3

State Feedback Receding Horizon Controls

3.1 Introduction

In this chapter, state feedback receding horizon controls for linear systems will be given for both quadratic and H_∞ performance criteria.

The state feedback receding horizon LQ controls will be extensively investigated because they are bases for the further developments of other receding controls. The receding horizon control with the quadratic performance criterion will be derived with detailed procedures. Time-invariant systems are dealt with with simple notations. The important monotonicity of the optimal cost will be introduced with different conditions, such as a free terminal state and a fixed terminal state. A nonzero terminal cost for the free terminal state is often termed a *free terminal state* thereafter, and a fixed terminal state as a *terminal equality constraint* thereafter. Stability of the receding horizon controls is proved under cost monotonicity conditions. Horizon sizes for guaranteeing the stability are determined regardless of terminal weighting matrices. Some additional properties of the receding horizon controls are presented.

Similar results are given for the H_∞ controls that are obtained from the minimax criterion. In particular, monotonicity of the saddle-point value and stability of the state feedback receding horizon H_∞ controls are discussed.

Since cost monotonicity conditions look difficult to obtain, we introduce easy computation of receding horizon LQ and H_∞ controls by the LMI.

In order to explain the concept of a receding horizon, we introduce the predictive form, say x_{k+j}, and the referenced predictive form, say $x_{k+j|k}$, in this chapter. Once the concept is clearly understood by using the reference predictive form, we will use the predictive form instead of the reference predictive form.

The organization of this chapter is as follows. In Section 3.2, predictive forms for systems and performance criteria are introduced. In Section 3.3, receding horizon LQ controls are extensively introduced with cost monotonicity, stability, and internal properties. A special case of input–output systems is investigated for GPC. In Section 3.4, receding horizon H_∞ controls are dealt with with cost monotonicity, stability, and internal properties. In Section 3.5, receding horizon LQ control and H_∞ control are represented via batch and LMI forms.

3.2 Receding Horizon Controls in Predictive Forms

3.2.1 Predictive Forms

Consider the following state-space model:

$$x_{i+1} = Ax_i + Bu_i \tag{3.1}$$
$$z_i = C_z x_i \tag{3.2}$$

where $x_i \in \Re^n$ and $u_i \in \Re^m$ are the state and the input respectively. z_i in (3.2) is called a controlled output. Note that the time index i is an arbitrary time point. This time variable will also be used for recursive equations.

With the standard form (3.1) and (3.2) it is not easy to represent the future time from the current time. In order to represent the future time from the current time, we can introduce a *predictive form*

$$x_{k+j+1} = Ax_{k+j} + Bu_{k+j} \tag{3.3}$$
$$z_{k+j} = C_z x_{k+j} \tag{3.4}$$

where k and j indicate the current time and the time distance from it respectively. Note that x_{k+j}, z_{k+j}, and u_{k+j} mean future state, future output, and future input at time $k+j$ respectively. In the previous chapter it was not necessary to identify the current time. However, in the case of the RHC the current time and the specific time points on the horizon should be distinguished. Thus, k is used instead of i for RHC, which offers a clarification during the derivation procedure. The time on the horizon denoted by $k+j$ means the time after j from the current time. This notation is depicted in Figure 3.1. However, the above predictive form also does not distinguish the current time if they are given as numbers. For example, $k = 10$ and $j = 3$ give $x_{k+j} = x_{13}$. When x_{13} is given, there is no way to know what the current time is. Therefore, in order to identify the current time we can introduce a *referenced predictive form*

$$x_{k+j+1|k} = Ax_{k+j|k} + Bu_{k+j|k} \tag{3.5}$$
$$z_{k+j|k} = C_z x_{k+j|k} \tag{3.6}$$

with the initial condition $x_{k|k} = x_k$. In this case, when $k = 10$ and $j = 3$, $x_{k+j|k}$ can be represented as $x_{13|10}$. We can see that the current time k is 10 and the distance j from the current time is 3. A referenced predictive form

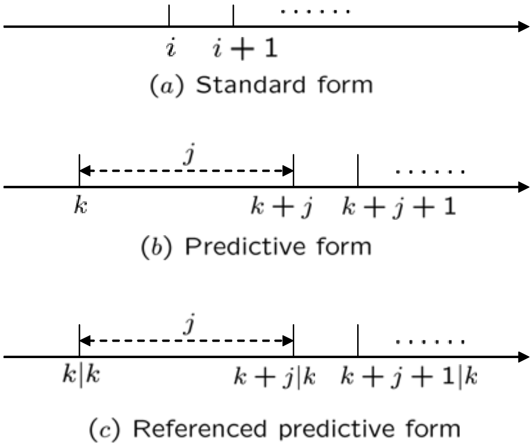

Fig. 3.1. Times in predictive forms

improves understanding. However, a predictive form will often be used in this book because the symbol k indicates the current time.

For a minimax problem, the following system is considered:

$$x_{i+1} = Ax_i + Bu_i + B_w w_i \tag{3.7}$$
$$z_i = C_z x_i \tag{3.8}$$

where w_i is a disturbance. In order to represent the future time we can introduce a *predictive form*

$$x_{k+j+1} = Ax_{k+j} + Bu_{k+j} + B_w w_{k+j} \tag{3.9}$$
$$z_{k+j} = C_z x_{k+j} \tag{3.10}$$

and a *referenced predictive form*

$$x_{k+j+1|k} = Ax_{k+j|k} + Bu_{k+j|k} + B_w w_{k+j} \tag{3.11}$$
$$z_{k+j|k} = C_z x_{k+j|k} \tag{3.12}$$

with the initial condition $x_{k|k} = x_k$.

In order to explain the concept of a receding horizon, we introduce the predictive form and the referenced predictive form. Once the concept is clearly understood by using the referenced predictive form, we will use the predictive form instead of the referenced predictive form for notational simplicity.

3.2.2 Performance Criteria in Predictive Forms

In the minimum performance criterion (2.31) for the free terminal cost, i_0 can be arbitrary and is set to k so that we have

$$J(x_k, x^r_\cdot, u_\cdot) = \sum_{i=k}^{i_f-1} \left[(x_i - x_i^r)^T Q (x_i - x_i^r) + u_i^T R u_i \right] + (x_{i_f} - x^r_{i_f})^T Q_f (x_{i_f} - x^r_{i_f}) \quad (3.13)$$

where x_i^r is given for $i = k, k+1, \cdots, i_f$.

The above minimum performance criterion (3.13) can be represented by

$$J(x_k, x^r_{k+\cdot}, u_{k+\cdot}) = \sum_{j=0}^{i_f-k-1} \left[(x_{k+j} - x^r_{k+j})^T Q (x_{k+j} - x^r_{k+j}) + u_{k+j}^T R u_{k+j} \right] + (x_{i_f} - x^r_{i_f})^T Q_f (x_{i_f} - x^r_{i_f}) \quad (3.14)$$

in a predictive form. The performance criterion (3.14) can be rewritten as

$$J(x_{k|k}, x^r, u_{k+\cdot|k}) = \sum_{j=0}^{i_f-k-1} \left[(x_{k+j|k} - x^r_{k+j|k})^T Q (x_{k+j|k} - x^r_{k+j|k}) + u_{k+j|k}^T R u_{k+j|k} \right] + (x_{i_f|k} - x^r_{i_f|k})^T Q_f (x_{i_f|k} - x^r_{i_f|k}) \quad (3.15)$$

in a referenced predictive form, where x^r is used instead of $x^r_{k+\cdot|k}$ for simplicity.

As can be seen in (3.13), (3.14), and (3.15), the performance criterion depends on the initial state, the reference trajectory, and the input on the horizon. If minimizations are taken for the performance criteria, then we denote them by $J^*(x_k, x^r)$ in a predictive form and $J^*(x_{k|k}, x^r)$ in a referenced predictive form. We can see that the dependency of the input disappears for the optimal performance criterion.

The performance criterion for the terminal equality constraint can be given as in (3.13), (3.14), and (3.15) without terminal costs, i.e. $Q_f = 0$. The terminal equality constraints are represented as $x_{i_f} = x^r_{i_f}$ in (3.13) and (3.14) and $x_{i_f|k} = x^r_{i_f|k}$ in (3.15).

In the minimax performance criterion (2.128) for the free terminal cost, i_0 can be arbitrary and is set to k so that we have

$$J(x_k, x^r, u_\cdot, w_\cdot) = \sum_{i=k}^{i_f-1} \left[(x_i - x_i^r)^T Q (x_i - x_i^r) + u_i^T R u_i - \gamma^2 w_i^T R_w w_i \right] + (x_{i_f} - x^r_{i_f})^T Q_f (x_{i_f} - x^r_{i_f}) \quad (3.16)$$

where x_i^r is given for $i = k, k+1, \cdots, i_f$.

The above minimax performance criterion (3.16) can be represented by

$$J(x_k, x^r, u_{k+\cdot}, w_{k+\cdot}) = \sum_{j=0}^{i_f-k-1} \left[(x_{k+j} - x^r_{k+j})^T Q (x_{k+j} - x^r_{k+j}) + u_{k+j}^T R u_{k+j} - \gamma^2 w_{k+j}^T R_w w_{k+j} \right] + (x_{i_f} - x^r_{i_f})^T Q_f (x_{i_f} - x^r_{i_f}) \quad (3.17)$$

in a predictive form. The performance criterion (3.17) can be rewritten as

$$J(x_{k|k}, x^r, u_{k+\cdot|k}, w_{k+\cdot|k}) = \sum_{j=0}^{i_f-k-1} \Big[(x_{k+j|k} - x^r_{k+j|k})^T Q (x_{k+j|k} - x^r_{k+j|k}) + u^T_{k+j|k} R u_{k+j|k} - \gamma^2 w^T_{k+j|k} R_w w_{k+j|k} \Big] + (x_{i_f|k} - x^r_{i_f|k})^T Q_f (x_{i_f|k} - x^r_{i_f|k}) \quad (3.18)$$

in a referenced predictive form.

Unlike the minimization problem, the performance criterion for the minimaxization problem depends on the disturbance. Taking the minimization and the maximization with respect to the input and the disturbance respectively yields the optimal performance criterion that depends only on the initial state and the reference trajectory. As in the minimization problem, we denote the optimal performance criterion by $J^*(x_k, x^r)$ in a predictive form and $J^*(x_{k|k}, x^r)$ in a referenced predictive form.

3.3 Receding Horizon Control Based on Minimum Criteria

3.3.1 Receding Horizon Linear Quadratic Control

Consider the following discrete time-invariant system of a referenced predictive form:

$$x_{k+j+1|k} = Ax_{k+j|k} + Bu_{k+j|k} \quad (3.19)$$

$$z_{k+j|k} = C_z x_{k+j|k} \quad (3.20)$$

A state feedback RHC for the system (3.19) and (3.20) is introduced in a tracking form. As mentioned before, the current time and the time distance from the current time are denoted by k and j for clarification. The time variable j is used for the derivation of the RHC.

Free Terminal Cost

The optimal control for the system (3.19) and (3.20) and the free terminal cost (3.18) can be rewritten in a referenced predictive form as

$$u^*_{k+j|k} = -R^{-1}B^T[I + K_{k+j+1,i_f|k}BR^{-1}B^T]^{-1} \times [K_{k+j+1,i_f|k}Ax_{k+j|k} + g_{k+j+1,i_f|k}] \quad (3.21)$$

where

$$K_{k+j,i_f|k} = A^T[I + K_{k+j+1,i_f|k}BR^{-1}B^T]^{-1}K_{k+j+1,i_f|k}A + Q$$
$$g_{k+j,i_f|k} = A^T[I + K_{k+j+1,i_f|k}BR^{-1}B^T]^{-1}g_{k+j+1,i_f|k} - Qx^r_{k+j|k}$$

with $K_{i_f,i_f|k} = Q_f$ and $g_{i_f,i_f|k} = -Q_f x^r_{i_f|k}$.

The receding horizon concept was introduced in the introduction chapter and is depicted in Figure 3.2. The optimal control is obtained first on the horizon $[k,\ k+N]$. Here, k indicates the current time and $k+N$, is the final time on the horizon. Therefore, $i_f = k+N$, where N is the horizon size. The

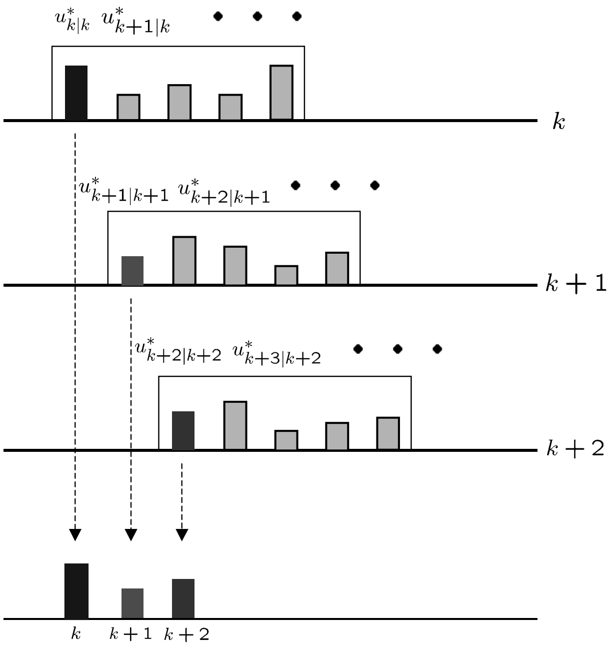

Fig. 3.2. Concept of receding horizon

performance criterion can be given in a referenced predictive form as

$$J(x_{k|k}, x^r, u_{k+\cdot|k}) = \sum_{j=0}^{N-1}\Big[(x_{k+j|k} - x^r_{k+j|k})^T Q(x_{k+j|k} - x^r_{k+j|k})$$
$$+ u^T_{k+j|k}Ru_{k+j|k}\Big] + (x_{k+N|k} - x^r_{k+N|k})^T Q_f(x_{k+N|k} - x^r_{k+N|k}) \qquad (3.22)$$

The optimal control on the interval $[k,\ k+N]$ is given in a referenced predictive form by

$$u^*_{k+j|k} = -R^{-1}B^T[I + K_{k+j+1,k+N|k}BR^{-1}B^T]^{-1}$$
$$\times\ [K_{k+j+1,k+N|k}Ax_{k+j|k} + g_{k+j+1,k+N|k}] \qquad (3.23)$$

where $K_{k+j+1,k+N|k}$ and $g_{k+j+1,k+N|k}$ are given by

$$\begin{aligned} K_{k+j,k+N|k} &= A^T[I + K_{k+j+1,k+N|k}BR^{-1}B^T]^{-1}K_{k+j+1,k+N|k}A \\ &\quad + Q \end{aligned} \tag{3.24}$$

$$\begin{aligned} g_{k+j,k+N|k} &= A^T[I + K_{k+j+1,k+N|k}BR^{-1}B^T]^{-1}g_{k+j+1,k+N|k} \\ &\quad - Qx^r_{k+j|k} \end{aligned} \tag{3.25}$$

with

$$K_{k+N,k+N|k} = Q_f \tag{3.26}$$

$$g_{k+N,k+N|k} = -Q_f x^r_{k+N|k} \tag{3.27}$$

The receding horizon LQ control at time k is given by the first control $u_{k|k}$ *among* $u_{k+i|k}$ *for* $i = 0, 1, \cdots, k+N-1$ *as in Figure 3.2.* It can be obtained from (3.23) with $j = 0$ as

$$\begin{aligned} u^*_{k|k} &= -R^{-1}B^T[I + K_{k+1,k+N|k}BR^{-1}B^T]^{-1} \\ &\quad \times [K_{k+1,k+N|k}Ax_k + g_{k+1,k+N|k}] \end{aligned} \tag{3.28}$$

where $K_{k+1,k+N|k}$ and $g_{k+1,k+N|k}$ are computed from (3.24) and (3.25).

The above notation in a referenced predictive form can be simplified to a predictive form by dropping the reference value.

It simply can be represented by a predictive form

$$\begin{aligned} u^*_{k+j} &= -R^{-1}B^T[I + K_{k+j+1,i_f}BR^{-1}B^T]^{-1} \\ &\quad \times [K_{k+j+1,i_f}Ax_{k+j} + g_{k+j+1,i_f}] \end{aligned} \tag{3.29}$$

where

$$K_{k+j,i_f} = A^T[I + K_{k+j+1,i_f}BR^{-1}B^T]^{-1}K_{k+j+1,i_f}A + Q \tag{3.30}$$

$$g_{k+j,i_f} = A^T[I + K_{k+j+1,i_f}BR^{-1}B^T]^{-1}g_{k+j+1,i_f} - Qx^r_{k+j} \tag{3.31}$$

with $K_{i_f,i_f} = Q_f$ and $g_{i_f,i_f} = -Q_f x^r_{i_f}$. Thus, $u_{k|k}$ and $K_{k+1,k+N|k}$ are replaced by u_k and $K_{k+1,k+N}$ so that we have

$$u^*_k = -R^{-1}B^T[I + K_{k+1,k+N}BR^{-1}B^T]^{-1}[K_{k+1,k+N}Ax_k + g_{k+1,k+N}] \tag{3.32}$$

where $K_{k+1,k+N}$ and $g_{k+1,k+N}$ are computed from

$$K_{k+j,k+N} = A^T[I + K_{k+j+1,k+N}BR^{-1}B^T]^{-1}K_{k+j+1,k+N}A + Q \tag{3.33}$$

$$g_{k+j,k+N} = A^T[I + K_{k+j+1,k+N}BR^{-1}B^T]^{-1}g_{k+j+1,k+N} - Qx^r_{k+j} \tag{3.34}$$

with

$$K_{k+N,k+N} = Q_f \tag{3.35}$$

$$g_{k+N,k+N} = -Q_f x^r_{k+N} \tag{3.36}$$

Note that $I+K_{k+j,k+N}BR^{-1}B^T$ is nonsingular since $K_{k+j,k+N}$ is guaranteed to be positive semidefinite and the nonsingularity of $I+MN$ implies that of $I+NM$ for any matrices M and N.

For the zero reference signal x_i^r becomes zero, so that for the free terminal state, we have

$$u_k^* = -R^{-1}B^T[I+K_{k+1,k+N}BR^{-1}B^T]^{-1}K_{k+1,k+N}Ax_k \tag{3.37}$$

from (3.32).

Terminal Equality Constraint

So far, the free terminal costs are utilized for the receding horizon tracking control (RHTC). The terminal equality constraint can also be considered for the RHTC. In this case, the performance criterion is written as

$$\begin{aligned} J(x_k, x^r, u_{k+\cdot|k}) = \sum_{j=0}^{N-1} \Big[&(x_{k+j|k} - x^r_{k+j|k})^T Q (x_{k+j|k} - x^r_{k+j|k}) \\ &+ u^T_{k+j|k} R u_{k+j|k} \Big] \end{aligned} \tag{3.38}$$

where

$$x_{k+N|k} = x^r_{k+N|k} \tag{3.39}$$

The condition (3.39) is often called the terminal equality condition. The RHC for the terminal equality constraint with a nonzero reference signal is obtained by replacing i and i_f by k and $k+N$ in (2.103) as follows:

$$\begin{aligned} u_k = -R^{-1}B^T(I+K_{k+1,k+N}BR^{-1}B^T)^{-1}\Big[&K_{k+1,k+N}Ax_k + M_{k+1,k+N} \\ \times S^{-1}_{k+1,k+N}(x^r_{k+N} - M^T_{k,k+N}x_k - h_{k,k+N}) &+ g_{k+1,k+N}\Big] \end{aligned} \tag{3.40}$$

where $K_{k+\cdot,k+N}$, $M_{k+\cdot,k+N}$, $S_{k+\cdot,k+N}$, $g_{k+\cdot,k+N}$, and $h_{k+\cdot,k+N}$ are as follows:

$$\begin{aligned} K_{k+j,k+N} &= A^T K_{k+j+1,k+N}(I+BR^{-1}B^T K_{k+j+1,k+N})^{-1}A + Q \\ M_{k+j,k+N} &= (I+BR^{-1}B^T K_{k+j+1,k+N})^{-T} M_{k+j+1,k+N} \\ S_{k+j,k+N} &= S_{k+j+1,k+N} \\ &\quad - M^T_{k+j+1,k+N} B(B^T K_{k+j+1,k+N}B + R)^{-1} B^T M_{k+j+1,k+N} \\ g_{k+j,k+N} &= A^T g_{k+j+1,k+N} \\ &\quad - A^T K_{k+j+1,k+N}(I+BR^{-1}B^T K_{k+j+1,k+N})^{-1} BR^{-1}B^T \\ &\quad \times g_{k+j+1,k+N} - Qx^r_{k+j} \\ h_{k+j,k+N} &= h_{k+j+1,k+N} \\ &\quad - M^T_{k+j+1,k+N}(I+BR^{-1}B^T K_{k+j+1,k+N})^{-1} BR^{-1}B^T g_{k+j+1,k+N} \end{aligned}$$

The boundary conditions are given by

$$K_{k+N,k+N} = 0, M_{k+N,k+N} = I, S_{k+N,k+N} = 0, g_{k+N,k+N} = 0, h_{k+N,k+N} = 0$$

For the regulation problem, (3.40) is reduced to

$$\begin{aligned} u_k^* = &-R^{-1}B^T(I + K_{k+1,k+N}BR^{-1}B^T)^{-1}\big[K_{k+1,k+N}A \\ &- M_{k+1,k+N}S_{k+1,k+N}^{-1}M_{k,k+N}^T\big]x_k \end{aligned} \quad (3.41)$$

From (2.68), u_k^* in (3.41) is represented in another form

$$u_k^* = -R^{-1}B^T P_{k+1,k+N+1}^{-1} A x_k \quad (3.42)$$

where $P_{k+1,k+N+1}$ is computed from (2.65)

$$\begin{aligned} P_{k+j,k+N+1} = &A^{-1}\big[I + P_{k+j+1,k+N+1}A^{-T}QA^{-1}\big]^{-1}P_{k+j+1,k+N+1}A \\ &+ BR^{-1}B^T \end{aligned} \quad (3.43)$$

with

$$P_{k+N+1,k+N+1} = 0 \quad (3.44)$$

Note that the system matrix A should be nonsingular in Riccati Equation (3.43). However, this requirement can be relaxed in the form of (3.41) or with the batch form, which is left as a problem at the end of this chapter.

3.3.2 Simple Notation for Time-invariant Systems

In previous sections the Riccati equations have had two arguments, one of which represents the terminal time. However, only one argument is used for time-invariant systems in this section for simplicity. If no confusion arises, then one argument will be used for Riccati equations throughout this book, particularly for Riccati equations for time-invariant systems.

Time-invariant homogeneous systems such as $x_{i+1} = f(x_i)$ have a special property known as shift invariance. If the initial condition is the same, then the solution depends on the distance from the initial time. Let x_{i,i_0} denote the solution at i with the initial i_0, as can be seen in Figure 3.3. That is

$$x_{i,i_0} = x_{i+N,i_0+N} \quad (3.45)$$

for any N with $x_{i_0,i_0} = x_{i_0+N,i_0+N}$.

Free Terminal Cost

Since (3.33) is also a time-invariant system, the following equation is satisfied:

$$K_{k+j,k+N} = K_{j,N} \quad (3.46)$$

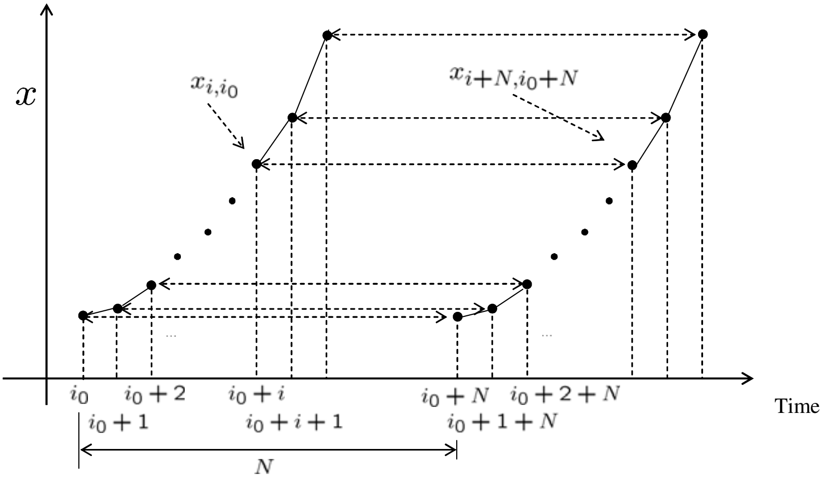

Fig. 3.3. Property of shift invariance

with $K_{N,N} = K_{k+N,k+N} = Q_f$.

Since N is fixed, we denote $K_{j,N}$ by simply K_j and K_j satisfies the following equation:

$$\begin{aligned} K_j &= A^T K_{j+1} A - A^T K_{i+1} B[R + B^T K_{j+1} B]^{-1} B^T K_{j+1} A + Q \\ &= A^T K_{j+1}[I + BR^{-1}B^T K_{j+1}]^{-1} A + Q \end{aligned} \tag{3.47}$$

with the boundary condition

$$K_N = Q_f \tag{3.48}$$

Thus, the receding horizon control (3.32) can be represented as

$$u_k^* = -R^{-1}B^T[I + K_1 BR^{-1}B^T]^{-1}[K_1 A x_k + g_{k+1,k+N}] \tag{3.49}$$

where K_1 is obtained from (3.47) and $g_{k+1,k+N}$ is computed from

$$g_{k+j,k+N} = A^T[I + K_{j+1}BR^{-1}B^T]^{-1} g_{k+j+1,k+N} - Q x_{k+j}^r \tag{3.50}$$

with the boundary condition

$$g_{k+N,k+N} = -Q_f x_{k+N}^r \tag{3.51}$$

It is noted that (3.34) is not a time-invariant system due to a time-varying signal, x_{k+j}^r. If x_{k+j}^r is a constant signal denoted by $\bar{x}^r$, then

$$g_j = A^T[I + K_{j+1}BR^{-1}B^T]^{-1} g_{j+1} - Q\bar{x}^r \tag{3.52}$$

with the boundary condition

$$g_N = -Q_f \bar{x}^r \tag{3.53}$$

The control can be written as

$$u_k = -R^{-1}B^T[I + K_1BR^{-1}B^T]^{-1}(K_1Ax_k + g_1) \tag{3.54}$$

It is noted that from shift invariance with a new boundary condition

$$K_{N-1} = Q_f \tag{3.55}$$

K_1 in (3.49) and (3.54) becomes K_0.

For the zero reference signal x_i^r becomes zero, so that for the free terminal cost we have

$$u_k^* = -R^{-1}B^T[I + K_1BR^{-1}B^T]^{-1}K_1Ax_k \tag{3.56}$$

from (3.32).

Terminal Equality Constraint

The RHC (3.40) for the terminal equality constraint with a nonzero reference can be represented as

$$\begin{aligned} u_k = -R^{-1}B^T[I + K_1BR^{-1}B^T]^{-1}\Big[&K_1Ax_k \\ &+ M_1S_1^{-1}(x_{k+N}^r - M_0^Tx_k - h_{k,k+N}) + g_{k+1,k+N}\Big] \end{aligned} \tag{3.57}$$

where K_j, M_j, S_j, $g_{k+j,k+N}$, and $h_{k+j,k+N}$ are as follows:

$$\begin{aligned}
K_j &= A^TK_{j+1}(I + BR^{-1}B^TK_{j+1})^{-1}A + Q \\
M_j &= (I + BR^{-1}B^TK_{j+1})^{-T}M_{j+1} \\
S_j &= S_{j+1} - M_{j+1}^TB(B^TK_{j+1}B + R)^{-1}B^TM_{j+1} \\
g_{k+j,k+N} &= A^Tg_{k+j+1,k+N} \\
&\quad - A^TK_{j+1}(I + BR^{-1}B^TK_{j+1})^{-1}BR^{-1}B^Tg_{k+j+1,k+N} \\
&\quad - Qx_{k+j}^r \\
h_{k+j,k+N} &= h_{k+j+1,k+N} \\
&\quad - M_{k+j+1,k+N}^T(I + BR^{-1}B^TK_{j+1})^{-1}BR^{-1}B^Tg_{k+j+1,k+N}
\end{aligned}$$

The boundary conditions are given by

$$K_N = 0,\ M_N = I,\ S_N = 0,\ g_{k+N,k+N} = 0,\ h_{k+N,k+N} = 0$$

For the regulation problem, (3.57) is reduced to

$$u_k^* = -R^{-1}B^T[I + K_1BR^{-1}B^T]^{-1}\big[K_1A - M_1S_1^{-1}M_0^T\big]x_k \tag{3.58}$$

From (3.42), u_k^* in (3.58) is represented in another form

$$u_k^* = -R^{-1}B^T P_1^{-1} A x_k \tag{3.59}$$

where P_1 is computed from

$$P_j = A^{-1}\big[I + P_{j+1}A^{-T}QA^{-1}\big]^{-1} P_{j+1}A + BR^{-1}B^T \tag{3.60}$$

with

$$P_{N+1} = 0 \tag{3.61}$$

Note that it is assumed that the system matrix A is nonsingular.

Forward Computation

The computation of (3.47) is made in a backward way and the following forward computation can be introduced by the transformation

$$\overrightarrow{K}_j = K_{N-j+1} \tag{3.62}$$

Thus, K_1 starting from $K_N = Q_f$ is obtained as

$$Q_f = \overrightarrow{K}_1 = K_N,\ \overrightarrow{K}_2 = K_{N-1},\ \cdots,\ \overrightarrow{K}_N = K_1$$

The Riccati equation can be written as

$$\overrightarrow{K}_{j+1} = A^T\overrightarrow{K}_jA - A^T\overrightarrow{K}_jB[R + B^T\overrightarrow{K}_jB]^{-1}B^T\overrightarrow{K}_jA + Q \tag{3.63}$$

$$= A^T\overrightarrow{K}_j[I + BR^{-1}B^T\overrightarrow{K}_j]^{-1}A + Q, \tag{3.64}$$

with the initial condition

$$\overrightarrow{K}_1 = Q_f \tag{3.65}$$

In the same way as the Riccati equation, g_1 starting from $g_N = -Q_f\bar{x}^r$ is obtained as

$$\overrightarrow{g}_{j+1} = A^T[I + \overrightarrow{K}_jBR^{-1}B^T]^{-1}\overrightarrow{g}_j - Q\bar{x}^r \tag{3.66}$$

where $\overrightarrow{g}_1 = -Q_f\bar{x}^r$. The relation and dependency among K_j, $\overrightarrow{K}_j$, g_j, and $\overrightarrow{g}_j$ are shown in Figure 3.4 and Figure 3.5.

The control is represented by

$$u_k = -R^{-1}B^T[I + \overrightarrow{K}_NBR^{-1}B^T]^{-1}(\overrightarrow{K}_NAx_k + \overrightarrow{g}_N) \tag{3.67}$$

For forward computation, the RHC (3.42) and Riccati Equation (3.43) can be written as

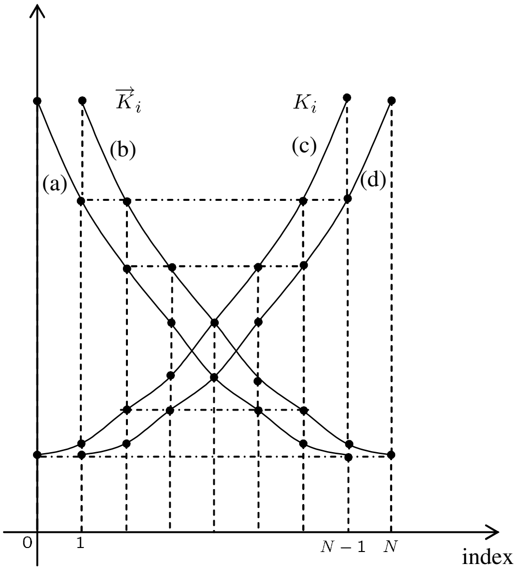

Fig. 3.4. Computation of K_i and $\hat{K}_i$. Initial conditions $i = 0$ in (a), $i = 1$ in (b), $i = N - 1$ in (c), and $i = N$ in (d)

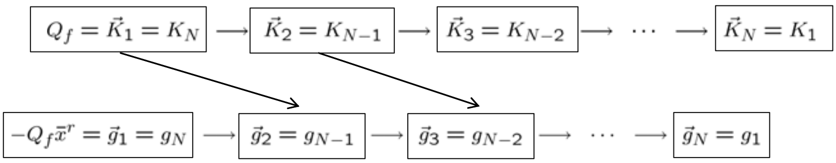

Fig. 3.5. Relation between K_i and g_i

$$u_k^* = -R^{-1}B^T\overrightarrow{P}_N^{-1}Ax_k \tag{3.68}$$

where $\overrightarrow{P}_N$ is computed by

$$\overrightarrow{P}_{j+1} = A^{-1}\overrightarrow{P}_j[I + A^{-T}QA^{-1}\overrightarrow{P}_j]^{-1}A^{-T} + BR^{-1}B^T \tag{3.69}$$

with $\overrightarrow{P}_1 = 0$.

3.3.3 Monotonicity of the Optimal Cost

In this section, some conditions are proposed for time-invariant systems which guarantee the monotonicity of the optimal cost. In the next section, under the proposed cost monotonicity conditions, the closed-loop stability of the RHC is shown. Since the closed-loop stability can be treated with the regulation problem, the g_i can be zero in this section.

It is noted that the cost function J (3.13)–(3.15) depends on several variables, such as the initial state x_i, input $u_{i+\cdot}$, initial time i, and terminal time i_f. Thus, it can be represented as $J(x_i, u_{i+\cdot}, i, i_f)$ and the optimal cost can be given as $J^*(x_i, i, i_f)$.

Define $\delta J^*(x_\tau, \sigma)$ as $\delta J^*(x_\tau, \sigma) = J^*(x_\tau, \tau, \sigma + 1) - J^*(x_\tau, \tau, \sigma)$. If $\delta J^*(x_\tau, \sigma) \leq 0$ or $\delta J^*(x_\tau, \sigma) \geq 0$ for any $\sigma > \tau$, then it is called a cost monotonicity. We will show first that the cost monotonicity condition can be easily achieved by the terminal equality condition. Then, the more general cost monotonicity condition is introduced by using a terminal cost.

For the terminal equality condition, i.e. $x_{i_f} = 0$, we have the following result.

Theorem 3.1. *For the terminal equality constraint, the optimal cost $J^*(x_i, i, i_f)$ satisfies the following cost monotonicity relation:*

$$J^*(x_\tau, \tau, \sigma + 1) \leq J^*(x_\tau, \tau, \sigma), \quad \tau \leq \sigma \tag{3.70}$$

If the Riccati solution exists for (3.70), then we have

$$K_{\tau,\sigma+1} \leq K_{\tau,\sigma} \tag{3.71}$$

Proof. This can be proved by contradiction. Assume that u_i^1 and u_i^2 are optimal controls to minimize $J(x_\tau, \tau, \sigma + 1)$ and $J(x_\tau, \tau, \sigma)$ respectively. If (3.70) does not hold, then

$$J^*(x_\tau, \tau, \sigma + 1) > J^*(x_\tau, \tau, \sigma)$$

Replace u_i^1 by u_i^2 up to $\sigma - 1$ and then $u_i^1 = 0$ at $i = \sigma$. In this case, $x_\sigma^1 = 0$, $u_\sigma^1 = 0$, and thus $x_{\sigma+1}^1 = 0$. Therefore, the cost for this control is $\bar{J}(x_\tau, \tau, \sigma + 1) = J^*(x_\tau, \tau, \sigma)$. Since this control may not be optimal for $J(x_\tau, \tau, \sigma + 1)$, we have $\bar{J}(x_\tau, \tau, \sigma + 1) \geq J^*(x_\tau, \tau, \sigma + 1)$, which implies that

$$J^*(x_\tau, \tau, \sigma) \geq J^*(x_\tau, \tau, \sigma + 1) \tag{3.72}$$

This is a contradiction to (3.70). This completes the proof. ∎

For the time-invariant systems we have

$$K_\tau \leq K_{\tau+1}$$

The above result is for the terminal equality condition. Next, the cost monotonicity condition using a free terminal cost is introduced.

Theorem 3.2. *Assume that Q_f in (3.15) satisfies the following inequality:*

$$Q_f \geq Q + H^T R H + (A - BH)^T Q_f (A - BH) \tag{3.73}$$

for some $H \in \Re^{m \times n}$.

For the free terminal cost, the optimal cost $J^*(x_i, i, i_f)$ then satisfies the following monotonicity relation:

$$J^*(x_\tau, \tau, \sigma+1) \leq J^*(x_\tau, \tau, \sigma), \quad \tau \leq \sigma \tag{3.74}$$

and thus

$$K_{\tau,\sigma+1} \leq K_{\tau,\sigma} \tag{3.75}$$

Proof. u_i^1 and u_i^2 are the optimal controls to minimize $J(x_\tau, \tau, \sigma+1)$ and $J(x_\tau, \tau, \sigma)$ respectively. If we replace u_i^1 by u_i^2 up to $\sigma - 1$ and $u_\sigma^1 = -Hx_\sigma$, then the cost for this control is given by

$$\begin{aligned}\bar{J}(x_\tau, \sigma+1) &\triangleq \sum_{i=\tau}^{\sigma-1} [x_i^{2T} Q x_i^2 + u_i^{2T} R u_i^2] + x_\sigma^{2T} Q x_\sigma^2 + x_\sigma^{2T} H^T R H x_\sigma^2 \\ &+ x_\sigma^{2T} (A - BH)^T Q_f (A - BH) x_\sigma^2 \\ &\geq J^*(x_\tau, \sigma+1)\end{aligned} \tag{3.76}$$

where the last inequality comes from the fact that this control may not be optimal. The difference between the adjacent optimal cost is less than or equal to zero as

$$\begin{aligned}J^*(x_\tau, \sigma+1) - J^*(x_\tau, \sigma) &\leq \bar{J}(x_\tau, \sigma+1) - J^*(x_\tau, \sigma) \\ &= x_\sigma^{2T} Q x_\sigma^2 + x_\sigma^{2T} H^T R H x_\sigma^2 \\ &+ x_\sigma^{2T} (A - BH)^T Q_f (A - BH) x_\sigma^2 - x_\sigma^{2T} Q_f x_\sigma^2 \\ &= x_\sigma^{2T} \{Q + H^T R H + (A - BH)^T Q_f (A - BH) - Q_f\} x_\sigma^2 \\ &\leq 0\end{aligned} \tag{3.77}$$

where

$$J^*(x_\tau, \sigma) = \sum_{i=\tau}^{\sigma-1} [x_i^{2T} Q x_i^2 + u_i^{2T} R u_i^2] + x_\sigma^{2T} Q_f x_\sigma^2 \tag{3.78}$$

From (3.77) we have

$$\triangle J^*(x_\tau, \sigma) = x_\tau^T [K_{\tau,\sigma+1} - K_{\tau,\sigma}] x_\tau \leq 0 \tag{3.79}$$

for all x_τ, and thus $K_{\tau,\sigma+1} - K_{\tau,\sigma} \leq 0$. This completes the proof. ∎

It looks difficult to find out Q_f and H satisfying (3.73). One approach is as follows: if H that makes $A - BH$ Hurwitz is given, then Q_f can be systematically obtained. First choose one matrix $M > 0$ such that $M \geq Q + H^T R H$. Then, calculate the solution Q_f to the following Lyapunov equation:

$$(A - BH)^T Q_f (A - BH) - Q_f = -M \tag{3.80}$$

It can be easily seen that Q_f obtained from (3.80) satisfies (3.73). Q_f can be explicitly expressed as

$$Q_f = \sum_{i=0}^{\infty} (A - BH)^{Ti} M (A - BH)^i \tag{3.81}$$

Another approach to find out Q_f and H satisfying (3.73) is introduced in Section 3.5.1, where LMIs are used.

It is noted that for time-invariant systems the inequality (3.75) implies

$$K_{\tau,\sigma+1} \leq K_{\tau+1,\sigma+1} \tag{3.82}$$

which leads to

$$K_{\tau} \leq K_{\tau+1} \tag{3.83}$$

The monotonicity of the optimal cost is presented in Figure 3.6. There are

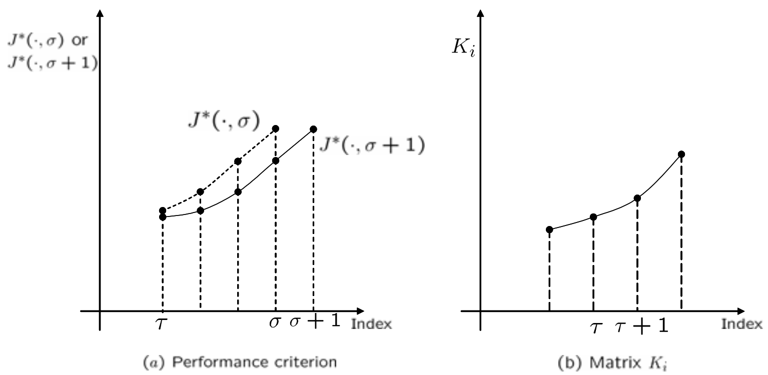

Fig. 3.6. Cost monotonicity of Theorem 3.1

several cases that satisfy the condition of Theorem 3.2.

Case 1:

$$Q_f \;\geq\; A^T Q_f [I + BR^{-1}B^T Q_f]^{-1} A + Q \tag{3.84}$$

If H is replaced by a matrix $H = [R + B^T Q_f B]^{-1} B^T Q_f A$, then we have

$$\begin{aligned} Q_f &\geq Q + H^T R H + (A - BH)^T Q_f (A - BH) \\ &= Q + A^T Q_f B [R + B^T Q_f B]^{-1} R [R + B^T Q_f B]^{-1} B^T Q A \\ &\quad + (A - B[R + B^T Q_f B]^{-1} B^T Q_f A)^T Q_f (A - B[R + B^T Q_f B]^{-1} B^T Q_f A) \\ &= Q + A^T Q_f A - A^T Q_f B [R + B^T Q_f B]^{-1} B^T Q_f A \\ &= A^T Q_f [I + BR^{-1} B^T Q_f]^{-1} A + Q \end{aligned} \tag{3.85}$$

which is a special case of (3.73). All Q_f values satisfying the inequality (3.85) are a subset of all Q_f values satisfying (3.73).

It can be seen that (3.73) implies (3.85) regardless of H as follows:

$$
\begin{aligned}
& Q_f - Q - A^T Q_f [I + BR^{-1}B^T Q_f]^{-1} A \\
& \geq -A^T Q_f [I + BR^{-1}B^T Q_f]^{-1} A + H^T R H + (A - BH)^T Q_f (A - BH) \\
& = \left[A^T Q_f B (R + B^T Q_f B)^{-1} - H\right]^T \left(R + B^T Q_f B\right) \\
& \times [(R + B^T Q_f B)^{-1} B^T Q_f A - H] \\
& \geq 0
\end{aligned}
\tag{3.86}
$$

Therefore, all Q_f values satisfying (3.73) also satisfy (3.85), and thus are a subset of all Q_f values satisfying (3.85). Thus, (3.73) is surprisingly equivalent to (3.85).

Q_f that satisfies (3.85) can also be obtained explicitly from the solution to the following Riccati equation:

$$
Q_f^* \;=\; \beta^2 A^T Q_f^* [I + \gamma B R^{-1} B^T Q_f^*]^{-1} A + \alpha Q \tag{3.87}
$$

with $\alpha \geq 1$, $\beta \geq 1$, and $0 \leq \gamma \leq 1$. It can be easily seen that Q_f^* satisfies (3.85), since

$$
\begin{aligned}
Q_f^* \;&=\; \beta^2 A^T Q_f^* [I + \gamma B R^{-1} B^T Q_f^*]^{-1} A + \alpha Q \\
&\geq\; A^T Q_f^* [I + B R^{-1} B^T Q_f^*]^{-1} A + Q
\end{aligned}
$$

Case 2:

$$
Q_f = Q + H^T R H + (A - BH)^T Q_f (A - BH) \tag{3.88}
$$

This Q_f is a special case of (3.73) and has the following meaning. Note that u_i is unknown for the interval $[\tau \ \sigma - 1]$ and defined as $u_i = -Hx_i$ on the interval $[\sigma, \ \ \infty]$. If a pair (A, B) is stabilizable and $u_i = -Hx_i$ is a stabilizing control, then

$$
\begin{aligned}
J &= \sum_{i=\tau}^{\infty} [x_i^T Q x_i + u_i^T R u_i] \\
&= \sum_{i=\tau}^{\sigma-1} [x_i^T Q x_i + u_i^T R u_i] \\
&\quad + x_\sigma^T \sum_{i=\sigma}^{\infty} (A^T - H^T B^T)^{i-\sigma} [Q + H^T R H] (A - BH)^{i-\sigma} x_\sigma \\
&= \sum_{i=\tau}^{\sigma-1} [x_i^T Q x_i + u_i^T R u_i] + x_\sigma^T Q_f x_\sigma
\end{aligned}
\tag{3.89}
$$

where Q_f satisfies $Q_f = Q + H^T R H + (A - BH)^T Q_f (A - BH)$. Therefore, Q_f is related to the steady-state performance with the control $u_i = -Hx_i$.

It is noted that, under (3.73), $u_i = -Hx_i$ will be proved to be a stabilizing control in Section 3.3.4.

Case 3:

$$Q_f = A^T Q_f [I + BR^{-1}B^T Q_f]^{-1} A + Q \tag{3.90}$$

This is actually the steady-state Riccati equation and is a special case of (3.85), and thus of (3.73). This Q_f is related to the steady-state optimal performance with the optimal control.

Case 4:

$$Q_f = Q + A^T Q_f A \tag{3.91}$$

If the system matrix A is stable and u_i is identically equal to zero for $i \geq \sigma \geq \tau$, then Q_f satisfies $Q_f = Q + A^T Q_f A$, which is also a special case of (3.73).

Proposition 3.3. *Q_f satisfying (3.73) has the following lower bound:*

$$Q_f \geq \bar{K} \tag{3.92}$$

where $\bar{K}$ is the steady-state solution to the Riccati equation in (3.90) and assumed to exist uniquely.

Proof. By the cost monotonicity condition, the solution to the recursive Riccati equation starting from Q_f satisfying Case 3 can be ordered

$$Q_f = K_{i_0} \geq K_{i_0+1} \geq K_{i_0+2} \geq \cdots \tag{3.93}$$

where

$$K_{i+1} = A^T K_i [I + BR^{-1}B^T K_i]^{-1} A + Q \tag{3.94}$$

with $K_{i_0} = Q_f$.

Since K_i is composed of two positive semidefinite matrices, K_i is also positive semidefinite, or bounded below, i.e. $K_i \geq 0$.

K_i is decreasing and bounded below, so that K_i has a limit value, which is denoted by $\bar{K}$. Clearly, it can be easily seen that

$$Q_f \geq K_i \geq \bar{K} \tag{3.95}$$

for any $i \geq i_0$.

The only thing we have to do is to guarantee that $\bar{K}$ satisfies the condition corresponding to Case 3. Taking the limitation on both sides of (3.94), we have

$$\lim_{i \to \infty} K_{i+1} = \lim_{i \to \infty} A^T K_i [I + BR^{-1}B^T K_i]^{-1} A + Q \tag{3.96}$$

$$\bar{K} = A^T \bar{K} [I + BR^{-1}B^T \bar{K}]^{-1} A + Q \tag{3.97}$$

The uniqueness of the solution to the Riccati equation implies that $\bar{K}$ is the solution that satisfies Case 3. This completes the proof. ∎

Theorem 3.2 discusses the nonincreasing monotonicity for the optimal cost. In the following, the nondecreasing monotonicity of the optimal cost can be obtained.

Theorem 3.4. *Assume that Q_f in (3.15) satisfies the inequality*

$$Q_f \leq A^T Q_f [I + BR^{-1} B^T Q_f]^{-1} A + Q \tag{3.98}$$

The optimal cost $J^*(x_i, i, i_f)$ then satisfies the relation

$$J^*(x_\tau, \tau, \sigma + 1) \geq J^*(x_\tau, \tau, \sigma), \quad \tau \leq \sigma \tag{3.99}$$

and thus

$$K_{\tau,\sigma+1} \geq K_{\tau,\sigma} \tag{3.100}$$

Proof. u_i^1 and u_i^2 are the optimal controls to minimize $J(x_\tau, \tau, \sigma + 1)$ and $J(x_\tau, \tau, \sigma)$ respectively. If we replace u_i^2 by u_i^1 up to $\sigma - 1$, then by the optimal principle we have

$$J^*(x_\tau, \sigma) = \sum_{i=\tau}^{\sigma-1} [x_i^{2T} Q x_i^2 + u_i^{2T} R u_i^2] + x_\sigma^{2T} Q_f x_\sigma^2 \tag{3.101}$$

$$\leq \sum_{i=\tau}^{\sigma-1} [x_i^{1T} Q x_i^1 + u_i^{1T} R u_i^1] + x_\sigma^{1T} Q_f x_\sigma^1 \tag{3.102}$$

The difference between the adjacent optimal cost can be expressed as

$$\begin{aligned} \delta J^*(x_\tau, \sigma) = & \sum_{i=\tau}^{\sigma} [x_i^{1T} Q x_i^1 + u_i^{1T} R u_i^1] + x_{\sigma+1}^{1T} Q_f x_{\sigma+1}^1 \\ & - \sum_{i=\tau}^{\sigma-1} [x_i^{2T} Q x_i^2 + u_i^{2T} R u_i^2] - x_\sigma^{2T} Q_f x_\sigma^2 \end{aligned} \tag{3.103}$$

Combining (3.102) and (3.103) yields

$$\begin{aligned} \delta J^*(x_\tau, \sigma) & \geq x_\sigma^{1T} Q x_\sigma^1 + u_\sigma^{1T} R u_\sigma^1 + x_{\sigma+1}^{1T} Q_f x_{\sigma+1}^1 - x_\sigma^{1T} Q_f x_\sigma^1 \\ & = x_\sigma^{1T} \{Q + A^T Q_f [I + BR^{-1} B^T Q_f]^{-1} A - Q_f\} x_\sigma^1 \\ & \geq 0 \end{aligned} \tag{3.104}$$

where $u_\sigma^1 = -Hx_\sigma^1$, $x_{\sigma+1}^1 = (A - BH)x_\sigma^1$ and $H = -(R + B^T Q_f B)^{-1} B^T Q_f A$. The second equality of (3.104) comes from the following fact:

$$\begin{aligned} & H^T R H + (A - BH)^T Q_f (A - BH) = \\ & A^T Q_f [I + BR^{-1} B^T Q_f]^{-1} A \end{aligned} \tag{3.105}$$

as can be seen in (3.85). The last inequality of (3.104) comes from (3.98). From (2.49) and (3.99) we have

$$\delta J^*(x_\tau, \sigma) = x_\tau^T [K_{\tau,\sigma+1} - K_{\tau,\sigma}] x_\tau \geq 0 \tag{3.106}$$

for all x_τ. Thus, $K_{\tau,\sigma+1} - K_{\tau,\sigma} \geq 0$. This completes the proof. ∎

It is noted that the relation (3.100) on the Riccati equation can be represented simply by one argument as

$$K_\tau \geq K_{\tau+1} \tag{3.107}$$

for time-invariant systems.

The monotonicity of the optimal cost in Theorem 3.4 is presented in Figure 3.7.

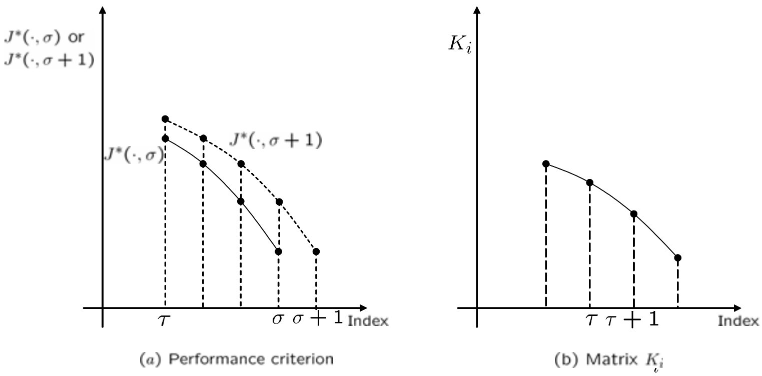

Fig. 3.7. Cost monotonicity of Theorem 3.2

We have at least one important case that satisfies the condition of Theorem 3.4.

Case 1: $Q_f = 0$

It is noted that the free terminal cost with the zero terminal weighting, $Q_f = 0$, satisfies (3.98). Thus, Theorem 3.4 includes the monotonicity of the optimal cost of the free terminal cost with the zero terminal weighting.

The terminal equality condition is more conservative than the free terminal cost. Actually, it is a strong requirement that the nonzero state must go to zero within a finite time. Thus, the terminal equality constraint has no solution for the small horizon size N, whereas the free terminal cost always gives a solution for any horizon size N. The free terminal cost requires less computation than the terminal equality constraint. However, the terminal equality constraint provides a simple approach for guaranteeing stability.

It is noted that the cost monotonicity in Theorems 3.1, 3.2 and 3.4 are obtained from the optimality. Thus, the cost monotonicity may hold even for nonlinear systems, which will be explained later.

In the following theorem, it will be shown that when the monotonicity of the optimal cost or the Riccati equations holds once, it holds for all subsequent times.

Theorem 3.5.

(a) If

$$J^*(x_{\tau'}, \tau', \sigma+1) \le J^*(x_{\tau'}, \tau', \sigma) \quad (or \ge J^*(x_{\tau'}, \tau', \sigma)) \tag{3.108}$$

for some τ', then

$$J^*(x_{\tau''}, \tau'', \sigma+1) \le J^*(x_{\tau''}, \tau'', \sigma) \quad (or \ge J^*(x_{\tau''}, \tau'', \sigma)) \tag{3.109}$$

where $\tau_0 \le \tau'' \le \tau'$.

(b) If

$$K_{\tau',\sigma+1} \le K_{\tau',\sigma} \quad (or \ge K_{\tau',\sigma}) \tag{3.110}$$

for some τ', then

$$K_{\tau'',\sigma+1} \le K_{\tau'',\sigma} \quad (or \ge K_{\tau'',\sigma}) \tag{3.111}$$

where $\tau_0 \le \tau'' \le \tau'$.

Proof. We first prove the part (a). u_i^1 and u_i^2 are the optimal controls to minimize $J(x_{\tau''}, \tau'', \sigma+1)$ and $J(x_{\tau''}, \tau'', \sigma)$ respectively. If we replace u_i^1 by u_i^2 up to $\tau' - 1$, then by the optimal principle we have

$$\begin{aligned} J^*(x_{\tau''}, \sigma+1) &= \sum_{i=\tau''}^{\tau'-1} [x_i^{1T} Q x_i^1 + u_i^{1T} R u_i^1] + J^*(x_{\tau'}^1, \tau', \sigma+1) \\ &\le \sum_{i=\tau''}^{\tau'-1} [x_i^{2T} Q x_i^2 + u_i^{2T} R u_i^2] + J^*(x_{\tau'}^2, \tau', \sigma+1) \end{aligned} \tag{3.112}$$

The difference between the adjacent optimal cost can be expressed as

$$\begin{aligned} \delta J^*(x_{\tau''}, \sigma) &= \sum_{i=\tau''}^{\tau'-1} [x_i^{1T} Q x_i^1 + u_i^{1T} R u_i^1] + J^*(x_{\tau'}^1, \tau', \sigma+1) \\ &- \sum_{i=\tau''}^{\tau'-1} [x_i^{2T} Q x_i^2 + u_i^{2T} R u_i^2] - J^*(x_{\tau'}^2, \tau', \sigma) \end{aligned} \tag{3.113}$$

Combining (3.112) and (3.113) yields

$$\begin{aligned} \delta J^*(x_{\tau''}, \sigma) &\le \sum_{i=\tau''}^{\tau'-1} [x_i^{2T} Q x_i^2 + u_i^{2T} R u_i^2] + J^*(x_{\tau'}^2, \tau', \sigma+1) \\ &- \sum_{i=\tau''}^{\tau'-1} [x_i^{2T} Q x_i^2 + u_i^{2T} R u_i^2] - J^*(x_{\tau'}^2, \tau', \sigma) \\ &= J^*(x_{\tau'}^2, \tau', \sigma+1) - J^*(x_{\tau'}^2, \tau', \sigma) \\ &= \delta J^*(x_{\tau'}^2, \sigma) \le 0 \end{aligned} \tag{3.114}$$

Replacing u_i^2 by u_i^1 up to $\tau^{'}-1$ and taking similar steps we have

$$\delta J^*(x_{\tau^{''}},\sigma) \geq \delta J^*(x^1_{\tau^{'}},\sigma) \tag{3.115}$$

from which $\delta J^*(x^1_{\tau^{'}},\sigma) \geq 0$ implies $\delta J^*(x_{\tau^{''}},\sigma) \geq 0$.

Now we prove the second part of the theorem. From (2.49), (3.108), and (3.109), the monotonicities of the Riccati equations hold. From the inequality

$$\begin{aligned} J^*(x_{\tau^{''}},\tau^{''},\sigma+1) - J^*(x_{\tau^{''}},\tau^{''},\sigma) &= x^T_{\tau^{''}}[K_{\tau^{''},\sigma+1} - K_{\tau^{''},\sigma}]x_{\tau^{''}} \\ &\leq (\geq) \;\; 0 \end{aligned}$$

$K_{\tau^{''},\sigma+1} \leq (\geq) K_{\tau^{''},\sigma}$ is satisfied. This completes the proof. ∎

For time-invariant systems the above relations can be simplified. If

$$K_{\tau^{'}} \leq K_{\tau^{'}+1} \ (\text{or} \geq K_{\tau^{'}+1}) \tag{3.116}$$

for some τ', then

$$K_{\tau^{''}} \leq K_{\tau^{''}+1} \ (\text{or} \geq K_{\tau^{''}+1}) \tag{3.117}$$

for $\tau_0 < \tau^{''} < \tau^{'}$.

Part (a) of Theorem 3.5 may be extended to constrained and nonlinear systems, whereas part (b) is only for linear systems.

Computation of the solutions of cost monotonicity conditions (3.73), (3.84), and (3.98) looks difficult to solve, but it can be easily solved by using LMI, as seen in Section 3.5.1.

3.3.4 Stability of Receding Horizon Linear Quadratic Control

For the existence of a stabilizing feedback control, we assume that the pair (A, B) is stabilizable. In this section it will be shown that the RHC is a stable control under cost monotonicity conditions.

Theorem 3.6. *Assume that the pairs (A,B) and $(A,Q^{\frac{1}{2}})$ are stabilizable and observable respectively, and that the receding horizon control associated with the quadratic cost $J(x_i,i,i+N)$ exists. If $J^*(x_i,i,i+N+1) \leq J^*(x_i,i,i+N)$, then asymptotical stability is guaranteed.*

Proof. For time-invariant systems, the system is asymptotically stable if the zero state is attractive. We show that the zero state is attractive. Since $J^*(x_i,i,\sigma+1) \leq J^*(x_i,i,\sigma)$,

$$\begin{aligned} J^*(x_i,i,i+N) &= x_i^TQx_i + u_i^{*T}Ru_i^* + J^*(x(i+1;x_i,i,u_i^*),i+1,i+N) \\ &\geq x_i^TQx_i + u_i^{*T}Ru_i^* \\ &\quad + J^*(x(i+1;x_i,i,u_i^*),i+1,i+N+1) \end{aligned} \tag{3.118}$$

Note that u_i is the receding horizon control since it is the first control on the interval $[i, i+N]$. From (3.118) we have

$$J^*(x_i, i, i+N) \geq J^*(x(i+1; x_i, i, u_i^*), i+1, i+N+1) \qquad (3.119)$$

Recall that a nonincreasing sequence bounded below converges to a constant. Since $J^*(x_i, i, i+N)$ is nonincreasing and $J^*(x_i, i, i+N) \geq 0$, we have

$$J^*(x_i, i, i+N) \rightarrow c \qquad (3.120)$$

for some nonnegative constant c as $i \rightarrow \infty$. Thus, as $i \rightarrow \infty$,

$$u_i^{*T} R u_i^* + x_i^T Q x_i \rightarrow 0 \qquad (3.121)$$

and

$$\sum_{j=i}^{i+l-1} x_j^T Q x_j + u_j^{*T} R u_j^* = x_i^T \sum_{j=i}^{i+l-1} (A^T - L_f^T B^T)^{j-i} (Q + L_f^T R L_f) \\ \times (A - BL_f)^{j-i} x_i = x_i^T G_{i,i+l}^o x_i \longrightarrow 0,$$

where L_f is the feedback gain of the RHC and $G_{i,i+l}^o$ is an observability Gramian of $(A - BL_f, (Q + L_f^T R L_f)^{\frac{1}{2}})$. However, since the pair $(A, Q^{\frac{1}{2}})$ is observable, it is guaranteed that $G_{i,i+l}^o$ is nonsingular for $l \geq n_c$ by Theorem B.5 in Appendix B. This means that $x_i \rightarrow 0$ as $i \rightarrow \infty$, independently of i_0. Therefore, the closed-loop system is asymptotically stable. This completes the proof. ∎

Note that if the condition $Q > 0$ is added in the condition of Theorem 3.6, then the horizon size N could be greater than or equal to 1.

The observability in Theorem 3.6 can be weakened with the detectability similarly as in [KK00].

It was proved in the previous section that the optimal cost with the terminal equality constraint has a nondecreasing property. Therefore, we have the following result.

Theorem 3.7. *Assume that the pairs (A, B) and $(A, Q^{\frac{1}{2}})$ are controllable and observable respectively. The receding horizon control (3.42) obtained from the terminal equality constraint is asymptotically stable for $n_c \leq N < \infty$.*

Proof. The controllability and $n_c \leq N < \infty$ are required for the existence of the optimal control, as seen in Figure 2.3. Then it follows from Theorem 3.1 and Theorem 3.6. ∎

Note that if the condition $Q > 0$ is added in the condition of Theorem 3.7, then the horizon size N could be $\max(n_c) \leq N < \infty$.

So far, we have discussed a terminal equality constraint. For the free terminal cost we have a cost monotonicity condition in Theorem 3.2 for the stability.

Theorem 3.8. *Assume that the pairs (A, B) and $(A, Q^{\frac{1}{2}})$ are stabilizable and observable respectively. For $Q_f \geq 0$ satisfying (3.73) for some H, the system (3.4) with the receding horizon control (3.56) is asymptotically stable for $1 \leq N < \infty$.* ∎

Theorem 3.8 follows from Theorems 3.2 and 3.6. Q_f in the four cases in Section 3.3.3 satisfies (3.73) and thus the condition of Theorem 3.8.

What we have talked about so far can be seen from a different perspective. The difference Riccati equation (3.47) for $j = 0$ can be represented as

$$K_1 = A^T K_1 A - A^T K_1 B[R + B^T K_1 B]^{-1} B^T K_1 A + \bar{Q} \tag{3.122}$$

$$\bar{Q} = Q + K_1 - K_0 \tag{3.123}$$

Equation (3.122) no longer looks like a recursion, but rather an algebraic equation for K_1. Therefore, Equation (3.122) is called the fake ARE (FARE).

The closed-loop stability of the RHC obtained from (3.122) and (3.123) requires the condition $\bar{Q} \geq 0$ and the detectability of the pair $(A, \bar{Q}^{\frac{1}{2}})$. The pair $(A, \bar{Q}^{\frac{1}{2}})$ is detectable if the pair $(A, Q^{\frac{1}{2}})$ is detectable and $K_1 - K_0 \geq 0$. The condition $K_1 - K_0 \geq 0$ is satisfied under the terminal inequality (3.73).

The free parameter H obtained in Theorem 3.8 is combined with the performance criterion to guarantee the stability of the closed-loop system. However, the free parameter H can be used itself as a stabilizing control gain.

Theorem 3.9. *Assume that the pairs (A, B) and $(A, Q^{\frac{1}{2}})$ are stabilizable and observable respectively. The system (3.4) with the control $u_i = -Hx_i$ is asymptotically stable where H is obtained from the inequality (3.73).*

Proof. Let

$$V(x_i) = x_i^T Q_f x_i \tag{3.124}$$

where we can show that Q_f is positive definite as follows. As just said before, Q_f of (3.73) satisfies the inequality (3.84). If $\triangle$ is defined as

$$\triangle = Q_f - A^T Q_f [I + BR^{-1} B^T Q_f]^{-1} A - Q \geq 0 \tag{3.125}$$

we can consider the following Riccati equation:

$$Q_f = A^T Q_f [I + BR^{-1} B^T Q_f]^{-1} A + Q + \triangle \tag{3.126}$$

The observability of $(A, Q^{\frac{1}{2}})$ implies the observability of $(A, (Q + \triangle)^{\frac{1}{2}})$, so that the unique positive solution Q_f comes from (3.126). Therefore, $V(x_i)$ can be considered to be a candidate of Lyapunov functions.

Subtracting $V(x_i)$ from $V(x_{i+1})$ yields

$$V(x_{i+1}) - V(x_i) = x_i^T[(A-BH)^T Q_f (A-BH) - Q_f]x_i$$
$$\leq x_i^T[-Q - H^T RH]x_i \leq 0$$

In order to use LaSalle's theorem, we try to find out the set $S = \{x_i | V(x_{i+l+1}) - V(x_{i+l}) = 0, l = 0, 1, 2, \cdots\}$. Consider the following equation:

$$x_i^T (A-BH)^{lT}[Q + H^T RH](A-BH)^l x_i = 0 \tag{3.127}$$

for all $l \geq 0$. According to the observability of $(A, Q^{\frac{1}{2}})$, the only solution that can stay identically in S is the trivial solution $x_i = 0$. Thus, the system driven by $u_i = -Hx_i$ is asymptotically stable. This completes the proof. ∎

Note that the control in Theorem 3.8 considers both the performance and the stability, whereas the one in Theorem 3.9 considers only the stability.

These results of Theorems 3.7 and 3.8 can be extended further. The matrix Q in Theorems 3.7 and 3.8 must be nonzero. However, it can even be zero in the extended result.

Let us consider the receding horizon control introduced in (3.59)

$$u_i = -R^{-1}B^T P_1^{-1} A x_i \tag{3.128}$$

where P_1 is computed from

$$P_i = A^{-1}P_{i+1}[I + A^{-T}QA^{-1}P_{i+1}]^{-1}A^{-T} + BR^{-1}B^T \tag{3.129}$$

with the boundary condition for the free terminal cost

$$P_N = Q_f^{-1} + BR^{-1}B^T \tag{3.130}$$

and $P_N = BR^{-1}B^T$ for the terminal equality constraint. However, we will assume that P_N can be arbitrarily chosen from now on and is denoted by P_f, $P_N = P_f$.

In the theorem to follow, we will show the stability of the closed-loop systems with the receding horizon control (3.128) under a certain condition that includes the well-known condition $P_f = 0$.

In fact, Riccati Equation (3.129) with the condition $P_f \geq 0$ can be obtained from the following system:

$$\hat{x}_{i+1} = A^{-T}\hat{x}_i + A^{-T}Q^{\frac{1}{2}}\hat{u}_i \tag{3.131}$$

with a performance criterion

$$\hat{J}(\hat{x}_{i_0}, i_0, i_f) = \sum_{i=i_0}^{i_f-1} [\hat{x}_i^T BR^{-1}B^T \hat{x}_i + \hat{u}_i^T \hat{u}_i] + \hat{x}_{i_f}^T P_f \hat{x}_{i_f} \tag{3.132}$$

The optimal performance criterion (3.132) for the system (3.131) is given by $\hat{J}^*(\hat{x}_i, i, i_f) = \hat{x}_i^T P_{i,i_f}\hat{x}_i$.

The nondecreasing monotonicity of (3.132) is given in the following corollary by using Theorem 3.4.

Assume that P_f in (3.132) satisfies the following inequality:

$$P_f \leq A^{-1}P_f[I + A^{-T}QA^{-1}P_f]^{-1}A^{-T} + BR^{-1}B^T \tag{3.133}$$

From Theorem 3.4 we have

$$P_{\tau,\sigma+1} \geq P_{\tau,\sigma} \tag{3.134}$$

For time-invariant systems we have

$$P_\tau \geq P_{\tau+1} \tag{3.135}$$

It is noted that Inequality (3.134) is the same as (3.71). The well-known condition for the terminal equality constraint $P_f = 0$ satisfies (3.133), and thus (3.134) holds.

Before investigating the stability under the condition (3.133), we need knowledge of an adjoint system. The two systems $x_{1,i+1} = Ax_{1,i}$ and $x_{2,i+1} = A^{-T}x_{2,i}$ are said to be adjoint to each other. They generate state trajectories while making the value of $x_i^T x_i$ fixed. If one system is shown to be unstable for any initial state the other system is guaranteed to be stable. Note that all eigenvalues of the matrix A are located outside the unit circle if and only if the system is unstable for any initial state. Additionally, it is noted that the eigenvalues of A are inverse to those of A^{-T}.

Now we are in a position to investigate the stability of the closed-loop systems with the control (3.128) under the condition (3.133) that includes the well-known condition $P_f = 0$.

Theorem 3.10. *Assume that the pair (A, B) is controllable and A is nonsingular.*

(1) If $P_{i+1} \leq P_i$ for some i, then the system (3.4) with the receding horizon control (3.128) is asymptotically stable for $n_c + 1 \leq N < \infty$.
(2) If $P_f \geq 0$ satisfies (3.133), then the system (3.4) with the receding horizon control (3.128) is asymptotically stable for $n_c + 1 \leq N < \infty$.

Proof. Consider the adjoint system of the system (3.4) with the control (3.128)

$$\hat{x}_{i+1} = [A - BR^{-1}B^TP_1^{-1}A]^{-T}\hat{x}_i \tag{3.136}$$

and the associated scalar-valued function

$$V(\hat{x}_i) = \hat{x}_i^T A^{-1}P_1A^{-T}\hat{x}_i \tag{3.137}$$

Note that the inverse of (3.136) is guaranteed to exist since, from (3.129), we have

$$
\begin{aligned}
P_1 &= A^{-1}P_2[I + A^{-T}QA^{-1}P_2]^{-1}A^{-T} + BR^{-1}B^T \\
&> BR^{-1}B^T
\end{aligned}
$$

for nonsingular, A and P_2. Note that $P_1 > 0$ and $(P_1 - BR^{-1}B^T)P_1^{-1}$ is nonsingular so that $A - BR^{-1}B^T P_1^{-1}A$ is nonsingular.

Taking the subtraction of functions at time i and $i+1$ yields

$$
\begin{aligned}
V(\hat{x}_i) &- V(\hat{x}_{i+1}) \\
&= \hat{x}_i^T A^{-1}P_1A^{-T}\hat{x}_i - \hat{x}_{i+1}^T A^{-1}P_1A^{-T}\hat{x}_{i+1} \\
&= \hat{x}_{i+1}^T \Big[(A - BR^{-1}B^TP_1^{-1}A)A^{-1}P_1A^{-T}(A - BR^{-1}B^TP_1^{-1}A)^T \\
&- A^{-1}P_1A^{-T}\Big]\hat{x}_{i+1}^T \\
&= -\hat{x}_{i+1}^T\Big[P_1 - 2BR^{-1}B^T + BR^{-1}B^TP_1^{-1}BR^{-1}B^T \\
&- A^{-1}P_1A^{-T}\Big]\hat{x}_{i+1}^T
\end{aligned} \tag{3.138}
$$

We have

$$
\begin{aligned}
P_1 &= (A^TP_2^{-1}A + Q)^{-1} + BR^{-1}B^T \\
&= A^{-1}(P_2^{-1} + A^{-T}QA^{-1})^{-1}A^{-T} + BR^{-1}B^T \\
&= A^{-1}\Big[P_2 - P_2A^{-T}Q^{\frac{1}{2}}(Q^{\frac{1}{2}}A^{-1}P_2A^{-T}Q^{\frac{1}{2}} + I)^{-1}Q^{\frac{1}{2}}A^{-1}P_2\Big] \\
&\times A^{-T} + BR^{-1}B^T \\
&= A^{-1}P_2A^{-T} + BR^{-1}B^T - Z
\end{aligned} \tag{3.139}
$$

where

$$
Z = A^{-1}P_2A^{-T}Q^{\frac{1}{2}}(Q^{\frac{1}{2}}A^{-1}P_2A^{-T}Q^{\frac{1}{2}} + I)^{-1}Q^{\frac{1}{2}}A^{-1}P_2A^{-T}
$$

Substituting (3.139) into (3.138) we have

$$
\begin{aligned}
V(\hat{x}_i) - V(\hat{x}_{i+1}) &= \hat{x}_{i+1}^T[-BR^{-1}B^T + BR^{-1}B^TP_1^{-1}BR^{-1}B^T]\hat{x}_{i+1} \\
&+ \hat{x}_{i+1}^T[A^{-1}(P_2 - P_1)A^{-T} - Z]\hat{x}_{i+1}
\end{aligned}
$$

Since $P_2 < P_1$ and $Z \geq 0$ we have

$$
\begin{aligned}
V(\hat{x}_i) - V(\hat{x}_{i+1}) &\leq \hat{x}_{i+1}^T[-BR^{-1}B^T + BR^{-1}B^TP_1^{-1}BR^{-1}B^T]\hat{x}_{i+1} \\
&= \hat{x}_{i+1}^T BR^{-\frac{1}{2}}[-I + R^{-\frac{1}{2}}B^TP_1^{-1}BR^{-\frac{1}{2}}]R^{-\frac{1}{2}}B^T\hat{x}_{i+1} \\
&= -\hat{x}_{i+1}^T BR^{-\frac{1}{2}}SR^{-\frac{1}{2}}B^T\hat{x}_{i+1}
\end{aligned} \tag{3.140}
$$

where $S = I - R^{-\frac{1}{2}}B^TP_1^{-1}BR^{-\frac{1}{2}}$. Note that S is positive definite, since the following equality holds:

$$\begin{aligned} S &= I - R^{-\frac{1}{2}}B^TP_1^{-1}BR^{-\frac{1}{2}} = I - R^{-\frac{1}{2}}B^T[\hat{P}_1 + BR^{-1}B^T]^{-1}BR^{-\frac{1}{2}} \\ &= [I + R^{-\frac{1}{2}}B^T\hat{P}_1^{-1}BR^{-\frac{1}{2}}]^{-1} \end{aligned}$$

where the second equality comes from the relation $P_1 = \hat{P}_1 + BR^{-1}B^T$. Note that $\hat{P}_1 > 0$ if $N \geq n_c + 1$. Summing both sides of (3.140) from i to $i + n_c - 1$, we have

$$\sum_{k=i}^{i+n_c-1}\Big[V(\hat{x}_{k+1}) - V(\hat{x}_k)\Big] \geq \sum_{k=i}^{i+n_c-1} \hat{x}_{k+1}^T BR^{-\frac{1}{2}}SR^{-\frac{1}{2}}B^T\hat{x}_{k+1} \tag{3.141}$$

$$V(\hat{x}_{i+n_c}) - V(\hat{x}_i) \geq \hat{x}_i^T\Theta\hat{x}_i \tag{3.142}$$

where

$$\begin{aligned} \Theta &= \sum_{k=i}^{i+n_c-1}\Big[\Psi^{(i-k-1)}W\Psi^{T(i-k-1)}\Big] \\ \Psi &= A - BR^{-1}B^TP_1^{-1}A \\ W &= BR^{-\frac{1}{2}}SR^{-\frac{1}{2}}B^T \end{aligned}$$

Recalling that $\lambda_{\max}(A^{-1}P_1A^{-1})|\hat{x}_i| \geq V(\hat{x}_i)$ and using (3.142), we obtain

$$\begin{aligned} V(\hat{x}_{i+n_c}) &\geq \hat{x}_i^T\Theta\hat{x}_i + V(\hat{x}_i) \\ &\geq \lambda_{\min}(\Theta)|\hat{x}_i|^2 + V(\hat{x}_i) \\ &\geq \varpi V(\hat{x}_i) + V(\hat{x}_i) \end{aligned} \tag{3.143}$$

where

$$\varpi = \lambda_{\min}(\Theta)\frac{1}{\lambda_{\max}(A^{-1}P_1A^{-1})} \tag{3.144}$$

Note that if(A, B) is controllable, then $(A - BH, B)$ and $((A - BH)^{-1}, B)$ are controllable. Thus, Θ is positive definite and its minimum eigenvalue is positive. ϖ is also positive. Therefore, from (3.143), the lower bound of the state is given as

$$\|\hat{x}_{i_0+m\times n_c}\|^2 \geq \frac{1}{\lambda_{\max}(A^{-1}P_1A^{-1})}(\varpi + 1)^m V(\hat{x}_{i_0}) \tag{3.145}$$

The inequality (3.145) implies that the closed-loop system (3.136) is exponentially increasing, i.e. the closed-loop system (3.4) with (3.128) is exponentially decreasing. The second part of this theorem can be easily proved from the first part. This completes the proof. ∎

It is noted that the receding horizon control (3.59) is a special case of controls in Theorem 3.10.

Theorem 3.7 requires the observability condition, whereas Theorem 3.10 does not. Theorem 3.10 holds for arbitrary $Q \geq 0$, including the zero matrix. When $Q = 0$, P_1 can be expressed as the following closed form:

$$P_1 = \sum_{j=i+1}^{i+N} A^{j-i-1} B R^{-1} B^T A^{(j-i-1)T} + A^N P_f (A^N)^T \tag{3.146}$$

where A is nonsingular. It is noted that, in the above equation, P_f can even be zero. This is the *simplest RHC*

$$u_i = -R^{-1} B^T \left[\sum_{j=i+1}^{i+N} A^{j-i-1} B R^{-1} B^T A^{(j-i-1)T} \right]^{-1} A x_i \tag{3.147}$$

that guarantees the closed-loop stability.

It is noted that P_f satifying (3.133) is equivalent to Q_f satisfying (3.84) in the relation of $P_f = Q_f^{-1} + BR^{-1}B^T$. Replacing P_f with $Q_f^{-1} + BR^{-1}B^T$ in (3.133) yields the following inequality:

$$\begin{aligned} Q_f^{-1} + BR^{-1}B^T &\leq A^{-1}[Q_f^{-1} + BR^{-1}B^T + A^{-T}QA^{-1}]^{-1}A^{-T} + BR^{-1}B^T \\ &= [A^T(Q_f^{-1} + BR^{-1}B^T)^{-1}A + Q]^{-1} + BR^{-1}B^T \end{aligned}$$

Finally, we have

$$Q_f \geq A^T (Q_f^{-1} + BR^{-1}B^T)^{-1} A + Q \tag{3.148}$$

Therefore, if Q_f satisfies (3.148), P_f also satisfies (3.133).

Theorem 3.11. *Assume that the pair (A, B) is controllable and A is nonsingular.*

(1) If $K_{i+1} \geq K_i > 0$ for some i, then the system (3.4) with receding horizon control (3.56) is asymptotically stable for $1 \leq N < \infty$.
(2) For $Q_f > 0$ satisfying (3.73) for some H, the system (3.4) with receding horizon control (3.56) is asymptotically stable for $1 \leq N < \infty$.

Proof. The first part is proved as follows. $K_{i+1} \geq K_i > 0$ implies $0 < K_{i+1}^{-1} \leq K_i^{-1}$, from which we have $0 < P_{i+1} \leq P_i$ satisfying the inequality (3.135). Thus, the control (3.128) is equivalent to (3.56). The second part is proved as follows. Inequalities $K_{i+1} \geq K_i > 0$ are satisfied for K_i generated from $Q_f > 0$ satisfying (3.73) for some H. Thus, the second result can be seen from the first one. This completes the proof. ∎

It is noted that (3.148) is equivalent to (3.73), as mentioned before.

So far, the cost monotonicity condition has been introduced for stability. Without this cost monotonicity condition, there still exists a finite horizon such that the resulting receding horizon control stabilizes the closed-loop systems.

Before proceeding to the main theorem, we introduce a matrix norm $\|A\|_{\rho,\epsilon}$, which satisfies the properties of the norm and $\rho(A) \leq \|A\|_{\rho,\epsilon} \leq \rho(A) + \epsilon$. Here, ϵ is a design parameter and $\rho(A)$ is the spectral radius of A, i.e. $\rho(A) = \max_{1 \leq i \leq n} |\lambda_i|$.

Theorem 3.12. *Suppose that* $Q \geq 0$ *and* $R > 0$*. If the pairs* (A, B) *and* $(A, Q^{\frac{1}{2}})$ *are controllable and observable respectively, then the receding horizon control (3.56) for the free terminal cost stabilizes the systems for the following horizon:*

$$N > 1 + \frac{1}{\ln \|A_c^T\|_{\rho,\epsilon} + \ln \|A_c\|_{\rho,\epsilon}} \ln \left[\frac{1}{\beta \|BR^{-1}B^T\|_{\rho,\epsilon}} \left\{ \frac{1}{\|A_c\|_{\rho,\epsilon}} - 1 - \epsilon \right\} \right] \tag{3.149}$$

where $\beta = \|Q_f - K_\infty\|_{\rho,\epsilon}$, $A_c = A - BR^{-1}B^T[I + K_\infty BR^{-1}B^T]^{-1}K_\infty A$, *and* K_∞ *is the solution to the steady-state Riccati equation.*

Proof. Denote $\triangle K_{i,N}$ by $K_{i,N} - K_\infty$, where $K_{i,N}$ is the solution at time i to the Riccati equation starting from time N, and K_∞ is the steady-state solution to the Riccati equation which is given by (2.108). In order to enhance the clarification, $K_{i,N}$ is used instead of K_i. $K_{N,N} = Q_f$ and $K_{1,N}$ of $i = 1$ are involved with the control gain of the RHC with a horizon size N. From properties of the Riccati equation, we have the following inequality:

$$\triangle K_{i,N} \leq A_c^T \triangle K_{i+1,N} A_c \tag{3.150}$$

Taking the norm $\| \cdot \|_{\rho,\epsilon}$ on both sides of (3.150), we obtain

$$\|\triangle K_{i,N}\|_{\rho,\epsilon} \leq \|A_c^T\|_{\rho,\epsilon} \|\triangle K_{i+1,N}\|_{\rho,\epsilon} \|A_c\|_{\rho,\epsilon} \tag{3.151}$$

where a norm $\| \cdot \|_{\rho,\epsilon}$ is defined just before this theorem. From (3.151), it can be easily seen that $\|\triangle K_{1,N}\|_{\rho,\epsilon}$ is bounded below as follows:

$$\|\triangle K_{1,N}\|_{\rho,\epsilon} \leq \|A_c^T\|_{\rho,\epsilon}^{N-1} \|\triangle K_{N,N}\|_{\rho,\epsilon} \|A_c\|_{\rho,\epsilon}^{N-1} = \|A_c^T\|_{\rho,\epsilon}^{N-1} \beta \|A_c\|_{\rho,\epsilon}^{N-1} \tag{3.152}$$

The closed-loop system matrix $A_{c,N}$ from the control gain $K_{1,N}$ is given by

$$A_{c,N} = A - BR^{-1}B^T[I + K_{1,N}BR^{-1}B^T]^{-1}K_{1,N}A \tag{3.153}$$

It is known that the steady-state closed-loop system matrices A_c and $A_{c,N}$ in (3.153) are related to each other as follows:

$$A_{c,N} = \left[I + BR_{o,N}^{-1} B^T \triangle K_{1,N} \right] A_c \tag{3.154}$$

where $R_{o,N} = R + B^T K_{1,N} B$. Taking the norm $\|\cdot\|_{\rho,\epsilon}$ on both sides of (3.154) and using the inequality (3.152), we have

$$\|A_{c,N}\|_{\rho,\epsilon} \leq \left[1 + \epsilon + \|BR^{-1}B^T\|_{\rho,\epsilon}\|\triangle K_{1,N}\|_{\rho,\epsilon}\right]\|A_c\|_{\rho,\epsilon}$$
$$\leq \left[1 + \epsilon + \|BR^{-1}B^T\|_{\rho,\epsilon}\|A_c^T\|_{\rho,\epsilon}^{N-1}\beta\|A_c\|_{\rho,\epsilon}^{N-1}\right]\|A_c\|_{\rho,\epsilon} \quad (3.155)$$

where ϵ should be chosen so that $\epsilon < \frac{1}{\|A_c\|_{\rho,\epsilon}} - 1$. In order to guarantee $\|A_{c,N}\|_{\rho,\epsilon} < 1$, we have only to make the right-hand side in (3.155) less than 1. Therefore, we have

$$\|A_c^T\|_{\rho,\epsilon}^{N-1}\|A_c\|_{\rho,\epsilon}^{N-1} \leq \frac{1}{\beta\|BR^{-1}B^T\|_{\rho,\epsilon}}\left[\frac{1}{\|A_c\|_{\rho,\epsilon}} - 1 - \epsilon\right] \quad (3.156)$$

It is noted that if the right-hand side of (3.156) is greater than or equal to 1, then the inequality (3.156) always holds due to the Hurwitz matrix A_c. Taking the logarithm on both sides of (3.156), we have (3.149). This completes the proof. ∎

Theorem 3.12 holds irrespective of Q_f. The determination of a suitable N is an issue.

The case of zero terminal weighting leads to generally large horizons and large terminal weighting to small horizons, as can be seen in the next example.

Example 3.1

We consider a scalar, time-invariant system and the quadratic cost

$$x_{i+1} = ax_i + bu_i \quad (3.157)$$
$$J = \sum_{j=0}^{N-1}[qx_{k+j}^2 + ru_{k+j}^2] + fx_{k+N}^2 \quad (3.158)$$

where $b \neq 0$, $r > 0$ and $q > 0$. It can be easily seen that (a, b) in (3.157) is stabilizable and $(a, \sqrt{q})$ is observable. In this case, the Riccati equation is simply represented as

$$p_k = a^2 p_{k+1} - \frac{a^2 b^2 p_{k+1}^2}{b^2 p_{k+1} + r} + q = \frac{a^2 r p_{k+1}}{b^2 p_{k+1} + r} + q \quad (3.159)$$

with $p_N = f$. The RHC with a horizon size N is obtained as

$$u_k = -Lx_k \quad (3.160)$$

where

$$L = \frac{abp_1}{b^2 p_1 + r} \tag{3.161}$$

Now, we investigate the horizon of the RHC for stabilizing the closed-loop system. The steady-state solution to the ARE and the system matrix of the closed-loop system can be written as

$$p_\infty = \frac{q}{2\Pi}\left[\pm\sqrt{(1-\Pi)^2 + \frac{4\Pi}{1-a^2}} - (1-\Pi)\right] \tag{3.162}$$

$$a^{cl} = a - bL = \frac{a}{1 + \frac{b^2}{r}p_\infty} \tag{3.163}$$

where

$$\Pi = \frac{b^2 q}{(1-a^2)r} \tag{3.164}$$

We will consider two cases. One is for a stable system and the other for an unstable system.

(1) Stable system ($|a| < 1$)
Since $|a| < 1$, $1 - a^2 > 0$ and $\Pi > 0$. In this case, we have the positive solution as

$$p_\infty = \frac{q}{2\Pi}\left[\sqrt{(1-\Pi)^2 + \frac{4\Pi}{1-a^2}} - (1-\Pi)\right] \tag{3.165}$$

From (3.163), we have $|a^{cl}| < |a| < 1$. So, the asymptotical stability is guaranteed for the closed-loop system.

(2) Unstable system ($|a| > 1$)
Since $|a| > 1$, $1 - a^2 < 0$ and $\Pi < 0$. In this case, we have the positive solution given by

$$p_\infty = -\frac{q}{2\Pi}\left[\sqrt{(1-\Pi)^2 + \frac{4\Pi}{1-a^2}} + (1-\Pi)\right] \tag{3.166}$$

The system matrix a^{cl} of the closed-loop system can be represented as

$$a^{cl} = \frac{a}{1 - \frac{1-a^2}{2}\left[\sqrt{(1-\Pi)^2 + \frac{4\Pi}{1-a^2}} + 1 - \Pi\right]} \tag{3.167}$$

We have $|a^{cl}| < 1$ from the following relation:

$$\begin{aligned} & |2 + \sqrt{((a^2-1)(1-\Pi)+2)^2 - 4a^2} + (a^2-1)(1-\Pi)| \\ > & |2 + \sqrt{(a^2+1)^2 - 4a^2} + (a^2-1)| \\ = & |2a^2| > 2|a|. \end{aligned}$$

From a^{cl}, the lower bound of the horizon size guaranteeing the stability is obtained as

$$N > 1 + \frac{1}{2\ln|a^{cl}|}\ln\left[\frac{r}{b^2|f - p_\infty|}\left\{\frac{1}{|a^{cl}|} - 1\right\}\right] \tag{3.168}$$

where $\epsilon = 0$ and absolute values of scalar values are used for $\|\cdot\|_{\rho,\epsilon}$ norm. ∎

As can be seen in this example, the gain and phase margins of the RHC are greater than those of the conventional steady-state LQ control. For the general result on multi-input–multi-output systems, this is left as an exercise.

3.3.5 Additional Properties of Receding Horizon Linear Quadratic Control

A Prescribed Degree of Stability

We introduce another performance criterion to make closed-loop eigenvalues smaller than a specific value. Of course, as closed-loop eigenvalues get smaller, the closed-loop system becomes more stable, probably with an excessive control energy cost.

Consider the following performance criterion:

$$J = \sum_{j=0}^{N-1} \alpha^{2j}(u_{k+j}^T R u_{k+j} + x_{k+j}^T Q x_{k+j}) + \alpha^{2N} x_{k+N}^T Q_f x_{k+N} \tag{3.169}$$

where $\alpha > 1$ and the pair (A, B) is stabilizable.

The first thing we have to do for dealing with (3.169) is to make transformations that convert the given problem to a standard LQ problem. Therefore, we introduce new variables such as

$$\hat{x}_{k+j} \triangleq \alpha^j x_{k+j} \tag{3.170}$$

$$\hat{u}_{k+j} \triangleq \alpha^j u_{k+j} \tag{3.171}$$

Observing that

$$\hat{x}_{k+j+1} = \alpha^{j+1} x_{k+j+1} = \alpha\alpha^j[Ax_{k+j} + Bu_{k+j}] = \alpha A\hat{x}_{k+j} + \alpha B\hat{u}_{k+j} \tag{3.172}$$

we may associate the system (3.172) with the following performance criterion:

$$J = \sum_{j=0}^{N-1} (\hat{u}_{k+j}^T R\hat{u}_{k+j} + \hat{x}_{k+j}^T Q\hat{x}_{k+j}) + \hat{x}_{k+N}^T Q_f \hat{x}_{k+N} \tag{3.173}$$

The receding horizon control for (3.172) and (3.173) can be written as

$$\hat{u}_k = -R^{-1}\alpha B^T[I + K_1\alpha BR^{-1}\alpha B^T]^{-1}K_1\alpha A\hat{x}_k \tag{3.174}$$

where K_1 is obtained from

$$K_j = \alpha A^T [I + K_{j+1} \alpha B R^{-1} \alpha B^T]^{-1} K_{j+1} \alpha A + Q \tag{3.175}$$

with $K_N = Q_f$. The RHC u_k can be written as

$$u_k = -R^{-1} B^T [I + K_1 \alpha B R^{-1} \alpha B^T]^{-1} K_1 \alpha A x_k \tag{3.176}$$

Using the RHC u_k in (3.176), we introduce a method to stabilize systems with a high degree of closed-loop stability. If α is chosen to satisfy the following cost monotonicity condition:

$$Q_f \geq Q + H^T R H + \alpha (A - BH)^T Q_f (A - BH) \alpha \tag{3.177}$$

then the RHC (3.176) stabilizes the closed-loop system. Note that since α is assumed to be greater than 1, the cost monotonicity condition holds even by replacing αA with A.

In order to check the stability of the RHC (3.176), the time index k is replaced with the arbitrary time point i and the closed-loop systems are constructed. The RHC (3.176) satisfying (3.177) makes $\hat{x}_i$ approach zero according to the following state-space model:

$$\hat{x}_{i+1} = \alpha (A \hat{x}_i + B \hat{u}_i) \tag{3.178}$$

$$= \alpha (A + B R^{-1} B^T [I + K_1 \alpha B R^{-1} \alpha B^T]^{-1} K_1 \alpha A) \hat{x}_i \tag{3.179}$$

where

$$\alpha \rho (A + B R^{-1} B^T [I + K_1 \alpha B R^{-1} \alpha B^T]^{-1} K_1 \alpha A) \leq 1 \tag{3.180}$$

From (3.178) and (3.179), the real state x_k and control u_k can be written as

$$x_{i+1} = A x_i + B u_i \tag{3.181}$$

$$= (A + B R^{-1} B^T [I + K_1 \alpha B R^{-1} \alpha B^T]^{-1} K_1 \alpha A) x_i \tag{3.182}$$

The spectral radius of the closed-loop eigenvalues for (3.181) and (3.182) is obtained from (3.180) as follows:

$$\rho (A + B R^{-1} B^T [I + K_1 \alpha B R^{-1} \alpha B^T]^{-1} K_1 \alpha A) \leq \frac{1}{\alpha} \tag{3.183}$$

Then, we can see that it is possible to define a modified receding horizon control problem which achieves a closed-loop system with a prescribed degree of stability α. That is, for a prescribed α, the state x_i approaches zero at least as fast as $|\frac{1}{\alpha}|^i$. The smaller that α is, the more stable is the closed-loop system. The same goes for the terminal equality constraint.

From now on we investigate the optimality of the RHC. The receding horizon control is optimal in the sense of the receding horizon concept. But this may not be optimal in other senses, such as the finite or infinite horizon

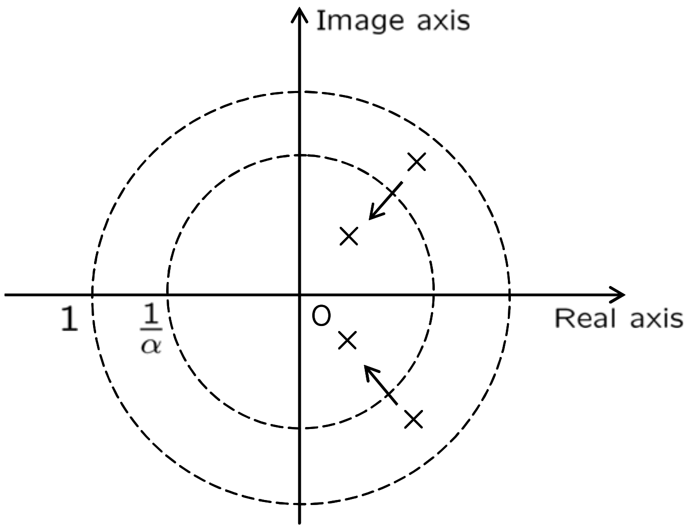

Fig. 3.8. Effect of parameter α

optimal control concept. Likewise, standard finite or infinite optimal control may not be optimal in the sense of the receding horizon control, whereas it is optimal in the sense of the standard optimal control. Therefore, it will be interesting to compare between them.

For simplicity we assume that there is no reference signal to track.

Theorem 3.13. *The standard quadratic performance criterion for the systems with the receding horizon control (3.59) under a terminal equality constraint has the following performance bounds:*

$$x_{i_0} K_{i_0,i_f} x_{i_0} \leq \sum_{i=i_0}^{i_f-1} [x_i^T Q x_i + u_i^T R u_i] \leq x_{i_0}^T P_0^{-1} x_{i_0} \tag{3.184}$$

Proof. We have the following inequality:

$$\begin{aligned} x_i^T P_0^{-1} x_i - x_{i+1}^T P_0^{-1} x_{i+1} &= x_i^T P_{-1}^{-1} x_i - x_{i+1}^T P_0^{-1} x_{i+1} + x_i^T [P_0^{-1} - P_{-1}^{-1}] x_i \\ &\geq x_i^T P_{-1}^{-1} x_i - x_{i+1}^T P_0^{-1} x_{i+1} \end{aligned} \tag{3.185}$$

which follows from the fact that $P_0^{-1} \geq P_{-1}^{-1}$. By using the optimality, we have

$$\begin{aligned} x_i^T P_{-1}^{-1} x_i - x_{i+1}^T P_0^{-1} x_{i+1} &= J^*(x_i, i, i+N+1) - J^*(x_{i+1}, i+1, i+N+1) \\ &\geq J^*(x_i, i, i+N+1) - J(x_{i+1}, i+1, i+N+1) \\ &\geq x_i^T Q x_i + u_i R u_i \end{aligned} \tag{3.186}$$

where $J(x_{i+1}, i+1, i+N+1)$ is a cost function generated from the state driven by the optimal control that is based on the interval $[i, i+N+1]$. From (3.186) we have

$$\sum_{i=i_0}^{i_f-1}[x_i^T Q x_i + u_i^T R u_i] \le x_{i_0}^T P_0^{-1} x_{i_0} - x_{i_f}^T P_0^{-1} x_{i_f} \le x_{i_0}^T P_0^{-1} x_{i_0} \quad (3.187)$$

The lower bound is obvious. This completes the proof. ∎

The next theorem introduced is for the case of the free terminal cost.

Theorem 3.14. *The standard quadratic performance criterion for the systems with the receding horizon control (3.56) under a cost monotonicity condition (3.73) has the following performance bounds:*

$$\begin{aligned} x_{i_0} K_{i_0,i_f} x_{i_0} &\le \sum_{i=i_0}^{i_f-1}[x_i^T Q x_i + u_i^T R u_i] + x_{i_f}^T Q_f x_{i_f} \\ &\le x_{i_0}^T [K_0 + \Theta^{(i_f-i_0)T} Q_f \Theta^{i_f-i_0}] x_{i_0} \end{aligned}$$

where

$$\Theta \triangleq A - BR^{-1}B^T K_1 [I + BR^{-1}B^T K_1]^{-1} A$$

K_0 is obtained from (3.47) starting from $K_N = Q_f$, and K_{i_0,i_f} is obtained by starting from $K_{i_f,i_f} = Q_f$.

Proof. The lower bound is obvious, since K_{i_0,i_f} is the cost incurred by the standard optimal control law. The gain of the receding horizon control is given by

$$\begin{aligned} L &\triangleq R^{-1}B^T K_1 [I + BR^{-1}B^T K_1]^{-1} A \\ &= [R + B^T K_1 B]^{-1} B^T K_1 A. \end{aligned}$$

As is well known, the quadratic cost for the feedback control $u_i = -Lx_i$ is given by

$$\sum_{i=i_0}^{i_f-1}[x_i^T Q x_i + u_i^T R u_i] + x_{i_f}^T Q_f x_{i_f} = x_{i_0}^T N_{i_0} x_{i_0}$$

where N_i is the solution of the following difference equation:

$$\begin{aligned} N_i &= [A - BL]^T N_{i+1} [A - BL] + L^T R L + Q \\ N_{i_f} &= Q_f \end{aligned}$$

From (3.47) and (3.48) we have

$$K_i = A^T K_{i+1} A - A^T K_{i+1} B [R + B^T K_{i+1} B]^{-1} B^T K_{i+1} A + Q$$

where $K_N = Q_f$. If we note that, for $i = 0$ in (3.188),

$$A^T K_1 B[R + B^T K_1 B]^{-1} B^T K_1 A = A^T K_1 BL = L^T B^T K_1 A \\ = L^T [R + B^T K_1 B] L$$

we can easily have

$$K_0 = [A - BL]^T K_1 [A - BL] + L^T RL + Q$$

Let

$$T_i \triangleq N_i - K_0$$

then T_i satisfies

$$T_i = [A - BL]^T [T_{i+1} - \tilde{T}_i][A - BL] \leq [A - BL]^T T_{i+1} [A - BL]$$

with the boundary condition $T_{i_f} = Q_{i_f} - K_0$, where $\tilde{T}_i = K_1 - K_0 \geq 0$ under a cost monotonicity condition. We can obtain T_{i_0} by evaluating recursively, and finally we have

$$T_{i_0} \leq \Theta^{(i_f - i_0)T} T_{i_f} \Theta^{i_f - i_0}$$

where $\Theta = A - BL$. Thus, we have

$$N_{i_0} \leq K_0 + \Theta^{(i_f - i_0)T} [Q_{i_f} - K_0] \Theta^{i_f - i_0}$$

from which follows the result. This completes the proof. ∎

Since $\Theta^{(i_f - i_0)T} \to 0$, the infinite time cost has the following bounds:

$$x_{i_0}^T K_{i_0,\infty} x_{i_0} \leq \sum_{i=0}^{\infty} x_i^T Q x_i + u_i^T R u_i \quad (3.188)$$

$$\leq x_{i_0}^T K_0 x_{i_0} \quad (3.189)$$

The receding horizon control is optimal in its own right. However, the receding horizon control can be used for a suboptimal control for the standard regulation problem. In this case, Theorem 3.14 provides a degree of suboptimality.

Example 3.2

In this example, it is shown via simulation that the RHC has good tracking ability for the nonzero reference signal. For simulation, we consider a two-dimensional free body system. This free body is moved by two kinds of forces, i.e. a horizontal force and a vertical force. According to Newton's laws, the following dynamics are obtained:

$$M\ddot{x} + B\dot{x} = u_x$$
$$M\ddot{x} + B\dot{y} = u_y$$

where M, B, x, y, u_x, and u_y are the mass of the free body, the friction coefficient, the horizontal position, the vertical position, the horizontal force, and the vertical force respectively. Through plugging the real values into the parameters and taking the discretization procedure, we have

$$x_{k+1} = \begin{bmatrix} 1 & 0.0550 & 0 & 0 \\ 0 & 0.9950 & 0 & 0 \\ 0 & 0 & 1 & 0.0550 \\ 0 & 0 & 0 & 0.9995 \end{bmatrix} x_k + \begin{bmatrix} 0.0015 & 0 \\ 0.0550 & 0 \\ 0 & 0.0015 \\ 0 & 0.0550 \end{bmatrix} u_k$$

$$y_k = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} x_k$$

where the first and second components of x_i denote the positions of x and y respectively, and the two components of u_i are for the horizontal and vertical forces.

The sampling time and the horizon size are taken as 0.055 and 3. The reference signal is given by

$$x_i^r = \begin{cases} 1 - \frac{i}{100} & 0 \leq i < 100 \\ 0 & 100 \leq i < 200 \\ \frac{i}{100} - 2 & 200 \leq i < 300 \\ 1 & 300 \leq i < 400 \\ 1 & i \geq 400 \end{cases} \quad y_i^r = \begin{cases} 1 & 0 \leq i < 100 \\ 2 - \frac{i}{100} & 100 \leq i < 200 \\ 0 & 200 \leq i < 300 \\ \frac{i}{100} - 3 & 300 \leq i < 400 \\ 1 & i \geq 400 \end{cases}$$

Q and R for the LQ and receding horizon controls are chosen to be unit matrices. The final weighting matrix for the RHC is set to $10^5 I$. In Figure 3.9, we can see that the RHC has the better performance for the given reference trajectory. Actually, the trajectory for the LQTC is way off the reference signal. However, one for the RHC keeps up with the reference well. ∎

Prediction Horizon

In general, the horizon N in the performance criterion (3.22) is divided into two parts, $[k,\ k+N_c-1]$ and $[k+N_c,\ k+N]$. The control on $[k,\ k+N_c-1]$ is obtained optimally to minimize the performance criterion on $[k,\ k+N_c-1]$, while the control on $[k+N_c,\ k+N]$ is usually a given control, say a linear control $u_i = Hx_i$ on this horizon. In this case, the horizon or horizon size N_c is called the control horizon and N is called the prediction horizon, the performance horizon, or the cost horizon. Here, N can be denoted as N_p to indicate the prediction horizon. In the previous problem we discussed so far, the control horizon N_c was the same as the prediction horizon N_p. In this case, we will use the term control horizon instead of prediction horizon. We consider the following performance criterion:

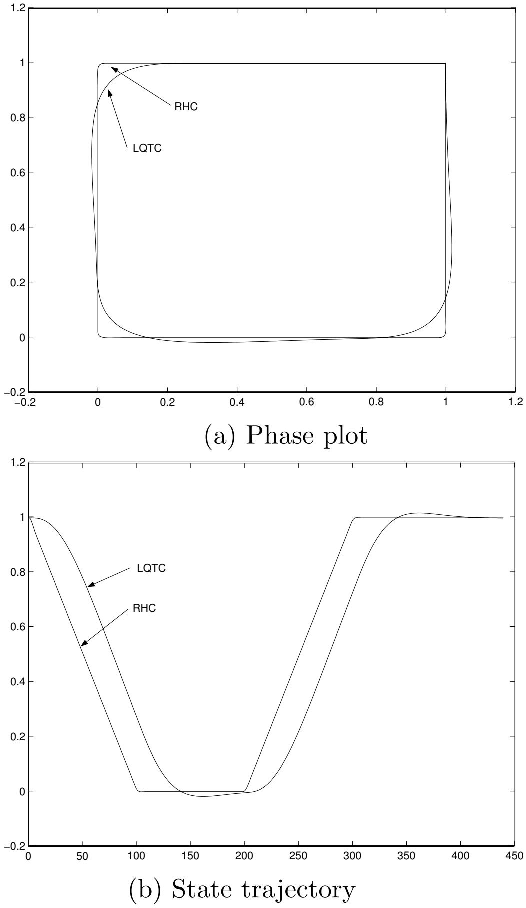

Fig. 3.9. Comparison RHC and LQTC

$$J = \sum_{j=0}^{N_c-1} (u_{k+j}^T R u_{k+j} + x_{k+j}^T Q x_{k+j}) + \sum_{j=N_c}^{N_p} (u_{k+j}^T R u_{k+j} + x_{k+j}^T Q x_{k+j}) \tag{3.190}$$

where the control horizon and the prediction horizon are $[k,\ k+N_c-1]$ and $[k,\ k+N_p]$ respectively. If we apply a linear control $u_i = Hx_i$ on $[k+N_c,\ k+N_p]$, we have

$$
\begin{aligned}
J &= \sum_{j=0}^{N_c-1} [x_{k+j}^T Q x_{k+j} + u_{k+j}^T R u_{k+j}] \\
&\quad + x_{k+N_c}^T \left\{ \sum_{j=N_c}^{N_p} ((A-HB)^T)^{j-N_c} [Q + H^T R H](A-BH)^{j-N_c} \right\} x_{k+N_c} \\
&= \sum_{j=0}^{N_c-1} [x_{k+j}^T Q x_{k+j} + u_{k+j}^T R u_{k+j}] + x_{k+N_c}^T Q_f x_{k+N_c} \qquad (3.191)
\end{aligned}
$$

where

$$
Q_f = \sum_{j=N_c}^{N_p} ((A-HB)^T)^{j-N_c} [Q + H^T R H](A-BH)^{j-N_c} \qquad (3.192)
$$

This is particularly important when $N_p = \infty$ with linear stable control, since this approach is sometimes good for constrained and nonlinear systems. But we may lose good properties inherited from a finite horizon. For linear systems, the infinite prediction horizon can be reduced to the finite one, which is the same as the control horizon. The infinite prediction horizon can be changed as

$$
\begin{aligned}
J &= \sum_{j=0}^{\infty} [x_{k+j}^T Q x_{k+j} + u_{k+j}^T R u_{k+j}] \\
&= \sum_{j=0}^{N_c-1} [x_{k+j}^T Q x_{k+j} + u_{k+j}^T R u_{k+j}] + x_{k+N_c}^T Q_f x_{k+N_c} \qquad (3.193)
\end{aligned}
$$

where Q_f satisfies $Q_f = Q + H^T R H + (A-BH)^T Q_f (A-BH)$. Therefore, Q_f is related to the terminal weighting matrix.

3.3.6 A Special Case of Input–Output Systems

In addition to the state-space model (3.1) and (3.2), GPC has used the following CARIMA model:

$$
A(q^{-1}) y_i = B(q^{-1}) \triangle u_{i-1} \qquad (3.194)
$$

where

$$
\begin{aligned}
A(q^{-1}) &= 1 + a_1 q^{-1} + \cdots + a_n q^{-n}, \quad a_n \neq 0 \qquad (3.195) \\
B(q^{-1}) &= b_1 + b_2 q^{-1} + \cdots + b_m q^{-m+1} \qquad (3.196)
\end{aligned}
$$

where q^{-1} is the unit delay operator, such as $q^{-1} y_i = y_{i-1}$, and $\triangle u_i = (1 - q^{-1}) u_i = u_i - u_{i-1}$. It is noted that $B(q^{-1})$ can be

$$b_1 + b_2 q^{-1} + \cdots + b_n q^{-n+1}, \quad m \leq n \tag{3.197}$$

where $b_i = 0$ for $i > m$. It is noted that (3.194) can be represented as

$$A(q^{-1}) y_i = \widetilde{B}(q^{-1}) \triangle u_i \tag{3.198}$$

where

$$\widetilde{B}(q^{-1}) = b_1 q^{-1} + b_2 q^{-2} \cdots + b_n q^{-n} \tag{3.199}$$

The above model (3.198) in an input–output form can be transformed to the state-space model

$$\begin{aligned} x_{i+1} &= \bar{A} x_i + \bar{B} \triangle u_i \\ y_i &= \bar{C} x_i \end{aligned} \tag{3.200}$$

where $x_i \in R^n$ and

$$\bar{A} = \begin{bmatrix} -a_1 & 1 \; 0 \cdots 0 \\ -a_2 & 0 \; 1 \cdots 0 \\ \vdots & \vdots \; \vdots \; \ddots \; \vdots \\ -a_{n-1} & 0 \; 0 \cdots 1 \\ -a_n & 0 \; 0 \cdots 0 \end{bmatrix} \quad \bar{B} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{n-1} \\ b_n \end{bmatrix} \tag{3.201}$$

$$\bar{C} = \begin{bmatrix} 1 \; 0 \; 0 \cdots 0 \end{bmatrix}$$

It is clear that $y_i = {x_i}^{(1)}$, where ${x_i}^{(1)}$ indicates the first element of x_i.

The common performance criterion for the CARIMA model (3.194) is given as

$$\sum_{j=1}^{N_c} \left[q(y_{k+j} - y^r_{k+j})^2 + r(\triangle u_{k+j-1})^2 \right] \tag{3.202}$$

Here, N_c is the control horizon.

Since y_k is given, the optimal control for (3.202) is also optimal for the following performance index:

$$\sum_{j=0}^{N_c-1} \left[q(y_{k+j} - y^r_{k+j})^2 + r(\triangle u_{k+j})^2 \right] + q(y_{k+N_c} - y^r_{k+N_c})^2 \tag{3.203}$$

The performance index (3.202) can be extended to include a free terminal cost such as

$$\sum_{j=1}^{N_c} \left[q(y_{k+j} - y^r_{k+j})^2 + r(\triangle u_{k+j-1})^2 \right] + \sum_{j=N_c+1}^{N_p} q_f (y_{k+j} - y^r_{k+j})^2 \tag{3.204}$$

We can consider a similar performance that generates the same optimal control for (3.204), such as

$$\sum_{j=0}^{N_c-1}\left[q(y_{k+j}-y^r_{k+j})^2+r(\triangle u_{k+j})^2\right]+\sum_{j=N_c}^{N_p}q_f^{(j)}(y_{k+j}-y^r_{k+j})^2 \quad (3.205)$$

where

$$q_f^{(j)}=\begin{cases} q, & j=N_c \\ q_f, & j>N_c \end{cases}$$

For a given $\bar{C}$, there exists always an L such that $\bar{C}L=I$. Let ${x_i}^r=L{y_i}^r$. The performance criterion (3.202) becomes

$$\begin{aligned}&\sum_{j=0}^{N_c-1}\left[(x_{k+j}-x^r_{k+j})^TQ(x_{k+j}-x^r_{k+j})+\triangle u^T_{k+j}R\triangle u_{k+j}\right]\\ &+\sum_{j=N_c}^{N_p}(x_{k+j}-x^r_{k+j})^TQ_f^{(j)}(x_{k+j}-x^r_{k+j}) \end{aligned} \quad (3.206)$$

where $Q=q\bar{C}^T\bar{C}$, $Q_f^{(j)}=q_f^{(j)}\bar{C}^T\bar{C}$, and $R=r$. GPC can be obtained using the state model (3.200) with the performance criterion (3.206), whose solutions are described in detail in this book. It is noted that the performance criterion (3.206) has two values in the time-varying state and input weightings. The optimal control is given in a state feedback form. From the special structure of the CARIMA model

$$x_i=\tilde{A}Y_{i-n}+\tilde{B}\triangle U_{i-n} \quad (3.207)$$

where

$$Y_{i-n}=\begin{bmatrix} y_{i-n}\\ \vdots \\ y_{i-1}\end{bmatrix} \quad \triangle U_{i-n}=\begin{bmatrix} \triangle u_{i-n}\\ \vdots \\ \triangle u_{i-1}\end{bmatrix} \quad (3.208)$$

$$\tilde{A}=\begin{bmatrix} -a_n & -a_{n-1} & \cdots & -a_2 & -a_1\\ 0 & -a_n & \cdots & -a_3 & -a_2\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & -a_n & -a_{n-1}\\ 0 & 0 & \cdots & 0 & -a_n \end{bmatrix} \quad (3.209)$$

$$\tilde{B}=\begin{bmatrix} b_n & b_{n-1} & \cdots & b_2 & b_1\\ 0 & b_n & \cdots & b_3 & b_2\\ \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 0 & \cdots & b_n & b_{n-1}\\ 0 & 0 & \cdots & 0 & b_n \end{bmatrix} \quad (3.210)$$

the state can be computed with input control and measured output. The optimal control can be given as an output feedback control.

If $\triangle u_{k+N_c} = ... = \triangle u_{k+N_p-1} = 0$ is assumed, for $N_p = N_c + n - 1$, then the terminal cost can be represented as

$$\begin{aligned} &\sum_{j=N_c}^{N_p} q_f^{(j)} (y_{k+j} - y_{k+j}^r)^2 \\ &= (Y_{k+N_c} - Y_{k+N_c}^r)^T \bar{Q}_f (Y_{k+N_c} - Y_{k+N_c}^r) \end{aligned} \tag{3.211}$$

where

$$\bar{Q}_f = \left[\mathrm{diag}(\overbrace{q \; q_f \; \cdots \; q_f}^{N_p - N_c + 1})\right], \quad Y_{k+N_c}^r = \begin{bmatrix} y_{k+N_c}^r \\ \vdots \\ y_{k+N_c+n-1}^r \end{bmatrix} \tag{3.212}$$

In this case the terminal cost becomes

$$\begin{aligned} &(x_{k+N_c+n} - \tilde{A} Y_{k+N_c}^r)^T (\tilde{A}^T)^{-1} \bar{Q}_f \tilde{A}^{-1} (x_{k+N_c+n} - \tilde{A} Y_{k+N_c}^r) \\ &= (x_{k+N_c} - x_{k+N_c}^r)^T Q_f (x_{k+N_c} - x_{k+N_c}^r) \end{aligned}$$

where

$$Q_f = q_f (\bar{A}^T)^n (\tilde{A}^T)^{-1} \bar{Q}_f \tilde{A}^{-1} \bar{A}^n \tag{3.213}$$

$$x_{k+N_c}^r = (\bar{A}^n)^{-1} \tilde{A} Y_{k+N_c}^r \tag{3.214}$$

It is noted that $\bar{A}$ and $\tilde{A}$ are all nonsingular.

The performance criterion (3.204) is for the free terminal cost. We can now introduce a terminal equality constraint, such as

$$y_{k+j} = y_{k+j}^r, \qquad j = N_c, \cdots, N_p \tag{3.215}$$

which is equivalent to $x_{k+N_c} = x_{k+N_c}^r$. GPC can be obtained from the results in state-space forms that have already been discussed.

3.4 Receding Horizon Control Based on Minimax Criteria

3.4.1 Receding Horizon H_∞ Control

In this section, a receding horizon H_∞ control in a tracking form for discrete time-invariant systems is obtained.

Based on the following system in a predictive form:

$$x_{k+j+1} = A x_{k+j} + B_w w_{k+j} + B u_{k+j} \tag{3.216}$$

$$z_{k+j} = C_z x_{k+j} \tag{3.217}$$

with the initial state x_k, the optimal control and the worst-case disturbance can be written in a predictive form as

$$u^*_{k+j} = -R^{-1}B^T\Lambda^{-1}_{k+j+1,i_f}[M_{k+j+1,i_f}Ax_{k+j} + g_{k+j+1,i_f}]$$
$$w^*_{k+j} = \gamma^{-2}R_w^{-1}B_w^T\Lambda^{-1}_{k+j+1,i_f}[M_{k+j+1,i_f}Ax_{k+j} + g_{k+j+1,i_f}]$$

M_{k+j,i_f} and g_{k+j,i_f} can be obtained from

$$M_{k+j,i_f} = A^T\Lambda^{-1}_{k+j+1,i_f}M_{k+j+1,i_f}A + Q, \quad i = i_0, \cdots, i_f - 1 \tag{3.218}$$
$$M_{i_f,i_f} = Q_f \tag{3.219}$$

and

$$g_{k+j,i_f} = -A^T\Lambda^{-1}_{k+j+1,i_f}g_{k+j+1,i_f} - Qx^r_{k+j} \tag{3.220}$$
$$g_{i_f,i_f} = -Q_f x^r_{i_f} \tag{3.221}$$

where

$$\Lambda_{k+j+1,i_f} = I + M_{k+j+1,i_f}(BR^{-1}B^T - \gamma^{-2}B_wR_w^{-1}B_w^T)$$

If we replace i_f with $k+N$, the optimal control on the interval $[k,\ k+N]$ is given by

$$u^*_{k+j} = -R^{-1}B^T\Lambda^{-1}_{k+j+1,k+N}[M_{k+j+1,k+N}Ax_{k+j} + g_{k+j+1,k+N}]$$
$$w^*_{k+j} = \gamma^{-2}R_w^{-1}B_w^T\Lambda^{-1}_{k+j+1,k+N}[M_{k+j+1,k+N}Ax_{k+j} + g_{k+j+1,k+N}]$$

The receding horizon control is given by the first control, $j = 0$, at each interval as

$$u^*_k = -R^{-1}B^T\Lambda^{-1}_{k+1,t+N}[M_{k+1,k+N}Ax_k + g_{k+1,k+N}]$$
$$w^*_k = \gamma^{-2}R_w^{-1}B_w^T\Lambda^{-1}_{k+1,k+N}[M_{k+1,k+N}Ax_k + g_{k+1,k+N}]$$

Replace k with i as an arbitrary time point for discrete-time systems to obtain

$$u^*_i = -R^{-1}B^T\Lambda^{-1}_{i+1,i+N}[M_{i+1,i+N}Ax_i + g_{i+1,i+N}]$$
$$w^*_i = \gamma^{-2}R_w^{-1}B_w^T\Lambda^{-1}_{i+1,i+N}[M_{i+1,i+N}Ax_i + g_{i+1,i+N}].$$

In case of time-invariant systems, the simplified forms are obtained as

$$u^*_i = -R^{-1}B^T\Lambda_1^{-1}[M_1Ax_i + g_{i+1,i+N}] \tag{3.222}$$
$$w^*_i = \gamma^{-2}R_w^{-1}B_w^T\Lambda_1^{-1}[M_1Ax_i + g_{i+1,i+N}] \tag{3.223}$$

M_1 and $g_{i,i+N}$ can be obtained from

$$M_j = A^T\Lambda^{-1}_{j+1}M_{j+1}A + Q, \quad j = 1, \cdots, N-1 \tag{3.224}$$
$$M_N = Q_f \tag{3.225}$$

and

$$g_{j,i+N} = -A^T \Lambda_{j+1}^{-1} g_{j+1,i+N} - Qx_j^r \tag{3.226}$$
$$g_{i+N,i+N} = -Q_f x_{i+N}^r \tag{3.227}$$

where

$$\Lambda_{j+1} = I + M_{j+1}(BR^{-1}B^T - \gamma^{-2}B_w R_w^{-1} B_w^T)$$

Recall through this chapter that the following condition is assumed to be satisfied:

$$R_w - \gamma^{-2} B_w^T M_i B_w > 0 \ , \ \ i = 1, \cdots, N \tag{3.228}$$

in order to guarantee the existence of the saddle point. Note that from (3.228), we have $M_i^{-1} > \gamma^{-2} B_w R_w^{-1} B_w^T$, from which the positive definiteness of $\Lambda_i^{-1} M_i$ is guaranteed. The positive definiteness of M_i is also guaranteed with the observability of $(A, Q^{\frac{1}{2}})$.

From (2.152) and (2.153) we have another form of the receding horizon H_∞ control:

$$u_i^* = -R^{-1}B^T P_1^{-1} A x_i \tag{3.229}$$
$$w_i^* = \gamma^{-2} R_w^{-1} B_w^T P_1^{-1} A x_i \tag{3.230}$$

where $\Pi = BR^{-1}B - \gamma^{-2} B_w R_w^{-1} B_w^T$,

$$P_i = A^{-1} P_{i+1} [I + A^{-1} Q A^{-1} P_{i+1}]^{-1} A^{-1} + \Pi \tag{3.231}$$

and the boundary condition $P_N = M_N^{-1} + \Pi = Q_f^{-1} + \Pi$.

We can use the following forward computation: by using the new variables $\overrightarrow{M}_j$ and $\overrightarrow{\Lambda}_j$ such that $\overrightarrow{M}_j = M_{N-j}$ and $\overrightarrow{\Lambda}_j = \Lambda_{N-j}$, (3.222) and (3.223) can be written as

$$u_i^* = -R^{-1}B^T \overrightarrow{\Lambda}_{N-1}^{-1} [\overrightarrow{M}_{N-1} A x_i + g_{i+1,i+N}] \tag{3.232}$$
$$w_i^* = \gamma^{-2} R_w^{-1} B_w^T \overrightarrow{\Lambda}_{N-1}^{-1} [\overrightarrow{M}_{N-1} A x_i + g_{i+1,i+N}] \tag{3.233}$$

where

$$\overrightarrow{M}_j = A^T \overrightarrow{\Lambda}_j^{-1} \overrightarrow{M}_{j-1} A + Q, \quad j = 1, \cdots, N-1$$
$$\overrightarrow{M}_0 = Q_f$$
$$\overrightarrow{\Lambda}_j = I + \overrightarrow{M}_j (BR^{-1}B^T - \gamma^{-2} B_w R_w^{-1} B_w^T)$$

3.4.2 Monotonicity of the Saddle-point Optimal Cost

In this section, terminal inequality conditions are proposed for linear discrete time-invariant systems which guarantee the monotonicity of the saddle-point value. In the next section, under the proposed terminal inequality conditions, the closed-loop stability of RHC is shown for linear discrete time-invariant systems.

Theorem 3.15. *Assume that Q_f in (3.219) satisfies the following inequality:*

$$Q_f \geq Q + H^T RH - \Gamma^T R_w \Gamma + A_{cl}^T Q_f A_{cl} \quad \text{for some } H \in \Re^{m\times n} \tag{3.234}$$

where

$$A_{cl} = A - BH + B_w \Gamma$$
$$\Gamma = \gamma^{-2} R_w^{-1} B_w^T \Lambda^{-1} Q_f A \tag{3.235}$$
$$\Lambda = I + Q_f (BR^{-1}B^T - \gamma^{-2} B_w R_w^{-1} B_w^T) \tag{3.236}$$

The saddle-point optimal cost $J^*(x_i, i, i_f)$ in (3.16) then satisfies the following relation:

$$J^*(x_\tau, \tau, \sigma+1) \leq J^*(x_\tau, \tau, \sigma), \quad \tau \leq \sigma \tag{3.237}$$

and thus $M_{\tau,\sigma+1} \leq M_{\tau,\sigma}$.

Proof. Subtracting $J^*(x_\tau, \tau, \sigma)$ from $J^*(x_\tau, \tau, \sigma+1)$, we can write

$$\delta J^*(x_\tau, \sigma) = \sum_{i=\tau}^{\sigma} [x_i^{1T} Q x_i^1 + u_i^{1T} R u_i^1 - \gamma^2 w_i^{1T} R_w w_i^1] + x_{\sigma+1}^{1T} Q_f x_{\sigma+1}^1 \tag{3.238}$$
$$- \sum_{i=\tau}^{\sigma-1} [x_i^{2T} Q x_i^2 + u_i^{2T} R u_i^2 - \gamma^2 w_i^{2T} R_w w_i^2] - x_\sigma^{2T} Q_f x_\sigma^2 \tag{3.239}$$

where the pair u_i^1 and w_i^1 is a saddle-point solution for $J(x_\tau, \tau, \sigma+1)$ and the pair u_i^2 and w_i^2 is one for $J(x_\tau, \tau, \sigma)$. If we replace u_i^1 by u_i^2 and w_i^2 by w_i^1 up to $\sigma - 1$, the following inequalities are obtained by $J(u^*, w^*) \leq J(u, w^*)$:

$$\sum_{i=\tau}^{\sigma} [x_i^{1T} Q x_i^1 + u_i^{1T} R u_i^1 - \gamma^2 w_i^{1T} R_w w_i^1] + x_{\sigma+1}^{1T} Q_f x_{\sigma+1}^1$$
$$\leq \sum_{i=\tau}^{\sigma-1} [\tilde{x}_i^T Q \tilde{x}_i + u_i^{2T} R u_i^2 - \gamma^2 w_i^{1T} R_w w_i^1] + \tilde{x}_\sigma^T Q \tilde{x}_\sigma + u_\sigma^{1T} R u_\sigma^1 - \gamma^2 w_\sigma^{1T} R_w w_\sigma^1$$
$$+ x_{\sigma+1}^{1T} Q_f x_{\sigma+1}^1$$

where $u_\sigma^1 = H\tilde{x}_\sigma$, and $w_\sigma^1 = \Gamma \tilde{x}_\sigma$. By $J(u^*, w^*) \geq J(u^*, w)$, we have

$$\sum_{i=\tau}^{\sigma-1}[x_i^{2T}Qx_i^2 + u_i^{2T}Ru_i^2 - \gamma^2 w_i^{2T}R_w w_i^2] + x_\sigma^T Q_f x_\sigma$$
$$\geq \sum_{i=\tau}^{\sigma-1}[\tilde{x}_i^T Q\tilde{x}_i + u_i^{2T}Ru_i^2 - \gamma^2 w_i^{1T}R_w w_i^1] + \tilde{x}_\sigma^T Q_f \tilde{x}_\sigma$$

Note that $\tilde{x}_i$ is a trajectory associated with u_i^2 and w_i^1. We have the following inequality:

$$\delta J^*(x_\tau, \sigma) \leq \tilde{x}_\sigma^T\{Q + H^T RH - \Gamma^T R_w \Gamma + A_{cl}^T Q_f A_{cl} - Q_f\}\tilde{x}_\sigma \leq 0 \quad (3.240)$$

where the last inequality comes from (3.234).

Since $\delta J^*(x_\tau, \sigma) = x_\tau^T[M_{\tau,\sigma+1} - M_{\tau,\sigma}]x_\tau \leq 0$ for all x_τ, we have that $M_{\tau,\sigma+1} - M_{\tau,\sigma} \leq 0$. For time-invariant systems we have

$$M_{\tau+1} \leq M_\tau \quad (3.241)$$

This completes the proof. ∎

Note that Q_f satisfying the inequality (3.234) in (3.15) should be checked for whether M_{i,i_f} generated from the boundary value Q_f satisfies $R_w - \gamma^{-2}B_w^T M_{i,i_f} B_w$. In order to obtain a feasible solution Q_f, R_w and γ can be adjusted.

Case 1: Γ in the inequality (3.234) includes Q_f, which makes it difficult to handle the inequality. We introduce the inequality without the variable Γ as follows:

$$\begin{aligned} & Q + H^T RH - \Gamma^T R_w \Gamma + A_{cl}^T Q_f A_{cl} \\ &= Q + H^T RH + \Sigma^T (B_w^T Q_f B_w - R_w)\Sigma \\ &- (A - BH)^T Q_f B_w (B_w^T Q_f B_w - R_w)^{-1} B_w Q_f (A - BH), \\ &\leq Q + H^T RH - (A - BH)^T (B_w^T Q_f B_w - R_w)^{-1}(A - BH) \leq Q_f \end{aligned} \quad (3.242)$$

where $\Sigma = \Gamma + (B_w^T Q_f B_w - R)^{-1} B_w^T Q_f (A - BH)$.

Case 2:

$$Q_f \geq A^T Q_f [I + \Pi Q_f]^{-1} A + Q \quad (3.243)$$

where $\Pi = BR^{-1}B - \gamma^{-2} B_w R_w^{-1} B_w$.

If H is replaced by an optimal gain $H = -R^{-1}B^T[I + Q_f \Pi]^{-1}\ Q_f\ A$, then by using the matrix inversion lemma in Appendix A, we can have (3.243). It is left as an exercise at the end of this chapter.

Case 3:

$$Q_f = Q + H^T R H - \Gamma^T R_w \Gamma + A_{cl}^T Q_f A_{cl} \tag{3.244}$$

which is a special case of (3.234). Q_f has the following meaning. If the pair (A, B) is stabilizable and the system is asymptotically stable with $u_i = -Hx_i$ and $w_i = \gamma^{-1} B_\gamma^T [I + M_{i+1,\infty} \hat{Q}]^{-1} M_{i+1,\infty} A x_i$ for $\sigma \geq i \geq \tau$, then

$$\begin{aligned} &\min_{u_i, i \in [\tau, \sigma-1]} \sum_{i=\tau}^{\infty} [x_i^T Q x_i + u_i^T R u_i - \gamma^2 w_i^T R_w w_i] \\ &= \min_{u_i, i \in [\tau, \sigma-1]} \sum_{i=\tau}^{\sigma-1} [x_i^T Q x_i + u_i^T R u_i - \gamma^2 w_i^T R_w w_i] + x_\sigma^T Q_f x_\sigma \end{aligned} \tag{3.245}$$

where Q_f can be shown to satisfy (3.244).

Case 4:

$$Q_f = Q - \Gamma^T R_w \Gamma + [A + B_w \Gamma]^T Q_f [A + B_w \Gamma] \tag{3.246}$$

which is also a special case of (3.234). If the system matrix A is stable with $u_i = 0$ and $w_i = \gamma^{-1} R_w^{-1} B_w^T [I + M_{i+1,\infty} \hat{Q}]^{-1} M_{i+1,\infty} A x_i$ for $\sigma \geq i \geq \tau$ then, Q_f satisfies (3.246).

In the following, the nondecreasing monotonicity of the saddle-point optimal cost is studied.

Theorem 3.16. *Assume that Q_f in (3.16) satisfies the following inequality:*

$$Q_f \leq A^T Q_f [I + \Pi Q_f]^{-1} A + Q \tag{3.247}$$

The saddle-point optimal cost $J^*(x_i, i, i_f)$ then satisfies the following relation:

$$J^*(x_\tau, \tau, \sigma + 1) \geq J^*(x_\tau, \tau, \sigma), \quad \tau \leq \sigma \tag{3.248}$$

and thus $M_{\tau,\sigma+1} \geq M_{\tau,\sigma}$.

Proof. In a similar way to the proof of Theorem 3.15, if we replace u_i^2 by u_i^1 and w_i^1 by w_i^2 up to $\sigma - 1$, then the following inequalities are obtained by $J(u^*, w^*) \geq J(u^*, w)$:

$$\begin{aligned} J^*(x_\tau, \tau, \sigma + 1) &= \sum_{i=\tau}^{\sigma} [x_i^{1T} Q x_i^1 + u_i^{1T} R u_i^1 - \gamma^2 w_i^{1T} R_w w_i^1] + x_{\sigma+1}^{1T} Q_f x_{\sigma+1}^1 \\ &\geq \sum_{i=\tau}^{\sigma-1} [\tilde{x}_i^T Q \tilde{x}_i + u_i^{1T} R u_i^1 - \gamma^2 w_i^{2T} R_w w_i^2] \\ &+ \tilde{x}_\sigma^T Q \tilde{x}_\sigma + u_\sigma^{1T} R u_\sigma^1 - \gamma^2 w_\sigma^{2T} R_w w_\sigma^2 + \tilde{x}_{\sigma+1}^T Q_f \tilde{x}_{\sigma+1} \end{aligned}$$

where

$$
\begin{aligned}
u_\sigma^1 &= H\tilde{x}_\sigma \\
w_\sigma^1 &= \Gamma\tilde{x}_\sigma \\
H &= -R^{-1}B^T[I + Q_f\Pi]^{-1}Q_fA \\
\Gamma &= \gamma^{-2}R_w^{-1}B_w^T\Lambda^{-1}Q_fA
\end{aligned}
$$

and $\tilde{x}_i$ is the trajectory associated with x_τ, u_i^1 and w_i^2 for $i \in [\tau, \sigma]$. By $J(u^*, w^*) \leq J(u, w^*)$, we have

$$
\begin{aligned}
J^*(x_\tau, \tau, \sigma) &= \sum_{i=\tau}^{\sigma-1}[x_i^{2T}Qx_i^2 + u_i^{2T}Ru_i^2 - \gamma^2 w_i^{2T}R_w w_i^2] + x_\sigma^{2T}Q_f x_\sigma^2 \\
&\leq \sum_{i=\tau}^{\sigma-1}[\tilde{x}_i^TQ\tilde{x}_i + u_i^{1T}Ru_i^1 - \gamma^2 w_i^{2T}R_w w_i^2] + \tilde{x}_\sigma^TQ_f\tilde{x}_\sigma
\end{aligned}
$$

The difference $\delta J^*(x_\tau, \sigma)$ between $J^*(x_\tau, \tau, \sigma+1)$ and $J^*(x_\tau, \tau, \sigma)$ is represented as

$$
\delta J^*(x_\tau, \sigma) \geq \tilde{x}_\sigma^T\{Q + H^TRH - \Gamma^TR_w\Gamma + A_{cl}^TQ_fA_{cl} - Q_f\}\tilde{x}_\sigma \geq 0 \quad (3.249)
$$

As in the inequality (3.243), (3.249) can be changed to (3.247). The relation $M_{\sigma+1} \geq M_\sigma$ follows from $J^*(x_i, i, i_f) = x_i^TM_{i,i_f}x_i$. This completes the proof. ∎

Case 1: $Q_f = 0$

The well-known free terminal condition, i.e. $Q_f = 0$ satisfies (3.247). Thus, Theorem 3.16 includes the monotonicity of the saddle-point value of the free terminal case.

In the following theorem based on the optimality, it will be shown that when the monotonicity of the saddle-point value or the Riccati equations holds once, it holds for all subsequent times.

Theorem 3.17. *The following inequalities for the saddle-point optimal cost and the Riccati equation are satisfied:*

(1) If

$$
J^*(x_{\tau'}, \tau', \sigma+1) \leq J^*(x_{\tau'}, \tau', \sigma) \quad (or \geq J^*(x_{\tau'}, \tau', \sigma)) \quad (3.250)
$$

for some τ', *then*

$$
J^*(x_{\tau''}, \tau'', \sigma+1) \leq J^*(x_{\tau''}, \tau'', \sigma) \quad (or \geq J^*(x_{\tau''}, \tau'', \sigma)) \quad (3.251)
$$

where $\tau_0 \leq \tau'' \leq \tau'$.

(2) If

$$M_{\tau',\sigma+1} \leq M_{\tau',\sigma} \quad (or \geq M_{\tau',\sigma}) \tag{3.252}$$

for some τ', *then,*

$$M_{\tau'',\sigma+1} \leq M_{\tau'',\sigma} \quad (or \geq M_{\tau'',\sigma}) \tag{3.253}$$

where $\tau_0 \leq \tau'' \leq \tau'$.

Proof. (a) Case of $J^*(x_{\tau'}, \tau', \sigma+1) \leq J^*(x_{\tau'}, \tau', \sigma)$:

The pair u_i^1 and w_i^1 is a saddle-point optimal solution for $J(x_{\tau''}, \tau'', \sigma+1)$ and the pair u_i^2 and w_i^2 for $J(x_{\tau''}, \tau'', \sigma)$. If we replace u_i^1 by u_i^2 and w_i^2 by w_i^1 up to τ', then

$$\begin{aligned} J^*(x_{\tau''}, \sigma+1) &= \sum_{i=\tau''}^{\tau'-1} [x_i^{1T} Q x_i^1 + u_i^{1T} R u_i^1 - \gamma^2 w_i^{1T} R_w w_i^1] + J^*(x_{\tau'}^1, \tau', \sigma+1) \\ &\leq \sum_{i=\tau''}^{\tau'-1} [\tilde{x}_i^T Q \tilde{x}_i + u_i^{2T} R u_i^2 - \gamma^2 w_i^{1T} R_w w_i^1] \\ &+ J^*(\tilde{x}_{\tau'}, \tau', \sigma+1) \end{aligned} \tag{3.254}$$

by $J(u^*, w^*) \leq J(u, w^*)$ and

$$\begin{aligned} J^*(x_{\tau''}, \sigma) &= \sum_{i=\tau''}^{\tau'-1} [x_i^{2T} Q x_i^2 + u_i^{2T} R u_i^2 - \gamma^2 w_i^{2T} R_w w_i^2] + J^*(x_{\tau'}^2, \tau', \sigma) \\ &\geq \sum_{i=\tau''}^{\tau'-1} [\tilde{x}_i^T Q \tilde{x}_i + u_i^{2T} R u_i^2 - \gamma^2 w_i^{1T} R_w w_i^1] \\ &+ J^*(\tilde{x}_{\tau'}, \tau', \sigma) \end{aligned} \tag{3.255}$$

by $J(u^*, w^*) \geq J(u^*, w)$. The difference between the adjacent optimal costs can be expressed as

$$\begin{aligned} \delta J^*(x_{\tau''}, \sigma) &= \sum_{i=\tau''}^{\tau'-1} [x_i^{1T} Q x_i^1 + u_i^{1T} R u_i^1] + J^*(x_{\tau'}^1, \tau', \sigma+1) \\ &- \sum_{i=\tau''}^{\tau'-1} [x_i^{2T} Q x_i^2 + u_i^{2T} R u_i^2] - J^*(x_{\tau'}^2, \tau', \sigma) \end{aligned} \tag{3.256}$$

Substituting (3.255) and (3.255) into (3.256), we have

$$\delta J^*(x_{\tau''},\sigma) \leq J^*(\tilde{x}_{\tau'},\tau',\sigma+1) - J^*(\tilde{x}_{\tau'},\tau',\sigma)$$
$$= \delta J^*(\tilde{x}_{\tau'},\sigma) \leq 0 \qquad (3.257)$$

Therefore,

$$\delta J^*(x_{\tau''},\sigma) \leq \delta J^*(\tilde{x}_{\tau'},\sigma) \leq 0$$

where $\tilde{x}_{\tau'}$ is the trajectory which consists of $x_{\tau''}$, u_i^2, and w_i^1 for $i \in [\tau'',\tau'-1]$.

(b) Case of $J^*(x_{\tau'},\tau',\sigma+1) \geq J^*(x_{\tau'},\tau',\sigma)$:
In a similar way to the case of (a), if we replace u_i^2 by u_i^1 and w_i^1 by w_i^2 up to τ', then

$$\delta J^*(x_{\tau''},\sigma) \ \geq \ \delta J^*(x_{\tau'},\sigma) \ \geq \ 0 \qquad (3.258)$$

The monotonicity of the Riccati equations follows from $J^*(x_i,i,i_f) = x_i^T M_{i,i_f} x_i$. This completes the proof. ∎

In the following section, stabilizing receding horizon H_∞ controls will be proposed by using the monotonicity of the saddle-point value or the Riccati equations for linear discrete time-invariant systems.

3.4.3 Stability of Receding Horizon H_∞ Control

In case of the conventional H_∞ control, the following two kinds of stability can be checked. H_∞ controls based on the infinite horizon are required to have the following properties:

1. Systems are stabilized in the case that there is no disturbance.
2. Systems are stabilized in the case that the worst-case disturbance enters the systems.

For the first case, we introduce the following result.

Theorem 3.18. *Assume that the pair (A,B) and $(A,Q^{\frac{1}{2}})$ are stabilizable and observable respectively, and that the receding horizon H_∞ control (3.222) associated with the quadratic cost $J(x_i,i,i+N)$ exists. If the following inequality holds:*

$$J^*(x_i,i,i+N+1) \leq J^*(x_i,i,i+N) \qquad (3.259)$$

then the asymptotic stability is guaranteed in the case that there is no disturbance.

Proof. We show that the zero state is attractive. Since $J^*(x_i,i,\sigma+1) \leq J^*(x_i,i,\sigma)$,

$$
\begin{aligned}
&J^*(x_i, i, i+N) && (3.260)\\
&= x_i^T Q x_i + u_i^{*T} R u_i^* - \gamma^2 w_i^{*T} R_w w_i^* \\
&+ J^*(x^1(i+1; (x_i, i, u_i^*)), i+1, i+N) \\
&\geq x_i^T Q x_i + u_i^{*T} R u_i^* + J^*(x^2(i+1; (x_i, i, u_i^*)), i+1, i+N) \\
&\geq x_i^T Q x_i + u_i^{*T} R u_i^* + J^*(x^2(i+1; (x_i, i, u_i^*)), i+1, i+N+1) && (3.261)
\end{aligned}
$$

where u_i^* is the optimal control at time i and x_{i+1}^2 is a state at time $i+1$ when $w_i = 0$ and the optimal control u_i^*. Therefore, $J^*(x_i, i, i+N)$ is nonincreasing and bounded below, i.e. $J^*(x_i, i, i+N) \geq 0$. $J^*(x_i, i, i+N)$ approaches some nonnegative constant c as $i \to \infty$. Hence, we have

$$x_i^T Q x_i + u_i^T R u_i \quad \longrightarrow \quad 0 \tag{3.262}$$

From the fact that the finite sum of the converging sequences also approaches zero, the following relation is obtained:

$$\sum_{j=i}^{i+l-1} \left[x_j^T Q x_j + u_j^T R u_j \right] \to 0, \tag{3.263}$$

leading to

$$x_i^T \left(\sum_{j=i}^{i+l-1} (A - BH)^{(j-i)T} [Q + H^T R H] (A - BH)^{j-i} \right) x_i \to 0 \tag{3.264}$$

However, since the pair $(A, Q^{\frac{1}{2}})$ is observable, $x_i \to 0$ as $i \to \infty$ independently of i_0. Therefore, the closed-loop system is asymptotically stable. This completes the proof. ∎

We suggest a sufficient condition for Theorem 3.18.

Theorem 3.19. *Assume that the pair (A, B) is stabilizable and the pair $(A, Q^{\frac{1}{2}})$ is observable. For $Q_f \geq 0$ satisfying (3.234), the system (3.216) with the receding horizon H_∞ control (3.222) is asymptotically stable for some N, $1 \leq N < \infty$.*

In the above theorem, Q must be nonzero. We can introduce another result as in a receding horizon LQ control so that Q could even be zero.

Suppose that disturbances show up. From (3.229) we have

$$u_i^* = -R^{-1} B^T P_1^{-1} A x_i \tag{3.265}$$

where

$$P_i = A^{-1} P_{i+1} [I + A^{-1} Q A^{-1} P_{i+1}]^{-1} A^{-1} + \Pi \tag{3.266}$$

$$P_N = M_N^{-1} + \Pi = Q_f^{-1} + \Pi \tag{3.267}$$

We will consider a slightly different approach. We assume that P_{i,i_f} in (2.154) is given from the beginning with a terminal constraint $P_{i_f,i_f} = P_f$ rather than P_{i_f,i_f} being obtained from (2.156).

In fact, Riccati Equation (2.154) with the boundary condition P_f can be obtained from the following problem. Consider the following system:

$$\hat{x}_{i+1} = A^{-T}\hat{x}_i + A^{-1}Q^{\frac{1}{2}}\hat{u}_i \tag{3.268}$$

where $\hat{x}_i \in \Re^n$, $\hat{u}_i \in \Re^m$, and a performance criterion

$$\hat{J}(\hat{x}_{i_0}, i_0, i_f) = \sum_{i=i_0}^{i_f-1}[\hat{x}_i^T \Pi \hat{x}_i + \hat{u}_i^T \hat{u}_i] + \hat{x}_{i_f}^T P_f \hat{x}_{i_f} \tag{3.269}$$

The optimal cost for the system (3.268) is given by $\hat{J}^*(\hat{x}_i, i, i_f) = \hat{x}_i^T P_{i,i_f}\hat{x}_i$. The optimal control $\hat{u}_i$ is

$$\hat{u}_{i,i_f} = -R^{-1}B^T P_{i+1,i_f}^{-1} A\hat{x}_i \tag{3.270}$$

From Theorem 3.16, it can be easily seen that $P_{\tau,\sigma+1} \geq P_{\tau,\sigma}$ if

$$P_f \leq A^{-1}P_f[I + A^{-T}QA^{-1}P_f]^{-1}A^{-T} + \Pi \tag{3.271}$$

Now, we are in a position to state the following result on the stabiltiy of the receding horizon H_∞ control.

Theorem 3.20. *Assume that the pair (A, B) is controllable and A is nonsingular. If the inequality (3.271) is satisfied, then the system (3.216) with the control (3.265) is asymptotically stable for $1 \leq N$.*

Proof. Consider the adjoint system of the system (3.216) with the control (3.270)

$$\hat{x}_{i+1} = [A - BR^{-1}B^T P_1^{-1}A]^{-T}\hat{x}_i \tag{3.272}$$

and the associated scalar-valued function

$$V(\hat{x}_i) = \hat{x}_i^T A^{-1}P_1 A^{-1}\hat{x}_i \tag{3.273}$$

Note that $P_1 - BR^{-1}B^T$ is nonsingular, which guarantees the nonsingularity of $A - BR^{-1}B^T P_1^{-1}A$ with a nonsingular A.

Subtracting $V(\hat{x}_{i+1})$ from $V(\hat{x}_i)$, we have

$$V(\hat{x}_i) - V(\hat{x}_{i+1}) = \hat{x}_i^T A^{-1}P_1 A^{-1}\hat{x}_i - \hat{x}_{i+1}^T A^{-1}P_1 A^{-1}\hat{x}_{i+1} \tag{3.274}$$

Recall the following relation:

$$\begin{aligned} P_0 &= (A^T P_1^{-1} A + Q)^{-1} + \Pi = A^{-1}(P_1^{-1} + A^{-T}QA^{-1})^{-1}A^{-T} + \Pi \\ &= A^{-1}\left[P_1 - P_1 A^{-T}Q^{\frac{1}{2}}(Q^{\frac{1}{2}}A^{-1}P_1 A^{-T}Q^{\frac{1}{2}} + I)^{-1}Q^{\frac{1}{2}}A^{-1}P_1\right]A^{-T} + \Pi \\ &= A^{-1}P_1 A^{-T} + \Pi - Z \end{aligned} \tag{3.275}$$

where

$$Z = A^{-1}P_1A^{-T}Q^{\frac{1}{2}}(Q^{\frac{1}{2}}A^{-1}P_1A^{-T}Q^{\frac{1}{2}} + I)^{-1}Q^{\frac{1}{2}}A^{-1}P_1A^{-T}$$

Replacing $\hat{x}_i$ with $[A - BR^{-1}B^TP_1^{-1}A]^T\hat{x}_{i+1}$ in (3.274) and plugging (3.275) into the second term in (3.274) yields

$$\begin{aligned} V(\hat{x}_i) - V(\hat{x}_{i+1}) &= \hat{x}_{i+1}^T[P_1 - 2BR^{-1}B^T + BR^{-1}B^TP_1^{-1}BR^{-1}B^T]\hat{x}_{i+1} \\ &\quad - \hat{x}_{i+1}^T[P_0 - \Pi + Z]\hat{x}_{i+1} \\ &= -\hat{x}_{i+1}^T[BR^{-1}B^T - BR^{-1}B^TP_1^{-1}BR^{-1}B^T]\hat{x}_{i+1} \\ &\quad - \hat{x}_{i+1}^T[P_0 - P_1 + \gamma^{-2}B_wR_w^{-1}B_w^T + Z]\hat{x}_{i+1} \end{aligned}$$

Since Z is positive semidefinite and $P_0 - P_1 \geq 0$, we have

$$V(\hat{x}_i) - V(\hat{x}_{i+1}) \leq -\hat{x}_{i+1}^T[BR^{-\frac{1}{2}}SR^{-\frac{1}{2}}B^T + \gamma^{-2}B_wR_w^{-1}B_w]\hat{x}_{i+1} \quad (3.276)$$

where $S = I - R^{-\frac{1}{2}}B^TP_1^{-1}BR^{-\frac{1}{2}}$.

In order to show the positive definiteness of S, we have only to prove $P_1 - BR^{-1}B^T > 0$ since

$$I - P_1^{-\frac{1}{2}}BR^{-1}B^TP_1^{-\frac{1}{2}} > 0 \Longleftrightarrow P_1 - BR^{-1}B^T > 0$$

Note that $I - AA^T > 0$ implies $I - A^TA > 0$ and vice versa for any rectangular matrix A. From the condition for the existence of the saddle point, the lower bound of P is obtained as

$$\begin{aligned} & R_w - \gamma^{-2}B_w^TM_iB_w = R_w - \gamma^{-2}B_w^T(P_i - \Pi)^{-1}B_w > 0 \\ \Longleftrightarrow\ & I - \gamma^{-2}(P_i - \Pi)^{-\frac{1}{2}}B_wR_w^{-1}B_w^T(P_i - \Pi)^{-\frac{1}{2}} > 0 \\ \Longleftrightarrow\ & P_i - \Pi - \gamma^{-2}B_wR_w^{-1}B_w^T = P_i - BR^{-1}B^T > 0 \\ \Longleftrightarrow\ & P_i > BR^{-1}B^T \end{aligned} \quad (3.277)$$

From (3.277), it can be seen that S in (3.276) is positive definite. Note that the left-hand side in (3.276) is always nonnegative. From (3.276) we have

$$V(\hat{x}(i+1;\hat{x}_{i_0},i_0)) - V(\hat{x}_{i_0},i_0) \geq \hat{x}_{i_0}^T\Theta\hat{x}_{i_0}$$

where

$$\begin{aligned} \Theta &\triangleq \left[\sum_{k=i_0}^{i}\Psi^{(i-i_0)T}W\Psi^{i-i_0}\right] \\ \Psi &\triangleq A - BR^{-1}B^TP_1^{-1}A \\ W &\triangleq BR^{-\frac{1}{2}}SR^{-\frac{1}{2}}B^T + \gamma^{-2}B_wR_w^{-1}B_w \end{aligned}$$

If (A, B) is controllable, then the matrix Θ is positive definite. Thus, all eigenvalues of Θ are positive and the following inequality is obtained:

$$V(\hat{x}(i+1;\hat{x}_{i_0},i_0)) - V(\hat{x}_{i_0}) \geq \lambda_{\min}(\Theta)\|\hat{x}_{i_0}\| \tag{3.278}$$

This implies that the closed-loop system (3.216) is exponentially increasing, i.e. the closed-loop system (3.216) with (3.270) is exponentially decreasing. This completes the proof. ∎

In Theorem 3.20, Q can be zero. If Q becomes zero, then P_1 can be expressed as the following closed form:

$$P_1 = \sum_{j=i+1}^{i+N} A^{j-i-1} \Pi A^{(j-i-1)T} + A^N P_f A^{TN} \tag{3.279}$$

where A is nonsingular.

It is noted that P_f satisfying (3.271) is equivalent to Q_f satisfying (3.243) in the relation of $P_f = Q_f^{-1} + \Pi$. Replacing P_f with $Q_f^{-1} + \Pi$ in (3.271) yields the following inequality:

$$\begin{aligned} Q_f^{-1} + \Pi &\leq A^{-1}[Q_f^{-1} + BR^{-1}B^T + A^{-T}QA^{-1}]^{-1}A^{-T} + \Pi \\ &= [A^T(Q_f^{-1} + BR^{-1}B^T)^{-1}A + Q]^{-1} + \Pi \end{aligned}$$

Finally, we have

$$Q_f \geq A^T(Q_f^{-1} + \Pi)^{-1}A + Q \tag{3.280}$$

Therefore, if Q_f satisfies (3.280), P_f also satisfies (3.271).

Theorem 3.21. *Assume that the pair (A, B) is controllable and A is nonsingular.*

(1) If $M_{i+1} \geq M_i > 0$ for some i, then the system (3.216) with the receding horizon H_∞ control (3.222) is asymptotically stable for $1 \leq N < \infty$.
(2) For $Q_f > 0$ satisfies (3.243) for some H, then the system (3.216) with the RH H_∞ control (3.222) is asymptotically stable for $1 \leq N < \infty$.

Proof. The first part is proved as follows. $M_{i+1} \geq M_i > 0$ implies $0 < M_{i+1}^{-1} \leq M_i^{-1}$, from which we have $0 < P_{i+1} \leq P_i$ satisfying the inequality (3.271). Thus, the control (3.265) is equivalent to the control (3.222). The second part is proved as follows: inequalities $K_{i+1} \geq K_i > 0$ are satisfied for K_i generated from $Q_f > 0$ satisfying (3.234) for some H. Thus, the second result can be seen from the first one. This completes the proof. ∎

It is noted that (3.280) is equivalent to (3.247), as mentioned before.

3.4.4 Additional Properties

Now, we will show that the stabilizing receding horizon controllers guarantee the H_∞ norm bound of the closed-loop system.

Theorem 3.22. *Under the assumptions given in Theorem 3.18, the H_∞ norm bound of the closed-loop system (3.216) with (3.222) is guaranteed.*

Proof. Consider the difference of the optimal cost between the time i and $i+1$:

$$\begin{aligned} & J^*(i+1, i+N+1) - J^*(i, i+N) \\ = & \sum_{j=i+1}^{i+N} \left[x_j^T Q x_j + u_j^T R u_j - \gamma^2 w_j^T R_w w_j \right] + x_{i+N+1}^T Q_f x_{i+N+1} \\ - & \sum_{j=i}^{i+N-1} \left[x_j^T Q x_j + u_j^T R u_j - \gamma^2 w_j^T R_w w_j \right] - x_{i+N}^T Q_f x_{i+N} \end{aligned} \quad (3.281)$$

Note that the optimal control and the worst-case disturbance on the horizon are time-invariant with respect to the moving horizon.

Applying the state feedback control $u_{i+N} = Hx_{i+N}$ at time $i+N$ yields the following inequality:

$$\begin{aligned} J^*(i+1, i+N+1) - J^*(i, i+N) \leq & -x_i^T Q x_i - u_i^T R u_i + \gamma^{-2} w_i^T R_w w_i \\ & + \begin{bmatrix} w_{i+N} \\ x_{i+N} \end{bmatrix}^T \Pi \begin{bmatrix} w_{i+N} \\ x_{i+N} \end{bmatrix} \end{aligned} \quad (3.282)$$

where

$$\Pi \triangleq \begin{bmatrix} -\gamma^2 R_w + B_w^T Q_f B_w & B_w^T Q_f (A + BH) \\ (A+BH)^T Q_f B_w & (A+BH)^T Q_f (A+BH) - Q_f + Q + H^T R H \end{bmatrix}$$

From the cost monotonicity condition, Π is guaranteed to be positive semidefinite. The proof is left as an exercise. Taking the summation on both sides of (3.282) from $i = 0$ to ∞ and using the positiveness of Π, we have

$$\begin{aligned} J^*(0, N) - J^*(\infty, \infty + N) = & \sum_{i=0}^{\infty} [J^*(i, i+N) - J^*(i+1, i+N+1)] \\ \geq & \sum_{i=0}^{\infty} [x_i^T Q x_i + u_i^T R u_i - \gamma^2 w_i^T R_w w_i] \end{aligned}$$

From the assumption $x_0 = 0$, $J^*(0, N) = 0$. The saddle-point optimal cost is guaranteed to be nonnegative, i.e. $J^*(\infty, \infty + N) \geq 0$. Therefore, it is guaranteed that

$$\sum_{i=0}^{\infty}[x_i^T Q x_i + u_i^T R u_i - \gamma^2 w_i^T R_w w_i] \le 0$$

which implies that

$$\frac{\sum_{i=0}^{\infty}[x_i^T Q x_i + u_i^T R u_i]}{\sum_{i=0}^{\infty} w_i^T R_w w_i} \le \gamma^2$$

This completes the proof. ∎

In the same way, under the assumptions given in Theorem 3.20, the H_∞ norm bound of the closed-loop system (3.216) with (3.222) is guaranteed with M_1 replaced by $[P_1 - \Pi]^{-1}$. The inverse matrices exist for $N \ge l_c + 1$ since $P_{i_f - i} - \Pi = [A^T P_{i_f-i-1}^{-1} A + Q]^{-1}$.

Example 3.3

In this example, the H_∞ RHC is compared with the LQ RHC through simulation. The target model and the reference signal are the same as those of Example 3.1. except that B_w is given by

$$B_w = \begin{bmatrix} 0.016 & 0.01 & 0.008 & 0 \\ 0.002 & 0.009 & 0 & 0.0005 \end{bmatrix}^T \tag{3.283}$$

For simulation, disturbances coming into the system are generated so that they become worst on the receding horizon. γ^2 is taken as 1.5.

As can be seen in Figure 3.10, the trajectory for the H_∞ RHC is less deviated from the reference signal than that for the LQ RHC.

The MATLAB® functions used for simulation are given in Appendix F.

3.5 Receding Horizon Control via Linear Matrix Inequality Forms

3.5.1 Computation of Cost Monotonicity Condition

Receding Horizon Linear Quadratic Control

It looks difficult to find H and Q_f that satisfy the cost monotonicity condition (3.73). However, this can be easily computed using LMI.

Pre- and post-multiplying on both sides of (3.73) by Q_f^{-1}, we obtain

$$X \ge XQX + XH^T RHX + (AX - BHX)^T X^{-1}(AX - BHX) \tag{3.284}$$

where $X = Q_f^{-1}$. Using Schur's complement, the inequality (3.284) is converted into the following:

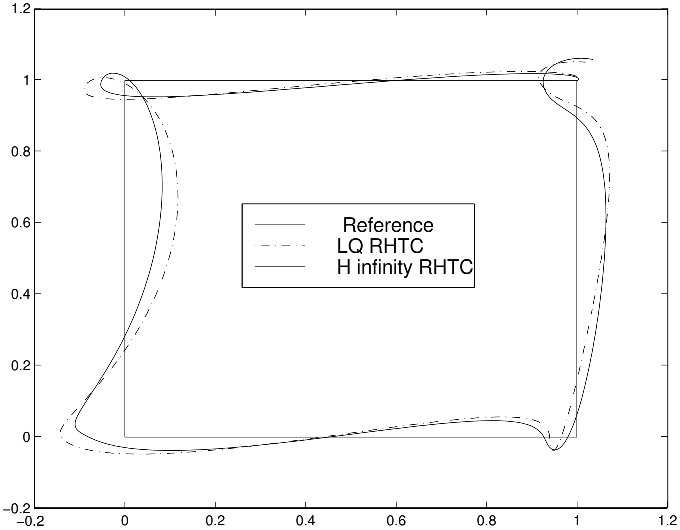

Fig. 3.10. Comparison between LQ RHTC and H_∞ RHTC

$$X - XQX - Y^T RY - (AX - BY)^T X^{-1}(AX - BY) \geq 0$$

$$\begin{bmatrix} X - XQX - Y^T RY & (AX - BY)^T \\ AX - BY & X \end{bmatrix} \geq 0 \tag{3.285}$$

where $Y = HX$. Partitioning the left side of (3.285) into two parts, we have

$$\begin{bmatrix} X & (AX - BY)^T \\ AX - BY & X \end{bmatrix} - \begin{bmatrix} XQX - Y^T RY & 0 \\ 0 & 0 \end{bmatrix} \geq 0 \tag{3.286}$$

In order to use Schur's complement, the second block matrix is decomposed as

$$\begin{bmatrix} X & (AX - BY)^T \\ AX - BY & X \end{bmatrix} - \begin{bmatrix} Q^{\frac{1}{2}}X & 0 \\ R^{\frac{1}{2}}Y & 0 \end{bmatrix}^T \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}^{-1} \begin{bmatrix} Q^{\frac{1}{2}}X & 0 \\ R^{\frac{1}{2}}Y & 0 \end{bmatrix} \geq 0 \tag{3.287}$$

Finally, we can obtain the LMI form as

$$\begin{bmatrix} X & (AX - BY)^T & (Q^{\frac{1}{2}}X)^T & (R^{\frac{1}{2}}Y)^T \\ AX - BY & X & 0 & 0 \\ Q^{\frac{1}{2}}X & 0 & I & 0 \\ R^{\frac{1}{2}}Y & 0 & 0 & I \end{bmatrix} \geq 0 \tag{3.288}$$

Once X and Y are found, Q_f and $H = YX^{-1}$ can be known.

Example 3.4

For the following systems and the performance criterion:

$$x_{k+1} = \begin{bmatrix} 0.6831 & 0.0353 \\ 0.0928 & 0.6124 \end{bmatrix} x_k + \begin{bmatrix} 0.6085 & 0.0158 \end{bmatrix} u_k \quad (3.289)$$

$$J(x_k, k, k+N) = \sum_{i=0}^{N-1} \left[x_{k+i}^T x_{k+i} + 3u_{k+i}^2 \right] + x_{k+N}^T Q_f x_{k+N} \quad (3.290)$$

The MATLAB® code for finding Q_f satisfying the LMI (3.288) is given in Appendix F. By using this MATLAB® program, we have one possible final weighting matrix for the cost monotonicity

$$Q_f = \begin{bmatrix} 0.4205 & -0.0136 \\ -0.0136 & 0.4289 \end{bmatrix} \quad (3.291)$$

Similar to (3.73), the cost monotonicity condition (3.85) can be represented as an LMI form. First, in order to obtain an LMI form, the inequality (3.85) is converted into the following:

$$Q_f - A^T Q_f [I + BR^{-1}B^T Q_f]^{-1} A - Q \geq 0 \quad (3.292)$$

$$\begin{bmatrix} Q_f - Q & A^T \\ A & Q_f^{-1} + BR^{-1}B^T \end{bmatrix} \geq 0 \quad (3.293)$$

Pre- and post-multiplying on both sides of (3.293) by some positive definite matrices, we obtain

$$\begin{bmatrix} Q_f^{-1} & 0 \\ 0 & I \end{bmatrix}^T \begin{bmatrix} Q_f - Q & A^T \\ A & Q_f^{-1} + BR^{-1}B^T \end{bmatrix} \begin{bmatrix} Q_f^{-1} & 0 \\ 0 & I \end{bmatrix} \geq 0 \quad (3.294)$$

$$\begin{bmatrix} X - XQX & XA^T \\ AX & X + BR^{-1}B^T \end{bmatrix} \geq 0 \quad (3.295)$$

where $Q_f^{-1} = X$

Partition the left side of (3.295) into two parts, we have

$$\begin{bmatrix} X & XA^T \\ AX & X + BR^{-1}B^T \end{bmatrix} - \begin{bmatrix} XQX & 0 \\ 0 & 0 \end{bmatrix} \geq 0 \quad (3.296)$$

In order to use Schur's complement, the second block matrix is decomposed as

$$\begin{bmatrix} X & (AX + BY)^T \\ AX + BY & X \end{bmatrix} - \begin{bmatrix} Q^{\frac{1}{2}} X & 0 \\ 0 & 0 \end{bmatrix}^T \begin{bmatrix} I & 0 \\ 0 & I \end{bmatrix}^{-1} \begin{bmatrix} Q^{\frac{1}{2}} X & 0 \\ 0 & 0 \end{bmatrix} \geq 0 \quad (3.297)$$

Finally, we can obtain the LMI form as

$$\begin{bmatrix} X & XA^T & (Q^{\frac{1}{2}} X)^T & 0 \\ AX & X + BR^{-1}B^T & 0 & 0 \\ Q^{\frac{1}{2}} X & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \geq 0 \quad (3.298)$$

Once X is obtained, Q_f is given by X^{-1}.

The cost monotonicity condition (3.98) in Theorem 3.4 can be easily obtained by changing the direction of the inequality of (3.298):

$$\begin{bmatrix} X & XA^T & (Q^{\frac{1}{2}}X)^T & 0 \\ AX & X+BR^{-1}B^T & 0 & 0 \\ Q^{\frac{1}{2}}X & 0 & I & 0 \\ 0 & 0 & 0 & I \end{bmatrix} \leq 0 \tag{3.299}$$

In the following section, stabilizing receding horizon controls will be obtained by LMIs.

Receding Horizon H_∞ Control

The cost monotonicity condition (3.234) can be written

$$\begin{bmatrix} \Gamma \\ I \end{bmatrix}^T \begin{bmatrix} R_w - B_w^T Q_f B_w & B_w^T Q_f (A-BH) \\ (A-BH)^T Q_f B_w & \Phi \end{bmatrix} \begin{bmatrix} \Gamma \\ I \end{bmatrix} \geq 0 \tag{3.300}$$

where

$$\Phi = Q_f - Q - H^T RH - (A-BH)^T Q_f (A-BH) \tag{3.301}$$

From (3.300), it can be seen that we have only to find Q_f such that

$$\begin{bmatrix} R_w - B_w^T Q_f B_w & B_w^T Q_f (A-BH) \\ (A-BH)^T Q_f B_w & \Phi \end{bmatrix} \geq 0 \tag{3.302}$$

where we have

$$\begin{bmatrix} R_w & 0 \\ 0 & Q_f - Q - H^T RH \end{bmatrix} - \begin{bmatrix} B_w^T \\ (A-BH)^T \end{bmatrix} Q_f \begin{bmatrix} B_w^T \\ (A-BH)^T \end{bmatrix}^T \geq 0 \tag{3.303}$$

By using Schur's complement, we can obtain the following matrix inequality:

$$\begin{bmatrix} R_w & 0 & B_w^T \\ 0 & Q_f - Q - H^T RH & (A-BH)^T \\ B_w & (A-BH) & Q_f^{-1} \end{bmatrix} \geq 0 \tag{3.304}$$

Multiplying both sides of (3.304) by the matrix $\mathrm{diag}\{I, Q_f^{-1}, I\}$ yields

$$\begin{bmatrix} R_w & 0 & B_w^T \\ 0 & X - XQX - XH^T RHX & X(A-BH)^T \\ B_w & (A-BH)X & X \end{bmatrix} \geq 0 \tag{3.305}$$

where $Q_f^{-1} = X$. Since the matrix in (3.305) is decomposed as

$$\begin{bmatrix} R_w & 0 & B_w^T \\ 0 & X & (AX-BY)^T \\ B_w & AX-BY & X \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ XQ^{\frac{1}{2}} & YR^{\frac{1}{2}} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ XQ^{\frac{1}{2}} & YR^{\frac{1}{2}} \\ 0 & 0 \end{bmatrix}^T$$

we have

$$\begin{bmatrix} R_w & 0 & B_w^T & 0 & 0 \\ 0 & X & (AX-BY)^T & XQ^{\frac{1}{2}} & YR^{\frac{1}{2}} \\ B_w & AX-BY & X & 0 & 0 \\ 0 & Q^{\frac{1}{2}}X & 0 & I & 0 \\ 0 & R^{\frac{1}{2}}Y^T & 0 & 0 & I \end{bmatrix} \geq 0 \tag{3.306}$$

where $Y = HX$.

3.5.2 Receding Horizon Linear Quadratic Control via Batch and Linear Matrix Inequality Forms

In the previous section, the receding horizon LQ control was obtained analytically in a closed form, and thus it can be easily computed. Here, how to achieve the receding horizon LQ control via an LMI is discussed, which will be utilized later in constrained systems.

Free Terminal Cost

The state equation in (3.3) can be written as

$$X_k = Fx_k + HU_k \tag{3.307}$$

$$U_k = \begin{bmatrix} u_k \\ u_{k+1} \\ \vdots \\ u_{k+N-1} \end{bmatrix}, \ X_k = \begin{bmatrix} x_k \\ x_{k+1} \\ \vdots \\ x_{k+N-1} \end{bmatrix}, \ F = \begin{bmatrix} I \\ A \\ \vdots \\ A^{N-1} \end{bmatrix} \tag{3.308}$$

$$H = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ B & 0 & 0 & \cdots & 0 \\ AB & B & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ A^{N-2}B & A^{N-3}B & \cdots & B & 0 \end{bmatrix} \tag{3.309}$$

The terminal state is given by

$$x_{k+N} = A^N x_k + \bar{B}U_k \tag{3.310}$$

where

$$\bar{B} = \begin{bmatrix} A^{N-1}B & A^{N-2}B & \cdots & B \end{bmatrix} \tag{3.311}$$

Let us define

$$\bar{Q}_N = \text{diag}\{\overbrace{Q, \cdots, Q}^{N}\}, \quad \bar{R}_N = \text{diag}\{\overbrace{R, \cdots, R}^{N}\} \tag{3.312}$$

Then, the cost function (3.22) can be rewritten by

$$\begin{aligned} J(x_k, U_k) &= [X_k - X_k^r]^T \bar{Q}_N [X_k - X_k^r] + U_k^T \bar{R}_N U_k \\ &\quad + (x_{k+N} - x_{k+N}^r)^T Q_f (x_{k+N} - x_{k+N}^r) \end{aligned}$$

where

$$X_k^r = \begin{bmatrix} x_k^r \\ x_{k+1}^r \\ \vdots \\ x_{k+N-1}^r \end{bmatrix}$$

From (3.307) and (3.310), the above can be represented by

$$\begin{aligned} J(x_k, U_k) &= [Fx_k + HU_k - X_k^r]^T \bar{Q}_N [Fx_k + HU_k - X_k^r] + U_k^T \bar{R}_N U_k \\ &+ [A^N x_k + \bar{B} U_k - x_{k+N}^r]^T Q_f [A^N x_k + \bar{B} U_k - x_{k+N}^r] \\ &= U_k^T [H^T \bar{Q}_N H + \bar{R}_N] U_k + 2[Fx_k - X_k^r]^T \bar{Q}_N H U_k \\ &+ [Fx_k - X_k^r]^T \bar{Q}_N [Fx_k - X_k^r] \\ &+ [A^N x_k + \bar{B} U_k - x_{k+N}^r]^T Q_f [A^N x_k + \bar{B} U_k - x_{k+N}^r] \\ &= U_k^T W U_k + w^T U_k + [Fx_k - X_k^r]^T \bar{Q}_N [Fx_k - X_k^r] \\ &+ [A^N x_k + \bar{B} U_k - x_{k+N}^r]^T Q_f [A^N x_k + \bar{B} U_k - x_{k+N}^r] \end{aligned} \tag{3.313}$$

where $W = H^T \bar{Q}_N H + \bar{R}_N$ and $w = 2H^T \bar{Q}_N^T [Fx_k - X_k^r]$. The optimal input can be obtained by taking $\frac{\partial J(x_k, U_k)}{\partial U_k}$. Thus we have

$$\begin{aligned} U_k &= -[W + \bar{B}^T Q_f \bar{B}]^{-1} [w + \bar{B}^T Q_f (A^N x_k - x_{k+N}^r)] \\ &= -[W + \bar{B}^T Q_f \bar{B}]^{-1} [H^T \bar{Q}_N (Fx_k - X_k^r) \\ &+ \bar{B}^T Q_f (A^N x_k - x_{k+N}^r)] \end{aligned} \tag{3.314}$$

The RHC can be obtained as

$$u_k = \begin{bmatrix} 1, 0, \cdots, 0 \end{bmatrix} U_k^* \tag{3.315}$$

In order to obtain an LMI form, we decompose the cost function (3.313) into two parts

$$J(x_k, U_k) = J_1(x_k, U_k) + J_2(x_k, U_k)$$

where

$$\begin{aligned} J_1(x_k, U_k) &= U_k^T W U_k + w^T U_k + [Fx_k - X_k^r]^T \bar{Q}_N [Fx_k - X_k^r] \\ J_2(x_k, U_k) &= (A^N x_k + \bar{B} U_k - x_{k+N}^r)^T Q_f (A^N x_k + \bar{B} U_k - x_{k+N}^r) \end{aligned}$$

Assume that

$$U_k^T W U_k + w^T U_k + [Fx_k - X_k^r]^T \bar{Q}_N [Fx_k - X_k^r] \le \gamma_1 \tag{3.316}$$
$$(A^N x_k + \bar{B}U_k - x_{k+N}^r)^T Q_f (A^N x_k + \bar{B}U_k - x_{k+N}^r) \le \gamma_2 \tag{3.317}$$

Note that

$$J(x_k, U_k) \le \gamma_1 + \gamma_2 \tag{3.318}$$

From Schur's complement, (3.316) and (3.317) are equivalent to

$$\begin{bmatrix} \gamma_1 - w^T U_k - [Fx_k - X_k^r]^T \bar{Q}_N [Fx_k - X_k^r] & U_k^T \\ U_k & W^{-1} \end{bmatrix} \ge 0 \tag{3.319}$$

and

$$\begin{bmatrix} \gamma_2 & [A^N x_k + \bar{B}U_k - x_{k+N}^r]^T \\ [A^N x_k + \bar{B}U_k - x_{k+N}^r] & Q_f^{-1} \end{bmatrix} \ge 0 \tag{3.320}$$

respectively. Finally, the optimal solution U_k^* can be obtained by an LMI problem as follows:

$$\min_{U_k} \quad \gamma_1 + \gamma_2 \quad \text{subject to} \quad (3.319) \text{ and } (3.320)$$

Therefore, the RHC in a batch form is obtained by

$$u_k = \begin{bmatrix} 1, 0, \cdots, 0 \end{bmatrix} U_k^* \tag{3.321}$$

Terminal Equality Constraint

The optimal control (3.314) can be rewritten by

$$\begin{aligned} U_k &= -\left[\begin{bmatrix} H \\ \bar{B} \end{bmatrix}^T \begin{bmatrix} \bar{Q}_N & 0 \\ 0 & Q_f \end{bmatrix} \begin{bmatrix} H \\ \bar{B} \end{bmatrix} + \bar{R}_N \right]^{-1} \begin{bmatrix} H \\ \bar{B} \end{bmatrix}^T \begin{bmatrix} \bar{Q}_N & 0 \\ 0 & Q_f \end{bmatrix} \\ &\quad \times \left[\begin{bmatrix} F \\ A^N \end{bmatrix} x_k - \begin{bmatrix} X_k^r \\ x_{k+N}^r \end{bmatrix} \right] \\ &= -\bar{R}_N^{-1} \left[\begin{bmatrix} H \\ \bar{B} \end{bmatrix}^T \begin{bmatrix} \bar{Q}_N & 0 \\ 0 & Q_f \end{bmatrix} \begin{bmatrix} H \\ \bar{B} \end{bmatrix} \bar{R}_N^{-1} + I \right]^{-1} \begin{bmatrix} H \\ \bar{B} \end{bmatrix}^T \begin{bmatrix} \bar{Q}_N & 0 \\ 0 & Q_f \end{bmatrix} \\ &\quad \times \left[\begin{bmatrix} F \\ A^N \end{bmatrix} x_k - \begin{bmatrix} X_k^r \\ x_{k+N}^r \end{bmatrix} \right] \end{aligned} \tag{3.322}$$

We define

$$\bar{H} = \begin{bmatrix} H \\ \bar{B} \end{bmatrix} \quad \bar{F} = \begin{bmatrix} F \\ A^N \end{bmatrix} \quad \bar{X}_k^r = \begin{bmatrix} X_k^r \\ x_{k+N}^r \end{bmatrix} \tag{3.323}$$

Then, using the formula $(I + AB)^{-1}A = A(I + BA)^{-1}$, we have

$$
\begin{aligned}
U_k &= -\bar{R}_N^{-1}\bar{H}^T[\widehat{Q}_N\bar{H}\bar{R}_N^{-1}\bar{H}^T + I]^{-1}\widehat{Q}_N[\bar{F} \ \ -I]\begin{bmatrix} x_k \\ \bar{X}_k^r \end{bmatrix} \\
&= -\bar{R}_N^{-1}\bar{H}^T[\widetilde{Q}_{N2}\bar{H}\bar{R}_N^{-1}\bar{H}^T + \widetilde{Q}_{N1}^{-1}]^{-1}\widetilde{Q}_{N2}[\bar{F} \ \ -I]\begin{bmatrix} x_k \\ \bar{X}_k^r \end{bmatrix} \quad (3.324)
\end{aligned}
$$

where

$$
\widehat{Q}_N = \begin{bmatrix} \bar{Q}_N & 0 \\ 0 & Q_f \end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & Q_f \end{bmatrix}\begin{bmatrix} \bar{Q}_N & 0 \\ 0 & I \end{bmatrix} = \widetilde{Q}_{N1}\widetilde{Q}_{N2} \quad (3.325)
$$

For terminal equality constraint, we take $Q_f = \infty I$ ($Q_f^{-1} = 0$). So U_k is given as (3.324) with $\widetilde{Q}_{N1}^{-1}$ replaced by $\begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$.

We introduce an LMI-based solution. In a fixed terminal case, (3.317) is not used. Instead, the condition $A^N x_k + \bar{B}U_k = x_{k+N}^r$ should be met. Thus, we need an equality condition together with an LMI. In order to remove the equality representation, we parameterize U_k in terms of known variables according to Theorem A.3. We can set U_k as

$$
U_k = -\bar{B}^{-1}(A^N x_k - x_{k+N}^r) + M\hat{U}_k \quad (3.326)
$$

where $\bar{B}^{-1}$ is the right inverse of $\bar{B}$ and columns of M are orthogonal to each other, spanning the null space of $\bar{B}$.

From (3.316) we have

$$
\begin{aligned}
J(x_k, U_k) &= U_k^T W U_k + w^T U_k + [Fx_k - X_k^r]^T\bar{Q}_N[Fx_k - X_k^r] \\
&= (-\bar{B}^{-1}(A^N x_k - x_{k+N}^r) + M\hat{U}_k)^T W(-\bar{B}^{-1}(A^N x_k - x_{k+N}^r) \\
&+ M\hat{U}_k) + w^T(-\bar{B}^{-1}(A^N x_k - x_{k+N}^r) + M\hat{U}_k) + [Fx_k - X_k^r]^T\bar{Q}_N \\
&\times [Fx_k - X_k^r] \\
&= \hat{U}_k^T\mathcal{V}_1\hat{U}_k + \mathcal{V}_2\hat{U}_k + \mathcal{V}_3
\end{aligned}
$$

where

$$
\begin{aligned}
\mathcal{V}_1 &= M^T W M \\
\mathcal{V}_2 &= -2(A^N x_k - x_{k+N}^r)^T\bar{B}^{-T}WM + w^T M \\
\mathcal{V}_3 &= (A^N x_k - x_{k+N}^r)^T\bar{B}^{-T}WB^{-1}(A^N x_k - x_{k+N}^r) \\
&\quad + [Fx_k - X_k^r]^T\bar{Q}_N[Fx_k - X_k^r] - w^T\bar{B}^{-1}(A^N x_k - x_{k+N}^r)
\end{aligned}
$$

The optimal input can be obtained by taking $\frac{\partial J(x_k,\hat{U}_k)}{\partial \hat{U}_k}$. Thus we have

$$
\hat{U}_k = -\mathcal{V}_1^{-1}\mathcal{V}_2^T
$$

The RHC in a batch form can be obtained as in (3.315). The optimal control for the fixed terminal case can be obtained from the following inequality:

$$
J(x_k, \hat{U}_k) = \hat{U}_k^T\mathcal{V}_1\hat{U}_k + \mathcal{V}_2\hat{U}_k + \mathcal{V}_3 \le \gamma_1
$$

which can be transformed into the following LMI:

$$\min \gamma_1$$
$$\begin{bmatrix} \gamma_1 - \mathcal{V}_2 \hat{U}_k - \mathcal{V}_3 & -\hat{U}_k^T \mathcal{V}_1^{\frac{1}{2}} \\ -\mathcal{V}_1^{\frac{1}{2}} \hat{U}_k & I \end{bmatrix} \geq 0$$

where $\hat{U}_k$ is obtained. U_k is computed from this according to (3.326). What remains to do is just to pick up the first one among U_k as in (3.321).

GPC for the CARIMA model (3.194) can be obtained in a batch form similar to that presented above. From the state-space model (3.200), we have

$$y_{k+j} = \bar{C}\bar{A}^j x_k + \sum_{i=0}^{j-1} \bar{C}\bar{A}^{j-i-1}\bar{B}\triangle u_{k+i} \tag{3.327}$$

The performance index (3.204) can be represented by

$$\begin{aligned} J = & \left[Y_k^r - V x_k - W\triangle U_k\right]^T \bar{Q} \left[Y_k^r - V x_k - W\triangle U_k\right] + \triangle U_k^T \bar{R} \triangle U_k \\ & + \left[Y_{k+N_c}^r - V_f x_k - W_f \triangle U_k\right]^T \bar{Q}_f \left[Y_{k+N_c}^r - V_f x_k - W_f \triangle U_k\right] \end{aligned} \tag{3.328}$$

where

$$Y_k^r = \begin{bmatrix} y_{k+1}^r \\ \vdots \\ y_{k+N_c}^r \end{bmatrix}, \; V = \begin{bmatrix} \bar{C}\bar{A} \\ \vdots \\ \bar{C}\bar{A}^{N_c} \end{bmatrix}, \; \triangle U_k = \begin{bmatrix} \triangle u_k \\ \vdots \\ \triangle u_{k+N_c-1} \end{bmatrix}$$

$$Y_{k+N_c}^r = \begin{bmatrix} y_{k+N_c+1}^r \\ \vdots \\ y_{k+N_p}^r \end{bmatrix}, \; V_f = \begin{bmatrix} \bar{C}\bar{A}^{N_c+1} \\ \vdots \\ \bar{C}\bar{A}^{N_p} \end{bmatrix}, \; W = \begin{bmatrix} \bar{C}\bar{B} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ \bar{C}\bar{A}^{N_c-1}\bar{B} & \cdots & \bar{C}\bar{B} \end{bmatrix}$$

$$W_f = \begin{bmatrix} \bar{C}\bar{A}^{N_c}\bar{B} & \cdots & \bar{C}\bar{A}\bar{B} \\ \vdots & \ddots & \vdots \\ \bar{C}\bar{A}^{N_p-1}\bar{B} & \cdots & \bar{C}\bar{A}^{N_p-N_c}\bar{B} \end{bmatrix}, \; \bar{R} = \left[\mathrm{diag}(\overbrace{r \; r \; \cdots \; r}^{N_c})\right]$$

$$\bar{Q}_f = \left[\mathrm{diag}(\overbrace{q_f \; q_f \; \cdots \; q_f}^{N_p - N_c})\right], \; \bar{Q} = \left[\mathrm{diag}(\overbrace{q \; q \; \cdots \; q}^{N_c})\right].$$

Using

$$\frac{\partial J}{\partial \triangle U_k} = 0$$

we can obtain

$$\begin{aligned} \triangle U_k = & \left[W^T \bar{Q} W + W_f^T \bar{Q}_f W_f + \bar{R}\right]^{-1} \Big\{ W^T \bar{Q} \left[Y_k^r - V x_k\right] \\ & + W_f^T \bar{Q}_f \left[Y_{k+N_c}^r - V_f x_k\right] \Big\} \end{aligned}$$

Therefore, $\triangle u_k$ is given by

$$\triangle u_k = \begin{bmatrix} I \ 0 \cdots \ 0 \end{bmatrix} \begin{bmatrix} W^T \bar{Q} W + W_f^T \bar{Q}_f W_f + \bar{R} \end{bmatrix}^{-1} \Big\{ W^T \bar{Q} \left[Y_k^r - V x_k \right] + W_f^T \bar{Q}_f \left[Y_{k+N_c}^r - V_f x_k \right] \Big\} \tag{3.329}$$

3.5.3 Receding Horizon H_∞ Control via Batch and Linear Matrix Inequality Forms

In the previous section, the receding horizon H_∞ control was obtained analytically in a closed form and thus it can be easily computed. Here, how to achieve the receding horizon H_∞ control via LMI is discussed.

The state equation (3.9) can be represented by

$$X_k = F x_k + H U_k + H_w W_k \tag{3.330}$$

where H_w is given by

$$H_w = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ G & 0 & 0 & \cdots & 0 \\ AG & G & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ A^{N-2}G & A^{N-3}G & \cdots & G & 0 \end{bmatrix} \tag{3.331}$$

and U_k, F, X_k, and H are defined in (3.308) and (3.309).

The H_∞ performance criterion can be written in terms of the augmented matrix as

$$\begin{aligned} J(x_k, U_k, W_k) = {} & [F x_k + H U_k + H_w W_k - X_k^r]^T \bar{Q}_N [F x_k + H U_k + H_w W_k \\ & - X_k^r] + [A^N x_k + \bar{B} U_k + \bar{G} W_k - x_{k+N}^r]^T Q_f [A^N x_k + \bar{B} U_k \\ & + \bar{G} W_k - x_{k+N}^r] + U_k^T \bar{R}_N U_k - \gamma^2 W_k^T W_k \end{aligned}$$

Representing $J(x_k, U_k, W_k)$ in quadratic form with respect to W_k yields the following equation:

$$\begin{aligned} J(x_k, U_k, W_k) = {} & W_k^T \mathcal{V}_1 W_k + 2 W_k^T \mathcal{V}_2 + [F x_k + H U_k - X_k^r]^T \bar{Q}_N [F x_k + H U_k \\ & - X_k^r] + U_k^T \bar{R}_N U_k + [A^N x_k + \bar{B} U_k - x_{k+N}^r]^T Q_f [A^N x_k + \bar{B} U_k \\ & - x_{k+N}^r] \\ = {} & [\mathcal{V}_1 W_k + \mathcal{V}_2]^T \mathcal{V}_1^{-1} [\mathcal{V}_1 W_k + \mathcal{V}_2] - \mathcal{V}_2^T \mathcal{V}_1^{-1} \mathcal{V}_2 + U_k^T \bar{R}_N U_k \\ & + [F x_k + H U_k - X_k^r]^T \bar{Q}_N [F x_k + H U_k - X_k^r] \\ & + [A^N x_k + \bar{B} U_k - x_{k+N}^r]^T Q_f [A^N x_k + \bar{B} U_k - x_{k+N}^r] \\ = {} & [\mathcal{V}_1 W_k + \mathcal{V}_2]^T \mathcal{V}_1^{-1} [\mathcal{V}_1 W_k + \mathcal{V}_2] + U_k^T \mathcal{P}_1 U_k + 2 U_k^T \mathcal{P}_2 \\ & + \mathcal{P}_3 \end{aligned} \tag{3.332}$$

where

$$\mathcal{V}_1 \triangleq -\gamma^2 I + \bar{G}^T Q_f \bar{G} + H_w^T \bar{Q}_N H_w \tag{3.333}$$

$$\mathcal{V}_2 \triangleq H_w^T \bar{Q}_N^T [F x_k + H U_k - X_k^r] + \bar{G}^T Q_f^T [A^N x_k + \bar{B} U_k - x_{k+N}^r] \tag{3.334}$$

$$\begin{aligned}\mathcal{P}_1 \triangleq & -(H_w^T \bar{Q}_N^T H + \bar{G}^T Q_f^T \bar{B})^T \mathcal{V}_1^{-1} (H_w^T \bar{Q}_N^T H + \bar{G}^T Q_f^T \bar{B}) \\ & + H^T \bar{Q}_N H + \bar{R}_N + \bar{B}^T Q_f \bar{B}\end{aligned} \tag{3.335}$$

$$\begin{aligned}\mathcal{P}_2 \triangleq & -(H_w^T \bar{Q}_N^T H + \bar{G}^T Q_f^T \bar{B})^T \mathcal{V}_1^{-1} (H_w^T \bar{Q}_N^T (F x_k - X_k^r) \\ & + \bar{G}^T Q_f^T (A^N x_k - x_{k+N}^r)) + H^T \bar{Q}_N F x_k + \bar{B}^T Q_f A^N x_k\end{aligned} \tag{3.336}$$

and $\mathcal{P}_3$ is a constant that is independent of U_k and W_k.

In order that the solution to the saddle point exists, $\mathcal{V}_1$ must be negative. Thus, we have

$$-\gamma^2 I + \bar{G}^T Q_f \bar{G} + H_w^T \bar{Q}_N H_w < 0$$

In order to maximize (3.332) with respect to W_k, we have only to maximize

$$[\mathcal{V}_1 W_k + \mathcal{V}_2]^T \mathcal{V}_1^{-1} [\mathcal{V}_1 W_k + \mathcal{V}_2] \tag{3.337}$$

to obtain

$$W_k = -\mathcal{V}_1^{-1} \mathcal{V}_2 \tag{3.338}$$

If we put (3.338) into (3.332), (3.332) can be represented by

$$J(x_k, U_k, W_k) = U_k^T \mathcal{P}_1 U_k + 2 U_k^T \mathcal{P}_2 + \mathcal{P}_3 \tag{3.339}$$

Then the optimal input can be obtained by taking $\frac{\partial J(x_k, U_k, W_k)}{\partial U_k}$. Thus we have

$$U_k = -\mathcal{P}_1^{-1} \mathcal{P}_2$$

Now we can introduce an LMI form for the receding horizon H_∞ control. In order to maximize (3.332) with respect to W_k, we have only to minimize

$$-[\mathcal{V}_1 W_k + \mathcal{V}_2]^T \mathcal{V}_1^{-1} [\mathcal{V}_1 W_k + \mathcal{V}_2] \tag{3.340}$$

Then we try to minimize (3.339). It follows finally that we have the following LMI:

$$\min_{U_k, W_k} \quad r_1 + r_2 \tag{3.341}$$

$$\begin{bmatrix} r_1 - \mathcal{P}_2^T U_k & U_k^T \\ U_k & \mathcal{P}_1^{-1} \end{bmatrix} \geq 0$$

$$\begin{bmatrix} r_2 & (\mathcal{V}_1 W_k + \mathcal{V}_2)^T \\ (\mathcal{V}_1 W_k + \mathcal{V}_2) & -\mathcal{V}_1 \end{bmatrix} \geq 0$$

The stabilizing RH H_∞ control can be obtained by solving the semidefinite program (3.306) and (3.341) where $Q_f = X^{-1}$. What remains to do is just to pick up the first one among U_k as in (3.321).

An LMI representation in this section would be useful for constrained systems.

3.6 References

In order to explain the receding horizon concept, the predictor form and the reference predictive form are introduced first in Section 3.2.1 of this chapter.

The primitive form of the RH control was given in [Kle70] [Kle74], where only input energy with fixed terminal constraint is concerned without the explicit receding horizon concept. The general form of the RHC was first given with receding horizon concepts in [KP77a], where state weighting is considered. The RHTC presented in Section 3.3.1 is similar to that in [KB89].

With a terminal equality constraint which corresponds to the infinite terminal weighting matrix, the closed-loop stability of the RHC was first proved in a primitive form [Kle70] and in a general form [KP77a]. There are also some other results in [Kle74] [KP78] [AM80] [NS97].

The terminal equality constraint in Theorem 3.1 is a well-known result.

Since the terminal equality constraint is somewhat strong, finite terminal weighting matrices for the free terminal cost have been investigated in [Yaz84] [BGP85] [KB89] [PBG88] [BGW90] [DC93] [NP97] [LKC98]. The monotone property of the Ricatti equation is used for the stability [KP77a]. Later, the monotone property of the optimal cost was introduced not only for linear, but also for nonlinear systems. At first, the cost monotonicity condition was used for the terminal equality constraint [KRC92] [SC94][RM93][KBM96] [LKL99]. The cost monotonicity condition for free terminal cost in Theorem 3.2 is first given in [LKC98]. The general proof of Theorem 3.2 is a discrete version of [KK00]. The inequality (3.84) is a special case of (3.73) and is partly studied in [KB89] [BGW90]. The terminal equality constraint comes historically before the free terminal cost. The inequality between the terminal weighting matrix and the steady-state Riccati solution in Proposition 3.3 appeared first in this book.

The opposite direction of the cost monotonicity in Theorem 3.4 is first introduced for discrete systems in this book. It is shown in [BGW90] that once the monotonicity of the Riccati equation holds at a certain point it holds for all subsequent times as in Theorem 3.5.

The stability of RHCs in Theorems 3.6 and 3.7 is first introduced in [LKC98] and the general proofs of these theorems in this book are discrete versions of [KK00].

The stability of the RHC in the case of the terminal equality constraint in Theorem 3.7 is derived by using Theorems 3.1 and 3.6.

A stabilizing control in Theorem 3.9 is first introduced in [LKC98] without a proof, and thus a proof is included in this book by using Lyapunov theory.

The observability in Theorems 3.6 and 3.9 can be weakened with detectability, similar to that in [KK00].

The results on Theorems 3.10 and 3.11 appear first in this book and are extensions of [KP77a]. For time-invariant systems, the controllability in Theorems 3.10 and 3.11 can be weakened with stabilizability, as shown in [RM93] and [KP77a]. The closed-loop stability of the RHC via FARE appears in [PBG88, BGW90]

The lower bound of the horizon size stabilizing the system in Theorem 3.12 appeared in [JHK04].

The RH LQ control with a prescribed degree of stability appeared in [KP77b] for continuous-time systems. In Section 3.3.5 of this book, slight modifications are made to obtain it for discrete-time systems.

The upper and lower bounds of the performance criteria in Theorems 3.13 and 3.14 are discrete versions of the result [KBK83] for continuous-time systems.

It was shown in [KBK83] that the RH LQ control stabilizes the system for a sufficiently large horizon size irrespective of the final weighting matrix.

The RH H_∞ control presented in Section 3.4.1 is a discrete version of the work by [KYK01]. The cost monotonicity condition of the RH H_∞ control in Theorems 3.15, 3.16, and 3.17 is a discrete version of the work by [KYK01]. The stability of the RH H_∞ control in Theorems 3.18 and 3.19 also appeared in [KYK01]. The free terminal cost in the above theorems was proposed in [LG94] [LKL99]. The relation between the free terminal cost and the monotonicity of the saddle point value was fully discussed in [KYK01].

The RH H_∞ control without requiring the observability of $(A, Q^{\frac{1}{2}})$, as in Theorems 3.20 and 3.21, is first discussed in this book in parallel with the RH LQ control.

The guaranteed H_∞ norm of the H_∞ RHC in Theorem 3.22 is first given in this book for discrete-time systems by a modification of the result on continuous-time systems in [KYK01].

In [LKC98], how to obtain the receding horizon control and a final weighting matrix satisfying the cost monotonicity condition was discussed by using LMIs. Sections 3.5.1 and 3.5.2 are mostly based on [LKC98].

The RHLQC with the equality constraint and the cost monotonicity condition for the H_∞ RHC in an LMI form appear first in Sections 3.5.2 and 3.5.3 of this book respectively.

3.7 Problems

3.1. Referring to Problem 2.6, make simulations for three kinds of planning based on Table 1.1. α, γ, β, $\bar{u}$ are set to 0.8, 1.3, 10, and 1 respectively. For long-term planning, use $N = 100$. For periodic and short-term planning, use $N = 5$ and a simulation time of 100.

3.2. Derive a cost monotonicity condition for the following performance criterion for the system (3.1):

$$J(x_{i_0}, u.) = \sum_{i=i_0}^{i_f-1} \begin{bmatrix} x_i \\ u_i \end{bmatrix}^T \begin{bmatrix} Q & S \\ S^T & R \end{bmatrix} \begin{bmatrix} x_i \\ u_i \end{bmatrix} + x_{i_f}^T Q_f x_{i_f}$$

3.3. (1) If Q_f satisfies a cost monotonicity condition, show that the RHC with this Q_f can be an infinite horizon optimal control (2.107) with some nonnegative symmetric Q and some positive definite R.

(2) Verify that the RHC with the equality constraint has the property that it is an infinite horizon optimal control (2.107) associated with some nonnegative symmetric Q and some positive definite R.

3.4. Consider the cost monotonicity condition (3.73).

(1) Show that the condition (3.73) can be represented as

$$Q_f \ \geq \ \min_{H} \left\{ Q + H^T R H + (A - BH)^T Q_f (A - BH) \right\} \qquad (3.342)$$

(2) Choose H so that the right side of (3.342) is minimized.

3.5. Consider a discrete-time system as

$$x_{i+1} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} x_i + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_i \qquad (3.343)$$

(1) Find an RHC for the following performance criterion:

$$x_{k+1|k}^T \begin{bmatrix} 1 & 2 \\ 2 & 6 \end{bmatrix} x_{k+1|k} + u_{k|k}^2 \qquad (3.344)$$

where the horizon size is 1. Check the stability.

(2) Find an RHC for the following performance criterion:

$$\sum_{j=0}^{1} \{ x_{k+j|k}^T \begin{bmatrix} 1 & 2 \\ 2 & 6 \end{bmatrix} x_{k+j|k} + u_{k+j|k}^2 \} + x_{k+2|k}^T \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} x_{k+2|k} \qquad (3.345)$$

where the horizon size is 2. Check the stability.

(3) In the problem (b), introduce the final weighting matrix as

$$\sum_{j=0}^{1} \{ x_{k+j|k}^T \begin{bmatrix} 1 & 2 \\ 2 & 6 \end{bmatrix} x_{k+j|k} + u_{k+j|k}^2 \} + x_{k+2|k}^T Q_f x_{k+2|k} \qquad (3.346)$$

and find Q_f such that the system is stabilized.

3.6. Suppose that Q_f is positive definite and the system matrix A is nonsingular.

(1) Prove that the solution to Riccati Equation (3.49) is positive definite.

(2) Let $V(x_i) = x_i^T A^{-1} (K_1^{-1} + B R^{-1} B^T) A^{-T} x_i$, where K_1 is obtained from the Riccati equation starting from $K_N = Q_f$, then show that the system can be stabilized. (Hint: use Lasalle's theorem and the fact that if A is Hurwitz, then so is A^T.)

Remark: in the above problem, the observability of $(A, Q^{\frac{1}{2}})$ is not required.

3.7. Prove the stability of the RHC (3.49) by using Lyapunov theory.

(1) Show that K_1 defined in (3.47) satisfies

$$K_1 \geq (A - BL_1)^T K_1 (A - BL_1) + L_1^T R L_1 + Q \tag{3.347}$$

starting from $K_N = Q_f$ satisfying (3.73) and $L_1 = [R + B^T K_1 B]^{-1} B^T K_1 A$.

(2) Show that $x_i^T K_1 x_i$ in (3.347) can be a Lyapunov function. Additionally, show the stability of the RHC under assumptions that (A, B) and $(A, Q^{\frac{1}{2}})$ are stabilizable and observable respectively.

3.8. Consider the FARE (3.122). Suppose that (A, B) is stabilizable, $\bar{Q} \geq 0$, and $(A, \bar{Q}^{\frac{1}{2}})$ is observable. If $K_{i_0+2} - 2K_{i_0+1} + K_{i_0} \leq 0$ for some i_0, then the system with the RHC (3.55) is stable for any $N \geq i_0$.

3.9. $*$ Denote the control horizon and the prediction horizon as N_c and N_p respectively. This book introduces various RHC design methods in the case of $N = N_c = N_p$. When we use different control and prediction horizons $(N_c \neq N_p)$:

(1) discuss the effect on the computational burden.
(2) discuss the effect on the optimal performance.

3.10. In this chapter, $\|A\|_{\rho,\epsilon}$ is introduced.

(1) Take an example that does not satisfy the following inequality

$$\rho(AB) \leq \rho(A)\rho(B)$$

where $\rho(A)$ is the spectral radius.

(2) Show that there always exists a matrix norm $\|A\|_{\rho,\epsilon}$ such that

$$\rho(A) \leq \|A\|_{\rho,\epsilon} \leq \rho(A) + \epsilon \tag{3.348}$$

for any $\epsilon > 0$.

(3) Disprove that $\rho(A) \leq 1$ implies $\|A\|_2 \leq 1$

3.11. Let K_i be the solution to the difference Riccati equation (2.45) and L_i its corresponding state feedback gain (2.57). K and L are the steady-state values of K_i and L_i.

(1) Show that

$$L_{i+1} - L = -R_{o,i+1}^{-1} B^T \triangle K_{i+1} A_c \tag{3.349}$$

$$A_{c,i+1} = A - BL_{i+1} = (I - BR_{o,i+1}^{-1} B^T \triangle K_{k+1}) A_c \tag{3.350}$$

where

$$R_{o,i+1} \triangleq R + B^T K_i B, \quad \triangle K_i \triangleq K_i - K, \quad A_c \triangleq A - BL$$

(2) Show that

$$\triangle K_i = A_c^T [\triangle K_{i+1} - \triangle K_{i+1} B R_{o,i+1}^{-1} B^T \triangle K_{i+1}] A_c \tag{3.351}$$

3.12. Suppose that the pair (A, B) and $(A, Q^{\frac{1}{2}})$ are controllable and observable respectively.

(1) Show that the closed-loop system can be written as

$$x_{i+1} = G_i x_i + BR^{-1}B^T \hat{K}^e_{i+1,N} A x_i \tag{3.352}$$

with

$$G_i = A - BR^{-1}B^T \hat{K}_{i+1,\infty} A \tag{3.353}$$

$$\hat{K}_{i+1,i+N} = [K^{-1}_{i+1,i+N} + BR^{-1}B^T]^{-1} \tag{3.354}$$

$$\hat{K}^e_{i+1,N} = \hat{K}_{i+1,\infty} - \hat{K}_{i+1,i+N} \tag{3.355}$$

where $K_{i+1,i+N}$ is given in (3.47) and $K_{i+1,\infty}$ is the steady-state solution of (3.47).

(2) Prove that, for all x,

$$\lim_{N\to\infty} \frac{|BR^{-1}B^T \hat{K}^e_{i+1,N} A x|}{|x|} = 0 \tag{3.356}$$

(3) Show that there exists a finite horizon size N such that the RHC (3.56) stabilizes the closed-loop system.
Hint. Use the following fact: suppose that $x_{i+1} = f(x_i)$ is asymptotically stable and $g(x_i, i)$ satisfies the equality $\lim_{i\to\infty} \frac{g(x_i,i)}{x_i} = 0$. Then, $x_{i+1} = f(x_i) + g(x_i, i)$ is also stable.

3.13. A state-space model is given as

$$x_{i+1} = \begin{bmatrix} 2 & 1 \\ 3 & 4 \end{bmatrix} x_i + \begin{bmatrix} 2 \\ 3 \end{bmatrix} u_i \tag{3.357}$$

where

$$Q = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}, \quad R = 2 \tag{3.358}$$

(1) According to the formula (3.149), find a lower bound of the horizon size N that guarantees the stability irrespective of the final weighting matrix Q_f.
(2) Calculate a minimum horizon size stabilizing the closed-loop systems by direct computation of the Riccati equation and closed-loop poles.

3.14. MAC used the following model:

$$y_k = \sum_{i=0}^{n-1} h_i u_{k-i}$$

(1) Obtain a state-space model.
(2) Obtain an RHC with the following performance:

$$J = \sum_{j=0}^{N-1} \{q[y_{k+j|k} - y^r_{k+j|k}]^2 + ru^2_{k+j|k}\}$$

3.15. DMC used the following model:

$$y_k = \sum_{i=0}^{n-1} g_i \triangle u_{k-i}$$

where $\triangle u_k = u_k - u_{k-1}$.

(1) Obtain a state-space model.
(2) Obtain an RHC with the following performance:

$$J = \sum_{j=0}^{N-1} \{q[y_{k+j|k} - y^r_{k+j|k}]^2 + r[\triangle u_{k+j|k}]^2\}.$$

3.16. Consider the CARIMA model (3.194). Find an optimal solution for the performance criterion (3.204).

3.17. (1) Show that

$$\begin{aligned} &Q_f - Q + H^T RH - \Gamma^T R_w \Gamma + (A - BH + B_w \Gamma)^T Q_f (A - BH + B_w \Gamma) \\ &\geq Q_f - Q + H^T RH + (A - BH)(Q_f^{-1} - B_w R_w^{-1} B_w^T)^{-1}(A - BH) \end{aligned} \quad (3.359)$$

holds irrespective of Γ.
(2) Find out Γ such that the equality holds in (3.359).
(3) Show that

$$Q_f - Q + H^T RH + (A - BH)(Q_f^{-1} - B_w R_w^{-1} B_w^T)^{-1}(A - BH) \geq 0$$

can be represented in the following LMI form:

$$\begin{bmatrix} X & (AX - BY)^T & (Q^{\frac{1}{2}} X)^T & (R^{\frac{1}{2}} Y)^T \\ AX - BY & X - B_w R_w^{-1} B_w^T & 0 & 0 \\ Q^{\frac{1}{2}} X & 0 & I & 0 \\ R^{\frac{1}{2}} Y & 0 & 0 & I \end{bmatrix} \geq 0 \quad (3.360)$$

where $X = Q_f^{-1}$ and $Y = HQ_f^{-1}$.

3.18. Consider the cost monotonicity condition (3.234) in the RH H_∞ control.

(1) Show that (3.234) is equivalent to the following performance criterion:

$$\max_w \; [(x^T Qx + u^T Ru - r^2 w^T w) - x^T Q_f x] \leq 0 \quad (3.361)$$

(2) Show that if (3.234) holds, then the following inequality is satisfied:

$$\begin{bmatrix} -\gamma^2 I + B_w^T Q_f B_w & B_1^T Q_f (A + B_2 H) \\ (A + B_2 H)^T Q_f B_1 & (A + BH)^T Q_f (A + BH) - Q_f + Q + H^T H \end{bmatrix} \leq 0$$

3.19. If H is replaced by an optimal gain $H = -R^{-1} B^T [I + Q_f \Pi]^{-1} Q_f A$, then show that we can have (3.243) by using the matrix inversion lemma.

3.20. As shown in Figure 3.11, suppose that there exists an input uncertainty $\triangle$ described by

$$\tilde{x}_{k+1} = \tilde{A}\tilde{x}_k + \tilde{B}\tilde{u}_k$$
$$\tilde{y}_k = \tilde{C}\tilde{x}_k$$

where the feedback interconnection is given by

$$\tilde{u}_k = u_k^{RHC}$$
$$u_k = -\tilde{y}_k$$

The input $\tilde{y}_k$ and output $\tilde{u}_k$ of the uncertainty $\triangle$ satisfy

$$\mathcal{V}(\tilde{x}_{k+1}) - \mathcal{V}(\tilde{x}_k) \leq \tilde{y}_k^T \tilde{u}_k - \rho \tilde{u}_k^T \tilde{u}_k$$

where $\mathcal{V}(x_k)$ is some nonnegative function (this is called the dissipative property) and ρ is a constant. If ρ is greater than $\frac{1}{4}$ and the H_∞ RHC (3.222) is adopted, show that the H_∞ norm bound of the closed-loop system with this input uncertainty is still guaranteed.

Hint: use the cost monotonicity condition.

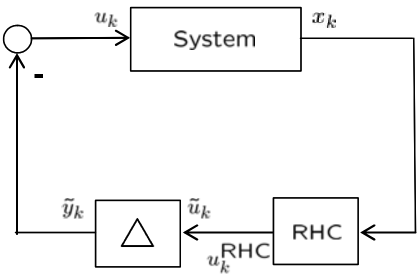

Fig. 3.11. Feedback Interconnection of Problem 3.20

3.21. The state equation (3.1) can be transformed into

$$X_{k+j} = F_j x_{k+j} + H_j U_{k+j}$$
$$x_{k+N} = A^{N-j} x_{k+j} + \bar{B}_j U_{k+j}$$
$$\bar{X}_{k+j} = \bar{F}_j x_{k+j} + \bar{H}_j U_{k+j}$$

where $0 \leq j \leq N-1$.

$$U_{k+j} = \begin{bmatrix} u_{k+j} \\ u_{k+j+1} \\ \vdots \\ u_{k+N-1} \end{bmatrix}, \quad X_{k+j} = \begin{bmatrix} x_{k+j} \\ x_{k+j+1} \\ \vdots \\ x_{k+N-1} \end{bmatrix}$$

$$F_j = \begin{bmatrix} I \\ A \\ \vdots \\ A^{N-1-j} \end{bmatrix} = \begin{bmatrix} I \\ F_{j+1}A \end{bmatrix}$$

$$H_j = \begin{bmatrix} 0 & 0 & 0 & \cdots & 0 \\ B & 0 & 0 & \cdots & 0 \\ AB & B & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ A^{N-2-j}B & A^{N-3-j}B & \cdots & B & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ F_{j+1}B & H_{j+1} \end{bmatrix}$$

$$\bar{B}_j = \begin{bmatrix} A^{N-1-j}B & A^{N-2-j}B & \cdots & AB & B \end{bmatrix}$$

$$\bar{X}_{k+j} = \begin{bmatrix} X_{k+j} \\ x_{k+N} \end{bmatrix}, \quad \bar{F}_j = \begin{bmatrix} F_j \\ A^{N-j} \end{bmatrix}, \quad \bar{H}_j = \begin{bmatrix} H_j \\ \bar{B}_j \end{bmatrix}$$

(1) We define

$$K_j = \bar{F}_j^T \hat{Q}_j \bar{F}_j - \bar{F}_j^T \hat{Q}_j \bar{H}_j (\bar{H}_j^T \hat{Q}_j \bar{H}_j + \bar{R}_j)^{-1} \bar{H}_j^T \hat{Q}_j \bar{F}_j$$

where

$$\hat{Q}_j = \text{diag}\{\overbrace{Q, \cdots, Q}^{N-j+1} \; Q_f\}, \quad \bar{R}_j = \text{diag}\{\overbrace{R, \cdots, R}^{N-j+1}\}.$$

Then, show that the optimal control (3.314) can be rewritten by

$$\begin{aligned} U_{k+j} &= -[\bar{H}_j^T \hat{Q}_j \bar{H}_j + \bar{R}_j]^{-1} \bar{H}_j^T \hat{Q}_j \bar{F}_j x_{k+j} \qquad (3.362) \\ &= \begin{bmatrix} -[R + B^T K_{j+1} B]^{-1} B^T K_{j+1} A x_{k+j} \\ -[\bar{R}_{j+1} + \bar{H}_{j+1}^T \hat{Q}_{j+1} \bar{H}_{j+1}]^{-1} \bar{H}_{j+1}^T \hat{Q}_{j+1} \bar{F}_{j+1} x_{k+j+1} \end{bmatrix} \\ &= \begin{bmatrix} u_{k+j} \\ U_{k+j+1} \end{bmatrix} \end{aligned}$$

(2) Show that the above-defined K_j satisfies (3.47), i.e. the recursive solution can be obtained from a batch form of solution.

3.22. Consider the GPC (3.329) for the CARIMA model (3.194).

(a) Using (3.329), obtain the GPC $\triangle u_k$ when $Q_f = \infty I$.
(b) Show that the above GPC is asymtotically stable.

3.23. In Section 2.5, the optimal control U_k on the finite horizon was obtained from the LMI approach. Derive an LMI for the control gain H of $U_k = Hx_k$, not the control U_k itself.