7

Important links

In this chapter we sketch some important links between ideas of the dressing Darboux transformation (DT), Bäcklund transformation (BT), etc. with related mathematical constructions. Firstly, it is the Hirota representation which originally produced many of the known families of multisoliton solutions, and these have often led to a disclosure of the underlying Lax systems and infinite sets of conserved quantities [209, 385]. In Sect. 7.1 we demonstrate a systematic derivation of the bilinear BTs from the so-called $\mathcal{Y}$-systems which are formulated in terms of the binary Bell polynomials. Taking as the example equations with the "sech^2" soliton solutions, we illustrate how to obtain the binary BTs for different weights of the $\mathcal{Y}$-polynomials. In Sect. 7.2 we represent the Darboux covariant Lax pairs in terms of the $\mathcal{Y}$-systems. In Sect. 7.3 we explain how to construct BTs from the explicit dressing formulas and, using the Noether theorem, how to derive discrete and continuous conservation laws. Next, in Sect. 7.4 the main formulas of the dressing theory are retrieved within the Weiss–Tabor–Carnevale procedure [449] of Painlevé analysis for partial differential equations (PDEs). In addition, we comment on a historical point connected with the appearance of the dressing method in the Zakharov–Shabat theory. Namely, we suggest in Sect. 7.5 an original revisiting of the technique of inverse scattering transform (IST) in terms of the Gel'fand–Levitan–Marchenko integral equation. Notice in connection with this that the search for perhaps the most general dressing scheme within the framework of the Zakharov and Shabat ideas is represented in [478].

7.1 Bilinear formalism. The Hirota method

A striking feature of the bilinear formalism is the ease with which direct insight can be gained into the nature of the eigenvalue problem associated with soliton equations (such as the KdV, Boussinesq, or Sawada–Kotera equations) derivable from the bilinear Hirota equation (representation) for a single Hirota function. The key element is the bilinear form of the BT which can be straightforwardly obtained from the Hirota representation of these equations, through decoupling of a related "two-field condition" by means of an appropriate constraint of minimal weight [262]. The main point is that bilinear BTs are obtained systematically, without the need for tricky *exchange formulas* [209]. They arise in the form of "$\mathcal{Y}$-systems," each equation within such a system belonging to a linear space spanned by the basis of binary Bell polynomials ($\mathcal{Y}$-polynomials) [187].

An important element is the logarithmic linearizability of $\mathcal{Y}$-systems, which implies that each bilinear BT can be mapped onto a corresponding linear system of the Lax type. However, it turns out that these linear systems involve differential operators which, even in the simplest case, do *not* constitute a Darboux covariant [265, 324] Lax pair . This fact prevents us from obtaining large classes of solutions by direct application of the powerful Darboux machinery to the systems which arise by straightforward linearization of the $\mathcal{Y}$-systems. Here we present a simple scheme to resolve this difficulty for a variety of soliton equations which allow a bilinear BT that comprises a constraint of the lowest possible weight (weight 2). Darboux covariant Lax pairs for the KdV, Boussinesq, and Lax equations are obtained in a unified manner, by exploiting the relations between the coefficients of linear differential operators connected by the classical DT. Exponential Bell polynomials [44] and generalized "multipotential" $\mathcal{Y}$-systems are found to be useful for this purpose. This approach reveals deep connections between the (1+1)-dimensional equations and the underlying (higher-dimensional) Kadomtsev–Petviashvili (KP) hierarchy. We start our discussion by recalling the basic properties of the $\mathcal{Y}$-polynomials (derived in [187]) and by indicating how the use of the $\mathcal{Y}$-basis can lead systematically from the original nonlinear PDEs to the associated linear systems. The example of the Lax equation is instructive since this fifth-order equation has no single bilinear Hirota representation. The content of this section follows [260].

7.1.1 Binary Bell polynomials

The class of exponential Bell polynomials, originally defined for the Abelian entries as

$$Y_{mx}(v) = Y_m(v_x, v_{xx}, ..., v_{mx}) \equiv e^{-v} \frac{\partial^m}{\partial x^m} e^v, \quad m \in \mathbb{Z}, \tag{7.1}$$

was introduced in Sect. 2.1. It keeps a balance between linear and quadratic terms of the (generalized) Burgers equation, for

$$Y_{mx}(\ln \psi) = \psi_{mx}/\psi. \tag{7.2}$$

Examples are easily derived and are given in Sect. 2.1. The property of x-homogeneity,

$$Y_{m(\lambda x)}(v) = \lambda^{-m}(v) Y_{mx}(v), \tag{7.3}$$

introduces the weight m.

The binary polynomials that we shall use in this section are defined in terms of the exponential Bell polynomials

$$Y_{mx,nt}(f) = \mathrm{e}^{-f}\partial_x^m \partial_t^n \mathrm{e}^f \tag{7.4}$$

as follows:

$$\mathcal{Y}_{mx,nt}(v,w) \equiv Y_{mx,nt}(f)\Big|_{f_{px,qt} = \begin{cases} v_{px,qt} & \text{if } p+q = \text{odd}, \\ w_{px,qt} & \text{if } p+q = \text{even}, \end{cases}} \tag{7.5}$$

with the understanding that $f_{px,qt} \equiv \partial_x^p \partial_t^q f$. They inherit the easily recognizable partition structure of the Bell polynomials (for a recurrent definition see Sect. 2.2):

$$\begin{aligned} \mathcal{Y}_x(v) &= v_x, \\ \mathcal{Y}_{2x}(v,w) &= w_{2x} + v_x^2, \\ \mathcal{Y}_{x,t}(v,w) &= w_{xt} + v_x v_t, \\ \mathcal{Y}_{3x}(v,w) &= v_{3x} + 3v_x w_{2x} + v_x^3, \ \cdots \end{aligned} \tag{7.6}$$

The link between the $\mathcal{Y}$-polynomials and the standard Hirota expression

$$D_x^p D_t^q G' \cdot G \equiv (\partial_x - \partial_{x'})^p \, (\partial_t - \partial_{t'})^q \, G'(x,t) G(x',t')\Big|_{x'=x,t'=t} \tag{7.7}$$

is given by the identity

$$\mathcal{Y}_{mx,nt}(v = \ln G'/G, \ w = \ln G'G) \equiv (G'G)^{-1} D_x^m D_t^n G' \cdot G. \tag{7.8}$$

In the particular case $G' = G$, one has

$$G^{-2} D_x^m D_t^n G \cdot G \equiv \mathcal{Y}_{mx,nt}(0, \ Q = 2\ln G) = \begin{cases} 0, & \text{if} \quad m+n = \text{odd}, \\ P_{mx,nt}(Q), & \text{if} \quad m+n = \text{even}, \end{cases} \tag{7.9}$$

the P-polynomials being characterized by an equally recognizable "even part" partition structure:

$$\begin{aligned} &P_{2x}(Q) = Q_{2x}, \quad P_{x,t}(Q) = Q_{xt}, \quad P_{4x}(Q) = Q_{4x} + 3Q_{2x}^2, \\ &P_{6x}(Q) = Q_{6x} + 15Q_{2x}Q_{4x} + 15Q_{2x}^3, \ldots. \end{aligned} \tag{7.10}$$

A crucial property of the $\mathcal{Y}$-polynomials relates to the transformation $w = v + Q$, $v = \ln\psi$:

$$\begin{aligned} &\mathcal{Y}_{px,qt}(v, w = v + Q)\Big|_{v=\ln\psi} \\ &= \psi^{-1} \sum_{j=0}^{p} \sum_{\substack{k=0 \\ j+k=\text{even}}}^{q} \binom{p}{q} \binom{q}{k} P_{jx,kt}(Q) \psi_{(p-j)x,(q-k)t} \end{aligned} \tag{7.11}$$

and originates from the addition formula for the polynomials $Y(v)$:

$$Y_{mx}(v_1 + v_2) = \sum_{j=0}^{m} \binom{m}{j} Y_{(m-j)x}(v_1) Y_{jx}(v_2). \tag{7.12}$$

The proof is performed by use of the Newton–Leibnitz formula.

It should also be noticed that polynomials $\mathcal{Y}_{px,qt}(v,w)$, constructed with the derivatives of dimensionless variables v and w, are homogeneous expressions of the weight $p+qr$, if r stands for the dimension of t (the dimension of x is chosen equal to 1).

7.1.2 $\mathcal{Y}$-systems associated with "sech^2" soliton equations

We consider four examples of "sech^2" soliton equations with the order ranging from 3 to 5: the KdV, Boussinesq, Lax, and Sawada–Kotera equations.

KdV equation

The invariance of the KdV equation

$$\mathrm{KdV}(u) \equiv u_t + u_{3x} + 6uu_x = 0 \tag{7.13}$$

under the scale transformation

$$x \to \lambda x, \quad t \to \lambda^3 t, \quad u \to \lambda^{-2} u \tag{7.14}$$

shows that u has the dimension -2. A dimensionless field Q can be introduced by setting $u = cQ_{2x}$, with c being a dimensionless parameter to be determined. The resulting equation for Q can be derived from the *potential* equation

$$Q_{xt} + Q_{4x} + 3cQ_{2x}^2 = 0, \tag{7.15}$$

which can be cast into the form

$$E(Q) \equiv P_{xt}(Q) + P_{4x}(Q) \equiv G^{-2}(D_x D_t + D_x^4) G \cdot G \Big|_{G=\exp(Q/2)} = 0 \tag{7.16}$$

by setting $c = 1$.

The well-known Hirota *two-field condition* on G and G', to be satisfied as a differential consequence of a bilinear BT (that we have to find), takes the form [209]

$$G'^{-2}(D_x D_t + D_x^4) G' \cdot G' - G^{-2}(D_x D_t + D_x^4) G \cdot G = 0. \tag{7.17}$$

It corresponds to the following condition on $Q = 2\ln G = w - v$ and $Q' = 2\ln G' = w + v$:

$$\begin{aligned} &E(w+v) - E(w-v) = 2(v_{xt} + v_{4x} + 6v_{2x}w_{2x}) \\ &\equiv 2\left\{\partial_x\left[\mathcal{Y}_t(v) + \mathcal{Y}_{3x}(v,w)\right] + 6W[\mathcal{Y}_{2x}(v,w), \mathcal{Y}_x(v)]\right\} = 0, \end{aligned} \tag{7.18}$$

where $W(\mathcal{Y}_1, \mathcal{Y}_2)$ is the Wronskian. This condition can easily be decoupled into a pair of equations in the form of linear combinations of the $\mathcal{Y}$-polynomials. It suffices to impose such a constraint on v and w (p_j and q_j are integers or zero, c_j is a constant),

$$\sum_j c_j \mathcal{Y}_{p_j x, q_j t}(v, w) = 0, \tag{7.19}$$

of the lowest possible order (or weight). The simplest choice is a constraint of weight 2:

$$\mathcal{Y}_{2x}(v, w) \equiv w_{2x} + v_x^2 = 0. \tag{7.20}$$

In order to obtain a parameter-dependent decomposition, we should impose the condition

$$\mathcal{Y}_{2x}(v, w) = \lambda, \tag{7.21}$$

where λ is an arbitrary parameter of weight 2. This leads to the following $\mathcal{Y}$-system

$$\mathcal{Y}_{2x}(v, w) - \lambda = 0, \qquad \mathcal{Y}_t(v) + \mathcal{Y}_{3x}(v, w) + 3\lambda \mathcal{Y}_x(v) = 0, \tag{7.22}$$

the compatibility of which is guaranteed by the corresponding system for ψ [setting $w = v + Q$, $v = \ln \psi$ and using (7.10)]:

$$(L_2 - \lambda)\psi \equiv \psi_{2x} + (Q_{2x} - \lambda)\psi = 0, \tag{7.23}$$

$$(\partial_t + \mathcal{L}_3)\psi \equiv \psi_t + \psi_{3x} + 3(Q_{2x} + \lambda)\psi_x = 0,$$

i.e., to the (λ-independent) condition:

$$(Q_{xt} + Q_{4x} + 3Q_{2x})_x \equiv \partial_x E(Q) = 0. \tag{7.24}$$

The bilinear equivalent of the $\mathcal{Y}$-system (7.22) is obtained by means of (7.8):

$$D_x^2 G' \cdot G = \lambda G' G, \qquad (D_t + D_x^3 + 3\lambda D_x) G' \cdot G = 0. \tag{7.25}$$

It is the bilinear BT for the KdV proposed by Hirota [209].

Boussinesq equation

A similar analysis can be applied to the Boussinesq equation

$$\mathrm{Bq}(u) \equiv u_{2t} - u_{4x} + 3(u^2)_{2x} = 0. \tag{7.26}$$

This equation can be derived from a potential version obtained by setting $u = -Q_{2x}$:

$$E(Q) \equiv P_{2t}(q) - P_{4x}(Q) \equiv G^{-2}(D_t^2 - D_x^4)G \cdot G\Big|_{G=\exp(Q/2)} = 0. \tag{7.27}$$

The corresponding two-field condition

$$E(Q' = w + v) - E(Q = w - v) \equiv 2(v_{2t} - v_{4x} - 6v_2 w_{2x})$$

$$= -2\partial_x \mathcal{Y}_{3x}(v, w) + 2v_{2t} + 6W[\mathcal{Y}_{2x}(v, w), \mathcal{Y}_x(v)] = 0 \tag{7.28}$$

can still be decoupled into a pair of equations of the form (7.19) by means of the $\mathcal{Y}$-constraint of weight 2 (notice that in this case the dimension of $t = 2$, so we dispose of two $\mathcal{Y}$-polynomials of weight 2):

$$\mathcal{Y}_t(v) + a\mathcal{Y}_{2x}(v, w) = 0, \tag{7.29}$$

where a is a dimensionless constant to be determined.

The decoupling requires $a^2 = -3$ and produces the following parameter-dependent $\mathcal{Y}$-system (λ is an integration constant):

$$\mathcal{Y}_t + a\mathcal{Y}_{2x}(v, w) = 0, \quad a\mathcal{Y}_{x,t}(v, w) + \mathcal{Y}_{3x}(v, w) = \lambda. \tag{7.30}$$

The corresponding bilinear system

$$(D_t + aD_x^2)G' \cdot G = 0, \qquad (aD_x D_t + D_x^3 - \lambda)G' \cdot G = 0 \tag{7.31}$$

is exactly the bilinear BT for the Boussinesq equation obtained by Nimmo and Freeman [350]. Its compatibility is subject to that of the linear equivalent to the system (7.30):

$$\psi_t + a\psi_{2x} + aQ_{2x}\psi = 0, \tag{7.32}$$

$$a\psi_{xt} + \psi_{3x} + 3Q_{2x}\psi_x + (aQ_{xt} - \lambda)\psi = 0,$$

i.e., to the following potential version of the Boussinesq equation:

$$\mathrm{PBq}(Q) \equiv (Q_{2t} - Q_{4x} - 3Q_{2x}^2)_x = 0. \tag{7.33}$$

Lax equation

We now consider the Lax equation

$$\mathrm{Lax}(u) \equiv u_t + u_{5x} + 10uu_{3x} + 20u_x u_{2x} + 30u^2 u_x = 0. \tag{7.34}$$

Setting $u = cQ_{2x}$ brings it to the potential equation:

$$E_c(Q) \equiv Q_{xt} + Q_{6x} + 10cQ_{2x}Q_{4x} + 5cQ_{3x}^2 + 10c^2Q_{2x}^3 = 0. \tag{7.35}$$

The left-hand side of this equation is homogeneous with weight 6, but there is no value of c such that (7.35) can be expressed as a linear combination of the weight 6 polynomials $P_{6x}(Q)$ and $P_{xt}(Q)$. Setting $c = 1$, we may nevertheless consider the two-field condition

$$E_1(w+v) - E_1(w-v) \equiv 2\left\{\partial_x\left[\mathcal{Y}_t(v) + \mathcal{Y}_{5x}(v,w)\right] + R(v,w)\right\} = 0, \tag{7.36}$$

with

$$\begin{aligned} R(v,w) = -5\big(&v_x w_{5x} - v_{2x}w_{4x} + 6v_x w_{2x}w_{3x} + 2v_x^3 w_{3x} - 3v_{2x}w_{2x}^2 \\ &+6v_x^2 v_{2x}w_{2x} + 4v_x v_{2x}v_{3x} + 2v_x^2 v_{4x} + v_x^4 v_{2x} - 2v_{2x}^3\big). \end{aligned} \tag{7.37}$$

Eliminating w_{2x} and its derivatives by means of the weight 2 constraint (7.21), we find that the condition (7.36) can be decoupled into the following $\mathcal{Y}$-system:

$$\mathcal{Y}_{2x}(v,w) = \lambda, \quad \mathcal{Y}(v) + \mathcal{Y}_{5x}(v,w) + 15\lambda^2\mathcal{Y}_x(v) = 0. \tag{7.38}$$

Its compatibility is subjected to that of the corresponding linear system:

$$\begin{gathered} \psi_{2x} + (Q_{2x} - \lambda)\psi = 0, \qquad \psi_t + \mathcal{L}_5\psi = 0, \\ \mathcal{L}_5 = \partial_x^5 + 10Q_{2x}\partial_x^3 + 5(Q_{4x} + 3Q_{2x}^2 + 3\lambda^2)\partial_x, \end{gathered} \tag{7.39}$$

i.e., to the condition

$$(Q_{xt} + Q_{6x} + 10Q_{2x}Q_{4x} + 5Q_{3x}^2 + 10Q_{2x}^3)_x \equiv \partial_x E_1(Q) = 0. \tag{7.40}$$

Sawada–Kotera equation

We finally consider the Sawada–Kotera equation

$$\mathrm{SK}(u) \equiv u_t + u_{5x} + 15uu_{3x} + 15u_x u_{2x} + 45u^2 u_x = 0, \tag{7.41}$$

which again can be derived from a potential equation by setting $u = Q_{2x}$, expressible in terms of $P_{6x}(Q)$ and $P_{xt}(Q)$:

$$E(Q) = P_{xt}(Q) + P_{6x}(Q) \equiv G^{-2}(D_x D_t + D_x^6)G\cdot G\Big|_{G=\exp(Q/2)} = 0. \tag{7.42}$$

It is easy to see that the corresponding two-field condition

$$E(w+v) - E(w-v) \equiv 2\partial_x\left[\mathcal{Y}_t(v) + \mathcal{Y}_{5x}(v,w)\right] + 10R(v,w) = 0, \tag{7.43}$$

with

$$
\begin{aligned}
R(v,w) = &-v_x w_{5x} + 2v_{2x}w_{4x} - 2v_{3x}w_{3x} + w_{2x}v_{4x} \\
&-2v_x^2 v_{4x} - 4v_x v_{2x} v_{3x} + 6v_{2x}w_{2x}^2 \\
&+3v_{2x}^3 - 6v_x w_{2x} w_{3x} - 2v_x^3 w_{3x} - 6v_x^2 v_{2x} w_{2x} - v_x^4 v_{2x},
\end{aligned} \tag{7.44}
$$

can no longer be decoupled into a $\mathcal{Y}$-system by means of a weight 2 constraint of the form (7.20).

Yet, the weight 3 constraint

$$
\mathcal{Y}_{3x}(v,w) \equiv v_{3x} + 3v_x w_{2x} + v_x^3 = \lambda \tag{7.45}
$$

enables us to express $R(v,w)$ as follows:

$$
R(v,w) = -\frac{1}{2}\partial_x \left[\mathcal{Y}_{5x}(v,w) + 3\lambda\mathcal{Y}_{2x}(v,w)\right]. \tag{7.46}
$$

This means that the condition (7.43) can be decoupled into the following (λ-dependent) $\mathcal{Y}$-system:

$$
\mathcal{Y}_{2x}(u,v) - \lambda = 0, \quad \mathcal{Y}_t(v) - \frac{3}{2}\mathcal{Y}_{5x}(v,w) - \frac{15}{2}\lambda\mathcal{Y}_{2x}(v,w) = 0. \tag{7.47}
$$

Its compatibility is subjected to that of the corresponding ψ-system ($w = v + Q,\ v = \ln\psi$):

$$
\begin{aligned}
&\psi_{3x} + 3Q_{2x}\psi_x - \lambda\psi = 0, \\
&\psi_t - \frac{3}{2}\psi_{5x} - 15Q_{2x}\psi_{3x} - \frac{15}{2}P_{4x}(Q)\psi_x - \frac{15}{2}\lambda(\psi_{2x} + Q_{2x}\psi) = 0,
\end{aligned} \tag{7.48}
$$

i.e., to the condition:

$$
\left(Q_{xt} + Q_{6x} + 15Q_{2x}Q_{4x} + 15Q_{2x}^3\right)_x \equiv \partial_x E(Q) = 0. \tag{7.49}
$$

The bilinear equivalent of the system (7.47),

$$
(D_x^3 - \lambda)G' \cdot G = 0, \qquad \left(D_t - \frac{3}{2}D_x^5 - \frac{15}{2}\lambda D_x^2\right) G' \cdot G = 0, \tag{7.50}
$$

is the bilinear BT for the Sawada–Kotera equation reported in [386].

7.2 Darboux-covariant Lax pairs in terms of $\mathcal{Y}$-functions

In Sect. 3.7 a joint covariance property of operators was defined and investigated. It results in some necessary conditions, e.g., the joint covariance equations, whose solutions yield restriction on a form of solvable equations. Let us now go back to the KdV equation (7.13) and the associated linear system (7.23). It comprises the second-order eigenvalue equation considered by Lax

[263], with the covariance property we study throughout this book. According to this property, (nonvanishing) solutions ϕ to the spectral equation produce transformations

$$G_\phi = \phi\partial_x\phi^{-1} = \partial_x - \sigma, \quad \sigma = \partial_x \ln\phi, \tag{7.51}$$

which map $L_2 = \partial_x^2 + Q_{2x}$ onto the similar operator

$$\widetilde{L}_2 \equiv G_\phi L_2(Q_{2x})G_\phi^{-1} \equiv L_2(\widetilde{Q}_{2x}), \tag{7.52}$$

with $\widetilde{Q}_{2x} = Q_{2x} + 2\sigma_x$. With the second-order eigenvalue equation obtained from the constraint (7.21) through the map $v = \ln\psi$,

$$\mathcal{Y}_{2x}(v,\ v+Q) = \lambda, \tag{7.53}$$

we may try to associate a Darboux-covariant third-order evolution equation. Note that any equation of the form

$$\sum_n c_n\mathcal{Y}_{p_n x, q_n t}\left(v, v+Q^{(n)}\right) = 0 \tag{7.54}$$

corresponds to a linear equation for ψ. In particular, there is a correspondence between the evolution equation (c_2 and c_3 are constants)

$$\mathcal{Y}_t(v) + c_2\mathcal{Y}_{2x}\left(v, v+Q^{(2)}\right) + c_3\mathcal{Y}_{3x}\left(v, v+Q^{(3)}\right) = 0 \tag{7.55}$$

and its linear counterpart

$$\psi_t + L_3\psi = 0, \quad L_3 = c_3\partial_x^3 + c_2\partial_x^2 + b_1\partial_x + b_0, \tag{7.56}$$

with

$$b_1 = 3c_3Q_{2x}^{(3)}, \quad b_0 = c_2Q_{2x}^{(2)}. \tag{7.57}$$

Let G_ϕ be a transformation (7.51) generated by a (nonvanishing) solution ϕ of the system

$$\begin{aligned}&(L_2-\lambda)\phi \equiv (\partial_x^2 + Q_{2x} - \lambda)\phi = 0,\\ &(\partial_t + L_3)\phi \equiv (\partial_t + c_3\partial_x^3 + c_2\partial_x^2 + b_1\partial_x + b_0)\phi = 0.\end{aligned} \tag{7.58}$$

It maps the operators $L_2 - \lambda$ and $\partial_t + L_3$ onto the similar operators

$$G_\phi(L_2-\lambda)G_\phi^{-1} = L_2(\widetilde{Q}_{2x}) - \lambda, \quad G_\phi(\partial_t + L_3)G_\phi^{-1} = \partial_t + \widetilde{L}_3,$$

$$\widetilde{L}_3 \equiv c_3\partial_x^3 + c_2\partial_x^2 + \widetilde{b}_1\partial_x + \widetilde{b}_0, \tag{7.59}$$

where

$$\widetilde{b}_1 = b_1 + \varDelta b_1, \qquad \varDelta b_1 = 3c_3\sigma_x, \tag{7.60}$$

$$\widetilde{b}_0 = b_0 + \varDelta b_0, \qquad \varDelta b_0 = b_{1,x} + \sigma\varDelta b_1 + 2c_2\sigma_x + 3c_3\sigma_{2x}, \tag{7.61}$$

and the following differential consequences of (7.58) have been taken into account:

$$\partial_x \left(\frac{\phi_{2x}}{\phi} + Q_{2x} - \lambda \right) = 0 \quad \Longleftrightarrow \quad \partial_x Y_{2x}(\ln \phi) + Q_{3x} \equiv (\sigma_x + \sigma^2)_x + Q_{3x} = 0,$$

$$\partial_x \left[Y_t(\ln \phi) + c_3 Y_{3x}(\ln \phi) + c_2 Y_{2x}(\ln \phi) + b_1 Y_x(\ln \phi) + b_0 \right] = 0 \tag{7.62}$$
$$\Longleftrightarrow \quad \sigma_t + c_3(\sigma_{2x} + 3\sigma\sigma_x + \sigma^3)_x + c_2(\sigma_x + \sigma^2)_x + (b_1\sigma)_x + b_{0,x} = 0.$$

In order that $\partial_t + L_3$ be the Darboux-covariant with $L_2 - \lambda$, we have to determine the coefficients b_i, $i = 0, 1$, as functions of Q_{2x} and its derivatives, in such a way that the covariance condition

$$\widetilde{L}_3(Q_{2x},\ Q_{3x},\ \ldots) = L_3(\widetilde{Q}_{2x},\ \widetilde{Q}_{3x}, \ldots) \tag{7.63}$$

be satisfied with

$$\Delta Q_{(r+1)x} \equiv \widetilde{Q}_{(r+1)x} - Q_{(r+1)x} = 2\sigma_{rx}, \quad r = 1, 2, \ldots. \tag{7.64}$$

Hence, we should look for expressions $b_i = F_i(Q_{2x}, Q_{3x}, \ldots)$ such that the differences Δb_i which appear in (7.58) and (7.59) are expressible as

$$\Delta b_i = F_i(Q_{2x} + \Delta Q_{2x}, Q_{3x} + \Delta Q_{3x}, \ldots) - F_i(Q_{2x}, Q_{3x}, \ldots), \quad i = 0, 1. \tag{7.65}$$

Because

$$\Delta b_1 = \frac{3}{2} c_3 \Delta Q_{2x}, \tag{7.66}$$

it is clear that we can find an expression F_i, linear in Q_{2x}, which satisfies condition (7.64), yielding

$$b_1 = \frac{3}{2} c_3 Q_{2x} + c_1, \tag{7.67}$$

c_1 being an arbitrary constant. The difference Δb_0 is now given by the relation

$$\Delta b_0 = \frac{3}{2} c_3 Q_{3x} + 3c_3 \sigma\sigma_x + 2c_2\sigma_x + 3c_3\sigma_{2x}, \tag{7.68}$$

which, on account of (7.62), becomes

$$\Delta b_0 = 2c_2\sigma_x + \frac{3}{2} c_3 \sigma_{2x} = c_2 \Delta Q_{2x} + \frac{3}{4} c_3 \Delta Q_{3x}. \tag{7.69}$$

It follows that we can find an expression F_0, linear in Q_{2x} and Q_{3x}, which satisfies condition (7.64), yielding

$$b_0 = c_2 Q_{2x} + \frac{3}{4} c_3 Q_{3x} + c_0, \tag{7.70}$$

where c_0 is an arbitrary constant. Setting $c_0 = c_1 = 0$, we obtain

$$L_3 = c_3\left(\partial_x^3 + \frac{3}{2}Q_{2x}\partial_x + \frac{3}{4}Q_{3x}\right) + c_2 L_2, \tag{7.71}$$

indicating that the simplest Darboux-covariant third-order evolution equation to be associated with (7.58) has the form (setting $c_2 = 0,\ c_3 = 4$)

$$(\psi_t + \bar{L}_3)\psi \equiv 0, \quad \bar{L}_3 = 4\partial_x^3 + 6Q_{2x}\partial_x + 3Q_{3x}. \tag{7.72}$$

Together with (7.54) it produces an equivalent version of our previous linear system (7.23) for the KdV equation, obtained by replacing the second equation by the combination

$$[\partial_t + \mathcal{L}_3 + 3\partial_x(L_2 - \lambda)]\psi = 0. \tag{7.73}$$

The operator $\bar{L}_3$ corresponds precisely to the third-order operator which gives rise to the KdV equation in the Lax formalism [350]:

$$[\partial_t + \bar{L}_3, L_2] = (Q_{xt} + Q_{4x} + 3Q_{2x}^2)_x = 0. \tag{7.74}$$

The full Darboux-covariant system obtained with expression (7.71) for L_3,

$$(L_2 - \lambda)\psi = 0, \qquad (\partial_t + L_3)\psi = 0, \tag{7.75}$$

corresponds, through the map $v = \ln\psi$, to the *multipotential* $\mathcal{Y}$-system

$$\mathcal{Y}_{2x}(v, v+Q) = \lambda, \tag{7.76}$$

$$\mathcal{Y}_t(v) + c_3\mathcal{Y}_{3x}\left(v, v+Q^{(3)}\right) + c_2\mathcal{Y}_{2x}\left(v, v+Q^{(2)}\right) = 0,$$

in which

$$Q_{2x}^{(3)} = \frac{1}{2}Q_{2x}, \quad Q_{2x}^{(2)} = Q_{2x} + \frac{3}{4}\frac{c_2}{c_3}Q_{3x}. \tag{7.77}$$

An interesting alternative to this system results from an interchange between $\mathcal{Y}_{2x}(v, v+Q)$ and $c_3\mathcal{Y}_{3x}\left(v, v+Q^{(3)}\right) + c_2\mathcal{Y}_{2x}\left(v, v+Q^{(2)}\right)$,

$$\begin{aligned} &c_3\mathcal{Y}_{3x}\left(v, v+Q^{(3)}\right) + c_2\mathcal{Y}_{2x}\left(v, v+Q^{(2)}\right) = \lambda, \\ &\mathcal{Y}_t(v) + \mathcal{Y}_{2x}(v, v+Q) = 0, \end{aligned} \tag{7.78}$$

which corresponds to an alternative Lax-like system with the third-order eigenvalue equation and second-order time evolution:

$$\begin{aligned} &L_3\psi \equiv (c_3\partial_x^3 + c_2\partial_x^2 + b_1\partial_x + b_0)\psi = \lambda\psi, \\ &(\partial_t + L_2)\psi \equiv \partial_t + \psi_{2x} + Q_{2x}\psi = 0, \end{aligned} \tag{7.79}$$

where the b_i, $i = 0, 1$, are given by (7.57).

Let G_ϕ be a transformation generated by a (nonvanishing) solution ϕ of the system (7.79). It still maps the operators $\partial_t + L_2$ and $L_3 - \lambda$ onto similar operators,

$$G_\phi(\partial_t + L_2)G_\phi^{-1} = \partial_t + \widetilde{L}_2, \quad \widetilde{L}_2 = L_2(\widetilde{Q}_{2x}), \tag{7.80}$$

$$G_\phi(L_3 - \lambda)G_\phi^{-1} = \widetilde{L}_3 - \lambda, \quad \widetilde{L}_3 = c_3\partial_x^3 + c_2\partial_x^2 + \widetilde{b}_1\partial_x + \widetilde{b}_0, \tag{7.81}$$

where the differences $\Delta b_i \equiv \widetilde{b}_i - b_i$ are given by (7.60) and (7.61) and where the following differential consequences of (7.79) have been taken into account:

$$\sigma_t + (\sigma_x + \sigma^2)_x + Q_{3x} = 0, \tag{7.82}$$

$$c_3(\sigma_{2x} + 3\sigma\sigma_x + \sigma^3)_x + c_2(\sigma_x + \sigma^2)_x + (b_1\sigma)_x + b_{0,x} = 0. \tag{7.83}$$

Extending the condition (7.64) to $r = 0$, we find by means of the above analysis that the covariance of $L_3 - \lambda$ with $\partial_t + L_2$ is guaranteed if

$$b_1 = \frac{3}{2}c_3Q_{2x} + c_1, \quad b_0 = c_2Q_{2x} + \frac{3}{4}c_3(Q_{3x} - Q_{xt}) + c_0, \tag{7.84}$$

where c_0 and c_1 are arbitrary constants. Setting $c_0 = c_1 = 0$, we find

$$L_3 = c_3\left(\partial_x^3 + \frac{3}{2}Q_{2x}\partial_x + \frac{3}{4}\left(Q_{3x} - Q_{xt}\right)\right) + c_2L_2, \tag{7.85}$$

yielding the simplest Darboux-covariant system of type (7.79):

$$\begin{aligned} &\widehat{L}_3\psi = \lambda\psi, \quad (\partial_t + L_2)\psi = 0, \\ &\widehat{L}_3 = 4\partial_x^3 + 6Q_{2x}\partial_x + 3\left(Q_{3x} - Q_{xt}\right). \end{aligned} \tag{7.86}$$

The operators $\widehat{L}_3$ and $\partial_t + L_2$ are found to constitute the Lax pair for an equation which is nothing other than the potential version of the Boussinesq equation (7.26) in which t has been rescaled ($t = a\tau,\ a^2 = -3$):

$$[\partial_t + L_2, \widehat{L}_3] = -(3Q_{2t} + Q_{4x} + 3Q_{2x}^2)_x = (Q_{2\tau} - Q_{4x} - 3Q_{2x}^2)_x. \tag{7.87}$$

It is easy to verify that the system (7.86) taken with $t = a\tau$ and $a^2 = -3$ is the equivalent version of our previous linear system (7.32) for the Boussinesq equation which results from subtracting a times the x-derivative of the first equation from the second one. The full Darboux-covariant system obtained with expression (7.85),

$$(L_3 - \lambda)\psi = 0, \qquad (\partial_t + L_2)\psi = 0, \tag{7.88}$$

corresponds, through the map $v = \ln\psi$, to a "covariant" version of the system (7.78) in which

$$Q_{2x}^{(3)} = \frac{1}{2}Q_{2x} \quad \text{and} \quad Q_{2x}^{(2)} = Q_{2x} + \frac{3}{4}\frac{c_2}{c_3}(Q_{3x} - Q_{xt}). \tag{7.89}$$

The striking similarity between the *covariant* $\mathcal{Y}$-systems associated with the KdV and Boussinesq equations reveals a deep connection between both soliton systems. It suffices to consider the next step which leads us from the system (7.78) to an alternative version with two evolution equations corresponding to two t-variables (t_p has the dimension p):

$$\mathcal{Y}_{t_2}(v) + \mathcal{Y}_{2x}(v, v+Q) = 0, \tag{7.90}$$

$$\mathcal{Y}_{t_3}(v) + c_2\mathcal{Y}_{2x}\left(v, v+Q^{(2)}\right) + c_3\mathcal{Y}_{3x}\left(v, v+Q^{(3)}\right) = 0.$$

It is clear from the above analysis that the Darboux covariance of the corresponding linear system for $\psi = \exp v$,

$$(\partial_{t_2} + L_2)\psi = 0, \quad \left(\partial_{t_3} + c_3\partial_x^3 + c_2\partial_x^2 + 3c_3Q_{2x}^{(3)}\partial_x + c_2Q_{2x}^{(2)}\right)\psi = 0, \tag{7.91}$$

is still guaranteed by the conditions (7.89) on $Q_{2x}^{(3)}$ and $Q_{2x}^{(2)}$. In particular, it is found that the compatibility of the simplest covariant system (setting $c_2 = 0, c_3 = 4$),

$$(\partial_t + L_2)\psi = 0, \quad (\partial_{t_3} + \widehat{L}_3)\psi = 0, \tag{7.92}$$
$$\widehat{L}_3 = 4\partial_x^3 + 6Q_{2x}\partial_x + 3(Q_{3x} - Q_{xt_2}),$$

is subjected to the condition

$$[\partial_{t_3} + \widehat{L}_3, \partial_{t_2} + L_2] = [P_{x,t_3}(Q) + 3P_{2t_2}(Q) + P_{4x}(Q)]_x = 0, \tag{7.93}$$

which is a potential version of the KP equation:

$$\mathrm{KP}(u) \equiv (u_{t_3} + u_{3x} + 6uu_x)_x + 3u_{2t_2} = 0, \tag{7.94}$$

obtained by setting $u = Q_{2x}$ and by integrating once with respect to x. We wish to stress that the above derivation of a covariant Lax pair for the KdV equation produced three closely related Darboux-covariant systems hinting in a direct manner at the (well-known) common origin of the KdV and Boussinesq equations as reductions of the KP equation.

We end our discussion with a direct derivation of a Darboux-covariant equivalent to the linear system (7.39) that we associated with the Lax equation (7.34). Our starting point is the multipotential $\mathcal{Y}$-system (c_i is a constant),

$$\mathcal{Y}_{2x}(v, v+Q) = \lambda, \quad \mathcal{Y}_t(v) + \sum_{i=2}^{5} c_i\mathcal{Y}_{ix}\left(v, v+Q^{(i)}\right) = 0, \tag{7.95}$$

or its linear version for $\psi = \exp v$,

$$(L_2 - \lambda)\psi = 0, \qquad (\partial_t + L_5)\psi = 0, \tag{7.96}$$

$$L_5 = c_5\partial_x^5 + c_4\partial_x^4 + b_3\partial_x^3 + b_2\partial_x^2 + b_1\partial_x + b_0,$$

with

$$
\begin{aligned}
&b_3 = 10c_5 Q_{2x}^{(5)} + c_3, \quad b_2 = 6c_4 Q_{2x}^{(4)} + c_2,\\
&b_1 = 3c_3 Q_{2x}^{(3)} + 5c_5 \left[Q_{4x}^{(5)} + 3\left(Q_{2x}^{(5)}\right)^2 \right] + c_1,\\
&b_0 = c_2 Q_{2x}^{(2)} + c_4 \left[Q_{4x}^{(4)} + 3\left(Q_{2x}^{(4)}\right)^2 \right] + c_0.
\end{aligned}
\tag{7.97}
$$

Let G_ϕ be a transformation (7.51) generated by a (nonvanishing) solution ϕ of the system (7.96) and (7.97). It maps $L_2 - \lambda$ and $\partial_t + L_5$ onto the similar operators (7.60) and $\partial_t + \widetilde{L}_5$, with

$$
\widetilde{L}_5 = c_5 \partial_x^5 + c_4 \partial_x^4 + \widetilde{b}_3 \partial_x^3 + \widetilde{b}_2 \partial_x^2 + \widetilde{b}_1 \partial_x + \widetilde{b}_0, \tag{7.98}
$$

where

$$
\begin{gathered}
\Delta b_3 \equiv \widetilde{b}_3 - b_3 = 5c_5\sigma_x = \tfrac{5}{2} c_5 \Delta Q_{2x},\\
\Delta b_2 \equiv \widetilde{b}_2 - b_2 = b_{3,x} + \sigma \Delta b_3 + 4c_4\sigma_x + 10c_5\sigma_{2x},\\
\Delta b_1 \equiv \widetilde{b}_1 - b_1 = b_{2,x} + \sigma \Delta b_2 + 3\sigma_x \widetilde{b}_3 + 6c_4\sigma_{2x} + 10c_5\sigma_{3x},\\
\Delta b_0 \equiv \widetilde{b}_0 - b_0 = b_{1,x} + \sigma \Delta b_1 + 2\sigma_x \widetilde{b}_2 + 3\sigma_{2x} \widetilde{b}_3 + 4c_4\sigma_{3x} + 5c_5\sigma_{4x}.
\end{gathered}
\tag{7.99}
$$

In order to ensure the Darboux covariance of $\partial_t + L_5$ with $L_2 - \lambda$, we must again determine expressions F_i for b_i, $i = 0, 1, 2, 3$, in terms of Q_{2x} and its derivatives, which are such that condition (7.65) is satisfied at $i = 0, 1, 2, 3$, with (7.64). It is clear from (7.98) that F_3 can be chosen to be linear in Q_{2x}, so

$$
b_3 = \frac{5}{2} c_5 Q_{2x} + c_3, \tag{7.100}
$$

where c_3 is an arbitrary constant. Equation (7.98) then becomes

$$
\Delta b_2 = \frac{5}{2} c_s Q_{3x} + 5c_5\sigma\sigma_x + 4c_4\sigma_x + 10c_5\sigma_{2x}. \tag{7.101}
$$

Using (7.62), we rewrite it as

$$
\Delta b_2 = 4c_4\sigma_x + \frac{15}{2} c_5\sigma_{2x} = 2c_4 \Delta Q_{2x} + \frac{15}{4} c_5 \Delta Q_{3x}, \tag{7.102}
$$

indicating that F_2 can be chosen to be linear in Q_{2x} and Q_{3x}, so

$$
b_2 = 2c_4 Q_{2x} + \frac{15}{2} c_5 Q_{3x} + c_2, \tag{7.103}
$$

where c_2 is an arbitrary constant. Hence, we obtain

$$
\begin{aligned}
\Delta b_1 = {} & 2c_4 Q_{3x} + \frac{15}{4} c_5 Q_{4x} + 4c_4\sigma\sigma_x + \frac{15}{2} c_5 \sigma\sigma_{2x} + \frac{15}{2} c_5 \sigma_x Q_{2x}\\
& + 3c_3\sigma_x + 15c_5\sigma_x^2 + 6c_4\sigma_{2x} + 10c_5\sigma_{3x},
\end{aligned}
\tag{7.104}
$$

or, using (7.62),

$$\begin{aligned}\Delta b_1 &= 2c_4\Delta Q_{3x} + \frac{25}{8}c_5\Delta Q_{4x} + \frac{15}{8}c_5\left(2Q_{2x}\Delta Q_{2x} + \Delta Q_{2x}\Delta Q_{2x}\right)\\ &+\frac{3}{2}c_3\Delta Q_{2x} + 2c_4\Delta Q_{3x} + \frac{25}{8}c_5\Delta Q_{4x} + \frac{15}{8}\Delta\left(Q_{2x}^2\right). \end{aligned} \tag{7.105}$$

It follows that F_1 can be chosen to be linear in $Q_{2x},\ Q_{3x},\ Q_{4x}$, and Q_{2x}^2, so

$$b_1 = \frac{3}{2}c_3Q_{2x} + 2c_4Q_{3x} + \frac{25}{8}c_5Q_{4x} + \frac{15}{8}c_5Q_{2x}^2 + c_1. \tag{7.106}$$

It is found from these results and (7.62) that Δb_0 becomes

$$\begin{aligned}\Delta b_0 &= \frac{15}{16}c_5\left[\Delta Q_{5x} + 2\Delta(Q_{2x}Q_{3x})\right]\\ &+ c_4\left[\Delta Q_{4x} + \Delta\left(Q_{2x}^2\right)\right] + \frac{3}{4}c_3\Delta Q_{3x} + c_2\Delta Q_{2x},\end{aligned} \tag{7.107}$$

indicating that the appropriate expression for b_0 is

$$b_0 = c_2Q_{2x} + \frac{3}{4}c_3Q_{3x} + c_4(Q_{4x} + Q_{2x}^2) + \frac{15}{16}c_5(Q_{5x} + 2Q_{2x}Q_{3x}) + c_0. \tag{7.108}$$

Setting $c_1 = c_0 = 0$, we obtain the following expression for L_5,

$$L_5 = c_4L_2^2 + c_3\left(\partial_x^3 + \frac{3}{2}Q_{2x}\partial_x + \frac{3}{4}Q_{3x}\right) + c_4L_2 + \widehat{L}_5, \tag{7.109}$$

with (choosing $c_5 = 16$)

$$\widehat{L}_5 = 16\partial_x^5 + 40Q_{2x}\partial_x^3 + 60Q_{3x}\partial_x^2 + (50Q_{4x} + 30Q_{2x}^2)\partial_x + 15(Q_{5x} + 2Q_{2x}Q_{3x}). \tag{7.110}$$

The relations between different potentials appearing in the *covariant* system (7.95) are determined by (7.97), (7.100), (7.103), (7.106), and (7.108). The simplest Darboux-covariant fifth-order evolution equation (7.97) to be associated with (7.96) has the form

$$\left(\partial_t + \widehat{L}_5\right)\psi = 0. \tag{7.111}$$

It is easy to see that the system (7.96) and (7.111) is equivalent to the original system (7.39):

$$\widehat{L}_5 = \mathcal{L}_5 + 15\left[\partial_x^3 + (Q_{2x} + \lambda)\partial_x + Q_{3x}\right](L_2 - \lambda). \tag{7.112}$$

Notice that the appearance of the third-order Darboux-covariant operator L_3 as a part of the general fifth-order covariant operator L_5 can be regarded as a direct confirmation of the close relationship between the KdV and Lax equations as the third- and fifth-order members of the same hierarchy.

7.3 Bäcklund transformations and Noether theorem

BTs naturally arise when the Darboux formalism is "projected" to solutions of nonlinear equations (the potentials of the corresponding Lax representation). The action is simple: "wave functions" of the Lax equations should be excluded [239].

7.3.1 BT and infinitesimal BT

In the previous section we showed that the bilinear BT is a DT covariant form of the equations of the Hirota method. For the KdV equation it is (7.25), which is obtained from (7.22). The second relation (7.22) is nothing more than the first equation of the classical BT, relating the fields $w = Q_x$ and $w' = Q'_x$:

$$(w + w')_x = (w - w')^2 - \kappa^2, \tag{7.113}$$

$$(w + w')_t = -2(w - w')(w - w')_{xx} + (w_x - w'_x)^2 + 3((w - w')^2 - \kappa^2)^2,$$

while the second equation (we take the form of [407], the appropriate change of notations is used) is derived from (7.18) in terms of the Q and Q' fields of the *potential* KdV equation (7.15); we denote $2\mu = -\kappa^2$. The form of this equation is not unique, because the first one can be used.

The famous consequence of the BT (7.113) is that both variables w and w' are solutions of the potential KdV equation

$$\Lambda w \equiv w_t - 6w_x^2 + w_{xxx} = 0. \tag{7.114}$$

Steudel [407, 408, 409] derived conservation laws for soliton equations by application of the Noether theorem, imposing the BT in a version of the extended interpretation of

$$w' = \mathbb{B}_\kappa w \equiv w + \kappa[1 + \kappa^{-2}(w'_x + w_x)]^{1/2}, \tag{7.115}$$

which is one of the solutions of the first relation in (7.113) with respect to $w' - w$. The real-valued w is in the realm of the extended BT transform, if $\inf[1 + \kappa^{-2}(w'_x + w_x)] \geq 0$, or $|w'_x + w_x| \leq M^2$, $|\kappa| \geq M$.

Theorem 7.1. *Let*

$$w_i = \mathbb{B}_{\kappa_i} w_0, \qquad w_3 = \mathbb{B}_{\kappa_2} w_1, \tag{7.116}$$

then

$$\mathbb{B}_{\kappa_1}\mathbb{B}_{\kappa_2} = \mathbb{B}_{\kappa_2}\mathbb{B}_{\kappa_1} \tag{7.117}$$

and

$$(w_3 - w_0)(w_2 - w_1) = \kappa_2^2 - \kappa_1^2. \tag{7.118}$$

The fundamental property of the extension basis is that (7.115) is valid not only for solutions of the potential KdV equation. In other words, the Laurent series

$$\delta w = \kappa + A_1 \kappa^{-1} + A_2 \kappa^{-2} + \dots \tag{7.119}$$

represents the infinitesimal transform at infinity on the κ-plane. Equating the x-derivative of the right-hand side of (7.119) and the right-hand side of the first relation in (7.113) yields

$$A_1 = w_x, \quad A_n = \frac{1}{2} A_{(n-1)x} - \frac{1}{2} \sum_{r=1}^{n-2} A_r A_{n-r-1}, \quad n = 2, 3, 4, \dots. \tag{7.120}$$

These formulas were first derived by Zakharov and Faddeev [469] in the context of the IST method; see also [385, 445]. Note also that the expansion (7.119) after differentiation in x gives an alternative representation of a DT as $\delta w_x \sim u[1] - u$. The recurrent relations (7.120) are solved explicitly:

$$A_2 = w_{xx}/2, \quad A_3 = w_{xxx} - w_x^2/2, \quad A_4 = w_{xxxx}/8 - w_x w_{xx}, \dots. \tag{7.121}$$

7.3.2 Noether identity and Noether theorem

A Lagrangian density for the KdV equation is chosen so that

$$\mathfrak{L} = \frac{1}{2} w_x w_t + \frac{1}{2} w_{xx}^2 - 2 w_x^2 \tag{7.122}$$

gives the potential KdV equation (7.114) as the Euler equation. A variant of the Noether theorem for the dependence of $\mathfrak{L}$ on w_{xx} is based on the following form for the variation (the Frechét differential on the prolonged space):

$$\delta \mathfrak{L} \equiv \frac{\partial \mathfrak{L}}{\partial w_t} \delta w_t + \frac{\partial \mathfrak{L}}{\partial w_x} \delta w_x + \frac{\partial \mathfrak{L}}{\partial w_{xx}} \delta w_{xx}. \tag{7.123}$$

A decomposition of the right-hand side of (7.134) into a divergence and a term proportional δw gives the Noether identity

$$\delta \mathfrak{L} = A_t + B_x - \Lambda \delta w, \tag{7.124}$$

where

$$A = \frac{\partial \mathfrak{L}}{\partial w_t} \delta w = \frac{1}{2} w_x \delta w, \tag{7.125}$$

$$B = \left[\frac{\partial \mathfrak{L}}{\partial w_x} - \left(\frac{\partial \mathfrak{L}}{\partial w_{xx}} \right)_x \right] \delta w + \frac{\partial \mathfrak{L}}{\partial w_{xx}} \delta w_x = \left(\frac{1}{2} w_t + w_{xxx} - 6 w_x^2 \right) \delta w - w_{xx} \delta w_x. \tag{7.126}$$

The expression for Λ is given by (7.114). A proof of the theorem follows from the identity

$$\left(\frac{\partial \mathfrak{L}}{\partial w_t} \delta w \right)_t = \left(\frac{\partial \mathfrak{L}}{\partial w_t} \right)_t \delta w + \frac{\partial \mathfrak{L}}{\partial w_t} \delta w_t, \tag{7.127}$$

and similar ones for other derivatives and the Euler equation. The identity (7.124) proves the Noether theorem:

Theorem 7.2. *If the Lagrangian changes by a divergence*

$$\delta\mathfrak{L} = \epsilon(\Theta_t + \Xi_x) \tag{7.128}$$

under the infinitesimal transformations $w \to w + \epsilon f$, then, for all solutions of the potential KdV equation $\Lambda w = 0$, the conservation law

$$T_t + X_x = 0 \tag{7.129}$$

exists, with

$$T = \epsilon^{-1}A - \Theta, \tag{7.130}$$

$$X = \epsilon^{-1}B - \Xi. \tag{7.131}$$

The following lemma occurs:

Lemma 7.3. *Let $d_{ik} = w_i - w_k$ and*

$$\mathfrak{L}[w_1] - \mathfrak{L}[w_0] = \Theta_t^{01} + \Xi_x^{01}, \tag{7.132}$$

with

$$\Theta^{10} = -\frac{1}{12}d_{10}d_{10}^2 + \frac{1}{4}\kappa^2, \tag{7.133}$$

$$\Xi^{10} = d_{10}\left(-\frac{4}{5}d_{10}^4 - 2\kappa^2 + w_x(d_{10}^2 - \kappa^2) - 2w_x^2 + \frac{1}{4}(w_1 + w_0)\right).$$

Then the transformation $\mathbb{B}_\kappa$ is the Noether transformation.

This is proved by the definitions of the Lagrangian (7.122) and w_1 (7.116) on the basis of (7.113). The product $\mathbb{B}_{\kappa+\epsilon}\mathbb{B}_{-\kappa}$, being the Noether transformation, generates the vector (Θ, Ξ) such that

$$\delta\mathfrak{L} = \mathfrak{L}[w_2] - \mathfrak{L}[w_1] = \epsilon(\Theta_t + \Xi_x) \tag{7.134}$$

determines the variation about the fixed w_0. Finally, the part of the vector (Θ, Ξ),

$$T = -\frac{\kappa}{2}d_{40}, \qquad X = 2\kappa d_{40}(w_x - \kappa^2), \tag{7.135}$$

which is symmetric with respect to $\kappa \to -\kappa$ (the symmetry of the BT is accounted for), contributes indeed to the Noether conservation law:

$$\frac{1}{2}d_{40t} + 2d_{40}(\kappa^2 - w_x)_x. \tag{7.136}$$

The substitution of expansion (7.119) into (7.136) produces the conservation laws

$$(A_{2r-1})_t + 4(A_{2r+1} - w_x A_{2r-1})_x = 0, \qquad r = 1, 2, \ldots \tag{7.137}$$

in the form of Wadati et al. [445].

7.3.3 Comment on Miura map

The first relation in (7.113) for imaginary $\kappa = \mathrm{i}k$ in terms of $d_{01} = w_0 - w_1 = d_{41}^*$ is nothing more than the Miura link for $u = 2w_x$,

$$\sigma_x = \sigma^2 + k^2 - u,$$

or, in the notation of this section,

$$d_{01} = \sigma = \phi_x/\phi,$$

where ϕ is a solution of

$$-\phi_{xx} + u\phi = -k^2\phi.$$

This link immediately leads to the continuum conservation law from the celebrated paper of Miura et al. [335]

$$(\phi^*\phi)_t + (\phi^*\phi_{xx} + \phi\phi_{xx}^* - 4|\phi_x^2| - 6k^2\phi^*\phi)_x = 0$$

in the context of the Noether theorem.

Quite similarly the sine–Gordon equation is treated in [409].

7.4 From singular manifold method to Moutard transformation

Paper [10] contains the so-called Ablowitz–Ramani–Segur conjecture that incorporated the Painlevé property [360]. This result was extended by Weiss et al. [449] as the Weiss–Tabor–Carnevale theory to check the Painlevé property for a PDE.

Estévez and Leble [145, 146] developed a procedure to derive the Moutard transformation (and hence the DTs) in the framework of the singular manifold method. The generalization of these ideas for the case of two Painlevé branches was made in [143].

We will illustrate the idea using the example of the singular manifold method analysis of a version of the 2+1 KdV (Boiti–Leon–Manna–Pempinelli 1) equation ([59]). Let us write this equation in the form [145]

$$m_{ty} = (m_{xxy} + m_y m_x)_x. \tag{7.138}$$

It is proved that (7.138) has the standard Painlevé property, i.e., its solutions can be locally expanded in terms of four arbitrary functions. The truncated expansion produces the auto-BT

$$m[1] = m + 6\frac{\phi_x}{\phi}, \tag{7.139}$$

which links two solutions of (7.138) by the "singular manifold" function ϕ. The substitution of (7.139) into (7.138) and application of the generalized procedure [146] leads to the Lax pair

$$\phi_{xxx} - \phi_t + m_x\phi_x = 0, \qquad 3\phi_{xy} + m_y\phi = 0. \tag{7.140}$$

A consideration of (7.139) as a transformation $m \to m[1]$ and the truncated expansion for the transformed function $\psi[1]$,

$$\psi[1] = \frac{p}{\phi}, \tag{7.141}$$

which is the solution of the Lax pair (7.140) with the transform $m[1]$, yields the following equations for p:

$$p_x = -2\psi\phi_x, \quad p_y = -2\phi\psi_y, \quad p_t = 2\psi_x\phi_{xx} - 2\phi_x\psi_{xx} - 2\psi\phi_t. \tag{7.142}$$

It can be proved that the form

$$d\Omega = -\psi\phi_x \mathrm{d}x - \phi\psi_y \mathrm{d}y + (\psi_x\phi_{xx} - \phi_x\psi_{xx} - \psi\phi_t)\mathrm{d}t \tag{7.143}$$

is exact (i.e., $\mathrm{d}p = -2\mathrm{d}\Omega$) on solutions ψ and ϕ of the Lax equations and hence there exists

$$\psi[1] = \psi - 2\frac{\Omega(\psi, \phi)}{\phi}, \tag{7.144}$$

which coincides with the Moutard transformation [340, 341]. The method seems to be an effective tool to derive the Moutard transformation formalism in 2+1 dimensions [140]. It was further applied to generate the DTs for the Bogoyavlenskii equation in $2+1$ dimensions [144]. The constructive elements of the theory are presented in [141].

7.5 Zakharov–Shabat dressing method via operator factorization

7.5.1 Sketch of IST method

In the "new history" of the soliton theory, half a century after the Bäcklund–Moutard–Darboux transformations, the notion of dressing appeared within the inverse scattering problem, when solving the Cauchy problem for the KdV equation [474]. To begin with, let us sketch the IST method and introduce scattering data for the one-dimensional Sturm–Liouville problem

$$-\partial_x^2\psi + u(x)\psi = k^2\psi \tag{7.145}$$

with a localized potential $u(x)$ ($\epsilon > 0, |x| \to \infty \Rightarrow |u(x)x^{(1+\epsilon)}| \to 0$) and the spectral parameter k^2. The scattering data comprise eigenvalues $k_n = \mathrm{i}\kappa_n$, normalization constants $a_n = \lim_{x\to\infty} \exp(\kappa_n)\psi_n$ for eigenfunctions ψ_n normalized as $\int_{-\infty}^{\infty} |\psi_n|^2 \mathrm{d}x = 1$, and the reflection coefficient $v(k)$. The last one is extracted from the asymptotic behavior of the continuum spectrum solutions

$$\psi(x,k) \simeq \begin{cases} \exp(-ikx) + v(k)\exp(ikx), & x \to \infty, \\ w(k)\exp(-ikx), & x \to -\infty. \end{cases} \tag{7.146}$$

Solving the scattering problem, we arrive at the function $F(x)$ [354]:

$$F(x) = \sum_m a_m \exp(-\kappa_m x) + \frac{1}{2\pi}\int_{-\infty}^{\infty} v(k)\exp(ikx)\,\mathrm{d}k, \tag{7.147}$$

which determines the kernel of the Gel'fand–Levitan–Marchenko (GLM) integral equation

$$K(x,y) + F(x+y) + \int_x^{\infty} K(x,s)F(s+y)\,\mathrm{d}s = 0, \quad x \le y. \tag{7.148}$$

Then the potential $u(x)$ is retrieved from the solution $K(x,y)$ of (7.148) as

$$u(x) = 2\frac{\mathrm{d}}{\mathrm{d}x}K(x,x). \tag{7.149}$$

Equation (7.148) links K and F; it maps the scattering data to the potential and is referred to as the inverse scattering transformation. The Gardner–Green–Kruskal–Miura theory, using the second operator of the Lax pair (see Chap. 3), gives explicit dependence of the scattering data on time, $a_m(t)$ and $v(k,t)$, via the initial values of $a_m(0)$ and $v(k,0)$.

The GLM equation (7.148) is solved explicitly in some of the simplest cases [354]. The multisoliton solutions correspond to zero v (reflectionless potentials). The kernel of the integral operator factorizes in this case and has a finite number of terms, as is seen from (7.147).

7.5.2 Dressible operators

The idea of the dressing method in its original IST version [474] (we follow the modification given in [466]) uses the fact that each function F generates the function K and hence a potential. Let us write (7.147) symbolically as

$$K + F + K^*F = 0, \tag{7.150}$$

where the asterisk denotes the action of the integral operator and the function $F(x,y)$ goes to $F(x+y)$ for the standard GLM equation. Consider a pair of operators M and $\widehat{M}$ which obey the equation

$$\widehat{M}K + MF + (\widehat{M}K)^*F + K^*(MF) = 0. \tag{7.151}$$

The operator M is named the "bare" operator, and $\widehat{M}$ is the "dressed" operator. Suppose the function F obeys the equation

$$MF = 0. \tag{7.152}$$

Then we have

$$\widehat{M}K = 0, \tag{7.153}$$

if the operator $\widehat{M}$ exists. The set of pairs $(M,\, \widehat{M})$ forms a vector space.

As an example, consider the operator

$$M = \partial_x + \partial_y. \tag{7.154}$$

In this case (7.151) takes the form

$$MK(x,y) + MF(x,y) + \partial_x \int_x^\infty K(x,s)F(s,y)\,\mathrm{d}s + \int_x^\infty K(x,s)\partial_y F(s,y)\,\mathrm{d}s = 0. \tag{7.155}$$

Evidently,

$$\partial_x \int_x^\infty K(x,s)F(s,y)\,\mathrm{d}s = -K(x,x)F(x,y) + \int_x^\infty [(\partial_x K(x,s)]\, F(s,y)\,\mathrm{d}s. \tag{7.156}$$

Integration by parts gives

$$I = \int_x^\infty [\partial_s K(x,s)]\, F(s,y)\,\mathrm{d}s = -\int_x^\infty K(x,s)\partial_s F(s,y)\,\mathrm{d}s - K(x,x)F(x,y). \tag{7.157}$$

In (7.155) take into account (7.156) and introduce I as

$$\begin{aligned} MK(x,y) + MF(x,y) - K(x,x)F(x,y) + \int_x^\infty [\partial_x K(x,s)]\, F(s,y)\,\mathrm{d}s \\ + \int_x^\infty K(x,s)\partial_y F(s,y)\,\mathrm{d}s + I - I = 0. \end{aligned} \tag{7.158}$$

For $+I$ substitute the mid-positioned term in (7.157), and for $-I$ the right-hand side of (7.157) with the opposite sign:

$$\begin{aligned} MK(x,y) + MF(x,y) - K(x,x)F(x,y) + \int_x^\infty [\partial_x K(x,s)]\, F(s,y)\,\mathrm{d}s \\ + \int_x^\infty K(x,s)\partial_y F(s,y)\,\mathrm{d}s + \int_x^\infty [\partial_s K(x,s)]\, F(s,y)\,\mathrm{d}s \\ + \int_x^\infty K(x,s)\partial_s F(s,y)\,\mathrm{d}s + K(x,x)F(x,y) = 0. \end{aligned} \tag{7.159}$$

Ordering the terms, we get

$$(\partial_x + \partial_y)K(x,y) + MF(x,y) \tag{7.160}$$
$$+ \int_x^\infty [(\partial_x + \partial_s) K(x,s)] F(s,y)\,\mathrm{d}s + \int_x^\infty K(x,s)(\partial_y + \partial_s) F(s,y)\,\mathrm{d}s = 0.$$

Hence, the following operator arises:

$$\widehat{M} = M = \partial_x + \partial_y. \tag{7.161}$$

The operator M is called *dressible.* A set of dressible operators forms linear space.

A connection between scattering data and a potential $U = U(x,t)$ with the additional (time) parameter is used in integrable equations via the Lax representation [335] and, directly, in quantum evolution problems. The problems in which potentials are functions of time can be studied by the present method because the operator of the time derivative ∂_t is dressible. As before, from the equation

$$MF(x,y,t) = 0$$

we obtain

$$\widehat{M}K(x,y,t) = 0.$$

Let a function ψ be a solution of two equations

$$(\partial_t - L[U])\psi = 0, \tag{7.162}$$

$$(\partial_y - A[U])\psi = 0. \tag{7.163}$$

If derivatives with respect to t and y commute, then the Lax representation is

$$A_t - L_y = [A, L]. \tag{7.164}$$

Proposition 7.4. *If two operators M and $\widehat{M}$ are such that there exists a solution of (7.151) (operator M is dressible) and if the operator M forms the Lax pair with N, then the operators $\widehat{M}$ and $\widehat{N}$ also form a Lax pair. If a pair of operators M and N produces a nonlinear system, then the pair $\widehat{M}$ and $\widehat{N}$ produces the same system.*

The next example is

$$M = \alpha\frac{\partial}{\partial t} + \partial_x^2 - \partial_y^2. \tag{7.165}$$

We want to dress the operator M, applying it to the GLM equation (7.151). Integrating by parts yields the operator

$$\widehat{M} = \alpha\frac{\partial}{\partial t} + \partial_x^2 - \partial_y^2 + U(x), \tag{7.166}$$

where

$$U(x) = -2\frac{\mathrm{d}}{\mathrm{d}x}K(x,x). \tag{7.167}$$

Note that a function U has appeared in the dressed operator, while for the first-order operator (7.154) the dressed operator is the same as the bare one (7.161).

In general, we put

$$L_0 = l_0(x, t, \ldots)\partial_x^n. \tag{7.168}$$

Consider the operator D of the following structure:

$$DF = \alpha\partial_t F + L_0 F - F L_0^+, \tag{7.169}$$

where L_0^+ is the Hermitian conjugate to L and acts to the left.

Proposition 7.5. *The operator (7.169) is dressible. The dressed operator $\widehat{D}$ is*

$$\widehat{D}K = \alpha\partial_y K + LK - K L_0^+, \tag{7.170}$$

where

$$L = L_0 + \widetilde{L} \tag{7.171}$$

and

$$\widetilde{L} = \hat{l}_0\, \partial_x^{n-1} + \ldots, \qquad \hat{l}_0 \sim (\partial_x - \partial_y)^i K\big|_{y=x}. \tag{7.172}$$

7.5.3 Example

Let us take

$$L_0 = \partial_x^2 \quad \Rightarrow \quad L = \partial_x^2 + U.$$

Solving the equation $MF = 0$ yields $\widehat{M}K = 0$; hence, some class of solvable equations appears, with some linear space.

Let us consider operators D_1 and D_2,

$$D_1 F = \alpha_1 \partial_{t_1} F + L_0^{(1)} - F L_0^{(1)+},$$

$$D_2 F = \alpha_2 \partial_{t_2} F + L_0^{(2)} - F L_0^{(2)+}.$$

This class of operators contains the Lax representation

$$\alpha_1 \partial_{t_1} L_0^{(2)} - \alpha_2 \partial_{t_2} L_0^{(1)} + \left[L_0^{(1)}, L_0^{(2)}\right] = 0.$$

For relevant forms of the operators $L_0^{(1)}$ and $L_0^{(2)}$ and for $\alpha_1 = \alpha$, $\alpha_2 = -1$, $t_1 = y$, and $t_2 = t$ we obtain the KP equation

$$\partial_x \left(u_t + 6uu_x + u_{xxx}\right) + \alpha^2 u_{yy} = 0.$$

In the case of $\alpha^2 = -1$ we have the KP I equation; otherwise, if $\alpha^2 = 1$ we have the KP II equation. The KP equation is the two-dimensional equation that contains the KdV equation as a y-independent reduction:

$$u_t + 6uu_x + u_{xxx} = 0.$$

It was demonstrated in [324] that the triangular (Volterra) factorization of the operator

$$F = (1 + K^{+})^{-1}(1 + K^{-})$$

proved by Zakharov and Shabat [474] links the Zakharov–Shabat dressing scheme to the DT dressing.