5

Dressing in 2+1 dimensions

In this chapter we speak again about the origin of the dressing technique, now in multidimensions. The important step was realized in the Moutard papers [340, 341] that the stabilization of the Laplace transformation chain can generate solutions. Notice again (see Chap. 1) that the net of points generated by the transform of the invariants of the *gauge transformations* has two possible symmetry reductions: the first reduction corresponds to the Moutard case and the second one was discovered by Goursat [192, 193]. The dressing procedure in two spatial dimensions opened a way to apply the Laplace equation in Lax pairs to solve some nonlinear 2+1 equations because their associated spectral problems are expressed in terms of the Laplace equation.

The celebrated 2+1 Kadomtsev–Petviashvili (KP) equation for surface water waves (there are lots of other applications [228]; see Chaps. 9, 10) and the corresponding dressing based on the direct extension of the Darboux theory (linear Schrödinger evolution as the first operator in the Lax pair) [313] have been the subject of intense studies [324]. The dressing methods for the Davey–Stewartson (DS) equation were introduced in [277], where, by means of eight Ablowitz–Kaup–Newell–Segur (AKNS) type pairs, ordinary and two-fold elementary Darboux transformations (DTs) were studied and used for construction of multisoliton solutions of both types (DS I and DS II) of the DS equation. The dressed potentials were expressed in terms of quasideterminants studied previously in [176]. It was proved that nonlinear superposition formulas have a symmetry structure that gives a possibility to build networks of DTs that can be used to solve boundary problems via the construction proposed in [199]. An important class of solutions of a general Zakharov-Shabat (ZS) hierarchy that was not mentioned in [324] is generated by the dressing formulas from [313, 314]. In particular, solutions of the KP equations are given by the relation [313]

$$u = -2\partial^2 \ln W(\varphi_1, \dots, \varphi_s),$$

where the Wronskian W is formed by the dressing functions φ_j depending on a parameter k and arbitrary function $g(k)$:

$$\varphi_j = [\partial_k + g(x)] \exp(kx + k^2 y + k^3 t)|_{k=k_j}.$$

This class of solutions contains the so-called general position solutions derived by Krichever [252] via the finite-gap formalism. Note also that these solutions generate the Calogero–Moser potentials

$$u = 2\sum_j \frac{1}{x - x_j(y,t)},$$

which can be extracted from the dressing formulas. For the N-particle problems and polynomial solutions of the ZS hierarchy we refer to [315]. The 2+1 theory of generalized AKNS equations, including the DS, the Boiti–Leon–Manna–Pempinelli (BLMP1 and BLMP2) [58, 65], and some other equations, is studied in [140, 141, 143, 144, 142].

Here we concentrate on studying a general theory of dressing based on combinations of the following transformations: Laplace, Darboux (Sects. 5.1, 5.2), Goursat (Sect. 5.3), and Moutard (Sect. 5.4). Among other things, we derive a new integrable equation (5.19) which can be treated as the two-dimensional generalization of the sinh–Gordon equation. Sections 5.5 and 5.6 illustrate applications of this theory to the two-dimensional Korteweg–de Vries (KdV), two-dimensional modified KdV (MKdV), Nizhnik–Veselov–Novikov, and BLMP1 equations.

5.1 Combined Darboux–Laplace transformations

In this section we formulate constraints to coefficients of the Laplace equation which reduce it to the Moutard and Goursat equations. We show that a number of integrable nonlinear equations arise as a consequences of the reduction equations for the DTs. The content of this section is based on [287].

5.1.1 Definitions

For the Laplace equation

$$\psi_{xy} + a\psi_y + b\psi = 0 \tag{5.1}$$

the following were introduced:

1. The Laplace transformations (LTs) (Sect. 1.5)

$$a \to a_{-1} = a - \partial_x \ln(b - a_y), \quad b \to b_{-1} = b - a_y, \quad \psi \to \psi_{-1} = \psi_x + a\psi, \tag{5.2}$$

$$a \to a_1 = a + \partial_x \ln b, \qquad b \to b_1 = b + \partial_y \left(a + \partial_x \ln b\right), \qquad \psi \to \psi_1 = \frac{\psi_y}{b}. \tag{5.3}$$

2. The DTs

$$a \to a_1 = a - \partial_x \ln(a+\sigma), \quad b \to b_1 = b + \sigma_y, \quad \psi \to \psi_1 = \psi_x - \sigma\psi, \tag{5.4}$$

$$a \to {}_1a = -(\sigma + b\rho), \qquad b \to {}_1b = b - (b\rho)_y, \qquad \psi \to {}_1\psi = \rho\psi_y - \psi. \tag{5.5}$$

where $\sigma = \sigma(x,y) = \phi_x/\phi$, $\rho = \phi/\phi_y$, and ψ and ϕ are particular solutions of (5.1) with predetermined a and b. We refer to ϕ as *the support function* of the DT.

5.1.2 Reduction constraints and reduction equations

A constraint for the coefficients a and b of (5.1) fixes a particular class of equations which we are interesting in. Namely, the condition

$$a = 0, \qquad b = u \tag{5.6}$$

yields the *Moutard equation*

$$\psi_{xy} + u(x,y)\psi = 0, \tag{5.7}$$

while

$$a = -\frac{1}{2}\partial_x \ln\lambda, \qquad b = -\lambda \tag{5.8}$$

leads to the *Goursat equation*

$$\zeta_{xy} = 2\sqrt{\lambda\,\zeta_x\zeta_y}. \tag{5.9}$$

After the substitution $\psi = \sqrt{\zeta_x}$ and $\chi = \sqrt{\zeta_y}$ we get

$$\psi_y = \sqrt{\lambda}\chi, \qquad \chi_x = \sqrt{\lambda}\psi$$

or, in the form of the Laplace equation,

$$\psi_{xy} = \frac{1}{2}(\ln\lambda)_x\psi_y + \lambda\psi \tag{5.10}$$

and a similar equation for χ; see also Sect. 5.1.3. The functions u and λ are solutions of the special equations which we call the *reduction equations*. In this section we will derive these equations for the LT and the DT. We study mostly the example of the Goursat equation, but the approach is directly reformulated for the Moutard equation.

Let us consider the LTs (5.2). The invariance of the reduction constraint (5.8) means

$$\lambda_{-1} = \lambda - \frac{1}{2}\partial_x\partial_y \ln\lambda = \frac{C}{2\lambda}, \quad C = \text{const.} \tag{5.11}$$

It is obvious that (5.11) is valid for the LT (5.3) as well because the last one is inverse to (5.2).

The reduction equation for this transformation is the well-known sinh–Gordon equation

$$\partial_x\partial_y \ln\lambda = 2\lambda - \frac{C}{\lambda}, \tag{5.12}$$

and the new potential λ_{-1} is a solution of (5.12) too. In the case $C = 0$ we obtain $\lambda_{-1} = 0$ and the Liouville equation, instead of (5.12). The general integral for the Liouville equation is well known:

$$\lambda = \frac{f'g'}{(f+g)^2},$$

where $f = f(x)$ and $g = g(y)$ are arbitrary differentiable functions. The Goursat equation is integrated as

$$\zeta = -\frac{1}{C_1^2}\partial_y \ln(f+g) + V, \quad C_1 = \text{const.}$$

The function $V = V(y)$ is determined by the equation

$$V' = \left(\frac{1}{2C_1}(\ln g')'\right)^2 = \frac{1}{4C_1^2}\left(\frac{g''}{g'}\right)^2$$

and

$$\psi = \frac{\sqrt{f'g'}}{C_1(f+g)}, \qquad \chi = \frac{1}{2C_1}\partial_y \ln\left(-\partial_y \frac{1}{f+g}\right).$$

Proposition 5.1. *Let M and L be two Laplace invariants of (5.1). This means that*

$$M = \frac{1}{2}\partial_x\partial_y \ln\lambda - \lambda, \qquad L = -\lambda.$$

Using the reduction equation (5.12) yields

$$M = -\frac{C}{2\lambda}, \qquad L = -\lambda$$

and

$$M_{-1} = M_1 = L, \qquad L_{-1} = L_1 = M.$$

Now we take the DT (5.4). Inserting both transforms into the reduction condition (5.8), we get

$$\lambda_1 = \lambda - \sigma_y = \lambda\left(\sigma - \frac{\lambda_x}{2\lambda}\right). \tag{5.13}$$

Denote $\alpha = \ln\phi$ and $\Lambda = \ln\lambda$. Since

$$\lambda - \sigma_y = \left(-\frac{1}{2}\Lambda_x + \alpha_x\right)\alpha_y$$

and $\sigma = \alpha_x$, we obtain from the transform (5.13) the condition for Λ:

$$\left(\alpha_x - \frac{1}{2}\Lambda_x\right)\left[\alpha_y - \exp(\Lambda)\left(\alpha_x - \frac{1}{2}\Lambda_x\right)\right] = 0. \tag{5.14}$$

Equating to zero the first parentheses yields

$$\Lambda_{xy} = 2\exp(\Lambda)$$

and $\alpha = \Lambda/2 - c(y)$, where $c(y)$ is an arbitrary function. But in this case we get $\lambda_1 = 0$, and the Liouville equation is in the realm of the reduction equation.

Equating to zero the brackets in (5.14), we arrive at the equation

$$[\exp(-2\alpha)\lambda]_x = [\exp(-2\alpha)]_y \ ; \tag{5.15}$$

therefore,

$$\theta_x = \psi^2 = \frac{1}{F_x + C_2}, \qquad \lambda = \frac{F_y + C_1}{F_x + C_2},$$

where $F = F(x, y)$ is any differentiable function and $C_{1,2} = \text{const}$. Substituting (5.15) into (5.10) yields

$$2(C_2 + F_x)C_1^2 + [(F_{yxx} + 4F_y)C_2 + F_xF_{yxx} + 4F_yF_x - F_{xx}F_{yx}]\,C_1 + 2F_y^2F_x + \left(F_{yxx}F_y - \frac{1}{2}F_{yx}^2 + 2F_y^2\right)C_2 - \frac{1}{2}F_{yx}^2F_x - F_yF_{xx}F_{yx} + F_xF_yF_{yxx} = 0. \tag{5.16}$$

Define new fields P and Q as

$$F_x = P - C_2, \qquad F_y = Q - C_1.$$

Then (5.16) can be split into the system

$$2Q_xQP_x - (2Q_{xx}Q - Q_x^2 + 4Q^2)P = 0, \qquad P_y = Q_x. \tag{5.17}$$

After integration of the first equation we get

$$P = \frac{C_3Q_x}{\sqrt{Q}}\exp G, \qquad G_x = 2\frac{Q}{Q_x},$$

where C_3 is the third constant of integration. It is necessary to obey the second equation in (5.17). Let

$$Q = n^2(x, y), \qquad G = \ln m(x, y).$$

Then the reduction equation is simplified:

$$\left(n^2\right)_x = 2C\left(mn_x\right)_y, \qquad m_xn_x = mn. \tag{5.18}$$

This system can be rewritten in more convenient form. Let

$$n_x = n\exp S, \qquad m_x = m\exp(-S),$$

$S = S(x, y)$. After substituting into (5.18) we get

$$S_y = \frac{1}{C}\frac{n}{m} - \partial_y \ln(mn);$$

therefore,

$$S_{xy} = 4(\sinh S)\,\partial_y \partial_x^{-1} \cosh S. \tag{5.19}$$

Equation (5.19) is the reduction equation for the DT (5.4). It looks like (5.12) and it is a generalization of the $d = 2$ sinh–Gordon equation. The Lax pair for (5.19) is introduced by means of the following:

Proposition 5.2. *The (L, A) pair for (5.19) is written as*

$$K\psi = 0, \qquad K_1 D\psi = 0,$$

where

$$D = \partial_x - \sigma, \quad K = \partial_x\partial_y - \frac{1}{2}\frac{\lambda_x}{\lambda}\partial_y - \lambda, \qquad K_1 = \partial_x\partial_y - \frac{1}{2}\frac{\lambda_{1,x}}{\lambda_1}\partial_y - \lambda_1,$$

and the variables λ and λ_1 are determined by

$$\lambda = \frac{(S_x + 2\cosh S)_y}{4\sinh S}\exp(-S), \qquad \lambda_1 = \frac{(S_x + 2\cosh S)_y}{4\sinh S}\exp S, \tag{5.20}$$

and $\sigma_y \equiv \lambda - \lambda_1$.

This statement is checked by direct substitution. Thus, the reduction equations for the DT (5.4) have either the form of (5.19) or the form of the Liouville equation.

The reduction equations for the DT (5.5) are obtained similarly. As a result, we get

$$\lambda = C_1\phi_y \exp F, \qquad {}_1\lambda = -\frac{C_1C_2\phi^2}{\phi_y}\exp F, \tag{5.21}$$

where ϕ is the support function of the DT (5.5) and the reduction equation can be written in the form of a system

$$\phi_{xy} = \phi_y[F_x + 2C_1\phi\exp F], \qquad F_y\phi_y = C_2\phi.$$

Proposition 5.3. *By the construction (5.20) for the DT (5.4) we get*

$$M = -\lambda_1, \qquad L = -\lambda$$

and

$$M_1 = M\exp(-2S), \qquad L_1 = L\exp(2S).$$

Similarly for the DT (5.5) the use of (5.21) gives

$$M = -\frac{C_2(-\phi_x + \phi F_x + C_1\phi^2 \exp F)}{\phi_y}, \qquad L = -C_1\phi_y \exp F$$

and

$${}_1M = -\frac{\phi_y^2}{C_2\phi^2}M, \qquad {}_1L = -\frac{C_2\phi^2}{\phi_y^2}L.$$

The product of the Laplace invariants ML is invariant in both cases. Combinations of LT and DT generate new equations and their Lax pairs.

5.1.3 Goursat equation, geometry, and two-dimensional MKdV equation

As shown in Sect. 5.1.2, the Goursat equation (5.9) is connected to the particular case of (5.1) with two potentials $a = a(x,y)$ and $b = b(x,y) = \lambda(x,y)$. We refer to λ as the potential function. The reduction (5.8) is valid only for special types of potentials if the form of the Laplace equation is maintained while transformations are performed. Our interest in the Goursat equation is caused by applications of this equation in geometry and in the soliton theory:

1. As regards geometry, let x be the complex coordinate, $y = -\overline{x}$, $\sqrt{\lambda}$ is the real-valued function, and ψ or χ as solutions of (5.10) are complex-valued functions. Then we define three real-valued functions X_i, $i = 1,2,3$ which are the coordinates of a surface in $\mathbb{R}^3$ [242]:

$$X_1 + \mathrm{i}X_2 = 2\mathrm{i}\int_\Gamma \left(\overline{\psi^2}\mathrm{d}y' - \overline{\chi^2}\mathrm{d}x'\right),$$

$$X_1 - \mathrm{i}X_2 = -2\mathrm{i}\int_\Gamma \left(\psi^2\mathrm{d}y' - \chi^2\mathrm{d}x'\right), \tag{5.22}$$

$$X_3 = -2\int_\Gamma \left(\overline{\psi}\chi\mathrm{d}y' + \overline{\chi}\psi\mathrm{d}x'\right),$$

where Γ is an arbitrary path of integration in the complex plane. The corresponding first fundamental form, the Gaussian curvature K, and the mean curvature H yield:

$$\mathrm{d}s^2 = 4U^2\mathrm{d}x\mathrm{d}y, \qquad K = \frac{1}{U^2}\partial_x\partial_y \ln U, \qquad H = \frac{\sqrt{\lambda}}{U}.$$

Here $U = \mid \psi \mid^2 + \mid \chi \mid^2$ and any analytic surface in $\mathbb{R}^3$ can be globally represented by (5.22) [244].

2. As an example of soliton equations, consider the system of the two-dimensional MKdV equations introduced by Boiti, Leon, Martina, and Pempinelli [58, 65]:

$$4\lambda^2(\lambda_t - A\lambda_x + B\lambda_y - \lambda_{xxx} - \lambda_{yyy}) + 4\lambda^3\left[(2\lambda + B)_y + (2\lambda - A)_x\right] +$$
$$+6\lambda(\lambda_y\lambda_{yy} + \lambda_x\lambda_{xx}) - 3(\lambda_x^3 + \lambda_y^3) = 0,$$
$$B_x = 3\lambda_y - \lambda_x, \qquad A_y = \lambda_y - 3\lambda_x. \tag{5.23}$$

Here $\lambda = \lambda(x, y, t)$, $A = A(x, y, t)$, and $B = B(x, y, t)$. If we introduce the function $u = \sqrt{\lambda}$, then we can rewrite (5.23) in the more customary form

$$u_t + 2u^2(u_x + u_y) + \tfrac{1}{2}\left(B_y - A_x\right)u + Bu_y - Au_x - u_{3y} - u_{3x} = 0,$$
$$B_x = \left(3\partial_y - \partial_x\right)u^2, \qquad A_y = \left(\partial_y - 3\partial_x\right)u^2. \tag{5.24}$$

The reduction conditions $A = -B = -2u^2$ and $u_y = u_x$ lead to the MKdV equation,

$$u_t + 12u^2u_x - 2u_{3x} = 0,$$

(here $u_{3x} \equiv u_{xxx}$) so we call (5.24) the two-dimensional MKdV equations. The two-dimensional MKdV equations (5.24) are the compatibility condition of the linear system comprising (5.10) and

$$\psi_t = \psi_{3x} + \psi_{3y} - \frac{3}{2}\frac{\lambda_y}{\lambda}\psi_{yy} + \left[\frac{3}{4}\left(\frac{\lambda_y}{\lambda}\right)^2 - \lambda - B\right]\psi_y + (A - \lambda)\psi_x + \frac{1}{2}(A_x - \lambda_x)\psi.$$

We will study (5.24) in Sect. 5.6.

Remark 5.4. Zenchuk [477] studied the chains of discrete transformations (5.2)–(5.5) of solutions and potentials in the general case of the linear second-order partial differential equation with two independent variables. Considering the simplest ($k = 2$) closed chains of these transformations, he obtained a novel *integrable* equation

$$\frac{1}{2}S_{xy} - \mathrm{e}^S - \mathrm{e}^{-S}\left[C_1 - C_2\partial_x^{-1}\left(\mathrm{e}^{-S}\right)_y\right] = 0,$$

where $C_2 > 0$.

In the present chapter we use the reduction restriction (5.8) as a (weak) condition of closure. In Sect. 5.1.2 we derived a new integrable equation (5.19), the two-dimensional generalization of the sinh–Gordon equation. In the next section we employ the Goursat transformation and the binary Goursat transformation to construct explicit solutions of the Goursat equation. These transformations allow us to obtain new solutions of the Goursat equation without solving the reduction equation. We also discuss the transformation for Laplace invariants.

5.2 Goursat and binary Goursat transformations

An analogy of the Moutard transformation for the Goursat equation was studied by Ganzha [169]. Such a Goursat transformation is valid without a reduction restriction and reduction equations. Many useful details can be found in the textbook of Ganzha and Tsarev [171], where the transformation is defined via two solutions of (5.9). The transformed function $\psi[1]$ and the potential $\lambda[1]$ are extracted by quadratures [169, 197].

Theorem 5.5. *Let the transform $\psi[1]$ be introduced by the relations*

$$(z_1\psi[1]/\psi_1)_x = z_1(\psi_2/\psi_1)_x, \tag{5.25}$$

$$(z_1\psi[1]/\psi_1)_y = [z_1 z_{1xy} - 2z_{1x}z_{1y}/z_{1xy}](\psi_2/\psi_1)_y,$$

where $z_{1,2}$ are solutions of (5.9) and $\psi_{1,2} = \sqrt{z_{1,2x}}$ solve (5.10). Then $\psi[1]$ is a solution of the (transformed) equation (5.10) with the potential

$$\lambda[1] = \lambda - (\ln z_1)_{xy}$$

and the transform $z[1]$ is found by a quadrature from

$$z[1]_x = \psi^2[1], \qquad z[1]_y = (\psi[1]_y)^2/\lambda[1]. \tag{5.26}$$

This transformation preserves the form of the Laplace–Goursat equation (5.10), e.g., possesses the covariance property. Below we introduce a twofold eDT for the Goursat equaton with the same property.

We introduce new variables $\xi = x + y$ and $\eta = x - y$ and rewrite (5.10) in matrix form,

$$\Psi_\eta = \sigma_3\Psi_\xi + U\Psi. \tag{5.27}$$

Here

$$\Psi = \begin{pmatrix} \psi_1 & \psi_2 \\ \chi_1 & \chi_2 \end{pmatrix}, \qquad U = \sqrt{\lambda}\sigma_1, \tag{5.28}$$

where $\psi_k = \psi_k(\xi,\eta)$ and $\chi_k = \chi_k(\xi,\eta)$, $k = 1,2$ are particular solutions of (5.10) with some $\lambda(\xi,\eta)$, and $\sigma_{1,3}$ are the Pauli matrices. Let Ψ_1 be some solution of (5.27) and $\Psi \neq \Psi_1$. We define a matrix function $\sigma \equiv \Psi_{1,\xi}\Psi_1^{-1}$. Equation (5.27) is covariant with respect to the classical DT:

$$\Phi[1] = \Phi_\xi - \sigma\Phi, \qquad U[1] = U + [\sigma_3, \sigma]. \tag{5.29}$$

It is a particular case of the general classical non-Abelian formula from Chap. 2, the Matveev Theorem 2.19.

Remark 5.6. It is not difficult to check that the DT (5.29) is the superposition formula for two simpler DTs given by (5.4) and (5.5).

Remark 5.7. Equation (5.27) is the spectral problem for the DS equation [13, 277]. The LT produces explicitly invertible Bäcklund autotransformations for the DS equation. It is shown in [459] that these transformations permit solutions to the DS equation to be constructed that fall off in all directions in the plane according to exponential and algebraic laws.

Next we consider a closed 1-form

$$\mathrm{d}\Omega = \mathrm{d}\xi\, \Phi\Psi + \mathrm{d}\eta\, \Phi\sigma_3\Psi, \qquad \Omega = \int \mathrm{d}\Omega,$$

where a 2×2 matrix function Φ solves the equation

$$\Phi_\eta = \Phi_\xi \sigma_3 - \Phi U. \tag{5.30}$$

Let us apply the DT. It can be verified by immediate substitution that (5.30) is covariant with respect to the transformation

$$\Phi[+1] = \Omega(\Phi, \Psi_1)\Psi_1^{-1}.$$

We can alternatively affect U (5.28) by the following transformation:

$$U[+1,-1] = U + [\sigma_3, \Psi_1\Omega^{-1}\Phi].$$

The particular solution of (5.30) has the form

$$\Phi_1 = \begin{pmatrix} s_1\psi_1 + s_2\psi_2 & -s_1\chi_1 - s_2\chi_2 \\ s_3\psi_1 + s_4\psi_2 & -s_3\chi_1 - s_4\chi_2 \end{pmatrix}, \tag{5.31}$$

where $s_k = \mathrm{const}$ $(k = 1, \ldots, 4)$. It is convenient to choose Φ_1 in the form

$$\Phi_1 = \Psi_1^{\mathrm{T}}\sigma_3, \tag{5.32}$$

where the superscript T stands for the transpose. Equation (5.32) is the particular case of (5.31). In this case

$$U[+1,-1] = U - 2A_F, \tag{5.33}$$

where A_F is the off-diagonal part of the matrix $A = \Psi_1\Omega^{-1}\Psi_1^{\mathrm{T}}$, $\Omega = \Omega(\Phi_1, \Psi_1)$ and

$$A_F^{\mathrm{T}} = A_F = f\sigma_1. \tag{5.34}$$

Here $f = f(\xi, \eta)$ is some function. Using (5.29), (5.33), and (5.34), we see that $U[+1,-1]$ has the same form as for the initial matrix U,

$$U[+1,-1] \equiv \begin{pmatrix} 0 & \sqrt{\lambda[+1,-1]} \\ \sqrt{\lambda[+1,-1]} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sqrt{\lambda} - 2f \\ \sqrt{\lambda} - 2f & 0 \end{pmatrix};$$

thus, the reduction restriction is valid without the reduction equations.

The new function $\Phi[+1,-1]$ has the form

$$\Phi[+1,-1] = \Phi - \Omega(\Phi, \Psi_1)\,[\Omega(\Phi_1, \Psi_1)]^{-1}\,\Phi_1, \tag{5.35}$$

where Φ is an arbitrary solution of (5.30).

Using the twofold DT (5.33) and (5.35), we can construct a new solution of the Goursat equaton by means of dressing a particular solution. As a result, we get the following theorem (returning to the former variables x and y):

Theorem 5.8. *Let*

$$\psi_{k,y} = \sqrt{\lambda}\chi_k, \qquad \chi_{k,x} = \sqrt{\lambda}\psi_k,$$

$$\alpha_{k,y} = -\sqrt{\lambda}\beta_k, \qquad \beta_{k,x} = -\sqrt{\lambda}\alpha_k,$$

where $k = 1, 2$. Then new functions

$$\alpha_1' = \alpha_1 - \frac{A_1\psi_1 + A_2\psi_2}{D}, \qquad \beta_1' = \beta_1 + \frac{A_1\chi_1 + A_2\chi_2}{D}$$

are solutions of the equations

$$\alpha_{1,y}' = \sqrt{\lambda'}\,\beta_1', \qquad \beta_{1,x}' = \sqrt{\lambda'}\,\alpha_1',$$

where

$$\sqrt{\lambda'} = -\sqrt{\lambda} + \frac{\psi_1\chi_1\Omega_{22} + \psi_2\chi_2\Omega_{11} - (\psi_1\chi_2 + \psi_2\chi_1)\Omega_{12}}{D}$$

and

$$\Omega_{11} = \int \mathrm{d}x\psi_1^2 + \mathrm{d}y\chi_1^2, \qquad \Omega_{12} = \Omega_{21} = \int \mathrm{d}x\psi_1\psi_2 + \mathrm{d}y\chi_1\chi_2,$$

$$\Omega_{22} = \int \mathrm{d}x\psi_2^2 + \mathrm{d}y\chi_2^2, \qquad D = \Omega_{11}\Omega_{22} - \Omega_{12}^2,$$

$$\Lambda_{11} = \int \mathrm{d}x\alpha_1\psi_1 + \mathrm{d}y\beta_1\chi_1, \qquad \Lambda_{12} = \int \mathrm{d}x\alpha_1\psi_2 + \mathrm{d}y\beta_1\chi_2,$$

$$\Lambda_{21} = \int \mathrm{d}x\alpha_2\psi_1 + \mathrm{d}y\beta_2\chi_1, \qquad \Lambda_{22} = \int \mathrm{d}x\alpha_2\psi_2 + \mathrm{d}y\beta_2\chi_2,$$

$$A_1 = \Lambda_{11}\Omega_{22} - \Lambda_{12}\Omega_{12}, \qquad A_2 = \Lambda_{12}\Omega_{11} - \Lambda_{11}\Omega_{12}.$$

Here $\int = \int_\Gamma$, where Γ is an arbitrary path of integration in the plane. The explicit expressions for the functions α_2' and β_2' are obtained by the direct picking up of the relations indicated.

Thus the twofold eDT allows us to construct explicit solutions of the Goursat equation without solving the reduction equation.

5.3 Moutard transformation

The Moutard transformation [340, 341] is a map of the DT type: it connects solutions and the coefficient $u(x,y)$ of the equation (5.7) so that if φ and ψ are different solutions of (5.7), then the solution of the twin equation with $\psi \to \psi[1]$ and $u(x,y) \to u[1](x,y)$ can be constructed by the solution of the system

$$(\psi[1]\varphi)_x = -\varphi^2(\psi\varphi^{-1})_x, \qquad (\psi[1]\varphi)_y = \varphi^2(\psi\varphi^{-1})_y.$$

In other terms,

$$\psi[1] = \psi - \varphi\Omega(\varphi,\psi)/\Omega(\varphi,\varphi), \tag{5.36}$$

where Ω is the integral of the exact differential form

$$\mathrm{d}\Omega = \varphi\psi_x \mathrm{d}x + \psi\varphi_y \mathrm{d}y. \tag{5.37}$$

The transformed coefficient (potential in mathematical physics) is given by

$$u[1] = u - 2(\log\varphi)_{xy} = -u + \varphi_x\varphi_y/\varphi^2.$$

The proof is straightforward; see [298] for details.

The important feature of the Moutard transformation is general for the DTs: the transform is parameterized by a pair of solutions of the equation and the transform vanishes if the solutions coincide. The Moutard equation is obviously transformed to the two-dimensional Schrödinger equation and studied in connection with the central problems of classical differential geometry [197].

In the soliton theory the Moutard equation enters the Lax pairs for nonlinear equations such as the KP equation [35, 168, 298, 430] (see Chaps. 9, 10 for more details).

5.4 Iterations of Moutard transformations

Analysis of the iteration sequences for the transformations of the form (5.36), where, in accordance with (5.37),

$$\Omega(\varphi,\varphi) = \int \Delta_2(\varphi,\varphi)\mathrm{d}x_i + c_\phi = \phi^2/2 \tag{5.38}$$

by the appropriate choice of the constant c_ϕ, is performed similarly to the algorithm given in [324] for the classical DT. Suppose the result of N iterations is a linear combination of the integrals $\Omega(\varphi_i,\psi)$ of (5.37):

$$\psi[N] = \psi + \sum_i s_i\Omega(\varphi_i,\psi). \tag{5.39}$$

This formula is proved by induction. The main property of the Mourtard transformation can be written as

$$\varphi_k + \sum_i s_i \Omega(\varphi_i, \varphi_k) = 0 \tag{5.40}$$

and gives

$$s_i = \Delta_i / \Delta \tag{5.41}$$

by Kramer's rule. Denoting

$$\Omega_i \equiv \Omega(\varphi_i, \psi) \quad \text{and} \quad \Omega_{ik} \equiv \Omega(\varphi_i, \varphi_k),$$

we get $\Delta = \det[\Omega_{ik}]$, and Δ_i is obtained from Δ by the known rule of action with the ith row. Hence, the results of the iterations can be presented in the compact determinant form as in the classical Crum case [324].

Differentiating (5.39) yields

$$\psi_{xy}[N] = \psi_{xy} + (s_i\Omega_i)_{xy} = -u[N]\psi[N] \tag{5.42}$$

$$= -u\psi + (s_{ix}\Omega_i + s_i\Omega_{ix})_y = -u[N](\psi + s_i\Omega_i),$$

and using the definition of the determinant Δ together with the properties $\Omega_{ix} = \varphi_i\psi_x$, $s_{ix} = -s_i \ln_x \varphi_i$ gives the DT for the iterated potential

$$u[N] = u + 6(\ln \Delta)_{xx}, \tag{5.43}$$

that is used for multikink (see the next section) and multidromions [145, 146] construction.

5.5 Two-dimensional KdV equation

Applications of the Moutard transformations for solution of the KP and DS equations are well known [324]; for the Nizhnik–Veselov–Novikov equation see [278]. Here we follow [145] concerning the equation

$$m_{ty} = (m_{xxy} + m_y m_x)_x, \tag{5.44}$$

which is the 2+1 version [281] of the KdV-like Hirota–Satsuma equation [211]. Equation (5.44) was integrated by inverse spectral transform in [58, 65]. Details of multisoliton (multikink) construction and asymptotic behavior are given in the next section. We also use this example in Sect. 7.3 to show how the singular manifold method generates the Moutard transformation.

5.5.1 Moutard transformations

Here we consider the asymptotic behavior of iterated solutions and the simplest example of repeated iterations from the zero seed potential that demonstrates the interaction of kinks. The formula for the N-times iterated solution is

$$m = 6(\ln \Delta)_x, \tag{5.45}$$

where, again, $\Delta = \det[\Delta_{ik}]$ and, like [277], the one-step transform was performed,

$$\Delta_{ik} = \int d\Omega(\phi_k, \phi_i) + C_{ik}, \quad C_{ik} + C_{ki} = \phi_k(0)\phi_i(0),$$

$$\Omega(\phi_k, \phi_i) = -2 \int [\delta_1 \mathrm{d}x + \delta_2 \mathrm{d}y + \delta_3 \mathrm{d}t], \tag{5.46}$$

$$\delta_1 = \phi_k \phi_{ix}, \quad \delta_2 = \phi_{ky}\phi_i, \quad \delta_3 = \phi_k \phi_{it} - \phi_{kx}\phi_{ixx} + \phi_{kxx}\phi_{ix}.$$

This way we fix the constants of integration. A similar combination of solutions leads to multidromions [145], the localized solitons in two dimensions (first appeared in [62]).

5.5.2 Asymptotics of multikink solutions of two-dimensional KdV equation

To demonstrate the possibilities of the technique in 2+1 dimensions, we consider the example of kink interaction and choose the seed Lax pair solution as

$$\phi_k = A_k \exp(a_k x + a_k^3 t) + B_k \exp(b_k y). \tag{5.47}$$

Introducing the notations

$$\alpha_{ik} = \frac{a_i}{a_i + a_k}, \qquad \beta_{ik} = \frac{b_i}{b_i + b_k},$$

$$\xi_k = a_k x + a_k^3 t, \qquad \xi_{i0} = a_i x_0 + a_i^3 t_0, \qquad A_i/B_i = p_i,$$

we perform integration from x_0, y_0, t_0 to x, y, t and obtain

$$\Delta_{ik} = C_{ik} + \alpha_{ik} p_i p_k [\exp(\xi_i + \xi_k) - \exp(\xi_{i0} + \xi_{k0})] + \tag{5.48}$$

$$+ p_i \left[\exp(\xi_i + b_k y) - \exp(\xi_{i0} + b_k y_0)\right] + \beta_{ik} \left[\exp(b_i + b_k)y - \exp(b_i + b_k)y_0\right].$$

We would stop at kinks within the choice $a_i > 0$, $b_i > 0$ for $x_0, y_0, t_0 \to -\infty$; hence,

$$\Delta_{ik} = [\alpha_{ik} p_i p_k \exp(\chi_i + \chi_k) + p_i \exp(\chi_i) + \beta_{ik}] \exp[(b_i + b_k)y] + C_{ik}, \tag{5.49}$$

where $\chi_i = a_i x + a_i^3 t - b_i y$. Notice that it is impossible to represent Δ_{ik} as a sum of two exponents with the opposite powers like for the multisoliton determinant representation for the KP equation [324]. Hence, we should develop an asymptotic calculation technique.

Let us consider the case

$$0 < \mathrm{Re}(a_1^2) < \ldots < \mathrm{Re}(a_N^2)$$

and go to the reference frame of the sth kink that means fixing the phase χ_s. Running at the level y, we shall derive the asymptotic at $t \to \pm\infty$. We shall also put $C_{ik} = 0$ and $\Delta'_{ik} = \Delta_{ik} \exp(b_i + b_k) y$ and account for the relation $(\ln \Delta)'_x = (\ln \Delta)_x$. Finally, let us investigate

$$\Delta'_{ik} = \alpha_{ik} \exp(\chi'_i + \chi'_k) + \exp(\chi'_i) + \beta_{ik}, \tag{5.50}$$

where

$$x = -a_s^2 t + b_s y/a_s + \chi_s/a_s, \tag{5.51}$$

$$\chi'_k = a_k(a_k^2 - a_s^2)t + (a_k b_s/a_s - b_k)y + \chi_k/a_s + ln[p_k].$$

Therefore, at $t \to \infty$ and $\chi_s = \mathrm{const}$,

$$\chi_k = \begin{cases} -\infty, \; k < s \\ +\infty, \; k > s \end{cases}$$

and the elements of the determinant matrix have the following asymptotic values:

1. $\Delta_{ik} \to \beta_{ik}, \quad i, k < s$
2. $\Delta_{ik} \to \alpha_{ik} \exp(\chi'_i + \chi'_k), \quad i, k > s$
3. $\Delta_{ik} \to \exp \chi'_i, \quad i > s, \quad k < s$
4. $\Delta_{ik} \to \alpha_{ik} \exp(\chi'_i + \chi'_k) + \beta_{ik}, \quad i < s, \quad k > s$

It can be shown that only the first term contributes to the determinant asymptotic. We list below the special cases:

$i = s$
$k < s, \quad \Delta_{sk} = \exp(\chi_s) + \beta_{sk},$
$k = s, \quad \Delta_{ss} = [\alpha_{ss} \exp(\chi_s) + 1] \exp(\chi_s) + \beta_{sk},$
$k > s, \quad \Delta_{sk} = \alpha_{sk} \exp(\chi'_s + \chi'_k).$

$k = s$
$i < s, \quad \Delta_{is} = \beta_{is},$
$i > s, \quad \Delta_{is} = \alpha_{is}[\exp(\chi_s) + 1] \exp(\chi_i) \exp(\chi_i).$

It is convenient to present the explicit form of the determinant via the supermatrix

	$k<s$	$k=s$	$k>s$
$i<s$	β_{ik}	β_{ik}	$\alpha_{ik}\exp(\chi_i'+\chi_k')$
$i=s$	$\Delta_{sk}=\exp(\chi_s)+\beta_{sk}$	Δ_{ss}	$\alpha_{sk}\exp(\chi_s+\chi_k)$
$i>s$	$\exp(\chi_i)$	$\exp(\chi_i)[\alpha_{is}\exp(\chi_s)+1]$	$\alpha_{ik}\exp(\chi_i+\chi_k)$

In this asymptotic determinant it is possible to extract $\exp\chi$ from rows $i>s$ and from columns $k>s$, i.e.,

$$\Delta=\exp\left(\sum_{i=1}^{n}\chi_i-\chi_s\right)\Delta_1. \tag{5.52}$$

Then

$$\Delta_1=\begin{vmatrix}\beta_{ik} & \beta_{is} & 0\\ \exp(\chi_s)+\beta & \Delta_{ss} & \alpha_{sk}\chi_s\\ 1 & \alpha_{is}\exp(\chi_s)+1 & \alpha_{ik}\end{vmatrix},$$

where 0 and 1 are matrices with zero and unit elements. Obviously, it follows from (5.45) that

$$m=\left[\sum_{i=1}^{n}\chi_i-\chi_s\right]_x+(\ln\Delta_1)_x=\sum_{i=1}^{n}a_i-a_s+(\ln\Delta_1)_x. \tag{5.53}$$

A Lagrange expansion by the row of number s,

$$[0,\dots,0,\alpha_{ss}\exp(2\chi_s),0,\dots,0]+[1,\dots,1,\alpha_{s,s+1}\exp(\chi_{s+1}),1,\dots,1]+\dots$$
$$+(\beta_{s1},\dots,\beta_{ss},0,\dots,0)$$

allows us to present the result for the asymptotic Δ_1 in a "kink" form:

$$\begin{aligned}\Delta_1=\alpha_{ss}\exp(2\chi_s)&\begin{vmatrix}\beta_{ik} & \beta_{is} & 0\\ 0 & 1 & 0\\ 1 & \alpha_{is}\exp(\chi_s)+1 & \alpha_{ik}\end{vmatrix}\\ +\exp(\chi_s)\begin{vmatrix}\beta_{ik} & \beta_{is} & 0\\ 1 & 1 & \alpha_{sk}\\ 1 & \alpha_{is}\exp(\chi_s)+1 & \alpha_{ik}\end{vmatrix}&+\begin{vmatrix}\beta_{ik} & \beta_{is} & 0\\ \beta_{sk} & \beta_{ss} & 0\\ 1 & \alpha_{is}\exp(\chi_s)+1 & \alpha_{ik}\end{vmatrix}.\end{aligned} \tag{5.54}$$

The first determinant is arranged via a sum of the columns with the number s terms:

$$\begin{vmatrix}\beta_{ik}\\ 1\\ 1\end{vmatrix}+\begin{vmatrix}0\\ 0\\ \alpha_{is}\exp(\chi_s)\end{vmatrix}.$$

The second determinant is zero because it has a zero row. Finally,

$$\Delta^a=\exp(2\chi_s)(\alpha_{ss}\Delta_1+\Delta_2)+\exp[\chi_s](\Delta_3+\Delta_4)+\Delta_5, \tag{5.55}$$

where

$$\Delta_1 = \begin{vmatrix} \beta_{ik} & \beta_{is} & 0 \\ 0 & 1 & 0 \\ 1^0 & 1^0 & \alpha_{ik} \end{vmatrix}, \qquad \Delta_2 = \begin{vmatrix} \beta_{ik} & 0 & 0 \\ 1^0 & 1 & \alpha_{sk} \\ 1^0 & \alpha_{is} & \alpha_{ik} \end{vmatrix}, \tag{5.56}$$

$$\Delta_3 = \begin{vmatrix} \beta_{ik} & \beta_{is} & 0 \\ 1^0 & 1 & \alpha_{sk} \\ 1^0 & 1^0 & \alpha_{ik} \end{vmatrix}, \qquad \Delta_4 = \begin{vmatrix} \beta_{ik} & \beta_{is} & 0 \\ \beta_{sk} & \beta_{ss} & \alpha_{sk} \\ 1^0 & \alpha_{is} & \alpha_{ik} \end{vmatrix}, \tag{5.57}$$

$$\Delta_5 = \begin{vmatrix} \beta_{ik} & \beta_{is} & 0 \\ \beta_{sk} & \beta_{ss} & 0 \\ 1^0 & 1^0 & \alpha_{ik} \end{vmatrix}. \tag{5.58}$$

The determination of the phase of the sth kink is performed in the following way. If we introduce the phase χ and rewrite Δ^a as

$$\Delta^a = (\exp\chi + a)^2 + b, \tag{5.59}$$

then

$$m = (\ln \Delta^a)_x = \Delta^a_x/\Delta^a = 2[\exp\chi + a]\alpha/[(\exp\chi + a)^2 + b], \tag{5.60}$$

where $\alpha = \chi_x$. As a result,

$$m = 0, \qquad \chi \to \infty, \tag{5.61}$$

$$m = 2a\alpha/(a^2 + b), \qquad \chi \to -\infty. \tag{5.62}$$

Equating powers of exponential terms,

$$2\chi = 2\chi_k + \ln(\alpha_{ss}\Delta_1 + \Delta_2), \tag{5.63}$$

$$2a \exp\left[\frac{1}{2}\ln(\alpha_{ss}\Delta_1 + \Delta_2)\right] = \Delta_3 + \Delta_4, \tag{5.64}$$

$$a^2 + b = \Delta_s, \tag{5.65}$$

we immediately determine the phase χ and asymptotic value of the sth kink taking into account (5.53), (5.60), and (5.45).

Concluding, though this note is rather technical, it contains ideas about a development of asymptotic construction in the "dromionic" case of 2+1 equations, as well as symmetry reductions of explicit solutions or the two-step equation reduction. It follows from Sects. 5.1 and 5.2 that there exists a direct possibility to construct solutions of (5.10) or (5.7) via forms like (5.37). More general asymptotic behavior can be analyzed similarly. For example, equating the phases of (5.53) and (5.48) and linear combinations of ξ and η of the form (5.50) and (5.51) with $Y = \text{const}$ yields

$$a_i x + a_i^3 t - b_i y = A_i\xi + B_i\eta,$$

$$A_i = a_i c_2, \qquad B_i = a_i c_2 - a_i^3 T = Y b_i.$$

The three-phase solutions are possible with one determinant condition on the parameters a_i and b_i, and so on.

5.6 Generalized Moutard transformation for two-dimensional MKdV equations

In this section we generate solutions of the two-dimensional MKdV equations, giving one more example of efficient applications of the technique which exploits the generalized Moutard transformation.

5.6.1 Definition of generalized Moutard transformation and covariance statement

The Lax pair for the two-dimensional MKdV equations (5.23) has the form

$$\psi_{xy} = \frac{u_x}{u}\psi_y + u^2\psi, \tag{5.66}$$
$$\psi_t = \psi_{xxx} + \psi_{yyy} - 3\frac{u_y}{u}\psi_{yy} + \left[3\left(\frac{u_y}{u}\right)^2 - u^2 - B\right]\psi_y$$
$$+ \left(A - u^2\right)\psi_x + \frac{1}{2}\left(A - u^2\right)_x\psi.$$

Ganzha [169] studied one type of the Moutard transformation for the Goursat equation. To use this transformation for obtaining exact solutions of (5.23), we should complete the definition of the Moutard transformation. It is easy to do that. Let ϕ be the second solution of (5.66) (the support function). Then we have a closed 1-form

$$\mathrm{d}\theta = \mathrm{d}x\,\theta_1 + \mathrm{d}y\,\theta_2 + \mathrm{d}t\,\theta_3, \qquad \theta = \int \mathrm{d}\theta,$$

where

$$\theta_1 = \phi^2, \qquad \theta_2 = \left(\frac{\phi_y}{u}\right)^2, \qquad \theta_3 = (A - u^2)\phi^2 - \phi_y^2 - \phi_x^2 + 2\phi\phi_{xx} +$$
$$+ u^{-4}\left[(2\phi_{3y}\phi_y - \phi_{yy}^2 - B\phi_y^2)u^2 - 2u\phi_y(u_y\phi_y)_y + 3\left(u_y\phi_y\right)^2\right].$$

We define the *generalized* Moutard transformation in the following way:

$$u \to \tilde{u} = u - \sqrt{(\ln\theta)_x(\ln\theta)_y}, \qquad A \to \tilde{A} = A - (\partial_x\partial_y - 3\partial_x^2)\ln\theta,$$
$$B \to \tilde{B} = B + (\partial_x\partial_y - 3\partial_y^2)\ln\theta, \qquad \psi \to \tilde{\psi} = \frac{\phi Q}{\theta}, \tag{5.67}$$

where

$$Q \equiv \int \mathrm{d}Q, \qquad \mathrm{d}Q = dx\,Q_1 + \mathrm{d}y\,Q_2 + \mathrm{d}t\,Q_3,$$

and ($w = \psi/\phi$)

$$Q_1 = \theta w_x, \qquad Q_2 = -\frac{\theta^3(1/\theta)_{xy} w_y}{\theta_{xy}},$$
$$Q_3 = \theta w_{xxx} + c_1 w_{yyy} + c_2 w_{xx} + c_3 w_{yy} + c_4 w_x + c_5 w_y$$

with

$$c_1 = -\frac{\theta_{xy}}{2u^2} + \theta, \qquad c_2 = \frac{3}{2}\theta(\ln\theta_x)_x - \theta_x,$$
$$c_3 = \frac{u_y\theta_{xy}}{2u^3} + \frac{\phi\phi_{yy}}{u^2} - \frac{3u_y\theta}{u} + 3\left(\frac{\theta}{2}(\ln\theta_x)_y - \theta_y\right),$$
$$c_4 = \left(\frac{3\phi_{xx}}{\phi} + A - u^2\right)\theta - \frac{\theta_{xx}}{2},$$
$$c_5 = -\frac{3u_y^2\theta_{xy}}{2u^4} + +\frac{1}{u^3}\left(\theta_{xy}u_{yy} + u_y\phi\phi_{yy}\right) + \frac{1}{u^2}\left[3\theta u_y^2 - \phi\phi_{3y} + \frac{1}{2}\left(B - \frac{\phi_{yy}}{\phi}\right)\theta_{xy}\right] + \left(\frac{3\phi_{yy}}{\phi} - B\right)\theta + \frac{u_y}{u}\left(2\theta_y - \frac{3\theta\theta_{xy}}{\theta_x}\right) + \frac{\theta_{xy}}{2} - u^2\theta.$$

The 1-form dQ is closed,

$$Q_{1,y} = Q_{2,x}, \qquad Q_{1,t} = Q_{3,x}, \qquad Q_{2,t} = Q_{3,y}.$$

It is easy to verify that the (L, A) pair (5.66) is covariant with respect to the generalized Mourtard transformation (5.67).

5.6.2 Solutions of two-dimensional MKdV (BLMP1) equations

Now we use these transformations to construct exact solutions of the two-dimensional MKdV equations (5.24). Let us choose $u = \text{const}$ and $A = B = 0$. We will consider two examples.

1. If we take the solution of (5.66) as $\phi = \sinh\xi$, where

$$\xi = ax + \frac{u^2}{a}y + \frac{(u^2 - a^2)(u^4 - a^4)}{a^3}t \tag{5.68}$$

with real $a = \text{const}$, then using (5.67) we get a new solution of the two-dimensional MKdV equations,

$$\tilde{u} = \frac{u\left[2\eta - a^3\sinh(2\xi)\right]}{2\eta + a^3\sinh(2\xi)}, \quad \tilde{A} = \frac{16a^3\sinh\xi\left[3a^5\sinh\xi - (u^2 - 3a^2)\eta\cosh\xi\right]}{\left[2\eta + a^3\sinh(2\xi)\right]^2},$$

$$\tilde{B} = \frac{16av^2\cosh\xi\left[3a^3u^2\cosh\xi - (3u^2 - a^2)\eta\sinh\xi\right]}{\left[2\eta + a^3\sinh(2\xi)\right]^2},$$

where

$$\eta = a^2(u^2 y - a^2 x) + (u^2 - a^2)(3u^4 + 3a^4 + 2a^2u^2)t \; . \qquad (5.69)$$

2. To construct the algebraic solutions of (5.44), we choose the solutions of (5.66) as

$$\phi = (-1)^n \int_\alpha^\beta \mathrm{d}k \zeta(k) \exp[\xi(k)] \frac{\mathrm{d}^n}{\mathrm{d}k^n} \delta(k - k_0),$$

with $\xi(k)$ from (5.68), $a = a(k)$, and $\beta > k_0 > \alpha > 0$, where $\zeta(k)$ is an arbitrary differentiable function. For $n = 1$, $\zeta = 1$ we get

$$\tilde{u} = \frac{u(a^6 - 2\eta^2 - 2a^3\eta)}{2\eta^2 + 2a^3\eta + a^6}, \qquad \tilde{A} = -\frac{8a^6(u^2 + 3a^2)\eta(\eta + a^3)}{(2\eta^2 + 2a^3\eta + a^6)^2},$$
$$\tilde{B} = \frac{8u^2a^4(3u^2 + a^2)\eta(\eta + a^3)}{(2\eta^2 + 2a^3\eta + a^6)^2}, \qquad (5.70)$$

with η from (5.69) and $a = a(k_0)$. Equation (5.70) is a simple nonsingular algebraic solution of the two-dimensional MKdV equations.

There is a group of equations for which the dressing technique is directly applied. The BLMP2 equation is a generalization of the Nizhnik–Veselov–Novikov equation [58]. There is another new integrable equation that is usually called the Boiti–Leon–Pempinelli (BLP) equation. It was proposed and studied in [65]. An integrable generalization of the sine and sinh–Gordon equations in two spatial dimensions was proposed in [64].