2

Factorization and classical Darboux transformations

In this chapter we describe the algebraical factorization-based method to dress solutions of (1+1)-dimensional equations. We also show how the Darboux transformation (DT) theory appears in this framework.

First, in Sect. 2.1, we introduce the non-Abelian Bell polynomials and then generalize them in Sect. 2.2 to formulate in Sect. 2.3 a problem of factorization of a polynomial differential operator in the form of division by a monomial from the right and from the left. The relation between the factorization rules and the classical Darboux theorem [102] generalized in [314] is described in Sect. 2.4: the formalism produces a compact form of the DT for non-Abelian coefficients of linear operators, polynomial in a differentiation on a ring. Section 2.5 is devoted to a representation of the iterated DTs in terms of quasideterminants. As a highly nontrivial example of the iterated DT formalism, we describe positon solutions of the Korteweg–de Vries (KdV) equation discovered by Matveev [318, 319].

The growing interest in discrete models appeals to wider classes of symmetry structures of the corresponding nonlinear problems [149, 196, 255, 256, 339]. Very recently a suitable basis for new searches in the field of differential-difference and difference-difference equations was discovered [321] in the framework of the classical DT theory such that the difference operator is replaced by an arbitrary automorphism transformation. In Sect. 2.6 we present the dressing method via factorization for such a kind of generalizations. Like in the case of differential operators, this approach demonstrates links with the Hirota bilinearization method [260] and the factorization theory [271], with similar applications. We reformulate the Darboux covariance theorem from the paper of Matveev [321] and introduce a kind of difference Bell polynomials. These polynomials correspond naturally to the differential (generalized) Bell polynomials in their non-Abelian version of Sect. 2.2.

The joint covariance principle is formulated in Sect. 2.7 for Abelian and in Sect. 2.8 for non-Abelian differential rings. The same construction for a pair of difference equations is elaborated in Sect. 2.9. The form of the DT presented here allows us to develop a classification scheme with respect to the DTs in connection with the generalized Bell polynomials [187, 260, 467]. If a pair of such operators determines the Lax equations, the joint covariance with respect to the DTs produces a symmetry for the compatibility condition [314, 324]. In Sects. 2.10 and 2.11 we illustrate the possibilities of the method by examples of specific nonlinear equations: the non-Abelian Hirota system [210] having promising applications [149], and the Nahm equations [344]. We introduce a lattice Lax pair for the Nahm equations which is covariant with respect to combined Darboux-gauge transformations that generate the dressing structure for the equations. Finally, in Sect. 2.12 we illustrate the formalism developed, solving a particular case of the Nahm equations.

2.1 Basic notations and auxiliary results. Bell polynomials

Let K be a differential ring of the zero characteristics with unit e (i.e., unitary ring) and with an involution denoted by a superscript asterisk. The differentiation is denoted as D. The differentiation and the involution are agreed with operations in K:

1. $(a^*)^* = a, \quad (a+b)^* = a^* + b^*, \quad (ab)^* = b^* a^*, \quad a, b \in K.$
2. $D(a+b) = Da + Db, \quad D(ab) = (Da)b + aDb.$
3. $(Da)^* = -Da^*.$
4. Operators D^n with different n form a basis in a K-module $\mathrm{Diff}(K)$ of differential operators. The subring of constants is K_0 and a multiplicative group of elements of K is G.
5. For any $s \in K$ there exists an element $\varphi \in K$ such that $D\varphi = s\varphi$; this also means the existence of a solution of the equation

$$D\phi = -\phi s, \tag{2.1}$$

owing to the involution properties.

There are lots of applications of the rings of square matrices in the theory of integrable nonlinear equations, as well as in classical and quantum linear problems. In this case matrices are parameterized by a variable x and D can be a derivative with respect to this variable or a combination of partial derivatives that satisfies conditions 1 and 2. If D is the standard differentiation, then the involution (asterisk) may be the Hermitian conjugation. In the case of a commutator, the operator D acts as $Da = [d, a]$ and $(Da)^* = -[d^*, a]$. Having in mind this or similar applications, we shall refer to the involution as conjugation. We do not restrict ourselves to the matrix-valued case; an appropriate operator ring is also suitable for our theory.

Below we introduce left and right non-Abelian Bell polynomials (see also [388]) and formulate the statements for them. The differential Bell polynomials are defined in Definition 2.1:

Definition 2.1. *The left and right non-Abelian Bell polynomials $B_n(s)$ are defined by the recurrence relations*

$$B_n(s) = DB_{n-1}(s) + B_{n-1}(s)s, \qquad n = 1, 2, \dots \tag{2.2}$$

for left Bell polynomials and

$$B_n^+(s) = -DB_{n-1}^+(s) + s, \qquad n = 1, 2, \dots \tag{2.3}$$

for right Bell polynomials with the "initial condition"

$$B_0(s) = e. \tag{2.4}$$

Proposition 2.2. *If an element $\varphi \in G$ satisfies the equation $D\varphi = s\varphi$, then*

$$D^n\varphi = B_n(s)\varphi, \quad n = 0, 1, 2, \dots .$$

Proposition 2.3. *If an element $\phi \in G$ satisfies (2.1), then*

$$D^n\phi = (-1)^n\phi B_n^+(s), \quad n = 0, 1, 2, \dots .$$

Proposition 2.4. *The left and right Bell polynomials are connected by the following relations:*

$$B_n(s)^* = B_n^+(s^*), \qquad B_n^+(s)^* = B_n(s^*).$$

If the ring is Abelian, left and right polynomials coincide.

Remark 2.5. Proposition 2.4 means that a duality takes place for the Bell polynomials: any relation for right polynomials can be transformed to the corresponding relation for left ones, and vice versa.

Let us denote

$$L_s = D - s. \tag{2.5}$$

Note that the recursion (2.3) may be written by means of L_s (2.5) as

$$B_{n+1}^+(s) = -L_sB_n^+(s), \quad n = 0, 1, 2, \dots ,$$

with the simple corollary

$$B_n^+(s) = (-1)^nL_s^ne, \quad n = 0, 1, 2, \dots .$$

2.2 Generalized Bell polynomials

In the next section a problem of division of an arbitrary operator L by the operator L_s will be studied. To this aim, for the right division we introduce here auxiliary operators H_n by means of Definition 2.6:

Definition 2.6. *The operators H_n are defined by the recurrence relation*

$$H_n = DH_{n-1} + B_n(s), \quad n = 1, 2, \ldots, \quad H_0 = e. \tag{2.6}$$

Proposition 2.7. *The following identity holds:*

$$D^n = H_{n-1}L_s + B_n(s), \quad n = 1, 2, \ldots .$$

Coefficients of the operators H_n are expressed via the generalized Bell polynomials that are defined in Definition 2.8:

Definition 2.8. *Generalized Bell polynomials are defined by the "initial conditions"*

$$B_{n,0}(s) = e, \quad n = 0, 1, 2, \ldots$$

and by the recurrence relations

$$B_{n,k}(s) = B_{n-1,k}(s) + DB_{n-1,k-1}(s), \quad k = 1, 2, \ldots n-1, \quad n = 2, 3, \ldots, \tag{2.7}$$

$$B_{n,n}(s) = DB_{n-1,n-1}(s) + B_n(s), \quad n = 1, 2, \ldots . \tag{2.8}$$

Proposition 2.7 is proved by acting with D from the left to (2.7) $n+1$ times and substituting (2.2) and (2.6) into the resulting equation because

$$D^{n+1} = H_nL_s + B_{n+1} = DH_{n-1}L_s + DB_n$$

$$= (DH_{n-1})L_s + H_{n-1}DL_s + B_n' + B_nD.$$

Proposition 2.9. *Generalized Bell polynomials are coefficients in the decomposition of the operators H_n, i.e.,*

$$H_n = \sum_{k=0}^{n} B_{n,n-k}(s)D^k, \quad n = 0, 1, 2, \ldots . \tag{2.9}$$

Since the recurrence relation (2.6) defines the operators H_n uniquely, (2.9) easily follows. Equations (2.7) and (2.8) are simple but not useful for evaluation of $B_{n,k}(s)$; therefore, we suggest a practically easier algorithm. For this reason we put (2.9) into (2.7). The following formulas are extracted:

$$B_{n,n-k+1}(s) = \sum_{i=k}^{n} \binom{i}{k} B_{n,n-i}(s)D^{i-k}s, \quad k = 1, 2, \ldots, n, \quad n = 0, 1, 2, \ldots \tag{2.10}$$

and

$$B_{n+1}(s) = \sum_{i=0}^{n} B_{n,n-i}(s)D^i s, \quad n = 0, 1, 2, \ldots . \tag{2.11}$$

Equation (2.11) expresses the standard (non-Abelian) Bell polynomials via the generalized ones:

$$B_{n+1}(s) = \sum_{i=0}^{n} B_{n,i}(s) D^{n-i} s, \quad n = 0, 1, 2, \dots .$$

Rearranging the summation in (2.10) as $k \to n - k + 1$ yields after simple calculation

$$B_{n,k}(s) = \sum_{i=0}^{k-1} \binom{n-i}{n-k+1} B_{n,i}(s) D^{k-i-1} s, \quad k = 1, \dots, n, \quad n = 0, 1, \dots . \tag{2.12}$$

Evaluation of the generalized Bell polynomials by (2.10) gives ($s' = Ds$)

$$B_{n,1}(s) = s, \; B_{n,2}(s) = s^2 + nDs, \; B_{n,3}(s) = s^3 + ns's + (n-1)sDs + \binom{n}{2} D^2 s,$$

$$B_{n,4}(s) = s^4 + ns's^2 + (n-1)ss's + (n-2)s^2 Ds$$
$$+ \binom{n}{2} s''s + n(n-2)(Ds)^2 + \binom{n-1}{2} sD^2 s + \binom{n}{3} D^3 s.$$

To solve the problem of the left division of L by L_s, a similar but somewhat simpler consideration is needed. The analog of Proposition 2.9 is as follows:

Proposition 2.10. *The following identity is valid:*

$$D^n = L_s H^+_{n-1} + B^+_n(s), \quad n = 1, 2, \dots , \tag{2.13}$$

where

$$H^+_n = \sum_{k=0}^{n} B^+_{n-k}(s) D^k, \quad n = 0, 1, 2, \dots . \tag{2.14}$$

2.3 Division and factorization of differential operators. Generalized Miura equations

Let

$$L = \sum_{n=0}^{N} a_n D^n, \quad a_n \in K \tag{2.15}$$

be a differential operator of order N. We shall study the right and left divisions of L by the operator L_s defined by (2.5). Suppose

$$L = ML_s + r, \qquad L = L_s M^+ + r^+, \tag{2.16}$$

where M and M^+ are the results of right and left divisions, r and r^+ being the remainders. Propositions 2.9 and 2.10 allow us to solve the problem of division in a simple way.

Proposition 2.11. *If the representation (2.16) is valid, then the remainder r and the result of division M are written as*

$$r = \sum_{n=0}^{N} a_n B_n(s),$$

$$M = \sum_{n=1}^{N} a_n H_{n-1} = \sum_{n=0}^{N-1} b_n D^n, \tag{2.17}$$

where

$$b_n = \sum_{k=n+1}^{N} a_k B_{k-1,k-n-1}(s), \quad n = 0, 1, \ldots, N-1. \tag{2.18}$$

For the proof it is enough to check

$$L = a_0 + \sum_{n=1}^{N} a_n [H_{n-1} L_s + B_n(s)] = \sum_{n=1}^{N} a_n H_{n-1} L_s + a_0 + \sum_{n=1}^{N} a_n B_n(s)$$

by the equality from Proposition 2.7 and to account for H_{n-1} given by (2.9). As a corollary we get the following:

Proposition 2.12. *For the linear operator L to be right-divisible by L_s without remainder, it is necessary and sufficient that s be a solution of the differential equation*

$$\sum_{n=0}^{N} a_n B_n(s) = 0. \tag{2.19}$$

If this condition holds, the operator L factorizes as $L = ML_s$, where M is given by (2.17) and (2.18).

Equation (2.19) is nonlinear. For $N = 2$ it is the Riccati-type equation known in the theory of the KdV equation as the Miura map. Therefore, it is natural to term it as a generalized right Miura equation . It links the function s and coefficients of the operator L. The left Miura equation is generalized by means of Proposition 2.2, giving the following theorem:

Theorem 2.13. *Let an invertible function φ be a solution to the linear differential equation*

$$\sum_{n=0}^{N} a_n D^n \varphi = 0. \tag{2.20}$$

Then the operator L, defined by (2.15), is right-divisible by L_s, where $s = \varphi' \varphi^{-1}$ and $\varphi' \equiv D\varphi$. Moreover, s is a solution of the right Miura equation (2.19).

To solve the left division problem, let us write the result of division in the form

$$M^+ = \sum_{n=0}^{N-1} b_n^+ D^n. \tag{2.21}$$

Now we should determine $b_n^+, n = 0, 1, \ldots, n-1$. To this aim we substitute (2.21) into the right-hand side of the second equation of (2.16). Following the lines of Proposition 2.11, we obtain

$$b_{N-1}^+ = a_N, \tag{2.22}$$

$$b_n^+ = a_{n+1} - L_s b_{n+1}^+, \quad n = 0, 1, \ldots, N-2, \tag{2.23}$$

and

$$r^+ = a_0 - L_s b_0^-. \tag{2.24}$$

Solving subsequently equations (2.23) and (2.24), we arrive at

$$b_n^+ = \sum_{k=n+1}^{N} (-1)^{k-n-1} L_s^{k-n-1} a_k, \quad n = 0, 1, \ldots, N-1 \tag{2.25}$$

and

$$r^+ = \sum_{k=0}^{N} (-1)^k L_s^k a_k. \tag{2.26}$$

The entities b_n^+, $n = 0, 1, \ldots, N-1$, and r^+ can be expressed in terms of the right Bell polynomials if we use (2.5) and take into account

$$L_s^k a = L_s^k e a = (-1)^k B_k^+(s) a.$$

Hence, (2.25) and (2.26) transform to

$$b_n^+ = \sum_{k=n+1}^{N} B_{k-n-1}^+(s) a_k, \quad n = 0, 1, \ldots, N-1 \tag{2.27}$$

and

$$r^+ = \sum_{k=0}^{N} B_k^+(s) a_k. \tag{2.28}$$

Formulas (2.16), (2.21), (2.25), and (2.26) give a solution of the left division problem of division of L by L_s. So, the following is proved:

Theorem 2.14. *For the operator L to be left-divisible by the operator L_s (without remainder), it is necessary and sufficient that s be a solution of the differential equation*

$$\sum_{k=0}^{N} B_k(s)^+ a_k = 0. \tag{2.29}$$

If this condition holds, the operator L factorizes as $L = L_s M^+$, where M^+ is given by (2.21) and (2.25) [or (2.27)]. For the reminder r^+ and the result of division M^+ equations (2.28) [or (2.26)] and (2.21) exist.

The nonlinear equation (2.29) is called the generalized left Miura equation, which is obviously linearized by Proposition 2.4. As a result, we have the following:

Proposition 2.15. *Let an invertible element φ satisfy the linear differential equation*

$$\sum_{n=0}^{N}(-1)^n B_n(s)^+ a_n D^n \varphi = 0.$$

Then the operator L determined by (2.15) is left-divisible by the operator L_s, where $s = -\varphi^{-1}\varphi'$. The function s is a solution to the generalized left Miura equation (2.29).

2.4 Darboux transformation. Generalized Burgers equations

The problem of the operator division is directly connected to the DT. To clarify this point, suppose that in the ring K there exists one more differentiation D_0 which commutes with the operator D. It may be a differentiation in a parameter t.

Let us introduce an auxiliary commutation relation

$$L_s r = r L_s + r' + [r, s]. \tag{2.30}$$

Indeed,

$$L_s r - r L_s = (D - s)r - r(D - s) = Dr - sr - rD + rs$$

$$= rD + Dr - sr - rD + rs = r' + [r, s].$$

Taking into account the equalities (2.30) and (2.16), we arrive at the relation

$$L_s(D_0 - L) = (D_0 - \tilde{L})L_s + D_0 s - r' - [r, s], \tag{2.31}$$

where

$$\tilde{L} = L_s M + r. \tag{2.32}$$

As the result, the following important conclusion can be drawn:

Proposition 2.16. *If a function s satisfies the equation*

$$D_0 s = r' + [r, s], \tag{2.33}$$

the operator L_s intertwines the operators $D_0 - L$ and $D_0 - \tilde{L}$,

$$L_s(D_0 - L) = (D_0 - \tilde{L})L_s. \tag{2.34}$$

The explicit expression for $\tilde{L}$ can be obtained in terms of (2.32) and (2.16) and has the form

$$\tilde{L} = a_0 + \sum_{n=1}^{N} (a'_n H_{n-1} + a_n H_n - s a_n H_{n-1}). \tag{2.35}$$

Let us write (2.33) explicitly using (2.16). It is established that for the intertwining relation (2.34) to be valid, it is necessary and sufficient that s be a solution of the equation

$$D_0 s = \sum_{n=0}^{N} [a'_n B_n(s) + a_n B_{n+1}(s) - s a_n B_n(s)]. \tag{2.36}$$

Remark 2.17. Equation (2.36) is nonlinear but linearizable. This equation (in a different form) was introduced in [388]. The form we suggest here is the most compact and convenient for further investigations, e.g., in the framework of the bilineraization technique of Hirota [210].

In the case of scalar functions and $L = D^2$ equation (2.36) is known as the Burgers equation. For this reason and owing to the integrability of (2.36) by the Cole–Hopf transformation, it is natural to refer to (2.36) as a generalized Burgers equation.

Proposition 2.18. *Suppose an invertible function φ is a solution to the linear differential equation*

$$D_0 \varphi = L \varphi.$$

Then the function s satisfies the generalized Burgers equation (2.36).

The obvious corollary of the intertwining relation (2.34) and Proposition 2.18 is as follows:

Theorem 2.19. *Let functions ψ and φ be solutions of the equations*

$$D_0 \psi = L \psi, \qquad D_0 \varphi = L \varphi \tag{2.37}$$

for an invertible function φ. Then the function

$$\tilde{\psi} = L_s \psi = D\psi - s\psi, \qquad s = (D\varphi)\varphi^{-1} \tag{2.38}$$

is a solution of the equation

$$D_0 \tilde{\psi} = \tilde{L} \tilde{\psi}. \tag{2.39}$$

The last statement accomplishes the proof of the Matveev theorem for differential polynomials [314] in its non-Abelian version.

The equality (2.35) gives a representation of the transformed operator in terms of the generalized Bell polynomials. The explicit expressions for the transformed coefficients are

$$a_N[1] = a_N, \tag{2.40}$$

$$a_k[1] = a_k + \sum_{n=k+1}^{N} [a_n B_{n,n-k} + (a_n' - s a_n) B_{n-1,n-1-k}], \tag{2.41}$$
$$k = 0, \dots, N-1.$$

2.5 Iterations and quasideterminants via Darboux transformation

Here we would like to revisit the non-Abelian iterated DT formulas following the ideas of the pioneering paper of Matveev [313], where the basic formulas were derived. Their Abelian counterpart is demonstrated in [324] and discussed also in [316, 322]. In fact, this approach goes back to the famous paper of Crum [94]. We will see, in the framework of a general non-Abelian DT theory, that the dressing procedure naturally produces the quasideterminants (Sect. 1.9). In the paper [191] this procedure is also properly analyzed for the matrix Schrödinger operator.

2.5.1 General statements

Let R be a differential algebra with a derivation $D : R \to R$ and $\phi \in R$ be an invertible element. Recall that we denote $D(g) = g'$ and $D^k(g) = g^{(k)}$. In particular, $D^{(0)}(g) = g$.

For $\psi \in R$ define $\mathcal{D}(\phi; \psi) = \psi' - \phi'\phi^{-1}\psi$. Following [321], we call $\mathcal{D}(\phi; \psi)$ the *DT* of ψ defined by ϕ.

Theorem 2.20. *Let $\phi_1, \dots, \phi_N \in R$. Define by induction the iterated DT $\mathcal{D}(\phi_N, \dots \phi_1; \psi)$ as follows. For $N = 1$, it coincides with the DT defined above. Assume $N > 1$. The expression $\mathcal{D}(\phi_N, \dots, \phi_1; \psi)$ is defined if $\mathcal{D}(\phi_N, \dots, \phi_2; \psi)$ is defined and invertible and $\mathcal{D}(\phi_N; \psi)$ is defined. In this case,*

$$\mathcal{D}(\phi_N, \dots \phi_1; \psi) = \mathcal{D}[\mathcal{D}(\phi_k, \dots \phi_2; \psi); \mathcal{D}(\phi_1; \psi)].$$

Theorem 2.21. *If all square submatrices of matrix $(\phi_i^{(j)})$, $i = 1, \dots, N$, $j = k-1, \dots, 0$ are invertible, then the Vandermond supermatrix defines the quasideterminant:*

$$\widehat{\mathcal{D}}(\phi_N, \dots, \phi_1; \psi) = \begin{vmatrix} \psi^{(k)} & \phi_1^{(k)} & \dots & \phi_k^{(k)} \\ \dots & \dots & \dots & \dots \\ \psi & \phi_1 & \dots & \phi_k \end{vmatrix}.$$

Recall that we use the "hat" symbol to denote the quasideterminant. This general statement first appeared in [313].

Proof. Iterations of the DT yield

$$\psi[N] = \psi^{(N)} + \sum_{m=0}^{N-1} s_m \psi^{(m)}, \tag{2.42}$$

which is a replica of an expression in [313, 322, 324] for the Abelian case. The iterated DT (2.42) as a function of ψ sends to zero any ϕ_i on which the transformation is constructed:

$$\phi_p[N] = \phi_p^{(N)} + \sum_{m=0}^{N-1} s_m \phi_p^{(m)} = 0, \quad p = 1, \ldots, N. \tag{2.43}$$

The successive excluding of s_m as a function of the derivatives $\psi_p^{(m)}$ from the system (2.43) yields the algorithm that results in the evaluation of s_m; the procedure was pointed out already in [314], applied in [277] and, as it is seen from a comparison with the quasideterminant definition, could define the Vandermond quasideterminant.

Let us illustrate the scheme with the case of $N = 2$. The first iteration is based on a set of ϕ_p, $p = 1, 2$. The equations for s_i, $i = 0, 1$

$$\phi_1^{(2)} + s_0\phi_1 + s_1\phi_1' = 0, \quad \phi_2^{(2)} + s_0\phi_2 + s_1\phi_2' = 0 \tag{2.44}$$

yield

$$s_0 = -\phi_1^{(2)}\phi_1^{-1} - s_1\phi_1'\phi_1^{-1}.$$

Inserting this into the second relation of (2.44) produces the equation for s_1:

$$s_1(\phi_2' - \phi_1'\phi_1^{-1}\phi_2) = -\phi_2^{(2)} + \phi_1^{(2)}\phi_1^{-1}.$$

It is solved as

$$s_1 = (-\phi_2^{(2)} + \phi_1^{(2)}\phi_1^{-1})(\phi_2' - \phi_1'\phi_1^{-1}\phi_2)^{-1},$$

and

$$s_0 = -\phi_1^{(2)}\phi_1^{-1} - (-\phi_2^{(2)} + \phi_1^{(2)}\phi_1^{-1})(\phi_2' - \phi_1'\phi_1^{-1}\phi_2)^{-1}\phi_1'\phi_1^{-1},$$

both recognized as quasideterminants. The final expression for $\psi[2]$ is given by (2.42).

As mentioned in [322], the comparison of the resulting formula for $\psi[N]$ (2.42) and the formula for the DT

$$\mathcal{D}(\phi_{N-1}, \ldots, \phi_1; \psi[N-1]) = \psi[N-1]' - \phi[N-1]'\phi[N-1]^{-1}\psi[N-1], \tag{2.45}$$

where $\phi[N-1] = \psi[N-1]|_{\psi=\phi_N}$, yields the non-Abelian Jacobi identity for quasi-Wronskians "for free." Recall that Crum [94] used the Jacobi identity to prove the determinant formulas for the iterated DT for Abelian entries.

Remark 2.22. The non-Abelian algorithm to exclude s_ms hints at the definition of quasideterminants as a function of submatrices a_{pm} (compare with the results in Sect. 1.9.1). Namely, it is enough to change $\psi_p^{(m)} \to a_{pm}$.

The solution of the system (2.43) with respect to s_p may be reinterpreted as the Vandermond-quasideterminant representation of the iterated DT for solutions (2.42) (with inserted s_m) and, next, linked to the DT for the potentials $a_k[N]$.

To check this proposition, let us substitute (2.42) into the evolution equation (2.37) for $\psi[N]$ (see Theorem 2.19):

$$\psi[N]_t = \sum_{k=0}^{n} a[N]_k \psi^{(N+k)} + \sum_{k=0}^{n} a[N]_k \sum_{m=0}^{N-1} (s_m \psi^{(m)})^{(k)}. \tag{2.46}$$

On the other hand,

$$\psi[N]_t = \psi_t^{(N)} + \sum_{m=0}^{N-1} (s_{m,t} \psi^{(m)} + s_m \psi_t^{(m)}) \tag{2.47}$$

$$= \sum_{k=0}^{n} (a_k \psi^{(k)})^{(N)} + \sum_{m=0}^{N-1} s_{m,t} \psi^{(m)} + \sum_{m=0}^{N-1} s_m \sum_{k=0}^{n} (a_k \psi^{(k)})^{(m)}.$$

Equating terms with the highest derivative $\psi^{(N+n)}$ gives

$$a[N]_n = a_n,$$

and, subsequently, for $\psi^{(N+n-1)}$ produces

$$a[N]_{n-1} = a_{n-1} + N a_n' + s_{N-1} a_n - a_n s_{N-1}.$$

For $\psi^{(N+n-2)}$ we obtain

$$a[N]_{n-2} = a_{n-2} + N a_{n-1}' + \frac{N(N+1)}{2} a_n'' - a_n (s_{N-2} + n s_{N-1}') \tag{2.48}$$

$$-a[N]_{n-1} s_{N-1} + (N-1) s_{N-1} a_n' + s_{N-1 a_{n-1}} + s_{N-2} a_n,$$

preserving the order of differentiation in (2.47) to keep the non-Abelian character. Substituting here $a[N]_{n-1}$ yields the explicit form of $a[N]_{n-2}$:

$$a[N]_{n-2} = a_{n-2} + N a_{n-1}' + \frac{N(N+1)}{2} a_n'' + [s_{N-1}, a_{n-1}] + [s_{N-2}, a_n]$$

$$-n a_n s_{N-1}' - (N a_n' + [s_{N-1}, a_n]) s_{N-1}. \tag{2.49}$$

One could compare the resulting expression (2.49) for commuting entries and for $a_n' = a_{n-1}' = 0$,

$$a[N]_{n-2} = a_{n-2} - n a_n s_{N-1}', \tag{2.50}$$

with (2.37) [322] when taking into account that $s_{N-1} = -\ln_{xx} W(\varphi_1, \ldots, \varphi_N)$, with φ_j being solutions of (2.38).

Corollary 2.23. [321] *In the commutative case, the iterated Darboux transformation is a ratio of two Wronskians, as follows by direct application of the Kramer rule to (2.43):*

$$\mathcal{D}(\phi_k,\dots,\phi_1;\psi)=\frac{W(\phi_1,\dots\phi_k,\psi)}{W(\phi_1,\dots,\phi_k)}. \tag{2.51}$$

All these results are naturally generalized to the cases when the main properties of the DTs are valid: the n-iterated transform is a linear function of $T^n\psi$ and there is an n-dimensional kernel of the transformation operator. This remark relates first of all to the next section (see also [321]) and to the Moutard/Goursat transformations (Chap. 6). The algorithm of the construction given here is easily transferred to the iterated Moutard/Goursat transformation because the "kernel property" (2.43) is also valid.

2.5.2 Positons

An interesting illustration of the application of (2.51) is concerned with *positons*. Positons were introduced by Matveev [318, 319] as a class of singular solutions of the KdV equation,

$$u_t-6uu_x+u_{xxx}=0, \tag{2.52}$$

that lead to a trivial scattering matrix for the associated spectral problem

$$-\psi_{xx}+u\psi=\lambda\psi. \tag{2.53}$$

Here we consider this topic in more detail, following [323].

The KdV equation can be written as the compatibility condition [13] of the linear system of equations comprising the spectral problem (2.53) and the evolutionary equation

$$\psi_t=-4\psi_{xxx}+6u\psi_x+3u_x\psi. \tag{2.54}$$

Note that the spectral problem (2.53) is a representative of the general equation (2.37).

Let $\phi(\lambda)$ solve the spectral equation (2.53). Differentiation in λ produces (in general, linearly independent) solutions $\phi^{[m]}=\partial^m\phi(\lambda)/\partial\lambda^m$ of the same equation. The set of solutions $\phi_1,\dots,\phi_1^{[m_1]},\phi_2,\dots,\phi_2^{[m_2]},\dots,\phi_n,\dots,\phi_n^{[m_n]}$, m_i are integers, generated by the λ-derivatives in the points λ_1, $\lambda_2,\dots,\lambda_n$, yields the iterated DT, which is the Abelian specification of the transform (2.42) and the quasideterminant formula (2.43).

Proposition 2.24. *Let $u(x,t)$ be a solution of the KdV equation (2.52). The Wronskians*

$$W_1=W(\phi_1,\dots,\phi_1^{[m_1]},\phi_2,\dots,\phi_2^{[m_2]},\dots,\phi_n,\dots,\phi_n^{[m_n]})$$

and

$$W_2 = W(\phi_1, \dots, \phi_1^{[m_1]}, \phi_2, \dots, \phi_2^{[m_2]}, \dots, \phi_n, \dots, \phi_n^{[m_n]}, \psi)$$

produce the DT

$$u[N] = u - 2\partial_x^2 \ln W_1, \qquad \psi[N] = W_2/W_1. \tag{2.55}$$

The Lax pair equations (2.53) and (2.54) are covariant with respect to the DT (2.55). In other words, the function $\psi[N]$ satisfies (2.53) and (2.54) with $u \to u[N]$ and $\psi \to \psi[N]$ and $u[N]$ is a new solution of the KdV equation.

Now we apply the DT (2.55) to dress the simplest seed solution $u = 0$ with the choice $\psi = \exp(\mathrm{i}kx + 4\mathrm{i}k^3 t)$, $k^2 = \lambda$. As a solution of the spectral equation with zero potential we choose an oscillating function

$$\phi = \sin \kappa(x + x_1(\kappa) + 4\kappa^2 t) \equiv \sin\theta.$$

Here κ is a real parameter and $x_1(\kappa)$ is an analytic function in the vicinity of the point κ. Therefore, a new solution to the KdV equation is given by

$$u[1] = -2\partial_x^2 \ln W(\phi, \partial_\kappa \phi) \tag{2.56}$$

and is written explicitly as

$$u = 32\kappa^2 \frac{\sin\theta - \kappa\gamma\cos\theta}{(\sin 2\theta - 2\kappa\gamma)^2} \sin\theta, \tag{2.57}$$

where

$$\gamma = \partial_\kappa \theta = x + x_2 + 12\kappa^2 t, \quad x_2 = x_1 + \kappa \partial_\kappa x_1, \quad W(\phi, \partial_\kappa\phi) = \sin 2\theta - 2\kappa\gamma. \tag{2.58}$$

The solution (2.57) is determined by three real parameters x_1, x_2, and κ and has a second-order pole in x. The precise pole position is found by solution of the nonlinear functional equation $W(\phi, \partial_\kappa\phi) = 0$. The corresponding solution of the Lax pair [or of (2.51)] takes the form

$$\psi(x,k) = \frac{W[\phi, \partial_\kappa\phi, \exp(\mathrm{i}kx + 4\mathrm{i}k^3 t)]}{W(\phi, \partial_\kappa\phi)} \tag{2.59}$$

$$= \left(-k^2 + \frac{4\mathrm{i}kx\sin^2\theta}{\sin 2\theta - 2\kappa\gamma} - \kappa^2 \frac{\sin 2\theta + 2\kappa\gamma}{\sin 2\theta - 2\kappa\gamma}\right) \mathrm{e}^{\mathrm{i}kx + 4\mathrm{i}k^3 t}.$$

In the point $k = \kappa$, this solution is simplified:

$$\psi(k,x) = -4\kappa^2 \frac{\sin\theta}{\sin 2\theta - 2\kappa\gamma}. \tag{2.60}$$

We see from (2.60) that the function ψ is localized near its pole but is not square-integrable on the whole x-axis. The point κ^2 is called the Wigner–von Neuman resonance [320].[1] Because $\kappa^2 > 0$, the solution (2.57) is called *positon*, as distinct from the soliton solution for which $\kappa^2 < 0$ (Sect. 8.7).

[1] Generic aspects of the scattering theory of the potentials leading to the Wigner–von Neumann resonances are discussed in [312, 311].

Asymptotic behavior of the function ψ is given by

$$\psi \to (-k^2 + \kappa^2)\mathrm{e}^{\mathrm{i}kx+4\mathrm{i}k^3 t}[1 + o(1)], \qquad x \to \pm\infty. \tag{2.61}$$

Let us compare (2.61) with the standard Jost solution $J(x, k, t)$ asymptotic of the linear Schrödinger equation with a decreasing potential:

$$\begin{aligned} &J(x, k, t) \to \mathrm{e}^{\mathrm{i}kx}(1 + o(1)), \quad x \to +\infty, \\ &J(x, k, t) \to a(k, t)\mathrm{e}^{\mathrm{i}kx} + b(k, t)\mathrm{e}^{-\mathrm{i}kx}, \quad x \to -\infty, \end{aligned}$$

where $a(k, t)$ and $b(k, t)$ are the transmission and reflection coefficients. For the positon potential we obtain

$$a(k, t) = 1, \qquad b(k, t) = 0.$$

Potentials for which $b(k, t) = 0$ are called reflectionless. The well-known example of the reflectionless potentials is provided by solitons. However, for solitons we have $a(k, t) \neq 1$. Hence, positons give a unique example of supertransparent (or superreflectionless) long-range potentials.

A two-positon solution is generated by the evident extension of (2.56),

$$u = -2\partial_x^2 \ln W(\phi_1, \partial_{\kappa_1}\phi_1, \phi_2, \partial_{\kappa_2}\phi_2),$$

with

$$\phi_1 = \sin \kappa_1(x + x_1 + 4\kappa_1^2 t), \qquad \phi_2 = \sin(x + x_2 + 4\kappa_2^2 t),$$

and is determined by six real parameters. For $x \to \infty$ the two-positon solution is decomposed into a sum of two free positons. It should be stressed that the positon scattering is not accompanied with a phase shift typical for the soliton scattering.[2]

Interesting suggestions concerning physical applications of positons can by found in the paper by Matveev [323].

2.6 Darboux transformations at associative ring with automorphism

In this section we reformulate and analyze the results from the paper of Matveev [321] for further use in the derivation of chain equations and joint covariance of operator pairs [265, 267, 271]. We begin with general notations. Let $\mathbf{R}$ be an associative ring with an automorphism, implying that there exists

[2] For singular potentials the scattering data are not uniquely defined. Different self-adjoint extensions of the same differential operators might lead to different scattering operators. The definition of the scattering coefficients given above is in agreement with the nonlinear picture of interaction between positons and solitons, although the latter can be analyzed independently of this definition.

a linear invertible map T, $\mathbf{R}\to\mathbf{R}$ such that for any $\psi(x,t)$ and $\varphi(x,t)\in\mathbf{R}$, $x\in R^n$, $t\in R$ we have

$$T(\psi\varphi)=T(\psi)T(\varphi),\qquad T(1)=1. \tag{2.62}$$

The automorphism with the defining property (2.62) allows us to write down a wide class of functional-differential-difference and difference-difference equations starting from

$$\psi_t(x,t)=\sum_{m=-M}^{N} U_m T^m\psi, \tag{2.63}$$

where M and N are integers. For example, the operator T can be chosen as

$$T\psi(x,t)=\psi(qx+\delta,t),$$

where $q\in GL(n,\mathbb{C})$, $\delta\in R^n$. Another choice gives

$$T\psi(x)=W\psi(x)W^{-1}\ ,\quad W\in GL(n,\mathbb{C}).$$

We will save the notations and conditions of the paper [321] discussing other potentials until the end of Sect. 4.9.

Let us consider two DTs for solutions of (2.63),

$$D^{\pm}f=f-\sigma^{\pm}T^{\pm1}f,\qquad \sigma^{\pm}=\varphi\left(T^{\pm1}\varphi\right)^{-1}, \tag{2.64}$$

where φ is a particular solution of the same equation (2.63). For the case of a differential ring and for $Tf(x,t)=f(x+\delta,t)$, $x,\delta\in R$ the limit $\partial f=\lim_{\delta\to0}\frac{1}{\delta}(T-1)f(x,t)$ gives the link to the classical DT.

To derive the DT of potentials U_m, it is necessary to evaluate the derivative of the elements $\sigma^{\pm}$ with respect to the variable t (say, time). We shall do it by introducing the special functions (analog of the differential Bell polynomials), similar to [467]. Let us start from the first version of the DT definition D^+, expressing $T\varphi$ from (2.64); hence, $T\varphi=(\sigma^+)^{-1}\varphi$. Acting on this relation by T and taking into account (2.62) yields $T^2\varphi=T\left[(\sigma^+)^{-1}\right]T\varphi=T\left[(\sigma^+)^{-1}\right](\sigma^+)^{-1}\varphi$. Repeating the action, we arrive at

$$T^m\varphi=\prod_{k=0}^{m-1}[T^k(\sigma^+)]^{-1}\varphi=B_m^+(\sigma^+)\varphi. \tag{2.65}$$

Here and below the product is ordered by the index k running from right to left.

Definition 2.25. *Equation (2.65) defines the function*

$$B_m^+(\sigma)=\prod_{k=0}^{m-1}\left[T^k(\sigma)\right]^{-1}.$$

It is convenient to write down the t-derivative of σ by means of the functions $B_m^+(\sigma^+)$ that are connected with the generalized Bell polynomials [467, 271]:

$$\sigma_t^+ = \sum_{m=-M}^{N} \left[U_m \, B_m^+\left(\sigma^+\right)\sigma^+ - \sigma^+ T\left(U_m\right) B_{m+1}^+\left(\sigma^+\right)\sigma^+\right]. \tag{2.66}$$

The resulting equation (2.66) is a nonlinear equation associated with (2.63) that is reduced to a generalized Miura transformation in the stationary case (Sect. 2.3).

The Matveev theorem for polynomials of the automorphism T provides far-reaching generalizations of the conventional Darboux theorem proved originally for the second-order differential equation (for generalizations see [324] as well) and can be formulated by means of the introduced entries in the following way:

Theorem 2.26. *Let the functions $\varphi \in \boldsymbol{R}$ and $\psi \in \boldsymbol{R}$ satisfy (2.63). Then the function $\psi^+ = D^+\psi$ satisfies the equation*

$$\psi_t^+(x,t) = \sum_{m=-M}^{N} U_m^+ T^m \psi^+,$$

where the coefficients are evaluated from the recurrence relations

$$U_{-M}^+ = U_{-M}, \tag{2.67}$$

$$U_1^+ - U_0^+\sigma^+ = U_1 - \sigma^+ T U_0 - \sigma_t^+, \tag{2.68}$$

$$U_m^+ - U_{m-1}^+ T^{m-1}\sigma^+ = U_m - \sigma^+ T U_{m-1}, \tag{2.69}$$

$$U_N^+ = \sigma^+\left(TU_N\right)\left(T^N\sigma^+\right)^{-1}. \tag{2.70}$$

Equations (2.67)–(2.70) define recursively the DTs of the coefficients (potentials) of the differential equation (2.63). Solving the recurrence (2.69) by means of (2.65) yields

$$U_m^+ = \sum_{l=0}^{m+M} U_{-M+l} - \sigma^+\left(TU_{-M+l-1}\right) B_{-M+l}^+\left(\sigma^+\right)\left[B_m^+\left(\sigma^+\right)\right]^{-1} \tag{2.71}$$

$$U_N^+ = \sigma^+\left(TU_N\right)\left(T^N\sigma^+\right)^{-1}. \tag{2.72}$$

Proof. For the proof it is necessary to check the additional equality that appears from the term $T^m\psi$ with essential use of the expression for σ_t^+ from (2.66).

This theorem establishes the covariance (form invariance) of (2.63) with respect to the DT (2.64).

The formalism for the second DT from (2.64) may be similarly constructed on the ground of the identity

$$T^m \varphi = \prod_{k=0}^{m} T^k \left(\sigma^-\right) T^{-1} \varphi = B_m^- \left(\sigma^-\right) T^{-1} \varphi. \tag{2.73}$$

The definition of the lattice Bell polynomials of the second type $B_m^- \left(\sigma^-\right)$ can be extracted from (2.73). The evolution equation for σ^- is similar to (2.66):

$$\sigma_t^- = \sum_{m=-M}^{N} \left[U_m \; B_m^- \left(\sigma^-\right) - \sigma^- T^{-1} \left(U_m \;\right) B_{m-1}^- \left(\sigma^-\right)\right].$$

It may be considered as a further generalization of the Burgers equation (2.36) and gives the second generalized Miura map for stationary solutions of (2.63). Explicit formulas for U_m^- are similar to (2.68)–(2.72).

2.7 Joint covariance of equations and nonlinear problems. Necessity conditions of covariance

If a pair of linear problems is simultaneously covariant with respect to a Darboux transformation, it generates Bäcklund transformations of the corresponding compatibility condition, or a nonlinear integrable equation. In the context of such an integrability, the joint covariance principle, used to construct solutions of nonlinear problems from the very beginning [313], can be considered as the origin of a classification scheme [265, 267]. In this book, we examine realizations of this scheme and seek the covariant form of equations and an appropriate basis with the simplest transformation properties. Note that a proof of the covariance theorems for the linear operators incorporates the generalized Burgers equations that in stationary versions reduce to the generalized Miura transformation. We give and examine the explicit form of the Miura equality in both the general and the stationary cases (see also [270]). This equality gives an additional nonlinear equation that is automatically solved by the Cole–Hopf substitution and is used to generate dressing t-chain equations [79]. We show how the form of the covariant operator can be found by comparing some kind of Frechét derivatives of the operator coefficients and the transforms.

2.7.1 Towards the classification scheme: joint covariance of one-field Lax pairs

The basis of the formalism introduced here has been elaborated in [265, 267] and the compact formulas with the generalized Bell polynomials are given in Sect. 2.2. The formalism is valid for non-Abelian coefficients a_n as well, and for solutions of (2.37); ϕ and ψ can be considered as matrices or operators. For simplicity, we start with the scalar case.

First we consider particular examples of the theory to derive the explicit expressions and show some details. We begin with a very simple analysis to clarify the integrability notion we introduce. Note first that the higher coefficients a_n (with $n = N$ and $n = N - 1$) are transformed almost trivially. It follows that the coefficients, in general, do not play the role of potentials to be dressed, or solutions of the nonlinear equation being the compatibility condition.

If $N = 2$, the general transformation (2.40) and (2.41) reduces to

$$\begin{aligned} &a_2[1] = a_2 \equiv a(x,t), \qquad a_1[1] = a_1(x,t) + Da(x,t), \\ &a_0[1] = a_0 + Da_1(x,t) + 2a(x,t)D\sigma + \sigma Da(x,t). \end{aligned} \tag{2.74}$$

Only the Abelian case is considered at this stage. The explicit form of the transformations clearly shows a difference between the coefficients $a(x,t)$ and $a_1(x,t)$, which transform irrespectively to solutions, on the one hand, and $a_0 = u(x,t)$, which will stand for an unknown function in a forthcoming nonlinear equation, on the other hand. We call $a_0 = u(x,t)$ the potential in the context of the Lax representation. The KdV case can be easily recognized here. Namely, when $a = \text{const}$ and $a_1 = 0$, a_0 plays the role of the only unknown function in the KdV equation (we call this situation the one-field case). We can therefore formulate the following:

Proposition 2.27. *The Abelian case with $N = 2$ is the first nontrivial example of a set of covariant operators with coefficients $a_{1,2}$ that depend only on x and an additional parameter (e.g., t), but their transformations contain only the functions $a_{1,2}$ and is hence said to be* trivial. *The transformation (generalized DT) for u is given by the last equation in (2.74) and depends on both $a_{1,2}$ and solutions of (2.36) via σ.*

Let us consider the third-order operator as the second one in the Lax pair. Letting $N = 3$ in (2.40) and (2.41) and changing $a_i \to b_i$, we have

$$b_3[1] = b_3, \quad b_2[1] = b_2 + Db_3', \quad b_1[1] = b_1 + Db_2 + 3b_3D\sigma + \sigma Db_3,$$

$$b_0[1] = b_0 + Db_1 + \sigma Db_2 + [\sigma^2 + (2D\sigma)]Db_3 + 3b_3(\sigma D\sigma + D^2\sigma). \tag{2.75}$$

We consider (2.74) and (2.75) as coefficients of the Lax pair of operators, both of which depend on the only variable u, and suppose that the coefficients of the operators and their derivatives with respect to x are analytic functions of u. We now choose $D \to \frac{\partial}{\partial y}$ and $L \to L_1$ in (2.37) corresponding to the case (2.74) and leave the parameter t, i.e., $D_0 \to \frac{\partial}{\partial t}$ for the second case, forming the Lax pair

$$\psi_y = L_1\psi, \tag{2.76}$$

$$\psi_t = L_2\psi. \tag{2.77}$$

Here $L_2 = \sum_{i=0}^{2} b_i D^i$.

Recall the KdV case. The general stationary version of (2.33) for $N = 2$ is

$$\sum_{n=0}^{2} a_n B_n = c = \text{const},$$

which yields

$$\sigma^2 + \sigma' + u = c. \tag{2.78}$$

Note that (2.33) for $N = 3$ is still valid for the same $\sigma = \phi_x \phi^{-1}$ if ϕ is a solution of the Lax pair (2.76) and (2.77). If we restrict ourselves to the case $b_2 = 0$ and $b_3 = b = \text{const}$ in (2.75), we obtain the second equation in the KdV Lax pair.

Returning to the general case and taking into account the triviality of transforms of $b_3 = b(x,t)$ and b_2 in the aforementioned sense, we find that the first nontrivial potential is $b_1 = F(u, u', \ldots)$. Suppose that the covariance principle holds or, equivalently, take the following equation for F:

$$\begin{aligned} b_1[1] &= F(u[1]) = F(u + Da_1 + 2aD\sigma + \sigma Da) \\ &= F(u) + Db_2 + 3bD\sigma + \sigma Db. \end{aligned} \tag{2.79}$$

The analyticity of F permits us to expand the left-hand side of (2.79) in a Taylor series:

$$F(u[1]) = F(u) + F_u(2aD\sigma + Da_1 + \sigma Da) + F_{Du}(\ldots) + \ldots . \tag{2.80}$$

Compare the transformation (2.79) with the Frechét differential (2.80) of the function F. Both equations are identical if the coefficients of σ, $D\sigma$, and the free term in both equations are the same. Introducing $F_u = c(x,t)$ yields

$$2ac = 3b, \tag{2.81}$$

or

$$F(u) = \frac{3bu}{2a}$$

with the additional conditions

$$cDa_1 = Db, \tag{2.82}$$

$$cDa = Db. \tag{2.83}$$

Substituting c from (2.81) in (2.83), we pass either to $3D(\ln a) = 2D(\ln b)$ and obtain $b = a^{3/2} c_1(t)$, or to $Da = Db = 0$. In the last case, (2.83) is valid with an arbitrary c or mutually independent $b(t)$ and $a(t)$, while (2.82) yields the equation for a_1 for both cases, $3Da_1 = 2aDb/3b$ with an arbitrary $c_1(t)$.

Further conditions follow from the last equation in (2.75), i.e., if we introduce a new analytic function G and set $b_0 = G(u, u', \ldots)$, the transformed b_0 gives

$$G(u + Da_1 + 2aD\sigma + \sigma Da) = G(u) + G_u(Da_1 + 2aD\sigma + \sigma Da) \\ + G_{Du}D(Da_1 + 2aD\sigma + \sigma Da) + \ldots . \tag{2.84}$$

The DT formula for the potential u is obviously used. The DT for the last coefficient b_0 [see (2.75)] yields

$$b_0[1] = G(u) + Db_1 + \sigma Db_2 + [\sigma^2 + 2(D\sigma)]Db + 3bD\left(\frac{\sigma^2}{2} + D\sigma\right). \tag{2.85}$$

We now consider a general version of the Miura transformation (2.78) which has the form

$$\sum_0^2 a_n B_n = u + a_1\sigma + a(\sigma^2 + D\sigma) \equiv \mu,$$

and can be used to express σ^2 in (2.85). Doing this and equating (2.84) and (2.85) yields

$$D\frac{3bu}{2a} + \sigma Db_2 + \left(\frac{\mu - u - a_1\sigma}{a} + D\sigma\right)Db + 3bD\left(\frac{\mu - u - a_1\sigma}{2a} + D\sigma\right) \\ = G_u(Da_1 + 2aD\sigma + \sigma Da) + G_{Du}D(Da_1 + 2aD\sigma + \sigma Da). \tag{2.86}$$

From (2.86) we obtain the coefficients

$$G_u(Du)2a = 3b \tag{2.87}$$

for $D^2\sigma$,

$$G_u 2a + \frac{9b(Da)}{2a} = \frac{Db - 3ba_1}{2a} \tag{2.88}$$

for $D\sigma$ taking (2.87) into account, and

$$G_u Da + \frac{3b}{2a}D^2a(x,t) = Db_2 - \frac{a_1}{a} - 3bD\left(\frac{a_1}{2a}\right) \tag{2.89}$$

for σ. The free term is

$$D\frac{3bu}{2a} + \left(\frac{\mu - u}{a}\right)Db + 3bD\left(\frac{\mu - u}{2a}\right) = G_u Da_1 + \frac{3b(D^2a_1)}{2a}. \tag{2.90}$$

From (2.87) and (2.88) we obtain

$$G_u = \frac{Db}{2a} - \frac{3ba_1}{4a^2} - \frac{9b(Da)}{4a^2}. \tag{2.91}$$

If G_u is nonzero, then it follows from (2.89) that

$$\left(\frac{Db}{2a} - \frac{3ba_1}{4a^2} - \frac{9b(Da)}{4a^2}\right)Da + \frac{3b}{2a}D^2a = Db_2 - \frac{a_1}{a} - 3bD\left(\frac{a_1}{2a}\right).$$

The free term (2.90) gives

$$u\frac{Db}{2a}+\mu\frac{Db}{a}-\frac{3bDa}{2a^2}=\left(\frac{Db}{2a}-\frac{3ba_1}{4a^2}-\frac{9bDa}{4a^2}\right)Da_1(x,t)+\frac{3b(D^2a_1)}{2a}. \quad (2.92)$$

If u is linearly independent of σ and its derivatives and we do not take into account higher terms in the Frechét differential, then the only choice $Db = 0$ eliminates the term with u, and (2.92) simplifies to

$$D^2a_1-\frac{a_1(Da_1)}{2a}=0.$$

The condition $Da = 0$ as a consequence of (2.83) has been used. Equation (2.89) also simplifies to

$$Db_2-\frac{a_1}{a}-\frac{3b(Da_1)}{2a}=0$$

and integration gives the expression for b_2.

Another possibility is $G_u = 0$, which gives

$$\frac{9b(Da)}{2a}=\frac{Db-3ba_1}{2a},$$

instead of (2.91). The free term transforms as

$$u\frac{Db}{2a}+\mu\left(\frac{Db}{a}-\frac{3bDa}{2a^2}\right)=\frac{3b(D^2a_1)}{2a}$$

and gives the conditions $Db = Da = 0$ for the same reasons. In turn, this means that $a_1 = 0$ and, finally, from (2.89), $Db_2 = 0$. Hence, this case contains the KdV equation with the (possibly, t-dependent) $a(t)$, $b(t)$, and $b_2(t)$.

Remark 2.28. The results for the single isolated equation (2.76) contain a rather wide class of coefficients, in comparison with the joint covariance of (2.76) and (2.77). Namely, a and a_1 are arbitrary functions of x and t. This may be useful for constructing potentials and solutions (e.g., special functions) for the linear Schrödinger equation and evolution equations in one-dimensional quantum mechanics [214].

The KdV case can be described separately (again using the notation $f' = Df$):

$$G_u\sigma'+G_{u'}\sigma''=\frac{3b(1-a)u'}{4a^2}+\frac{3b\sigma''}{4a}.$$

The only possible choice, if we consider σ, σ', and u' as independent variables, is

$$G_u=0,\qquad G_{u'}=\frac{3b}{4a},$$

or taking into account the condition of zero coefficient for u', $a = 1$, we obtain

$$G(u, u', \dots) = \frac{3bu'}{4}.$$

This result leads directly to one of the equivalent Lax pairs for the KdV equation.

2.7.2 Covariance equations

First we reproduce the "Abelian" scheme, generalizing the study of the Boussinesq equation [270]. To start with, we should fix the number of fields. Let us consider the third-order operator (2.20) with coefficients b_k, $k = 0, 1, 2, 3$, reserving a_k for the coefficients in the second operator in a Lax pair. Suppose, both operators depend on the only potential function w. The problem we consider now can be formulated as follows: to find restrictions on the coefficients $b_3(t)$, $b_2(x,t)$, $b_1 = b(w,t)$, and $b_0 = G(w,t)$ compatible with the DT rules of the potential function w induced by the DT for b_i. The classical DT for the third order operator coefficients (Matveev generalization [314]) yields

$$b_2[1] = b_2 + b_3', \tag{2.93}$$

$$b_1[1] = b_1 + b_2' + 3b_3\sigma', \tag{2.94}$$

$$b_0[1] = b_0 + b_1' + \sigma b_2' + 3b_3(\sigma\sigma' + \sigma''), \tag{2.95}$$

having in mind that the highest coefficient b_3 does not transform. Note also that $b_3' = 0$ yields invariance of b_2.

The general idea of the DT form invariance can be realized considering transformations of the coefficients consistent with respect to the fixed transform of w. Generalizing the analysis of the third order operator transformation [270], we arrive at the equations for the functions $b_2(x,t)$, $b(w,t)$, and $G(w)$. The covariance of the spectral equation

$$b_3\psi_{xxx} + b_2(x,t)\psi_{xx} + b(w,t)\psi_x + G(w,t)\psi = \lambda\psi \tag{2.96}$$

can be considered separately and leads to the link between b_i only. We, however, study the problem (2.96) in the context of the Lax representation for some nonlinear equation; hence, the covariance of the second Lax equation is taken into account from the very beginning. We refer to such an approach as the *principle of joint covariance* [265, 267]. The second (evolution) equation is written as

$$\psi_t = a_2(x,t)\psi_{xx} + a_1(x,t)\psi_x + w\psi, \tag{2.97}$$

with the operator on the right-hand side having again the general polynomial form of (2.20).

If we consider the operators L and A of the form $\sum a_i D^i$, specified in equations (2.96) and (2.97) as the Lax pair equations, the DT of w implied by the covariance of (2.97) should be compatible with DT formulas of both w-dependent coefficients of (2.96):

$$a_2[1] = a_2 = a(x,t), \quad a_1[1] = a_1(x,t) + Da(x,t),$$

$$a_0[1] = w[1] = w + a_1' + 2a_2\sigma' + \sigma a_2'.$$

The following important relations being in fact the identities in the DT theory [467] are the particular cases of the generalized Burgers equation for σ (2.36):

$$\sigma_t = [a_2(\sigma^2 + \sigma_x) + a_1\sigma + w]_x \tag{2.98}$$

for the problem (2.97) and

$$b_3(\sigma^3 + 3\sigma_x\sigma + \sigma_{xx}) + b_2(\sigma^2 + \sigma_x) + b(w,t)\sigma + G(w) = \mathrm{const}$$

for (2.96), where ϕ is a solution of both Lax equations.

Suppose now that the coefficients of the operators are analytic functions of w together with its derivatives (or integrals) with respect to x (such functions are named functions on the prolonged space [33]). For the coefficient $b_0 = G(w,t)$ this means

$$G = G(\partial^{-1}w, w, w_x, \ldots, \partial^{-1}w_t, w_t, w_{tx}, \ldots). \tag{2.99}$$

The covariance condition is formulated for the Frechét derivative of the function G on the prolonged space. In other words, the first terms of a multidimensional Taylor series for (2.99) read

$$G(w + a_1' + 2a_2\sigma' + \sigma a_2') = G(w) + G_{w_x}(a_1' + 2a_2\sigma' + \sigma a_2')' + \ldots. \tag{2.100}$$

We show only the terms which enter the “minimal” equations of the hierarchy.

In full analogy with (2.94) and (2.100), quite similar expansion arises for the coefficient $b_1 = b(w,t)$. Equating the DT and the expansion, we obtain the condition

$$b_2' + 3b_3\sigma' = b_w(a_1' + 2a_2\sigma' + \sigma a_2') + b_{w'}(a_1' + 2a_2\sigma' + \sigma a_2')' \ldots. \tag{2.101}$$

We call this equation as the (first) *joint covariance equation* that guarantees consistency between transformations of the coefficients of the Lax pair (2.96) and (2.97). In the frame of our choice $a_2' = 0$, the equation simplifies and linear independence of the derivatives $\sigma^{(n)}$ yields two constraints

$$3b_3 = 2b_w a_2, \qquad b_2' = b_w a_1',$$

or, solving the second and plugging into the first, results in

$$b_w = 3b_3/2a_2, \qquad b_2' = 3b_3 a_1'/2a_2. \tag{2.102}$$

So, if one wants to save the form of the standard DT for the variable w (potential), simple comparison of both transformation formulas gives the following connection for $b(w)$ [with arbitrary function $\alpha(t)$]:

$$b(w,t) = 3b_3 w/2a_2 + \alpha(t). \tag{2.103}$$

Equating the expansion (2.100) with the transform of the $b_0 = G(w,t)$ yields

$$b_1' + \sigma b_2' + 3b_3(\sigma^2/2 + \sigma')' \tag{2.104}$$
$$= G_{w_x}(a_1' + 2a_2\sigma' + \sigma a_2')' + G_{\partial^{-1}w_t}[a_{1t} + 2\partial^{-1}(a_2\sigma_t') + \partial^{-1}(\sigma a_2')_t] + \dots.$$

This second joint covariance equation also simplifies when $a_2' = 0$ and (2.103) is accounted for:

$$3b_3 w'/2a_2 + \sigma b_2' + 3b_3(\partial^{-1}\sigma_t - w)'/2a_2 + 3b_3\sigma''/2 \tag{2.105}$$
$$= G_{w_x}(a_1' + 2a_2\sigma')' + G_{\partial^{-1}w}(a_1 + 2a_2\sigma) + G_{\partial^{-1}w_t}(a_{1t} + 2a_2\sigma_t) + \dots.$$

Note that the "Miura" transform (2.98) is used on the left-hand side of (2.105) and linearizes the Frechét derivative with respect to σ; therefore, the derivatives of the function G,

$$G_{w_x} = 3b_3/4a_2, \quad G_{\partial^{-1}w_t} = 3b_3/4a_2^2, \quad G_{\partial^{-1}w} = b_2'/2a_2,$$

are accompanied by the constraint

$$a_{1t} + a_2 a_1'' + a_1 a_1' = 0, \tag{2.106}$$

which acquires the form of the Burgers equation after using (2.102). Finally, the integration of (2.102) gives

$$b_2 = 3b_3 a_1/2a_2 + \beta(t) \tag{2.107}$$

and the "lower" coefficient of the third-order operator is expressed by

$$G(w,t) = 3b_3 w_x/2a_2 + 3b_3 a_1'\partial^{-1}w/2(a_2)^2 + 3b_3\partial^{-1}w_t/2a_2^2.$$

Proposition 2.29. *The expressions (2.97), (2.96), (2.103), and (2.107) define the covariant Lax pair when the constraints (2.102) and (2.106) hold.*

Remark 2.30. We cut the Frechét differential formulas on the level that is necessary for the minimal flows. The account of higher terms leads to the whole hierarchy, similarly to [260, 261].

2.7.3 Compatibility condition

In the case $a_2' = 0$ the Lax system (2.96) and (2.97) produces the following compatibility conditions:

$$\begin{aligned}
2a_2b_3' &= 3b_3a_2',\\
b_{3t} &= 2a_2b_2' - 3b_3a_1'',\\
b_{2t} &= a_2b_2'' + 2a_2b_1' + a_1b_2' - 3b_3a_1'' - 2b_2a_1' - 3b_3a_0',\\
b_{1t} &= a_2b_1'' + a_1b_1' - b_3a_1''' - b_2a_1'' - b_1a_1' - 3b_3a_0'' - 2b_2a_0' + 2a_2b_0',\\
b_{0t} &= a_1b_0' + a_2b_0'' - b_1a_0' - b_2a_0'' - b_3a_0'''.
\end{aligned} \tag{2.108}$$

In the particular case $a_2 = 0$ we derive from the first of the equalities (2.108) the constraint $b_3' = 0$. The direct consequence of (2.107) is $b_{3t} = 0$. In the rest of the equations the links (2.108) and (2.107) are taken into account. Hence, (2.106) in combination with the expression for b_{2t} produces $\beta_t = -2\beta a_1'$ with $\beta(t)$ from (2.107). The last two equations (for $b_3 = 1$ and $a_2 = -1$) become

$$\alpha w + \alpha_t + 3a_1''\partial^{-1}w/2 + (2\beta - 3a_1/2)w' + a_1''' + 3a_1a_1''/2 = 0,$$

$$3\partial^{-1}(w_t + a_1w)_t/4 = (\alpha - 3w/2)w' - w'''/4 + 3a_1w_t/4$$
$$+3a_1a_1''\partial^{-1}w/4 + 3a_1a_1'w/4 - 3a_1'w'/4 + (\beta + 3a_1/4)w''.$$

In the simplest case of constant coefficients ($b_2' = a_1' = 0$), one goes down to

$$3b_3(w_t + a_1w)_t/4a_2^2 \tag{2.109}$$
$$- \left[(3b_3w/2a_2 + \alpha)w' - b_3w'''/4 + 3b_3a_1w_t/4a_2^2 + (\beta - 3b_3a_1/4a_2)w''\right]'.$$

This equation reduces to the standard Boussinesq equation when $b_1 = a_1 = 0$, $b_3 = 1$, and $a_2 = -1$.

We should stress once again that the results given in Sect. 2.2 have been simplified to show more clearly the algorithm of the derivation of the covariant Lax pair. A more general study can be developed if $a_2' \neq 0$.

2.8 Non-Abelian case. Zakharov–Shabat problem

In this section we consider linear equations comprising the Lax pair with the coefficients from the non-Abelian differential ring A (for details of the definitions of the mathematical objects, see [467]) and apply for them the joint covariance principle.

2.8.1 Joint covariance conditions for general Zakharov–Shabat equations

Let us change the notations for the first-order ($n = 1$) equation (2.39) as follows:

$$\psi_t = (J + u\partial)\psi. \tag{2.110}$$

Here the operator $J \in A$ does not depend on x, y, t and the potential $a_0 \equiv u(x, y, t) \in A$ is a function of the variables indicated. The operator $\partial = \partial/\partial x$ can be considered as a general differentiation, as in [467]. The transformed potential

$$\tilde{u} = u + [J, \sigma], \tag{2.111}$$

where $\sigma = \phi_x \phi^{-1}$ and ϕ is another solution of (2.110), is defined by the same formula as before, but the order of the elements is important. The covariance of the operator in (2.110) follows from the general transformations of the coefficients in the polynomial (2.41). The coefficient J is not transformed.

Suppose the second operator of a Lax pair has the same form but with different entries and derivatives:

$$\psi_y = (Y + w\partial)\psi, \qquad Y \in A, \tag{2.112}$$

where the potential $w = F(u) \in A$ is a function of the potential of the first equation (2.110). The principle of joint covariance [265, 267] hence reads

$$\tilde{w} = w + [Y, \sigma] = F(u + [J, \sigma]),$$

with the direct consequence

$$F(u) + [Y, \sigma] = F(u + [J, \sigma]). \tag{2.113}$$

So, the joint covariance equation (2.113) defines the function $F(u)$. In the case of the Abelian algebra we use the Taylor series (generalized by use of the Frechét derivative) to determine this function. Now some generalization is necessary. Let us make some remarks.

An operator-valued function $F(u)$ of an operator u in a Banach space may be considered as a generalized Taylor series with coefficients that are expressed in terms of Frechèt derivatives. The linear in u part of the series approximates (in a sense of the space norm) the function

$$F(u) = F(0) + F'(0)u + \dots .$$

This representation is not unique and a similar expression

$$F(u) = F(0) + u\hat{F}'(0) + \dots$$

may be introduced (definitions are given similarly to those in [33]). Both expressions, however, are not Hermitian; hence, they are not suitable for the majority of physical models. It means that the class of such operator functions is too restrictive. To explain how a more general class of functions could be introduced, let us consider some examples.

2.8.2 Covariant combinations of symmetric polynomials

The first natural example is the generalized Euler top equation with the Hamiltonian $Hu+uH$ which is discussed in Sect. 3.9. The covariant Lax pair for this case consists of two equations (2.110) and (2.112); the entries of the operators satisfy the joint covariance condition (2.113) and the compatibility condition if $J = H$ and $Y = H^2$.

The next example is related to the operator polynomial

$$P_2(H, u) = H^2u + HuH + uH^2,$$

whereas the choice $F(u) = P_2(H, u)$ satisfies the link (2.113). The direct substitution in the covariance and compatibility equations leads to a covariant constraint that turns out to be the identity if $Y = H^3$ and $J = H$.

More general connection $Y = J^n$ and $J = H$ leads to the covariance of the function

$$P_n(H, u) = \sum_{p=0}^{n} H^{n-p}uH^p.$$

This observation was exhibited in [276]. For further generalization let us consider combinations of polynomials,

$$f(H, u) = Hu + uH + S^2u + SuS + uS^2. \tag{2.114}$$

Plugging (2.114) as $F(u) = f(H, u)$ into (2.113) hints at a choice $Y = AB + CDE$ that yields

$$A[B, \sigma] + [A, \sigma]B + CD[E, \sigma] + C[D, \sigma]E + [C, \sigma]DE$$
$$= H[J, \sigma] + [J, \sigma]H + S^2[J, \sigma] + S[J, \sigma]S + [J, \sigma]S^2.$$

The last expression turns out to be the identity if $A = B = J = H$, $C = \alpha H$, $D = \alpha H$, $D = \alpha H$, $S = \beta H$, and $[\alpha, H] = 0$, $[\beta, H] = 0$ with the link $\alpha^3 = \beta^2$. Continuing this analysis, we arrive at the following:

Proposition 2.31. *The joint covariance principle defines a class of homogeneous polynomials $P_n(H, u)$, symmetric with respect to cyclic permutations, as possible Hamiltonians $h(\rho) = P_n(H, u)$ for the Liouville–von Neumann type evolution (Sect. 3.9). A linear combination of polynomials $\sum_{n=1}^{N} \beta_n P_n(H, u)$ with the coefficients commuting with u and H also yields the covariant pair if the conditions $Y = \sum_{n=1}^{N} \alpha_n H^{n+1}$, $\alpha_1 = \beta_1 = 1$, $\alpha_n^{n+2} = \beta^{n+1}$, and $n \neq 1$ hold.*

A proof could be performed by induction that is based on homogeneity of P_n and linearity of the constraints with respect to u. The functions $F_H(u) = \sum_0^\infty a_n P_n(H, u)$ also satisfy the constraints if the corresponding series converges.

2.9 A pair of difference operators

Let us consider a pair of equations of the same type (2.63) for a function ψ:

$$\psi_t(x,t) = \sum_{m=-M}^{N} U_m T^m \psi, \tag{2.115}$$

$$\psi_y(x,t) = \sum_{m=-M'}^{N'} V_m T^m \psi. \tag{2.116}$$

The compatibility condition for them is the nonlinear equation

$$U_{sy} - V_{st} = \sum_k \left[V_k T^k \left(U_{s-k}\right) - U_{s-k} T^{s-k} \left(V_k\right)\right] \tag{2.117}$$

for $s = -M-M', ..., N+N',\ k \in \{k' = -M', ..., N'\} \cap \{s-k = -M, ..., N\}$.

In the simplest case of the Zakharov–Shabat operators in both (2.115) and (2.116) with the subclass of stationary in y solutions we obtain three conditions:

$$\begin{aligned} U_{0t} &= V_0 U_0 - U_0 V_0, \\ U_{1t} &= V_0 U_1 - U_0 V_1 + V_1 T\left(U_0\right) - U_1 T\left(V_0\right), \end{aligned}$$

and

$$V_1 T\left(U_1\right) = U_1 T\left(U_1\right).$$

The connection with polynomials of a differential operator and hence with the theory of classical Bell polynomials can be revealed if we change the definition of potentials. It is clear that if the automorphism T is the shift operator $Tf(x) = f(x+\delta)$, the coefficients of the polynomials in T should be arranged as follows:

$$\psi_t(x,t) = \sum_{m=-M}^{N} \frac{u_m}{\delta^m} \sum_{r=0}^{m} \binom{m}{m-r} (-1)^{m-r} T^r \psi. \tag{2.118}$$

The recursion equation that defines classical differential Bell polynomials in commutative variables $y_1, y_2, \ldots$ [388],

$$B_{m+1} = \sum_{r=0}^{m} \binom{m}{r} B_{m-r} y_{r+1},$$

together with the definition (2.65) of B_m^+, connects these special functions. Let us remark that the transformations for U_m found in Sect. 2.6 give the transforms for u_m defined by (2.118). The possibility of inverse transition depends on the independence of functions $(T-1)^n f$ for a given T and the set of functions ψ under consideration. The joint covariance of the system (2.115) and (2.116) hence may be investigated along the guidelines of [260] and [270], where the so-called binary Bell polynomials are used to form a convenient basis.

2.10 Non-Abelian Hirota system

Let us consider a pair of the Zakharov–Shabat type equations,

$$\psi_t(x,y,t) = \quad (V_0 + V_1 T)\,\psi \tag{2.119}$$

and

$$\psi_y(x,y,t) = \quad \left(U_0 + U_{-1}T^{-1}\right)\psi. \tag{2.120}$$

It differs from that used in the previous section by the change $T \to T^{-1}$ on the right-hand side of (2.120).

In a t-lattice version of equation (2.63) with $j \in \mathbb{Z}$ we go to

$$f(x, j+1) = \sum_{m=-M}^{N} U_m T^m f(x,j).$$

The case of a lattice in all variables is generated by the transition to the discrete variables $x, y, t \to n, j, r \in \mathbb{Z}$, $f(x,y,t) \to f_n(j,r)$, defined as in [321]. The operator T acts as the shift of n: $Tf_n(j,r) = f_{n+1}(j,r)$. The corresponding equations (2.119) and (2.120) are written as

$$f_n(j-1,r) = f_{n+1}(j,r) + v(n,j,r)f_n(j,r) \tag{2.121}$$

and

$$f_n(j,r-1) = f_n(j,r) + u(n,j,r)f_{n-1}(j,r) \tag{2.122}$$

with the potentials indicated. The compatibility condition of the linear equations (2.121) and (2.122) has the form

$$\begin{aligned} u(n,j-1,r) - u(n+1,j,r) &= v(n,j,r-1) - v(n,j,r),\\ v(n,j,r-1)u(n,j,r) &= u(n,j-1,r)v(n-1,j,r). \end{aligned} \tag{2.123}$$

The second equation in (2.123) is automatically valid if

$$\begin{aligned} u(n,j,r) &= \tau_{n+1}(j,r-1)\tau_n^{-1}(j,r-1)\tau_{n-1}(j,r)\tau_n^{-1}(j,r),\\ v(n,j,r) &= \tau_{n+1}(j-1,r)\tau_n^{-1}(j-1,r)\tau_n(j,r)\tau_{n+1}^{-1}(j,r). \end{aligned} \tag{2.124}$$

It should be stressed that the order of the entries in these expressions is important. The substitution of (2.124) in the first equation in (2.123) leads to the generalized Hirota bilinear equation [210] (compare also with the generalizations in [336]):

$$\begin{gathered} \tau_{n+1}(j-1,r-1)\tau_n^{-1}(j-1,r-1)\tau_{n-1}(j-1,r)\tau_n^{-1}(j-1,r)\\ -\tau_{n+1}(j-1,r-1)\tau_n^{-1}(j-1,r-1)\tau_{n-1}(j,r-1)\tau_n^{-1}(j,r-1)\\ -\tau_{n+2}(j,r-1)\tau_{n+1}^{-1}(j,r-1)\tau_n(j,r)\tau_{n+1}^{-1}(j,r)\\ +\tau_{n+1}(j-1,r)\tau_n^{-1}(j-1,r)\tau_n(j,r)\tau_{n+1}^{-1}(j,r) = 0. \end{gathered} \tag{2.125}$$

In the scalar case the system reduces to the Hirota bilinear equation [321]

$$\tau_n(j+1,r)\tau_n(j,r+1)-\tau_n(j,r)\tau_n(j+1,r\ +1)$$
$$+\tau_{n+1}(j+1,r)\tau_{n-1}(j,r+1)=0. \tag{2.126}$$

Using (2.124) and the DT formalism, we could elaborate a non-Abelian version of these equations that can be useful for applications in the theory of quantum transfer matrices for fusion rules [255, 256] and of quantum correlation functions [36, 37]. Note that the non-Abelian Hirota–Miwa equation is discussed by Nimmo [351].

Let us return to the DT theory. Equations (2.119) and (2.120) are jointly covariant; hence, solving equations (2.123) or (2.125) is based on the symmetry that is generated by the joint covariance of (2.121) and (2.122) with respect to the transformations of the type (2.111), namely,

$$\psi^-(j,r)=\psi-\sigma^-T^{-1}f, \qquad \sigma^-=\varphi\left(T^{-1}\varphi\right)^{-1}.$$

As can be easily seen, the form of both linear equations (2.121) and (2.122) represents reductions of (2.119) and (2.120) with $V_1=1$, $V_0=v$, $U_0=1$, and $U_{-1}=u$. We show further some details in the proof of the covariance theorem because it demonstrates important features in the procedure of the derivation of the chain equation. Let us start, say, from (2.122). The covariance conditions are obtained from the coefficients by ψ, $T^{-1}\psi$, and $T^{-2}\psi$. The first one is valid automatically,

$$u^-=u-\sigma^-\left(r-1\right)+\sigma^-\left(r\right), \tag{2.127}$$

$$u^-T^{-1}\sigma^-\left(r\right)=\sigma^-\left(r-1\right)u. \tag{2.128}$$

2.11 Nahm equations

The Nahm equations [344] appear in conformal field theory in connection with the monopole problem. They are solved by the variational method in [129], producing a parameterization of the Bogomolny equations. Their generalizations attract great attention in mathematical physics [101, 345].

In the following example, we change the DT formulas a bit, showing the alternative version, similar to [381]. We stress, however, that the formulas from Sect. 2.1 give an equivalent result. Some generalization will be needed within the reduction constraints related to an additional (gauge) transformation denoted by g. This is expressed by the following:

Theorem 2.32. *The equation*

$$\psi_y=uT\psi+v\psi+wT^{-1}\psi \tag{2.129}$$

is covariant with respect to the combined gauge–DT

$$\psi[1] = g(T - \sigma)\psi. \tag{2.130}$$

Here $\sigma = (T\phi)\phi^{-1}$*, where* ϕ *is a solution of the same equation (2.129) and* g *is an invertible element of the ring. The transforms of the equation coefficients are*

$$u[1] = gT(u)[T(g)]^{-1}, \tag{2.131}$$
$$v[1] = gT(v))g^{-1} - g\sigma ug^{-1} + gT(u)T(\sigma)g^{-1} + g_y g^{-1}, \tag{2.132}$$
$$w[1] = g\sigma w[T^{-1}(g\sigma)]^{-1}. \tag{2.133}$$

Proof. The substitution of (2.130) into the transformed equation (2.129) gives four equations assuming $T^n\psi$ are independent. Three of them yield transformed potentials (2.131)–(2.133). The fourth equation after use of the transforms takes the form

$$\sigma_y = \sigma F - (TF)\sigma, \tag{2.134}$$

where

$$F = u\sigma + v + w[T^{-1}(\sigma)]^{-1}.$$

One can check the condition (2.134) by direct substitution of the operator σ and by use of the equation for ϕ.

Remark 2.33. Theorem 2.32 is evidently valid for the spectral problem

$$\lambda\psi = uT\psi + v\psi + wT^{-1}\psi \tag{2.135}$$

with the only correction being that the last term for the transform $v[1]$ is absent. The equation goes to the "Riccati equation" analog for the function σ:

$$\mu = u\sigma + v + w[T^{-1}(\sigma)]^{-1}. \tag{2.136}$$

Note that inserting the element $\sigma = (T\phi)\phi^{-1}$ into (2.136) transforms it to the spectral problem for ϕ (2.135) with the spectral parameter μ.

The Nahm equations can be written by means of the Lax representation using the spectral equation (2.135) and the evolution equation

$$\psi_y = (q + pT)\psi \tag{2.137}$$

with potentials p and q. The covariance of this equation with respect to the DT (2.130) can be established similarly to Theorem 2.32 with account of the y-evolution of $\sigma(y)$:

$$\sigma_y = T(q)\sigma - \sigma p\sigma + T(p)T(\sigma)\sigma - \sigma q = 0, \tag{2.138}$$

which proves the following transformation formulas for the coefficients in (2.137):

$$p[1] = gT(p)\,[T(g)]^{-1}$$

and

$$q[1] = g\left[T(q) - \sigma p + T(p)T(\sigma)\right]g^{-1} + g_y g^{-1}.$$

The joint covariance principle (Sect. 2.7 and [265]) defines the connection between potentials p and q and u and v:

$$p = u + \beta I, \qquad q = v/2. \tag{2.139}$$

Hence, the joint DT covariance means integrability of the compatibility condition of equations (2.137) and (2.129), e.g., of the Nahm equations:

$$\begin{aligned} u_y &= \frac{1}{2}[uT(v) - vu] + \beta[T(v) - v], \\ v_y &= uT(w) - wT^{-1}u + \beta[T(w) - w], \\ w_y &= \frac{1}{2}vw - wT - 1(v). \end{aligned}$$

One more possible specification is the use of periodic potentials in the problem (2.137) with the evolution (2.129) with account for the connections (2.139) that result in the appearance of commutators on the right-hand sides of the equations. Some linear transformations and rescaling

$$\begin{aligned} u &= \alpha(-\mathrm{i}\varphi_1/2 - \varphi_3), \qquad v = \varphi_3, \\ w &= \alpha^{-1}(-\mathrm{i}\varphi_1/2 + \varphi_3), \qquad q = \varphi_3/2, \\ p &= \alpha(-\mathrm{i}\varphi_1/2 - \varphi_3) + \beta I \end{aligned}$$

produce the Nahm equations for the periodic functions $T\varphi_i = \varphi_i$ [periodicity of φ_i does not mean a periodicity of solutions ψ and ϕ of the Lax pair and the corresponding $\sigma = (T\phi)\phi^{-1}$]:

$$\varphi_{iy} = \mathrm{i}\epsilon_{ikl}[\varphi_k, \varphi_l]. \tag{2.140}$$

α and β are free parameters. This system is covariant with respect to the combined DT–gauge transformations if the gauge transformation $g = \exp G$ is chosen as follows:

$$G_y = \alpha\left[(\varphi_3 + \varphi_1/2)T(\sigma) - \sigma(\varphi_3 + \varphi_1/2)\right]/2. \tag{2.141}$$

Finally, the following theorem can be formulated:

Theorem 2.34. *For $T\varphi_i = \varphi_i$ the system (2.140) is invariant with respect to the transformations*

$$\begin{aligned} \varphi_1[1] &= g\left[(\varphi_1/2 - \mathrm{i}\varphi_3)T(g)^{-1} + \sigma(\varphi_1/2 + \mathrm{i}\varphi_3)[T^{-1}(g\sigma)]^{-1}\right], \\ \varphi_2[1] &= g\left[\varphi_2 + \alpha(\mathrm{i}\sigma\varphi_1/2 - \mathrm{i}\varphi_1 T(\sigma)/2 + \sigma\varphi_3 - T(\varphi_3\sigma))\right]g^{-1}, \\ \varphi_3[1] &= g\left[(-\mathrm{i}\varphi_1/2 - \varphi_3)T(g)^{-1} + \sigma(-\mathrm{i}\varphi_1/2 + \varphi_3)[T^{-1}(g\sigma)]^{-1}\right] \end{aligned} \tag{2.142}$$

with the function $g = \exp G$, where G is obtained by integrating (2.141), if the element σ is a solution of the system

$$\mu = \alpha(-\mathrm{i}\varphi_1/2 - \varphi_3)\sigma + \varphi_3 + \alpha^{-1}(-\mathrm{i}\varphi_1/2 + \varphi_3)[T^{-1}(\sigma)]^{-1}, \tag{2.143}$$

$$\sigma_y = [\varphi_3, \sigma]/2 - \sigma[\alpha(-\mathrm{i}\varphi_1/2 - \varphi_3) + \beta I]\sigma + [\alpha(-\mathrm{i}\varphi_1/2 - \varphi_3) + \beta I]T(\sigma)\sigma = 0.$$

The system (2.143) follows from (2.138) and (2.136).

Remark 2.35. A similar statement can be formulated for the discrete version [342] of the Nahm system (2.140), as may be seen from the previous section.

2.12 Solutions of Nahm equations

Making use of the construction described in the previous section, we consider a simple example. Let T be a shift operator $T\psi(x,y) = \psi(x+1,y)$. As a seed solution of the Nahm equations (2.140) take commuting constant matrices $\varphi_i = A_i$, $i = 1,2,3$, which means constant u, v, and w. First of all we should generate a solution of the Lax pair (2.135) and (2.137) that can be found in the form $\phi = \xi(t)\Phi(x)$ (all elements are supposed to be invertible). The equation for ξ is obtained as

$$\xi_t = [v/2 + (u+\beta I)T]\xi = Z\xi,$$

which is solved by

$$\xi = \exp(Zt)\xi_0.$$

Plugging Φ into (2.135) yields the spectral problem for the difference shift operators:

$$\mu\Phi(x) = \xi^{-1}[u\xi\Phi(x+1) + v\xi\Phi + w\xi\Phi(x-1)].$$

Separating variables again, a class of particular solutions is built as

$$\Phi = \eta\exp(\Sigma x)\ ;$$

hence, we arrive at the matrix spectral problem for η:

$$\mu\eta = \xi^{-1}\left[u\xi\eta\exp(\Sigma) + v\xi\eta + w\xi\eta\exp(-\Sigma)\right],$$

with the operator on the right-hand side and, therefore, spectral parameter μ parameterized by t. Finally, the matrix σ is composed as

$$\sigma = \xi(t)\eta\exp(\Sigma)\eta^{-1}\xi^{-1}(t).$$

An appropriate choice of commutator algebra for A_i, Σ, and η allows us to obtain an explicit form of σ and, hence, to construct and solve the following equation for G:

$$G_t = \frac{\alpha}{2}\left[\left(\varphi_3 + \frac{1}{2}\varphi_1\right)\xi(t)\eta\exp(\Sigma)\eta^{-1}\xi^{-1}(t)\right.$$
$$\left. -\xi(t)\eta\exp(\Sigma)\eta^{-1}\xi^{-1}(t)\left(\varphi_3 + \frac{1}{2}\varphi_1\right)\right].$$

Its exponent (the matrix g) is necessary for the dressing formulas (2.142). We would like to stress that the matrices σ and g do not depend on x; hence, the dressed $\varphi[i]$ also does not.

Starting from the known solution of (2.140), we arrive at the Euler system for $f_i(y)$ that is solved in the Jacobi functions [129]. The solutions are dressed by the transformations (2.142). A more general possibility is a direct series solution of (2.138).