar{x} \in S$ such that $f(\bar{x}) = \alpha = \inf\{f(x) : x \in S\}$, and so $\bar{x}$ is a minimising solution. This completes the proof. □

2.3 Convex Analysis

One of the main concepts that underpins optimisation theory is that of *convexity*. Indeed, the *big divide* in optimisation is between convex problems and nonconvex problems, rather than between, say, linear and nonlinear problems. Thus, understanding the notion of convexity can be a crucial step in solving many real world problems.

2.3.1 Convex Sets

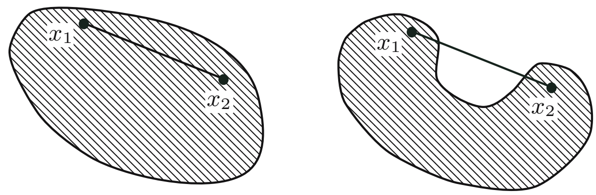
We have the following definition of a convex set.

Definition 2.3.1 (Convex Set) *A set $S \subset \mathbb{R}^n$ is* convex *if the line segment joining any two points of the set also belongs to the set. In other words, if $x_1, x_2 \in S$ then $\lambda x_1 + (1-\lambda)x_2$ must also belong to S for each $\lambda \in [0, 1]$.* ◦

Figure 2.1 below illustrates the notions of convex and nonconvex sets. Note that in Figure 2.1 (b), the line segment joining x_1 and x_2 does not lie entirely in the set.

The following are some examples of convex sets:

(i) **Hyperplane**. $S = \{x : p^{\mathrm{T}} x = \alpha\}$, where p is a nonzero vector in $\mathbb{R}^n$, called the *normal* to the hyperplane, and α is a scalar.



(a) Convex (b) Nonconvex

Figure 2.1. Illustration of a convex and a nonconvex set.

(ii) **Half-space**. $S = \{x : p^{\mathrm{T}}x \leq \alpha\}$, where p is a nonzero vector in $\mathbb{R}^n$, and α is a scalar.
(iii) **Open half-space**. $S = \{x : p^{\mathrm{T}}x < \alpha\}$, where p is a nonzero vector in $\mathbb{R}^n$ and α is a scalar.
(iv) **Polyhedral set**. $S = \{x : Ax \leq b\}$, where A is an $m \times n$ matrix, and b is an m vector. (Here and in the remainder of the book the inequality should be interpreted *elementwise*.)
(v) **Polyhedral cone**. $S = \{x : Ax \leq 0\}$, where A is an $m \times n$ matrix.
(vi) **Cone spanned by a finite number of vectors**. $S = \{x : x = \sum_{j=1}^{m} \lambda_j a_j,\ \lambda_j \geq 0,\ \text{for } j = 1, \ldots, m\}$, where $a_1, \ldots, a_m$ are given vectors in $\mathbb{R}^n$.
(vii) **Neighbourhood**. $N_\varepsilon(\bar{x}) = \{x \in \mathbb{R}^n : ||x - \bar{x}|| < \varepsilon\}$, where $\bar{x}$ is a fixed vector in $\mathbb{R}^n$ and $\varepsilon > 0$.

Some of the geometric optimality conditions presented in this chapter use *convex cones*, defined below.

Definition 2.3.2 (Convex Cone) *A nonempty set C in $\mathbb{R}^n$ is called a* cone *with vertex zero if $x \in C$ implies that $\lambda x \in C$ for all $\lambda \geq 0$. If, in addition, C is convex, then C is called a* convex cone*.* ◦

Figure 2.2 shows an example of a convex cone and an example of a nonconvex cone.

2.3.2 Separation and Support of Convex Sets

Almost all optimality conditions and duality relationships use some sort of separation or support of convex sets. We begin by stating the geometric facts that, given a closed convex set S and a point $y \notin S$, there exists a unique point $\bar{x} \in S$ with minimum distance from y (Theorem 2.3.1) and a hyperplane that separates y and S (Theorem 2.3.2).

Theorem 2.3.1 (Closest Point Theorem) *Let S be a nonempty, closed convex set in $\mathbb{R}^n$ and $y \notin S$. Then, there exists a unique point $\bar{x} \in S$ with*

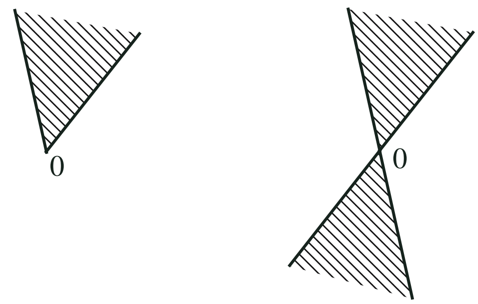


Figure 2.2. Examples of cones.

minimum distance from y. Furthermore, $\bar{x}$ is the minimising point, or closest point to y, if and only if $(y-\bar{x})^{\mathrm{T}}(x-\bar{x}) \leq 0$ for all $x \in S$.

Proof. We first establish the existence of a closest point. Since $S \neq \emptyset$, there exists a point $\hat{x} \in S$, and we can confine our attention to the set $\bar{S} = S \cap \{x : \|y-x\| \leq \|y-\hat{x}\|\}$ in seeking the closest point. In other words, the closest point problem $\inf\{\|y-x\| : x \in S\}$ is equivalent to $\inf\{\|y-x\| : x \in \bar{S}\}$. However, the latter problem involves finding the minimum of a continuous function over a nonempty, compact set $\bar{S}$, and so by Weierstrass' theorem (Theorem 2.2.2) we know that there exists a minimising point $\bar{x}$ in S that is closest to the point y.

Next, we prove that the closest point is unique. Suppose that there is an $\bar{x}' \in S$ such that $\|y-\bar{x}\| = \|y-\bar{x}'\| = \gamma$. By convexity of S, $\frac{1}{2}\bar{x} + \frac{1}{2}\bar{x}' \in S$. By the triangle inequality, we obtain

$$\left\| y - \left(\frac{1}{2}\bar{x} + \frac{1}{2}\bar{x}'\right)\right\| \leq \frac{1}{2}\|y-\bar{x}\| + \frac{1}{2}\|y-\bar{x}'\| = \gamma.$$

If strict inequality holds, we have a contradiction to $\bar{x}$ being the closest point to y. Therefore, equality holds, and we must have $y - \bar{x} = \lambda(y - \bar{x}')$ for some λ. Since $\|y-\bar{x}\| = \|y-\bar{x}'\| = \gamma$, $|\lambda| = 1$. Clearly, $\lambda \neq -1$, because otherwise we would have $y = \frac{1}{2}\bar{x} + \frac{1}{2}\bar{x}' \in S$, contradicting the assumption that $y \notin S$. So, $\lambda = 1$, $\bar{x}' = \bar{x}$, and uniqueness is established.

Finally, we prove that $(y-\bar{x})^{\mathrm{T}}(x-\bar{x}) \leq 0$ for all $x \in S$ is both a necessary and sufficient condition for $\bar{x}$ to be the point in S closest to y. To prove sufficiency, let $x \in S$. Then,

$$\|y-x\|^2 = \|y-\bar{x}+\bar{x}-x\|^2 = \|y-\bar{x}\|^2 + \|\bar{x}-x\|^2 + 2(\bar{x}-x)^{\mathrm{T}}(y-\bar{x}).$$

Since $\|\bar{x}-x\|^2 \geq 0$ and $(\bar{x}-x)^{\mathrm{T}}(y-\bar{x}) \geq 0$ by assumption, $\|y-x\|^2 \geq \|y-\bar{x}\|^2$, and $\bar{x}$ is the minimising point. Conversely, assume that $\|y-x\|^2 \geq \|y-\bar{x}\|^2$

for all $x \in S$. Let $x \in S$ and note that $\bar{x} + \lambda(x - \bar{x}) \in S$ for all $0 \leq \lambda \leq 1$ by the convexity of S. Therefore,

$$\|y - \bar{x} - \lambda(x - \bar{x})\|^2 \geq \|y - \bar{x}\|^2. \tag{2.2}$$

Also

$$\|y - \bar{x} - \lambda(x - \bar{x})\|^2 = \|y - \bar{x}\|^2 + \lambda^2\|x - \bar{x}\|^2 - 2\lambda(y - \bar{x})^{\mathrm{T}}(x - \bar{x}). \tag{2.3}$$

From (2.2) and (2.3), we obtain

$$2\lambda(y - \bar{x})^{\mathrm{T}}(x - \bar{x}) \leq \lambda^2\|x - \bar{x}\|^2, \tag{2.4}$$

for all $0 \leq \lambda \leq 1$. Dividing (2.4) by any such $\lambda > 0$ and letting $\lambda \to 0^+$, the result follows. □

The above theorem is illustrated in Figure 2.3.

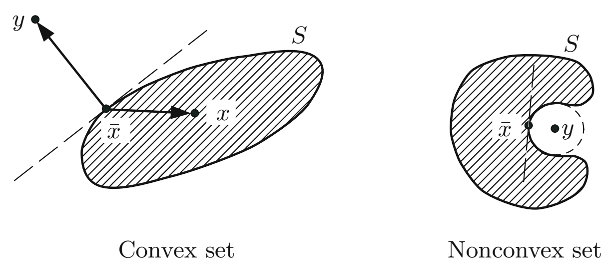


Figure 2.3. Closest point to a closed convex set.

Definition 2.3.3 (Separation of Sets) *Let S_1 and S_2 be nonempty sets in $\mathbb{R}^n$. A hyperplane $H = \{x : p^{\mathrm{T}}x = \alpha\}$* separates *$S_1$ and S_2 if $p^{\mathrm{T}}x \geq \alpha$ for each $x \in S_1$ and $p^{\mathrm{T}}x \leq \alpha$ for each $x \in S_2$. If, in addition, $p^{\mathrm{T}}x \geq \alpha + \varepsilon$ for each $x \in S_1$ and $p^{\mathrm{T}}x \leq \alpha$ for each $x \in S_2$, where ε is a positive scalar, then the hyperplane H is said to* strongly separate *the sets S_1 and S_2. (Notice that strong separation implies separation of sets.)*

Figure 2.4 illustrates the concepts of separation and strong separation of sets.

The following is the most fundamental separation theorem.

Theorem 2.3.2 (Separation Theorem) *Let S be a nonempty closed convex set in $\mathbb{R}^n$ and $y \notin S$. Then, there exists a nonzero vector p and a scalar α such that $p^{\mathrm{T}}y > \alpha$ and $p^{\mathrm{T}}x \leq \alpha$ for each $x \in S$.*

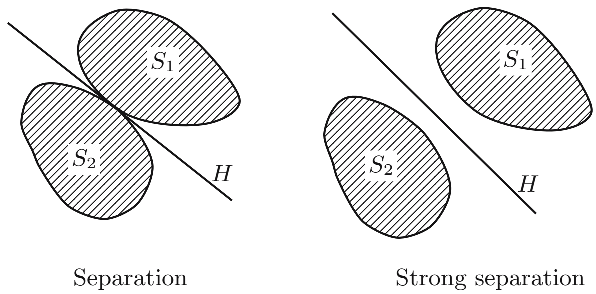


Figure 2.4. Separation and strong separation of sets.

Proof. S is a nonempty closed convex set and $y \notin S$. Hence, by Theorem 2.3.1 there exists a unique minimising point $\bar{x} \in S$ such that $(y - \bar{x})^{\mathrm{T}}(x - \bar{x}) \leq 0$ for each $x \in S$. Letting $p = (y - \bar{x}) \neq 0$ and $\alpha = (y - \bar{x})^{\mathrm{T}}\bar{x} = p^{\mathrm{T}}\bar{x}$, we obtain $p^{\mathrm{T}}x \leq \alpha$ for each $x \in S$. We also have $p^{\mathrm{T}}y - \alpha = (y - \bar{x})^{\mathrm{T}}(y - \bar{x}) = \|y - \bar{x}\|^2 > 0$ and, hence, $p^{\mathrm{T}}y > \alpha$. This completes the proof. □

Closely related to the above concept is the notion of a *supporting hyperplane.*

Definition 2.3.4 (Supporting Hyperplane at a Boundary Point) *Let S be a nonempty set in $\mathbb{R}^n$, and let $\bar{x} \in \partial S$. A hyperplane $H = \{x : p^{\mathrm{T}}(x - \bar{x}) = 0\}$ is called a* supporting hyperplane *of S at $\bar{x}$ if either $p^{\mathrm{T}}(x - \bar{x}) \geq 0$ for each $x \in S$, or else, $p^{\mathrm{T}}(x - \bar{x}) \leq 0$ for each $x \in S$.*

Figure 2.5 shows an example of a supporting hyperplane.

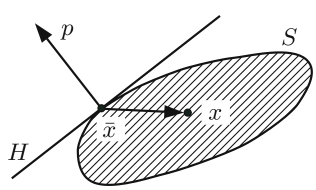


Figure 2.5. Supporting hyperplane.

The next result shows that a convex set has a supporting hyperplane at each boundary point. As a corollary, a result similar to Theorem 2.3.2, where S is not required to be closed, follows.

Theorem 2.3.3 (Supporting Hyperplane) *Let S be a nonempty convex set in $\mathbb{R}^n$, and let $\bar{x} \in \partial S$. Then there exists a hyperplane that supports S at*

$\bar{x}$; that is, there exists a nonzero vector p such that $p^{\mathrm{T}}(x-\bar{x}) \leq 0$ for each $x \in \operatorname{cl} S$.

Proof. Since $\bar{x} \in \partial S$, there exists a sequence $\{y_k\}$ not in cl S such that $y_k \to \bar{x}$. By Theorem 2.3.2, corresponding to each y_k there exists a p_k such that $p_k^{\mathrm{T}} y_k > p_k^{\mathrm{T}} x$ for each $x \in \operatorname{cl} S$. Without loss of generality, we can normalise the vector in Theorem 2.3.2 by dividing it by its norm, such that $\|p_k\| = 1$. Since $\{p_k\}$ is bounded, it has a convergent subsequence $\{p_k\}_K$ with limit p whose norm is also equal to 1. Considering this subsequence, we have $p_k^{\mathrm{T}} y_k > p_k^{\mathrm{T}} x$ for each $x \in \operatorname{cl} S$. Fixing $x \in \operatorname{cl} S$ and taking limits as $k \in K$ approaches ∞, we obtain, $p^{\mathrm{T}}(x-\bar{x}) \leq 0$. Since this is true for each $x \in \operatorname{cl} S$, the result follows. □

Corollary 2.3.4 *Let S be a nonempty convex set in $\mathbb{R}^n$ and $\bar{x} \notin \operatorname{int} S$. Then there is a nonzero vector p such that $p^{\mathrm{T}}(x-\bar{x}) \leq 0$ for each $x \in \operatorname{cl} S$.*

Proof. If $\bar{x} \notin \operatorname{cl} S$, then the result follows from Theorem 2.3.2 choosing $y = \bar{x}$. On the other hand, if $\bar{x} \in \partial S$, the result follows from Theorem 2.3.3. □

The next theorem shows that, if two convex sets are disjoint, then they can be separated by a hyperplane.

Theorem 2.3.5 (Separation of Two Disjoint Convex Sets) *Let S_1 and S_2 be nonempty convex sets in $\mathbb{R}^n$ and suppose that $S_1 \cap S_2$ is empty. Then there exists a hyperplane that separates S_1 and S_2; that is, there exists a nonzero vector p in $\mathbb{R}^n$ such that*

$$\inf\{p^{\mathrm{T}} x : x \in S_1\} \geq \sup\{p^{\mathrm{T}} x : x \in S_2\}.$$

Proof. Consider the set $S = S_1 \ominus S_2 \triangleq \{x_1 - x_2 : x_1 \in S_1 \text{ and } x_2 \in S_2\}$. Note that S is convex. Furthermore, $0 \notin S$, because otherwise $S_1 \cap S_2$ would be nonempty. By Corollary 2.3.4, there exists a nonzero $p \in \mathbb{R}^n$ such that $p^{\mathrm{T}} x \geq 0$ for all $x \in S$. This means that $p^{\mathrm{T}} x_1 \geq p^{\mathrm{T}} x_2$ for all $x_1 \in S_1$ and $x_2 \in S_2$, and the result follows. □

The following corollary shows that the above result holds true even if the two sets have some points in common, as long as their interiors are disjoint.

Corollary 2.3.6 *Let S_1 and S_2 be nonempty convex sets in $\mathbb{R}^n$. Suppose that* int S_2 *is not empty and that $S_1 \cap \operatorname{int} S_2$ is empty. Then, there exists a hyperplane that separates S_1 and S_2; that is, there exists a nonzero p such that*

$$\inf\{p^{\mathrm{T}} x : x \in S_1\} \geq \sup\{p^{\mathrm{T}} x : x \in S_2\}.$$

Proof. Replace S_2 by int S_2, apply Theorem 2.3.5, and note that

$$\sup\{p^{\mathrm{T}} x : x \in S_2\} = \sup\{p^{\mathrm{T}} x : x \in \operatorname{int} S_2\}.$$

The result then follows. □

2.3.3 Convex Functions

Convex functions have many important properties for optimisation problems. For example, any local minimum of a convex function over a convex set is also a global minimum. We present here some properties of convex functions, beginning with their definition.

Definition 2.3.5 (Convex Function) *Let* $f : S \to \mathbb{R}$*, where* S *is a nonempty convex set in* $\mathbb{R}^n$*. The function* f *is* convex *on* S *if*

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \lambda f(x_1) + (1-\lambda)f(x_2)$$

for each $x_1, x_2 \in S$ *and for each* $\lambda \in (0,1)$*.*

The function f *is* strictly convex *on* S *if the above inequality is true as a strict inequality for each distinct* $x_1, x_2 \in S$ *and for each* $\lambda \in (0,1)$*.*

The function f *is* (strictly) concave *on* S *if* $-f$ *is (strictly) convex on* S*.*

○

The geometric interpretation of a convex function is that the value of f at the point $\lambda x_1 + (1-\lambda)x_2$ is less than the height of the chord joining the points $[x_1, f(x_1)]$ and $[x_2, f(x_2)]$. For a concave function, the chord is below the function itself. Figure 2.6 shows some examples of convex and concave functions.

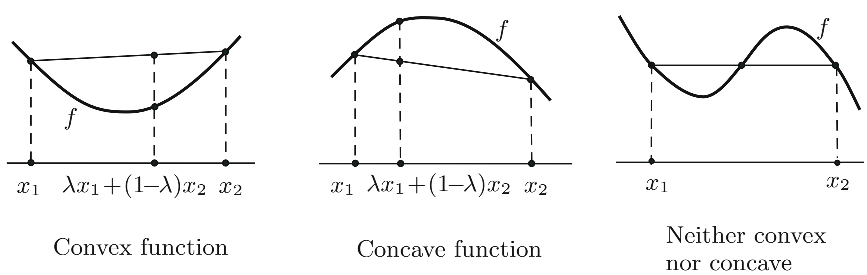


Figure 2.6. Examples of convex and concave functions.

The following are useful properties of convex functions.

(i) Let $f_1, f_2, \ldots, f_k : \mathbb{R}^n \to \mathbb{R}$ be convex functions. Then
- $f(x) = \sum_{j=1}^{k} \alpha_j f_j(x)$, where $\alpha_j > 0$ for $j = 1, 2, \ldots k$, is a convex function;
- $f(x) = \max\{f_1(x), f_2(x), \ldots, f_k(x)\}$ is a convex function.

(ii) Suppose that $g : \mathbb{R}^n \to \mathbb{R}$ is a concave function. Let $S = \{x : g(x) > 0\}$, and define $f : S \to \mathbb{R}$ as $f(x) = 1/g(x)$. Then f is convex over S.

(iii) Let $g : \mathbb{R} \to \mathbb{R}$ be a nondecreasing, univariate, convex function, and let $h : \mathbb{R}^n \to \mathbb{R}$ be a convex function. Then the composite function $f : \mathbb{R}^n \to \mathbb{R}$ defined as $f(x) = g(h(x))$ is a convex function.

(iv) Let $g : \mathbb{R}^m \to \mathbb{R}$ be a convex function, and let $h : \mathbb{R}^n \to \mathbb{R}^m$ be an affine function of the form $h(x) = Ax + b$, where A is an $m \times n$ matrix, and b is an $m \times 1$ vector. Then the composite function $f : \mathbb{R}^n \to \mathbb{R}$ defined as $f(x) = g(h(x))$ is a convex function.

Associated with a convex function f is the *level set* S_α defined as $S_\alpha = \{x \in S : f(x) \leq \alpha\}$, $\alpha \in \mathbb{R}$. We then have:

Lemma 2.3.7 (Convexity of Level Sets) *Let S be a nonempty convex set in $\mathbb{R}^n$ and let $f : S \to \mathbb{R}$ be a convex function. Then the level set $S_\alpha = \{x \in S : f(x) \leq \alpha\}$, where $\alpha \in \mathbb{R}$, is a convex set.*

Proof. Let $x_1, x_2 \in S_\alpha$. Thus, $x_1, x_2 \in S$, and $f(x_1) \leq \alpha$ and $f(x_2) \leq \alpha$. Now, let $\lambda \in (0, 1)$ and $x = \lambda x_1 + (1 - \lambda)x_2 \in S$ (by the convexity of S). Furthermore, by convexity of f,

$$f(x) \leq \lambda f(x_1) + (1 - \lambda)f(x_2) \leq \lambda\alpha + (1 - \lambda)\alpha = \alpha.$$

Hence, $x \in S_\alpha$, and we conclude that S_α is convex. □

An important property of convex functions is that they are continuous on the interior of their domain, as we prove next.

Theorem 2.3.8 (Continuity of Convex Functions) *Let S be a nonempty convex set in $\mathbb{R}^n$ and let $f : S \to \mathbb{R}$ be a convex function. Then f is continuous on the interior of S.*

Proof. Let $\bar{x} \in \text{int } S$. Hence, there exists a $\delta' > 0$ such that $||x - \bar{x}|| \leq \delta'$ implies that $x \in S$. Consider the vector $e_i \in \mathbb{R}^n$ having all elements equal to zero except for a 1 in the ith position. Now, construct

$$\theta \triangleq \max_{1 \leq i \leq n} \{\max [f(\bar{x} + \delta' e_i) - f(\bar{x}), f(\bar{x} - \delta' e_i) - f(\bar{x})]\}. \tag{2.5}$$

Note, from the convexity of f, that we have:

$$f(\bar{x}) = f\left[\frac{1}{2}(\bar{x} + \delta' e_i) + \frac{1}{2}(\bar{x} - \delta' e_i)\right] \leq \frac{1}{2}f(\bar{x} + \delta' e_i) + \frac{1}{2}f(\bar{x} - \delta' e_i),$$

for all $1 \leq i \leq n$, from where we conclude that $\theta \geq 0$.

Now, for any given $\epsilon > 0$, define:

$$\delta \triangleq \min\left\{\frac{\delta'}{n}, \frac{\epsilon\delta'}{n\theta}\right\}. \tag{2.6}$$

Choose an x with $||x - \bar{x}|| \leq \delta$. Let v_i denote the ith element of a vector v. If $x_i - \bar{x}_i \geq 0$, define $z_i = \delta' e_i$, otherwise define $z_i = -\delta' e_i$. Then, $x - \bar{x} = \sum_{i=1}^{n} \alpha_i z_i$, for some $\alpha_i \geq 0$, $1 \leq i \leq n$. Furthermore,

$$\|x-\bar{x}\| = \delta' \left(\sum_{i=1}^{n} \alpha_i^2\right)^{\frac{1}{2}} \leq \delta. \tag{2.7}$$

It follows from (2.6) and (2.7) that $\alpha_i \leq 1/n$ and $\alpha_i \leq \epsilon/n\theta$, for $i = 1, 2, \ldots, n$. From the convexity of f, and since $0 \leq n\alpha_i \leq 1$, we obtain

$$\begin{aligned} f(x) = f\left(\bar{x} + \sum_{i=1}^{n} \alpha_i z_i\right) &= f\left(\frac{1}{n}\sum_{i=1}^{n}(\bar{x} + n\alpha_i z_i)\right) \leq \frac{1}{n}\sum_{i=1}^{n} f(\bar{x} + n\alpha_i z_i) \\ &= \frac{1}{n}\sum_{i=1}^{n} f[(1-n\alpha_i)\bar{x} + n\alpha_i(\bar{x}+z_i)] \\ &\leq \frac{1}{n}\sum_{i=1}^{n}[(1-n\alpha_i)f(\bar{x}) + n\alpha_i f(\bar{x}+z_i)]. \end{aligned}$$

Therefore, $f(x) - f(\bar{x}) \leq \sum_{i=1}^{n} \alpha_i[f(\bar{x}+z_i) - f(\bar{x})]$. From (2.5) and the definition of z_i it follows that $f(\bar{x}+z_i) - f(\bar{x}) \leq \theta$ for each i; and since $\alpha_i \geq 0$, it follows that

$$f(x) - f(\bar{x}) \leq \theta \sum_{i=1}^{n} \alpha_i. \tag{2.8}$$

As noted above, $\alpha_i \leq \epsilon/n\theta$, for $i = 1, 2, \ldots, n$ and, thus, it follows from (2.8) that $f(x) - f(\bar{x}) \leq \epsilon$.

Now, let $y = 2\bar{x} - x$ and note that $\|y - \bar{x}\| \leq \delta$. Hence, as above, we have $f(y) - f(\bar{x}) \leq \epsilon$. But, $\bar{x} = \frac{1}{2}y + \frac{1}{2}x$, and by the convexity of f, we have $f(\bar{x}) \leq \frac{1}{2}f(y) + \frac{1}{2}f(x)$. Combining the last two inequalities, it follows that $f(\bar{x}) - f(x) \leq \epsilon$.

Summarising, we have shown that for any $\epsilon > 0$ there exists a $\delta > 0$ (defined as in (2.6)) such that $\|x - \bar{x}\| \leq \delta$ implies that $f(x) - f(\bar{x}) \leq \epsilon$ and that $f(\bar{x}) - f(x) \leq \epsilon$; that is, that $|f(x) - f(\bar{x})| \leq \epsilon$. Hence, f is continuous at $\bar{x} \in \text{int } S$, and the proof is complete. □

2.3.4 Generalisations of Convex Functions

We present various types of functions that are similar to convex or concave functions but share only some of their desirable properties.

Definition 2.3.6 (Quasiconvex Function) *Let $f : S \to \mathbb{R}$, where S is a nonempty convex set in $\mathbb{R}^n$. The function f is* quasiconvex *if, for each $x_1, x_2 \in S$, the following inequality is true:*

$$f(\lambda x_1 + (1-\lambda)x_2) \leq \max\{f(x_1), f(x_2)\} \quad \textit{for each}\, \lambda \in (0,1).$$

The function f is quasiconcave *if $-f$ is quasiconvex.* ◦

Note, from the definition, that a convex function is quasiconvex.

From the above definition, a function f is quasiconvex if, whenever $f(x_2) \geq f(x_1)$, $f(x_2)$ is greater than or equal to f at all convex combinations of x_1 and x_2. Hence, if f increases locally from its value at a point along any direction, it must remain nondecreasing in that direction. Figure 2.7 shows some examples of quasiconvex and quasiconcave functions.

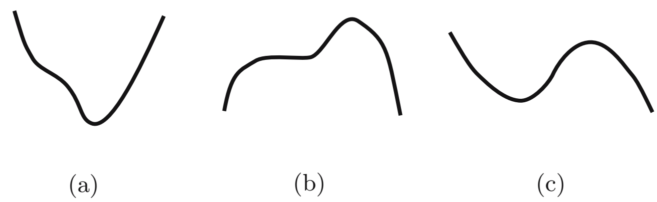


Figure 2.7. (a) Quasiconvex function. (b) Quasiconcave function. (c) Neither quasiconvex nor quasiconcave.

The following result states that a quasiconvex function is characterised by the convexity of its level sets.

Theorem 2.3.9 (Level Sets of a Quasiconvex Function) *Let $f : S \to \mathbb{R}$, where S is a nonempty convex set in $\mathbb{R}^n$. The function f is quasiconvex if and only if $S_\alpha = \{x \in S : f(x) \leq \alpha\}$ is convex for each real number α.*

Proof. Suppose that f is quasiconvex, and let $x_1, x_2 \in S_\alpha$. Therefore, $x_1, x_2 \in S$ and $\max\{f(x_1), f(x_2)\} \leq \alpha$. Let $\lambda \in (0,1)$, and let $x = \lambda x_1 + (1-\lambda)x_2$. By the convexity of S, $x \in S$. Furthermore, by the quasiconvexity of f, $f(x) \leq \max\{f(x_1), f(x_2)\} \leq \alpha$. Hence, $x \in S_\alpha$, and thus S_α is convex. Conversely, suppose that S_α is convex for each real number α. Let $x_1, x_2 \in S$ and take $\alpha = \max\{f(x_1), f(x_2)\}$. Hence, $x_1, x_2 \in S_\alpha$. Furthermore, let $\lambda \in (0,1)$ and $x = \lambda x_1 + (1-\lambda)x_2$. By assumption, S_α is convex, so that $x \in S_\alpha$. Therefore, $f(x) \leq \alpha = \max\{f(x_1), f(x_2)\}$. Hence, f is quasiconvex, and the proof is complete. □

We will next define *strictly quasiconvex functions*.

Definition 2.3.7 (Strictly Quasiconvex Function) *Let $f : S \to \mathbb{R}$, where S is a nonempty convex set in $\mathbb{R}^n$. The function f is* strictly quasiconvex *if, for each $x_1, x_2 \in S$ with $f(x_1) \neq f(x_2)$, the following inequality is true*

$$f(\lambda x_1 + (1-\lambda)x_2) < \max\{f(x_1), f(x_2)\} \quad \textit{for each}\, \lambda \in (0,1).$$

The function f is strictly quasiconcave *if $-f$ is strictly quasiconvex.* ∘

Note from the above definition that a convex function is also strictly quasiconvex. Figure 2.8 shows some examples of quasiconvex and strictly quasiconvex functions.

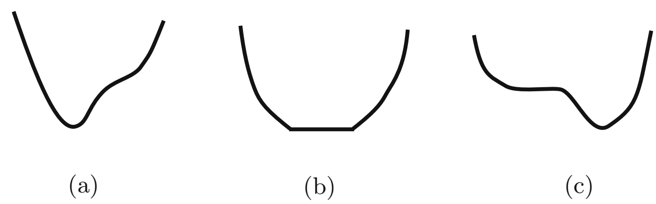


Figure 2.8. (a) Strictly quasiconvex function. (b) Strictly quasiconvex function. (c) Quasiconvex function but not strictly quasiconvex.

Notice that the definition precludes any *flat spots* from occurring anywhere except at extremising points. This, in turn, implies that a local optimal solution of a strictly quasiconvex function over a convex set is also a global optimal solution. (Local and global optima for constrained optimisation problems will be formally defined in Definition 2.5.1.)

We observe that strictly quasiconvex functions are not necessarily quasiconvex. However, if f is *lower semicontinuous*[2], then it can be shown that strict quasiconvexity implies quasiconvexity.

We will next introduce another type of function that generalises the concept of a convex function, called a *pseudoconvex function.* Pseudoconvex functions share the property of convex functions that, if $\nabla f(\bar{x}) = 0$ at some point $\bar{x}$, then $\bar{x}$ is a global minimum of f. (See Theorem 2.4.5.)

Definition 2.3.8 (Pseudoconvex Function) *Let S be a nonempty open set in $\mathbb{R}^n$, and let $f : S \to \mathbb{R}$ be differentiable on S. The function f is* pseudoconvex *if, for each $x_1, x_2 \in S$ with $\nabla f(x_1)(x_2 - x_1) \geq 0$, then $f(x_2) \geq f(x_1)$; or, equivalently, if $f(x_2) < f(x_1)$, then $\nabla f(x_1)(x_2 - x_1) < 0$.*

The function f is pseudoconcave *if $-f$ is pseudoconvex.*

The function f is strictly pseudoconvex *if, for each distinct $x_1, x_2 \in S$ with $\nabla f(x_1)(x_2 - x_1) \geq 0$, then $f(x_2) > f(x_1)$; or, equivalently, if for each distinct $x_1, x_2 \in S$, $f(x_2) \leq f(x_1)$, then $\nabla f(x_1)(x_2 - x_1) < 0$.* ◦

Note that the definition asserts that if the directional derivative of a pseudoconvex function at any point x_1 in the direction $x_2 - x_1$ is nonnegative,

[2] A function $f : S \to \mathbb{R}$, where S is a nonempty set in $\mathbb{R}^n$, is *lower semicontinuous* at $\bar{x} \in S$ if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that $x \in S$ and $\|x - \bar{x}\| < \delta$ imply that $f(x) - f(\bar{x}) > -\varepsilon$. Obviously, a continuous function at $\bar{x}$ is also lower semicontinuous at $\bar{x}$.

then the function values are nondecreasing in that direction. Figure 2.9 shows examples of pseudoconvex and pseudoconcave functions.

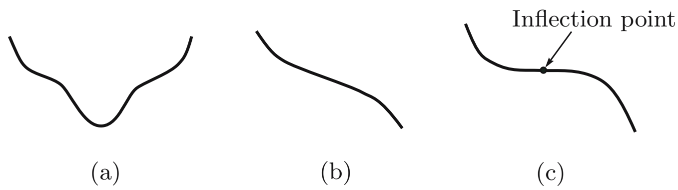


Figure 2.9. (a) Pseudoconvex function. (b) Both pseudoconvex and pseudoconcave. (c) Neither pseudoconvex nor pseudoconcave.

Several relationships among the different types of convexity can be established. For example, one of these relationships is that every pseudoconvex function is both strictly quasiconvex and quasiconvex. Figure 2.10 summarises the implications among the different types of convexity. In the particular case of a *quadratic* function f it can be shown that f is pseudoconvex if and only if f is strictly quasiconvex, which holds true if and only if f is quasiconvex.

Convexity at a Point

In some optimisation problems, the requirement of convexity may be too strong and not essential, and convexity at a point may be all that is needed. Hence, we present below several types of convexity at a point that are relaxations of the various forms of convexity presented so far.

Definition 2.3.9 (Various Types of Convexity at a Point) *Let S be a nonempty convex set in $\mathbb{R}^n$, and $f : S \to \mathbb{R}$. We then have the following definitions:*

Convexity at a point. The function f is said to be convex at $\bar{x} \in S$ if $f(\lambda\bar{x} + (1-\lambda)x) \leq \lambda f(\bar{x}) + (1-\lambda)f(x)$ for each $\lambda \in (0,1)$ and each $x \in S$.

Strict convexity at a point. The function f is strictly convex at $\bar{x} \in S$ if $f(\lambda\bar{x} + (1-\lambda)x) < \lambda f(\bar{x}) + (1-\lambda)f(x)$ for each $\lambda \in (0,1)$ and each $x \in S$, $x \neq \bar{x}$.

Quasiconvexity at a point. The function f is quasiconvex at $\bar{x} \in S$ if $f(\lambda\bar{x} + (1-\lambda)x) \leq \max\{f(\bar{x}), f(x)\}$ for each $\lambda \in (0,1)$ and each $x \in S$.

Strict quasiconvexity at a point. The function f is strictly quasiconvex at $\bar{x} \in S$ if $f(\lambda\bar{x} + (1-\lambda)x) < \max\{f(\bar{x}), f(x)\}$ for each $\lambda \in (0,1)$ and each $x \in S$ such that $f(x) \neq f(\bar{x})$.

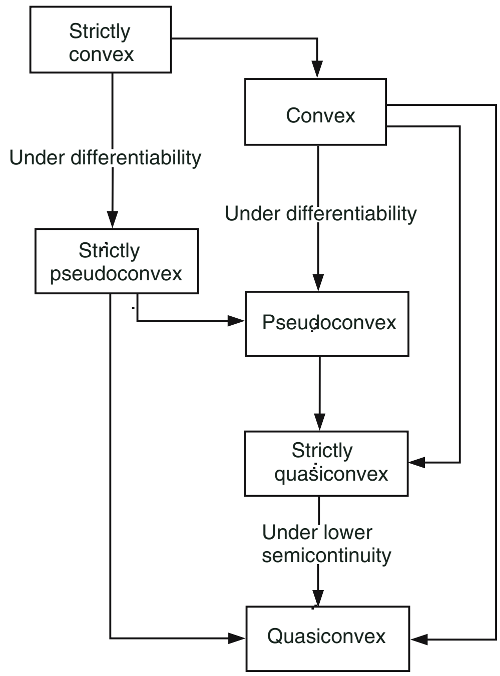


Figure 2.10. Relationship among various types of convexity. The arrows mean implications and, in general, the converses do not hold. (See Bazaraa et al. (1993) for a more complete picture of the relationships among the types of convexity.)

Pseudoconvexity at a point. Suppose f is differentiable at $\bar{x} \in \operatorname{int} S$. Then f is pseudoconvex at $\bar{x}$ if $\nabla f(\bar{x})(x-\bar{x}) \geq 0$ for $x \in S$ implies that $f(x) \geq f(\bar{x})$.

Strict pseudoconvexity at a point. Suppose f is differentiable at $\bar{x} \in \operatorname{int} S$. Then f is strictly pseudoconvex at $\bar{x}$ if $\nabla f(\bar{x})(x-\bar{x}) \geq 0$ for $x \in S$, $x \neq \bar{x}$, implies that $f(x) > f(\bar{x})$. ○

Figure 2.11 illustrates some types of convexity at a point.

2.4 Unconstrained Optimisation

An unconstrained optimisation problem is a problem of the form

$$\text{minimise } f(x), \tag{2.9}$$

without any constraint on the vector x. Our ultimate goal in this book is constrained optimisation problems. However, we start by reviewing unconstrained problems because optimality conditions for constrained problems are a logical extension of the conditions for unconstrained problems.

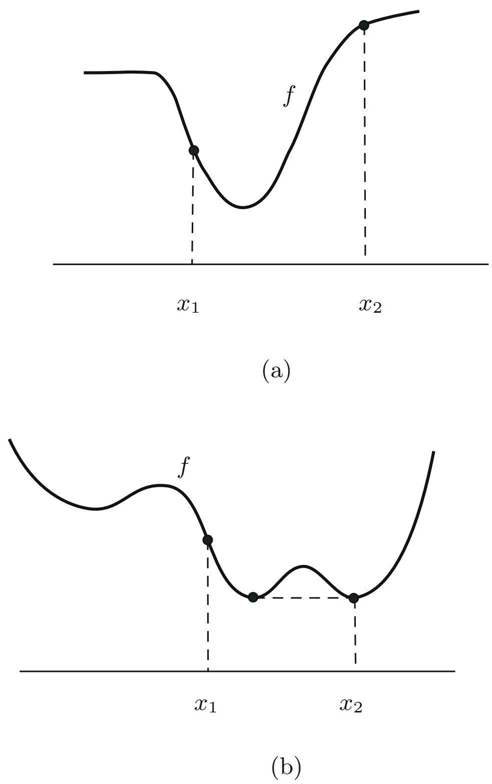


Figure 2.11. Convexity at a point. (a) f is quasiconvex but not strictly quasiconvex at x_1; f is both quasiconvex and strictly quasiconvex at x_2. (b) f is both pseudoconvex and strictly pseudoconvex at x_1; f is pseudoconvex but not strictly pseudoconvex at x_2.

Let us first define *local and global minima* for unconstrained problems.

Definition 2.4.1 (Local and Global Minima) *Consider the problem of minimising $f(x)$ over $\mathbb{R}^n$ and let $\bar{x} \in \mathbb{R}^n$. If $f(\bar{x}) \leq f(x)$ for all $x \in \mathbb{R}^n$, then $\bar{x}$ is called a* global minimum. *If there exists an ε-neighbourhood $N_\varepsilon(\bar{x})$ around $\bar{x}$ such that $f(\bar{x}) \leq f(x)$ for each $x \in N_\varepsilon(\bar{x})$, then $\bar{x}$ is called a* local minimum, *whilst if $f(\bar{x}) < f(x)$ for all $x \in N_\varepsilon(\bar{x})$, $x \neq \bar{x}$, for some $\varepsilon > 0$, then $\bar{x}$ is called a* strict local minimum. *Clearly, a global minimum is also a local minimum.* ◦

Given a point $x \in \mathbb{R}^n$, we wish to determine, if possible, whether or not the point is a local or global minimum of a function f. For differentiable functions, there exist conditions that provide this characterisation, as we will see below. We will first present a result that allows the characterisation of *descent directions* of differentiable functions.

Theorem 2.4.1 (Descent Direction) *Suppose that* $f : \mathbb{R}^n \to \mathbb{R}$ *is differentiable at* $\bar{x}$*. If there exists a vector* d *such that* $\nabla f(\bar{x})d < 0$*, then there exists a* $\delta > 0$ *such that* $f(\bar{x} + \lambda d) < f(\bar{x})$ *for each* $\lambda \in (0, \delta)$*, so that* d *is a* descent direction *of* f *at* $\bar{x}$*.*

Proof. By the differentiability of f at $\bar{x}$, we have

$$f(\bar{x} + \lambda d) = f(\bar{x}) + \lambda \nabla f(\bar{x})d + \lambda \|d\| \alpha(\bar{x}, \lambda d),$$

where $\alpha(\bar{x}, \lambda d) \to 0$ as $\lambda \to 0$. Rearranging the terms and dividing by λ, $\lambda \neq 0$, we obtain

$$\frac{f(\bar{x} + \lambda d) - f(\bar{x})}{\lambda} = \nabla f(\bar{x})d + \|d\| \alpha(\bar{x}, \lambda d).$$

Since $\nabla f(\bar{x})d < 0$ and $\alpha(\bar{x}, \lambda d) \to 0$ as $\lambda \to 0$, there exists a $\delta > 0$ such that the right hand side above is negative for all $\lambda \in (0, \delta)$. The result then follows. □

Corollary 2.4.2 *Suppose that* $f : \mathbb{R}^n \to \mathbb{R}$ *is differentiable at* $\bar{x}$*. If* $\bar{x}$ *is a local minimum, then* $\nabla f(\bar{x}) = 0$*.*

Proof. Suppose that $\nabla f(\bar{x}) \neq 0$. Then, letting $d = -\nabla f(\bar{x})^{\mathrm{T}}$, we get $\nabla f(\bar{x})d = -\|\nabla f(\bar{x})\|^2 < 0$, and by Theorem 2.4.1 there is a $\delta > 0$ such that $f(\bar{x} + \lambda d) < f(\bar{x})$ for each $\lambda \in (0, \delta)$, contradicting the assumption that $\bar{x}$ is a local minimum. Hence, $\nabla f(\bar{x}) = 0$. □

The above condition uses the gradient vector, whose components are the first partial derivatives of f; hence, it is called a *first-order condition*. Necessary conditions can also be stated in terms of the Hessian matrix H, which comprises the second derivatives of f, and are then called *second-order conditions*. One such condition is given below.

Theorem 2.4.3 (Necessary Condition for a Minimum) *Suppose that* $f : \mathbb{R}^n \to \mathbb{R}$ *is twice-differentiable at* $\bar{x}$*. If* $\bar{x}$ *is a local minimum, then* $\nabla f(\bar{x}) = 0$ *and* $H(\bar{x})$ *is positive semidefinite.*

Proof. Consider an arbitrary direction d. Then, since by assumption f is twice-differentiable at $\bar{x}$, we have

$$f(\bar{x} + \lambda d) = f(\bar{x}) + \lambda \nabla f(\bar{x})d + \frac{1}{2}\lambda^2 d^{\mathrm{T}} H(\bar{x})d + \lambda^2 \|d\|^2 \alpha(\bar{x}, \lambda d), \qquad (2.10)$$

where $\alpha(\bar{x}, \lambda d) \to 0$ as $\lambda \to 0$. Since $\bar{x}$ is a local minimum, from Corollary 2.4.2 we have $\nabla f(\bar{x}) = 0$. Rearranging the terms in (2.10) and dividing by $\lambda^2 > 0$, we obtain

$$\frac{f(\bar{x}+\lambda d)-f(\bar{x})}{\lambda^2} = \frac{1}{2}d^{\mathrm{T}}H(\bar{x})d + \|d\|^2\alpha(\bar{x},\lambda d)\,. \tag{2.11}$$

Since $\bar{x}$ is a local minimum, $f(\bar{x}+\lambda d) \geq f(\bar{x})$ for sufficiently small λ . From (2.11), it is thus clear that $\frac{1}{2}d^{\mathrm{T}}H(\bar{x})d + \|d\|^2\alpha(\bar{x},\lambda d) \geq 0$ for sufficiently small λ. By taking the limit as $\lambda \to 0$, it follows that $d^{\mathrm{T}}H(\bar{x})d \geq 0$; and, hence, $H(\bar{x})$ is positive semidefinite. □

The conditions presented so far are necessary conditions for a local minimum. We now give a sufficient condition for a local minimum.

Theorem 2.4.4 (Sufficient Condition for a Local Minimum) *Suppose that $f : \mathbb{R}^n \to \mathbb{R}$ is twice-differentiable at $\bar{x}$. If $\nabla f(\bar{x}) = 0$ and $H(\bar{x})$ is positive definite, then $\bar{x}$ is a strict local minimum.*

Proof. Since f is twice-differentiable at $\bar{x}$, we must have, for each $x \in \mathbb{R}^n$,

$$f(x) = f(\bar{x}) + \nabla f(\bar{x})(x-\bar{x}) + \frac{1}{2}(x-\bar{x})^{\mathrm{T}}H(\bar{x})(x-\bar{x}) + \|x-\bar{x}\|^2\alpha(\bar{x},x-\bar{x})\,, \tag{2.12}$$

where $\alpha(\bar{x},x-\bar{x}) \to 0$ as $x \to \bar{x}$. Suppose, by contradiction, that $\bar{x}$ is not a strict local minimum; that is, suppose there exists a sequence $\{x_k\}$ converging to $\bar{x}$ such that $f(x_k) \leq f(\bar{x})$, $x_k \neq \bar{x}$, for each k. Considering this sequence, noting that $\nabla f(\bar{x}) = 0$ and $f(x_k) \leq f(\bar{x})$, and denoting $(x_k - \bar{x})/\|x_k - \bar{x}\|$ by d_k, (2.12) then implies that

$$\frac{1}{2}d_k^{\mathrm{T}}H(\bar{x})d_k + \alpha(\bar{x},x_k-\bar{x}) \leq 0 \qquad \text{for each}\,k. \tag{2.13}$$

But $\|d_k\| = 1$ for each k; and, hence, there exists an index set K such that $\{d_k\}_K$ converges to d, where $\|d\| = 1$. Considering this subsequence, and the fact that $\alpha(\bar{x},x_k-\bar{x}) \to 0$ as $k \in K$ approaches infinity, then (2.13) implies that $d^{\mathrm{T}}H(\bar{x})d \leq 0$. This contradicts the assumption that $H(\bar{x})$ is positive definite since $\|d\| = 1$. Therefore, $\bar{x}$ is indeed a strict local minimum. □

As is generally the case with optimisation problems, more powerful results exist under (generalised) convexity conditions. The following result shows that the necessary condition $\nabla f(\bar{x}) = 0$ is also sufficient for $\bar{x}$ to be a global minimum if f is pseudoconvex at $\bar{x}$.

Theorem 2.4.5 (Necessary and Sufficient Condition for Pseudoconvex Functions) *Let $f : \mathbb{R}^n \to \mathbb{R}$ be pseudoconvex at $\bar{x}$. Then $\bar{x}$ is a global minimum if and only if $\nabla f(\bar{x}) = 0$.*

Proof. By Corollary 2.4.2, if $\bar{x}$ is a global minimum then $\nabla f(\bar{x}) = 0$. Now, suppose that $\nabla f(\bar{x}) = 0$, so that $\nabla f(\bar{x})(x-\bar{x}) = 0$ for each $x \in \mathbb{R}^n$. By the pseudoconvexity of f at $\bar{x}$, it then follows that $f(x) \geq f(\bar{x})$ for each $x \in \mathbb{R}^n$, and the proof is complete. □

2.5 Constrained Optimisation

We now proceed to the main topic of interest in this book; namely, constrained optimisation. We first derive optimality conditions for a problem of the following form:

$$\begin{aligned} &\text{minimise } f(x), \\ &\text{subject to:} \\ &x \in S. \end{aligned} \tag{2.14}$$

We will first consider a general constraint set S. Later, the set S will be more explicitly defined by a set of equality and inequality constraints. For constrained optimisation problems we have the following definitions.

Definition 2.5.1 (Feasible and Optimal Solutions) *Let $f : \mathbb{R}^n \to \mathbb{R}$ and consider the constrained optimisation problem* (2.14)*, where S is a nonempty set in $\mathbb{R}^n$.*

- *A point $x \in S$ is called a* feasible solution *to problem* (2.14).
- *If $\bar{x} \in S$ and $f(x) \geq f(\bar{x})$ for each $x \in S$, then $\bar{x}$ is called an* optimal solution, *a* global optimal solution, *or simply a* solution *to the problem.*
- *The collection of optimal solutions is called the set of* alternative optimal solutions.
- *If $\bar{x} \in S$ and if there exists an ε-neighbourhood $N_\varepsilon(\bar{x})$ around $\bar{x}$ such that $f(x) \geq f(\bar{x})$ for each $x \in S \cap N_\varepsilon(\bar{x})$, then $\bar{x}$ is called a* local optimal solution.
- *If $\bar{x} \in S$ and if $f(x) > f(\bar{x})$ for each $x \in S \cap N_\varepsilon(\bar{x})$, $x \neq \bar{x}$, for some $\varepsilon > 0$, then $\bar{x}$ is called a* strict local optimal solution. ◦

Figure 2.12 illustrates examples of local and global minima for problem (2.14).

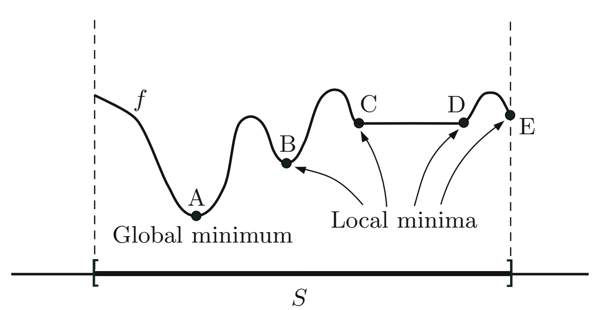


Figure 2.12. Local and global minima.

The function f and the constraint set S are shown in the figure. The points in S corresponding to A, B and E are also strict local minima, whereas those corresponding to the flat segment of the graph between C and D are local minima that are not strict.

In Chapter 13 we will treat a class of problems in which the constraint set S is not convex. However, in most of the book we will be concerned with problems in which the function f and set S in problem (2.14) are, respectively, a convex function and a convex set. Such a problem is known as a *convex programming problem*. The following result shows that each local minimum of a convex program is also a global minimum.

Theorem 2.5.1 (Local Minima of Convex Programs are Global Minima) *Consider problem* (2.14)*, where S is a nonempty convex set in $\mathbb{R}^n$, and $f : S \to \mathbb{R}$ is convex on S. If $\bar{x} \in S$ is a local optimal solution to the problem, then $\bar{x}$ is a global optimal solution. Furthermore, if either $\bar{x}$ is a strict local minimum, or if f is strictly convex, then $\bar{x}$ is the unique global optimal solution.*

Proof. Since $\bar{x}$ is a local optimal solution, there exists an ε-neighbourhood $N_\varepsilon(\bar{x})$ around $\bar{x}$ such that

$$f(x) \geq f(\bar{x}) \quad \text{for each } x \in S \cap N_\varepsilon(\bar{x}). \tag{2.15}$$

By contradiction, suppose that $\bar{x}$ is not a global optimal solution so that $f(\hat{x}) < f(\bar{x})$ for some $\hat{x} \in S$. By the convexity of f, we have that:

$$f(\lambda\hat{x} + (1-\lambda)\bar{x}) \leq \lambda f(\hat{x}) + (1-\lambda)f(\bar{x}) < \lambda f(\bar{x}) + (1-\lambda)f(\bar{x}) = f(\bar{x}),$$

for each $\lambda \in (0,1)$. But, for $\lambda > 0$ and sufficiently small, $\lambda\hat{x} + (1-\lambda)\bar{x} = \bar{x} + \lambda(\hat{x} - \bar{x}) \in S \cap N_\varepsilon(\bar{x})$. Hence, the above inequality contradicts (2.15), and we conclude that $\bar{x}$ is a global optimal solution.

Next, suppose that $\bar{x}$ is a strict local minimum. Then, as just proven, $\bar{x}$ is a global minimum. Now, suppose that $\bar{x}$ is not the unique global optimal solution. That is, suppose that there exist an $\hat{x} \in S$ such that $f(\hat{x}) = f(\bar{x})$. Then, defining $x_\lambda = \lambda\hat{x} + (1-\lambda)\bar{x}$ for $0 \leq \lambda \leq 1$, we have, by the convexity of f and S, that $f(x_\lambda) \leq \lambda f(\hat{x}) + (1-\lambda)f(\bar{x}) = f(\bar{x})$, and $x_\lambda \in S$ for all $0 \leq \lambda \leq 1$. By taking $\lambda \to 0^+$ we can make $x_\lambda \in N_\varepsilon(\bar{x}) \cap S$ for any $\varepsilon > 0$. However, this contradicts the strict local optimality of $\bar{x}$ and, hence, $\bar{x}$ is the unique global minimum.

Finally, suppose that $\bar{x}$ is a local optimal solution and that f is strictly convex. Since strict convexity implies convexity then, as proven earlier, $\bar{x}$ is a global optimal solution. By contradiction, suppose that $\bar{x}$ is not the unique global optimal solution so that there exists an $\tilde{x} \in S$, $\tilde{x} \neq \bar{x}$, such that $f(\tilde{x}) = f(\bar{x})$. By strict convexity, we have that $f(\frac{1}{2}\tilde{x} + \frac{1}{2}\bar{x}) < \frac{1}{2}f(\tilde{x}) + \frac{1}{2}f(\bar{x}) = f(\bar{x})$. Since S is convex, $\frac{1}{2}\tilde{x} + \frac{1}{2}\bar{x} \in S$, and the above inequality contradicts global optimality of $\bar{x}$. Hence, $\bar{x}$ is the unique global minimum, and this completes the proof. □

2.5.1 Geometric Necessary Optimality Conditions

In this section we give a necessary optimality condition for problem (2.14) using the *cone of feasible directions* defined below. Note that, in the sequel and in Sections 2.5.2– 2.5.4, we do not assume problem (2.14) to be a convex program. As a consequence of this generality, only *necessary* conditions for optimality will be derived. In a later section, Section 2.5.5, we will impose suitable convexity conditions to the problem in order to obtain sufficiency conditions for optimality.

Definition 2.5.2 (Cones of Feasible Directions and of Improving Directions) *Let S be a nonempty set in $\mathbb{R}^n$ and let $\bar{x} \in \operatorname{cl} S$. The* cone of feasible directions *of S at $\bar{x}$, denoted by D, is given by*

$$D = \{d : d \neq 0, \textit{ and } \bar{x} + \lambda d \in S \textit{ for all } \lambda \in (0, \delta) \textit{ for some } \delta > 0\}.$$

Each nonzero vector $d \in D$ is called a feasible direction*. Moreover, given a function $f : \mathbb{R}^n \to \mathbb{R}$, the* cone of improving directions *at $\bar{x}$, denoted by F, is given by*

$$F = \{d : f(\bar{x} + \lambda d) < f(\bar{x}) \textit{ for all } \lambda \in (0, \delta) \textit{ for some } \delta > 0\}.$$

Each direction $d \in F$ is called an improving direction*, or a* descent direction *of f at $\bar{x}$.* ◦

We will now consider the function f to be differentiable at the point $\bar{x}$. We can then define the sets

$$F_0 \triangleq \{d : \nabla f(\bar{x})d < 0\}, \tag{2.16}$$

$$F_0' \triangleq \{d \neq 0 : \nabla f(\bar{x})d \leq 0\}. \tag{2.17}$$

Observe that the set F_0 defined in (2.16) is an open half-space defined in terms of the gradient vector. Note also that, from Theorem 2.4.1, if $\nabla f(\bar{x})d < 0$, then d is an improving direction. It then follows that $F_0 \subseteq F$. Hence, the set F_0 gives an algebraic description of the set of improving directions F. Also, if $d \in F$, we must have $\nabla f(\bar{x})d \leq 0$, or else, analogous to Theorem 2.4.1, $\nabla f(\bar{x})d > 0$ would imply that d is an *ascent direction*. Hence, we have

$$F_0 \subseteq F \subseteq F_0'. \tag{2.18}$$

The following theorem states that a necessary condition for local optimality is that every improving direction in F_0 is not a feasible direction.

Theorem 2.5.2 (Geometric Necessary Condition for Local Optimality Using the Sets F_0 and D) *Consider the problem to minimise $f(x)$ subject to $x \in S$, where $f : \mathbb{R}^n \to \mathbb{R}$ and S is a nonempty set in $\mathbb{R}^n$. Suppose that f is differentiable at a point $\bar{x} \in S$. If $\bar{x}$ is a local optimal solution then $F_0 \cap D = \emptyset$, where $F_0 = \{d : \nabla f(\bar{x})d < 0\}$ and D is the cone of feasible directions of S at $\bar{x}$.*

Proof. Suppose, by contradiction, that there exists a vector $d \in F_0 \cap D$. Since $d \in F_0$, then, by Theorem 2.4.1, there exists a $\delta_1 > 0$ such that

$$f(\bar{x} + \lambda d) < f(\bar{x}) \quad \text{for each } \lambda \in (0, \delta_1). \tag{2.19}$$

Also, since $d \in D$, by Definition 2.5.2, there exists a $\delta_2 > 0$ such that

$$\bar{x} + \lambda d \in S \quad \text{for each } \lambda \in (0, \delta_2). \tag{2.20}$$

The assumption that $\bar{x}$ is a local optimal solution is not compatible with (2.19) and (2.20). Thus, $F_0 \cap D = \emptyset$. □

The necessary condition for local optimality of Theorem 2.5.2 is illustrated in Figure 2.13, where the vertices of the cones F_0 and D are translated from the

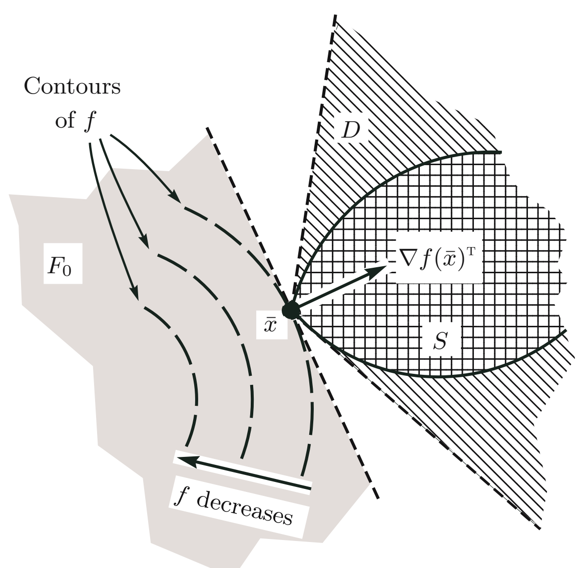


Figure 2.13. Illustration of the necessary condition for local optimality of Theorem 2.5.2: $F_0 \cap D = \emptyset$.

origin to $\bar{x}$ for convenience.

2.5.2 Problems with Inequality and Equality Constraints

We next consider a specific description for the feasible region S as follows:

$$S = \{x \in X : g_i(x) \leq 0,\ i = 1, \ldots, m,\ h_i(x) = 0,\ i = 1, \ldots, l\},$$

where $g_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 1, \ldots, m$, $h_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 1, \ldots, \ell$, and X is a nonempty open set in $\mathbb{R}^n$. This gives the following *nonlinear programming* problem with inequality and equality constraints:

$$\begin{aligned}
&\text{minimise } f(x),\\
&\text{subject to:}\\
&g_i(x) \le 0 \quad \text{for } i = 1, \ldots, m,\\
&h_i(x) = 0 \quad \text{for } i = 1, \ldots, \ell,\\
&x \in X.
\end{aligned} \tag{2.21}$$

The following theorem shows that if $\bar{x}$ is a local optimal solution to problem (2.21), then either the gradients of the equality constraints are linearly dependent at $\bar{x}$, or else $F_0 \cap G_0 \cap H_0 = \emptyset$, where F_0 is defined as in (2.16) and the sets G_0 and H_0 are defined in the statement of the theorem.

Theorem 2.5.3 (Geometric Necessary Condition for Problems with Inequality and Equality Constraints) *Let X be a nonempty open set in $\mathbb{R}^n$, and let $f : \mathbb{R}^n \to \mathbb{R}$, $g_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 1, \ldots, m$, $h_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 1, \ldots, \ell$. Consider the problem defined in* (2.21)*. Suppose that $\bar{x}$ is a local optimal solution, and let $I = \{i : g_i(\bar{x}) = 0\}$ be the index set for the* binding *or* active *constraints. Furthermore, suppose that each g_i for $i \notin I$ is continuous at $\bar{x}$, that f and g_i for $i \in I$ are differentiable at $\bar{x}$, and that each h_i for $i = 1, \ldots, \ell$ is continuously differentiable at $\bar{x}$. If $\nabla h_i(\bar{x})^{\mathrm{T}}$ for $i = 1, \ldots, \ell$ are linearly independent, then $F_0 \cap G_0 \cap H_0 = \emptyset$, where*

$$\begin{aligned}
F_0 &= \{d : \nabla f(\bar{x})d < 0\},\\
G_0 &= \{d : \nabla g_i(\bar{x})d < 0 \quad \textit{for } i \in I\},\\
H_0 &= \{d : \nabla h_i(\bar{x})d = 0 \quad \textit{for } i = 1, \ldots, \ell\}.
\end{aligned} \tag{2.22}$$

Proof. We use contradiction. Suppose there exists a vector $y \in F_0 \cap G_0 \cap H_0$; that is, $\nabla f(\bar{x})y < 0$, $\nabla g_i(\bar{x})y < 0$ for each $i \in I$, and $\nabla h(\bar{x})y = 0$, where $\nabla h(\bar{x})$ is the $\ell \times n$ Jacobian matrix whose ith row is $\nabla h_i(\bar{x})$. We now construct a feasible arc from $\bar{x}$. For $\lambda \ge 0$, define $\alpha : \mathbb{R} \to \mathbb{R}^n$ by the following differential equation and boundary condition:

$$\frac{d\alpha(\lambda)}{d\lambda} = \mathbf{P}(\lambda)y, \qquad \alpha(0) = \bar{x}, \tag{2.23}$$

where $\mathbf{P}(\lambda)$ is the matrix that projects any vector into the null space of $\nabla h(\alpha(\lambda))$. For sufficiently small λ, the above equation is well-defined and solvable because $\nabla h(\bar{x})$ has full row rank and h_i, $i = 1, \ldots, \ell$, are continuously differentiable at $\bar{x}$, so that $\mathbf{P}$ is continuous in λ. Obviously, $\alpha(\lambda) \to \bar{x}$ as $\lambda \to 0^+$.

We now show that for sufficiently small $\lambda > 0$, $\alpha(\lambda)$ is feasible and $f(\alpha(\lambda)) < f(\bar{x})$, thus contradicting local optimality of $\bar{x}$. By the chain rule of differentiation and using (2.23), we obtain

$$\frac{dg_i(\alpha(\lambda))}{d\lambda} = \nabla g_i(\alpha(\lambda))\mathbf{P}(\lambda)y, \tag{2.24}$$

for each $i \in I$. In particular, y is in the null space of $\nabla h(\bar{x})$, and so for $\lambda = 0$, we have $\mathbf{P}(0)y = y$. Hence, from (2.24) and the fact that $\nabla g_i(\bar{x})y < 0$, we obtain

$$\left.\frac{dg_i(\alpha(\lambda))}{d\lambda}\right|_{\lambda=0} = \nabla g_i(\bar{x})y < 0, \tag{2.25}$$

for $i \in I$. Recalling that $g_i(\alpha(0)) = g_i(\bar{x}) = 0$ for all $i \in I$, this and (2.25) further imply that $g_i(\alpha(\lambda)) < 0$ for sufficiently small $\lambda > 0$, and for each $i \in I$. For $i \notin I$, $g_i(\bar{x}) < 0$, and g_i is continuous at $\bar{x}$, and thus $g_i(\alpha(\lambda)) < 0$ for sufficiently small λ. By the mean value theorem (Theorem 2.2.1), we have

$$h_i(\alpha(\lambda)) = h_i(\alpha(0)) + \lambda \left.\frac{dh_i(\alpha(\lambda))}{d\lambda}\right|_{\lambda=\mu} = \lambda \left.\frac{dh_i(\alpha(\lambda))}{d\lambda}\right|_{\lambda=\mu}, \tag{2.26}$$

for some $\mu \in (0, \lambda)$. However, by the chain rule of differentiation and similarly to (2.24), we obtain

$$\left.\frac{dh_i(\alpha(\lambda))}{d\lambda}\right|_{\lambda=\mu} = \nabla h_i(\alpha(\mu))\mathbf{P}(\mu)y.$$

By construction, $\mathbf{P}(\mu)y$ is in the null space of $\nabla h_i(\alpha(\mu))$ and, hence, from the above equation we obtain $\left.\frac{dh_i(\alpha(\lambda))}{d\lambda}\right|_{\lambda=\mu} = 0$. Substituting in (2.26), it follows that $h_i(\alpha(\lambda)) = 0$ for all i. Also, since X is open, $\alpha(\lambda) \in X$ for sufficiently small λ.

We have, so far, established that the arc $\alpha(\lambda)$ defined by (2.23) is a feasible solution to the problem (2.21) for each sufficiently small $\lambda > 0$, since $g_i(\alpha(\lambda)) < 0$ for all $i = 1, \ldots, m$, $h_i(\alpha(\lambda)) = 0$ for all $i = 1, \ldots, \ell$, and $\alpha(\lambda) \in X$. To complete the proof by contradiction we next prove that such a feasible arc $\alpha(\lambda)$ would constitute an arc of improving solutions. By an argument similar to that leading to (2.25), we obtain

$$\left.\frac{df(\alpha(\lambda))}{d\lambda}\right|_{\lambda=0} = \nabla f(\bar{x})y < 0,$$

and, hence, $f(\alpha(\lambda)) < f(\bar{x})$ for sufficiently small $\lambda > 0$. This contradicts local optimality of $\bar{x}$. Hence, $F_0 \cap G_0 \cap H_0 = \emptyset$, and the proof is complete. □

2.5.3 The Fritz John Necessary Conditions

In this section we express the geometric optimality condition $F_0 \cap G_0 \cap H_0 = \emptyset$ of Theorem 2.5.3 in a more usable algebraic form known as the Fritz John conditions.

Theorem 2.5.4 (The Fritz John Necessary Conditions) *Let X be a nonempty open set in $\mathbb{R}^n$, and let $f : \mathbb{R}^n \to \mathbb{R}$, $g_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 1, \ldots, m$, $h_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 1, \ldots, \ell$. Consider the optimisation problem defined in (2.21). Let $\bar{x}$ be a feasible solution, and let $I = \{i : g_i(\bar{x}) = 0\}$. Furthermore, suppose that each g_i for $i \notin I$ is continuous at $\bar{x}$, that f and g_i for $i \in I$ are differentiable at $\bar{x}$, and that each h_i for $i = 1, \ldots, \ell$ is continuously differentiable at $\bar{x}$. If $\bar{x}$ locally solves problem (2.21), then there exist scalars u_0 and u_i for $i \in I$, and v_i for $i = 1, \ldots, \ell$, such that*

$$\begin{aligned} u_0 \nabla f(\bar{x})^{\mathrm{T}} + \sum_{i \in I} u_i \nabla g_i(\bar{x})^{\mathrm{T}} + \sum_{i=1}^{\ell} v_i \nabla h_i(\bar{x})^{\mathrm{T}} &= 0, \\ u_0, u_i \geq 0 \quad \text{for } i \in I&, \\ (u_0, u_I, v) \neq (0, 0, 0)&, \end{aligned} \tag{2.27}$$

where u_I and v are vectors whose components are u_i, $i \in I$, and v_i, $i = 1, \ldots, \ell$, respectively. Furthermore, if g_i, $i \notin I$ are also differentiable at $\bar{x}$, then the above conditions can be written as

$$\begin{aligned} u_0 \nabla f(\bar{x})^{\mathrm{T}} + \sum_{i=1}^{m} u_i \nabla g_i(\bar{x})^{\mathrm{T}} + \sum_{i=1}^{\ell} v_i \nabla h_i(\bar{x})^{\mathrm{T}} &= 0, \\ u_i g_i(\bar{x}) = 0 \quad \text{for } i = 1, \ldots, m&, \\ u_0, u_i \geq 0 \quad \text{for } i = 1, \ldots, m&, \\ (u_0, u, v) \neq (0, 0, 0)&, \end{aligned} \tag{2.28}$$

where u and v are vectors whose components are u_i, $i = 1, \ldots, m$, and v_i, $i = 1, \ldots, \ell$, respectively.

Proof. In the case where the vectors $\nabla h_i(\bar{x})^{\mathrm{T}}$ for $i = 1, \ldots, \ell$ are linearly dependent, then one can find scalars $v_1, \ldots, v_\ell$, not all zero, such that $\sum_{i=1}^{\ell} v_i \nabla h_i(\bar{x})^{\mathrm{T}} = 0$. Letting u_0 and u_i for $i \in I$ equal to zero, conditions (2.27) hold trivially.

Now suppose that $\nabla h_i(\bar{x})^{\mathrm{T}}$ for $i = 1, \ldots, \ell$ are linearly independent. Then, from Theorem 2.5.3, local optimality of $\bar{x}$ implies that the sets defined in (2.22) satisfy:

$$F_0 \cap G_0 \cap H_0 = \emptyset. \tag{2.29}$$

Let A_1 be the matrix whose rows are $\nabla f(\bar{x})$ and $\nabla g_i(\bar{x})$ for $i \in I$, and let A_2 be the matrix whose rows are $\nabla h_i(\bar{x})$ for $i = 1, \ldots, \ell$. Then, it is easy to see that condition (2.29) is satisfied if and only if the system:

$$\begin{aligned} A_1 d &< 0, \\ A_2 d &= 0, \end{aligned}$$

is inconsistent. Now consider the following two sets:

$$S_1 = \{(z_1, z_2) : z_1 = A_1 d,\ z_2 = A_2 d,\ d \in \mathbb{R}^n\},$$
$$S_2 = \{(z_1, z_2) : z_1 < 0,\ z_2 = 0\}.$$

Note that S_1 and S_2 are nonempty convex sets and, since the system $A_1 d < 0$, $A_2 d = 0$ has no solution, then $S_1 \cap S_2 = \emptyset$. Then, by Theorem 2.3.5, there exists a nonzero vector $p^{\mathrm{T}} = (p_1^{\mathrm{T}}, p_2^{\mathrm{T}})$ such that

$$p_1^{\mathrm{T}} A_1 d + p_2^{\mathrm{T}} A_2 d \geq p_1^{\mathrm{T}} z_1 + p_2^{\mathrm{T}} z_2,$$

for each $d \in \mathbb{R}^n$ and $(z_1, z_2) \in \operatorname{cl} S_2$. Noting that $z_2 = 0$ and since each component of z_1 can be made an arbitrarily large negative number, it follows that $p_1 \geq 0$. Also, letting $(z_1, z_2) = (0, 0) \in \operatorname{cl} S_2$, we must have $(p_1^{\mathrm{T}} A_1 + p_2^{\mathrm{T}} A_2) d \geq 0$ for each $d \in \mathbb{R}^n$. Letting $d = -(A_1^{\mathrm{T}} p_1 + A_2^{\mathrm{T}} p_2)$, it follows that $-\|A_1^{\mathrm{T}} p_1 + A_2^{\mathrm{T}} p_2\|^2 \geq 0$, and thus $A_1^{\mathrm{T}} p_1 + A_2^{\mathrm{T}} p_2 = 0$. Summarising, we have found a nonzero vector $p^{\mathrm{T}} = (p_1^{\mathrm{T}}, p_2^{\mathrm{T}})$ with $p_1 \geq 0$ such that $A_1^{\mathrm{T}} p_1 + A_2^{\mathrm{T}} p_2 = 0$. Denoting the components of p_1 by u_0 and u_i, $i \in I$, and letting $p_2 = v$, conditions (2.27) follow. The equivalent form (2.28) is readily obtained by letting $u_i = 0$ for $i \notin I$, and the proof is complete. □

In the Fritz John conditions (2.28) the scalars u_0, u_i for $i = 1, \ldots, m$, and v_i for $i = 1, \ldots, \ell$, are called the *Lagrange multipliers* associated, respectively, with the objective function, the inequality constraints $g_i(x) \leq 0$, $i = 1, \ldots, m$, and the equality constraints $h_i(x) = 0$, $i = 1, \ldots, \ell$. Observe that the v_i are *unrestricted* in sign. The condition that $\bar{x}$ be feasible for the optimisation problem (2.21) is called the *primal feasibility* [PF] condition. The requirements $u_0 \nabla f(\bar{x})^{\mathrm{T}} + \sum_{i=1}^{m} u_i \nabla g_i(\bar{x})^{\mathrm{T}} + \sum_{i=1}^{\ell} v_i \nabla h_i(\bar{x})^{\mathrm{T}} = 0$, with $u_0, u_i \geq 0$ for $i = 1, \ldots, m$, and $(u_0, u, v) \neq (0, 0, 0)$ are called the *dual feasibility* [DF] conditions. The condition $u_i g_i(\bar{x}) = 0$ for $i = 1, \ldots, m$ is called the *complementary slackness* [CS] condition; it requires that $u_i = 0$ if the corresponding inequality is nonbinding (that is, $g_i(\bar{x}) < 0$), and allows for $u_i > 0$ only for those constraints that are binding. Together, the PF, DF and CS conditions are called the *Fritz John* [FJ] *optimality conditions*. Any point $\bar{x}$ for which there exist Lagrange multipliers $\bar{u}_0$, $\bar{u}_i$, $i = 1, \ldots, m$, $\bar{v}_i$, $i = 1, \ldots, \ell$, such that the FJ conditions are satisfied is called an *FJ point*.

The FJ conditions can also be written in vector form as follows:

$$\begin{aligned} \nabla f(\bar{x})^{\mathrm{T}} u_0 + \nabla g(\bar{x})^{\mathrm{T}} u + \nabla h(\bar{x})^{\mathrm{T}} v &= 0, \\ u^{\mathrm{T}} g(\bar{x}) &= 0, \\ (u_0, u) &\geq (0, 0), \\ (u_0, u, v) &\neq (0, 0, 0), \end{aligned} \tag{2.30}$$

where $\nabla g(\bar{x})$ is the $m \times n$ Jacobian matrix whose ith row is $\nabla g_i(\bar{x})$, $\nabla h(\bar{x})$ is the $\ell \times n$ Jacobian matrix whose ith row is $\nabla h_i(\bar{x})$, and $g(\bar{x})$ is the m vector function whose ith component is $g_i(\bar{x})$. Also, u and v are, respectively, an m

vector and an ℓ vector, whose elements are the Lagrange multipliers associated with, respectively, the inequality and equality constraints.

At this point it is important to note that, given an optimisation problem, there might be points that satisfy the FJ conditions trivially. For example, if a feasible point $\bar{x}$ (not necessarily an optimum) satisfies $\nabla f(\bar{x}) = 0$, or $\nabla g_i(\bar{x}) = 0$ for some $i \in I$, or $\nabla h_i(\bar{x}) = 0$ for some $i = 1, \ldots, \ell$, then we can let the corresponding Lagrange multiplier be any positive number, set all the other multipliers equal to zero, and satisfy conditions (2.27). In fact, given *any* feasible solution $\bar{x}$ we can always add a redundant constraint to the problem to make $\bar{x}$ an FJ point. For example, we can add the constraint $\|x - \bar{x}\|^2 \geq 0$, which holds true for all $x \in \mathbb{R}^n$, is a binding constraint at $\bar{x}$ and whose gradient is zero at $\bar{x}$.

2.5.4 Karush–Kuhn–Tucker Necessary Conditions

In the previous section we stated the FJ necessary conditions for optimality. We saw that the FJ conditions relate to the existence of scalars $u_0, u_i \geq 0$ and v_i, not all zero, such that the conditions (2.27) are satisfied. We also saw that there are instances where there are points that satisfy the conditions trivially, for example, when the gradient of some binding constraint (which might even be redundant) vanishes.

It is also possible that, at some feasible point $\bar{x}$, the FJ conditions (2.27) are satisfied with Lagrange multiplier associated with the objective function $u_0 = 0$. In such cases, the FJ conditions become virtually useless since the objective function gradient does not play a role in the optimality conditions (2.27) and the conditions merely state that the gradients of the binding inequality constraints and of the equality constraints are linearly dependent. Thus, when $u_0 = 0$, the FJ conditions are of no practical value in locating an optimal point. Under suitable assumptions, referred to as *constraint qualifications* [CQ], u_0 is guaranteed to be positive and the FJ conditions become the Karush–Kuhn–Tucker [KKT] conditions, which will be presented next. There exist various constraint qualifications for problems with inequality and equality constraints. Here, we use a typical constraint qualification that requires that the gradients of the inequality constraints for $i \in I$ and the gradients of the equality constraints at $\bar{x}$ be linearly independent.

Theorem 2.5.5 (Karush–Kuhn–Tucker Necessary Conditions) *Let X be a nonempty open set in $\mathbb{R}^n$, and let $f : \mathbb{R}^n \to \mathbb{R}$, $g_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 1, \ldots, m$, $h_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 1, \ldots, \ell$. Consider the problem defined in* (2.21). *Let $\bar{x}$ be a feasible solution, and let $I = \{i : g_i(\bar{x}) = 0\}$. Suppose that f and g_i for $i \in I$ are differentiable at $\bar{x}$, that each g_i for $i \notin I$ is continuous at $\bar{x}$, and that each h_i for $i = 1, \ldots, \ell$ is continuously differentiable at $\bar{x}$. Furthermore, suppose that $\nabla g_i(\bar{x})^{\mathrm{T}}$ for $i \in I$ and $\nabla h_i(\bar{x})^{\mathrm{T}}$ for $i = 1, \ldots, \ell$ are linearly independent. If $\bar{x}$ is a local optimal solution, then there exist unique scalars u_i for $i \in I$, and v_i for $i = 1, \ldots, \ell$, such that*

$$\nabla f(\bar{x})^{\mathrm{T}} + \sum_{i \in I} u_i \nabla g_i(\bar{x})^{\mathrm{T}} + \sum_{i=1}^{\ell} v_i \nabla h_i(\bar{x})^{\mathrm{T}} = 0, \qquad u_i \geq 0 \quad \textit{for } i \in I. \tag{2.31}$$

Furthermore, if g_i, $i \notin I$ *are also differentiable at* $\bar{x}$, *then the above conditions can be written as*

$$\begin{aligned} \nabla f(\bar{x})^{\mathrm{T}} + \sum_{i=1}^{m} u_i \nabla g_i(\bar{x})^{\mathrm{T}} + \sum_{i=1}^{\ell} v_i \nabla h_i(\bar{x})^{\mathrm{T}} &= 0, \\ u_i g_i(\bar{x}) = 0 \quad &\textit{for } i = 1, \dots, m, \\ u_i \geq 0 \quad &\textit{for } i = 1, \dots, m. \end{aligned} \tag{2.32}$$

Proof. We have, from the FJ conditions (Theorem 2.5.4), that there exist scalars $\hat{u}_0$ and $\hat{u}_i$, $i \in I$, and $\hat{v}_i$, $i = 1, \dots, \ell$, not all zero, such that

$$\hat{u}_0 \nabla f(\bar{x})^{\mathrm{T}} + \sum_{i \in I} \hat{u}_i \nabla g_i(\bar{x})^{\mathrm{T}} + \sum_{i=1}^{\ell} \hat{v}_i \nabla h_i(\bar{x})^{\mathrm{T}} = 0, \qquad \hat{u}_0, \hat{u}_i \geq 0 \quad \text{for } i \in I. \tag{2.33}$$

Note that the assumption of linear independence of $\nabla g_i(\bar{x})^{\mathrm{T}}$ for $i \in I$ and $\nabla h_i(\bar{x})^{\mathrm{T}}$ for $i = 1, \dots, \ell$, together with (2.33) and the fact that at least one of the multipliers is nonzero, implies that $\hat{u}_0 > 0$. Then, letting $u_i = \hat{u}_i / \hat{u}_0$ for $i \in I$, and $v_i = \hat{v}_i / \hat{u}_0$ for $i = 1, \dots, \ell$ we obtain conditions (2.31). Furthermore, the linear independence assumption implies the uniqueness of these Lagrange multipliers. The equivalent form (2.32) follows by letting $u_i = 0$ for $i \notin I$. This completes the proof. □

As in the FJ conditions, the scalars u_i and v_i are called the *Lagrange multipliers*. Observe that the v_i are *unrestricted* in sign. The condition that $\bar{x}$ be feasible for the optimisation problem (2.21) is called the *primal feasibility* [PF] condition. The requirement that $\nabla f(\bar{x})^{\mathrm{T}} + \sum_{i=1}^{m} u_i \nabla g_i(\bar{x})^{\mathrm{T}} + \sum_{i=1}^{\ell} v_i \nabla h_i(\bar{x})^{\mathrm{T}} = 0$, with $u_i \geq 0$ for $i = 1, \dots, m$ is called the *dual feasibility* [DF] condition. The condition $u_i g_i(\bar{x}) = 0$ for $i = 1, \dots, m$ is called the *complementary slackness* [CS] condition; it requires that $u_i = 0$ if the corresponding inequality is nonbinding (that is, $g_i(\bar{x}) < 0$), and it permits $u_i > 0$ only for those constraints that are binding. Together, the PF, DF and CS conditions are called the Karush–Kuhn–Tucker [KKT] optimality conditions. Any point $\bar{x}$ for which there exist Lagrange multipliers $\bar{u}_i$, $i = 1, \dots, m$, $\bar{v}_i$, $i = 1, \dots, \ell$, that, together with $\bar{x}$, satisfy the KKT conditions is called a *KKT point*.

The KKT conditions can also be written in vector form as follows:

$$\begin{aligned} \nabla f(\bar{x})^{\mathrm{T}} + \nabla g(\bar{x})^{\mathrm{T}} u + \nabla h(\bar{x})^{\mathrm{T}} v = 0, \\ u^{\mathrm{T}} g(\bar{x}) = 0, \\ u \geq 0, \end{aligned} \tag{2.34}$$

where $\nabla g(\bar{x})$ is the $m \times n$ Jacobian matrix whose ith row is $\nabla g_i(\bar{x})$, $\nabla h(\bar{x})$ is the $\ell \times n$ Jacobian matrix whose ith row is $\nabla h_i(\bar{x})$, and $g(\bar{x})$ is the m vector function whose ith component is $g_i(\bar{x})$. Also, u and v are, respectively, an m vector and an ℓ vector, whose elements are the Lagrange multipliers associated with, respectively, the inequality and equality constraints.

2.5.5 Karush–Kuhn–Tucker Sufficient Conditions

In the previous section we derived the KKT necessary conditions for optimality from the FJ optimality conditions. This derivation was done by asserting that the multiplier associated with the objective function is positive at a local optimum whenever a linear independence constraint qualification is satisfied. It is important to notice that the linear independence constraint qualification is only a *sufficient condition*[3] placed on the behaviour of the constraints to ensure that an FJ point (and hence, from Theorem 2.5.4, any local optimum) be a KKT point. Thus, the importance of the constraint qualifications is to guarantee that, by examining only KKT points, we do not lose out on optimal solutions. There is an important special case; namely, when the constraints are linear, in which case the KKT conditions are always necessary optimality conditions irrespective of the behaviour of the objective function. (Although we will not prove this result here, it comes from the fact that a more general constraint qualification to that of linear independence, known as *Abadie's constraint qualification*—see Abadie 1967—is automatically satisfied when the constraints are linear.) However, we are still left with the problem of determining, among all the points that satisfy the KKT conditions, which ones constitute local optimal solutions. The following result shows that, under moderate convexity assumptions, the KKT conditions are also sufficient for local optimality.

Theorem 2.5.6 (Karush–Kuhn–Tucker Sufficient Conditions) *Let X be a nonempty open set in $\mathbb{R}^n$, and let $f : \mathbb{R}^n \to \mathbb{R}$, $g_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 1, \ldots, m$, $h_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 1, \ldots, \ell$. Consider the problem defined in* (2.21)*. Let $\bar{x}$ be a feasible solution, and let $I = \{i : g_i(\bar{x}) = 0\}$. Suppose that the KKT conditions hold at $\bar{x}$; that is, there exist scalars $\bar{u}_i \geq 0$ for $i \in I$, and $\bar{v}_i$ for $i = 1, \ldots, \ell$, such that*

$$\nabla f(\bar{x})^{\mathrm{T}} + \sum_{i \in I} \bar{u}_i \nabla g_i(\bar{x})^{\mathrm{T}} + \sum_{i=1}^{\ell} \bar{v}_i \nabla h_i(\bar{x})^{\mathrm{T}} = 0. \tag{2.35}$$

[3] It is possible, in some optimisation problems, for a local optimum to be a KKT point and yet not satisfy the linear independence constraint qualification.

Let $J = \{i : \bar{v}_i > 0\}$ and $K = \{i : \bar{v}_i < 0\}$. Further, suppose that f is pseudoconvex at $\bar{x}$, g_i is quasiconvex at $\bar{x}$ for $i \in I$, h_i is quasiconvex at $\bar{x}$ for $i \in J$, and h_i is quasiconcave at $\bar{x}$ (that is, $-h_i$ is quasiconvex at $\bar{x}$) for $i \in K$. Then $\bar{x}$ is a global optimal solution to problem (2.21). In particular, if these generalised convexity assumptions on the objective and constraint functions are restricted to the domain $N_\varepsilon(\bar{x})$ for some $\varepsilon > 0$, then $\bar{x}$ is a local minimum for problem (2.21).

Proof. Let x be any feasible solution to problem (2.21). (In the case where we need to restrict the domain to $N_\varepsilon(\bar{x})$, then let x be a feasible solution to problem (2.21) that also lies within $N_\varepsilon(\bar{x})$.) Then, for $i \in I$, $g_i(x) \leq g_i(\bar{x})$, since $g_i(x) \leq 0$ and $g_i(\bar{x}) = 0$. By the quasiconvexity of g_i at $\bar{x}$ it follows that

$$g_i(\bar{x} + \lambda(x - \bar{x})) = g_i(\lambda x + (1 - \lambda)\bar{x}) \leq \max\{g_i(x), g_i(\bar{x})\} = g_i(\bar{x}),$$

for all $\lambda \in (0, 1)$. This implies that g_i does not increase by moving from $\bar{x}$ along the direction $x - \bar{x}$. Thus, by an analogous result to that of Theorem 2.4.1, we must have

$$\nabla g_i(\bar{x})(x - \bar{x}) \leq 0 \qquad \text{for } i \in I. \tag{2.36}$$

Similarly, since h_i is quasiconvex at $\bar{x}$ for $i \in J$ and h_i is quasiconcave at $\bar{x}$ for $i \in K$, we have

$$\nabla h_i(\bar{x})(x - \bar{x}) \leq 0 \qquad \text{for } i \in J, \tag{2.37}$$

$$\nabla h_i(\bar{x})(x - \bar{x}) \geq 0 \qquad \text{for } i \in K. \tag{2.38}$$

Multiplying (2.36), (2.37) and (2.38) by $\bar{u}_i \geq 0$, $\bar{v}_i > 0$, and $\bar{v}_i < 0$, respectively, and adding the terms, we obtain

$$\sum_{i \in I} \bar{u}_i \nabla g_i(\bar{x})(x - \bar{x}) + \sum_{i \in J \cup K} \bar{v}_i \nabla h_i(\bar{x})(x - \bar{x}) \leq 0. \tag{2.39}$$

Transposing (2.35), multiplying by $(x - \bar{x})$ and noting that $\bar{v}_i = 0$ for $i \notin J \cup K$, then (2.39) implies that

$$\nabla f(\bar{x})(x - \bar{x}) \geq 0.$$

By the pseudoconvexity of f at $\bar{x}$, we must have $f(x) \geq f(\bar{x})$, and the proof is complete. □

An important point to note is that, despite the sufficiency of the KKT conditions under the generalised convexity assumptions of Theorem 2.5.6, the KKT conditions are *not necessary* for *optimality* for these problems. (This situation, however, only arises when the constraint qualification does not hold at a local optimal solution, and hence the local solution is not captured by the KKT conditions.)

2.5.6 Quadratic Programs

Quadratic programs represent a special class of nonlinear programs in which the objective function is quadratic and the constraints are linear. Thus, a quadratic programming [QP] problem can be written as

$$\begin{aligned}&\text{minimise } \frac{1}{2}x^{\mathrm{T}}Hx + x^{\mathrm{T}}c, \\ &\text{subject to:} \\ &A_I^{\mathrm{T}}x \leq b_I, \\ &A_E^{\mathrm{T}}x = b_E,\end{aligned} \tag{2.40}$$

where H is an $n \times n$ matrix, c is an n vector, A_I is an $n \times m_I$ matrix, b_I is an m_I vector, A_E is an $n \times m_E$ matrix and b_E is an m_E vector.

As mentioned in the previous section, since the constraints are linear we have that a constraint qualification, known as Abadie's constraint qualification, is automatically satisfied and, hence, a local minimum $\bar{x}$ is necessarily a KKT point. Also, since the constraints are linear, the constraint set $S = \{x : A_I^{\mathrm{T}}x \leq b_I,\, A_E^{\mathrm{T}}x = b_E\}$ is a (polyhedral) convex set. Thus, the QP problem (2.40) is a convex program if and only if the objective function is convex; that is, if and only if H is symmetric and positive semidefinite. In this case we have, from Theorem 2.5.1, that $\bar{x}$ is a local minimum if and only if $\bar{x}$ is a global minimum. And, from Theorems 2.5.5 and 2.5.6 (and from the automatic fulfilment of Abadie's constraint qualification), we have that the above is true if and only if $\bar{x}$ is a KKT point. Furthermore, if H is positive definite, then we have that the objective function is strictly convex and we can conclude from Theorem 2.5.1 that $\bar{x}$ is the unique global minimum for problem (2.40).

The KKT conditions (2.34) for the QP problem defined in (2.40) are:

$$\begin{aligned}\text{PF:} \quad && A_I^{\mathrm{T}}\bar{x} &\leq b_I, \\ && A_E^{\mathrm{T}}\bar{x} &= b_E, \\ \text{DF:} \quad && H\bar{x} + c + A_I u + A_E v &= 0, \\ && u &\geq 0, \\ \text{CS:} \quad && u^{\mathrm{T}}(A_I^{\mathrm{T}}\bar{x} - b_I) &= 0,\end{aligned} \tag{2.41}$$

where u is an m_I vector of Lagrange multipliers corresponding to the inequality constraints and v is an m_E vector of Lagrange multipliers corresponding to the equality constraints.

2.6 Lagrangian Duality

In this section we present the concept of *Lagrangian duality*. Given a nonlinear programming problem, known as the *primal problem*, there exists another nonlinear programming problem, closely related to it, that receives the name of the *Lagrangian dual problem.* As we will see later in Section 2.6.3, under certain convexity assumptions and suitable constraint qualifications, the primal and dual problems have equal optimal objective values.

2.6.1 The Lagrangian Dual Problem

We will first define the primal and dual problems as separate optimisation problems. Later we will see that these two problems are closely related.

Thus, first consider the following nonlinear programming problem, called the *primal problem.*

Primal Problem P

$$\begin{aligned}&\text{minimise } f(x), \\ &\text{subject to:} \\ &g_i(x) \le 0 \quad \text{for } i = 1, \dots, m, \\ &h_i(x) = 0 \quad \text{for } i = 1, \dots, \ell, \\ &x \in X.\end{aligned} \tag{2.42}$$

Then the *Lagrangian dual problem* is defined as the following nonlinear programming problem.

Lagrangian Dual Problem D

$$\begin{aligned}&\text{maximise } \theta(u, v), \\ &\text{subject to:} \\ &u \ge 0,\end{aligned} \tag{2.43}$$

where

$$\theta(u, v) = \inf\{f(x) + \sum_{i=1}^{m} u_i g_i(x) + \sum_{i=1}^{\ell} v_i h_i(x) : x \in X\} \tag{2.44}$$

is the *Lagrangian dual function.*

In the dual problem (2.43)–(2.44), the vectors u and v have as their components the Lagrange multipliers u_i for $i = 1, \dots, m$, and v_i for $i = 1, \dots, \ell$. Note that the Lagrange multipliers u_i, corresponding to the inequality constraints $g_i(x) \le 0$, are restricted to be nonnegative, whereas the Lagrange multipliers v_i, corresponding to the equality constraints $h_i(x) = 0$, are unrestricted in sign.

Given the primal problem P (2.42), several Lagrangian dual problems D of the form of (2.43)–(2.44) can be devised, depending on which constraints are handled as $g_i(x) \le 0$ and $h_i(x) = 0$, and which constraints are handled by the set X. Hence, an appropriate selection of the set X must be made, depending on the nature of the problem and the goal of formulating or solving the dual problem D.

The primal and dual problems can also be written in *vector* form. Consider the function $f : \mathbb{R}^n \to \mathbb{R}$ and the vector functions $g : \mathbb{R}^n \to \mathbb{R}^m$ and $h : \mathbb{R}^n \to \mathbb{R}^\ell$, whose ith components are g_i and h_i, respectively. Then, we can write:

Primal Problem P

$$\begin{aligned} &\text{minimise } f(x), \\ &\text{subject to:} \\ &g(x) \leq 0, \\ &h(x) = 0, \\ &x \in X. \end{aligned} \tag{2.45}$$

Lagrangian Dual Problem D

$$\begin{aligned} &\text{maximise } \theta(u, v), \\ &\text{subject to:} \\ &u \geq 0, \end{aligned} \tag{2.46}$$

where $\theta(u, v) = \inf\{f(x) + u^{\mathrm{T}} g(x) + v^{\mathrm{T}} h(x) : x \in X\}$.

The relationship between the primal and dual problems will be explored below.

2.6.2 Geometric Interpretation of the Lagrangian Dual

An interesting geometric interpretation of the dual problem can be made by considering a simpler problem with only one inequality constraint and no equality constraint. Consider the following primal problem P:

Primal Problem P

$$\begin{aligned} &\text{minimise } f(x), \\ &\text{subject to:} \\ &g(x) \leq 0, \\ &x \in X, \end{aligned} \tag{2.47}$$

where $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$, and define the following set in $\mathbb{R}^2$:

$$G = \{(y, z) : y = g(x), z = f(x) \text{ for some } x \in X\},$$

that is, G is the image of X under the (g, f) map. Figure 2.14 shows an example of the set G. Then, the primal problem consists of finding a point in G with $y \leq 0$ that has minimum ordinate z. Obviously this point in Figure 2.14 is $(\bar{y}, \bar{z})$.

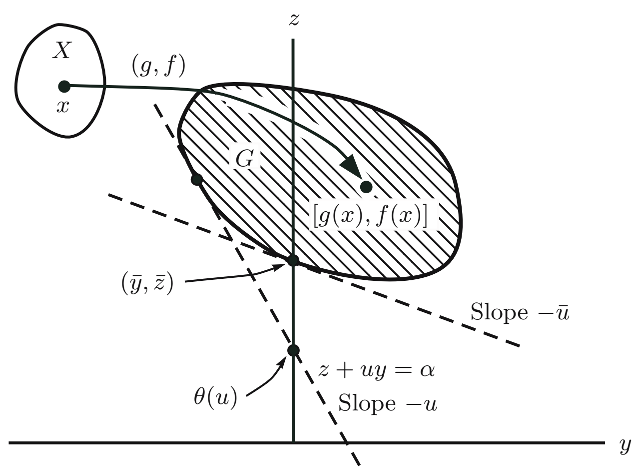


Figure 2.14. Geometric interpretation of Lagrangian duality: case with no duality gap.

Now, consider the Lagrangian dual problem D:

Lagrangian Dual Problem D

$$\begin{aligned} &\text{maximise } \theta(u), \\ &\text{subject to:} \\ &u \geq 0. \end{aligned} \tag{2.48}$$

The solution of the Lagrangian dual problem (2.48) requires one to first solve the following *Lagrangian dual subproblem*:

$$\theta(u) = \inf\{f(x) + ug(x) : x \in X\}. \tag{2.49}$$

Given $u \geq 0$, problem (2.49) is equivalent to minimise $z + uy$ over points (y, z) in G. Note that $z + uy = \alpha$ is the equation of a straight line with slope $-u$ that intercepts the z-axis at α. Thus, in order to minimise $z + uy$ over G we need to move the line $z + uy = \alpha$ parallel to itself as far down as possible, whilst it remains in contact with G. The last intercept on the z-axis thus obtained is the value of $\theta(u)$ corresponding to the given $u \geq 0$, as shown in Figure 2.14. Finally, to solve the dual problem (2.48), we have to find the line with slope $-u$ ($u \geq 0$) such that the last intercept on the z-axis, $\theta(u)$, is maximal. Such a line is shown in Figure 2.14. It has slope $-\bar{u}$ and supports the set G (recall Definition 2.3.4) at the point $(\bar{y}, \bar{z})$. Thus, the solution to the dual problem (2.48) is $\bar{u}$, and the optimal dual objective value is $\bar{z}$. It can be seen that, in the example illustrated in Figure 2.14, the optimal primal and dual objective values are equal. In such cases, it is said that there is no *duality gap*. In the next section we will develop conditions such that no duality gap exists.

2.6.3 Weak and Strong Duality

In this section we explore the relationships between the primal problem P and its Lagrangian dual problem D. In particular, we are interested in the conditions that the primal problem P must satisfy for the primal and dual objective values to be equal; this situation is known as *strong duality*. The first result shows that the objective value of any feasible solution to the dual problem constitutes a lower bound for the objective value of any feasible solution to the primal problem.

Theorem 2.6.1 (Weak Duality Theorem) *Consider the primal problem* P *given by* (2.45) *and its Lagrangian dual problem* D *given by* (2.46). *Let* x *be a feasible solution to* P*; that is,* $x \in X$*,* $g(x) \le 0$*, and* $h(x) = 0$*. Also, let* (u, v) *be a feasible solution to* D*; that is,* $u \ge 0$*. Then:*

$$f(x) \ge \theta(u, v).$$

Proof. We use the definition of θ given in (2.44), and the facts that $x \in X$, $u \ge 0$, $g(x) \le 0$ and $h(x) = 0$. We then have

$$\begin{aligned}\theta(u, v) &= \inf\{f(\tilde{x}) + u^{\mathrm{T}} g(\tilde{x}) + v^{\mathrm{T}} h(\tilde{x}) : \tilde{x} \in X\} \\ &\le f(x) + u^{\mathrm{T}} g(x) + v^{\mathrm{T}} h(x) \le f(x),\end{aligned}$$

and the result follows. □

Corollary 2.6.2

$$\inf\{f(x) : x \in X, g(x) \le 0, h(x) = 0\} \ge \sup\{\theta(u, v) : u \ge 0\}. \tag{2.50}$$

◦

Note from (2.50) that the optimal objective value of the primal problem is greater than or equal to the optimal objective value of the dual problem. If (2.50) holds as a *strict* inequality, then it is said that there exists a *duality gap*. Figure 2.15 shows an example for the primal and dual problems defined in (2.47) and (2.48)–(2.49), respectively. Notice that, in the case shown in the figure, there exists a duality gap. We see, by comparing Figure 2.15 with Figure 2.14, that the presence of a duality gap is due to the nonconvexity of the set G. As we will see in Theorem 2.6.4 below, if some suitable convexity conditions are satisfied, then there is no duality gap between the primal and dual optimisation problems. Before stating the conditions that guarantee the absence of a duality gap, we need the following result.

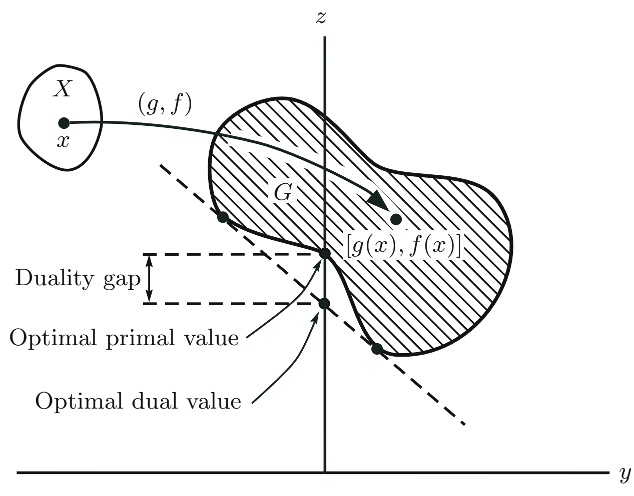


Figure 2.15. Geometric interpretation of Lagrangian duality: case with duality gap.

Lemma 2.6.3 *Let X be a nonempty convex set in $\mathbb{R}^n$. Let $\alpha : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}^m$ be convex,*[4] *and $h : \mathbb{R}^n \to \mathbb{R}^\ell$ be affine (that is, assume h is of the form $h(x) = Ax - b$). Also, let u_0 be a scalar, $u \in \mathbb{R}^m$ and $v \in \mathbb{R}^\ell$. Consider the following two systems:*

System 1: $\alpha(x) < 0, \quad g(x) \leq 0, \quad h(x) = 0 \quad$ *for some* $x \in X$.

System 2: $u_0\alpha(x) + u^{\mathrm{T}}g(x) + v^{\mathrm{T}}h(x) \geq 0$ *for some* $(u_0, u, v) \neq (0, 0, 0)$, $(u_0, u) \geq (0, 0)$ *and for all* $x \in X$.

If System 1 has no solution x, then System 2 has a solution (u_0, u, v). Conversely, if System 2 has a solution (u_0, u, v) with $u_0 > 0$, then System 1 has no solution.

Proof. Assume first that System 1 has no solution. Define the set:

$$S = \{(p, q, r) : p > \alpha(x), q \geq g(x), r = h(x) \quad \text{for some } x \in X\}.$$

The reader can easily verify that, since X, α and g are convex and h is affine, the set S is convex. Since System 1 has no solution, we have that $(0, 0, 0) \notin S$. We then have, from Corollary 2.3.4, that there exists a nonzero vector (u_0, u, v) such that

$$(u_0, u, v)^{\mathrm{T}}[(p, q, r) - (0, 0, 0)] = u_0 p + u^{\mathrm{T}}q + v^{\mathrm{T}}r \geq 0, \tag{2.51}$$

[4] That is, each component of the vector-valued function g is a convex function.

for each $(p,q,r) \in \operatorname{cl} S$. Now, fix an $x \in X$. Noticing, from the definition of S, that p and q can be made arbitrarily large, we have that in order to satisfy (2.51), we must have $u_0 \geq 0$ and $u \geq 0$. Also, note that $[\alpha(x), g(x), h(x)] \in \operatorname{cl} S$ and we have from (2.51) that

$$u_0\alpha(x) + u^{\mathrm{T}} g(x) + v^{\mathrm{T}} h(x) \geq 0.$$

Since the above inequality is true for each $x \in X$, System 2 has a solution.

To prove the converse, assume that System 2 has a solution (u_0, u, v) such that $u_0 > 0$ and $u \geq 0$, and $u_0\alpha(x) + u^{\mathrm{T}} g(x) + v^{\mathrm{T}} h(x) \geq 0$ for each $x \in X$. Suppose that $x \in X$ is such that $g(x) \leq 0$ and $h(x) = 0$. From the previous inequality we conclude that $u_0\alpha(x) \geq -u^{\mathrm{T}} g(x) \geq 0$, since $u \geq 0$ and $g(x) \leq 0$. But, since $u_0 > 0$, we must then have that $\alpha(x) \geq 0$. Hence, System 1 has no solution. This completes the proof. □

The following result, known as the *strong duality theorem*, shows that, under suitable convexity assumptions and under a constraint qualification, there is no *duality gap* between the primal and dual optimal objective function values.

Theorem 2.6.4 (Strong Duality Theorem) *Let X be a nonempty convex set in $\mathbb{R}^n$. Let $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}^m$ be convex, and $h : \mathbb{R}^n \to \mathbb{R}^\ell$ be affine. Suppose that the following constraint qualification is satisfied. There exists an $\hat{x} \in X$ such that $g(\hat{x}) < 0$ and $h(\hat{x}) = 0$, and $0 \in \operatorname{int} h(X)$, where $h(X) = \{h(x) : x \in X\}$. Then,*

$$\inf\{f(x) : x \in X, g(x) \leq 0, h(x) = 0\} = \sup\{\theta(u,v) : u \geq 0\}, \tag{2.52}$$

where $\theta(u,v) = \inf\{f(x) + u^{\mathrm{T}} g(x) + v^{\mathrm{T}} h(x) : x \in X\}$. Furthermore, if the inf *is finite, then* $\sup\{\theta(u,v) : u \geq 0\}$ *is achieved at $(\bar{u}, \bar{v})$ with $\bar{u} \geq 0$. If the* inf *is achieved at $\bar{x}$, then $\bar{u}^{\mathrm{T}} g(\bar{x}) = 0$.*

Proof. Let $\gamma = \inf\{f(x) : x \in X, g(x) \leq 0, h(x) = 0\}$. By assumption there exists a feasible solution $\hat{x}$ for the primal problem and hence $\gamma < \infty$. If $\gamma = -\infty$, we then conclude from Corollary 2.6.2 that $\sup\{\theta(u,v) : u \geq 0\} = -\infty$ and, hence, (2.52) is satisfied. Thus, suppose that γ is finite, and consider the following system:

$$f(x) - \gamma < 0, \qquad g(x) \leq 0 \qquad h(x) = 0, \qquad \text{for some } x \in X.$$

By the definition of γ, this system has no solution. Hence, from Lemma 2.6.3, there exists a nonzero vector (u_0, u, v) with $(u_0, u) \geq (0, 0)$ such that

$$u_0[f(x) - \gamma] + u^{\mathrm{T}} g(x) + v^{\mathrm{T}} h(x) \geq 0 \qquad \text{for all } x \in X. \tag{2.53}$$

We will next show that $u_0 > 0$. Suppose, by contradiction that $u_0 = 0$. By assumption, there exists an $\hat{x} \in X$ such that $g(\hat{x}) < 0$ and $h(\hat{x}) = 0$. Substituting in (2.53) we obtain $u^{\mathrm{T}} g(\hat{x}) \geq 0$. But, since $g(\hat{x}) < 0$ and $u \geq 0$, $u^{\mathrm{T}} g(\hat{x}) \geq 0$ is only possible if $u = 0$. From (2.53), $u_0 = 0$ and $u = 0$ imply that $v^{\mathrm{T}}h(x) \geq 0$ for all $x \in X$. But, since $0 \in \operatorname{int} h(X)$, we can choose an $x \in X$ such that $h(x) = -\lambda v$, where $\lambda > 0$. Therefore, $0 \leq v^{\mathrm{T}}h(x) = -\lambda\|v\|^2$, which implies that $v = 0$. Thus, it has been shown that $u_0 = 0$ implies that $(u_0, u, v) = (0, 0, 0)$, which is a contradiction. We conclude, then, that $u_0 > 0$. Dividing (2.53) by u_0 and denoting $\bar{u} = u/u_0$ and $\bar{v} = v/u_0$, we obtain

$$f(x) + \bar{u}^{\mathrm{T}}g(x) + \bar{v}^{\mathrm{T}}h(x) \geq \gamma \qquad \text{for all } x \in X. \tag{2.54}$$

This implies that $\theta(\bar{u}, \bar{v}) = \inf\{f(x) + \bar{u}^{\mathrm{T}}g(x) + \bar{v}^{\mathrm{T}}h(x) : x \in X\} \geq \gamma$. We then conclude, from Theorem 2.6.1, that $\theta(\bar{u}, \bar{v}) = \gamma$ and, from Corollary 2.6.2, that $(\bar{u}, \bar{v})$ solves the dual problem. Finally, to complete the proof, assume that $\bar{x}$ is an optimal solution to the primal problem; that is, $\bar{x} \in X$, $g(\bar{x}) \leq 0$, $h(\bar{x}) = 0$ and $f(\bar{x}) = \gamma$. From (2.54), letting $x = \bar{x}$, we get $\bar{u}^{\mathrm{T}}g(\bar{x}) \geq 0$. Since $\bar{u} \geq 0$ and $g(\bar{x}) \leq 0$, we get $\bar{u}^{\mathrm{T}}g(\bar{x}) = 0$. This completes the proof. □

2.7 Multiconvex Problems

We have emphasised convex optimisation problems since these have many desirable properties, for example, all local minima are global minima, absence of duality gap, and so on. Sometimes a problem is nonconvex but can be partitioned into a finite number of subproblems, each of which is convex within a convex region. In this case, we can solve each of the convex problems using constraints to restrict the solution to the appropriate region. Then one can simply compare the resulting objective values and decide which is best. Of course, the disadvantage of this idea is that one has to solve as many convex problems as there are convex regions. Nonetheless, this is a useful strategy in many problems of interest in practice (see, for example, Chapter 9).

This completes our brief introduction to optimisation theory. Of course, this is a rich topic and many more results are available in the literature. We refer the reader to some of the books listed in Section 2.8. However, our brief introduction will suffice for the problems addressed here. Indeed, we will make extensive use of the concepts outlined in this chapter. As a prelude of what is to follow, we note that in Chapter 3 we will use the KKT optimality conditions in the context of nonlinear optimal control; and in Chapter 10 we will utilise strong Lagrangian duality to connect constrained control and estimation.

2.8 Further Reading

For complete list of references cited, see References section at the end of book.

General

This chapter is mainly based on Bazaraa et al. (1993).

The following books complement and extend the material presented in this chapter: Boyd and Vandenberghe (2003), Nocedal and Wright (1999), Nash and Sofer (1996), Floudas (1995), Fiacco and McCormick (1990), Fletcher (2000), Luenberger (1984), (1989), Fiacco (1983), Gill, Murray and Wright (1981), Abadie (1967).