Evgeny Doktorov
Sergey Leble

MATHEMATICAL PHYSICS STUDIES 28

A Dressing Method in Mathematical Physics

Springer

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MATHEMATICAL PHYSICS STUDIES

VOLUME 28

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by

Evgeny V. Doktorov
Institute of Physics, Minsk, Belarus

and

Sergey B. Leble
University of Technology, Gdansk, Poland

Springer

A C.I.P. Catalogue record for this book is available from the Library of Congress.

ISBN 978-1-4020-6138-7 (HB)
ISBN 978-1-4020-6140-0 (e-book)

Published by Springer,
P.O. Box 17, 3300 AA Dordrecht, The Netherlands.
www.springer.com

Printed on acid-free paper

55 udir convienmi ancor come l'essemplo
56 e l'essemplare non vanno d'un modo,
57 ché io per me indarno a ciò contemplo.

Dante Alighieri, *Divina Commedia*
Paradiso, Canto XXVIII

55 then I still have to hear just how the model
56 and copy do not share in one same plan
57 for by myself I think on this in vain.

Translated by A. Mandelbaum

Contents

Preface

The emergence of a new paradigm in science offers vast perspectives for future investigations, as well as providing fresh insight into existing areas of knowledge, discovering hitherto unknown relations between them. We can observe this kind of process in connection with the appearance of the concept of *solitons* [465]. Understanding the fact that nonlinear modes are as fundamental as linear ones, with the advent of a rigorous formalism making it possible to find exact solutions of a wide class of physically important nonlinear equations, gave rise to "a revolution that has quietly transformed the realm of science over the past quarter century" [392].

The *inverse spectral* (or *scattering*) *transform* (IST) method serves as the mathematical background for the soliton theory. The development of the IST formalism affects many fields of mathematics, revealing on frequent occasions unexpected links between them. For example, the theory of surfaces in $\mathbb{R}^3$ can be considered as a chapter of the theory of solitons [468]. The modern version of IST is based on the *dressing method* proposed by Zakharov and Shabat, first in terms of the factorization of integral operators on a line into a product of two Volterra integral operators [474] and then using the Riemann–Hilbert (RH) problem [475]. The most powerful version of the dressing method incorporates the $\bar{\partial}$ problem formalism. The $\bar{\partial}$ problem was put forward by Beals and Coifman [39, 40] as a generalization of the RH problem and was applied to the study of first-order one-dimensional spectral problems. The full-scale opportunities provided by this formalism came to be clear after the paper by Ablowitz et al. [1] devoted to solving the Kadomtsev–Petviashvili II equation. The main achievements within this subject have been summarized in the excellent books by Novikov et al. [354], Faddeev and Takhtajan [148], Ablowitz and Clarkson [3], and Belokolos et al. [45], published more than a decade ago. Experimental aspects of the soliton physics are presented in the book by Remoissenet [373]. The elegant group-theoretical approach to integrable systems was presented in a recent book by Reyman and Semenov-tyan-Shansky [374].

Generally, the term "dressing" implies a construction that contains a transformation from a simpler (*bare, seed*) state of a system to a more advanced, *dressed* state. In particular cases, dressing transformations, as the purely algebraic construction, are realized in terms of the Bäcklund transformations which act in the space of solutions of the nonlinear equation, or the Darboux transformations (DTs) acting in the space of solutions of the associated linear problem.

At the same time, it should be stressed that the term "dressed" has appeared for the first time perhaps in quantum field theory that operates with the states of bare and dressed particles or quasiparticles. These states are interconnected by operators whose properties have much in common, no matter whether we speak about electrons or phonons. The study of these operators, which goes back to Heisenberg and Fock, was in due course one of the stimuli for active promotion of the methods of the Lie groups and algebras in physics.

In mathematical physics, the operators of this sort occur under different names, like creation–annihilation, raising–lowering, or ladder operators. The factorization method [214] widely applicable in quantum mechanics consists in fact in dressing of the vacuum state by the creation operators which are obtained as a result of the factorization of the Schrödinger operator. The property of intertwining of the dressing operators is ultimately connected with the algebraic construction known as supersymmetry.

Hence, the concept of dressing is in fact considerably wider than if we were to take into account its application in soliton theory alone. Evidently, an attempt to span all the diversity of dressing applications treated in the aforementioned extended sense under the cover of a single book seems too ambitious. With regard to the authors' scientific interests, we restrict our consideration to essentially two global aspects of the dressing method. The first one is mostly algebraical and relates to an extension of the possibilities of the DTs and Moutard transformations invoking new constructions and enhancing classes of objects used. In essence, we aim to go beyond the traditional scope of the Darboux–Bäcklund transformations towards the modern development like dressing chains, operator factorization on associative rings, a nonlinear von Neumann equation for the density matrix, and so on. Following our extended understanding of dressing, we demonstrate efficient use of the Darboux-like transformations for the discrete spectrum management in linear quantum mechanics. The second aspect of the dressing concept is largely analytical and is based on the RH and ∂ formalisms following most closely the Zakharov and Shabat ideas.

The DTs, as the representative of the direct methods in soliton theory, provide a powerful tool to analyze and solve nonlinear equations [324] and allow far-reaching generalizations. On the other hand, direct methods are not very suitable for solving the initial-value (Cauchy) problems or to describe interaction of radiation with localized objects. Therefore, the second main topic of our book is devoted to solving the Cauchy problem and finding localized

solution of various nonlinear integrable equations in both 1+1 and 2+1 dimensions by means of the RH and $\bar{\partial}$ problems.

Let us briefly comment on a modern state of the art of the subjects our book is devoted to. If $\psi(x,\lambda)$ and $\varphi(x,\mu) \in \mathbb{C}$ are linearly independent solutions of the linear equation

$$-\psi_{xx} + u(x)\psi = \lambda\psi$$

associated with the parameters λ and μ, then

$$\psi[1] = \psi_x + \sigma\psi \ , \qquad \sigma = -\varphi_x/\varphi$$

is the solution of the equation

$$-\psi[1]_{xx} + u[1]\psi[1] = \lambda\psi[1] \ ,$$

with

$$u[1](x) = u + 2\sigma_x \ .$$

They are the analytic expressions of $\psi[1]$ and $u[1]$ in terms of ψ, φ, and u that determine the DT.

Already the pioneering papers of Matveev [313, 314, 315] have shown that the DT represents in fact a universal algebraic operation up to the most advanced one [321] for associative rings. The Matveev theorem provides a natural generalization of the DTs in the spirit of the classical approach of Darboux [102] with a great variety of applications. Let us start with the class of functional-differential equations for some function $f(x,t)$ and coefficients $u_m(x,t)$ belonging to the ring,

$$f_t(x,t) = \sum_{m=-M}^{N} u_m(x,t)T^m(f) \ , \qquad t \in \mathbb{R} \ ,$$

where T is an automorphism. This equation is covariant with respect to the DT:

$$D^{\pm}f = f - \sigma^{\pm}T^{\pm 1}f,$$

with $\sigma^{\pm} = \varphi[T^{\pm 1}(\varphi)]^{-1}$. It is possible to reformulate the result for differential-difference or difference-difference equations and give the explicit expressions for the transformed coefficients [321]. From this result, the lattice and q-deformation DTs for matrix-valued functions follow in a straightforward way:

$$T(f)(x,t) = f(x+\delta,t) \ , \qquad x,\delta \in \mathbb{R}$$

or

$$T(f)(x,t) = f(qx,t) \ , \qquad x,q \in \mathbb{R} \ , \qquad q \neq 0 \ .$$

It is sufficient to take the limit

$$\sigma f - f_x = \lim_{\delta\to 0} \frac{1}{\delta} D^{\pm} f = \lim_{q\to 1} (x - xq)^{-1} D^{\pm} f$$

to reproduce the formalism in the case of classical differential operators [321].

The general form of the DT permits us to incorporate the Combesqure and Levy transforms of *conjugate nets* in classical differential geometry [138], as well as the vectorial DTs for *quadrilateral lattices* [128, 307].

Being the covariance transformation, the DT can be iterated and this thus constitutes an important feature of the dressing procedure. The result of the iterations is expressed through determinants of the *Wronskian* type [94]. The universal way to generate the iterated transforms for different versions of the DT including those containing integral operators is given in [324]; e.g., the Abelian lattice DT results in the *Casorati determinants* [314, 322].

The DT theory is strictly connected with the problem of the factorization of differential and difference T operators [271] and hence with the technique of symbolic manipulations [298, 429, 431]. Namely, let $Q^{\pm} = \pm D + \sigma$ and

$$H^{(0)} = -D^2 + u = Q^- Q^+ , \qquad H^{(1)} = Q^+ Q^- = -D^2 + u[1] .$$

The operators $H^{(i)}$ play an important role in quantum mechanics as the one-dimensional energy operators. The *spectral parameter* λ stands for the energy and the relation $Q^+Q^-(Q^+\psi_\lambda) = \lambda(Q^+\psi_\lambda)$ shows the property of DTs $Q^{\pm}$ to be the *ladder operators*. The majority of explicitly solvable models of quantum mechanics are connected with those properties that allow us to generate new potentials together with eigenfunctions [190, 214, 324]. The operator of the DT deletes the energy level that corresponds to the solution φ. Conversely, the inverse transformation adds a level. So, there is a possibility to manage the spectrum by a sequence of DTs. The intertwining relation $H^{(1)}Q^+ = Q^+H^{(0)}$ gives rise to *supersymmetry algebra* that is an example of infinite-dimensional *graded Lie algebras* or, more generally, the *Kac–Moody algebras*. The Moutard transformation is a map of the DT type: it connects solutions and potentials of the equation

$$\psi_{xy} + u(x, y)\psi = 0 ,$$

so that if φ and ψ are different solutions, then the solution of the twin equation with $\psi \to \psi[1]$ and $u(x, y) \to u[1](x, y)$ can be constructed solving the system

$$(\psi[1]\varphi)_x = -\varphi^2(\psi\varphi^{-1})_x , \quad (\psi[1]\varphi)_y = \varphi^2(\psi\varphi^{-1})_y .$$

The transformed potential is given by

$$u[1] = u - 2(\log\varphi)_{xy} = -u + \varphi_x\varphi_y/\varphi^2$$

together with the transformation of the wave function

$$\psi[1] = \psi - \varphi\Omega(\varphi, \psi)/\Omega(\varphi, \varphi) ,$$

where Ω is the integral of the exact differential form

$$\mathrm{d}\Omega = \varphi\psi_x \mathrm{d}x + \psi\varphi_y \mathrm{d}y \ .$$

The Moutard equation, by a complexification of independent variables, is transformed to the two-dimensional Schrödinger equation and studied in connection with problems of classical differential geometry [242]. In the *soliton theory* it enters the *Lax pairs* for some (2+1)-dimensional nonlinear equations [3, 58]. Another generalization of the Moutard transformations leads after iterations to multidimensional Toda-like lattice models [435]. Note that there is a possibility of local approximation of solutions by a sequence of Moutard and Ribacour transformations [170]. Other applications of the DT theory in multidimensions can be found in [26, 228, 278, 281, 287, 277]. A useful chronological survey of DTs, intertwining relations, and the factorization method is given by Rosu [377].

A wide class of geometrical ideas and particular results of *soliton surfaces* [417] in real semisimple Lie algebras is connected with the concept of the *Darboux matrix* that seems to be the most "Darboux-like" approach in the whole of DT theory. Note also in this connection the application of the DTs in vortex and relativistic string problems initiated by the paper of Nahm [344].

In searching for alternative formulations of the method containing the principal ideas of the Darboux approach, the so-called elementary DT [279] on a *differential ring* was introduced [467] . Its particular case that does not depend on solutions (only on potentials) is referred to as the *Schlesinger transformation* [389, 467]. The elementary DT in combination with a conjugate to it generates a new transformation. This construction was named the *binary DT* in [267, 270, 281]. Such a name intersects with the notion introduced in [317]; for details, see [324]. Therefore, we use the new term of *twofold elementary DT* throughout this book. This transformation strictly realizes the *dressing procedure* for solutions of integrable nonlinear equations. Namely, the twofold elementary DT solves the matrix RH problem with zeros.

One of the main purposes in introducing the concept of the twofold DT directly concerns the problem of reductions [331]. The properties of the Zakharov–Shabat (ZS) spectral problem and its conjugate give the possibility to establish a class of reductions by solving the simple conditions for parameters of the elementary DTs which comprise the twofold combination [279, 280, 434]. The symmetric form of the resulting expressions for potentials and wave functions make almost obvious the heredity of reduction restrictions [281] and underlying authomorphisms [181, 331, 361] of the generating ZS problem. In [276] an application to some operator problem (Liouville–von Neumann equation) is studied. Examples of transformations of different kinds and in different contexts were introduced in [317] (see again [324]) under the name "binary." The binary transformations in [317, 324] are a 2+1 construction based on alternative Lax pairs. This is a combination of the classical DTs for the time-dependent Schrödinger equation and a special one for a conjugate problem. Combinations of twofold elementary DTs were used

to obtain multisolitons and other solutions of the three-level Maxwell–Bloch equation [279]. A natural generalization of this construction consists in replacing matrix elements by appropriate matrices. The most promising applications of the technique are related to operator rings. Such an example was considered in [267].

As regards the RH problem, its application to the study of spectral equations goes back to the 1975 paper by Shabat [394], though Zakharov and Shabat [473] in their classic paper used in fact a formalism closely related to that of the RH problem. A status of the "keystone" of the soliton theory was acquired by the RH problem as a result of the 1979 paper by Zakharov and Shabat [475]. The next important step is associated with Manakov [305], who put forward a concept of the *nonlocal* RH problem. This idea turned out to be very profitable for integration of (2+1)-dimensional nonlinear equations (and some integro-differential equations in 1+1 dimensions as well). In addition to the results described in the aforementioned monographs, mention should be made of more recent papers devoted to the application of the RH problem to the soliton theory. This includes integration of equations associated with more complicated spectral problems than the ZS one (e.g., the modified Manakov equation [125] and the Ablowitz–Ladik equation [122, 185]). Results of principal importance were obtained by Shchesnovich and Yang [400, 401], who derived a novel class of solitons in 1+1 dimensions that corresponds to higher-order zeros of the RH problem data. The soliton solutions associated with multiple-pole eigenfunctions of the spectral problems for (2+1)-dimensional nonlinear equations were obtained by Ablowitz and Villarroel [14, 439, 440].

The RH problem has been proved to be efficient for analysis of nearly integrable systems as well as when solitons are subjected to small perturbations. The soliton perturbation theory has been elaborated on the basis of the RH formalism in a number of papers [122, 123, 237, 398, 397, 399]. A connection between the RH problem and the approximation theory and random matrix ensembles is demonstrated in [113], where the steepest descent analysis for the matrix RH problem was performed, and in [160], where the matrix RH problem was associated with the problem of reconstructing orthogonal polynomials. A closely related area of problems focuses on finding the semiclassical limit of the N-soliton solution for large N [302, 333].

As is known, solving the RH problem amounts to reconstructing a sectionally meromorphic function from a given jump condition at some contour (or contours) of the domains of meromorphy and discrete data given at the prescribed singularities. Studying some nonlinear equations in 2+1 dimensions reveals a situation when we cannot formulate the RH problem because of the absence of domains of meromorphy. In other words, functions we work with are nowhere meromorphic. Beals and Coifman [41] and Ablowitz et al. [1] invoked a new tool for studying nonlinear equations, the $\bar{\partial}$ problem, which amounts to overcoming the difficulty with meromorphy. The *$\bar{\partial}$-dressing method* constitutes now a true foundation of the soliton theory. As the latest development of the $\bar{\partial}$-dressing formalism, a derivation of the quasiclassical limit of the

scalar nonlocal $\bar{\partial}$-dressing problem should be mentioned [245]. Besides, the $\bar{\partial}$ problem with conjugation has been analyzed within the dressing approach by Bogdanov and Zakharov [57].

The book is organized as follows. We begin in Chap. 1 with the introduction of some mathematical notions used throughout the book. This chapter reviews concisely the operator technique that can be considered as one of the sources of the dressing ideas. We discuss its origin in Lie algebra theory and applications in quantum mechanics (creation–annihilation operators, angular momentum, and spin theory), as well as in classical mechanics in the Poisson representation. We also give the main definitions and results concerning the RH boundary-value problem, both scalar and matrix, and the $\bar{\partial}$ problem.

The other important idea of the dressing technology goes back to factorization of differential and difference operators discussed in Chap. 2. The story of the factorization of operators of linear equations starts perhaps from the classic papers by Euler [147] and Jacobi [218] (see the historical essay in [52]). We present here a rather general construction of the factorization [467], necessary from the point of view of the dressing theory. Of course, the result of a right/left division of the differential operators strongly depends on the ring/field used in the construction, but the link between factors and the eigenstates is universal. To explain the thesis, note that the factorization of the second-order differential operator produces the DT by the operator $L_\sigma = (D - \sigma)$ [324]. The factorization of $L = (-D - \sigma)(D - \sigma) = L_\sigma^+ L_\sigma$ yields a new operator $L[1] = L_\sigma L_\sigma^+$ that is *intertwined* with L:

$$L[1]L_\sigma = L_\sigma L\,. \tag{0.1}$$

This relation is the basis of the algebraic dressing procedure, when applied to some eigenstate of L. The theory was developed in [102] in connection with applications in geometry [103]; it has been attracting more and more attention from researchers since its introduction (for many developments, see [197, 376]).

We elaborate a compact form of the solution of the factorization problem by introducing special (Bell) polynomials for a general non-Abelian case. It gives a direct link to the DT derivation, a covariance theorem formulation, and proof. Some examples complementary to those used in the books mentioned are demonstrated. A natural connection with supersymmetry is shown.

In Chap. 3 we introduce a general non-Abelian version of the elementary and twofold elementary DT constructed by means of an arbitrary number of orthogonal projectors p_i. The order of the elements in determining the equations is therefore essential. The resulting expressions for transformations may be represented both in general operator form and by means of "matrix elements" $x_{ik} = p_i x p_k$ of the ring element x (x stands for either a potential or a solution of the linear problem).

A comparison with the relations originating from the matrix RH problem with zeros demonstrates the possibility to generate the projectors that connect solutions of the RH problem in a simple algebraic way. More detailed

exposition of this subject is given in Chap. 8. Moreover, for the same reason, the limiting procedures may be explicitly performed without any reference to analytic properties of the entries. Note that there are lots of other (advanced in comparison with twofold) possibilities to combine elementary DTs as well as to use them directly. It is shown how the non-Abelian geometry is induced by the DT on a differential ring.

In the last part of Chap. 3 we study a generalization of the theory of small deformations of iterated transforms with respect to intermediate parameters that appear within the iteration procedure of twofold elementary DTs. The perturbation formulas allow us to define and investigate generators of the corresponding group, being a symmetry group of a given hierarchy associated with the ZS problem. Then we give examples that generalize the N-wave system as a zero-curvature condition of an appropriate pair of the ZS problems. This case is chosen to show the importance of this approach in both geometry and applied mathematics, with a perspective to apply the DT theory to computations of eigenfunctions and eigenvalues.

The nontrivial development of methods aimed at solving spectral problems and nonlinear equations is associated with dressing chain equations produced by iterated DTs (Chap. 4). It is first of all a link of the DT theory to the finite-gap potentials (also as solutions of integrable equations) and to the investigation of asymptotic behavior. The role of the complete set of the DT-covariance conditions (the so-called Miura maps) is studied. As the new object, t-chains are constructed and superposed with the x-chains in 1+1 dimensions.

In Chap. 5 we show in detail recent results on integrable nonlinear equations in two space and one time variables that could be solved by the Moutard-like and the Goursat-like transformations. We use examples of (2+1)-dimensional Boiti–Leon–Manna–Pempinelli and Boiti–Leon–Pempinelli equations. The asymptotic formulas for the multikink solutions are analyzed.

Chapter 6 is devoted to applications of the dressing method to linear problems of quantum and classical mechanics, exemplifying thereby the "inverse" influence of the nonlinear theory on the linear one. We briefly review exactly solvable quantum-mechanical problems on a line with potentials from the review paper by Infeld and Hull [214] subjected to algebraic deformations. Next we report results concerned with the radial Schrödinger equation and treat via the dressing procedure the popular model of zero-range potentials. In particular, we dress the zero-range potentials and consider the dressing of scattering data. Considering the DT that preserves a potential, we can conclude about the spectrum and eigenfunctions of the spectral problem. Going to the problem of dressing of differential equations with matrix coefficients, we show links to relativistic quantum equations. Some classical wave and heat-conduction equations can be solved by the Green function constructed via the dressing procedure. For the classical n-point system, we can associate the Poisson bracket with a differentiation, which leads to the possibility to treat the dressing of classical evolution as a generalized DT.

In Chap. 7 we connect the dressing method with the Hirota formalism. We also explain how to construct in a general way Bäcklund transformations proceeding from the explicit form of the DT. One more aspect of the dressing theory appears within the Weiss–Tabor–Carnevale procedure of Painlevé analysis for partial differential equations. We derive DT formulas using the singular manifold method. At the end of this chapter we comment on the historical point connected with the appearance of the dressing method in the ZS theory and suggest some revision of the technique.

The last three chapters deal with a realization of the dressing approach in terms of complex analysis. In Chap. 8 we apply the local RH problem for finding soliton (and some other) solutions of (1+1)-dimensional nonlinear integrable equations. The distinctive feature of the formalism used is the vector parameterization of the discrete spectral data of the RH problem. Such a parameterization arises naturally within the RH problem. Using an example of the classical nonlinear Schrödinger equation, we demonstrate in detail the dressing of the bare (trivial) solution which leads to the soliton. Each subsequent section in this chapter demonstrates a new peculiarity in the application of the matrix RH problem. Besides, our formalism turns out to be efficient to obtain another class of solutions associated with the notion of *homoclinic orbits* which arise in the case of periodic boundary conditions. The last section contains the description of the well-known Korteweg–de Vries (KdV) equation. A purpose of this section is rather methodological: we discuss the KdV equation in the manner most suitable for treating in the next chapter nonlinear equations in terms of the nonlocal RH problem. We hope the content of this chapter is useful to newcomers as a concise introduction to the modern machinery of the theory of solitons.

Dressing by means of the nonlocal RH problem is the main topic of Chap. 9. We consider three featured examples: the Benjamin–Ono (BO) equation, the Kadomtsev–Petviashvili I (KP I) equation, and the Davey–Stewartson I (DS I) equation. Despite the fact that all these equations are well known, most of the results of Chap. 9 cannot be found in monographic literature. Namely, for the BO equation we pose the reality condition from the very beginning and account for important reductions in the space of spectral data. For the KP I equation we describe a class of localized solutions which arise from the eigenfunctions with multiple poles. The consideration of the DS I equation is more traditional and aims to demonstrate peculiarities which occur when using the matrix nonlocal RH problem.

Finally, Chap. 10 is devoted to the description of the $\bar{\partial}$ method, as applied to nonlinear integrable equations. First we develop in detail the technique, which is based on a rather unusual symbolic calculation, and prove its efficiency. We apply this formalism for the analysis of nonlinear equations with a self-consistent source (or with a nonanalytic dispersion relation) both in 1+1 and in 2+1 dimensions. The classic example of equations with a self-consistent source is the Maxwell–Bloch equation. Following our approach, we obtain the main results concerning the Lax pairs, the recursion operators,

gauge-equivalent counterparts, and so on. The KP II equation was historically the first one to be successfully analyzed by means of the $\bar{\partial}$ formalism. We briefly outline the main steps of such an analysis. The DS II equation is considered in more detail. In particular, we describe a recently developed method aimed at incorporating multiple-pole eigenfunctions for generating a new class of localized solutions.

Some words about possible linkages of our book with those recently published and devoted to similar subjects are in order. The part devoted to the DT theory is complementary to the book of Matveev and Salle [324]. We include mostly the results obtained after their book was published. We also avoided discussing matters dealt with in the book of Rogers and Schief [376] and the quite new book of Gu et al. [197] where the geometrical problems are discussed from the scope of the Darboux approach. We almost do not touch classical one-dimensional integrability discussed in the books of Perelomov [366, 367].

We are very grateful to our colleagues Pilar Estévez, Nadya Matsuka, Yury Brezhnev, Marek Czachor, Vladimir Gerdjikov, Maciej Kuna, Franklin Lambert, Vassilis Rothos, Mikhail Salle, Valery Shchesnovich, Johann Springael, Nikolai Ustinov, Rafael Vlasov, Jianke Yang, Artem Yurov, and Anatoly Zaitsev for fruitful collaboration and exciting discussions. We are also indebted very much to Vladimir Matveev for valuable criticism and friendly recommendations. Some figures were kindly provided by Robert Milson and Javier Villarroel. E.V.D. is particularly thankful to the eJDS service of the Abdus Salam International Centre for Theoretical Physics (Trieste) for information support. Of course, we are greatly indebted to our wives Tania and Ania. They offered us encouragement and support when we needed it most and never failed to remind us that there is more to life than the dressing method and solitons.

Evgeny V. Doktorov

Sergey B. Leble

1

Mathematical preliminaries

In this chapter we sketch the basic mathematical notions used in this book, starting from general relations and illustrating them by the simplest examples. We also briefly review the ideas of the dressing from the viewpoint of intertwining relations under the scope of Lie algebras [151]. There is a long history of the applications of (semisimple) Lie algebras for determination of operator spectra. One line dating back to Weyl [450] relates to the explicit algebraic solution of an eigenvalue problem; an overview has been given by Joseph and Coulson [223, 224, 225]. Perhaps the best known example of such a construction is the quantum theory of angular momentum, including its development for many-particle systems (from three particles to aggregates) in terms of hyperspherical harmonics [154, 456]. The good old geometry of surfaces and conjugate nets uses the Laplace equations and transformations as a starting point [138]. The challenging problem of the Laplace operator factorization, perhaps first addressed by Laplace, created something like an "undressing" procedure which, being cut at some step, leads to the complete integrability. The direct attempt to extend the technique of the Laplace transformations and invariants to higher-order operators was made in [264]. In [405] this technique was generalized under the name of the Darboux integrability including nonlinearity up to the first derivatives. The search is still going on; see the very recent paper of Tsarev [431]. It is not yet the Darboux transformation (DT) but it is precisely in this way that Moutard [340, 341] found its transference.

Then we are concerned with the modern development of the determinant theory related to non-Abelian rings. It appears under the name of quasideterminant [174]. Quasideterminants defined for matrices over free skew-fields are not an analog of the commutative determinants but rather of a ratio of the determinant of $n \times n$ matrices to the determinants of $(n-1) \times (n-1)$ submatrices. Such a definition is natural for the Darboux dressing. In the last two sections we give basic notions of the Riemann–Hilbert (RH) problem and $\bar{\partial}$ problem which will be used in chapters devoted to solving soliton equations.

1.1 Intertwining relation

We start from the notion of *intertwining relation*. Let us consider three operators L, L_1, and A, denoting $D(L)$, $D(L_1)$, and $D(A)$ their domains of definition. Consider the equality

$$L_1 A = AL, \tag{1.1}$$

named as an intertwining relation.

Proposition 1.1. *Generally, if*

$$L\psi = 0, \qquad \psi \in D(A), \tag{1.2}$$

then

$$L_1(A\psi) = 0. \tag{1.3}$$

In other words, the operator A maps a solution of (1.2) onto a solution of (1.3), if $A\psi \neq 0$ and $A\psi \in D(L_1)$. The case of $A\psi = 0$ means that ψ belongs to the kernel of the operator A.

Consider next an eigenvalue problem for the operator L which acts in a Hilbert space $\mathbb{H}$:

$$L\psi = \lambda\psi, \qquad \psi \in \mathbb{H}. \tag{1.4}$$

Then, owing to (1.1), $L_1(A\psi) = AL\psi = \lambda(A\psi)$. This means that the map $A : \psi \to \psi_1$, $\psi_1 = A\psi$ links eigenspaces of operators L and L_1, leaving eigenvalues unchanged. If $A\psi \in \mathbb{H}$ for any λ and ψ, the operator A is referred to as an isospectral transformation.

Remark 1.2. If for some ψ, $A\psi = 0$, then the eigenvalue λ of A does not belong to the spectrum of A_1.

Remark 1.3. If the operator L is factorizable, i.e., $L = SA$, then A intertwines L and

$$L_1 = AS. \tag{1.5}$$

For Hermitian L we have $S = A^+$, A^+ is a Hermitian conjugate to A, i.e., the intertwining relation takes place automatically for $L_1 = AA^+$.

Given an operator algebra, we can derive comprehensive statements about eigenvalues and eigenstates of operators. The important example of such a construction (ladder operators) is given in the following section.

1.2 Ladder operators

Dressing by means of ladder operators is perhaps the most familiar example of generating new solutions from the seed one. In this section we recall the definition of ladder operators, discuss their Hermitian properties, and demonstrate the diagonalization of the model Jaynes–Cummings (JC) Hamiltonian by means of a unitary dressing operator.

1.2.1 Definitions and Lie algebra interpretation

The concept of *ladder operators* is widely used; they are discussed in [223, 224, 225], where their self-adjoint version is reviewed. Let us start from the commutation relations

$$[M, A^+] = A^+, \qquad [M, A^-] = -A^-, \tag{1.6}$$

where A^+ and A^- are mutually adjoint operators. The link to the factorization method (Chap. 2) is immediately seen. Rewriting, for example, the first relation in (1.6) as $MA^+ = A^+(M+1)$, one can easily check that $MA^+A^- = A^+A^-M$. So, the operators M and A^+A^- commute; hence, spectral problems for both can be considered together and there exists a link between the spectral parameters [80]. Such a property is often referred to as supersymmetry [204].

The important link to the Lie algebra representation theory can be illustrated by the simplest example. The algebra $su(1,1)$ is generated by (1.6) and

$$[A^-, A^+] = 2M. \tag{1.7}$$

The Casimir operator C is constructed as the second-order Hermitian operator

$$C = M^2 - \frac{1}{2}(A^-A^+ + A^+A^-), \tag{1.8}$$

whose eigenvalues are equal to $k(k-1)$ for the unitary irreducible representations. This set defines the representation [positive discrete series $D^+(k)$]

$$M|m,k> = (m+k)|m,k>, \tag{1.9}$$

$$\begin{aligned} A^+|m,k> &= \sqrt{(m+1)(m+2k)}|m+1,k>, \\ A^-|m,k> &= \sqrt{m(m+2k-1)}|m-1,k>, \end{aligned} \tag{1.10}$$

where $m = 0, 1, 2, \ldots$. The operators $A^\pm$ act as lowering and raising ones for m.

Generally the ideas expressed by relations (1.7)–(1.10) are used in the Cartan–Weyl representation theory of Lie algebras [205].

1.2.2 Hermitian ladder operators

The operators in (1.6), being mutually adjoint, cannot be Hermitian; however, some modification of the theory is possible as mentioned in the previous subsection in connection with [223, 224, 225].

Considering the same example with the operators M, A^+, and A^-, let us define two matrix operators $\widetilde{M}$ and $\widetilde{A}$:

$$\widetilde{M} = \begin{pmatrix} M & 0 \\ 0 & -M \end{pmatrix}, \qquad \widetilde{A} = \begin{pmatrix} 0 & A^- \\ A^+ & 0 \end{pmatrix}. \tag{1.11}$$

Both operators are Hermitian by definition and their anticommutator is

$$[\widetilde{M}, \widetilde{A}]_+ \equiv (\widetilde{M}\widetilde{A} + \widetilde{A}\widetilde{M}) = -\widetilde{A}, \tag{1.12}$$

where the intertwining relation is also recognized: $\widetilde{A}$ intertwines $\widetilde{M}$ and $\mathbb{1} - \widetilde{M}$.

With every eigenfunction ψ_m of the operator M, a pair of eigenfunctions $\psi_m^{(a)}$ and $\psi_m^{(b)}$ of the operator $\widetilde{M}$ can be associated. The space of eigenfunctions of this operator is decomposed into a direct sum of two subspaces designated by (a) and (b). The functions are spinorlike vectors,

$$\psi_m^{(a)} = \begin{pmatrix} \psi_m \\ \\ 0 \end{pmatrix}, \qquad \psi_m^{(b)} = \begin{pmatrix} 0 \\ \\ \psi_m \end{pmatrix}, \tag{1.13}$$

that give solutions to the eigenvalue problems

$$\widetilde{M}\psi_m^{(a)} = m\psi_m^{(a)}, \qquad \widetilde{M}\psi_m^{(b)} = -m\psi_m^{(b)}. \tag{1.14}$$

Applying the operator $\widetilde{A}$ to the eigenfunctions (1.13) and taking into account the anticommutator (1.12) yields

$$\widetilde{A}\psi_m^{(a)} \sim \psi_{m+1}^{(b)}, \qquad \widetilde{A}\psi_{m+1}^{(b)} \sim \psi_m^{(a)}.$$

Hence, we encounter again the ladder operators which move the eigenfunctions from one subspace to the other, either increasing or decreasing the eigenvalue m by 1.

Consider an example in which the Hermitian ladder operators appear. Let M be the operator of the projection of the angular momentum on the z-axis:

$$M = L_z. \tag{1.15}$$

Then we take

$$A^{\pm} = L_x \pm \mathrm{i}L_y. \tag{1.16}$$

As a consequence, the operators $\widetilde{M}$ and $\widetilde{A}$ are written in the form

$$\widetilde{M} = \begin{pmatrix} L_z & 0 \\ \\ 0 & -L_z \end{pmatrix} = \sigma_z L_z \tag{1.17}$$

and

$$\widetilde{A} = \begin{pmatrix} 0 & L_x - \mathrm{i}L_y \\ L_x + \mathrm{i}L_y & 0 \end{pmatrix} = \sigma_x L_x + \sigma_y L_y. \tag{1.18}$$

Here σ_x, σ_y, and σ_z are the Pauli matrices

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_y = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.19}$$

The operators $A^{\pm}$ are often rather complicated, while the Hermitian ones $\widetilde{A}$ and $\widetilde{M}$ are simpler. The applications studied in [223, 224, 225] cover many quantum problems, from the harmonic oscillator to the relativistic Kepler problem; see also more recent papers [107, 301, 379, 457].

1.2.3 Jaynes–Cummings model

One more example of the dressing technique with the use of ladder operators is concerned with the multimode JC model [204, 221, 300, 402].

Many phenomena of matter–radiation interaction can be described by a model of (nearly) resonant interaction of a linearly coupled quantum radiation field and a two-level atomic system. The corresponding Hamiltonian is written as

$$H = \omega_0 S_z + \sum_k \omega_k a_k^+ a_k + \sum_q [\epsilon_q a_q^+ S_- + \epsilon_q^* a_q S_+], \tag{1.20}$$

where ω and ϵ_q are c-numbers. The creation and annihilation operators a_p^+ and a_q obey the standard commutator algebra $[a_q, a_p^+] = \delta_{pq}$. The atomic spin matrices S_z, S_+, and S_- are expressed by the Pauli matrices (1.19) as

$$S_z = \sigma_z, \qquad S_{\pm} = \sigma_x \pm \mathrm{i}\sigma_y.$$

This canonical approach has been previously applied to the original JC model, and has proved itself to be much more effective than the algebraic approach in elucidating the physical origin of the dressing processes [82]. In [299, 300] the dressing was applied for the two-photon-interaction Hamiltonian

$$H = \omega_0 S_z + \sum_k \omega_k a_k^+ a_k + \sum_{k,q} [\epsilon_{k,q} a_k^+ a_q^+ S_- + \epsilon_{k,q}^* a_k a_q S_+]. \tag{1.21}$$

This Hamiltonian, like the original one-mode JC one (1.20), is diagonalizable by a transformation in the operator space. The eigenstates are known as states of a *dressed* atom, a new physical object.

The ladder operators

$$A^+ = \frac{1}{2} \sum_{k,q} \alpha_{k,q} a_k^+ a_q^+, \quad A^- = \frac{1}{2} \sum_{k,q} \alpha_{k,q}^* a_k a_q, \quad M = \frac{1}{4} \sum_k (2a_k^+ a_k + 1) \tag{1.22}$$

with the commutators (1.6) and (1.7) generate the Lie algebra $su(1,1)$. The corresponding group elements can be used to define generalized coherent states [366]. If the coefficients in the last term of the Hamiltonian (1.21) are related to $\alpha_{k,q}$, $\epsilon_{k,q} = \epsilon\alpha_{k,q}$ with real ϵ, it could be rewritten as

$$H = \omega_0 S_z + 2\omega(M - \frac{m}{4}) + V, \tag{1.23}$$

where m is a boson mode number and

$$V = 2\epsilon(A_+ S_- + A_- S_+).$$

This Hamiltonian is reduced to the familiar two-mode two-quantum JC Hamiltonian [221] if $\alpha_{k,q} = 1 - \delta_{k,q}$:

$$H_{2\mathrm{JC}} = \omega_0 S_z + \omega(a_k^+ a_k + a_q^+ a_q) + \epsilon(a_k^+ a_q^+ S_- + a_k a_q S_+). \tag{1.24}$$

The Hamiltonian (1.21) is diagonalized by the dressing unitary operator

$$T = \exp\left(\frac{\theta}{2\beta}(A_+ S_- - A_- S_+)\right), \tag{1.25}$$

where the parameters θ and β are defined by

$$\tan\theta = -\frac{4\epsilon\beta}{2\omega - \omega_0}, \quad \beta = \sqrt{(N - 1 + m/4)(N + m/4) - C},$$

where C is the Casimir operator.

In the expression for the β, there is an eigenvalue N of the operator

$$\aleph = (M - m/4) + S_z + 1/2$$

that commutes with both H_0 and V and, owing to (1.22), has the sense of the total numbers of excitations.

The other (group theoretical) aspect of this approach is connected with a pioneering paper by Fock [153] reviewed in [150] and generalized in [139] to a nonlinear case.

1.3 Results for differential operators

The greatest part of this book is directed to operator algebra aspects. We consider mainly the so-called correspondences [165] of operators polynomial in the differentiation and shift operators with matrix coefficients. The particular case of $L = L_1$ in (1.1) means simply that L and A commute. Such a "zero" level of the study from the point of view of the intertwining relations was initiated by Burchnal and Chaundy [80].

1.3.1 Commuting ordinary differential operators

Consider two commuting operators P_m and Q_n, polynomial in the differentiation operator D and having finite-dimensional eigenspaces. There exists a common solution η of these equations:

$$(P_m - \lambda)y = 0, \qquad (Q_n - \mu)y = 0. \tag{1.26}$$

Eliminating $m + n$ derivatives y, $Dy, \ldots$, $D^{m+n-1}y$ in $m + n$ linear homogeneous equations

$$D^r(P_m - \lambda)y = 0, \qquad r = 0, 1, \ldots, n-1$$

and

$$D^s(Q_n - \mu)y = 0, \qquad s = 0, 1, \ldots, m-1$$

yields the connection (spectral curve) $f(\lambda, \mu) = 0$. Analysis of terms with the highest order in λ and μ and evaluation of the commutator $P_mQ_n - Q_nP_m$ that contains powers of D from "0" to "$m+n-1$" lead to the conclusion that the highest-order term of $f(\lambda, \mu)$ in λ is n and in μ is m.

The operator $f(P_m, Q_n)$ maps to zero any linear combination of eigenfunctions η_i of different eigenvalues of P; hence, it is identically zero. Otherwise, the relation $f(P_m, Q_n) = 0$ is fundamental. It could be shown that the inverse statement is valid if m and n are *interprime*, i.e., when the highest (in D) order of $aQ_n^m - bP_m^n$ is equal to mn.

The algebraic construction of the "dressing" type was investigated in [80]. The existence of common solutions of (1.26) implies the existence of an operator T such that the common solutions form the kernel of T; hence, the operators factorize, or $P - h \equiv RT$ and $Q - k \equiv ST$. The commutativity of P and Q yields $RTST = STRT$ and the operator R, say, intertwines TS and ST, i.e., $RTS = STR$, or S intertwines RT and TR. Such a phenomenon is called a transference of the common factor because the operators $P' = TR$ and $Q' = TS$ commute. In Chap. 2 this phenomenon is used to introduce an analog of the classical DT. By the transference of a new common factor (T_1), one gets $P_1 - h_1 \equiv T_1R_1$, generating a sequence of operators.

Remark 1.4. The characteristic identity $f(P, Q) = 0$ is invariant with respect to the transform $P \to P', \quad Q \to Q'$.

The proof is based on the intertwining relation

$$Tf(P, Q) = f(P', Q')T,$$

which follows directly from the expansion $f(P, Q) = \sum_{rs}(P - h)^r(Q - k)^s$ after substitution of the factorized form of P and Q.

Investigation of a general form of commuting operators begins with the standard form that is obtained by a gauge transformation with a function θ:

$$\widetilde{P} = \theta^{-1} P \theta, \qquad \widetilde{Q} = \theta^{-1} Q \theta,$$

which leads to a unit leading coefficient and to a constant second coefficient. Irreducibility of P and Q means that these operators are not represented as functions of other commuting operators. The fact that $Q - P^r$ commutes with P if $n = mr$ excludes combinations of multiple orders of D; hence, the least possible order is $m = 2$, $m > n$, $m \neq nr$ and the minimal noncommutative duet comprises the operator P of the order 2 and the operator Q of the order 3.

The case $m = 2$ is connected with the famous relation to stationary solutions of the Korteweg–de Vries equation [353]. Here

$$P = D^2 + u. \tag{1.27}$$

The operator Q should be of odd order:

$$Q_{2n+1} = D^{2n+1} + \beta_2 D^{2n-1} + ... + \theta_n. \tag{1.28}$$

The commutativity condition yields the recurrent formula for θ_n:

$$\theta_{n+1} = \theta_n'''/4 + \theta_1' \theta_n + 2\theta_1 \theta_n', \tag{1.29}$$

which leads to hyperelliptic functions. For example, for $m = 2$ and $2n + 1 = 3$ the differential condition which provides commutativity gives $\theta_{n+1}' = 0$. From (1.29) we have

$$\theta_1'''/4 + 3\theta_1' \theta_1 = 0.$$

Integration gives $\theta'^2 + 4\theta_1^3 = g_1 \theta_1 - g_3$. As a result, the explicit expressions for P and Q are obtained in terms of the Weierstrass function $\wp(x)$:

$$P = D^2 - 2\wp(x), \qquad Q = D^3 - 3\wp(x) D - 3\wp'(x)/2.$$

1.3.2 Direct consequences of intertwining relations in the matrix case and multidimensions

A link between the one-dimensional Schrödinger operators $L_0 = -\partial^2/\partial x^2$ and $L = -\partial^2/\partial x^2 + u(x)$ in terms of the intertwining relations was established in [137] for a potential with regular singularities (the order of poles is less than 3), vanishing at infinity. We give here some details following [191].

A monodromy $\psi \to (\partial - \sigma)\psi$, $\sigma = \sigma(x)$ is called trivial if all the solutions of the equation

$$\left(-\frac{\partial^2}{\partial x^2} + u(x) \right) \psi = \lambda \psi$$

are single-valued in the λ-plane. The monodromy is proved trivial iff the operator L is intertwined with L_0 by a finite product of the DTs. This means

that a given operator $A = D^n + ... + a_0$ having the kernel K intertwines operators L and L_0. The proof is based on the property of the operator A (Proposition 1.1) and the statement about division of L_0 by A [191]. Then the theorem about the important property of the potential having only regular singularities generalizes the statement of [137] for the matrix potentials.

The paper [191] describes the matrix-valued potentials $U(x) \in \mathrm{Mat}_d(\mathbb{C})$ of the Schrödinger operator

$$L_U = -\frac{\partial^2}{\partial x^2} + U(x) \tag{1.30}$$

having trivial monodromy. The monodromy operator is expressed via the quasideterminant (see the definition in Sect. 1.9) as

$$A\Psi = |\widehat{W}(\Psi_1, ..., \Psi_n, \Psi)|_{n+1,n+1}. \tag{1.31}$$

The $d \times d$ matrices Ψ_i, $i = 1, \ldots, n$ and Ψ are eigenmatrices of the operator L_0 and the Vandermond supermatrix $\widehat{W}(\Psi_1, ..., \Psi_n, \Psi)$ is defined by its first superrow. Hence,

$$U = U_0 - 2\partial_x(Y_{nn}\widehat{W}_{nn}^{-1}), \tag{1.32}$$

where Y is obtained from $\widehat{W}(\Psi_1, ..., \Psi_n)$ by deletion of the $(n-1)$th superrow. The special case of $U_0 = 0$ yields

$$U(z) = \sum_{i=1}^{N_d} \frac{A_i}{(z - z_i)^2}, \qquad N_d = \deg \det \widehat{W},$$

where the singularities of the potential, owing to (1.32), are the roots of $\det \widehat{W} = 0$, and the matrices A_i have rank 1 in the generic case of the simple roots z_i.

The intertwining relation for a dressing in *multidimensions* in the case of the zero potential was studied from a general mathematical point of view for $x \in \mathbb{C}^n$ [85]. The intertwining relation was used for studying maps between solutions of the Laplace equations with the seed operator

$$L_0 = \Delta = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$$

and the "dressed" operator

$$L_u = \Delta + u(x)$$

with a rational potential u,

$$u(x) = \sum_{\alpha \in S} \frac{m_\alpha(m_\alpha + 1)(\alpha_\perp, \alpha_\perp)}{[\alpha(x)]^2},$$

for $\alpha_\perp = \{a_i\}$, $a_i \in \mathbb{C}$. The set S contains a finite number of hyperplanes $\alpha(x) = (\alpha_\perp, x) = a_0$. Relation (1.1) is considered with $A(x, \frac{\partial}{\partial x})$ and is named as the DT.

The paper [85] formulates the sufficient condition (the results from [47] are used) for the existence of the intertwining operator and describes *all* rational potentials linked to Δ by the intertwining relation.

1.4 Hyperspherical coordinate systems and ladder operators

In the review paper [150] the hyperspherical coordinates [153, 154] are used to describe quantum dynamical evolution of atomic and molecular aggregates, ranging from their compact states to fragments. Using such a type of coordinates is directly related to a generalization of the ladder operators' structure to many degrees of freedom.

In this approach, in contrast to the traditional independent-particle theory [304], a quantum-mechanical multiparticle problem is parameterized by the single collective radial parameter. The hyperradius

$$R = \left(\sum_{i=1}^{N} \frac{M_i r_i^2}{M} \right)^{1/2}, \qquad M = \sum_{i=1}^{N} M_i$$

serves as the basic aggregate coordinate of a wave function. Subsequent separation of variables results in the equation for hyperspherical harmonics which is a product of the Legendre functions of subaggregate angle variables and the Jacobi function.

The invariance of the total kinetic energy under multidimensional rotations hints at an analogy with the angular momentum theory and hence with ladder operators, new eigenvalues, and new quantum numbers. A new basis is formed by hyperspherical harmonics constructed by a kind of the "laddering" or, as we call it here, the purely algebraic dressing procedure [456], without solving differential equations.

A corresponding theory follows from the angular momentum theory of quantum mechanics, where the components L_x, L_y, and L_z of one-particle angular momentum combine to reproduce the ladder operators' algebra as in (1.15) and (1.16). In spherical coordinates, $l_3 = L_z = -\mathrm{i}(\partial/\partial\phi)$, $l_\pm = L_x \pm \mathrm{i}L_y$. Generalizing to the case of arbitrary number of particles, we take

$$l_{zi} = -\mathrm{i}\left(x_i \frac{\partial}{\partial y_i} - y_i \frac{\partial}{\partial x_i} \right) = -\mathrm{i}\frac{\partial}{\partial \phi_i}. \tag{1.33}$$

The celebrated example of the helium atom [154] involves three independent rotations of two Cartesian sets in the center-of-mass system. As a

result, two one-particle operators l_{z1} and l_{z2} are combined with the third ("mixed") one,

$$-\mathrm{i}\left(z_1\frac{\partial}{\partial z_2}-z_2\frac{\partial}{\partial z_1}\right). \tag{1.34}$$

The operators (1.33) and (1.34) commute; hence, they form the Cartan subalgebra $h_i \in \mathcal{H}$ [205] in the Lie algebra of angular-momentum-like operators formed by $x_i(\partial/\partial y_j)$ and $x_i(\partial/\partial x_j)$. The ladder operators in d dimensions

$$J_{ij}^{xy}=-\mathrm{i}\left(x_i\frac{\partial}{\partial y_j}-y_j\frac{\partial}{\partial x_i}\right), \quad J_{ij}^{xx}=-\mathrm{i}\left(x_i\frac{\partial}{\partial x_j}-x_j\frac{\partial}{\partial x_i}\right) \tag{1.35}$$

raise or lower eigenvalues (or "weights", in the nomenclature of the semisimple Lie algebra representation theory) of the operators h_i. In the whole algebra, some commutator relations can be read as eigenvalue problems of the adjoint representation of h_i:

$$\mathrm{ad}(h_i)e_\alpha=[h_i,e_\alpha]=\alpha_i e_\alpha, \qquad i=1,\ldots,l=\dim\mathcal{H}. \tag{1.36}$$

The ladder operators are then the eigenvectors (the Cartan–Weyl basis), and the lowering and raising properties are the direct consequences of (1.36). The relations (1.36) generalize (1.6). So, the hyperspherical harmonics form the basis of a representation of the Lie algebra of rank l.

The transition from the angular vector $\mathbf{R}$ to homogeneous components of the Jacobi coordinates [404] is performed by means of recursive change $\mathbf{r}_i \to \boldsymbol{\xi}_i$. The recursion looks as follows: a single vector

$$\boldsymbol{\xi}_{pq}=\sqrt{\frac{M_pM_q}{M_p+M_q}}(\mathbf{r}_p+\mathbf{r}_q)$$

is formed for a pair of the Jacobi vectors $\boldsymbol{\xi}_p$ and $\boldsymbol{\xi}_q$ of two subaggregates of particles centered at $\mathbf{r}_p$ and $\mathbf{r}_q$ with masses M_p and M_q. The construction is called a timber transformation by association with the Jacobi tree. For the comprehensive outlook, details, and examples, see again [150].

1.5 Laplace transformations

The general (hyperbolic) Laplace equation

$$\phi_{xy}+\alpha(x,y)\phi_x+\beta(x,y)\phi_y+\gamma(x,y)\phi=0$$

goes to

$$\psi_{xy}+a\psi_y+b\psi=0 \tag{1.37}$$

after the gauge transformation

$$\phi = g\psi, \qquad g = \exp\left(\int^x [a(x',y) - \beta(x',y)]dx' - \int^y \alpha(x,y')dy'\right),$$

if the Laplace invariants of both equations have the same values [168], or $-\alpha_x = a_y - \beta_y = b - \gamma + \alpha\beta$. The Laplace transformation (LT) for (1.37) has the form

$$a \to a_{-1} = a - \partial_x \ln(b - a_y), \quad b \to b_{-1} = b - a_y, \quad \psi \to \psi_{-1} = \psi_x + a\psi,$$

$$a \to a_1 = a + \partial_x \ln b, \quad b \to b_1 = b + \partial_y\left(a + \partial_x \ln b\right), \quad \psi \to \psi_1 = \frac{\psi_y}{b}$$

and can be taken as a starting point in the theory of soliton equations in 2+1 dimensions [34, 168]. The LT is also a kind of a dressing procedure; it leads to a "partial" factorization of the operator of (1.37) and in the case of zero *Laplace invariants* at some step of the LT iterations allows us to build explicit solutions.

Important progress in the development of the LT theory was achieved in [431]. Consider the operator

$$L = \sum_{i=0}^{2} p_i D_x^i D_y^{2-i} + a_1(x,y)D_x + a_2(x,y)D_y + c(x,y)$$

with arbitrary $p_i = p_i(x,y)$. Solutions of the *characteristic equation* $m_i^2 p_0 - m_i n_i p_1 + n_i^2 p_2 = 0$ define the first-order characteristic operators $X_i = m_i(x,y)D_x + n_i(x,y)D_y$, which are strictly hyperbolic if the roots are different. The equation $Lu = 0$ can be rewritten in the *characteristic form*

$$(X_1X_2 + \alpha_1X_1 + \alpha_2X_2 + \alpha_3)u = 0, \quad (X_2X_1 + \beta_1X_1 + \beta_2X_2 + \alpha_3)u = 0, \quad (1.38)$$

where $\alpha_i = \alpha_i(x,y)$ and $\beta_i = \beta_i(x,y)$, the coefficients of the first-order characteristic operators X_i, can be found [up to a rescaling $X_i \to \gamma_i(x,y)X_i$]. Since the operators X_i do not commute, we have to take into consideration the *commutation rule*

$$[X_1, X_2] = X_1X_2 - X_2X_1 = P(x,y)X_1 + Q(x,y)X_2. \qquad (1.39)$$

Using the *Laplace invariants* of (1.38),

$$h = X_1(\alpha_1) + \alpha_1\alpha_2 - \alpha_3, \quad k = X_2(\beta_2) + \beta_1\overline{\alpha}_2 - \alpha_3,$$

we represent the original operator L in two partially factorized forms

$$L = (X_1 + \alpha_2)(X_2 + \alpha_1) - h = (X_2 + \beta_1)(X_1 + \beta_2) - k. \qquad (1.40)$$

Each of them allows us to introduce the LT and leads to a factorization if either h or k equals zero. Note also that the equation $Lu = 0$ is equivalent to any of the first-order systems (for this idea see also [34, 35])

$$\begin{cases} X_2 u = -\alpha_1 u + v, \\ X_1 v = hu - \alpha_2 v \end{cases} \Leftrightarrow \begin{cases} X_1 u = -\beta_2 u + w, \\ X_2 w = ku - \beta_1 w. \end{cases} \tag{1.41}$$

Making use of the matrix nomenclature, we read a pair of u and w as columns, introducing the coefficient matrices $\alpha_{ik}, \overline{\alpha}_{ik}$ for the right-hand side of (1.41). The generalization of the classical LT originates from the central idea of swapping the operators X_1 and X_2: after using the commutator (1.39), the coefficients of the first-order operators are changed, which produces the transformed operator L[1]. The explicit procedure is described in [431] and is summarized in the following steps:

1. If we have to solve an equation $Lu = 0$, transform it into the characteristic form (1.41).
2. If the matrix $[\alpha_{ij}(x, y)]$ of the characteristic system is upper- or lower-triangular, solve the equations consecutively.
3. If the matrix is block-triangular, the system *factors* into several lower-order systems; try for each subsystem step 2.
4. In the general case of a nontriangular matrix $[\alpha_{ij}(x, y)]$, perform several (consecutive) generalized LTs , using different choices of the *pivot element* $\alpha_{ik} \neq 0$. The goal is to obtain a block-triangular matrix for one of the transformed systems.

In [405] the general hyperbolic quasilinear equation $u_{xy} = F(x, y, u, u_x, u_y)$ is treated from the Laplace theory point of view. The Laplace invariants H_i are introduced via the recurrence

$$D_x D_y[\log H_i] = H_{i+1} + H_{i-1} - 2H_i, \quad i \in \mathbb{Z},$$

where $D_{x,y}$ are total derivatives and the first terms of the recurrence are [359]

$$H_0 = D_x(F_{u_x}) - F_{u_x} F_{u_y} - F_u, \quad H_1 = D_y(F_{u_y}) - F_{u_x} F_{u_y} - F_u.$$

The recurrence obviously simplifies in the case of (1.37). The following theorem is the result of joint efforts of the authors of [22] and [405].

Theorem 1.5. *A break off of the recurrent sequence at both sides, i.e., $\exists n, m$, such as $H_n = H_m = 0$ means the Darboux integrability, i.e., there exists a pair of functions P and Q on prolonged space such that $P_y = 0$ and $Q_x = 0$.*

The famous example of such a (nonlinear) equation is the Liouville equation $u_{xy} = \exp(u)$. The other one is concerned with the linear equation (1.37).

Recent important results are reported in [431], where a matrix version of the classical LT is given. Let us reproduce the main proposition of [431]:

Proposition 1.6. *Suppose the linear partial differential equation of order* $n \geq 2$,

$$Lu = \sum_{i+j \leq n} p_{i,j}(x,y) D_x^i D_y^j u = 0 \tag{1.42}$$

is strictly hyperbolic, i.e., the characteristic equation $\sum_{i+j=n} p_{i,j} \lambda^i = 0$ *has* n *simple real roots* $\lambda_k(x,y)$. *Any strictly hyperbolic linear partial differential equation (1.42) is equivalent to an* $n \times n$ *first-order system in characteristic form*

$$X_i u_i = \sum_k \alpha_{ik}(x,y) u_k. \tag{1.43}$$

This statement opens the way for a generalization of the Laplace theory by means of the procedure described above by steps 1–3.

The recurrences introduced in this section are a kind of dressing in the sense of a procedure that algebraically connects equations of the same form with different coefficients. It resembles the Schlesinger transformation (degenerate elementary DT) that we will deal with in Chap. 3.

1.6 Matrix factorization

In this section we establish the link between the matrix factorization and the intertwining relations and recall basic facts of matrix factorization in terms of dressing procedures.

1.6.1 Example

It was shown that a factorization (1.5) yields the intertwining relation (1.1) automatically. Taking the simplest example of 2×2 matrices, let us consider a Hermitian matrix L as a product of mutually conjugate matrices A and A^+:

$$L = A^+ A = \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} |a|^2 + |c|^2 & a^* b + c^* d \\ b^* a + d^* c & |b|^2 + |d|^2 \end{pmatrix}.$$

Introducing the column vectors $\psi = \begin{pmatrix} a \\ c \end{pmatrix}$ and $\phi = \begin{pmatrix} b \\ d \end{pmatrix}$, we obtain the following relations for these vectors:

$$|\psi|^2 = L_{11}, \quad |\phi|^2 = L_{22}, \quad (\psi, \phi) = L_{12}.$$

It is easy to check that the $n \times n$ matrix can be factorized in the similar way. For a diagonal matrix L, the scalar product (ψ, ϕ) is zero and one has an orthogonal set of vectors. The matrix A made of the orthogonal vectors

intertwines the diagonal matrix $L = A^+A$ and the matrix AA^+ built from norms of the row vectors and their scalar products:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix} = \begin{pmatrix} |a|^2 + |b|^2 & ac^* + bd^* \\ cb^* + db^* & |c|^2 + |d|^2 \end{pmatrix}.$$

Generally, for an $n \times n$ matrix it is a set of n columns ψ_i that forms factorizing matrices.

1.6.2 *QR* algorithm

Another example of the factorization of an (invertible) $n \times n$ matrix M into a product of an orthogonal matrix Q and an upper-triangular matrix R,

$$M = QR, \tag{1.44}$$

is well known because it produces an algorithm for computing eigenvalues of the matrix M [403]. A proof is provided by the Gramm–Schmidt orthogonalization procedure [453].

The algorithm is the following. We start from (1.44) and factorize the result of the transposition

$$M_1 = RQ = Q^{-1}MQ = Q^{\mathrm{T}}MQ$$

as $M_1 = Q_1R_1$, which allows us to construct $M_2 = Q_1^{\mathrm{T}}M_1Q_1$. The repetition of the factorization produces the chain

$$M_{k+1} = Q_k^{\mathrm{T}}M_kQ_k. \tag{1.45}$$

This iterative process (which is also named as "dressing") is applied in the theory of integrable Toda systems.

1.6.3 Factorization of the λ matrix

A λ matrix is determined as the polynomial

$$L(\lambda) = \sum_{k=0}^{n} L_k\lambda^k \tag{1.46}$$

with matrix coefficients. Among the problems connected with operators of the form (1.46), there are the eigenvalue problem for the equation $L_0\psi + \lambda L_1\psi = 0$ when $n = 1$ and "bundle" problems [338] with arbitrary n. A special factorization

$$L(\lambda) = M(\lambda)D_\lambda \tag{1.47}$$

introduces a right divisor of the matrix L with respect to the first-order polynomial $D_\lambda = A\lambda + B$, where

$$M(\lambda) = \sum_{k=0}^{n-1} M_k \lambda^k.$$

The substitution of D_λ and $M(\lambda)$ into (1.47),

$$\sum_{k=0}^{n-1} M_k A \lambda^{k+1} + \sum_{k=0}^{n-1} M_k B \lambda^k = \sum_{k=0}^{n} L_k \lambda^k,$$

yields the recurrence

$$M_{k-1}A + M_k B = L_k, \quad k = 1, \ldots, n-1, \quad M_0 B = L_0, \quad M_{n-1}A = L_n. \tag{1.48}$$

To calculate recurrently M_k by means of (1.48), the existence of A^{-1} and B^{-1} is necessary.

1.7 Elementary factorization of matrix

For future use of the theory, the special (idempotent) case of the degenerate matrix D_λ necessitates special attention. Namely, let two orthogonal idempotents p and q be given by $p^2 = p$, $q^2 = q$, and $p+q = e$, where e is the identity element. Hence, the space $\mathcal{A}$ of the matrices allows a natural splitting into the column subspaces $\mathcal{A}_p$ and $\mathcal{A}_q$. The splitting is realized by the following identity for an arbitrary matrix M: $M(p+q) = Mp+Mq$. Consider a λ matrix $D_\lambda = p\lambda - \sigma$ that intertwines the polynomials of the first order:

$$(A_1\lambda + B_1)(p\lambda - \sigma) = (p\lambda - \sigma)(A\lambda + B).$$

It leads to the conditions

$$A_1 p = pA, \quad B_1 p - A_1\sigma = -\sigma A + pB, \quad B_1\sigma = \sigma B.$$

The equivalent set corresponding to the decomposition of the unit matrix by the projectors p and q looks as follows:

$$pAp = pA_1p, \quad qA_1p = 0, \quad pAq = 0\,, \tag{1.49}$$

$$pB_1p - pA_1(p+q)\sigma p = -p\sigma(p+q)Ap + pBp, \tag{1.50}$$

$$qB_1p - qA_1(p+q)\sigma p = -q\sigma(p+q)Ap, \tag{1.51}$$

$$-pA_1(p+q)\sigma q = -p\sigma qAq + pBq, \tag{1.52}$$

$$-qA_1q\sigma q = -q\sigma qAq\,, \tag{1.53}$$

$$pB_1(p+q)\sigma p = p\sigma(p+q)Bp, \quad qB_1(p+q)\sigma p = q\sigma(p+q)Bp\,, \tag{1.54}$$

$$(pB_1p + pB_1q)\sigma q = p\sigma(pBq + qBq), \quad (qB_1p + qB_1q)\sigma q = q\sigma(pBq + qBq). \tag{1.55}$$

Let us consider the particular case of $A = A_1 = ap + bq$, $a, b \in \mathbb{C}$ and commuting elements of σ, B, and B_1. The conditions (1.49) and (1.53) hold automatically, while (1.50) gives

$$pB_1p = pBp. \tag{1.56}$$

Then (1.51) and (1.52) connect elements of B, B_1, and σ:

$$qB_1p = (b-a)q\sigma p, \tag{1.57}$$

$$p\sigma q = (b-a)^{-1}pBq. \tag{1.58}$$

Equations (1.56) and (1.57) can be read as a part of the transformation $B \to B_1$ for which we are searching.

Excluding pB_1p from (1.54) yields

$$pBp\,\sigma p + pB_1q\sigma p = p\sigma pBp + p\sigma qBp. \tag{1.59}$$

Plugging the transform (1.57) and the link (1.58) into the second relation of (1.55) produces

$$(b-a)q\sigma p\,\sigma p + qB_1q\sigma p = q\sigma pBp + q\sigma qBp. \tag{1.60}$$

In the case of Abelian elements, the condition (1.59) is simplified:

$$pB_1q\sigma p = p\sigma qBp. \tag{1.61}$$

If (1.55) and (1.61) can be solved with respect to elements of B_1 [we assume $\exists (q\sigma p)^{-1}$], then (1.61) accomplishes a construction of the transformation $B \to B_1$, together with (1.56) and (1.57).

An important case of intertwining operators with degenerate coefficients was mentioned in the previous section. Indeed, let there exist matrices $\phi_p \in A_p$ such that $\exists\, p\,\phi_p^{-1}$. Let us fix the kernel of D_{λ_0} by means of the elements of ϕ_p as

$$(p\lambda_0 - \sigma)\phi_p = 0. \tag{1.62}$$

It can be read as

$$p\,\sigma(p+q)\phi_p = \lambda_0 p\phi_p, \quad q\sigma(p+q)\phi_p = 0.$$

These equations connect the matrix elements of σ with elements of ϕ_p:

$$p\,\sigma p = \lambda_0 p\,\phi_p - (b-a)^{-1}pBq\phi_p, \quad q\,\sigma p = -q\phi_p(p\phi_p)^{-1}, \tag{1.63}$$

where (1.58) for $p\sigma q$ was used.

Plugging the second equation in (1.63) into (1.57) gives the expressions for qB_1p:

$$qB_1p = (a-b)q\phi_p(p\phi_p)^{-1}, \quad pB_1q = -(b-a)^{-1}pB_1qBp\phi_p(q\phi_p)^{-1}. \quad (1.64)$$

The equation (1.60), after substitutions from (1.64), determines qB_1q, the last element of B. For the Abelian elements, we have

$$-q\phi_p(p\phi_p)^{-1}[\lambda_0(b-a)p\phi_p - pBq\phi_p + qB_1q - pBp] = q\sigma qB_1p.$$

Finally,

$$qB_1q = -\lambda_0(b-a)p\phi_p + pBq\phi_p + pBp - (p\phi_p)(q\phi_p)^{-1}q\sigma qBp. \quad (1.65)$$

The matrix elements of B_1 are expressed through elements of B and elements of ϕ_p, up to the "free parameter" $q\,\sigma q$.

The general setting of the elementary DT theory is as follows:

1. One begins with
$$(A\lambda_0 + B_0)\phi_p = 0 \quad (1.66)$$
and note that owing to (1.62) the intertwining relation is valid identically.
2. For any λ and $\psi(\lambda)$, if
$$(A\lambda + B)\psi(\lambda) = 0, \quad (1.67)$$
the intertwining relation means that
$$(A\lambda + B_1)D_\lambda\psi(\lambda) = 0, \quad (1.68)$$
where B_1 is determined by a solution of (1.66) and the parameter $q\sigma q$ is given via (1.56), (1.64), and (1.65). This procedure links the known problem (1.66) with another one (1.68) in a "covariant" way.

Remark 1.7. All matrices in this section could be considered as elements of a ring as in Chaps. 2 and 3.

Remark 1.8. The case of higher polynomials in λ is studied similarly.

1.8 Matrix factorizations and integrable systems

The title of this section coincides with the title of the paper [111]. We give here a concise overview of this paper and two preceding ones [109, 110] devoted to the Toda lattice [152] and its applications in algorithms for computing matrix eigenvalues [189]. The main idea of the QR algorithm is based on the factorization of a matrix. Namely, if the chain of factorizations (1.45) yields

$$M_{k+1} = Q_k^t M_k Q_k, \quad (1.69)$$

it is realized as a transformation that could be interpreted as a shift along an auxiliary axis, say t [419]. The next idea goes back to the Moser theorem [338] that states that a nondiagonal part of an initial matrix tends to zero if this t-transition is considered. More exactly, if a flow is generated by the Hamiltonian

$$H_{QR} = -\mathrm{tr}(M \log M - M) \tag{1.70}$$

on $(\mathbb{R}^{2n}, \omega)$, where $\omega = \frac{1}{4}\sum_{i=1}^{n} dp_i \wedge dq_i$, then

$$M_{QR}(k) = M_k, \qquad k = 1, 2, \dots .$$

An important role in this theory is played by the Toda chain written in the Flashka form

$$M_t = [B(M), M], \tag{1.71}$$

where M is a tridiagonal matrix

$$M = \begin{pmatrix} a_1 & b_1 & 0 & . \\ b_1 & . & & \\ & & . & b_{n-1} \\ & & b_{n-1} & a_n \end{pmatrix} \tag{1.72}$$

and $B(M)$ contains zeros at the diagonal:

$$B(M) = \begin{pmatrix} 0 & -b_1 & 0 & . \\ b_1 & 0 & & \\ & & . & -b_{n-1} \\ & & b_{n-1} & 0 \end{pmatrix} \tag{1.73}$$

Let us rephrase the theorem given in the introduction of [111]:

Theorem 1.9. (The QR dressing chain integrability). *The Hamiltonian (1.70) generates an integrable flow (1.71) that interpolates the QR dressing chain at integer t.*

The statement about asymptotic values of the off-diagonal part of the matrix M reads

Theorem 1.10. (Moser Toda chain theorem). *The matrix elements $b_k(t)$ go to zero at both infinities $t \to \pm\infty$.*

This theorem could be applied directly to many problems of difference approximations of the Schrödinger equation [202].

The QR algorithm deals with invertible matrices only, but the question of integrability of the flow (1.71) with

$$B(M) = (M_+)^t - M_+,$$

where M_+ is strictly an upper part of M, is solved positively for the symmetric initial M_0. Adler [16] and Kostant [250] proved that the Hamiltonian

$H_I(M) = (\mathrm{tr}M^2)/2$ yields the equations of motion (1.71) on the symplectic manifold and Deift et al. [110] found expressions for conservation laws as the eigenvalues of the problems:

$$\widetilde{M}_k\beta_{rk} = \lambda_{rk}\beta_{rk}, \quad k = 0, \ldots, [n/2], \quad r = 1, \ldots, n-2k,$$

where $\widetilde{M}_k$ is the result of deleting the first k rows and the last k columns of M.

Further generalization was achieved by Deift et al. [111] in the form of the integrability theorem for a generic matrix M by means of the Lax representation with

$$B(M) = [(M^t)_+]^t - (M^t)_+.$$

The additional (to the symmetric case) $(n-1)^2/4$ integrals of motion J_{rk} are extracted from the invariant spectral curve by means of the identity transformation

$$\det[(1-h)M + hM - z] = \sum_{r,k}[h(1-h)]^k z^{M-r} J_{rk}.$$

The explicit expressions for the corresponding solutions of the flow (1.71) are given in terms of theta functions.

The paper [111] also contains proofs of the integrability theorems for the other types of factorization using the Cholesky algorithm

$$M_t = [((M^t)_- + (M^t)_0/2)^t, M] \tag{1.74}$$

and LU flow (the LU factorization algorithm, where L is the lower triangular matrix with unit diagonal entries and U is the upper triangular matrix)

$$M_t = [((M^t)_- + (M^t)_0)^t, M], \tag{1.75}$$

where M_- is the lower part of the matrix M and M_0 is the diagonal one, as well as algorithms based on the factorization [189]. The possibility of blowing up in finite time [447] should also be mentioned, as this is important for applications.

Deift et al. [111], extending the results of the previous paper [110], used the Lie group theory to construct orbits and symplectic structures on the basis of the unique QL factorization

$$g = g_0 g_L, \quad g_0 \in O^+(n, R), \quad g_L \in L^+(n, R).$$

1.9 Quasideterminants

This section contains a novel tool to manage block matrices (supermatrices) which appears in the dressing theory. The history of the classical theory of determinants and its extension related to the notion of quasideterminants are reviewed in the excellent and profound paper [174]. Let us start with a quotation from [174]:

> Our experience shows that in dealing with noncommutative objects one should not imitate the classical commutative mathematics, but follow "the way it is" starting with basics.

The purpose of this section is to give a brief introduction to the theory of quasideterminants based on the text of [174] (see also [191]).

1.9.1 Definition of quasideterminants

Let A be a matrix with numbers as entries. We write

$$|A|_{ij} = \begin{vmatrix} a_{11} & \dots & a_{1j} & \dots & a_{1n} \\ & \dots & & \dots & \\ a_{i1} & \dots & a_{ij} & \dots & a_{in} \\ & \dots & & \dots & \\ a_{n1} & \dots & a_{nj} & \dots & a_{nn} \end{vmatrix}. \tag{1.76}$$

For a 2×2 block matrix $A = (a_{ij})$, $i, j = 1, 2$, there are four quasideterminants:

$$\begin{aligned} |\widehat{A}|_{11} &= a_{11} - a_{12} \cdot a_{22}^{-1} \cdot a_{21}, \\ |\widehat{A}|_{12} &= a_{12} - a_{11} \cdot a_{21}^{-1} \cdot a_{22}, \\ |\widehat{A}|_{21} &= a_{21} - a_{22} \cdot a_{12}^{-1} \cdot a_{11}, \\ |\widehat{A}|_{22} &= a_{22} - a_{21} \cdot a_{11}^{-1} \cdot a_{12}. \end{aligned}$$

We see that each of the quasideterminants $|\widehat{A}|_{11}$, $|\widehat{A}|_{12}$, $|\widehat{A}|_{21}$, and $|\widehat{A}|_{22}$ is defined whenever the corresponding elements a_{22}, a_{21}, a_{12}, and a_{11} are invertible.

For a generic $n \times n$ matrix (in the sense that all square submatrices of A are invertible) there exist n^2 quasideterminants of A. A nongeneric matrix may have k quasideterminants where $0 \le k \le n^2$.

Generally, the definition of quasideterminants is given over a ring R with a unit element. Let $A = (a_{ij})$, $i \in I$, $j \in J$, be a matrix over R. Denote by r_i^j the row submatrix of length $n-1$ obtained from the ith row of A by deleting the element a_{ij}, and by c_j^i the column submatrix of height $n-1$ obtained from the jth column of A by deleting the element a_{ij}.

Denote by A^{ij}, $i \in I$, $j \in J$ the submatrix of A obtained from A by deleting its ith row and jth column. Then we can formulate the following.

Definition 1.11. *Let I and J be finite sets with the same number of elements. If $I = \{i\}$, $J = \{j\}$, put $|\widehat{A}|_{ij} = a_{ij}$. If $|I|, |J| > 1$, the quasideterminant $|A|_{ij}$ is defined whenever the submatrix A^{ij} is invertible over R and in this case put*

$$|\widehat{A}|_{ij} = a_{ij} - r_i^j (A^{ij})^{-1} c_j^i.$$

The term "quasideterminant," as it is used in, e.g., [191], denotes rather a fraction of determinants.

In the context of our book, it is important to note that it is the iterated non-Abelian DT that is written in terms of quasideterminants .

1.9.2 Noncommutative Sylvester–Toda lattices

Let R be a division algebra with a derivation $D : R \to R$. Let $\phi \in R$ and the quasideterminants

$$\widehat{T}_n(\phi) = \begin{vmatrix} \phi & D\phi & \dots & D^{n-1}\phi \\ D\phi & D^2\phi & \dots & D^n\phi \\ \dots & \dots & \dots & \dots \\ D^{n-1}\phi & D^n\phi & \dots & D^{2n-2}\phi \end{vmatrix}$$

are defined and invertible. Set $\phi_1 = \phi$ and $\phi_n = T_n(\phi)$, $n = 2, 3, \dots$.

Theorem 1.12. *Elements ϕ_n, $n = 1, 2, \dots$ satisfy the following equations:*

$$D[(D\phi_1)\phi_1^{-1}] = \phi_2\phi_1^{-1}, \quad D[(D\phi_n)\phi_n^{-1}] = \phi_{n+1}\phi_n^{-1} - \phi_n\phi_{n-1}^{-1}. \tag{1.77}$$

If R is commutative, the determinants of the matrices used in $\widehat{T}_n(\phi)$ satisfy a nonlinear system of differential equations. In the modern literature this system is called the Toda lattice [356] but in fact it was discovered by Sylvester [416] in 1862 and probably, should be called the Sylvester–Toda lattice. Our system can be viewed as a noncommutative generalization of the Sylvester–Toda lattice. Theorem 1.12 appeared in [177, 178] and was generalized in [372]. The following theorem is a noncommutative analog of the famous Hirota identities.

Theorem 1.13. *For $n \geq 2$*

$$T_{n+1}(\phi) = T_n(D^2\phi) - T_n(D\phi) \cdot [(T_{n-1}(D^2\phi)^{-1} - T_n(\phi)^{-1}]^{-1} \cdot T_n(D\phi).$$

The proof follows from the noncommutative Sylvester identity [174].

1.9.3 Noncommutative orthogonal polynomials

The results described in this subsection were obtained in [175]. Let S_0, S_1, $S_2, \dots$ be elements of a skew field R and x be a commutative variable. Define a sequence of elements $P_i(x) \in R[x]$, $i = 0, 1, \dots$ by setting $P_0 = S_0$ and

$$P_n(x) = \begin{vmatrix} S_n & \dots & S_{2n-1} & x^n \\ S_{n-1} & \dots & S_{2n-2} & x^{n-1} \\ \dots & \dots & \dots & \dots \\ S_0 & \dots & S_{n-1} & 1 \end{vmatrix}$$

for $n \geq 1$. The expansion of the right column implies that $P_n(x)$ is a polynomial of degree n. If R is commutative, then $P_n, n \geq 0$, are orthogonal polynomials defined by the moments S_n, $n \geq 0$. We are going to show that if R is a free division algebra generated by S_n, $n \geq 0$, then polynomials P_n are indeed orthogonal with regard to a natural noncommutative R-valued product on $R[x]$.

Let R be a free skew field generated by c_n, $n \geq 0$. Define on R a natural anti-involution $a \mapsto a^*$ by setting $c_n^* = c_n$ for all n and extend the involution to $R[x]$ by setting $(\sum a_i x^i)^* = \sum a_i x^i$. Define the R-valued inner product on $R[x]$ by setting

$$\left\langle \sum a_i x^i, \sum b_j x^j \right\rangle = \sum a_i c_{i+j} b_j^*.$$

Theorem 1.14. *For $n \neq m$ we have*

$$\langle P_n(x), P_m(x) \rangle = 0.$$

1.10 The Riemann–Hilbert problem

This section is devoted to a brief review of basic facts concerning the RH boundary value problem which will be used in this book. More detailed exposition of the RH problem can be found in [4, 167, 375, 464].

1.10.1 The Cauchy-type integral

Let us consider a class of complex functions $f(\ell)$

1. Which are defined for all ℓ belonging to a contour γ. The contour γ is a smooth closed counterclockwise oriented curve dividing the extended complex k-plane $\mathbb{C}$ (including the infinite point ∞) into two domains $\mathbb{C}_+$ and $\mathbb{C}_-$.
2. Which obey the Hölder condition on γ:

$$|f(\ell_2) - f(\ell_1)| \leq A|\ell_2 - \ell_1|^\mu, \quad A = \text{const}, \quad 0 < \mu \leq 1. \tag{1.78}$$

3. Where $f(\ell) \to 0$ at $\ell \to \infty$.

Consider a point k in the k-plane and define a function (the Cauchy-type integral)

$$\phi(k) = \frac{1}{2\pi \mathrm{i}} \int_\gamma \mathrm{d}\ell \frac{f(\ell)}{\ell - k}. \tag{1.79}$$

The function $\phi(k)$ is analytic in $\mathbb{C}$, except for points on γ, and tends to zero at $k \to \infty$. For simplicity we choose γ to be the real axis of the k-plane. Then $\mathbb{C}_+ (\mathbb{C}_-)$ corresponds to the upper (lower) half planes of $\mathbb{C}$. It should be noted that it is the Hölder condition that ensures the existence of the integral (1.79). Indeed, we can write (1.79) as

$$\frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \mathrm{d}\ell \frac{f(\ell)}{\ell - k} = \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \mathrm{d}\ell \frac{f(\ell) - f(k)}{\ell - k} + \frac{f(k)}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \frac{\mathrm{d}\ell}{\ell - k}. \tag{1.80}$$

The last integral on the right-hand side of (1.80) is well defined, while the first integral exists owing to the Hölder condition.

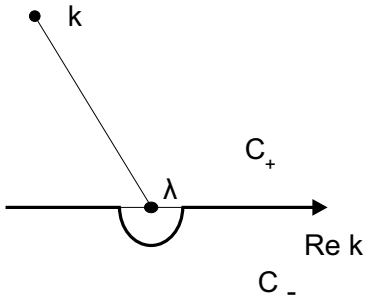

Fig. 1.1. Contour of integration (*bold line*) when the point k reaches the real axis

An important question is how to define the function $\phi(k)$ for real k, i.e., when k belongs to the contour γ. Suppose $k \in \mathbb{C}_+$ tends to the point λ on the real axis $\mathrm{Im}k = 0$, being still left in $\mathbb{C}_+$ (Fig. 1.1). Then $\phi(k)$ tends to $\phi_+(\lambda)$. To evaluate $\phi_+(\lambda)$, we deform the contour in such a way that it passes the point λ from below. Then

$$\phi_+(\lambda) = \frac{1}{2\pi\mathrm{i}} \lim_{\epsilon\to 0} \left(\int_{-\infty}^{\lambda-\epsilon} \mathrm{d}\ell \frac{f(\ell)}{\ell - k} + \int_{\lambda+\epsilon}^{\infty} \mathrm{d}\ell \frac{f(\ell)}{\ell - k} \right) + \frac{1}{2\pi\mathrm{i}} \int_{\lambda-\epsilon}^{\lambda+\epsilon} \mathrm{d}\ell \frac{f(\ell)}{\ell - k}.$$

The first term in the parentheses defines the principal value of the Cauchy integral $\mathrm{v.p.} \int_{-\infty}^{\infty} \mathrm{d}\ell f(\ell)/(\ell - \lambda)$ and the last integral is easily calculated after a change $\ell - \lambda = \epsilon \exp(\mathrm{i}\theta)$ and integrating in θ from π to 2π. The result is

$$\phi_+(\lambda) = \frac{1}{2\pi\mathrm{i}} \mathrm{v.p.} \int_{-\infty}^{\infty} \mathrm{d}\ell \frac{f(\ell)}{\ell - \lambda} + \frac{1}{2} f(\lambda), \qquad \mathrm{Im}\lambda = +0. \tag{1.81}$$

Similarly, we can calculate the function $\phi_-(\lambda)$, which is the limit of $\phi(k)$ when k located in $\mathbb{C}_-$ tends to $\lambda \in \mathrm{Im}k = -0$:

$$\phi_-(\lambda) = \frac{1}{2\pi\mathrm{i}} \mathrm{v.p.} \int_{-\infty}^{\infty} \mathrm{d}\ell \frac{f(\ell)}{\ell - \lambda} - \frac{1}{2} f(\lambda). \tag{1.82}$$

Hence, we derive the Sokhotsky–Plemelj formulas (1.81) and (1.82), which are usually written as

$$\phi_+(\lambda) - \phi_-(\lambda) = f(\lambda), \qquad \mathrm{Im}\lambda = 0, \tag{1.83}$$

$$\phi_+(\lambda) + \phi_-(\lambda) = \frac{1}{\pi\mathrm{i}} \mathrm{v.p.} \int_{-\infty}^{\infty} \mathrm{d}\ell \frac{f(\ell)}{\ell - \lambda}. \tag{1.84}$$

Therefore, the Cauchy-type integral defines a sectionally continuous function which is regular off the contour and continuous when tending to the contour both from above and from below.

To recognize analytic properties of $\phi(k)$, we allow the point $k \in \mathbb{C}_+$ to move to the point λ on the real axis, to cross the axis,and to move below the

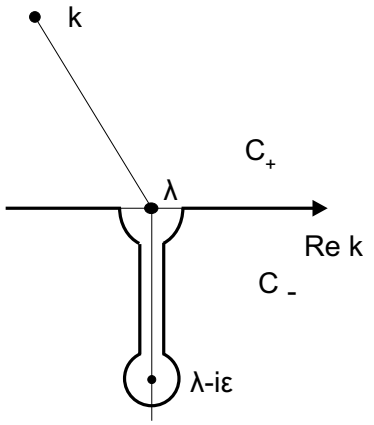

Fig. 1.2. Contour of integration (*bold line*) when the point k crosses the real axis

axis to the point $\lambda - \mathrm{i}\epsilon$, $\epsilon > 0$. To evaluate $\phi_+(\lambda - \mathrm{i}\epsilon)$, we deform the contour γ as in Fig. 1.2. The calculation yields

$$\phi_+(\lambda - \mathrm{i}\epsilon) = \frac{1}{2\pi\mathrm{i}} \int_{-\infty}^{\infty} \mathrm{d}\ell \frac{f(\ell)}{\ell - \lambda + \mathrm{i}\epsilon} + f(\lambda - \mathrm{i}\epsilon) = \phi_-(\lambda - \mathrm{i}\epsilon) + f(\lambda - \mathrm{i}\epsilon).z$$

Note that the last term does not contain the factor $1/2$ because the contour encloses the point $\lambda - \mathrm{i}\epsilon$ almost entirely. We see that $\phi_+(\lambda - \mathrm{i}\epsilon)$ does not coincide with $\phi_-(\lambda - \mathrm{i}\epsilon)$. Therefore, the Cauchy-type integral defines two different analytic functions: $\phi_+(k)$, $k \in \mathbb{C}_+$ and $\phi_-(k)$, $k \in \mathbb{C}_-$. Accordingly, we can write

$$\begin{aligned}
\phi_+(k) &= \frac{1}{2\pi\mathrm{i}} \int_{-\infty}^{\infty} \mathrm{d}\ell \frac{f(\ell)}{\ell - k}, \qquad k \in \mathbb{C}_+, \\
\phi_+(k) &= \frac{1}{2\pi\mathrm{i}} \int_{-\infty}^{\infty} \mathrm{d}\ell \frac{f(\ell)}{\ell - k} + \frac{1}{2} f(k), \quad \mathrm{Im} k = +0, \\
\phi_+(k) &= \frac{1}{2\pi\mathrm{i}} \int_{-\infty}^{\infty} \mathrm{d}\ell \frac{f(\ell)}{\ell - k} + f(k), \quad k \in \mathbb{C}_-.
\end{aligned} \tag{1.85}$$

Several conclusions follow from (1.85). First, $\phi_+(k)$ can be analytically continued to $\mathbb{C}_-$ if $f(k)$ allows such a continuation. Second, the function $\phi_+(k)$, being regular in $\mathbb{C}_+$, acquires singularities in $\mathbb{C}_-$, otherwise it will be constant in entire $\mathbb{C}$ as a regular analytic function everywhere, in accordance with the Liouville theorem. Third, if we know a jump $\Delta(k) = \phi_+(k) - \phi_-(k)$ of analytic functions $\phi_\pm$ across the real axis $\mathrm{Im} k = 0$, we can restore $\phi_\pm$ as $\phi_\pm(k) = (P_\pm \Delta)(k)$, where projectors $P_\pm$ act as follows:

$$(P_\pm \Delta)(k) = \frac{1}{2\pi\mathrm{i}} \int\limits_{-\infty}^{\infty} \frac{\mathrm{d}\ell}{\ell - (k \pm \mathrm{i}0)} \Delta(\ell). \tag{1.86}$$

If $\phi_\pm$ tend to some limits as $|k| \to \infty$, these limits should be added to the integral (1.86).

1.10.2 Scalar RH problem

The scalar RH problem can be formulated as a problem of analytic factorization of a scalar function $g(k)$ given on a contour γ,

$$\phi_-(k)\phi_+(k) = g(k), \quad k \in \gamma, \tag{1.87}$$

in a product of functions $\phi_\pm(k)$ analytic in $\mathbb{C}_\pm$. A solution of this problem is not unique: functions $\tilde{\phi}_+ = \phi_+ r_+$ and $\tilde{\phi}_-/r_+$, when $r_+(k)$ is a rational function with all zeros being in $\mathbb{C}_+$ and all poles being in $\mathbb{C}_-$, are solutions of (1.87) as well. To fix the solution, we shall pose a normalization condition. The condition $\phi_-(\infty) = 1$ is called the canonical normalization.

Now we define the index of the RH problem:

$$\operatorname{ind}_\gamma\{g(k)\} = \frac{1}{2\pi\mathrm{i}} \int_\gamma \mathrm{d}\{\ln g(k)\}.$$

The index measures a change of the phase of $g(k)$ over the contour γ. For analytic functions the index gives the difference between the number of zeros and poles of this function (accounting for their multiplicities) in the domain bounded by the contour. If the index is zero, both functions $\phi_\pm(k)$ have the same number of zeros in their domains of analyticity.

The scalar RH problem with zero index can be easily solved. Let k_j and $\bar{k}_j$, $j = 1, \ldots, \mathcal{N}$ be simple zeros of $\phi_+(k)$ and $\phi_-(k)$, respectively. First we regularize these functions. This means that the functions

$$\phi_+^{(0)}(k) = \phi_+(k) \prod_{j=1}^{\mathcal{N}} \frac{k - \bar{k}_j}{k - k_j}, \quad \phi_-^{(0)}(k) = \phi_-(k) \prod_{j=1}^{\mathcal{N}} \frac{k - k_j}{k - \bar{k}_j} \tag{1.88}$$

solve the same RH problem and obey the same normalization condition as $\phi_\pm(k)$ but have no zeros. The regularized RH problem subjected to a given normalization condition has a unique solution. Indeed, for the functions $p_\pm(k) = \pm \ln \phi_\pm^{(0)}(k)$ equation (1.87) is written as

$$p_+(k) - p_-(k) = \ln g(k), \qquad k \in \gamma.$$

Hence, we have two holomorphic functions in $\mathbb{C}_\pm$ which are both zero at infinity and have the jump $\ln g(k)$ across the contour γ. Using the Sokhotsky–Plemelj formula (1.83) and assuming the Hölder condition for $\ln g(k)$, we get the solution

$$p(k) = \frac{1}{2\pi\mathrm{i}} \int_\gamma \frac{\mathrm{d}\ell}{\ell - k} \ln g(\ell).$$

Accordingly,

$$\phi_\pm^{(0)}(k) = \exp\left(\frac{\pm 1}{2\pi\mathrm{i}} \int_\gamma \frac{\mathrm{d}\ell}{\ell - k} \ln g(\ell) \right), \qquad k \in \mathbb{C}_\pm.$$

Therefore, the general solution to the scalar RH problem has the form

$$\phi_{\pm}(k) = \prod_{j=1}^{\mathcal{N}} \left(\frac{k - k_j}{k - \bar{k}_j} \right)^{\pm 1} \exp\left(\frac{\pm 1}{2\pi \mathrm{i}} \int_{\gamma} \frac{\mathrm{d}\ell}{\ell - k} \ln g(\ell) \right), \quad k \in C_{\pm}. \tag{1.89}$$

1.10.3 Matrix RH problem

Consider the problem of analytic factorization of a matrix function $G(k)$ given on a contour γ such that $\det G(k) \neq 0$, $G(\infty) = \mathbb{1}$ if $\infty \in \gamma$, and $\mathrm{ind}_{\gamma}\{\det G(k)\} = 0$, in a product of two analytic functions $\Phi_{+}(k)$ and $\Phi_{-}^{-1}(k)$ in $\mathbb{C}_{+}$ and $\mathbb{C}_{-}$, respectively:

$$\Phi_{-}^{-1}(k)\Phi_{+}(k) = G(k), \qquad k \in \gamma, \tag{1.90}$$

with the normalization condition $\Phi(\infty) = \mathbb{1}$ (i.e., one or both of $\Phi_{\pm}$ obey this condition). The zero index condition ensures that $\det\Phi_{+}(k)$ and $\det\Phi_{-}^{-1}(k)$ have an equal number of zeros in $\mathbb{C}_{+}$ and $\mathbb{C}_{-}$. A regularization of the matrix RH problem is performed by some matrix functions rational in k. The regularized matrix RH problem has a unique solution similar to the scalar case. The significant difference, however, lies in the fact that the matrix case does not allow an explicit general solution. The investigation of the solvability of the regularized matrix RH problem can be reduced to that of some matrix linear integral equation of the Fredholm type. Indeed, let $\Phi_{\pm}^{(0)}(k)$ be matrix functions analytic in $\mathbb{C}_{\pm}$ that determine the regularized RH problem:

$$\Phi_{-}^{(0)-1}\Phi_{+}^{(0)} = G(k). \tag{1.91}$$

Then the Sokhotsky–Plemelj formula (1.83) can be applied to $\Psi_{\pm} = \Phi_{\pm}^{(0)} - \mathbb{1}$, giving

$$\Psi_{\pm}(k) = \frac{\pm 1}{2\pi \mathrm{i}} \int_{\gamma} \frac{\mathrm{d}\ell}{\ell - k} \Psi_{\pm}(\ell). \tag{1.92}$$

In terms of $\Psi_{\pm}$ the problem (1.91) is written as

$$\Psi_{+}(k) = \Psi_{-}(k)G(k) + G(k) - \mathbb{1}, \qquad k \in \gamma. \tag{1.93}$$

Then we obtain from (1.92) and (1.93) the following equation for $\Psi_{-}(k)$ for $k \in \gamma$:

$$\Psi_{-}(k) = \frac{1}{2\pi \mathrm{i}} \int_{\gamma} \mathrm{d}\ell \Psi_{-}(\ell) K(\ell, k) + H(k). \tag{1.94}$$

The kernel $K(\ell, k)$ and inhomogeneous term $H(k)$ have the forms

$$K(\ell, k) = \frac{G(\ell)G^{-1}(k) - \mathbb{1}}{\ell - k} \tag{1.95}$$

and

$$H(k) = \frac{1}{2}\left[G^{-1}(k) - \mathbb{1}\right] + \frac{1}{2\pi\mathrm{i}}\int_\gamma \frac{\mathrm{d}\ell}{\ell - k}\left[G(k) - \mathbb{1}\right]G^{-1}(k).$$

Note the kernel $K(\ell, k)$ (1.95) is regular for $\ell = k$; hence, the integral equation (1.94) is of the Fredholm type. Gohberg and Krein [188] have formulated the sufficient condition for the solvability of the matrix RH problem. Let us introduce "real" and "imaginary" parts of the matrix G,

$$G_{\mathrm{R}}(k) = \frac{1}{2}\left[G(k) + G^\dagger(k)\right], \quad G_{\mathrm{I}}(k) = \frac{1}{2\mathrm{i}}\left[G(k) - G^\dagger(k)\right],$$

where $G^\dagger$ stands for the Hermitian conjugation. Then the regularized matrix RH problem has a solution if the real or the imaginary part of $G(k)$ is positive (or negative) definite. The definiteness of G means that $\mathbf{x}^\dagger G\mathbf{x}$ is real and sign-definite for all nonzero vectors $\mathbf{x}$. Our choice of the normalization of the RH problem is compatible with positive definiteness. In applications to the soliton theory we will encounter only the solvable matrix RH problems.

1.11 $\bar\partial$ Problem

The analytic function $f(x, y) = u(x, y) + \mathrm{i}v(x, y)$ defined on the extended complex plane (the Riemann sphere) $\mathbb{C}$ with the coordinates x and y obeys the Cauchy–Riemann condition

$$u_x = v_y, \qquad u_y = -v_x. \tag{1.96}$$

In the complex coordinates $z = x + \mathrm{i}y$ and $\bar z = x - \mathrm{i}y$ equation (1.96) takes a compact form

$$\bar\partial f(z, \bar z) = 0,$$

where $\bar\partial \equiv \partial_{\bar z} = (1/2)(\partial_x + \mathrm{i}\partial_y)$. The Cauchy–Riemann operator $\bar\partial$ measures the "departure from analyticity" for the function $f(z, \bar z)$ and the equation

$$\bar\partial f(z, \bar z) = g(z, \bar z) \tag{1.97}$$

is referred to as the $\bar\partial$ problem. The Cauchy formula (1.79) for analytic functions is generalized to the case of nonanalytic functions which satisfy (1.97) as

$$f(z, \bar z) = \frac{1}{2\pi\mathrm{i}}\int\limits_\gamma \frac{\mathrm{d}\zeta}{\zeta - z} f(\zeta, \bar\zeta) + \frac{1}{2\pi\mathrm{i}}\iint\limits_{\mathcal{D}} \frac{\mathrm{d}\zeta \wedge \mathrm{d}\bar\zeta}{\zeta - z} g(\zeta, \bar\zeta). \tag{1.98}$$

Here $f(z, \bar z)$ is any function which has smooth derivatives with respect to both z and $\bar z$ in some domain $\mathcal{D}$ in the complex plane and is continuous in the closed domain $\mathcal{D} \cup \gamma$ with a counterclockwise-oriented boundary γ. The exterior product $\mathrm{d}z \wedge \mathrm{d}\bar z$ is skew-symmetric, $\mathrm{d}z \wedge \mathrm{d}\bar z = -\mathrm{d}\bar z \wedge \mathrm{d}z$, and can be written as $\mathrm{d}z \wedge \mathrm{d}\bar z = -2\mathrm{i}\mathrm{d}x\mathrm{d}y$. If the domain $\mathcal{D}$ is the entire complex

plane, then the boundary γ can be taken as a circle with arbitrarily large radius. For the canonically normalized function $f(z,\bar{z}) \to 1$ at $|z| \to \infty$ the Cauchy–Green formula (1.98) is reduced to

$$f(z,\bar{z}) = 1 + \frac{1}{2\pi \mathrm{i}} \iint\limits_{\mathcal{D}} \frac{\mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta}}{\zeta - z} g(\zeta,\bar{\zeta}). \tag{1.99}$$

Evidently, the function

$$f(z,\bar{z}) = a(z) + \frac{1}{2\pi \mathrm{i}} \iint\limits_{\mathcal{D}} \frac{\mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta}}{\zeta - z} g(\zeta,\bar{\zeta})$$

with arbitrary analytic function $a(z)$ is a solution to the $\bar{\partial}$ problem (1.97) as well.

Acting on (1.98) with the $\bar{\partial}$ operator, we encounter the expression $\bar{\partial}(z-\zeta)^{-1}$. It is seen from (1.98) that we should put

$$\bar{\partial}\left(\frac{1}{z-\zeta}\right) = \pi\delta(z-\zeta), \tag{1.100}$$

where $\delta(z-\zeta) = \delta(x-\zeta_{\mathrm{R}})\delta(y-\zeta_{\mathrm{I}})$ is the Dirac delta function. Hence, the integral with the delta function gives

$$\iint\limits_{\mathcal{C}} \mathrm{d}z \wedge \mathrm{d}\bar{z} f(z,\bar{z})\delta(z-z_0) = -2\mathrm{i} f(z_0,\bar{z}_0). \tag{1.101}$$

We know from Sect. 1.10.1 that the Cauchy-type integral

$$f(z) = \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \mathrm{d}\zeta \frac{g(\zeta)}{\zeta - z}$$

determines a sectionally analytic function with a jump across the real axis $\mathrm{Im} z = 0$. Applying the $\bar{\partial}$ operator to this integral gives in accordance with (1.100) and (1.83)

$$\bar{\partial} f(z) = \bar{\partial} \frac{1}{2\pi \mathrm{i}} \int\limits_{-\infty}^{\infty} \mathrm{d}\zeta \frac{g(\zeta)}{\zeta - z} = \frac{\mathrm{i}}{2} \int\limits_{-\infty}^{\infty} \mathrm{d}\zeta g(\zeta)\delta(\zeta - z) = \frac{\mathrm{i}}{2} g(x)\delta(y)$$

$$= \frac{\mathrm{i}}{2} \left[f_+(x) - f_-(x)\right] \delta(y).$$

Hence, for the case of sectionally analytic functions with a jump discontinuity the $\bar{\partial}$ problem reduces to the conjugation (RH) problem.

In the theory of nonlinear equations, a nonlocal $\bar{\partial}$ problem

$$\bar{\partial} f(z,\bar{z}) = \iint_{\mathcal{C}} \mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta} f(\zeta,\bar{\zeta}) R(\zeta,\bar{\zeta},z,\bar{z}) \tag{1.102}$$

plays the key role. The functions f and R can be scalar or matrix ones. Following (1.99), a solution of the nonlocal $\bar{\partial}$ problem (1.102) is given by

$$f(z,\bar{z}) = 1 + \frac{1}{2\pi \mathrm{i}} \iint_{\mathcal{C}} \frac{\mathrm{d}\zeta \wedge \mathrm{d}\bar{\zeta}}{\zeta - z} \iint_{\mathcal{C}} \mathrm{d}\zeta' \wedge \mathrm{d}\bar{\zeta}' f(\zeta',\bar{\zeta}') R(\zeta',\bar{\zeta}',\zeta,\bar{\zeta}). \quad (1.103)$$

A non-canonical normalization of the function f, $f(z,\bar{z}) \neq 1$ at $|z| \to \infty$, leads to an inhomogeneous $\bar{\partial}$ problem [56] .

For particular structures of the kernel $R(\zeta,\bar{\zeta},z,\bar{z})$ the nonlocal $\bar{\partial}$ problem allows explicit solutions. Namely, the kernel R is called degenerate if it can be represented in a factorized form

$$R(\zeta,\bar{\zeta},z,\bar{z}) = \sum_{j=1}^{\mathcal{N}} \mu_j(\zeta,\bar{\zeta}) \nu_j(z,\bar{z})$$

with arbitrary functions μ_j and ν_j. Various particular choices of the functions μ_j and ν_j are considered in the book by Konopelchenko [241].

2

Factorization and classical Darboux transformations

In this chapter we describe the algebraical factorization-based method to dress solutions of (1+1)-dimensional equations. We also show how the Darboux transformation (DT) theory appears in this framework.

First, in Sect. 2.1, we introduce the non-Abelian Bell polynomials and then generalize them in Sect. 2.2 to formulate in Sect. 2.3 a problem of factorization of a polynomial differential operator in the form of division by a monomial from the right and from the left. The relation between the factorization rules and the classical Darboux theorem [102] generalized in [314] is described in Sect. 2.4: the formalism produces a compact form of the DT for non-Abelian coefficients of linear operators, polynomial in a differentiation on a ring. Section 2.5 is devoted to a representation of the iterated DTs in terms of quasideterminants. As a highly nontrivial example of the iterated DT formalism, we describe positon solutions of the Korteweg–de Vries (KdV) equation discovered by Matveev [318, 319].

The growing interest in discrete models appeals to wider classes of symmetry structures of the corresponding nonlinear problems [149, 196, 255, 256, 339]. Very recently a suitable basis for new searches in the field of differential-difference and difference-difference equations was discovered [321] in the framework of the classical DT theory such that the difference operator is replaced by an arbitrary automorphism transformation. In Sect. 2.6 we present the dressing method via factorization for such a kind of generalizations. Like in the case of differential operators, this approach demonstrates links with the Hirota bilinearization method [260] and the factorization theory [271], with similar applications. We reformulate the Darboux covariance theorem from the paper of Matveev [321] and introduce a kind of difference Bell polynomials. These polynomials correspond naturally to the differential (generalized) Bell polynomials in their non-Abelian version of Sect. 2.2.

The joint covariance principle is formulated in Sect. 2.7 for Abelian and in Sect. 2.8 for non-Abelian differential rings. The same construction for a pair of difference equations is elaborated in Sect. 2.9. The form of the DT presented here allows us to develop a classification scheme with respect to the DTs in

connection with the generalized Bell polynomials [187, 260, 467]. If a pair of such operators determines the Lax equations, the joint covariance with respect to the DTs produces a symmetry for the compatibility condition [314, 324]. In Sects. 2.10 and 2.11 we illustrate the possibilities of the method by examples of specific nonlinear equations: the non-Abelian Hirota system [210] having promising applications [149], and the Nahm equations [344]. We introduce a lattice Lax pair for the Nahm equations which is covariant with respect to combined Darboux-gauge transformations that generate the dressing structure for the equations. Finally, in Sect. 2.12 we illustrate the formalism developed, solving a particular case of the Nahm equations.

2.1 Basic notations and auxiliary results. Bell polynomials

Let K be a differential ring of the zero characteristics with unit e (i.e., unitary ring) and with an involution denoted by a superscript asterisk. The differentiation is denoted as D. The differentiation and the involution are agreed with operations in K:

1. $(a^*)^* = a, \quad (a+b)^* = a^* + b^*, \quad (ab)^* = b^* a^*, \quad a, b \in K.$
2. $D(a+b) = Da + Db, \quad D(ab) = (Da)b + aDb.$
3. $(Da)^* = -Da^*.$
4. Operators D^n with different n form a basis in a K-module $\mathrm{Diff}(K)$ of differential operators. The subring of constants is K_0 and a multiplicative group of elements of K is G.
5. For any $s \in K$ there exists an element $\varphi \in K$ such that $D\varphi = s\varphi$; this also means the existence of a solution of the equation

$$D\phi = -\phi s, \tag{2.1}$$

owing to the involution properties.

There are lots of applications of the rings of square matrices in the theory of integrable nonlinear equations, as well as in classical and quantum linear problems. In this case matrices are parameterized by a variable x and D can be a derivative with respect to this variable or a combination of partial derivatives that satisfies conditions 1 and 2. If D is the standard differentiation, then the involution (asterisk) may be the Hermitian conjugation. In the case of a commutator, the operator D acts as $Da = [d, a]$ and $(Da)^* = -[d^*, a]$. Having in mind this or similar applications, we shall refer to the involution as conjugation. We do not restrict ourselves to the matrix-valued case; an appropriate operator ring is also suitable for our theory.

Below we introduce left and right non-Abelian Bell polynomials (see also [388]) and formulate the statements for them. The differential Bell polynomials are defined in Definition 2.1:

Definition 2.1. *The left and right non-Abelian Bell polynomials $B_n(s)$ are defined by the recurrence relations*

$$B_n(s) = DB_{n-1}(s) + B_{n-1}(s)s, \qquad n = 1, 2, \dots \tag{2.2}$$

for left Bell polynomials and

$$B_n^+(s) = -DB_{n-1}^+(s) + s, \qquad n = 1, 2, \dots \tag{2.3}$$

for right Bell polynomials with the "initial condition"

$$B_0(s) = e. \tag{2.4}$$

Proposition 2.2. *If an element $\varphi \in G$ satisfies the equation $D\varphi = s\varphi$, then*

$$D^n\varphi = B_n(s)\varphi, \quad n = 0, 1, 2, \dots.$$

Proposition 2.3. *If an element $\phi \in G$ satisfies (2.1), then*

$$D^n\phi = (-1)^n\phi B_n^+(s), \quad n = 0, 1, 2, \dots.$$

Proposition 2.4. *The left and right Bell polynomials are connected by the following relations:*

$$B_n(s)^* = B_n^+(s^*), \qquad B_n^+(s)^* = B_n(s^*).$$

If the ring is Abelian, left and right polynomials coincide.

Remark 2.5. Proposition 2.4 means that a duality takes place for the Bell polynomials: any relation for right polynomials can be transformed to the corresponding relation for left ones, and vice versa.

Let us denote

$$L_s = D - s. \tag{2.5}$$

Note that the recursion (2.3) may be written by means of L_s (2.5) as

$$B_{n+1}^+(s) = -L_sB_n^+(s), \quad n = 0, 1, 2, \dots,$$

with the simple corollary

$$B_n^+(s) = (-1)^nL_s^ne, \quad n = 0, 1, 2, \dots.$$

2.2 Generalized Bell polynomials

In the next section a problem of division of an arbitrary operator L by the operator L_s will be studied. To this aim, for the right division we introduce here auxiliary operators H_n by means of Definition 2.6:

Definition 2.6. *The operators H_n are defined by the recurrence relation*

$$H_n = DH_{n-1} + B_n(s), \quad n = 1, 2, \ldots, \quad H_0 = e. \tag{2.6}$$

Proposition 2.7. *The following identity holds:*

$$D^n = H_{n-1}L_s + B_n(s), \quad n = 1, 2, \ldots.$$

Coefficients of the operators H_n are expressed via the generalized Bell polynomials that are defined in Definition 2.8:

Definition 2.8. *Generalized Bell polynomials are defined by the "initial conditions"*

$$B_{n,0}(s) = e, \quad n = 0, 1, 2, \ldots$$

and by the recurrence relations

$$B_{n,k}(s) = B_{n-1,k}(s) + DB_{n-1,k-1}(s), \quad k = 1, 2, \ldots n-1, \quad n = 2, 3, \ldots, \tag{2.7}$$

$$B_{n,n}(s) = DB_{n-1,n-1}(s) + B_n(s), \quad n = 1, 2, \ldots. \tag{2.8}$$

Proposition 2.7 is proved by acting with D from the left to (2.7) $n+1$ times and substituting (2.2) and (2.6) into the resulting equation because

$$D^{n+1} = H_nL_s + B_{n+1} = DH_{n-1}L_s + DB_n$$

$$= (DH_{n-1})L_s + H_{n-1}DL_s + B_n' + B_nD.$$

Proposition 2.9. *Generalized Bell polynomials are coefficients in the decomposition of the operators H_n, i.e.,*

$$H_n = \sum_{k=0}^{n} B_{n,n-k}(s)D^k, \quad n = 0, 1, 2, \ldots. \tag{2.9}$$

Since the recurrence relation (2.6) defines the operators H_n uniquely, (2.9) easily follows. Equations (2.7) and (2.8) are simple but not useful for evaluation of $B_{n,k}(s)$; therefore, we suggest a practically easier algorithm. For this reason we put (2.9) into (2.7). The following formulas are extracted:

$$B_{n,n-k+1}(s) = \sum_{i=k}^{n} \binom{i}{k} B_{n,n-i}(s)D^{i-k}s, \quad k = 1, 2, \ldots, n, \quad n = 0, 1, 2, \ldots \tag{2.10}$$

and

$$B_{n+1}(s) = \sum_{i=0}^{n} B_{n,n-i}(s)D^i s, \quad n = 0, 1, 2, \ldots. \tag{2.11}$$

Equation (2.11) expresses the standard (non-Abelian) Bell polynomials via the generalized ones:

$$B_{n+1}(s) = \sum_{i=0}^{n} B_{n,i}(s) D^{n-i} s, \quad n = 0, 1, 2, \ldots .$$

Rearranging the summation in (2.10) as $k \to n - k + 1$ yields after simple calculation

$$B_{n,k}(s) = \sum_{i=0}^{k-1} \binom{n-i}{n-k+1} B_{n,i}(s) D^{k-i-1} s, \quad k = 1, \ldots, n, \quad n = 0, 1, \ldots . \tag{2.12}$$

Evaluation of the generalized Bell polynomials by (2.10) gives ($s' = Ds$)

$$B_{n,1}(s) = s, \; B_{n,2}(s) = s^2 + nDs, \; B_{n,3}(s) = s^3 + ns's + (n-1)sDs + \binom{n}{2} D^2 s,$$

$$B_{n,4}(s) = s^4 + ns'^2 s + (n-1)ss's + (n-2)s^2 Ds$$
$$+ \binom{n}{2} s''s + n(n-2)(Ds)^2 + \binom{n-1}{2} sD^2 s + \binom{n}{3} D^3 s.$$

To solve the problem of the left division of L by L_s, a similar but somewhat simpler consideration is needed. The analog of Proposition 2.9 is as follows:

Proposition 2.10. *The following identity is valid:*

$$D^n = L_s H^+_{n-1} + B^+_n(s), \quad n = 1, 2, \ldots , \tag{2.13}$$

where

$$H^+_n = \sum_{k=0}^{n} B^+_{n-k}(s) D^k, \quad n = 0, 1, 2, \ldots . \tag{2.14}$$

2.3 Division and factorization of differential operators. Generalized Miura equations

Let

$$L = \sum_{n=0}^{N} a_n D^n, \quad a_n \in K \tag{2.15}$$

be a differential operator of order N. We shall study the right and left divisions of L by the operator L_s defined by (2.5). Suppose

$$L = ML_s + r, \qquad L = L_s M^+ + r^+, \tag{2.16}$$

where M and M^+ are the results of right and left divisions, r and r^+ being the remainders. Propositions 2.9 and 2.10 allow us to solve the problem of division in a simple way.

Proposition 2.11. *If the representation (2.16) is valid, then the remainder r and the result of division M are written as*

$$r = \sum_{n=0}^{N} a_n B_n(s),$$

$$M = \sum_{n=1}^{N} a_n H_{n-1} = \sum_{n=0}^{N-1} b_n D^n, \tag{2.17}$$

where

$$b_n = \sum_{k=n+1}^{N} a_k B_{k-1,k-n-1}(s), \quad n = 0, 1, \ldots, N-1. \tag{2.18}$$

For the proof it is enough to check

$$L = a_0 + \sum_{n=1}^{N} a_n [H_{n-1} L_s + B_n(s)] = \sum_{n=1}^{N} a_n H_{n-1} L_s + a_0 + \sum_{n=1}^{N} a_n B_n(s)$$

by the equality from Proposition 2.7 and to account for H_{n-1} given by (2.9). As a corollary we get the following:

Proposition 2.12. *For the linear operator L to be right-divisible by L_s without remainder, it is necessary and sufficient that s be a solution of the differential equation*

$$\sum_{n=0}^{N} a_n B_n(s) = 0. \tag{2.19}$$

If this condition holds, the operator L factorizes as $L = ML_s$, where M is given by (2.17) and (2.18).

Equation (2.19) is nonlinear. For $N = 2$ it is the Riccati-type equation known in the theory of the KdV equation as the Miura map. Therefore, it is natural to term it as a generalized right Miura equation . It links the function s and coefficients of the operator L. The left Miura equation is generalized by means of Proposition 2.2, giving the following theorem:

Theorem 2.13. *Let an invertible function φ be a solution to the linear differential equation*

$$\sum_{n=0}^{N} a_n D^n \varphi = 0. \tag{2.20}$$

Then the operator L, defined by (2.15), is right-divisible by L_s, where $s = \varphi' \varphi^{-1}$ and $\varphi' \equiv D\varphi$. Moreover, s is a solution of the right Miura equation (2.19).

To solve the left division problem, let us write the result of division in the form

$$M^+ = \sum_{n=0}^{N-1} b_n^+ D^n. \tag{2.21}$$

Now we should determine $b_n^+, n = 0, 1, \ldots, n-1$. To this aim we substitute (2.21) into the right-hand side of the second equation of (2.16). Following the lines of Proposition 2.11, we obtain

$$b_{N-1}^+ = a_N, \tag{2.22}$$

$$b_n^+ = a_{n+1} - L_s b_{n+1}^+, \quad n = 0, 1, \ldots, N-2, \tag{2.23}$$

and

$$r^+ = a_0 - L_s b_0^-. \tag{2.24}$$

Solving subsequently equations (2.23) and (2.24), we arrive at

$$b_n^+ = \sum_{k=n+1}^{N} (-1)^{k-n-1} L_s^{k-n-1} a_k, \quad n = 0, 1, \ldots, N-1 \tag{2.25}$$

and

$$r^+ = \sum_{k=0}^{N} (-1)^k L_s^k a_k. \tag{2.26}$$

The entities b_n^+, $n = 0, 1, \ldots, N-1$, and r^+ can be expressed in terms of the right Bell polynomials if we use (2.5) and take into account

$$L_s^k a = L_s^k e a = (-1)^k B_k^+(s) a.$$

Hence, (2.25) and (2.26) transform to

$$b_n^+ = \sum_{k=n+1}^{N} B_{k-n-1}^+(s) a_k, \quad n = 0, 1, \ldots, N-1 \tag{2.27}$$

and

$$r^+ = \sum_{k=0}^{N} B_k^+(s) a_k. \tag{2.28}$$

Formulas (2.16), (2.21), (2.25), and (2.26) give a solution of the left division problem of division of L by L_s. So, the following is proved:

Theorem 2.14. *For the operator L to be left-divisible by the operator L_s (without remainder), it is necessary and sufficient that s be a solution of the differential equation*

$$\sum_{k=0}^{N} B_k(s)^+ a_k = 0. \tag{2.29}$$

If this condition holds, the operator L factorizes as $L = L_s M^+$, where M^+ is given by (2.21) and (2.25) [or (2.27)]. For the reminder r^+ and the result of division M^+ equations (2.28) [or (2.26)] and (2.21) exist.

The nonlinear equation (2.29) is called the generalized left Miura equation, which is obviously linearized by Proposition 2.4. As a result, we have the following:

Proposition 2.15. *Let an invertible element φ satisfy the linear differential equation*

$$\sum_{n=0}^{N}(-1)^n B_n(s)^+ a_n D^n \varphi = 0.$$

Then the operator L determined by (2.15) is left-divisible by the operator L_s, where $s = -\varphi^{-1}\varphi'$. The function s is a solution to the generalized left Miura equation (2.29).

2.4 Darboux transformation. Generalized Burgers equations

The problem of the operator division is directly connected to the DT. To clarify this point, suppose that in the ring K there exists one more differentiation D_0 which commutes with the operator D. It may be a differentiation in a parameter t.

Let us introduce an auxiliary commutation relation

$$L_s r = r L_s + r' + [r, s]. \tag{2.30}$$

Indeed,

$$L_s r - r L_s = (D - s)r - r(D - s) = Dr - sr - rD + rs$$

$$= rD + Dr - sr - rD + rs = r' + [r, s].$$

Taking into account the equalities (2.30) and (2.16), we arrive at the relation

$$L_s(D_0 - L) = (D_0 - \tilde{L})L_s + D_0 s - r' - [r, s], \tag{2.31}$$

where

$$\tilde{L} = L_s M + r. \tag{2.32}$$

As the result, the following important conclusion can be drawn:

Proposition 2.16. *If a function s satisfies the equation*

$$D_0 s = r' + [r, s], \tag{2.33}$$

the operator L_s intertwines the operators $D_0 - L$ and $D_0 - \tilde{L}$,

$$L_s(D_0 - L) = (D_0 - \tilde{L})L_s. \tag{2.34}$$

The explicit expression for $\tilde{L}$ can be obtained in terms of (2.32) and (2.16) and has the form

$$\tilde{L} = a_0 + \sum_{n=1}^{N} (a'_n H_{n-1} + a_n H_n - s a_n H_{n-1}). \tag{2.35}$$

Let us write (2.33) explicitly using (2.16). It is established that for the intertwining relation (2.34) to be valid, it is necessary and sufficient that s be a solution of the equation

$$D_0 s = \sum_{n=0}^{N} [a'_n B_n(s) + a_n B_{n+1}(s) - s a_n B_n(s)]. \tag{2.36}$$

Remark 2.17. Equation (2.36) is nonlinear but linearizable. This equation (in a different form) was introduced in [388]. The form we suggest here is the most compact and convenient for further investigations, e.g., in the framework of the bilineraization technique of Hirota [210].

In the case of scalar functions and $L = D^2$ equation (2.36) is known as the Burgers equation. For this reason and owing to the integrability of (2.36) by the Cole–Hopf transformation, it is natural to refer to (2.36) as a generalized Burgers equation.

Proposition 2.18. *Suppose an invertible function φ is a solution to the linear differential equation*

$$D_0 \varphi = L\varphi.$$

Then the function s satisfies the generalized Burgers equation (2.36).

The obvious corollary of the intertwining relation (2.34) and Proposition 2.18 is as follows:

Theorem 2.19. *Let functions ψ and φ be solutions of the equations*

$$D_0 \psi = L\psi, \qquad D_0 \varphi = L\varphi \tag{2.37}$$

for an invertible function φ. Then the function

$$\tilde{\psi} = L_s \psi = D\psi - s\psi, \qquad s = (D\varphi)\varphi^{-1} \tag{2.38}$$

is a solution of the equation

$$D_0 \tilde{\psi} = \tilde{L} \tilde{\psi}. \tag{2.39}$$

The last statement accomplishes the proof of the Matveev theorem for differential polynomials [314] in its non-Abelian version.

The equality (2.35) gives a representation of the transformed operator in terms of the generalized Bell polynomials. The explicit expressions for the transformed coefficients are

$$a_N[1] = a_N, \tag{2.40}$$

$$a_k[1] = a_k + \sum_{n=k+1}^{N} [a_n B_{n,n-k} + (a'_n - s a_n) B_{n-1,n-1-k}], \tag{2.41}$$

$$k = 0, \dots, N-1.$$

2.5 Iterations and quasideterminants via Darboux transformation

Here we would like to revisit the non-Abelian iterated DT formulas following the ideas of the pioneering paper of Matveev [313], where the basic formulas were derived. Their Abelian counterpart is demonstrated in [324] and discussed also in [316, 322]. In fact, this approach goes back to the famous paper of Crum [94]. We will see, in the framework of a general non-Abelian DT theory, that the dressing procedure naturally produces the quasideterminants (Sect. 1.9). In the paper [191] this procedure is also properly analyzed for the matrix Schrödinger operator.

2.5.1 General statements

Let R be a differential algebra with a derivation $D : R \to R$ and $\phi \in R$ be an invertible element. Recall that we denote $D(g) = g'$ and $D^k(g) = g^{(k)}$. In particular, $D^{(0)}(g) = g$.

For $\psi \in R$ define $\mathcal{D}(\phi; \psi) = \psi' - \phi' \phi^{-1} \psi$. Following [321], we call $\mathcal{D}(\phi; \psi)$ the *DT* of ψ defined by ϕ.

Theorem 2.20. *Let $\phi_1, \dots, \phi_N \in R$. Define by induction the iterated DT $\mathcal{D}(\phi_N, \dots \phi_1; \psi)$ as follows. For $N = 1$, it coincides with the DT defined above. Assume $N > 1$. The expression $\mathcal{D}(\phi_N, \dots, \phi_1; \psi)$ is defined if $\mathcal{D}(\phi_N, \dots, \phi_2; \psi)$ is defined and invertible and $\mathcal{D}(\phi_N; \psi)$ is defined. In this case,*

$$\mathcal{D}(\phi_N, \dots \phi_1; \psi) = \mathcal{D}[\mathcal{D}(\phi_k, \dots \phi_2; \psi); \mathcal{D}(\phi_1; \psi)].$$

Theorem 2.21. *If all square submatrices of matrix $(\phi_i^{(j)})$, $i = 1, \dots, N$, $j = k-1, \dots, 0$ are invertible, then the Vandermond supermatrix defines the quasideterminant:*

$$\widehat{\mathcal{D}}(\phi_N, \dots, \phi_1; \psi) = \begin{vmatrix} \psi^{(k)} & \phi_1^{(k)} & \dots & \phi_k^{(k)} \\ \dots & \dots & \dots & \dots \\ \psi & \phi_1 & \dots & \phi_k \end{vmatrix}.$$

Recall that we use the "hat" symbol to denote the quasideterminant. This general statement first appeared in [313].

Proof. Iterations of the DT yield

$$\psi[N] = \psi^{(N)} + \sum_{m=0}^{N-1} s_m \psi^{(m)}, \tag{2.42}$$

which is a replica of an expression in [313, 322, 324] for the Abelian case. The iterated DT (2.42) as a function of ψ sends to zero any ϕ_i on which the transformation is constructed:

$$\phi_p[N] = \phi_p^{(N)} + \sum_{m=0}^{N-1} s_m \phi_p^{(m)} = 0, \quad p = 1, \dots, N. \tag{2.43}$$

The successive excluding of s_m as a function of the derivatives $\psi_p^{(m)}$ from the system (2.43) yields the algorithm that results in the evaluation of s_m; the procedure was pointed out already in [314], applied in [277] and, as it is seen from a comparison with the quasideterminant definition, could define the Vandermond quasideterminant.

Let us illustrate the scheme with the case of $N = 2$. The first iteration is based on a set of ϕ_p, $p = 1, 2$. The equations for s_i, $i = 0, 1$

$$\phi_1^{(2)} + s_0\phi_1 + s_1\phi_1' = 0, \quad \phi_2^{(2)} + s_0\phi_2 + s_1\phi_2' = 0 \tag{2.44}$$

yield

$$s_0 = -\phi_1^{(2)}\phi_1^{-1} - s_1\phi_1'\phi_1^{-1}.$$

Inserting this into the second relation of (2.44) produces the equation for s_1:

$$s_1(\phi_2' - \phi_1'\phi_1^{-1}\phi_2) = -\phi_2^{(2)} + \phi_1^{(2)}\phi_1^{-1}.$$

It is solved as

$$s_1 = (-\phi_2^{(2)} + \phi_1^{(2)}\phi_1^{-1})(\phi_2' - \phi_1'\phi_1^{-1}\phi_2)^{-1},$$

and

$$s_0 = -\phi_1^{(2)}\phi_1^{-1} - (-\phi_2^{(2)} + \phi_1^{(2)}\phi_1^{-1})(\phi_2' - \phi_1'\phi_1^{-1}\phi_2)^{-1}\phi_1'\phi_1^{-1},$$

both recognized as quasideterminants. The final expression for $\psi[2]$ is given by (2.42).

As mentioned in [322], the comparison of the resulting formula for $\psi[N]$ (2.42) and the formula for the DT

$$\mathcal{D}(\phi_{N-1}, \dots, \phi_1; \psi[N-1]) = \psi[N-1]' - \phi[N-1]'\phi[N-1]^{-1}\psi[N-1], \tag{2.45}$$

where $\phi[N-1] = \psi[N-1]|_{\psi=\phi_N}$, yields the non-Abelian Jacobi identity for quasi-Wronskians "for free." Recall that Crum [94] used the Jacobi identity to prove the determinant formulas for the iterated DT for Abelian entries.

Remark 2.22. The non-Abelian algorithm to exclude s_ms hints at the definition of quasideterminants as a function of submatrices a_{pm} (compare with the results in Sect. 1.9.1). Namely, it is enough to change $\psi_p^{(m)} \to a_{pm}$.

The solution of the system (2.43) with respect to s_p may be reinterpreted as the Vandermond-quasideterminant representation of the iterated DT for solutions (2.42) (with inserted s_m) and, next, linked to the DT for the potentials $a_k[N]$.

To check this proposition, let us substitute (2.42) into the evolution equation (2.37) for $\psi[N]$ (see Theorem 2.19):

$$\psi[N]_t = \sum_{k=0}^{n} a[N]_k \psi^{(N+k)} + \sum_{k=0}^{n} a[N]_k \sum_{m=0}^{N-1} (s_m \psi^{(m)})^{(k)}. \tag{2.46}$$

On the other hand,

$$\psi[N]_t = \psi_t^{(N)} + \sum_{m=0}^{N-1} (s_{m,t}\psi^{(m)} + s_m \psi_t^{(m)}) \tag{2.47}$$

$$= \sum_{k=0}^{n} (a_k \psi^{(k)})^{(N)} + \sum_{m=0}^{N-1} s_{m,t}\psi^{(m)} + \sum_{m=0}^{N-1} s_m \sum_{k=0}^{n} (a_k \psi^{(k)})^{(m)}.$$

Equating terms with the highest derivative $\psi^{(N+n)}$ gives

$$a[N]_n = a_n,$$

and, subsequently, for $\psi^{(N+n-1)}$ produces

$$a[N]_{n-1} = a_{n-1} + N a_n' + s_{N-1} a_n - a_n s_{N-1}.$$

For $\psi^{(N+n-2)}$ we obtain

$$a[N]_{n-2} = a_{n-2} + N a_{n-1}' + \frac{N(N+1)}{2} a_n'' - a_n (s_{N-2} + n s_{N-1}') \tag{2.48}$$

$$-a[N]_{n-1} s_{N-1} + (N-1) s_{N-1} a_n' + s_{N-1 a_{n-1}} + s_{N-2} a_n,$$

preserving the order of differentiation in (2.47) to keep the non-Abelian character. Substituting here $a[N]_{n-1}$ yields the explicit form of $a[N]_{n-2}$:

$$a[N]_{n-2} = a_{n-2} + N a_{n-1}' + \frac{N(N+1)}{2} a_n'' + [s_{N-1}, a_{n-1}] + [s_{N-2}, a_n]$$

$$-n a_n s_{N-1}' - (N a_n' + [s_{N-1}, a_n]) s_{N-1}. \tag{2.49}$$

One could compare the resulting expression (2.49) for commuting entries and for $a_n' = a_{n-1}' = 0$,

$$a[N]_{n-2} = a_{n-2} - n a_n s_{N-1}', \tag{2.50}$$

with (2.37) [322] when taking into account that $s_{N-1} = -\ln_{xx} W(\varphi_1, \ldots, \varphi_N)$, with φ_j being solutions of (2.38).

Corollary 2.23. [321] *In the commutative case, the iterated Darboux transformation is a ratio of two Wronskians, as follows by direct application of the Kramer rule to (2.43):*

$$\mathcal{D}(\phi_k, \dots, \phi_1; \psi) = \frac{W(\phi_1, \dots \phi_k, \psi)}{W(\phi_1, \dots, \phi_k)}. \tag{2.51}$$

All these results are naturally generalized to the cases when the main properties of the DTs are valid: the n-iterated transform is a linear function of $T^n\psi$ and there is an n-dimensional kernel of the transformation operator. This remark relates first of all to the next section (see also [321]) and to the Moutard/Goursat transformations (Chap. 6). The algorithm of the construction given here is easily transferred to the iterated Moutard/Goursat transformation because the "kernel property" (2.43) is also valid.

2.5.2 Positons

An interesting illustration of the application of (2.51) is concerned with *positons*. Positons were introduced by Matveev [318, 319] as a class of singular solutions of the KdV equation,

$$u_t - 6uu_x + u_{xxx} = 0, \tag{2.52}$$

that lead to a trivial scattering matrix for the associated spectral problem

$$-\psi_{xx} + u\psi = \lambda\psi. \tag{2.53}$$

Here we consider this topic in more detail, following [323].

The KdV equation can be written as the compatibility condition [13] of the linear system of equations comprising the spectral problem (2.53) and the evolutionary equation

$$\psi_t = -4\psi_{xxx} + 6u\psi_x + 3u_x\psi. \tag{2.54}$$

Note that the spectral problem (2.53) is a representative of the general equation (2.37).

Let $\phi(\lambda)$ solve the spectral equation (2.53). Differentiation in λ produces (in general, linearly independent) solutions $\phi^{[m]} = \partial^m\phi(\lambda)/\partial\lambda^m$ of the same equation. The set of solutions $\phi_1, \dots, \phi_1^{[m_1]}, \phi_2, \dots, \phi_2^{[m_2]}, \dots, \phi_n, \dots, \phi_n^{[m_n]}$, m_i are integers, generated by the λ-derivatives in the points λ_1, $\lambda_2, \dots, \lambda_n$, yields the iterated DT, which is the Abelian specification of the transform (2.42) and the quasideterminant formula (2.43).

Proposition 2.24. *Let $u(x,t)$ be a solution of the KdV equation (2.52). The Wronskians*

$$W_1 = W(\phi_1, \dots, \phi_1^{[m_1]}, \phi_2, \dots, \phi_2^{[m_2]}, \dots, \phi_n, \dots, \phi_n^{[m_n]})$$

and

$$W_2 = W(\phi_1, \dots, \phi_1^{[m_1]}, \phi_2, \dots, \phi_2^{[m_2]}, \dots, \phi_n, \dots, \phi_n^{[m_n]}, \psi)$$

produce the DT

$$u[N] = u - 2\partial_x^2 \ln W_1, \qquad \psi[N] = W_2/W_1. \tag{2.55}$$

The Lax pair equations (2.53) and (2.54) are covariant with respect to the DT (2.55). In other words, the function $\psi[N]$ satisfies (2.53) and (2.54) with $u \to u[N]$ and $\psi \to \psi[N]$ and $u[N]$ is a new solution of the KdV equation.

Now we apply the DT (2.55) to dress the simplest seed solution $u = 0$ with the choice $\psi = \exp(\mathrm{i}kx + 4\mathrm{i}k^3t)$, $k^2 = \lambda$. As a solution of the spectral equation with zero potential we choose an oscillating function

$$\phi = \sin\kappa(x + x_1(\kappa) + 4\kappa^2 t) \equiv \sin\theta.$$

Here κ is a real parameter and $x_1(\kappa)$ is an analytic function in the vicinity of the point κ. Therefore, a new solution to the KdV equation is given by

$$u[1] = -2\partial_x^2 \ln W(\phi, \partial_\kappa \phi) \tag{2.56}$$

and is written explicitly as

$$u = 32\kappa^2 \frac{\sin\theta - \kappa\gamma\cos\theta}{(\sin 2\theta - 2\kappa\gamma)^2} \sin\theta, \tag{2.57}$$

where

$$\gamma = \partial_\kappa \theta = x + x_2 + 12\kappa^2 t, \quad x_2 = x_1 + \kappa\partial_\kappa x_1, \quad W(\phi, \partial_\kappa\phi) = \sin 2\theta - 2\kappa\gamma. \tag{2.58}$$

The solution (2.57) is determined by three real parameters x_1, x_2, and κ and has a second-order pole in x. The precise pole position is found by solution of the nonlinear functional equation $W(\phi, \partial_\kappa\phi) = 0$. The corresponding solution of the Lax pair [or of (2.51)] takes the form

$$\psi(x,k) = \frac{W[\phi, \partial_\kappa\phi, \exp(\mathrm{i}kx + 4\mathrm{i}k^3t)]}{W(\phi, \partial_\kappa\phi)} \tag{2.59}$$

$$= \left(-k^2 + \frac{4\mathrm{i}kx\sin^2\theta}{\sin 2\theta - 2\kappa\gamma} - \kappa^2 \frac{\sin 2\theta + 2\kappa\gamma}{\sin 2\theta - 2\kappa\gamma}\right) \mathrm{e}^{\mathrm{i}kx + 4\mathrm{i}k^3 t}.$$

In the point $k = \kappa$, this solution is simplified:

$$\psi(k,x) = -4\kappa^2 \frac{\sin\theta}{\sin 2\theta - 2\kappa\gamma}. \tag{2.60}$$

We see from (2.60) that the function ψ is localized near its pole but is not square-integrable on the whole x-axis. The point κ^2 is called the Wigner–von Neuman resonance [320].[1] Because $\kappa^2 > 0$, the solution (2.57) is called *positon*, as distinct from the soliton solution for which $\kappa^2 < 0$ (Sect. 8.7).

[1] Generic aspects of the scattering theory of the potentials leading to the Wigner–von Neumann resonances are discussed in [312, 311].

Asymptotic behavior of the function ψ is given by

$$\psi \to (-k^2 + \kappa^2)\mathrm{e}^{\mathrm{i}kx+4\mathrm{i}k^3t}[1 + o(1)], \qquad x \to \pm\infty. \tag{2.61}$$

Let us compare (2.61) with the standard Jost solution $J(x, k, t)$ asymptotic of the linear Schrödinger equation with a decreasing potential:

$$J(x, k, t) \to \mathrm{e}^{\mathrm{i}kx}(1 + o(1)), \quad x \to +\infty,$$
$$J(x, k, t) \to a(k, t)\mathrm{e}^{\mathrm{i}kx} + b(k, t)\mathrm{e}^{-\mathrm{i}kx}, \quad x \to -\infty,$$

where $a(k, t)$ and $b(k, t)$ are the transmission and reflection coefficients. For the positon potential we obtain

$$a(k, t) = 1, \qquad b(k, t) = 0.$$

Potentials for which $b(k, t) = 0$ are called reflectionless. The well-known example of the reflectionless potentials is provided by solitons. However, for solitons we have $a(k, t) \neq 1$. Hence, positons give a unique example of supertransparent (or superreflectionless) long-range potentials.

A two-positon solution is generated by the evident extension of (2.56),

$$u = -2\partial_x^2 \ln W(\phi_1, \partial_{\kappa_1}\phi_1, \phi_2, \partial_{\kappa_2}\phi_2),$$

with

$$\phi_1 = \sin\kappa_1(x + x_1 + 4\kappa_1^2 t), \qquad \phi_2 = \sin(x + x_2 + 4\kappa_2^2 t),$$

and is determined by six real parameters. For $x \to \infty$ the two-positon solution is decomposed into a sum of two free positons. It should be stressed that the positon scattering is not accompanied with a phase shift typical for the soliton scattering.[2]

Interesting suggestions concerning physical applications of positons can by found in the paper by Matveev [323].

2.6 Darboux transformations at associative ring with automorphism

In this section we reformulate and analyze the results from the paper of Matveev [321] for further use in the derivation of chain equations and joint covariance of operator pairs [265, 267, 271]. We begin with general notations. Let $\mathbf{R}$ be an associative ring with an automorphism, implying that there exists

[2] For singular potentials the scattering data are not uniquely defined. Different self-adjoint extensions of the same differential operators might lead to different scattering operators. The definition of the scattering coefficients given above is in agreement with the nonlinear picture of interaction between positons and solitons, although the latter can be analyzed independently of this definition.

a linear invertible map T, $\mathbf{R}\to\mathbf{R}$ such that for any $\psi(x,t)$ and $\varphi(x,t)\in\mathbf{R}$, $x\in R^n$, $t\in R$ we have

$$T(\psi\varphi)=T(\psi)T(\varphi),\qquad T(1)=1. \tag{2.62}$$

The automorphism with the defining property (2.62) allows us to write down a wide class of functional-differential-difference and difference-difference equations starting from

$$\psi_t(x,t)=\sum_{m=-M}^{N}U_mT^m\psi, \tag{2.63}$$

where M and N are integers. For example, the operator T can be chosen as

$$T\psi(x,t)=\psi(qx+\delta,t),$$

where $q\in GL(n,\mathbb{C})$, $\delta\in R^n$. Another choice gives

$$T\psi(x)=W\psi(x)W^{-1}\ ,\quad W\in GL(n,\mathbb{C}).$$

We will save the notations and conditions of the paper [321] discussing other potentials until the end of Sect. 4.9.

Let us consider two DTs for solutions of (2.63),

$$D^{\pm}f=f-\sigma^{\pm}T^{\pm1}f,\qquad \sigma^{\pm}=\varphi\left(T^{\pm1}\varphi\right)^{-1}, \tag{2.64}$$

where φ is a particular solution of the same equation (2.63). For the case of a differential ring and for $Tf(x,t)=f(x+\delta,t)$, $x,\delta\in R$ the limit $\partial f=\lim_{\delta\to0}\frac{1}{\delta}(T-1)f(x,t)$ gives the link to the classical DT.

To derive the DT of potentials U_m, it is necessary to evaluate the derivative of the elements $\sigma^{\pm}$ with respect to the variable t (say, time). We shall do it by introducing the special functions (analog of the differential Bell polynomials), similar to [467]. Let us start from the first version of the DT definition D^+, expressing $T\varphi$ from (2.64); hence, $T\varphi=(\sigma^+)^{-1}\varphi$. Acting on this relation by T and taking into account (2.62) yields $T^2\varphi=T\left[(\sigma^+)^{-1}\right]T\varphi=T\left[(\sigma^+)^{-1}\right](\sigma^+)^{-1}\varphi$. Repeating the action, we arrive at

$$T^m\varphi=\prod_{k=0}^{m-1}[T^k(\sigma^+)]^{-1}\varphi=B_m^+(\sigma^+)\varphi. \tag{2.65}$$

Here and below the product is ordered by the index k running from right to left.

Definition 2.25. *Equation (2.65) defines the function*

$$B_m^+(\sigma)=\prod_{k=0}^{m-1}\left[T^k(\sigma)\right]^{-1}.$$

It is convenient to write down the t-derivative of σ by means of the functions $B_m^+(\sigma^+)$ that are connected with the generalized Bell polynomials [467, 271]:

$$\sigma_t^+ = \sum_{m=-M}^{N} \left[U_m\ B_m^+\left(\sigma^+\right)\sigma^+ - \sigma^+ T\left(U_m\ \right) B_{m+1}^+\left(\sigma^+\right)\sigma^+\right]. \tag{2.66}$$

The resulting equation (2.66) is a nonlinear equation associated with (2.63) that is reduced to a generalized Miura transformation in the stationary case (Sect. 2.3).

The Matveev theorem for polynomials of the automorphism T provides far-reaching generalizations of the conventional Darboux theorem proved originally for the second-order differential equation (for generalizations see [324] as well) and can be formulated by means of the introduced entries in the following way:

Theorem 2.26. *Let the functions $\varphi \in \boldsymbol{R}$ and $\psi \in \boldsymbol{R}$ satisfy (2.63). Then the function $\psi^+ = D^+\psi$ satisfies the equation*

$$\psi_t^+(x,t) = \sum_{m=-M}^{N} U_m^+ T^m \psi^+,$$

where the coefficients are evaluated from the recurrence relations

$$U_{-M}^+ = U_{-M}, \tag{2.67}$$

$$U_1^+ - U_0^+\sigma^+ = U_1 - \sigma^+ T U_0 - \sigma_t^+, \tag{2.68}$$

$$U_m^+ - U_{m-1}^+ T^{m-1}\sigma^+ = U_m - \sigma^+ T U_{m-1}, \tag{2.69}$$

$$U_N^+ = \sigma^+\left(TU_N\right)\left(T^N\sigma^+\right)^{-1}. \tag{2.70}$$

Equations (2.67)–(2.70) define recursively the DTs of the coefficients (potentials) of the differential equation (2.63). Solving the recurrence (2.69) by means of (2.65) yields

$$U_m^+ = \sum_{l=0}^{m+M} U_{-M+l} - \sigma^+\left(TU_{-M+l-1}\right) B_{-M+l}^+\left(\sigma^+\right)\left[B_m^+\left(\sigma^+\right)\right]^{-1} \tag{2.71}$$

$$U_N^+ = \sigma^+\left(TU_N\right)\left(T^N\sigma^+\right)^{-1}. \tag{2.72}$$

Proof. For the proof it is necessary to check the additional equality that appears from the term $T^m\psi$ with essential use of the expression for σ_t^+ from (2.66).

This theorem establishes the covariance (form invariance) of (2.63) with respect to the DT (2.64).

The formalism for the second DT from (2.64) may be similarly constructed on the ground of the identity

$$T^m\varphi = \prod_{k=0}^{m} T^k\left(\sigma^-\right) T^{-1}\varphi = B_m^-\left(\sigma^-\right) T^{-1}\varphi. \tag{2.73}$$

The definition of the lattice Bell polynomials of the second type $B_m^-\left(\sigma^-\right)$ can be extracted from (2.73). The evolution equation for σ^- is similar to (2.66):

$$\sigma_t^- = \sum_{m=-M}^{N} \left[U_m\ B_m^-\left(\sigma^-\right) - \sigma^- T^{-1}\left(U_m\ \right) B_{m-1}^-\left(\sigma^-\right)\right].$$

It may be considered as a further generalization of the Burgers equation (2.36) and gives the second generalized Miura map for stationary solutions of (2.63). Explicit formulas for U_m^- are similar to (2.68)–(2.72).

2.7 Joint covariance of equations and nonlinear problems. Necessity conditions of covariance

If a pair of linear problems is simultaneously covariant with respect to a Darboux transformation, it generates Bäcklund transformations of the corresponding compatibility condition, or a nonlinear integrable equation. In the context of such an integrability, the joint covariance principle, used to construct solutions of nonlinear problems from the very beginning [313], can be considered as the origin of a classification scheme [265, 267]. In this book, we examine realizations of this scheme and seek the covariant form of equations and an appropriate basis with the simplest transformation properties. Note that a proof of the covariance theorems for the linear operators incorporates the generalized Burgers equations that in stationary versions reduce to the generalized Miura transformation. We give and examine the explicit form of the Miura equality in both the general and the stationary cases (see also [270]). This equality gives an additional nonlinear equation that is automatically solved by the Cole–Hopf substitution and is used to generate dressing t-chain equations [79]. We show how the form of the covariant operator can be found by comparing some kind of Frechét derivatives of the operator coefficients and the transforms.

2.7.1 Towards the classification scheme: joint covariance of one-field Lax pairs

The basis of the formalism introduced here has been elaborated in [265, 267] and the compact formulas with the generalized Bell polynomials are given in Sect. 2.2. The formalism is valid for non-Abelian coefficients a_n as well, and

for solutions of (2.37); ϕ and ψ can be considered as matrices or operators. For simplicity, we start with the scalar case.

First we consider particular examples of the theory to derive the explicit expressions and show some details. We begin with a very simple analysis to clarify the integrability notion we introduce. Note first that the higher coefficients a_n (with $n = N$ and $n = N - 1$) are transformed almost trivially. It follows that the coefficients, in general, do not play the role of potentials to be dressed, or solutions of the nonlinear equation being the compatibility condition.

If $N = 2$, the general transformation (2.40) and (2.41) reduces to

$$\begin{aligned} &a_2[1] = a_2 \equiv a(x,t), \qquad a_1[1] = a_1(x,t) + Da(x,t), \\ &a_0[1] = a_0 + Da_1(x,t) + 2a(x,t)D\sigma + \sigma Da(x,t). \end{aligned} \tag{2.74}$$

Only the Abelian case is considered at this stage. The explicit form of the transformations clearly shows a difference between the coefficients $a(x,t)$ and $a_1(x,t)$, which transform irrespectively to solutions, on the one hand, and $a_0 = u(x,t)$, which will stand for an unknown function in a forthcoming nonlinear equation, on the other hand. We call $a_0 = u(x,t)$ the potential in the context of the Lax representation. The KdV case can be easily recognized here. Namely, when $a = \text{const}$ and $a_1 = 0$, a_0 plays the role of the only unknown function in the KdV equation (we call this situation the one-field case). We can therefore formulate the following:

Proposition 2.27. *The Abelian case with $N = 2$ is the first nontrivial example of a set of covariant operators with coefficients $a_{1,2}$ that depend only on x and an additional parameter (e.g., t), but their transformations contain only the functions $a_{1,2}$ and is hence said to be* trivial. *The transformation (generalized DT) for u is given by the last equation in (2.74) and depends on both $a_{1,2}$ and solutions of (2.36) via σ.*

Let us consider the third-order operator as the second one in the Lax pair. Letting $N = 3$ in (2.40) and (2.41) and changing $a_i \to b_i$, we have

$$b_3[1] = b_3, \quad b_2[1] = b_2 + Db_3', \quad b_1[1] = b_1 + Db_2 + 3b_3D\sigma + \sigma Db_3,$$

$$b_0[1] = b_0 + Db_1 + \sigma Db_2 + [\sigma^2 + (2D\sigma)]Db_3 + 3b_3(\sigma D\sigma + D^2\sigma). \tag{2.75}$$

We consider (2.74) and (2.75) as coefficients of the Lax pair of operators, both of which depend on the only variable u, and suppose that the coefficients of the operators and their derivatives with respect to x are analytic functions of u. We now choose $D \to \frac{\partial}{\partial y}$ and $L \to L_1$ in (2.37) corresponding to the case (2.74) and leave the parameter t, i.e., $D_0 \to \frac{\partial}{\partial t}$ for the second case, forming the Lax pair

$$\psi_y = L_1\psi, \tag{2.76}$$

$$\psi_t = L_2\psi. \tag{2.77}$$

Here $L_2 = \sum_{i=0}^{2} b_i D^i$.

Recall the KdV case. The general stationary version of (2.33) for $N = 2$ is

$$\sum_{n=0}^{2} a_n B_n = c = \text{const},$$

which yields

$$\sigma^2 + \sigma' + u = c. \tag{2.78}$$

Note that (2.33) for $N = 3$ is still valid for the same $\sigma = \phi_x \phi^{-1}$ if ϕ is a solution of the Lax pair (2.76) and (2.77). If we restrict ourselves to the case $b_2 = 0$ and $b_3 = b = \text{const}$ in (2.75), we obtain the second equation in the KdV Lax pair.

Returning to the general case and taking into account the triviality of transforms of $b_3 = b(x,t)$ and b_2 in the aforementioned sense, we find that the first nontrivial potential is $b_1 = F(u, u', \ldots)$. Suppose that the covariance principle holds or, equivalently, take the following equation for F:

$$\begin{aligned} b_1[1] = F(u[1]) &= F(u + Da_1 + 2aD\sigma + \sigma Da) \\ &= F(u) + Db_2 + 3bD\sigma + \sigma Db. \end{aligned} \tag{2.79}$$

The analyticity of F permits us to expand the left-hand side of (2.79) in a Taylor series:

$$F(u[1]) = F(u) + F_u(2aD\sigma + Da_1 + \sigma Da) + F_{Du}(\ldots) + \ldots . \tag{2.80}$$

Compare the transformation (2.79) with the Frechét differential (2.80) of the function F. Both equations are identical if the coefficients of σ, $D\sigma$, and the free term in both equations are the same. Introducing $F_u = c(x,t)$ yields

$$2ac = 3b, \tag{2.81}$$

or

$$F(u) = \frac{3bu}{2a}$$

with the additional conditions

$$cDa_1 = Db, \tag{2.82}$$

$$cDa = Db. \tag{2.83}$$

Substituting c from (2.81) in (2.83), we pass either to $3D(\ln a) = 2D(\ln b)$ and obtain $b = a^{3/2} c_1(t)$, or to $Da = Db = 0$. In the last case, (2.83) is valid with an arbitrary c or mutually independent $b(t)$ and $a(t)$, while (2.82) yields the equation for a_1 for both cases, $3Da_1 = 2aDb/3b$ with an arbitrary $c_1(t)$.

Further conditions follow from the last equation in (2.75), i.e., if we introduce a new analytic function G and set $b_0 = G(u, u', \ldots)$, the transformed b_0 gives

$$G(u + Da_1 + 2aD\sigma + \sigma Da) = G(u) + G_u(Da_1 + 2aD\sigma + \sigma Da) + G_{Du}D(Da_1 + 2aD\sigma + \sigma Da) + \ldots . \tag{2.84}$$

The DT formula for the potential u is obviously used. The DT for the last coefficient b_0 [see (2.75)] yields

$$b_0[1] = G(u) + Db_1 + \sigma Db_2 + [\sigma^2 + 2(D\sigma)]Db + 3bD\left(\frac{\sigma^2}{2} + D\sigma\right). \tag{2.85}$$

We now consider a general version of the Miura transformation (2.78) which has the form

$$\sum_0^2 a_n B_n = u + a_1\sigma + a(\sigma^2 + D\sigma) \equiv \mu,$$

and can be used to express σ^2 in (2.85). Doing this and equating (2.84) and (2.85) yields

$$D\frac{3bu}{2a} + \sigma Db_2 + \left(\frac{\mu - u - a_1\sigma}{a} + D\sigma\right)Db + 3bD\left(\frac{\mu - u - a_1\sigma}{2a} + D\sigma\right) = G_u(Da_1 + 2aD\sigma + \sigma Da) + G_{Du}D(Da_1 + 2aD\sigma + \sigma Da). \tag{2.86}$$

From (2.86) we obtain the coefficients

$$G_u(Du)2a = 3b \tag{2.87}$$

for $D^2\sigma$,

$$G_u 2a + \frac{9b(Da)}{2a} = \frac{Db - 3ba_1}{2a} \tag{2.88}$$

for $D\sigma$ taking (2.87) into account, and

$$G_u Da + \frac{3b}{2a}D^2 a(x,t) = Db_2 - \frac{a_1}{a} - 3bD\left(\frac{a_1}{2a}\right) \tag{2.89}$$

for σ. The free term is

$$D\frac{3bu}{2a} + \left(\frac{\mu - u}{a}\right)Db + 3bD\left(\frac{\mu - u}{2a}\right) = G_u Da_1 + \frac{3b(D^2 a_1)}{2a}. \tag{2.90}$$

From (2.87) and (2.88) we obtain

$$G_u = \frac{Db}{2a} - \frac{3ba_1}{4a^2} - \frac{9b(Da)}{4a^2}. \tag{2.91}$$

If G_u is nonzero, then it follows from (2.89) that

$$\left(\frac{Db}{2a} - \frac{3ba_1}{4a^2} - \frac{9b(Da)}{4a^2}\right)Da + \frac{3b}{2a}D^2 a = Db_2 - \frac{a_1}{a} - 3bD\left(\frac{a_1}{2a}\right).$$

The free term (2.90) gives

$$u\frac{Db}{2a}+\mu\frac{Db}{a}-\frac{3bDa}{2a^2}=\left(\frac{Db}{2a}-\frac{3ba_1}{4a^2}-\frac{9bDa}{4a^2}\right)Da_1(x,t)+\frac{3b(D^2a_1)}{2a}. \quad (2.92)$$

If u is linearly independent of σ and its derivatives and we do not take into account higher terms in the Frechét differential, then the only choice $Db=0$ eliminates the term with u, and (2.92) simplifies to

$$D^2a_1-\frac{a_1(Da_1)}{2a}=0.$$

The condition $Da=0$ as a consequence of (2.83) has been used. Equation (2.89) also simplifies to

$$Db_2-\frac{a_1}{a}-\frac{3b(Da_1)}{2a}=0$$

and integration gives the expression for b_2.

Another possibility is $G_u=0$, which gives

$$\frac{9b(Da)}{2a}=\frac{Db-3ba_1}{2a},$$

instead of (2.91). The free term transforms as

$$u\frac{Db}{2a}+\mu\left(\frac{Db}{a}-\frac{3bDa}{2a^2}\right)=\frac{3b(D^2a_1)}{2a}$$

and gives the conditions $Db=Da=0$ for the same reasons. In turn, this means that $a_1=0$ and, finally, from (2.89), $Db_2=0$. Hence, this case contains the KdV equation with the (possibly, t-dependent) $a(t)$, $b(t)$, and $b_2(t)$.

Remark 2.28. The results for the single isolated equation (2.76) contain a rather wide class of coefficients, in comparison with the joint covariance of (2.76) and (2.77). Namely, a and a_1 are arbitrary functions of x and t. This may be useful for constructing potentials and solutions (e.g., special functions) for the linear Schrödinger equation and evolution equations in one-dimensional quantum mechanics [214].

The KdV case can be described separately (again using the notation $f'=Df$):

$$G_u\sigma'+G_{u'}\sigma''=\frac{3b(1-a)u'}{4a^2}+\frac{3b\sigma''}{4a}.$$

The only possible choice, if we consider σ, σ', and u' as independent variables, is

$$G_u=0,\qquad G_{u'}=\frac{3b}{4a},$$

or taking into account the condition of zero coefficient for u', $a = 1$, we obtain

$$G(u, u', \ldots) = \frac{3bu'}{4}.$$

This result leads directly to one of the equivalent Lax pairs for the KdV equation.

2.7.2 Covariance equations

First we reproduce the "Abelian" scheme, generalizing the study of the Boussinesq equation [270]. To start with, we should fix the number of fields. Let us consider the third-order operator (2.20) with coefficients b_k, $k = 0, 1, 2, 3$, reserving a_k for the coefficients in the second operator in a Lax pair. Suppose, both operators depend on the only potential function w. The problem we consider now can be formulated as follows: to find restrictions on the coefficients $b_3(t)$, $b_2(x, t)$, $b_1 = b(w, t)$, and $b_0 = G(w, t)$ compatible with the DT rules of the potential function w induced by the DT for b_i. The classical DT for the third order operator coefficients (Matveev generalization [314]) yields

$$b_2[1] = b_2 + b_3', \tag{2.93}$$

$$b_1[1] = b_1 + b_2' + 3b_3\sigma', \tag{2.94}$$

$$b_0[1] = b_0 + b_1' + \sigma b_2' + 3b_3(\sigma\sigma' + \sigma''), \tag{2.95}$$

having in mind that the highest coefficient b_3 does not transform. Note also that $b_3' = 0$ yields invariance of b_2.

The general idea of the DT form invariance can be realized considering transformations of the coefficients consistent with respect to the fixed transform of w. Generalizing the analysis of the third order operator transformation [270], we arrive at the equations for the functions $b_2(x, t)$, $b(w, t)$, and $G(w)$. The covariance of the spectral equation

$$b_3\psi_{xxx} + b_2(x, t)\psi_{xx} + b(w, t)\psi_x + G(w, t)\psi = \lambda\psi \tag{2.96}$$

can be considered separately and leads to the link between b_i only. We, however, study the problem (2.96) in the context of the Lax representation for some nonlinear equation; hence, the covariance of the second Lax equation is taken into account from the very beginning. We refer to such an approach as the *principle of joint covariance* [265, 267]. The second (evolution) equation is written as

$$\psi_t = a_2(x, t)\psi_{xx} + a_1(x, t)\psi_x + w\psi, \tag{2.97}$$

with the operator on the right-hand side having again the general polynomial form of (2.20).

If we consider the operators L and A of the form $\sum a_i D^i$, specified in equations (2.96) and (2.97) as the Lax pair equations, the DT of w implied

by the covariance of (2.97) should be compatible with DT formulas of both w-dependent coefficients of (2.96):

$$a_2[1] = a_2 = a(x,t), \quad a_1[1] = a_1(x,t) + Da(x,t),$$

$$a_0[1] = w[1] = w + a_1' + 2a_2\sigma' + \sigma a_2'.$$

The following important relations being in fact the identities in the DT theory [467] are the particular cases of the generalized Burgers equation for σ (2.36):

$$\sigma_t = [a_2(\sigma^2 + \sigma_x) + a_1\sigma + w]_x \tag{2.98}$$

for the problem (2.97) and

$$b_3(\sigma^3 + 3\sigma_x\sigma + \sigma_{xx}) + b_2(\sigma^2 + \sigma_x) + b(w,t)\sigma + G(w) = \text{const}$$

for (2.96), where ϕ is a solution of both Lax equations.

Suppose now that the coefficients of the operators are analytic functions of w together with its derivatives (or integrals) with respect to x (such functions are named functions on the prolonged space [33]). For the coefficient $b_0 = G(w,t)$ this means

$$G = G(\partial^{-1}w, w, w_x, \ldots, \partial^{-1}w_t, w_t, w_{tx}, \ldots). \tag{2.99}$$

The covariance condition is formulated for the Frechét derivative of the function G on the prolonged space. In other words, the first terms of a multidimensional Taylor series for (2.99) read

$$G(w + a_1' + 2a_2\sigma' + \sigma a_2') = G(w) + G_{w_x}(a_1' + 2a_2\sigma' + \sigma a_2')' + \ldots. \tag{2.100}$$

We show only the terms which enter the "minimal" equations of the hierarchy.

In full analogy with (2.94) and (2.100), quite similar expansion arises for the coefficient $b_1 = b(w,t)$. Equating the DT and the expansion, we obtain the condition

$$b_2' + 3b_3\sigma' = b_w(a_1' + 2a_2\sigma' + \sigma a_2') + b_{w'}(a_1' + 2a_2\sigma' + \sigma a_2')' \ldots. \tag{2.101}$$

We call this equation as the (first) *joint covariance equation* that guarantees consistency between transformations of the coefficients of the Lax pair (2.96) and (2.97). In the frame of our choice $a_2' = 0$, the equation simplifies and linear independence of the derivatives $\sigma^{(n)}$ yields two constraints

$$3b_3 = 2b_w a_2, \qquad b_2' = b_w a_1',$$

or, solving the second and plugging into the first, results in

$$b_w = 3b_3/2a_2, \qquad b_2' = 3b_3 a_1'/2a_2. \tag{2.102}$$

So, if one wants to save the form of the standard DT for the variable w (potential), simple comparison of both transformation formulas gives the following connection for $b(w)$ [with arbitrary function $\alpha(t)$]:

$$b(w,t) = 3b_3 w/2a_2 + \alpha(t). \tag{2.103}$$

Equating the expansion (2.100) with the transform of the $b_0 = G(w,t)$ yields

$$b_1' + \sigma b_2' + 3b_3(\sigma^2/2 + \sigma')' \tag{2.104}$$
$$= G_{w_x}(a_1' + 2a_2\sigma' + \sigma a_2')' + G_{\partial^{-1}w_t}[a_{1t} + 2\partial^{-1}(a_2\sigma_t') + \partial^{-1}(\sigma a_2')_t] + \dots.$$

This second joint covariance equation also simplifies when $a_2' = 0$ and (2.103) is accounted for:

$$3b_3 w'/2a_2 + \sigma b_2' + 3b_3(\partial^{-1}\sigma_t - w)'/2a_2 + 3b_3\sigma''/2 \tag{2.105}$$
$$= G_{w_x}(a_1' + 2a_2\sigma')' + G_{\partial^{-1}w}(a_1 + 2a_2\sigma) + G_{\partial^{-1}w_t}(a_{1t} + 2a_2\sigma_t) + \dots.$$

Note that the "Miura" transform (2.98) is used on the left-hand side of (2.105) and linearizes the Frechét derivative with respect to σ; therefore, the derivatives of the function G,

$$G_{w_x} = 3b_3/4a_2, \quad G_{\partial^{-1}w_t} = 3b_3/4a_2^2, \quad G_{\partial^{-1}w} = b_2'/2a_2,$$

are accompanied by the constraint

$$a_{1t} + a_2 a_1'' + a_1 a_1' = 0, \tag{2.106}$$

which acquires the form of the Burgers equation after using (2.102). Finally, the integration of (2.102) gives

$$b_2 = 3b_3 a_1/2a_2 + \beta(t) \tag{2.107}$$

and the "lower" coefficient of the third-order operator is expressed by

$$G(w,t) = 3b_3 w_x/2a_2 + 3b_3 a_1'\partial^{-1}w/2(a_2)^2 + 3b_3\partial^{-1}w_t/2a_2^2.$$

Proposition 2.29. *The expressions (2.97), (2.96), (2.103), and (2.107) define the covariant Lax pair when the constraints (2.102) and (2.106) hold.*

Remark 2.30. We cut the Frechét differential formulas on the level that is necessary for the minimal flows. The account of higher terms leads to the whole hierarchy, similarly to [260, 261].

2.7.3 Compatibility condition

In the case $a_2' = 0$ the Lax system (2.96) and (2.97) produces the following compatibility conditions:

$$\begin{aligned}
2a_2b_3' &= 3b_3a_2', \\
b_{3t} &= 2a_2b_2' - 3b_3a_1'', \\
b_{2t} &= a_2b_2'' + 2a_2b_1' + a_1b_2' - 3b_3a_1'' - 2b_2a_1' - 3b_3a_0', \\
b_{1t} &= a_2b_1'' + a_1b_1' - b_3a_1''' - b_2a_1'' - b_1a_1' - 3b_3a_0'' - 2b_2a_0' + 2a_2b_0', \\
b_{0t} &= a_1b_0' + a_2b_0'' - b_1a_0' - b_2a_0'' - b_3a_0'''.
\end{aligned} \tag{2.108}$$

In the particular case $a_2 = 0$ we derive from the first of the equalities (2.108) the constraint $b_3' = 0$. The direct consequence of (2.107) is $b_{3t} = 0$. In the rest of the equations the links (2.108) and (2.107) are taken into account. Hence, (2.106) in combination with the expression for b_{2t} produces $\beta_t = -2\beta a_1'$ with $\beta(t)$ from (2.107). The last two equations (for $b_3 = 1$ and $a_2 = -1$) become

$$\alpha w + \alpha_t + 3a_1''\partial^{-1}w/2 + (2\beta - 3a_1/2)w' + a_1''' + 3a_1a_1''/2 = 0,$$

$$\begin{aligned}
3\partial^{-1}(w_t + a_1w)_t/4 &= (\alpha - 3w/2)w' - w'''/4 + 3a_1w_t/4 \\
&+3a_1a_1''\partial^{-1}w/4 + 3a_1a_1'w/4 - 3a_1'w'/4 + (\beta + 3a_1/4)w''.
\end{aligned}$$

In the simplest case of constant coefficients ($b_2' = a_1' = 0$), one goes down to

$$3b_3(w_t + a_1w)_t/4a_2^2 \tag{2.109}$$

$$-\left[(3b_3w/2a_2 + \alpha)w' - b_3w'''/4 + 3b_3a_1w_t/4a_2^2 + (\beta - 3b_3a_1/4a_2)w''\right]'.$$

This equation reduces to the standard Boussinesq equation when $b_1 = a_1 = 0$, $b_3 = 1$, and $a_2 = -1$.

We should stress once again that the results given in Sect. 2.2 have been simplified to show more clearly the algorithm of the derivation of the covariant Lax pair. A more general study can be developed if $a_2' \neq 0$.

2.8 Non-Abelian case. Zakharov–Shabat problem

In this section we consider linear equations comprising the Lax pair with the coefficients from the non-Abelian differential ring A (for details of the definitions of the mathematical objects, see [467]) and apply for them the joint covariance principle.

2.8.1 Joint covariance conditions for general Zakharov–Shabat equations

Let us change the notations for the first-order ($n = 1$) equation (2.39) as follows:

$$\psi_t = (J + u\partial)\psi. \tag{2.110}$$

Here the operator $J \in A$ does not depend on x, y, t and the potential $a_0 \equiv u(x, y, t) \in A$ is a function of the variables indicated. The operator $\partial = \partial/\partial x$ can be considered as a general differentiation, as in [467]. The transformed potential

$$\tilde{u} = u + [J, \sigma], \tag{2.111}$$

where $\sigma = \phi_x \phi^{-1}$ and ϕ is another solution of (2.110), is defined by the same formula as before, but the order of the elements is important. The covariance of the operator in (2.110) follows from the general transformations of the coefficients in the polynomial (2.41). The coefficient J is not transformed.

Suppose the second operator of a Lax pair has the same form but with different entries and derivatives:

$$\psi_y = (Y + w\partial)\psi, \qquad Y \in A, \tag{2.112}$$

where the potential $w = F(u) \in A$ is a function of the potential of the first equation (2.110). The principle of joint covariance [265, 267] hence reads

$$\tilde{w} = w + [Y, \sigma] = F(u + [J, \sigma]),$$

with the direct consequence

$$F(u) + [Y, \sigma] = F(u + [J, \sigma]). \tag{2.113}$$

So, the joint covariance equation (2.113) defines the function $F(u)$. In the case of the Abelian algebra we use the Taylor series (generalized by use of the Frechét derivative) to determine this function. Now some generalization is necessary. Let us make some remarks.

An operator-valued function $F(u)$ of an operator u in a Banach space may be considered as a generalized Taylor series with coefficients that are expressed in terms of Frechèt derivatives. The linear in u part of the series approximates (in a sense of the space norm) the function

$$F(u) = F(0) + F'(0)u + \dots.$$

This representation is not unique and a similar expression

$$F(u) = F(0) + u\hat{F}'(0) + \dots$$

may be introduced (definitions are given similarly to those in [33]). Both expressions, however, are not Hermitian; hence, they are not suitable for the majority of physical models. It means that the class of such operator functions is too restrictive. To explain how a more general class of functions could be introduced, let us consider some examples.

2.8.2 Covariant combinations of symmetric polynomials

The first natural example is the generalized Euler top equation with the Hamiltonian $Hu+uH$ which is discussed in Sect. 3.9. The covariant Lax pair for this case consists of two equations (2.110) and (2.112); the entries of the operators satisfy the joint covariance condition (2.113) and the compatibility condition if $J = H$ and $Y = H^2$.

The next example is related to the operator polynomial

$$P_2(H,u) = H^2u + HuH + uH^2,$$

whereas the choice $F(u) = P_2(H,u)$ satisfies the link (2.113). The direct substitution in the covariance and compatibility equations leads to a covariant constraint that turns out to be the identity if $Y = H^3$ and $J = H$.

More general connection $Y = J^n$ and $J = H$ leads to the covariance of the function

$$P_n(H,u) = \sum_{p=0}^{n} H^{n-p}uH^p.$$

This observation was exhibited in [276]. For further generalization let us consider combinations of polynomials,

$$f(H,u) = Hu + uH + S^2u + SuS + uS^2. \tag{2.114}$$

Plugging (2.114) as $F(u) = f(H,u)$ into (2.113) hints at a choice $Y = AB + CDE$ that yields

$$A[B,\sigma] + [A,\sigma]B + CD[E,\sigma] + C[D,\sigma]E + [C,\sigma]DE$$

$$= H[J,\sigma] + [J,\sigma]H + S^2[J,\sigma] + S[J,\sigma]S + [J,\sigma]S^2.$$

The last expression turns out to be the identity if $A = B = J = H$, $C = \alpha H$, $D = \alpha H$, $D = \alpha H$, $S = \beta H$, and $[\alpha, H] = 0$, $[\beta, H] = 0$ with the link $\alpha^3 = \beta^2$. Continuing this analysis, we arrive at the following:

Proposition 2.31. *The joint covariance principle defines a class of homogeneous polynomials $P_n(H,u)$, symmetric with respect to cyclic permutations, as possible Hamiltonians $h(\rho) = P_n(H,u)$ for the Liouville–von Neumann type evolution (Sect. 3.9). A linear combination of polynomials $\sum_{n=1}^{N} \beta_n P_n(H,u)$ with the coefficients commuting with u and H also yields the covariant pair if the conditions $Y = \sum_{n=1}^{N} \alpha_n H^{n+1}$, $\alpha_1 = \beta_1 = 1$, $\alpha_n^{n+2} = \beta^{n+1}$, and $n \neq 1$ hold.*

A proof could be performed by induction that is based on homogeneity of P_n and linearity of the constraints with respect to u. The functions $F_H(u) = \sum_0^\infty a_n P_n(H,u)$ also satisfy the constraints if the corresponding series converges.

2.9 A pair of difference operators

Let us consider a pair of equations of the same type (2.63) for a function ψ:

$$\psi_t(x,t) = \sum_{m=-M}^{N} U_m T^m \psi, \tag{2.115}$$

$$\psi_y(x,t) = \sum_{m=-M'}^{N'} V_m T^m \psi. \tag{2.116}$$

The compatibility condition for them is the nonlinear equation

$$U_{sy} - V_{st} = \sum_k \left[V_k T^k (U_{s-k}) - U_{s-k} T^{s-k} (V_k) \right] \tag{2.117}$$

for $s = -M-M', ..., N+N',\ k \in \{k' = -M', ..., N'\} \cap \{s-k = -M, ..., N\}$.

In the simplest case of the Zakharov–Shabat operators in both (2.115) and (2.116) with the subclass of stationary in y solutions we obtain three conditions:

$$\begin{aligned} U_{0t} &= V_0 U_0 - U_0 V_0, \\ U_{1t} &= V_0 U_1 - U_0 V_1 + V_1 T(U_0) - U_1 T(V_0), \end{aligned}$$

and

$$V_1 T(U_1) = U_1 T(U_1).$$

The connection with polynomials of a differential operator and hence with the theory of classical Bell polynomials can be revealed if we change the definition of potentials. It is clear that if the automorphism T is the shift operator $Tf(x) = f(x+\delta)$, the coefficients of the polynomials in T should be arranged as follows:

$$\psi_t(x,t) = \sum_{m=-M}^{N} \frac{u_m}{\delta^m} \sum_{r=0}^{m} \binom{m}{m-r} (-1)^{m-r} T^r \psi. \tag{2.118}$$

The recursion equation that defines classical differential Bell polynomials in commutative variables y_1, y_2, ... [388],

$$B_{m+1} = \sum_{r=0}^{m} \binom{m}{r} B_{m-r} y_{r+1},$$

together with the definition (2.65) of B_m^+, connects these special functions. Let us remark that the transformations for U_m found in Sect. 2.6 give the transforms for u_m defined by (2.118). The possibility of inverse transition depends on the independence of functions $(T-1)^n f$ for a given T and the set of functions ψ under consideration. The joint covariance of the system (2.115) and (2.116) hence may be investigated along the guidelines of [260] and [270], where the so-called binary Bell polynomials are used to form a convenient basis.

2.10 Non-Abelian Hirota system

Let us consider a pair of the Zakharov–Shabat type equations,

$$\psi_t(x,y,t) = (V_0 + V_1 T)\,\psi \tag{2.119}$$

and

$$\psi_y(x,y,t) = \left(U_0 + U_{-1}T^{-1}\right)\psi. \tag{2.120}$$

It differs from that used in the previous section by the change $T \to T^{-1}$ on the right-hand side of (2.120).

In a t-lattice version of equation (2.63) with $j \in \mathbb{Z}$ we go to

$$f(x,j+1) = \sum_{m=-M}^{N} U_m T^m f(x,j).$$

The case of a lattice in all variables is generated by the transition to the discrete variables $x,y,t \to n,j,r \in \mathbb{Z}$, $f(x,y,t) \to f_n(j,r)$, defined as in [321]. The operator T acts as the shift of n: $Tf_n(j,r) = f_{n+1}(j,r)$. The corresponding equations (2.119) and (2.120) are written as

$$f_n(j-1,r) = f_{n+1}(j,r) + v(n,j,r)f_n(j,r) \tag{2.121}$$

and

$$f_n(j,r-1) = f_n(j,r) + u(n,j,r)f_{n-1}(j,r) \tag{2.122}$$

with the potentials indicated. The compatibility condition of the linear equations (2.121) and (2.122) has the form

$$\begin{aligned} u(n,j-1,r) - u(n+1,j,r) &= v(n,j,r-1) - v(n,j,r),\\ v(n,j,r-1)u(n,j,r) &= u(n,j-1,r)v(n-1,j,r). \end{aligned} \tag{2.123}$$

The second equation in (2.123) is automatically valid if

$$\begin{aligned} u(n,j,r) &= \tau_{n+1}(j,r-1)\tau_n^{-1}(j,r-1)\tau_{n-1}(j,r)\tau_n^{-1}(j,r),\\ v(n,j,r) &= \tau_{n+1}(j-1,r)\tau_n^{-1}(j-1,r)\tau_n(j,r)\tau_{n+1}^{-1}(j,r). \end{aligned} \tag{2.124}$$

It should be stressed that the order of the entries in these expressions is important. The substitution of (2.124) in the first equation in (2.123) leads to the generalized Hirota bilinear equation [210] (compare also with the generalizations in [336]):

$$\begin{aligned} &\tau_{n+1}(j-1,r-1)\tau_n^{-1}(j-1,r-1)\tau_{n-1}(j-1,r)\tau_n^{-1}(j-1,r)\\ &-\tau_{n+1}(j-1,r-1)\tau_n^{-1}(j-1,r-1)\tau_{n-1}(j,r-1)\tau_n^{-1}(j,r-1)\\ &-\tau_{n+2}(j,r-1)\tau_{n+1}^{-1}(j,r-1)\tau_n(j,r)\tau_{n+1}^{-1}(j,r)\\ &+\tau_{n+1}(j-1,r)\tau_n^{-1}(j-1,r)\tau_n(j,r)\tau_{n+1}^{-1}(j,r) = 0. \end{aligned} \tag{2.125}$$

In the scalar case the system reduces to the Hirota bilinear equation [321]

$$\tau_n(j+1,r)\tau_n(j,r+1)-\tau_n(j,r)\tau_n(j+1,r\ +1)$$
$$+\tau_{n+1}(j+1,r)\tau_{n-1}(j,r+1)=0. \tag{2.126}$$

Using (2.124) and the DT formalism, we could elaborate a non-Abelian version of these equations that can be useful for applications in the theory of quantum transfer matrices for fusion rules [255, 256] and of quantum correlation functions [36, 37]. Note that the non-Abelian Hirota–Miwa equation is discussed by Nimmo [351].

Let us return to the DT theory. Equations (2.119) and (2.120) are jointly covariant; hence, solving equations (2.123) or (2.125) is based on the symmetry that is generated by the joint covariance of (2.121) and (2.122) with respect to the transformations of the type (2.111), namely,

$$\psi^-(j,r)=\psi-\sigma^-T^{-1}f,\qquad \sigma^-=\varphi\left(T^{-1}\varphi\right)^{-1}.$$

As can be easily seen, the form of both linear equations (2.121) and (2.122) represents reductions of (2.119) and (2.120) with $V_1=1$, $V_0=v$, $U_0=1$, and $U_{-1}=u$. We show further some details in the proof of the covariance theorem because it demonstrates important features in the procedure of the derivation of the chain equation. Let us start, say, from (2.122). The covariance conditions are obtained from the coefficients by ψ, $T^{-1}\psi$, and $T^{-2}\psi$. The first one is valid automatically,

$$u^-=u-\sigma^-\left(r-1\right)+\sigma^-\left(r\right), \tag{2.127}$$

$$u^-T^{-1}\sigma^-\left(r\right)=\sigma^-\left(r-1\right)u. \tag{2.128}$$

2.11 Nahm equations

The Nahm equations [344] appear in conformal field theory in connection with the monopole problem. They are solved by the variational method in [129], producing a parameterization of the Bogomolny equations. Their generalizations attract great attention in mathematical physics [101, 345].

In the following example, we change the DT formulas a bit, showing the alternative version, similar to [381]. We stress, however, that the formulas from Sect. 2.1 give an equivalent result. Some generalization will be needed within the reduction constraints related to an additional (gauge) transformation denoted by g. This is expressed by the following:

Theorem 2.32. *The equation*

$$\psi_y=uT\psi+v\psi+wT^{-1}\psi \tag{2.129}$$

is covariant with respect to the combined gauge–DT

$$\psi[1] = g(T - \sigma)\psi. \tag{2.130}$$

Here $\sigma = (T\phi)\phi^{-1}$*, where* ϕ *is a solution of the same equation (2.129) and* g *is an invertible element of the ring. The transforms of the equation coefficients are*

$$u[1] = gT(u)[T(g)]^{-1}, \tag{2.131}$$
$$v[1] = gT(v))g^{-1} - g\sigma u g^{-1} + gT(u)T(\sigma)g^{-1} + g_y g^{-1}, \tag{2.132}$$
$$w[1] = g\sigma w[T^{-1}(g\sigma)]^{-1}. \tag{2.133}$$

Proof. The substitution of (2.130) into the transformed equation (2.129) gives four equations assuming $T^n\psi$ are independent. Three of them yield transformed potentials (2.131)–(2.133). The fourth equation after use of the transforms takes the form

$$\sigma_y = \sigma F - (TF)\sigma, \tag{2.134}$$

where

$$F = u\sigma + v + w[T^{-1}(\sigma)]^{-1}.$$

One can check the condition (2.134) by direct substitution of the operator σ and by use of the equation for ϕ.

Remark 2.33. Theorem 2.32 is evidently valid for the spectral problem

$$\lambda\psi = uT\psi + v\psi + wT^{-1}\psi \tag{2.135}$$

with the only correction being that the last term for the transform $v[1]$ is absent. The equation goes to the "Riccati equation" analog for the function σ:

$$\mu = u\sigma + v + w[T^{-1}(\sigma)]^{-1}. \tag{2.136}$$

Note that inserting the element $\sigma = (T\phi)\phi^{-1}$ into (2.136) transforms it to the spectral problem for ϕ (2.135) with the spectral parameter μ.

The Nahm equations can be written by means of the Lax representation using the spectral equation (2.135) and the evolution equation

$$\psi_y = (q + pT)\psi \tag{2.137}$$

with potentials p and q. The covariance of this equation with respect to the DT (2.130) can be established similarly to Theorem 2.32 with account of the y-evolution of $\sigma(y)$:

$$\sigma_y = T(q)\sigma - \sigma p\sigma + T(p)T(\sigma)\sigma - \sigma q = 0, \tag{2.138}$$

which proves the following transformation formulas for the coefficients in (2.137):

$$p[1] = gT(p)\,[T(g)]^{-1}$$

and

$$q[1] = g\left[T(q) - \sigma p + T(p)T(\sigma)\right] g^{-1} + g_y g^{-1}.$$

The joint covariance principle (Sect. 2.7 and [265]) defines the connection between potentials p and q and u and v:

$$p = u + \beta I, \qquad q = v/2. \tag{2.139}$$

Hence, the joint DT covariance means integrability of the compatibility condition of equations (2.137) and (2.129), e.g., of the Nahm equations:

$$u_y = \frac{1}{2}[uT(v) - vu] + \beta[T(v) - v],$$
$$v_y = uT(w) - wT^{-1}u + \beta[T(w) - w],$$
$$w_y = \frac{1}{2}vw - wT - 1(v).$$

One more possible specification is the use of periodic potentials in the problem (2.137) with the evolution (2.129) with account for the connections (2.139) that result in the appearance of commutators on the right-hand sides of the equations. Some linear transformations and rescaling

$$u = \alpha(-\mathrm{i}\varphi_1/2 - \varphi_3), \qquad v = \varphi_3,$$
$$w = \alpha^{-1}(-\mathrm{i}\varphi_1/2 + \varphi_3), \qquad q = \varphi_3/2,$$
$$p = \alpha(-\mathrm{i}\varphi_1/2 - \varphi_3) + \beta I$$

produce the Nahm equations for the periodic functions $T\varphi_i = \varphi_i$ [periodicity of φ_i does not mean a periodicity of solutions ψ and ϕ of the Lax pair and the corresponding $\sigma = (T\phi)\phi^{-1}$]:

$$\varphi_{iy} = \mathrm{i}\epsilon_{ikl}[\varphi_k, \varphi_l]. \tag{2.140}$$

α and β are free parameters. This system is covariant with respect to the combined DT–gauge transformations if the gauge transformation $g = \exp G$ is chosen as follows:

$$G_y = \alpha\left[(\varphi_3 + \varphi_1/2)T(\sigma) - \sigma(\varphi_3 + \varphi_1/2)\right]/2. \tag{2.141}$$

Finally, the following theorem can be formulated:

Theorem 2.34. *For $T\varphi_i = \varphi_i$ the system (2.140) is invariant with respect to the transformations*

$$\begin{aligned}
\varphi_1[1] &= g\left[(\varphi_1/2 - \mathrm{i}\varphi_3)T(g)^{-1} + \sigma(\varphi_1/2 + \mathrm{i}\varphi_3)[T^{-1}(g\sigma)]^{-1}\right], \\
\varphi_2[1] &= g\left[\varphi_2 + \alpha(\mathrm{i}\sigma\varphi_1/2 - \mathrm{i}\varphi_1 T(\sigma)/2 + \sigma\varphi_3 - T(\varphi_3\sigma))\right] g^{-1}, \\
\varphi_3[1] &= g\left[(-\mathrm{i}\varphi_1/2 - \varphi_3)T(g)^{-1} + \sigma(-\mathrm{i}\varphi_1/2 + \varphi_3)[T^{-1}(g\sigma)]^{-1}\right]
\end{aligned} \tag{2.142}$$

with the function $g = \exp G$, where G is obtained by integrating (2.141), if the element σ is a solution of the system

$$\mu = \alpha(-\mathrm{i}\varphi_1/2 - \varphi_3)\sigma + \varphi_3 + \alpha^{-1}(-\mathrm{i}\varphi_1/2 + \varphi_3)[T^{-1}(\sigma)]^{-1}, \tag{2.143}$$
$$\sigma_y = [\varphi_3, \sigma]/2 - \sigma[\alpha(-\mathrm{i}\varphi_1/2 - \varphi_3) + \beta I]\sigma + [\alpha(-\mathrm{i}\varphi_1/2 - \varphi_3) + \beta I]T(\sigma)\sigma = 0.$$

The system (2.143) follows from (2.138) and (2.136).

Remark 2.35. A similar statement can be formulated for the discrete version [342] of the Nahm system (2.140), as may be seen from the previous section.

2.12 Solutions of Nahm equations

Making use of the construction described in the previous section, we consider a simple example. Let T be a shift operator $T\psi(x,y) = \psi(x+1,y)$. As a seed solution of the Nahm equations (2.140) take commuting constant matrices $\varphi_i = A_i$, $i = 1,2,3$, which means constant u, v, and w. First of all we should generate a solution of the Lax pair (2.135) and (2.137) that can be found in the form $\phi = \xi(t)\Phi(x)$ (all elements are supposed to be invertible). The equation for ξ is obtained as

$$\xi_t = [v/2 + (u + \beta I)T]\xi = Z\xi,$$

which is solved by

$$\xi = \exp(Zt)\xi_0.$$

Plugging Φ into (2.135) yields the spectral problem for the difference shift operators:

$$\mu\Phi(x) = \xi^{-1}[u\xi\Phi(x+1) + v\xi\Phi + w\xi\Phi(x-1)].$$

Separating variables again, a class of particular solutions is built as

$$\Phi = \eta\exp(\Sigma x)\ ;$$

hence, we arrive at the matrix spectral problem for η:

$$\mu\eta = \xi^{-1}\left[u\xi\eta\exp(\Sigma) + v\xi\eta + w\xi\eta\exp(-\Sigma)\right],$$

with the operator on the right-hand side and, therefore, spectral parameter μ parameterized by t. Finally, the matrix σ is composed as

$$\sigma = \xi(t)\eta\exp(\Sigma)\eta^{-1}\xi^{-1}(t).$$

An appropriate choice of commutator algebra for A_i, Σ, and η allows us to obtain an explicit form of σ and, hence, to construct and solve the following equation for G:

$$G_t = \frac{\alpha}{2}\left[\left(\varphi_3 + \frac{1}{2}\varphi_1\right)\xi(t)\eta\exp(\Sigma)\eta^{-1}\xi^{-1}(t)\right.$$

$$\left.-\xi(t)\eta\exp(\Sigma)\eta^{-1}\xi^{-1}(t)\left(\varphi_3 + \frac{1}{2}\varphi_1\right)\right].$$

Its exponent (the matrix g) is necessary for the dressing formulas (2.142). We would like to stress that the matrices σ and g do not depend on x; hence, the dressed $\varphi[i]$ also does not.

Starting from the known solution of (2.140), we arrive at the Euler system for $f_i(y)$ that is solved in the Jacobi functions [129]. The solutions are dressed by the transformations (2.142). A more general possibility is a direct series solution of (2.138).

3

From elementary to twofold elementary Darboux transformation

In this chapter we extend the results of Chap. 2 related to the classical Darboux transformation (DT), by means of more detailed analysis of algebraic aspects of general theory. Indeed, already in the pioneering paper by Matveev [314] it was shown that the DT represents a universal algebraic operation. We start from the intertwining relations (Sect. 1.1) and formulate in Sect. 3.1 a general definition of the DT, as well as its connection with gauge transformations. We introduce a concept of the elementary DT (eDT) [278] which will play a similar role for constructing particular solutions of nonlinear equations as the classical DT does (for a comprehensive study of the method see [433]). In Sect. 3.2 we begin the development of a purely algebraic construction of a matrix DT on the basis of two projectors [289]. The extension of the eDT covariance based on the existence of idempotents and skew fields in an associative differential ring is discussed in Sect. 3.3 using an example of three basic projectors [267]. We stress that the twofold DT widely used as a dressing tool represents a sequence of two eDTs defined for mutually conjugated Zakharov–Shabat (ZS) problems. After a detailed consideration of particular cases in the preceding sections, we formulate in Sect. 3.4 the definitions of the eDTs and twofold DTs for an arbitrary number of projectors [269]. Explicit formulas are given for both eDTs and twofold DTs. A special case of the Schlesinger transformation is defined in Sect. 3.5. The usefulness of this transformation lies in the fact that it directly connects the seed and transformed potentials. In Sect. 3.6 we demonstrate a generalization of the known Bianchi–Lie formula to the non-Abelian case. Section 3.7 is devoted to the non-Abelian N-wave equations with linear terms for illustrating the general case of an arbitrary number of idempotents. We describe reduction constraints and soliton solutions for $N = 3$ that account for damping and space asynchronism. In Sect. 3.8 we show that a particular form of the twofold DTs determines a Lie group. Hence, we can determine infinitesimal transforms for the iterated DTs which can be applied to study the stability of soliton solutions. In Sect. 3.9 we demonstrate an interesting example of using the Darboux-integration technique for solving a class of nonlinear von Neumann equations. In particular,

we show that the existence of self-scattering solutions to these equations represents a generic consequence of the nonlinearities involved. In Sect. 3.10 we introduce a notion of a compound eDT joining the structures of the classical DTs and the eDTs. As an example, in the framework of this approach we produce explicit solutions to the integrable Korteweg–de Vries (KdV)–modified KdV (MKdV) system.

3.1 Gauge transformations and general definition of Darboux transformation

When dealing with general dressing procedures, two prolonged spaces are usually used on which the problem can be posed: the first one is spanned by the derivatives $\psi^{(n)}$, while the second space is determined by the successive action of automorphism powers $T^n\psi$. Both constructions were used in the previous chapter. Here we restrict ourselves to the first type of prolonged space.

Let again A be a differential ring with a differentiation D. The generic transform [466]

$$f = \epsilon D\psi + \Sigma\psi \,, \qquad \epsilon,\ \Sigma,\ \psi \in A \tag{3.1}$$

gives the classical DT in the case of $\epsilon = 1$, the gauge transformation if $\epsilon = 0$, and the combination of the DT and gauge transformation if $\Sigma = \epsilon\sigma$. The first two cases have been well studied; the third one has been used for integration of the Nahm system (Sect. 2.11). In this chapter we will consider a degenerate operator ϵ which is proportional to a projector (idempotent). In Sects. 3.2–3.5 such a case is studied under the name of eDT. Next, a definite combination of the eDTs produces a twofold DT, or more complicated transformations (Sects. 3.6–3.9). In Sect. 3.10 the whole space is taken, so the form of the DT, named a combined eDT, looks like a generic transform (3.1), with ϵ being a projector.

Following [466], we assume a covariance of the evolution equation

$$\psi_t = \sum_0^n u_k D^k\psi \,, \tag{3.2}$$

with respect to the transformation (3.1). The transformed potentials for the ZS problem ($n = 1$)

$$\begin{aligned} u_1[1] &= \epsilon u_1\epsilon^{-1} \,, \\ u_0[1] &= \epsilon[u_0 + u_1' + [\sigma u_1] - u_1\epsilon^{-1}\epsilon' + \epsilon_t]\epsilon^{-1} \end{aligned} \tag{3.3}$$

demonstrate new possibilities of the combined transformations. Additional transformation of the independent variable leads to a possibility to widen the class of covariant operators. The combinations of this sort will be used in Chap. 4 for studying the shape-invariant potentials.

Further, if ϵ depends on a solution of (3.2), e.g., $\epsilon = A\phi B\phi^{-1}D$, where A, B, and D are constants [466], the transform of the potentials u_k, $k \neq 0, 1$ for arbitrary n looks like a recurrence

$$u_{k-1}[1] = \epsilon(u_k + u'_{k-1})\epsilon^{-1} + \left(\sigma u_k - \sum_{p=k}^{n} u_p[1](C_p^k \sigma^{(p-k)} - C_p^{k-1}\epsilon^{(p-k-1)})\right)\epsilon^{-1} ,$$

$$u_0[1] = \epsilon(u_0 + u'_1)\epsilon^{-1} + \left(\sigma u_1 - \sum_{p=1}^{n} u_p[1](C_p^1 \sigma^{(p-1)} - C_p^0 \epsilon^{(p)})\right)\epsilon^{-1} + \epsilon_t \epsilon^{-1} . \quad (3.4)$$

This recurrence reduces to the known Matveev formulas from [314] when $A = B = D = \mathbb{1}$. Let us also mention the search for a general dressing scheme performed in [478] along the lines of the ZS method.

3.2 Zakharov–Shabat equations for two projectors. Elementary Darboux transformation

In the algebra A we fix the element

$$J = a_1 p + a_2 q , \qquad a_1 \neq a_2 ,$$

where a_1 and a_2 belong to the field K and p and q are projectors ($p^2 = p$, $q^2 = q$, $pq = qp = 0$, $p + q = e$, e is the identity). It is easy to verify that

$$pJ = J_p = a_1 p , \qquad qJ = J_q = a_2 q . \quad (3.5)$$

Moreover, for every element $x \in A$ the following equality holds:

$$[J, x] = a(x_{12} - x_{21}) \in A_{12} \oplus A_{21} , \qquad a = a_1 - a_2 . \quad (3.6)$$

Here $A_{12} = pAq$ and $A_{21} = qAp$ in accordance with the decomposition $A = pAp + pAq + qAp + qAq$.

Definition 3.1. *The ZS operator in a module M is a K-linear operator*

$$L_u : \psi \to D\psi - (\lambda J + u)\psi ,$$

where $\lambda \in K$, $u \in A$, and $\psi \in M$.

In the same way, the ZS operators are defined in modules A_p and A_q, where $A_p = A_{pp} \oplus A_{qp}$ and $A_q = A_{pq} \oplus A_{qq}$. Taking into account standard applications (spectral problems for linear operators and nonlinear evolution equations of mathematical physics), we shall refer to λ as a spectral parameter and u as a potential.

Let us suppose that the potential u of the ZS operator satisfies the restriction

$$u_{pp} = pup = 0, \qquad u_{qq} = quq = 0 \ , \tag{3.7}$$

i.e., $u \in A_{pq} \oplus A_{qp}$.

It is not necessary to pose this restriction at the very beginning; instead it can be realized by means of an appropriate "gauge" transformation (see the previous section).

Definition 3.2. *For the potential u let there exist a new potential $\tilde{u}$, satisfying the condition (3.2), and the elements $\sigma \in A$ such that for all $\lambda \in K$ the following intertwining relation holds:*

$$\ell L_u = L_{\tilde{u}} \ell, \tag{3.8}$$

where the action of a K-linear operator ℓ in module M is determined by the equality

$$\ell\psi = (\lambda p - \sigma)\psi \ , \qquad \psi \in M \ .$$

Then the transformation

$$u \to \tilde{u} \ , \qquad \psi \to \tilde{\psi} = \ell\psi \tag{3.9}$$

is referred to as the elementary DT (eDT) connected with the projector p.

The simplest consequence of the intertwining relation (3.8) represents the well-known fact that if ψ is a solution (partial or general) of the ZS equation $L_u\psi = 0$, then $\tilde{\psi}$ (3.9) is a solution of the transformed equation $L_{\tilde{u}}\tilde{\psi} = 0$. Evidently, the significance of the DT is not exhausted by this property.

Taking into account (3.5), we can easily verify the identity

$$(\ell L_u - L_{\tilde{u}}\ell)\psi = \lambda(\tilde{u}p - pu - [J, \sigma]) + (\partial\sigma - \tilde{u}\sigma + \sigma u))\psi \ , \qquad \psi \in M \ . \tag{3.10}$$

From this and the condition of nondegeneracy of the module M we have the following:

Lemma 3.3. *The intertwining relation (3.8) identically holds for all $\lambda \in K$ and $\psi \in M$ if and only if*

$$\tilde{u}p = pu + [J, \sigma] \ , \qquad \partial\sigma = \tilde{u}\sigma - \sigma u \ .$$

For the proof of this assertion it is sufficient to derive the identity (3.10) or to calculate

$$\ell L_u = (\lambda - \sigma)(D - \lambda J - u) = \lambda pD - \sigma D - \lambda^2 pJ - \lambda pu + \sigma u \ ,$$

accounting for the definition of the differentiation D and the equality (3.5). The equalities from the statement of Lemma 3.3 nullify the right-hand side of (3.10) identically; therefore, (3.8) holds. The left-hand side of (3.10) is obviously reconstructed from the equalities of Lemma 3.3.

The problem of constructing the eDT and intertwining operators is therefore reduced to the solution of the equations of Lemma 3.3 with respect to $\tilde{u}$ and $\tilde{\sigma}$. Splitting them with account of (3.6), we obtain after simplifications

$$\tilde{u}_{qp} = -a\sigma_{qp} \ , \qquad \sigma_{pq} = -a^{-1}u_{pq} \ , \tag{3.11}$$

$$\partial\sigma_{pp} = \tilde{u}_{pq}\sigma_{qp} + a^{-1}u_{pq}u_{qp} \ , \ \tilde{u}_{pq}\sigma_{qq} = \sigma_{pp}u_{pq} - a^{-1}\partial u_{pq} \ ,$$
$$\partial\sigma_{qp} = -a\sigma_{qp}\sigma_{pp} - \sigma_{qq}u_{pp} \ , \qquad \partial\sigma_{qq} = 0 \ . \tag{3.12}$$

Two cases should be discriminated:

1. $\sigma_{qq} \neq 0$. Designate $\sigma_{qq} = c$ and suppose that $c \in \ker D \setminus \{0\}$. Then (3.11) and (3.12) are transformed to the following equations:

$$\tilde{u}_{pq} = (\partial\sigma_{pq} + \sigma_{pp}u_{pq})c^{-1} \ , \tag{3.13}$$

$$\partial\sigma_{pp} = (\sigma_{pp}u_{pq} - a^{-1}\partial u_{pq})c^{-1}\sigma_{qp} + a^{-1}u_{pq}u_{qp} \ ,$$
$$\partial\sigma_{qp} = -a\sigma_{qp}\sigma_{pp} - cu_{qp} \ . \tag{3.14}$$

Now we consider the auxiliary ZS problem in a module A_p:

$$\partial\varphi = (\lambda_0 J + u)\varphi \ , \qquad \lambda_0 \in K \ , \qquad \varphi \in A_p \ . \tag{3.15}$$

Lemma 3.4. *Assume that the ring A contains the division ring B and denote $C = B \bigcap \ker\partial$, i.e., $C \setminus 0$ is the set of invertible constants of the ring A. Let the element $\varphi \in A_p$ satisfy (3.15) and $\varphi_p = p\varphi \in B_p \setminus \{0\}$, i.e., φ_p has the inverse element φ_p^{-1} in algebra A_{pp}. Then the element $\xi = \varphi_q\varphi_p^{-1}$, $\varphi_q = q\varphi$ satisfies the Riccati equation*

$$\partial\xi = -\xi u_{pq}\xi - a\lambda_0\xi + u_{qp} \ . \tag{3.16}$$

Proof. Using once again the definition of D and equalities that directly follow from the ZS equation (3.15) in module A_p,

$$\partial\varphi_q = (q\lambda_0 J + qu)\varphi$$

and similarly with p, we calculate

$$\partial\xi = (q\lambda_0 J\varphi)\varphi_p^{-1} + (qu\varphi)\varphi_p^{-1} - \varphi_q\varphi_p^{-1}(p\lambda_0 J\varphi)\varphi_p^{-1} - \varphi_q\varphi_p^{-1}(pu\varphi)\varphi_p^{-1} \ .$$

As $[q, \lambda_0] = 0$ and $qJ = a_2 q$ [see (3.6)], it follows that

$$\partial\xi = \lambda_0 a_2\xi + (qu(q+p)\varphi)\varphi_p^{-1} - \xi\lambda_0 a_1 - \xi pu(q+p)\varphi\varphi_p^{-1} \ ,$$

where $p + q = e$ is inserted. Taking into account the definition of the potential $u_{qp} = puq$ and gauge conditions (3.2), we go to the statement of Lemma 3.4.

Theorem 3.5. *Let the condition of Lemma 3.4 be valid. Then the formulas*

$$\sigma_{pp} = \lambda_0 + a^{-1}u_{pq}\xi \qquad \text{and} \qquad \sigma_{qp} = -c\xi \tag{3.17}$$

give a solution of equations (3.14).

The proof of Theorem 3.5 can be performed by a direct substitution of (3.17) in (3.14) with account for (3.16).
Substituting (3.17) in (3.13) and (3.11), we obtain the transformed potential $\tilde{u}$ in the form:

$$\tilde{u} = ac\xi + a^{-1}(a\lambda_0 u_{pq} + u_{pq}\xi u_{pq} - \partial u_{pq})c^{-1}. \tag{3.18}$$

Taking together (3.11) and (3.16), we get

$$\sigma = \lambda_0 + a^{-1}u_{pq}\xi - a^{-1}u_{pq} - c\xi + c\,. \tag{3.19}$$

Thus, transformations (3.9), where $\tilde{u}$ and σ are defined by (3.18) and (3.19), give a (one possible) DT that corresponds to the projector p. Interchanging p and q, we obtain the DT corresponding to the projector q.

2. $\sigma_{qq} = 0$. Now equations (3.12) take the following form:

$$\sigma_{pp}u_{pq} = a^{-1}\partial u_{pq}\,, \qquad \partial\sigma_{qp} = -a\sigma_{qp}\sigma_{pp}\ , \tag{3.20}$$

and

$$\tilde{u}_{pq}\sigma_{qp} = \partial\sigma_{pp} - a^{-1}u_{pq}u_{qp}\,. \tag{3.21}$$

If there exists the reciprocal element u_{pq}^{-1}, then the solution of (3.20) is given by the formulas

$$\sigma_{pp} = a^{-1}(\partial u_{pq})u_{pq}^{-1}\,, \qquad \sigma_{qp} = c_0 u_{pq}^{-1}\,, \qquad c_0 \in C_q \setminus \{0\}\,.$$

Substituting them in (3.21) and taking into account (3.11), we go to Theorem 3.6:

Theorem 3.6. *Let the condition $\sigma_{pp} = 0$ for a definition of l hold. Then the transformed potential of the ZS operator L_u is given by*

$$\tilde{u}_{pq} = a^{-1}(\partial^2 u_{pq} - u_{pq}^{-1}\partial u_{pq}u_{pq}^{-1} - u_{pq}u_{qp}u_{pq})c_0^{-1}\,,$$

$$\tilde{u}_{qp} = -ac_0 u_{pq}^{-1}\,. \tag{3.22}$$

The proof of the theorem is a chain of equalities before its formulation.

Hence, in case 2 we have built once again the DT corresponding to the projector p. Its peculiarity is that in this case it is not necessary to use a solution of the auxiliary ZS equation (3.15) and the transformed potential $\tilde{u}$ is expressed explicitly via the seed potential u (unlike the DT in case 1), as the formulas (3.22) show. The relations (3.22) generalize the well-known Schlesinger transformation [389].

Both cases may be effectively used when we go down from the abstract level to specific examples. Such examples can be explicitly constructed for the differential rings of matrices. If matrix elements are functions of parameters, differentiation may be defined as a derivative with respect to a parameter. Such a matrix realization of the eDT was introduced and applied in [278] for an arbitrary matrix dimension. Similar realizations were used in [281] and the combinations of the eDTs leading to a binary DT were constructed and used to obtain multisoliton formulas.

3.3 Elementary and twofold Darboux transformations for ZS equation with three projectors

We continue to develop a rather abstract extension of the eDT covariance based on the existence of idempotents and division rings (skew fields) in an associative differential ring A over the field K [267]. Let D be the differentiation map on A and let us fix orthogonal ($pq = qp = 0$) idempotents (projectors) p and q, such that $p, q \in \ker D$. Then the element $s = e - p - q$ is the third orthogonal projector. We choose here the case of three basic projectors for it covers features of a general formulation but nevertheless has a clear explicit form.

For every $x \in A$ we denote $x_{\alpha\beta} = \alpha x \beta$, where $\alpha, \beta \in \ p, q, s$, so we split the ring into the direct sum

$$A = \oplus_{\alpha,\beta} A_{\alpha\beta} \ .$$

We fix the element $J = a_1 p + a_2 q + a_3 s$, $a_1, a_2, a_3 \in K$, $a_1 \neq a_2 \neq a_3 \neq a_1$. The degenerate case of equal a_i can be considered in a similar manner [278].

Definition 3.7. *The ZS operator L_u is the linear operator in A,*

$$L_u : \psi \to D\psi + (\lambda J - u)\psi \ ,$$

where $\lambda \in K$, $u, \psi \in A$.

Suppose the potential u of the ZS operator satisfies the gauge restrictions $u_{pp} = pup = 0, \quad u_{qq} = quq = 0, \quad u_{ss} = sus = 0$.

Definition 3.8. *Let for the potential u there exist a new potential u^e and the element $\sigma \in A$ such that for all $\lambda \in K$ the following intertwining identity holds*

$$EL_u = L_{u^e} E \ , \tag{3.23}$$

where the action of a K-linear operator E is determined by the equality

$$E\psi = (\lambda p + \sigma)\psi \ . \tag{3.24}$$

Then the transformation

$$u \to u^e\ , \qquad \psi \to \psi^e = E\psi$$

is referred to as the eDT connected with the projector p.

It follows from (3.23) that

$$D\sigma = u^e\sigma - \sigma u \tag{3.25}$$

and

$$u^e p = pu + [J, \sigma]\ . \tag{3.26}$$

Let a solution of the ZS equation for $\lambda = \mu$ be φ and $\varphi_{pp} = p\varphi p \equiv \varphi_p$ has the inverse element φ_p^{-1} in A_{pp} . Then the elements $\xi = \varphi_q\varphi_p^{-1}$ and $\eta = \varphi_s\varphi_p^{-1}$ with $\varphi_q = q\varphi$ and $\varphi_s = s\varphi$ satisfy the system of the Riccati-type equations [240] and the following theorems may be proved as in [324].

Theorem 3.9. *Let $\sigma_{qq} = c \neq 0$, $\sigma_{ss} = d \neq 0$, $c, d \in \ker D$. Then for the transformed potential u^e the following formulas are valid:*

$$\begin{gathered}
u^e_{pq} = (\sigma_{pp}u_{pq} - b^{-1}u_{ps}u_{sq} - a^{-1}Du_{pq})c^{-1}\ , \quad u^e_{qp} = -a\sigma_{qp}\ , \\
u^e_{ps} = (\sigma_{pp}u_{ps} - a^{-1}u_{pq}u_{qs} - b^{-1}Du_{ps})d^{-1}\ , \quad u^e_{sp} = -b\sigma_{sp}\ , \\
u^e_{qs} = [(1 - a/b)\sigma_{qp}u_{ps} + cu_{qs}]d^{-1}\ , \\
u^e_{sq} = [(1 - b/a)\sigma_{sp}u_{pq} + du_{sq}]c^{-1}\ ,
\end{gathered} \tag{3.27}$$

where $a = a_1 - a_2$ and $b = a_1 - a_3$, and the relations

$$\begin{gathered}
\sigma_{pp} = -\mu p + a^{-1}u_{pq}\xi + b^{-1}u_{ps}\eta\ , \\
\sigma_{pq} = -a^{-1}u_{pq}\ , \qquad \sigma_{qp} = -c\xi\ , \\
\sigma_{ps} = -b^{-1}u_{ps}\ , \qquad \sigma_{sp} = -d\eta\ , \qquad \sigma_{qs} = \sigma_{sq} = 0
\end{gathered} \tag{3.28}$$

define the eDT. Each additional projector introduces an eDT (obviously a different one) in a similar way.

We mention here additional possibilities of the Schlesinger transformations [389] in the case of the constraints $\sigma_{qq} = 0$ or $\sigma_{ss} = 0$. For the Schlesinger transformations it is not necessary to use the auxiliary solutions of the ZS equation: the transformed potential u^e is directly expressed via the seed potential u. The choice $\sigma_{qq} = 0$ and $\sigma_{ss} = 0$ introduces constraints for the potential u [240].

For the conjugate problem

$$D^*\phi + \phi(\kappa J - u) = 0 \tag{3.29}$$

the equations for σ^c and u^c determine the transform (the upper index c labels the conjugate transform)

$$\phi^c = \phi E^c \,, \qquad E^c = \kappa p - \sigma^c \,. \tag{3.30}$$

The equations for σ^c and u^c differ from (3.25) and (3.26) only in the order of the operators and the form of D^*.

The solution of these equations is determined again by solutions with projectors multiplied from the right, $\phi_p = \phi p$ and so on, with the spectral parameter μ. Namely,

Theorem 3.10. *Let $\sigma_{qq} = c_* \neq 0, \sigma_{ss} = d_* \neq 0$, and $\phi_p^{-1} = (p\phi p)^{-1} \in A_{pp}$ exists. Then the transform (3.30) is determined by the elements*

$$\sigma^c_{pp} = -\mu + a^{-1}\phi_p^{-1}\phi_q u_{qp} + b^{-1}\phi_p^{-1}\phi_s u_{sp} \,,$$

$$\sigma^c_{pq} = -\phi_p^{-1}\phi_q c_* \,, \qquad \sigma^c_{qp} = -a^{-1}u_{qp} \,, \tag{3.31}$$

$$\sigma^c_{ps} = -\phi_p^{-1}\phi_s d_* \,, \qquad \sigma^c_{sp} = -b^{-1}u_{sp} \,, \qquad \sigma^c_{sq} = \sigma^c_{qs} = 0 \,.$$

The set of formulas (3.31) in turn defines the eDT of eigenfunctions (3.30) for the conjugate problem (3.29) via constants $c_*, d_* \in \ker D$ entering equations (3.31).

The potential is transformed similarly to (3.27):

$$\begin{aligned} u^c_{pq} &= -a\sigma^c_{pq} \,, \qquad u^c_{ps} = -b\sigma^c_{ps} \,, \\ u^c_{qp} &= c_*^{-1}(u_{qp}\sigma^c_{pp} - b^{-1}u_{qs}u_{sp} - a^{-1}Du_{qp}) \,, \\ u^c_{sp} &= d_*^{-1}(u_{sp}\sigma^c_{pp} - a^{-1}u_{sq}u_{qp} - b^{-1}Du_{sp}) \,, \\ u^c_{qs} &= c_*^{-1}[u_{cp}\sigma^c_{ps}(1 - b/a) - u_{qs}d_*^{-1}] \,, \\ u^c_{sq} &= d_*^{-1}[u_{sp}\sigma^c_{pq}(1 - a/b) - u_{sq}c_*^{-1}] \,. \end{aligned} \tag{3.32}$$

We consider the case $D^* = -D$; hence, ζE^{-1}, where ζ stands for another solution of the ZS spectral problem, is the solution of the conjugate equation (3.29) and the transformed potential is identical to u^e. Namely, this point allows us to carry out the twofold transformations. Elements of E^{-1} associated with the corresponding projectors can be subsequently calculated. After that the twofold DT is constructed and the respective theorem may be proved. We introduce the twofold transformation as a sequence of two elementary ones in both spaces. In the case of the conjugate space, the first transformation is made by the inverse operator to E with the spectral parameter μ and the respective solution φ of the direct problem. The second map is generated by the resulting function χ^e expressed in the same way but from a linearly independent seed solution χ of the equation (3.29) with the parameter ν . For the direct case, the order is the opposite. The final formulation is given by Theorem 3.11:

Theorem 3.11. *Let the inverse of the element $p\chi\varphi p$ exist. The twofold DTs for solution of the direct ZS problem and for the conjugate one (3.29) with $D^* = -D$ are given by the following equalities:*

$$\psi^{ec} = (p + c_*^{-1}c + d_*^{-1}d)\left(\psi + \frac{\nu - \mu}{\lambda - \nu}\varphi_(\chi, \varphi)_p^{-1}\chi\psi\right) \tag{3.33}$$

and

$$\zeta^{ec} = \zeta^e E^{ce} = \zeta^e(\kappa p + \sigma^{ce}) = \left(\zeta - \frac{\nu - \mu}{\kappa - \mu}\zeta\varphi(\chi, \varphi)_p^{-1}\chi\right)(p + c^{-1}c_* + d^{-1}d_*)\ . \tag{3.34}$$

As regards the potentials, we have for example

$$u_{pq}^{ce} = u_{pq}c_1 + a\varphi_p(\chi, \varphi)_p^{-1}\chi_q c_1(\nu - \mu)$$

and

$$u_{qp}^{ce} = c_1^{-1}\left[u_{qp} + a\varphi_q(\chi, \varphi)_p^{-1}\chi_p\right](\nu - \mu)\ , \tag{3.35}$$

where $c_1 = c^{-1}c_*$ (for other elements it is sufficient to change the relevant indices).

The proof of (3.34) may be provided by the substitution of σ^{ec} from formulas for the conjugate problem (3.31) and (3.32) with $u \to u^e$ and $\phi \to \chi^e$. The derivation of (3.33) is performed first by (3.27) and (3.28) and then by the inverse of the transformation (3.31) based on the results of the previous step. For (3.35) the Riccati-like system for ξ and η is useful. Here the scalar product analog is introduced as

$$(\zeta, \varphi)_p = p\zeta\varphi p \in A_{pp}$$

and $c_{1,2}$ appear owing to an arbitrary choice of the constant diagonal elements of σ for both elementary transformations (Theorems 3.9 and 3.10).

It is easy to check that the transform of the potential may be rewritten in terms of idempotents

$$P = \phi(\chi, \varphi)_p^{-1}\chi \tag{3.36}$$

up to the choice $c_1 = 1$. Hence,

$$u[1] = u + \delta[J, P]\ . \tag{3.37}$$

The twofold-like DT has been used to create multisolitons and other solutions of nonlinear evolution equations [35, 240, 314, 354].

3.4 Elementary and twofold Darboux transformations. General case

Let D be the differentiation map on an associative differential ring A over the field K and assume there are idempotents (projectors) $p_i \in \ker D$ such that $\sum_i p_i = e$ and $p_i p_k = p_k p_i = 0$. For every $x \in A$ we denote $x_{ik} = p_i x p_k$, so we can split the ring into the direct sum of A_{ik}, $x_{ik} \in A_{ik}$.

Let us fix the element $J = \sum_i a_i p_i$, $a_i \in K$, $a_i \neq a_k$. The degenerate case of equal a_i is considered in a similar manner (see [449] where the matrix example is studied).

Definition 3.12. *The ZS operator is the linear operator in A acting in accordance with*

$$L_u : \psi \to D\psi + (\lambda J - u)\psi \ ,$$

where $\lambda \in K$, $\psi, u \in A$.

Suppose the potential u of the ZS operator satisfies the restriction $u_{ii} = 0$.

Definition 3.13. *Let for the potential u there exist a new potential u^e and the element $\sigma \in A$ such that for all $\lambda \in K$ the following intertwining relation holds:*

$$EL_u = L_{u^e} E \ , \tag{3.38}$$

where the action of a K-linear operator E is determined by the equality

$$E\psi = (\lambda p + \sigma)\psi \ .$$

Then the transformation

$$u \to u^e, \qquad \psi \to \psi^e = E\psi \ ,$$

where ψ is a solution of the equation

$$D\psi + (\lambda J - u)\psi = 0 \ , \tag{3.39}$$

is referred to as the eDT connected with the projector p.

It follows from (3.38) that

$$D\sigma = u^e \sigma - \sigma u \tag{3.40}$$

and

$$u^e p = pu + [J, \sigma] \ . \tag{3.41}$$

Theorem 3.14. *Let φ be a solution of the auxiliary ZS equation for $\lambda = \mu$ and $\varphi_{pp} = p\varphi p := \varphi_p$ has the inverse element φ_p^{-1} in A_{pp}. Assume also that $\sigma_{ii} = c_i \in A_{ii}$ and c_i^{-1} exists (this means that $c_i c_i^{-1} = c_i^{-1} c_i = p$). Then the operator of the eDT is defined by*

$$\sigma = -\mu p + \sum_{i=2}^{n} \left\{ \left[(a_1 - a_i)^{-1} u_{1i} - c_i p_i \right] \xi_i - (a_1 - a_i)^{-1} u_{1i} \right\} \ , \tag{3.42}$$

where $\xi_i = \varphi_i\varphi_p^{-1}$. *For the elements of the transformed potential* u^e *the following formulas are valid:*

$$u^e_{1i} = \left(\sigma_{pp}u_{1i} - \sum_{k=2}^{n}(a_1 - a_k)^{-1}u_{1k}u_{ki} - (a_1 - a_i)^{-1}Du_{1i}\right)c_i^{-1},$$

$$u^e_{i1} = (a_1 - a_i)c_i\xi_i\ , \qquad u^e_{ik} = c_i\left(u_{ik} + \frac{a_i - a_k}{a_k - a_1}\xi_i u_{1k}\right)c_k^{-1}. \tag{3.43}$$

The expressions (3.43) define the eDT, $i, k = 2, \ldots, n$. Of course, any other projector introduces the eDT in a similar way. First we note that $c_i \in \ker D$ and the expressions for u^e can be found in the spirit of the original Darboux approach. Then the elements $\xi_i = \varphi_i\varphi_p^{-1}$ with $\varphi_i = p_i\varphi$, $i = 2, \ldots, n$ satisfy the system of the Riccati-type equations and the relevant theorems may be proved along the lines of [314].

For the conjugate problem

$$D_*\phi + \phi(\kappa J - u) = 0 \tag{3.44}$$

the analog of Theorem 3.14 can be formulated and equations for σ^c and u^c differ from (3.40) and (3.41) only in the order of the operators. Hence, similarly to (3.30) we define

$$\phi^c = \phi E^c\ , \qquad E^c = \kappa p - \sigma^c\ , \tag{3.45}$$

where

$$\sigma^c = -\mu p + \sum_{i=2}^{n}\left[(a_1 - a_i)^{-1}(\xi_{*i} - 1)u_{i1} - \xi_{*i}c_{*i}p_i\right], \tag{3.46}$$

and

$$u^c_{i1} = c_{*i}^{-1}\left(u_{i1}\sigma^c_{pp} - \sum_{k=2}^{n}(a_1 - a_k)^{-1}u_{ik}u_{k1} + (a_1 - a_i)^{-1}D^*u_{i1}\right),$$

$$u^c_{1i} = (a_1 - a_i)\xi_{*i}c_{*i}\ , \qquad u^c_{ik} = c_{*i}^{-1}(u_{ik} + \frac{a_i - a_k}{a_k - a_1}u_{1k}\xi_{*i})c_{*k}\ , \tag{3.47}$$

with the same a_i, $c_{*i} = \sigma^c_{ii}$, but $\xi_{*i} = \phi_p^{-1}\phi_i, \phi_i = \phi p_i$; the existence of c_{*i}^{-1} and ϕ_p^{-1} in A_{ii} is implied.

For the constraints $\sigma_{ii} = 0$, for all or some i, the transformed potential u^e is expressed via the seed potential u only. We denote this additional possibility as the Schlesinger transformation. The case $n = 2$ was described by Matveev [314].

It is important to note that ζE^{-1} is the solution of the conjugate equation (3.44) when $D_* = -D$ and the eDT-transformed potential is identical to u^e. The inverse operator of E can be written in terms of projectors $p_iE^{-1}p_i = E_{ii}^{-1}$ so that for the first idempotent $p = p_1$ we have

$$E_{pp}^{-1} = (\lambda - \mu)^{-1}p\ ,$$

and the remaining ones are subsequently found:

$$E_{i1}^{-1} = (\lambda - \mu)^{-1}\xi_i \ , \quad E_{1i}^{-1} = (\lambda - \mu)^{-1}u_{1i}c_i/a_i \ , \quad E_{ik}^{-1} = \xi_i u_{1k}c_k^{-1} \ . \tag{3.48}$$

Now the twofold transformations and the respective DT theorem can be written and formulated. We introduce the twofold transformation as a sequence of two elementary ones in conjugate spaces. The first one is made by the inverse operator to E with the spectral parameter μ and the relevant solution φ of the direct problem. The second map is generated by the resulting function χ^e expressed in the same way but from a linearly independent seed solution χ of the equation (3.44) with the parameter ν. Another possibility exists for the opposite order of actions. The formulation of the results is given by Theorem 3.15:

Theorem 3.15. *If the scalar product*

$$(\zeta, \varphi)_p = p\zeta\varphi p \in A_{pp} \tag{3.49}$$

and c_i, c_{*i} are invertible in the subalgebra A_{pp}, and P is defined by

$$P = \varphi(\chi, \varphi)_p^{-1}\chi \ , \tag{3.50}$$

then the twofold DTs for solutions of the direct ZS problem and for the conjugate one (3.29) with $D^* = -D$ are given by the following equalities:

$$\psi^{ec} = \left(p + \sum_2^n c_{*i}^{-1}c_i\right)\left(\psi + \frac{\nu - \mu}{\lambda - \nu}P\psi\right) \tag{3.51}$$

$$\zeta^{ec} = \zeta^e E^{ce} = \zeta^e(\kappa p + \sigma^{ce}) = \left[\zeta - \frac{\nu - \mu}{\kappa - \mu}\zeta P\right]\left(p + \sum_2^n c_i^{-1}c_{*i}\right) \ . \tag{3.52}$$

The potentials are given by

$$u^{ec} = \left(p + \sum_2^n c_{*i}^{-1}c_i\right)(u + (\nu - \mu)[J, P])\left(p + \sum_2^n c_i^{-1}c_{*i}\right) \ . \tag{3.53}$$

Note that the operator P is a projector.

The proof of covariance of the ZS equations can be performed by substitution of σ^{ec} from formulas for the conjugate problem analogous to (3.46) with $u \to u^e$ and $\phi \to \chi^e$. The structure of the twofold DT (3.51)–(3.53) in the Abelian case and for $c_i = p_i$ resembles the known results [354]. This means that the iterated bDTs give a solution of the Riemann–Hilbert problem with zeros. The iterations are generated by combinations of (3.51) and (3.52). By the direct computation, the product of them provides

$$(\zeta^{ec}, \psi^{ec}) = p\zeta^{ec}\psi^{ec}p = (\zeta, \psi) + \frac{(\nu - \mu)(\kappa - \lambda)}{(\kappa - \mu)(\lambda - \nu)}\zeta\phi(\chi\phi)_p^{-1}\chi\psi \ . \tag{3.54}$$

This formula facilitates the iteration process.

One of the main purposes for the introduction of the twofold DT directly is concerned with the problem of reductions [331]. The properties of the ZS problem and its conjugate allow us to establish a class of reductions solving the simple conditions for the eDT parameters that enter the binary combination [280, 278, 281, 433] or go to the Lie algebra level [181, 361]. Combinations of the twofold DTs were used to obtain multisolitons and other solutions of the three-level Maxwell–Bloch equations [449]. A straightforward generalization can be obtained by replacing matrix elements by appropriate matrices. The most promising applications of the technique are related to operator rings. Such an example was developed in [265].

3.5 Schlesinger transformation as a special case of elementary Darboux transformation. Chains and closures

We begin with recalling the definition of the eDT and its combinations. The form we choose [265] combines results of the $n \times n$ matrix representation with a somewhat abstract extension of it based on the existence of idempotents and the respective division ring (skew field) B in the associative differential ring A over the field K. Let D be a differentiation map on A and two idempotents (projectors) p, $q = e - p$, $e = \mathrm{id} \in A$ be fixed by $p = p^2$, $p\,q = 0$. The projectors are rather general and all we should know about them is that both do not depend on the parameters of the theory and commute with D.

Consider the ZS problem $L_u\psi = (D + \lambda J - u)\psi = 0$, where $\lambda \in K, u, \psi \in A$, connected with the element $J = a_1 p + a_2 q$, $a_1 - a_2 = a \neq 0$. The general eDT $\psi \to \psi[1] = E\psi = (\lambda p - \sigma)\psi$ is defined by the element $\sigma \in A$ via intertwining relation $EL_u = L_{u[1]}E$. Analyzing the operator equations that follow from the intertwining relation, one arrives at the important consequence $q\sigma q = c$. It can be shown that within this choice of the eDT (another eDT appears if one interchanges $p \to q$ in the definition of the operator E) the element $q\sigma q = c$ commutes with D; therefore, c is a constant. Denote

$$puq = u_{pq} = v_n\ , \qquad qup = u_{qp} = w_n\ . \tag{3.55}$$

Here the index n marks the iteration number. We will consider equations (3.55) as determining the chain equations. This chain is infinite; therefore, the choice of origin ($n = 0$) is arbitrary. Suppose there is a solution of the ZS problem $\phi \in A_p = pAp \oplus qAp$, $p\,\phi = \phi_p \in B$, that corresponds to the spectral parameter μ; suppose next that $\exists \phi_p^{-1}$ and the gauge $c = qeq$ are adopted. The transforms

$$v_{n+1} = ac\xi_n + \mu_n v_n + v_n\xi_n v_n - Dv_n\ , \qquad w_{n+1} = a\xi_n\ , \tag{3.56}$$

and the additional "Miura" equation

$$D\xi_n = -\xi_n v_n \xi_n - \mu_n a\xi_n + w_n \tag{3.57}$$

form a closed set of connections defining the chain. It is enough to substitute the eDT connections (3.56) into the Miura links (3.57) and express the potentials v_n via ξ. We obtain the potentials

$$v_n = a\xi_n^{-1}\xi_{n-1}\xi_n^{-1} - a\xi_n^{-1}\mu_n - \xi_n^{-1}(D\xi_n)\xi_n^{-1} , \qquad w_n = a\xi_{n-1} , \tag{3.58}$$

which yield the chain equation

$$\begin{aligned} & a\xi_{n+1}^{-1}\xi_n\xi_{n+1}^{-1} - a\xi_{n+1}^{-1}\mu_{n+1} - \xi_{n+1}^{-1}(D\xi_{n+1})\xi_{n+1}^{-1} \\ &= ac\xi_n + \mu_n[a\xi_n^{-1}\xi_{n-1}\xi_n^{-1} - a\xi_n^{-1}\mu_n - \xi_n^{-1}(D\xi_n)\xi_n^{-1}] \\ &+ [a\xi_n^{-1}\xi_{n-1}\xi_n^{-1} - a\xi_n^{-1}\mu_n - \xi_n^{-1}(D\xi_n)\xi_n^{-1}]\xi_n \\ &\times [a\xi_n^{-1}\xi_{n-1}\xi_n^{-1} - a\xi_n^{-1}\mu_n - \xi_n^{-1}(D\xi_n)\xi_n^{-1}] \\ &- D[a\xi_n^{-1}\xi_{n-1}\xi_n^{-1} - a\xi_n^{-1}\mu_n - \xi_n^{-1}(D\xi_n)\xi_n^{-1}] . \end{aligned} \tag{3.59}$$

Remark 3.16. A straightforward consequence of (3.58) and (3.59) leads to a definite link between elements of the potential u. The link permits only such constraints that are compatible with the definitions of ξ and ϕ. The use of the second eDT ($p \leftrightarrow q$) immediately allows us to put constraints with the whole powerful set of algebraic tools [181] based on automorphisms of the underlying Lie algebra [331] with grading [361]. It is obvious that the scope of the whole theory is much broader than what we can present here.

The Schlesinger transformation for nonzero elements of the potential is defined by the limiting case $q\sigma q = 0$; this condition is degenerate for the initial system of intertwining relations. Therefore one should solve the seed equations from the very beginning [278, 289]. The advantage of employing the Schlesinger transformation consists in the fact that in this case there is no need to use the solution of the auxiliary ZS problem, since the transformed potential u^e is expressed via the seed potential u only. Finally, the elements of u are transformed by the Schlesinger transformation as

$$u^s_{pq} = (D^2u_{pq} - Du_{pq}u_{pq}^{-1}Du_{pq} - u_{pq}u_{qp}u_{pq})(ac_0)^{-1} , \quad u^s_{qp} = -ac_0u_{pq}^{-1} . \tag{3.60}$$

Suppose the inverse element u_{pq}^{-1} exists. The 2×2 matrix ZS problem enters the KdV and nonlinear Schrödinger (NLS) theories together with the appropriate choice of the second (covariant) Lax operator. As an illustration denote $u_{12} = v, u_{21} = w$. After the nth iteration we arrive at the chain system

$$v_{n+1} = [v_n'' + (v_n')^2/v_n - v_n^2w_n]/ac_0 , \qquad w_{n+1} = ac_0v_n^{-1} . \tag{3.61}$$

Reductions for the KdV and NLS theories are given by $w_n = 1$ and $v_n = w_n$, respectively. The simplest heredity condition $v_{n+1} = w_{n+1}$ closes the chain for the NLS case:

$$v'' - (v')^2/v = v^3 - ac_0/v .$$

This equation is integrated by the substitution $v' = F(v)$. In terms of $s = F^2$ we have

$$vs_v/2 - s = v^4 - a^2c_0^2 \, .$$

Integrating, we arrive at

$$v' = \sqrt{v^4 + a^2c_0^2 + c_1v^2} \, ,$$

where c_0 and c_1 are constants. The resulting differential equation is integrated in elliptic functions. The study of a compatible Lax pair, e.g., for the NLS equation, demands a "time" variable, additional to x. Let the "time" dependence be defined via the second Lax equation of the same (ZS) form as the spectral equation. Then one arrives at the t-chain equation of the form (3.61) but with different constants. Combining both chains gives equations of a hydrodynamic type.

In the richer case of three projectors p, q, and s, we redefine J as $J = a_1p + a_2q + a_3s$. General equations for the eDT with the same form of intertwining relations lead again to the constant elements $q\sigma q = c$, $s\sigma s = d$. In the generic case of nonzero c and d the eDT transforms are determined in [269].

Consider now the Schlesinger transformations with restrictions to the potential given by $\sigma_{qq} = c \neq 0$, but $\sigma_{ss} = 0$. The covariance theorem has the following formulation:

Theorem 3.17. *Let $\sigma_{ss} = 0$ and assume u_{ps}^{-1} and σ_{sp}^{-1} exist and the condition $[c, \sigma_{sp}] = 0$ holds. Then the equations for σ can be solved directly and the transform of the potential of the ZS operator L_u $u \to u^s$ is determined by*

$$u_{pq}^s = (Du_{pq}/a + \sigma_{pp}u_{pq} - u_{ps}u_{sq}/b)c^{-1} \, ,$$

$$u_{qp}^s = -abcu_{qp}u_{ps}^{-1}/(a-b) \, , \qquad u_{sp}^s = -b\sigma_{sp} \, ,$$

$$u_{ps}^s = -[(b\sigma_{pp}u_{pq} - u_{ps}u_{sq} - bDu_{pq}/a)u_{qs}u_{ps}^{-1} + u_{pq}u_{qp}/a + u_{ps}u_{sp}/b - D\sigma_{pp}]/\sigma_{sp} \, ,$$

$$u_{sq}^s = (1-a/b)\sigma_{sp}u_{pq}c^{-1} \, , \quad u_{qs}^s = [(1-a/b)\sigma_{qp}u_{ps} + cu_{qs}]d^{-1} \, ,$$

where

$$\sigma_{pp} = (Du_{ps} + u_{pq}u_{qs})u_{ps}^{-1}/b \, , \quad \sigma_{qp} = bcu_{qs}u_{ps}^{-1}/(a-b) \, ,$$

and σ_{sp} is found from the equation

$$\sigma_{sp}^{-1}D\sigma_{sp} = -\left(Du_{ps} + \frac{b}{a}u_{qs}\right)u_{ps}^{-1} + \left(1 - \frac{a}{b}\right)c^{-1}u_{pq} \, .$$

In general, where there are more than two projectors, we have the additional possibility of nonzero σ_{ss}.

The chain equations with the new possibilities to further construct solutions may be derived by the algorithm that is described at the beginning of this section and leads to the analog of (3.59). The simplest applications concern 3×3 matrix problems with known reductions to N-wave, KdV–MKdV, Hirota–Satsuma, and Oikawa–Yajima equations [211, 269, 278, 458].

3.6 Twofold Darboux transformation and Bianchi–Lie formula

We have introduced the twofold transformation as a sequence of two elementary ones in conjugate spaces. The first transformation is performed by the inverse operator to E with the spectral parameter μ and the corresponding solution of the direct problem φ. The second map is generated by the resulting functions $\chi^e = E^{-1}\chi$ expressed in the same way but from a linearly independent seed solution χ of the conjugate ZS equation with the parameter ν. The final form of the transformation is written as [269]

$$\psi^{ec} = \psi + \beta\varphi_(\chi,\varphi)_p^{-1}\chi\psi\ , \qquad \beta = \frac{\nu-\mu}{\lambda-\nu}\ . \tag{3.62}$$

The analog of a scalar product is introduced by

$$(\chi,\varphi)_p = p\chi\varphi p \in A_{pp} = pAp$$

and the inverse exists in A_{pp}. It is easy to check that the transform of the potential may be rewritten in terms of the idempotent $P = \phi(\chi,\varphi)_p^{-1}\chi$ as

$$U^{ec} = U + (\nu-\mu)[J,P]\ ,$$

for example,

$$u_{pq}^{ec} = u_{pq} + a\varphi_p(\chi,\varphi)_p^{-1}\chi_q(\nu-\mu)\ .$$

Remember that $J = a_1p + a_2q$, $a_1 - a_2 = a \neq 0$. An analog of the position vector at the ring under consideration can be defined as before. Let us denote $\gamma(\lambda) = (\lambda-\mu)(\lambda-\nu)^{-1}$ and $s = \psi^{-1}P\psi$. Then the solutions obtained by the twofold transformations (3.62) yield

$$r_1 = r + \frac{\partial\gamma(\lambda)}{\partial\lambda}\gamma^{-1}s\ .$$

The element s is determined by the seed solution only. This formula generalizes the Bianchi–Lie transformation for the non-Abelian entries. The existence of the inverse element ψ^{-1} is supposed and the identity $(1+\beta P)^{-1} = 1 - \beta P/(\beta+1)$ is taken into account.

As for the complete set of projectors, the form $\sum_i p_iABp_i = (A,B) = (B,A)$ is symmetric; therefore, it may be regarded as an analog of the Killing–Cartan metrics. The length of the vector s is then unity. The transform may be generalized further for the case of the iterated twofold DTs as

$$r[n] = r + \sum_i^n \gamma_{i,\lambda}s[i]/\gamma_i.$$

We conclude with the remark that dealing with a few projectors [269] may produce various versions of the Schlesinger transformation whose non-Abelian version seems interesting from the point of view of quantum problems.

3.7 *N*-wave equations: example

The general dressing by the DT allows us to solve non-Abelian three-wave systems when a linear term is added. Among these systems there is the matrix equation that goes after reduction to the classical N-wave systems with linear terms. The linear terms can account for such an important phenomenon as asynchronism. In this case the so-called inclined solitons occur, with a loss of symmetry between the leading and back wavefronts [269, 272], see the next subsection for pictures. The classical (Abelian) three-wave system is discussed in Sect. 8.5.

3.7.1 Twofold DT of *N*-wave equations with linear term

Let us consider the set of n idempotents p_i. The elements $a_i, b_k \in K$ define combinations

$$M = \sum_i a_i p_i \ , \qquad N = \sum_i b_i p_i \ .$$

If elements of the ring are functions of parameters t and y, for arbitrary $\Psi \in A$ the internal derivative $\mathrm{ad}_x \Psi = [x, \Psi]$ may define the first-order problem via the combined differential operators $D_{t,y}$. The general idea of the twofold DT as a symmetry of a nonlinear system is demonstrated by two linear equations called the Lax pair (3.63) for an auxiliary matrix function $\Phi(t, y)$,

$$\begin{aligned} D_t \Phi &= -\lambda M \Phi + \big[H, M\big]\Phi \ , \\ D_y \Phi &= -\lambda N \Phi + \big[H, N\big]\Phi \ , \end{aligned} \tag{3.63}$$

with the following definitions of matrices (n = 3):

$$M = \mathrm{diag}\{a_1, a_2, a_3\}, \quad N = \mathrm{diag}\{b_1, b_2, b_3\},$$

$$R = \mathrm{diag}\{r_1, r_2, r_3\}, \quad S = \mathrm{diag}\{s_1, s_2, s_3\}..$$

Matrix elements of the off-diagonal matrix H are identified as being proportional to the components of the wavetrain envelopes. Operators D_t and D_y are defined as

$$D_t \Phi = \Phi_t + R\big[x, \Phi\big] \ , \tag{3.64}$$

$$D_y \Phi = \Phi_y + S\big[z, \Phi\big] \tag{3.65}$$

with the constant diagonal matrices x and z. The compatibility condition for the Lax pair (3.63) is the following (matrix) equation:

$$\big[H_t, N\big] - \big[H_y, M\big] = \big[\big[H, M\big], \big[H, N\big]\big] - \big[x, \big[RH, N\big]\big] + \big[z, \big[SH, M\big]\big] \ , \tag{3.66}$$

which represents the general N-wave system. If, for $N = 3$, we use the (unreduced) system of equations (3.66), we get six equations. Assuming that matrix H is Hermitian, the desired reduction to three equations is obtained. Finally,

$$(b_1 - b_2)H_{21,t} + (a_2 - a_1)H_{21,y} = [a_3(b_1 - b_2) + a_1(b_2 - b_3)$$
$$+a_2(b_3 - b_1)]H^*_{32}H_{31} + [r_2(b_1 - b_2)(x_1 - x_2) - s_2(a_1 - a_2)(z_1 - z_2)]H_{21} ,$$
$$(b_1-b_3)H_{31,t}+(a_3-a_1)H_{31,y} = [a_3(b_2-b_1)+a_2(b_1-b_3)+a_1(b_3-b_2)]H_{21}H_{32}$$
$$+[r_3(b_1 - b_3)(x_1 - x_3) - s_3(a_1 - a_3)(z_1 - z_3)]H_{31} , \qquad (3.67)$$
$$(b_2 - b_3)H_{32,t} + (-a_2 + a_3)H_{32,y} = [a_3(b_1 - b_2) + a_1(b_2 - b_3)$$
$$+a_2(-b_1 + b_3)]H^*_{21}H_{31} + [r_3(b_2 - b_3)(x_2 - x_3) - s_3(a_2 - a_3)(z_2 - z_3)]H_{32} .$$

Equations (3.67) have the form of the three-wave system with asynchronism.

3.7.2 Inclined soliton by twofold DT dressing of the "zero seed solution"

Plugging (3.64) and (3.65) into (3.63), we can write the Lax pair for (3.67):

$$\Phi_t + R[x, \Phi] = -\lambda M\Phi + [H, M]\Phi , \qquad (3.68)$$

$$\Phi_y + S[z, \Phi] = -\lambda N\Phi + [H, N]\Phi .$$

Taking into account the explicit forms of matrices R, S, x, and z, we obtain

$$\phi_{ij,t} + r_i(x_i - x_j)\phi_{ij} = -\lambda a_i\phi_{ij} + (a_j - a_i)H_{ij}\phi_{jk} , \qquad (3.69)$$

$$\phi_{ij,y} + s_i(z_i - z_j)\phi_{ij} = -\lambda b_i\phi_{ij} + (b_j - b_i)H_{ij}\phi_{jk} ,$$

where ϕ_{ij} are matrix elements of Φ. The general formula for the twofold DT (the dressing formula) takes the form

$$H[1] = H + (\mu + \mu^*)\chi , \qquad (3.70)$$

where $\chi_{ij} = \phi^*_j\phi_i/(\phi, \phi)$ is built from the columns ϕ_j of the matrix Φ and (ϕ, ϕ) is a scalar product. With $H = 0$ as a trivial solution of the three-wave system, i.e., "zero seed solution," the "dressed" solution of (3.69) is given by

$$\phi_{ij} = D_{ij} \exp [-\lambda(a_i t + b_i y) - r_i(x_i - x_j)t - s_i(z_i - z_j)y] , \qquad (3.71)$$

where D_{ij} are constants.

We get a soliton solution using the first column of Φ:

$$\phi_{i1} = D_{i1} \exp [-\lambda(a_i t + b_i y) - r_i(x_i - x_1)t - s_i(z_i - z_1)y] . \qquad (3.72)$$

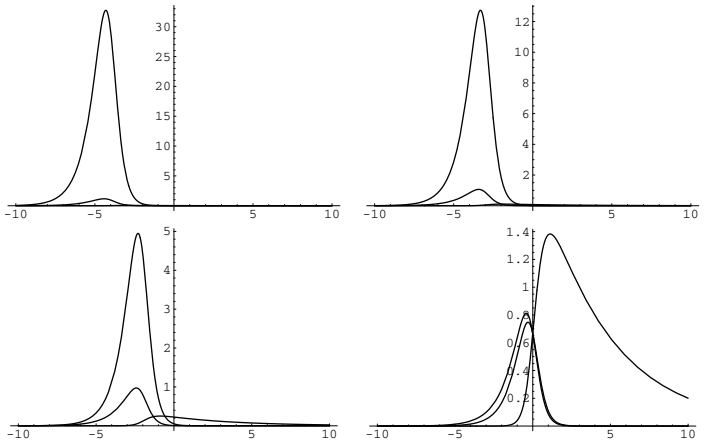

Fig. 3.1. Generation of the third wave with account for asynchronism for the time $t = 0, 1, 2, 3$ (in dimensionless units)

When we have a solution of the twofold DT in the form of

$$H_{ij} = (\lambda + \lambda^*)\frac{\phi_j^* \phi_i}{(\phi, \phi)} ,$$

we can numerically analyze the three-wave interaction. Because the matrix H is Hermitian it is enough to consider only the coefficients H_{21}, H_{31}, and H_{32}. Taking the parameters as $\lambda = 1$, $a_1 = -1$, $a_2 = 1$, $a_3 = 0$, $b_1 = 1$, $b_2 = -1$, $b_3 = 0$, $r_1 = 0.2$, $r_2 = -0.5$, $r_3 = 0.6$, $s_1 = 1$, $s_2 = 0.5$, $s_3 = 0$, $x_1 = 0.45$, $x_2 = 0.1$, $x_3 = 0.6$, $z_1 = -0.54$, $z_2 = 1$, and $z_3 = 0.6$, we get a simulation of the three-wave interaction in one dimension by the explicit formula (3.72) (Fig. 3.1). Here y stands for the propagation direction.

It is worth noting that the most important parameter to determine the properties of the solution is λ, the spectral parameter. Physically, λ has the meaning of an amplitude. Much smaller contributions are provided by the other coefficients (a_i, b_i, r_i, z_i, x_i, and s_i). Let us stress once again that the interaction without asynchronism produces only symmetric envelope pulses. For more details see [272].

3.7.3 Application of classical DT to three-wave system

Let us compare the above results with those that can be obtained by the method in the previous chapter. Consider an $n \times n$ matrix set $\{A\}$ and choose n projectors $p_i^2 = p_i \in \{A\}$, $p_i p_k = 0$, $i, k = 1, 2, \ldots, n$. The simplest example

of such a matrix is a diagonal one with the only ith nonzero element that is equal to 1. A choice of numbers a_i, b_k defines matrices

$$M = \sum_i a_i p_i \ , \qquad N = \sum_i b_i p_i \ .$$

The representation of this form is convenient for generalizations to the operator case [267]. The nonlinear equations for interacting waves appear as a compatibility condition if we start from the pair of ZS equations of the form

$$\Psi_t = MD\Psi + [H, M]\Psi \ ,$$

$$\Psi_y = ND\Psi + [H, N]\Psi \ , \tag{3.73}$$

where the operator D can play the role of abstract differentiation, realized here as the commutator with some constant matrix x: $D\Psi = [x, \Psi]$. In analogy with [354] we represent the potentials of the ZS equations as commutators $[H, M]$ and $[H, N]$. Consider first the standard DT [324]. The existence of the classical DT may restrict the choice of the matrices x, M, and N. For elements $H_{ik} = p_i H p_k$ and the obvious choice $x = \sum_i x_i p_i$, we obtain the system

$$(a_k - a_i)H_{ik,y} - (b_k - b_i)H_{ik,t} \tag{3.74}$$

$$= [(a_s - a_i)(b_k - b_s) - (b_s - b_i)(a_k - a_s)]H_{is}H_{sk} - (H_{ik}x_i - x_k H_{ik})(b_i a_k - a_i b_k) \ ,$$

which differs from (3.67) in the last (linear) terms for non-Abelian matrix elements. The solutions of the system may be constructed by means of the following proposition [272]:

Proposition 3.18. *The system (3.74) is integrable by means of the matrix DT*

$$\Psi[1] = D\Psi - (D\Phi)\Phi^{-1}\Psi \ ,$$

if Φ and Ψ are solutions of (3.73), when the equality $MN = NM$ holds. The DT of the matrix H in combination with some gauge transformation is written as

$$H[1] = H + (D\Phi)\Phi^{-1} + A \ , \tag{3.75}$$

where A is a matrix that commutes with both matrices M and N. This matrix is the gauge one guaranteeing $H_{ii}[1] = 0$. The possible choice of A for diagonal matrices M and N consists of

$$A = -\mathrm{diag}(D\Phi)\Phi^{-1} \ .$$

Proof. The standard transforms for the potentials $[H, M]$ and $[H, N]$ are $[H, M] + [(D\Phi)\Phi^{-1}, M]$ and $[H, N] + [(D\Phi)\Phi^{-1}, N]$, with the obvious possibility to add A to H.

If one treats the simplest three-wave case, the compatibility condition (3.74) is written with more detail. For example, the first equation is

$$H_{12,t} - v_{12}H_{12,y} = n_{12}H_{13}H_{32} - k_{12}H_{12}$$

and similar expressions exist for the others, with group velocity $v_{12} = (b_2 - b_1)/(a_2 - a_1)$ and nonlinear constant $n_{12} = [(a_3 - a_1)(b_2 - b_3)/(a_2 - a_1) - (b_3 - b_1)]$. The coefficient $k_{12} = (x_2 - x_1)(b_1a_2 - a_1b_2)/(a_2 - a_1)$ either defines attenuation or can be identified with the asynchronism parameter $\Delta k = k_1 - k_2 - k_3$ [266, 358]. The last term seems to be interesting for N-wave systems even in a matrix case. For example, such a linear term may account for the stimulated Brillouin scattering/stimulated Raman scattering effects and phase differences of waves that appear from the wave asynchronism. Some damping may be accounted for as well.

The physical systems mentioned appear if the reduction constraint is chosen as $H^+ = H$. To preserve the reduction constraint during the process of iterations, we should provide the existence of the hereditary property of iterations. In other words, Φ should be chosen to satisfy the condition

$$x - x^+ = \Phi x \Phi^{-1} - (\Phi^+)^{-1} x^+ \Phi^+ \ . \tag{3.76}$$

For the unitary matrix $\Phi^+\Phi = I$, the condition (3.76) simplifies:

$$x - x^+ = \Phi(x - x^+)\Phi^{-1} \ , \tag{3.77}$$

which is satisfied when the matrix x is Hermitian. If, further, $D\Phi = \Phi\Lambda$, then

$$H[1] = H + \Phi\Lambda\Phi^{-1} + A \ . \tag{3.78}$$

Another choice of the matrix Φ is possible via stationary solutions of the basic Lax equations with a matrix spectral parameter $\Lambda = \mathrm{diag}\{\lambda_1, \ldots, \lambda_n\}$ so that $\Phi_t = \Phi\Lambda$.

The one-step transformation generates solutions that are illustrated by Figs. 3.2 and 3.3, which give $H_{ik}(t)$ for different values of the parameters x_i.

3.8 Infinitesimal transforms for iterated Darboux transformations

The linear dependence on the spectral parameter difference $\delta = \mu - \nu$ [see (3.53)] offers an interesting possibility to generate new hierarchies of a combined nonlinear system of potentials and eigenfunctions. The corresponding equations are obtained by the limiting procedure $\delta \to 0$, e.g.,

$$u_\delta = [J, P_0] \ , \qquad \Psi_\delta = \frac{P_0\Psi}{\lambda - \mu} \ , \qquad \zeta_\delta = \frac{\zeta P_0}{\kappa - \mu} \ , \tag{3.79}$$

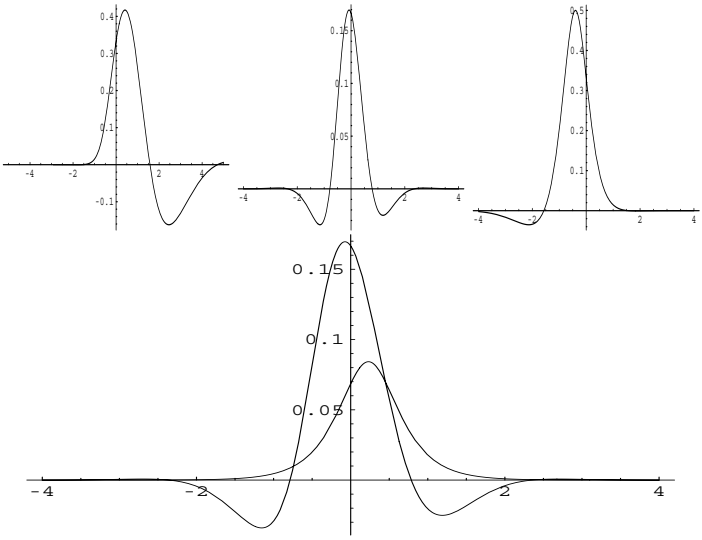

Fig. 3.2. The "inclined" solitons of a three-wave system. Plots are given for the components H_{12}, H_{23}, and H_{31} as functions of t at $y = 0$. The parameters of the equations are $a_1 = 1$, $a_2 = -1$, $b_1 = 1$, $b_2 = -1$, $b_3 = 0$, and $a_3 = 0$. The bottom plot shows the component H_{12} with different (real) values of the parameter k_{12}. The symmetric one has $k_{12} = 0$. The asymmetric one corresponds to $x_2 = 0.5$ and $x_3 = 0.6$

where $P_0 = \lim_{\delta \to 0} P$ and ζ and κ are solution and spectral parameters of the conjugate equation. This system can be studied within the DT technique as the formalism contains its solutions. It results in some hierarchy after the expansion of P_0 in the Laurent series with respect to free parameters λ and κ [269].

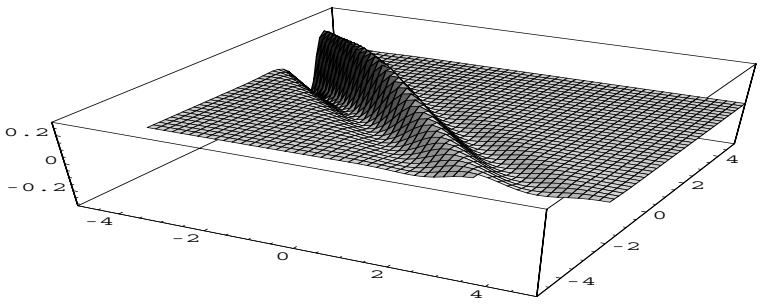

Fig. 3.3. The dependence of the perturbed H_{12} component on both variables (ty-plane)

Following the results of [267, 434], we take the transform of solutions of the ZS problem with intermediate function φ and conjugate function χ that both correspond now to μ, and consider the difference $\delta = \nu - \mu$ as a small parameter. The expression (3.51) takes the form

$$\Psi[1] = \Psi + \delta P\Psi/(\lambda - \mu) \; . \tag{3.80}$$

From (3.53) with the choice of $c_i = p_i$, the infinitesimal transformation of the potential element follows:

$$u[1] = u + \delta \, [\, J \, , \, P_0 \,] \; . \tag{3.81}$$

If the nonzero scalar product $(\varphi, \chi)_p$ can be normalized in A_{pp}, we may simplify the projector as $P^{(0)} = \varphi p \chi$, with φ and χ being solutions of the direct and conjugate (3.44) problems with the same spectral parameter μ, such that $(\chi, \varphi)_p = p$.

Theorem 3.19. *Let the solutions $\varphi^{(i)}$ and $\chi^{(i)}$ of the direct and conjugate problems corresponding to the spectral parameters μ_i, ν_i, and $(\varphi^{(i)}, \chi^{(i)})_p^{-1}$ exist. For the iterated twofold DT with the parameters $\nu_i = \mu_i + \delta_i$ we have*

$$U[N] = U + \left[J, \sum_{i=1}^{N} \delta_i \varphi^{(i)} p \chi^{(i)} \right] \; ,$$

if

$$\delta_i = \mathcal{O}\left(\delta_1\right) \; , \quad i = 1, \ldots, N \; , \qquad |\delta_1| << |\mu_k| \; , \; |\lambda - \mu_k| \; .$$

The proof can be performed by induction using (3.79) and (3.81).

The dependence of the solution on an arbitrary function gives the possibility to use it for the construction of general solutions of initial-boundary problems, looking for a sequence of infinitesimal symmetries. One can also study the stability of soliton solutions of the corresponding nonlinear equations.

The main observation that allows us to develop the technique, is that the binary combination of the nonzero eDTs (3.45)–(3.47) can generate a new potential that coincides with the seed one. Namely, this transform is $u[1] = u$ if $\nu = \mu$ and $(\chi, \phi) \neq 0$. This means that the twofold DTs determine a Lie group and a corresponding Lie algebra, according to the linear appearance of the parameter $\mu - \nu$ in the one-parametrical subgroups. All iterated solutions of the ZS problem determine independent elements of the Lie algebra.

All items in (3.79) have poles of the first order. Hence, going from sums to integrals, we obtain the following representations of the linear part of the iterated potential:

$$u^{(1)} = \left[J \, , \int_S \psi(\lambda) p \phi(\lambda) \mathrm{d}\alpha(\lambda) \right] .$$

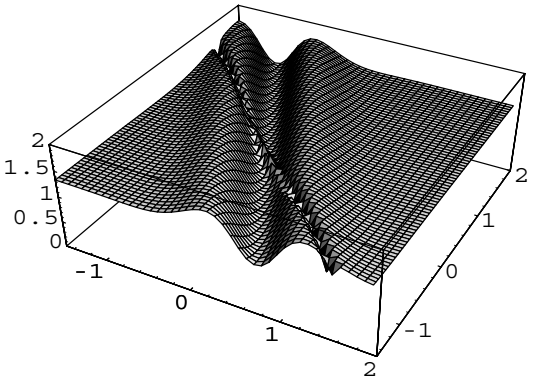

Fig. 3.4. The dependence of the perturbed H_{12} component on both variables (ty-plane). There is no instability development

Here S and $\mathrm{d}\alpha$ are the path of integration in the α-plane and the measure, respectively.

We mentioned before the possibility to analyze the stability of soliton solutions. Going down to the usual matrix case for illustration, e.g., to the classic three-wave problem (3.67) with the asynchronism, let us consider the second iteration formula for the perturbed soliton. The solutions of the linear system (3.69) (over zero seed potential) are given by (3.51) and (3.52) from Sect. 3.4. The perturbation of the soliton solution H[1] [see (3.70) for $H = 0$] is given by $-(2\mathrm{Re}\,\lambda)P$ and contains the projector $P_{ij} = \phi_j^*\phi_i/(\phi, \phi)$ multiplied by $\delta = 2\mathrm{Re}\lambda$, which can be chosen as a small parameter.. The analysis shows the stability of the soliton solution in this class of perturbations (Fig. 3.4).

3.9 Darboux integration of $\mathrm{i}\dot{\boldsymbol{\rho}} = [\boldsymbol{H}, \boldsymbol{f}(\boldsymbol{\rho})]$

The Darboux-type method of solving a class of nonlinear von Neumann equations $\mathrm{i}\dot{\rho} = [H, f(\rho)]$ is exposed below following [437]. The explicit construction demonstrates that the presence of self-scattering solutions constitutes a generic property of the nonlinearities considered. A solution for an infinite-dimensional case is presented as well.

3.9.1 General remarks

The equation

$$\mathrm{i}\dot{X} = [Y, f(X)] \tag{3.82}$$

plays an important role in different branches of quantum and classical physics. First, if $f(x)$ $(x \in \mathbb{R})$ satisfies $f(0) = 0$, $f(1) = 1$ and $X = X^\dagger = X^2$, then $f(X) = X$ and the equation is simply the linear von Neumann equation. In

addition, if $X = |\psi\rangle\langle\psi|$ and $Y = H$ is a Hamiltonian, (3.82) with *nonlinear* functions $f(X)$ obeying the above properties is physically equivalent to the Schrödinger equation

$$\mathrm{i}|\dot{\psi}\rangle = H|\psi\rangle \ . \tag{3.83}$$

Second, equation (3.82) can be written for a large class of nonlinearities in the form of the Heisenberg equation of motion

$$\mathrm{i}\dot{X} = [X, h(X)] \tag{3.84}$$

with Hamiltonian $h(X)$. [Assuming, to take an example, $f(X) = -X^n$, we obtain $h(X) = X^{n-1}Y + X^{n-2}YX + ... + YX^{n-1}$]. The equations of this form are often used in nonlinear quantum optics.

Third, it is known that the linear von Neumann equation can be associated with a Lie–Poisson Hamiltonian system. In the case of the usual "extensive" statistics, the Hamiltonian function is the average energy $H_1 = \mathrm{tr}(H\rho)$. Yet, there are statistical problems that are naturally described by *nonextensive* statistics in terms of the Tsallis $q \neq 1$ entropies and q-averaged energies $H_q = \mathrm{tr}(H\rho^q)/\mathrm{tr}\rho^q$ [426, 427]. One of the remarkable properties of q-statistics is the q-independence of standard geometric structures associated with equilibrium thermodynamics. Extending this observation from equilibria to nonequilibria, one finds that H_q is a Hamiltonian function for Lie–Poisson dynamics [99, 347]

$$\mathrm{i}\dot{\rho} = [H, f(\rho)], \tag{3.85}$$

where $f(\rho) = C_q(\rho)\rho^q$ and $C_q(\rho)$ is a Casimir function satisfying $C_1(\rho) = 1$. The von Neumann equation (3.85) with arbitrary nonlinearity is Lie–Poissonian with the Hamiltonian function $H_f = \mathrm{tr}[Hf(\rho))]$.

Fourth, the equation

$$\mathrm{i}\dot{X} = [Y, X^2] \tag{3.86}$$

appears in several contexts. The most familiar physical example (with $Y_{kl} = \delta_{kl}/I_l$, $X_{kl} = \mathrm{i}\varepsilon_{kml}J_m$) is the Euler equation of a freely rotating rigid body. Less known and often more abstract versions of this equation are related to the Lie–Poisson equations occurring in fluid dynamics [32, 309], the Nahm equations in non-Abelian gauge theories [212], and the N-wave equations for electromagnetic waves in nonlinear media [471, 472].

Quite recently the equation

$$\mathrm{i}\dot{X} = [Y, X^3] \tag{3.87}$$

was discovered in connection with symmetries of (3.86) [420].

Further, general equations of the form (3.85) appeared in the context of nonlinear Nambu-type theories [95]. Nonlinear Lie–Poisson density matrix equations were applied in quantum mechanics with mean-field backgrounds [74, 72, 73] and nonlinear quantum mechanics [75, 76, 222]. Solutions of these equations were used as models of non-completely-positive nonlinear maps

which nevertheless satisfy physical conditions widely believed to be equivalent to complete positivity [97]. Finally, the Lie–Poisson density matrix techniques for extending the nonlinear evolutions of subsystems to entangled states proved to have applications in quantum information theory [15].

Although the literature devoted to the Euler equation is quite extensive [18, 32, 309, 371], analytical methods were only recently applied to its density matrix analog (Euler–von Neumann equation) [96, 98, 254, 276]. The constraints imposed on density matrices ($\rho^\dagger = \rho$, $\rho \geq 0$, $\mathrm{tr}\rho = 1$) and Hamiltonians ($H^\dagger = H$, $H \geq 0$, unboundedness) require techniques which are not based on standard integration via quadratures and the similar, since the systems in question are generically infinite-dimensional. The technique used in [96, 98, 254, 276] is an appropriate modification of the dressing method [87, 88, 354] or, rather, of its twofold elementary DT version [267, 434].

The Darboux-type method of integration of the Euler–von Neumann equation

$$\mathrm{i}\dot{\rho} = [H, \rho^2] \tag{3.88}$$

introduced in [276] led to discovery of the so-called self-scattering solutions [96, 98]. The process of self-scattering continuously interpolates between two asymptotically linear evolutions. Equation (3.88) possesses a class of solutions of the form $\rho(t) = \mathrm{e}^{-\mathrm{i}Ht}\rho(0)\mathrm{e}^{-\mathrm{i}Ht}$ which occur whenever $[H, f(\rho)] = [H, \rho]$. We regard such solutions as "trivial" solutions of (3.88) constructed by means of the dressing method.

A problem remained open in all the previous papers was how to obtain solutions of (3.85) with other values of the Tsallis parameter q. In fact, the case $q = 2$ was not very interesting from the point of view of nonextensive statistics applications, since the parameters involved in analysis of actual physical situations were either close to 1 or $0 < q < 1$. The case $q = 1/2$ turned out to be of special interest owing to its significance in plasma physics. Next we present an extension of the Darboux technique to a wide class of nonlinear von Neumann equations.

3.9.2 Lax pair and Darboux covariance

We begin with the overdetermined linear system (Lax pair)

$$z_\lambda \langle\psi| = \langle\psi|(\rho - \lambda H)\,, \tag{3.89}$$

$$-\mathrm{i}\dot{\langle\psi|} = \frac{1}{\lambda}\langle\psi|A\,. \tag{3.90}$$

Here A, ρ, and H are operators acting on a "bra" vector $\langle\psi|$ associated with an element of a Hilbert space; the dot denotes the time derivative $\mathrm{d}/\mathrm{d}t$, and complex numbers λ and z_λ are independent of t. The operators ρ and H will typically play the roles of density matrices and Hamiltonians, respectively, but one can also think of them as just some operators without any particular

quantum-mechanical connotations. The system (3.89) and (3.90) is compatible, if the equations

$$\mathrm{i}\dot{\rho} = [H, A] \,, \tag{3.91}$$
$$\dot{H} = 0 \,, \qquad [A, \rho] = 0$$

are fulfilled. Equation (3.91) reduces to (3.85) if $A = f(\rho)$. A generalization of the von Neumann equation that includes a class of "non-Abelian nonlinearities" (i.e., $[\rho, f(\rho)] \neq 0$) was discussed in the previous chapter; see also [89, 436].

For further consideration we introduce two additional Lax pairs

$$z_\nu \langle\chi| = \langle\chi|(\rho - \nu H) \,, \tag{3.92}$$
$$-\mathrm{i}\langle\dot{\chi}| = \frac{1}{\nu}\langle\chi|A \,, \tag{3.93}$$
$$z_\mu |\varphi\rangle = (\rho - \mu H)|\varphi\rangle \,, \tag{3.94}$$
$$\mathrm{i}|\dot{\varphi}\rangle = \frac{1}{\mu} A|\varphi\rangle \,. \tag{3.95}$$

The method of solving (3.85) is based on the following theorem establishing the Darboux covariance of the Lax pair (3.89) and (3.90).

Theorem 3.20. *Assume $\langle\psi|$, $\langle\chi|$, and $|\varphi\rangle$ are solutions of (3.89), (3.90), and (3.92)–(3.95) and $\langle\psi_1|$, ρ_1, A_1 are defined by*

$$\langle\psi_1| = \langle\psi|\Big(1 + \frac{\nu - \mu}{\mu - \lambda} P\Big) \,, \tag{3.96}$$
$$\rho_1 = \Big(1 + \frac{\mu - \nu}{\nu} P\Big)\rho\Big(1 + \frac{\nu - \mu}{\mu} P\Big) \,, \tag{3.97}$$
$$A_1 = \Big(1 + \frac{\mu - \nu}{\nu} P\Big) A \Big(1 + \frac{\nu - \mu}{\mu} P\Big) \,, \tag{3.98}$$
$$P = \frac{|\varphi\rangle\langle\chi|}{\langle\chi|\varphi\rangle} \,. \tag{3.99}$$

Then

$$z_\lambda \langle\psi_1| = \langle\psi_1|(\rho_1 - \lambda H) \,, \tag{3.100}$$
$$-\mathrm{i}\langle\dot{\psi}_1| = \frac{1}{\lambda}\langle\psi_1|A_1 \,. \tag{3.101}$$

Proof. Equation (3.100) is checked immediately. To prove (3.101), one first notices that the operator P given by (3.99) satisfies the nonlinear equation

$$\mathrm{i}\dot{P} = \frac{1}{\mu}(1 - P)AP - \frac{1}{\nu}PA(1 - P) \tag{3.102}$$

and (3.101) follows from a straightforward calculation.

Lemma 3.21. *If $A = f(\rho)$, then*

$$\mathrm{i}\dot{\rho}_1 = [H, f(\rho_1)] \,. \tag{3.103}$$

Proof. The statement is a consequence of equations (3.97) and (3.102), if one takes into account

$$f(\rho_1) = A_1 = \Big(1 + \frac{\mu - \nu}{\nu} P\Big) f(\rho) \Big(1 + \frac{\nu - \mu}{\mu} P\Big) \,. \tag{3.104}$$

Remark 3.22. Having a solution ρ, we can generate a new solution ρ_1 of the nonlinear von Neumann equation (3.85). The procedure can be further iterated, $\rho \to \rho_1 \to \rho_2 \to \ldots$, in direct analogy to the Darboux method of generating multisoliton solutions [324] or supersymmetric quantum mechanics [93].

Remark 3.23. If ρ is a density matrix and $\mu = \bar{\nu}$, one can put $|\varphi\rangle = |\chi\rangle = \langle\chi|^\dagger$. In this case ρ_1 is also a density matrix and the spectra of ρ and ρ_1 are identical.

3.9.3 Self-scattering solutions

We will now show that self-scattering solutions obtained in [276] for (3.88) are a generic property of the nonlinear von Neumann equations considered here.

We begin with a seed solution obeying the condition

$$f(\rho) - a\rho = \Delta_a \,,$$

where $[\Delta_a, H] = 0$ and Δ_a is not a multiple of the identity. The solution satisfies

$$\mathrm{i}\dot{\rho} = [H, f(\rho)] = a[H, \rho]$$

and

$$\rho(t) = \mathrm{e}^{-\mathrm{i}aHt} \rho(0) \mathrm{e}^{\mathrm{i}aHt} \,.$$

Taking the Lax pairs with $\mu = \bar{\nu}$ and repeating the construction from [96, 98, 276], we get the self-scattering solution

$$\begin{aligned} \rho_1(t) &= \mathrm{e}^{-\mathrm{i}aHt} \Big(\rho(0) + (\bar{\nu} - \nu) F_a(t)^{-1} \mathrm{e}^{-\mathrm{i}\Delta_a t/\bar{\nu}} \\ &= \times [|\chi(0)\rangle\langle\chi(0)|, H] \mathrm{e}^{\mathrm{i}\Delta_a t/\nu} \Big) \mathrm{e}^{\mathrm{i}aHt}, \end{aligned} \tag{3.105}$$

where

$$F_a(t) = \langle\chi(0)| \exp\Big(\mathrm{i}\frac{\bar{\nu} - \nu}{|\nu|^2} \Delta_a t\Big) |\chi(0)\rangle \tag{3.106}$$

and $\langle\chi(0)|$ is an initial condition for the solution of the Lax pair equations.

As an example, let us consider the harmonic oscillator Hamiltonian $H = \omega \sum_{n=0}^{\infty} n|n\rangle\langle n|$ and

$$f(\rho) = \rho^q - 2\rho^{q-1} \tag{3.107}$$

with $q \in \mathbb{R}$. To construct the operator Δ_a we first note that for any q the equation

$$x^q - 2x^{q-1} - x + 2 = (x^{q-1} - 1)(x - 2) = 0 \tag{3.108}$$

has two positive solutions, $x = 1$ and $x = 2$, which will be used in the construction of an appropriate seed ρ. Putting $a = 1$, we define

$$\Delta_1 = f(\rho) - \rho = \rho^q - 2\rho^{q-1} - \rho \,. \tag{3.109}$$

The next task is to choose a seed ρ such that $[\rho, H] \neq 0$ and $[\Delta_1, H] = 0$. We do this as follows. Take any three eigenstates of H corresponding to three equally spaced eigenvalues. For example, let $|0\rangle$, $|1\rangle$, $|2\rangle$ be the three lowest energy eigenstates of H. Taking

$$\rho(0) = \frac{3}{2}\Big(|0\rangle\langle 0| + |2\rangle\langle 2|\Big) = +\frac{7}{4}|1\rangle\langle 1| = -\frac{1}{2}\Big(|2\rangle\langle 0| + |0\rangle\langle 2|\Big) \,, \tag{3.110}$$

we find $[\rho(0), H] \neq 0$ and

$$\Delta_1 = -2\Big(|0\rangle\langle 0| + |2\rangle\langle 2|\Big) = \Big(-2 + \tfrac{1}{4}\big[1 - \big(\tfrac{4}{7}\big)^{1-q}\big]\Big)|1\rangle\langle 1| \,.$$

Take $\nu = -\mathrm{i}\sqrt{3}/(4\omega)$. The left eigenvalues of $\rho - \nu H$ are $5/4 - \mathrm{i}\sqrt{3}/4$ and $7/4 - \mathrm{i}\sqrt{3}/4$, the latter being twice degenerated. The initial condition for the solution of the Lax pair is chosen to be a linear combination of the two orthonormal eigenvectors corresponding to $7/4 - \mathrm{i}\sqrt{3}/4$:

$$\langle\chi(0)| = \frac{\sqrt{3} - \mathrm{i}}{4}\langle 0| + \frac{1}{\sqrt{2}}\langle 1| - \frac{\sqrt{3} + \mathrm{i}}{4}\langle 2| \,. \tag{3.111}$$

As a rule, the self-scattering solutions can occur only for $\langle\chi(0)|$ which is not an eigenvector of Δ_a. Here this means that $q \neq 1$, which is consistent with the fact that for $q = 1$ the von Neumann equation is linear.

We are now in position to explicitly write the self-scattering solution for any real q. We start with the seed solution

$$\rho(t) = \mathrm{e}^{-\mathrm{i}Ht}\rho(0)\mathrm{e}^{\mathrm{i}Ht} \tag{3.112}$$

and, combining (3.105), (3.110), and (3.111), obtain

$$\rho_1(t) = \mathrm{e}^{-\mathrm{i}Ht}\rho_{\mathrm{int}}(t)\mathrm{e}^{\mathrm{i}Ht}, \tag{3.113}$$

where

$$\rho_{\mathrm{int}}(t) = \sum_{m,n=0}^{2} r(t)_{mn}|m\rangle\langle n| \,,$$

$$\begin{pmatrix} r_{00} & r_{01} & r_{02} \\ r_{10} & r_{11} & r_{12} \\ r_{20} & r_{21} & r_{22} \end{pmatrix} = \begin{pmatrix} 3/2 & -\xi(t) & \zeta(t) \\ -\bar{\xi}(t) & 7/4 & \xi(t) \\ \bar{\zeta}(t) & \bar{\xi}(t) & 3/2 \end{pmatrix} ,$$

$$\xi(t) = \frac{(-3\mathrm{i} + \sqrt{3})\mathrm{e}^{\omega_q t}}{4\sqrt{2}\big(1 + \mathrm{e}^{2\omega_q t}\big)} \,, \qquad \zeta(t) = \frac{1 - \mathrm{i}\sqrt{3} - 2\mathrm{e}^{2\omega_q t}}{4\big(1 + \mathrm{e}^{2\omega_q t}\big)} \,,$$

$\omega_q = [(4/7)^{1-q} - 1]\omega/\sqrt{3}$, $\omega_q \geq 0$ for $q \geq 1$, $\omega_q \leq 0$ for $q \leq 1$. The self-scattering asymptotics are

$$\begin{aligned}
&\xi_{q\neq 1}(\pm\infty) = 0 \,, \\
&\zeta_{q>1}(+\infty) = \zeta_{q<1}(-\infty) = -1/2 \,, \\
&\zeta_{q>1}(-\infty) = \zeta_{q<1}(+\infty) = (1 - \mathrm{i}\sqrt{3})/4 \,.
\end{aligned}$$

Let us note that the seed solution $\rho(t)$ we started with reappears in the asymptotic states

$$\begin{aligned}
&\rho_1 \to \rho \,, \quad \text{for } q > 1,\ t \to +\infty \,, \\
&\rho_1 \to \rho \,, \quad \text{for } q < 1,\ t \to -\infty \,.
\end{aligned}$$

It is interesting that, somewhat counterintuitively, the equations with very different q may lead to evolutions which are practically indistinguishable. Indeed, for $q \to +\infty$ we get $\omega_q \to \infty$ and, therefore, the transition around $t = 0$ between the asymptotic linear evolutions takes less time the greater the value of q. By the same method but with a different choice of $\langle\chi(0)|$ one can generate solutions whose self-scattering takes place in the neighborhood of an arbitrarily chosen $t = t_0$ (for details see the discussion of the case $f(\rho) = \rho^2$ given in [96]).

The time scales involved in self-scattering are best illustrated by the average position of the harmonic oscillator as a function of time. Figure 3.5 shows the evolution of $\langle x\rangle = \mathrm{tr}(\hat{x}\rho_1)/\mathrm{tr}\rho_1$, $\hat{x} = (a + a^\dagger)/\sqrt{2}$ for different values of q. In Figs. 3.5 and 3.6 the self-scattering is explicitly seen in the contour plot of a Harzian [420], a 3D surface representing the self-scattering probability density in position space as a function of time.

3.9.4 Infinite-dimensional example

It should be stressed that the technique presented in previous sections is not limited to matrix cases. The example given below is, perhaps, rather artificial but at least clearly demonstrates the possibility of constructing self-scattering solutions involving infinite-dimensional subspaces. We are not interested in the *reducible*, trivial situation of the dynamics decomposable into a direct sum of finite-dimensional evolutions.

Let the spectrum of the Hamiltonian H contains a discrete part $\{E_n\}_{n=1}^{\infty}$. One of the technical assumptions we will need[1] is concerned with the symmetry of this part of the spectrum. Namely, assume that the spectral representation of the Hamiltonian has the form

$$H = \sum_{n=1}^{\infty} E_n\big(|n,+\rangle\langle n,+| \,-\, |n,-\rangle\langle n,-|\big) + \ldots, \tag{3.114}$$

[1] The assumption is caused by the 2×2 block-diagonal form of H. Taking higher-dimensional blocks (say 3×3), we do not need this restriction anymore.

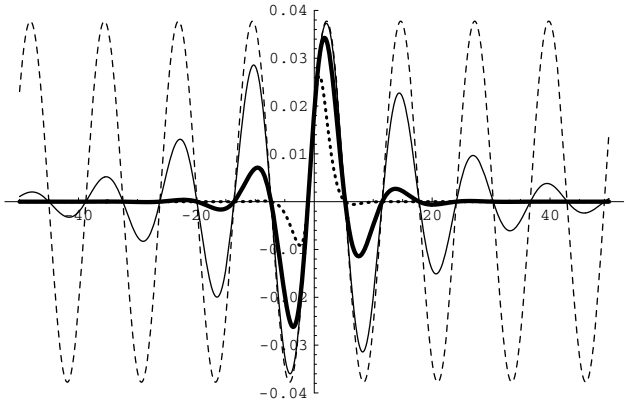

Fig. 3.5. $\langle x \rangle$ for $q = 1$ (*dashed line*), $q = 2^{1/2}$ (*thin solid line*), $q = \pi$ (*dotted line*), and $q = -2$ (*thick solid line*). All evolutions for $q > 1$ ($q < 1$) have identical asymptotic states and the same initial condition (all curves intersect at $t = 0$). The solution for $q = 1$ satisfies the same linear equation as the asymptotic states for $q \neq 1$

where the dots represent the remaining part of the spectrum. Also suppose in the sequel that $f(-x) = f(x)$. [For the Tsallis-type description one can put $f(x) = x^q$, where $q = 2n/(2n \pm 1)$, $n \in \mathbb{N}$, and the root is arithmetic].

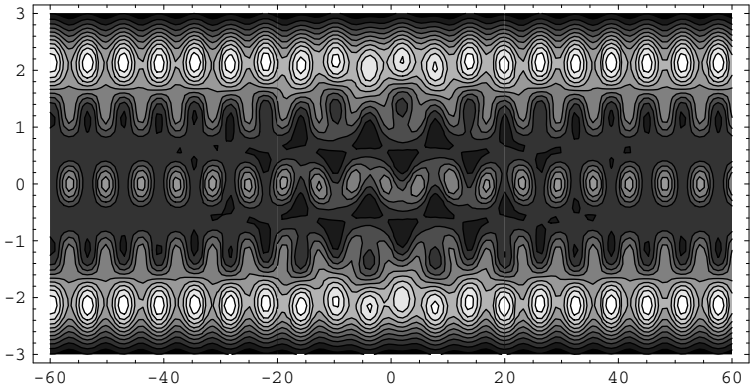

Fig. 3.6. Contour plot of a 3D surface representing probability density in position space $\langle x|\rho_1|x\rangle$ as a function of time for $q = 1/2$, $\omega = 1/2$, and $-60 < t < 60$. A continuous transition (self-scattering) between two solutions of the linear equation is clearly visible. To make the plot clearer, we have illustrated the effect by ρ_1 which has nonvanishing matrix elements in the subspace spanned by $|2\rangle$, $|3\rangle$, and $|4\rangle$

Under the above conditions we can take the seed solution ρ in the form of an infinite-dimensional matrix

$$\rho = \begin{pmatrix} a_1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & -a_1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & a_2 & 0 & 0 & 0 & \dots \\ 0 & 0 & 0 & -a_2 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & a_3 & 0 & \dots \\ 0 & 0 & 0 & 0 & 0 & -a_3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

and the Hamiltonian as

$$H = \begin{pmatrix} b_1 & c_1 & 0 & 0 & 0 & 0 & \dots \\ \bar{c}_1 & -b_1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & b_2 & c_2 & 0 & 0 & \dots \\ 0 & 0 & \bar{c}_2 & -b_2 & 0 & 0 & \dots \\ 0 & 0 & 0 & 0 & b_3 & c_3 & \dots \\ 0 & 0 & 0 & 0 & \bar{c}_3 & -b_3 & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} .$$

Any Hamiltonian with eigenvalues $E_k = \pm\sqrt{b_k^2 + |c_k|^2}$ can be written in this way in some basis. Let us note that for $c_k \neq 0$ one finds $[\rho, H] \neq 0$ but, nevertheless, $[f(\rho), H] = 0$, which means that the seed solution ρ is stationary. The next step is to choose the parameters a_k and ν in a way guaranteeing that the projector P given by (3.99) will have nonzero matrix elements between any two eigenvectors of H. By construction, the same will hold for ρ_1 and the nonlinear evolution will be *infinite-dimensional and irreducible*, i.e., will involve the transitions between all the eigenvectors of H spanning the infinite-dimensional subspace.

Take two real constants α and β satisfying $b_k = \alpha a_k$ and $|c_k|^2 = \beta^2 a_k^2$. Then the eigenvalues of H are $E_k^{\pm} = \pm a_k\sqrt{\alpha^2 + \beta^2}$. It turns out that with the above choice of ρ the condition of *infinite dimensionality and irreducibility* can be fulfilled only if $z_\nu = 0$. In this case ν has to satisfy

$$(\alpha^2 + \beta^2)\nu^2 - 2\alpha\nu + 1 = 0 ,$$

which gives $\nu_\pm = (\alpha \pm \mathrm{i}\beta)/(\alpha^2 + \beta^2)$.

The eigenvector $\langle\chi|$ corresponding to $z_\nu = 0$ is

$$\langle\chi| = (u_1 w,\ u_2 w,\ \dots,\ u_k w,\ \dots) , \tag{3.115}$$

with $w = (1/\sqrt{2})\,(1,\ \mathrm{i})$ and $u_n \in \mathbb{C}$, $\sum_{n=1}^{\infty} |u_n|^2 < \infty$. We finally obtain

$$\rho_1 = \begin{pmatrix} \rho_{1,11} & \rho_{1,12} & \rho_{1,13} & \cdots \\ \rho_{1,21} & \rho_{1,22} & \rho_{1,23} & \cdots \\ \rho_{1,31} & \rho_{1,32} & \rho_{1,33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

where

$$\rho_{1,kl} = a_k \delta_{kl} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = +\frac{\beta F_{kl}}{G} \left[\frac{a_k}{\alpha - \mathrm{i}\beta} \begin{pmatrix} \mathrm{i} & 1 \\ 1 & -\mathrm{i} \end{pmatrix} + \frac{a_l}{\alpha + \mathrm{i}\beta} \begin{pmatrix} -\mathrm{i} & 1 \\ 1 & \mathrm{i} \end{pmatrix} \right] ,$$

$$G = \sum_{n=1}^{\infty} |u_n|^2 \mathrm{e}^{2\beta f(a_n)t} , \tag{3.116}$$

$$F_{kl} = u_k \bar{u}_l \exp\left(-\mathrm{i}\alpha[f(a_k) - f(a_l)]t + \beta[f(a_k) + f(a_l)]t\right) .$$

3.9.5 Comments

We have shown that:

1. The eigenvalues of ρ_1 are the same as those of ρ.
2. The definitions of G and F_{kl} imply that ρ_1 is again a self-scattering solution. To our knowledge, this is the first example of *infinite-dimensional and irreducible* nonlinear dynamics one can find in the literature. The formalism can be applied to genuinely infinite-dimensional systems.
3. Physically nontrivial solutions of the von Neumann equations with a large class of f-nonlinearities can be obtained by the dressing method.
4. These nonlinear equations possess self-scattering solutions whose behavior is qualitatively similar for different nonlinearities.
5. The nonlinear effects in the evolution of these solutions are well localized in time; transiently and asymptotically the solutions correspond to those of linear von Neumann equations.
6. Even large modifications of nonlinearity can lead to small and very short-lived modifications of standard linear dynamics for a given initial state.
7. All nonlinearities $\sim \rho^q$ which are expected to be related to nonextensive statistics can be treated within the proposed formalism.

In light of these findings one may wonder whether is it possible to experimentally distinguish between a general f and a *linear* f. Indeed, the fact that some experimental data are well fitted by linear dynamics may only mean that a self-scattering has taken place in the past, or will take place in the future. If, in addition, the state is pure, then its dynamics is given by the linear von Neumann (or Schrödinger) equation even in case of a highly nonlinear function f. It follows that not only the results we have reported may prove useful as a technical tool in many branches of classical and quantum

physics, but they may also shed new light on the negative results of experiments searching for quantum nonlinearity.

Applications to the dynamics of biological molecules appear in a well-illustrated article [19] with some more references therein.

3.10 Further development. Definition and application of compound elementary DT

In this section we combine the structures of the classical and elementary DTs, preserving the main ideas of the method: the intertwining relation that leads to incorporation of dressing and the assumption of the existence of a nonzero kernel of the transformation operator.

3.10.1 Definition of compound elementary DT

The general extensions of the DT definition are described in Sect. 3.1. Here we study the case when a degenerate operator (idempotent projector) stands for the operator of a derivative. Such a transformation was introduced in [278]; we will name it the compound eDT. We restrict ourselves to the example of a differential equation of the second order with 2×2 matrix coefficients (generalizations are produced as in Sects. 3.2–3.5). Let us consider the equation

$$\Phi_{xx} + F\Phi_x + U\Phi = \lambda\sigma_3\Phi, \tag{3.117}$$

where the spectral parameter is λ and the vector $\Phi = (\varphi_1, \varphi_2)^{\mathrm{T}}$ is a solution. The matrix potentials are $U = \{u_{ij}\}$, $F = \{f_{ij}, f_{ii} = 0\}$, $i = 1, 2$ and σ_3 is the Pauli matrix.

Following [200], we perform the compound eDT for the differential equation (3.117) as

$$\Phi[1] = P\,\Phi_x + K\Phi\,, \tag{3.118}$$

where $P^2 = P$ is a projection operator, say, $P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. The matrix $K = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & 1 \end{pmatrix}$ represents a matrix potential function, which is defined by the corresponding intertwining relation and the auxiliary condition of existence of a nonzero kernel

$$\exists\Psi : P\Psi_x + K\Psi = 0\,. \tag{3.119}$$

On the right-hand side of (3.118) we see a combination of the differentiation with respect to x as in the classical DT (Chap. 2) and of the projector P intrinsically related to the eDT, the central notion of this chapter (Sect. 3.1). The condition (3.119) implements the auxiliary solutions in the transformation and is necessary when the iterative Crum-like formulas are derived [278].

Replacing Φ in (3.117) by $\Phi[1]$ and collecting coefficients of similar terms, we arrive at the intertwining relation for the operators entering (3.117) and its transform. Taking into account the condition for the transformation (3.119), we get the first eDT:

$$\begin{aligned}
&\varphi_1[1] = \varphi_{1x} + k_{11}\,\varphi_1 + k_{12}\,\varphi_2\ ,\\
&\varphi_2[1] = \varphi_2 + \ k_{21}\,\varphi_1\ ,\\
&\psi_1[1] = (\partial_x + k_{11})\,\psi_3 + k_{12}\,\psi_4\ ,\\
&\psi_2[1] = \psi_4 + k_{21}\,\psi_3\ ,\\
&k_{11} = -\left(\psi_{1x} + \tfrac{1}{2} f_{12}\,\psi_2\right)/\psi_1\ ,\\
&k_{12} = f_{12}/2\ ,\\
&k_{21} = -\ \psi_2/\psi_1\ ,
\end{aligned} \tag{3.120}$$

where $(\psi_1,\, \psi_2)^{\mathrm{T}}$ and $(\psi_3,\, \psi_4)^{\mathrm{T}}$ are two solutions of (3.119) corresponding to different values of the spectral parameter $\lambda \to \mu_{1,2}$.

The new potentials are found to have the following expressions:

$$\begin{aligned}
&f_{12}[1] = u_{12} + f_{12}\ k_{11}\ ,\\
&f_{21}[1] = -2\,k_{21}\ ,\\
&u_{11}[1] = u_{11} - 2k_{11x} - f_{12}[1]\,k_{21} - f_{21}\,k_{12}\ ,\\
&u_{12}[1] = u_{12x} - k_{12xx} + k_{11}\,u_{12} - k_{12}\,(u_{11}[1] + u_{22})\ \ ,\\
&u_{21}[1] = f_{21} - 2k_{21x} - f_{21}[1]\,k_{11}\ ,\\
&u_{22}[1] = u_{22} - k_{21}\,u_{12} - u_{21}[1]\,k_{12} - f_{21}[1]\,k_{12x}\ .
\end{aligned} \tag{3.121}$$

The second eDT

$$\Phi[1] = Q\,\Phi_x + K\,\Phi\ , \qquad Q = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

is performed after the first one by similar formulas (only the interchange of indices $1 \rightleftharpoons 2$ is necessary, see again [278]) and is found to have the following potentials and transformed K:

$$\begin{aligned}
&f_{21}\,[2] = u_{21}[1] + f_{21}[1]\,k_{22}[1]\ ,\\
&f_{12}[2] = -2\,k_{12}[1]\ ,\\
&u_{22}[2] = u_{22}[1] - 2k_{22x}[1] - f_{21}[2]\,k_{12}[1] - f_{12}[1]\,k_{21}[1]\ ,\\
&u_{21}[2] = u_{21x}[1] - k_{21xx}[1] + k_{22}[1]\,u_{21}[1] - k_{21}[1]\,(u_{22}[2] + u_{11}[1])\ \ ,\\
&u_{12}[2] = f_{12}[1] - 2k_{12x}[1] - f_{12}[2]\,k_{22}[1]\ ,\\
&u_{11}[2] = u_{11}[1] - k_{12}[1]\,u_{21}[1] - u_{12}[2]\,k_{21}[1] - f_{12}[2]\,k_{21x}[1]\ ,\\
&k_{12}[1] = -\ \psi_1[1]/\psi_2[1]\ ,\\
&k_{21}[1] = f_{21}[1]/\,2\ ,\\
&k_{22}[1] = -\left(\psi_{2x}[1] + \tfrac{1}{2} f_{21}[1]\,\psi_1[1]\right)/\psi_2[1]\ .
\end{aligned} \tag{3.122}$$

Such a combination of two compound eDTs allows us to account for reductions similarly to the twofold DT theory.

3.10.2 Solution of coupled KdV–MKdV system via compound elementary DTs

The spectral equation (3.117) is considered as the first equation of the Lax pair. Take the second one as

$$\Phi_t = \Phi_{xxx} + B\Phi_x + C\Phi \,, \tag{3.123}$$

where

$$B = \frac{3}{2}\mathrm{diag}U + \frac{3}{2}F_x + \frac{3}{4}F^2 \,,$$

$$C = \frac{3}{2}U_x - \frac{3}{4}\mathrm{diag}U_x - \frac{3}{4}(f_{12}u_{21} + f_{21}u_{12})I$$

$$+\frac{3}{8}(f_{12,x}f_{21} - f_{12}f_{21,x})\sigma_3 + \frac{3}{4}(u_{11} - u_{22})\sigma_3 F \,.$$

Equation (3.123) is also covariant under transformations (3.120) and (3.121). The compatibility conditions have the following form

$$F_t - F_{3x} + B_{2x} - 3U_{2x} + 2C_x + FB_x - \sigma_3 B\sigma_3 F_x + UB$$

$$-\sigma_3 B\sigma_3 U + FC - \sigma_3 C\sigma_3 F = 0 \,,$$

$$U_t - U_{3x} + C_{2x} + UC - \sigma_3 C\sigma_3 U + FC_x - \sigma_3 B\sigma_3 U_x = 0 \tag{3.124}$$

and the transformations (3.120) and (3.121) determine a discrete symmetry of (3.124). The existence of different kinds of automorphism causes special constraints [181]. Multiplying (3.117) by $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ to get

$$\sigma_1\Phi_{xx} + \sigma_1 F\Phi_x + \sigma_1 U\Phi = \lambda\sigma_1\sigma_3\Phi \,, \tag{3.125}$$

and accounting for $\sigma_1\sigma_3 = -\sigma_3\sigma_1$, as well as considering the conditions $\sigma_1 F = F\sigma_1$ and $\sigma_1 U = U\sigma_1$, we obtain

$$f_{12} = f_{21} = f \,, \quad u_{11} = u_{22} = u \,, \quad u_{12} = u_{21} = v \,. \tag{3.126}$$

So (3.125) becomes

$$(\sigma_1\Phi)_{xx} + F\,(\sigma_1\Phi)_x + U\,(\sigma_1\Phi) = -\lambda\sigma_3\,(\sigma_1\Phi) \,.$$

The above automorphism $\Phi(\lambda) \leftarrow \sigma_1\Phi(-\lambda)$ relates two pairs of solutions (ψ_1, ψ_2) and (ψ_3, ψ_4) of (3.117) corresponding to different values of the spectral parameter λ and $-\lambda$ as

$$\begin{pmatrix} \psi_3(-\lambda) \\ \psi_4(-\lambda) \end{pmatrix} = \sigma_1 \begin{pmatrix} \psi_1(\lambda) \\ \psi_2(\lambda) \end{pmatrix} = \begin{pmatrix} \psi_2(\lambda) \\ \psi_1(\lambda) \end{pmatrix} \,.$$

Using this result in the compound eDTs (3.120)–(3.122), one obtains the expressions for new potentials f, u, and v. Here we study combinations of such transforms that do not coincide with twofold ones (see the previous chapter), to produce explicit solutions to the integrable coupled KdV–MKdV system of the form

$$\begin{aligned} &f_t + \frac{1}{2}f_{xxx} + \frac{3}{2}(uf)_x - \frac{3}{4}f_x f^2 = 0\,, \\ &u_t - \frac{1}{4}u_{xxx} - \frac{3}{2}u_x u + 3vv_x + \frac{3}{4}u_x f^2 - \frac{3}{2}(f_x v)_x = 0\,, \\ &v_t + \frac{1}{2}v_{xxx} + \frac{3}{2}v_x u - \frac{3}{4}(vf^2)_x + \frac{3}{4}u_{xx}f + \frac{3}{2}u_x f_x = 0\,. \end{aligned} \tag{3.127}$$

The Lax pair of this system is given in [278]. Equations (3.127) exhibit two integrable reductions, the Hirota–Satsuma equation [118, 211] and a two-component KdV–MKdV system [278].

In this section we derive explicit two-parameter solutions of the system (3.127) that were been specified in [278]. We demonstrate the use of two arbitrary eDTs and the special choice that holds the heredity of the reduction to build an infinite set of explicit solutions to the KdV–MKdV system (3.127). The influence of choosing the parameters on the solution properties will be demonstrated as well.

In the case of zero seed potential, these new potentials have the following forms:

$$\begin{aligned} f &= 2\frac{\psi_1(\psi_2)_x - \psi_2(\psi_1)_x}{(\psi_1)^2 - (\psi_2)^2}\,, \\ u &= \left(\frac{(\psi_1^2)_x - (\psi_2^2)_x}{(\psi_1)^2 - (\psi_2)^2}\right)_x + 2\left(\frac{\psi_1(\psi_2)_x - \psi_2(\psi_1)_x}{(\psi_1)^2 - (\psi_2)^2}\right)^2\,, \\ v &= 2\left(\frac{\psi_1(\psi_2)_x - \psi_2(\psi_1)_x}{(\psi_1)^2 - (\psi_2)^2}\right)_x + \frac{\left[(\psi_1(\psi_2)_x - \psi_2(\psi_1)_x\right]\left[(\psi_1^2)_x - (\psi_2^2)_x\right]}{\left[(\psi_1)^2 - (\psi_2)^2\right]^2}\,, \end{aligned} \tag{3.128}$$

where ψ_1 and ψ_2 are the seed solutions of the system (3.117) and (3.123):

$$\psi_1 = c_1 \mathrm{e}^{ax+a^3t} + c_2 \mathrm{e}^{-(ax+a^3t)}, \quad \psi_2 = d_1 \mathrm{e}^{\mathrm{i}ax+(\mathrm{i}a)^3t} + d_2 \mathrm{e}^{-[\mathrm{i}ax+(\mathrm{i}a)^3t]}\,, \tag{3.129}$$

and c_1, c_2, d_1, and d_2 are arbitrary constants, and $a = \sqrt{\lambda}$. The above expressions are solutions of the system (3.124) that is reduced under the reduction conditions (3.126) to (3.127). The appearance of the imaginary unit in ψ_2 allows us to obtain combined (hyperbolic-oscillatory) behavior in the denominators of (3.128), hence demonstrating new specific features of the solitonic solutions.

The choice of arbitrary constants (c_1, c_2, d_1, d_2) affects the behavior of the solution (3.128). For example, choosing equal constants$c_1 = c_2 = d_1 =$

$d_2 = 0.5$ (we choose the value 0.5 to simplify the resulting formula, but other values could be chosen as well), we obtain the solutions in the form

$$f = 2a\frac{\sin\eta_1\ \cosh\eta_2 - \cos\eta_1\ \sinh\eta_2}{\cosh^2\eta_2 - \cos^2\eta_1}\ , \tag{3.130}$$

$$u = 2a^2\frac{\sin\eta_1\ \cosh\eta_2 + \cos\eta_1\ \sinh\eta_2)^2}{(\cosh^2\eta_2 - \cos^2\eta_1)^2}\ ,$$

$$v = 2a^2$$
$$\times\frac{\cos 3\eta_1\ \cosh\eta_2 - 2\sin\eta_1\ \sinh\eta_2(\cos 2\eta_1 + \cosh 2\eta_2 + 2) - \cos\eta_1\ \cosh 3\eta_2}{(\cosh^2\eta_2 - \cos^2\eta_1)^2}\ ,$$

where $\eta_1 = a^3t - ax$, $\eta_2 = a^3t + ax$, and $a = \sqrt{\lambda}$ is real. We see that (3.130) is singular at $\eta_2 = 0$, $\eta_1 = n\pi$, $n = 0, 1, 2, \ldots$. Hence, the solution has singularity at $x = (n\pi)/(2a)$, $t = (n\pi)/(2a^3)$.

To obtain continuous solutions we can choose $c_1 = c_2$, $d_1 = d_2 = rc_1$, and r is a real constant. We again choose $c_1 = 0.5$ (for simplification), so potentials (3.128) have the form

$$f = 2ar\frac{\cosh\eta_2\sin\eta_1 - \cos\eta_1\sinh\eta_2}{\cosh^2\eta_2 - r^2\cos^2\eta_1}\ ,$$

$$u = a^2\frac{1 - r^4 - r^4\cos 2\eta_1 + \cosh 2\eta_2 + r^2\sin 2\eta_1\sinh 2\eta_2}{\left(\cosh^2\eta_2 - r^2\cos^2\eta_1\right)^2}\ ,$$

$$\begin{aligned} v = \ & 2a^2r\{[(-7 + 6r^2 + 2r^2\cos 2\eta_1)\cos\eta_1\cosh\eta_2 - \cos\eta_1\cosh 3\eta_2] - 2(1 + r^2 \\ & + r^2\cos 2\eta_1 + \cosh 2\eta_2)\sin\eta_1\sinh\eta_2)\}/(-1 + r^2 + r^2\cos 2\eta_1 - \cosh 2\eta_2)^2\ , \end{aligned} \tag{3.131}$$

where a, η_1, and η_2 are the same as in (3.130).

Choosing $r < 1$ gives real nonsingular solutions, but for $r \geq 1$ poles appear. This behavior is illustrated in Figs. 3.7 and 3.8.

In transition from the continuous solution to the singular one, the first mode u is the most sensitive and is firstly affected as Figs. 3.9 and 3.10 clearly show for $r = 1$ and $r = 2$, respectively. Formula (3.131) is built from hyperbolic and periodic functions, so solutions do not preserve their shape but remain localized. Figure 3.11 shows the result of the evolution of the configuration that corresponds to Fig. 3.7, for $t = 2$.

Moreover, the choice of these arbitrary constants (c_1, c_2, d_1, d_2) as well as the spectral parameter λ affects the reality of the resulting solution. Indeed, for $\lambda = -2im^2$ with real m and choosing $c_1 = c_2 = d_1 = d_2 = 0.5$, we get the real solution

$$f = 4m\frac{\cos 2\zeta_1\sinh\zeta_2 - \sinh\zeta_2 - \sin\zeta_1\cosh 2\zeta_2 + \sin\zeta_1}{(\cosh 2\zeta_2 - 1)(1 - \cos 2\zeta_1)}\ ,$$

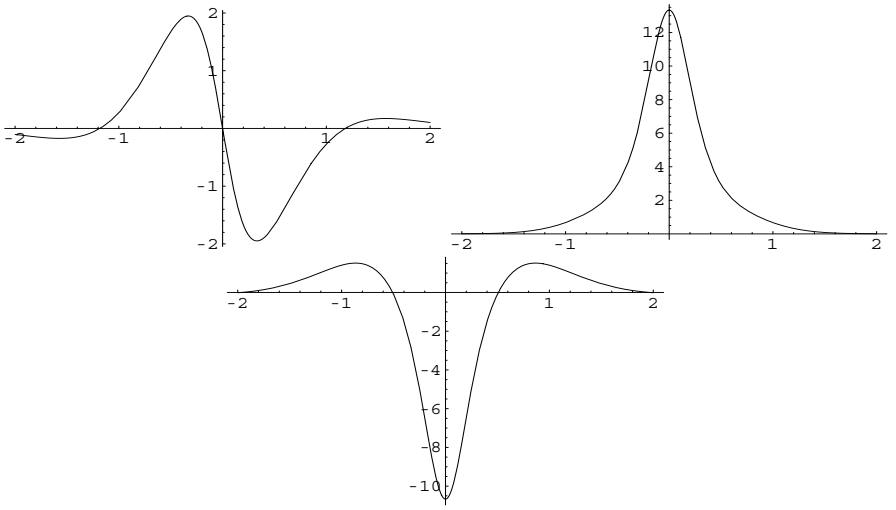

Fig. 3.7. Nonsingular solutions f, u, and v ($r = 0.5$, $a = 2$, $t = 0$)

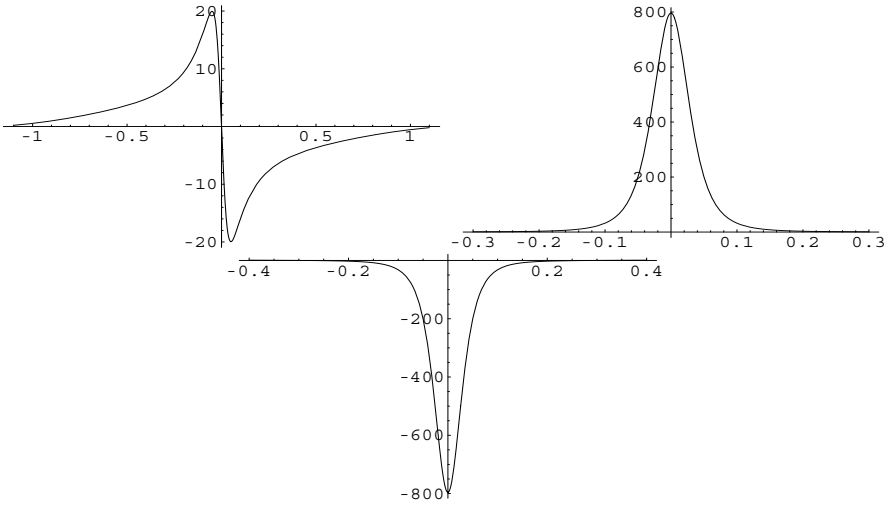

Fig. 3.8. Nonsingular solutions f, u, and v ($r = 0.99$, $a = 2$, $t = 0$)

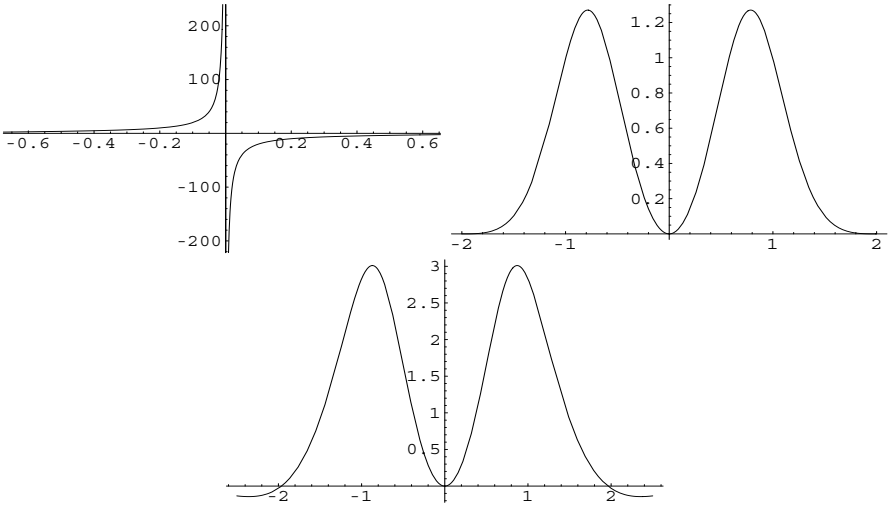

Fig. 3.9. Solutions f, u, and v for $r = 1$ ($a = 2$, $t = 0$). The solution u firstly demonstrates singular behavior

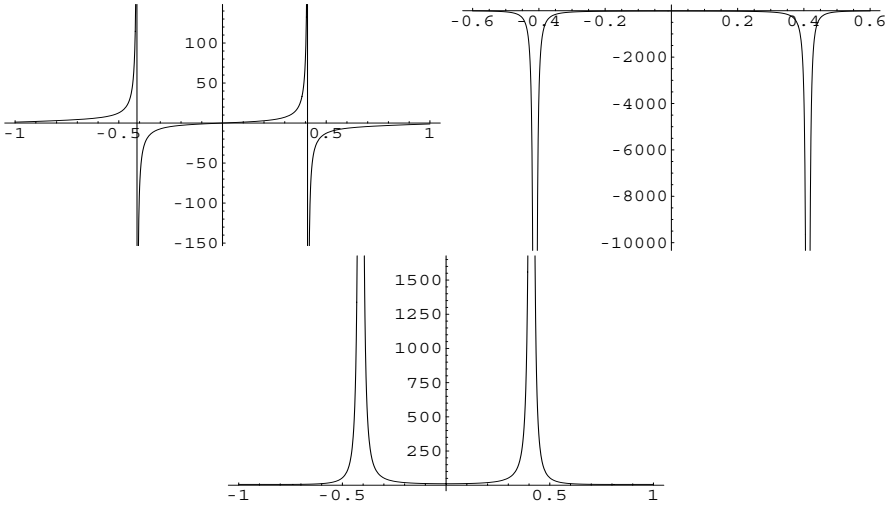

Fig. 3.10. Singular solutions f, u, and v for $r = 2$ ($a = 2$, $t = 0$)

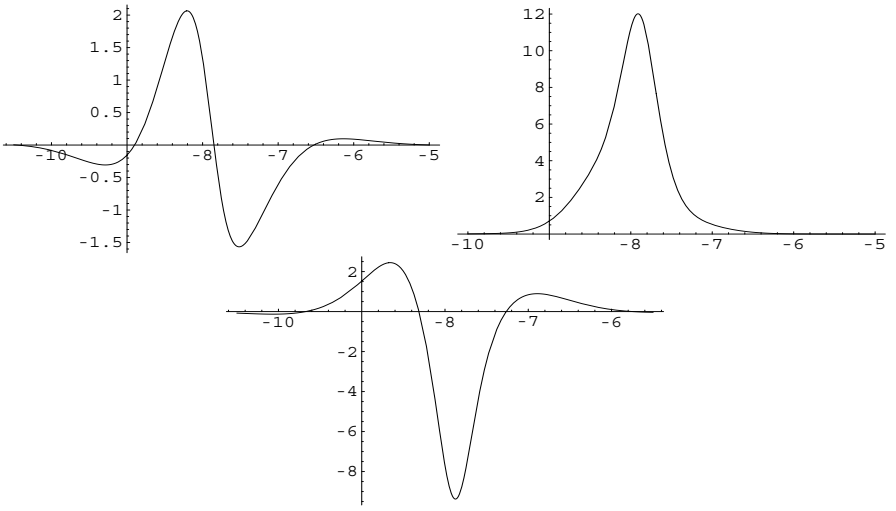

Fig. 3.11. Propagation of solutions f, u, and v (3.131) at $r = 0.5$, $a = 2$, and $t = 2$. The solitonic solutions are built from elliptic and periodic functions so do not preserve their shape with time

where $\zeta_1 = 2mx + 4m^3t$ and $\zeta_2 = 2mx - 4m^3t$, while choosing $c_1 = c_2 = 1$ and $d_1 = d_2 = 2$ gives the following complex solution:

$$f = -8m\big[5\sin\zeta_1(\cosh 2\zeta_2 - 1) + 5\sinh\zeta_2(1 - \cos 2\zeta_1)$$

$$+3\mathrm{i}(2\sin\zeta_1 + 2\sinh\zeta_2 + \cos\zeta_1\sinh 2\zeta_2 + \sin 2\zeta_1\cosh\zeta_2)\big]$$

$$\times(17\cosh 2\zeta_2 + 10 + 36\cos\zeta_1\cosh\zeta_2 - 8\cos 2\zeta_1\cosh 2\zeta_2 + 17\cos 2\zeta_1)^{-1}\ .$$

Some generalized KdV–MKdV systems have solitary wave solutions with the property that increasing nonlinearity of one variable affects the very existence of solitary waves [253]; for explicit solutions in terms of the Jacobi elliptic functions see [198].

The general coupled KdV–MKdV system arises in many problems of mathematical physics. Some integrable systems are associated with a polynomial spectral problem and have the Virasoro symmetry algebras [303]. A dispersive system describing a vector multiplet interacting with the KdV field is a member of the bi-Hamiltonian integrable hierarchy [257]. Multisymplectic geometry connected with the systems under consideration is studied in [166]. See [200] for a convergent stable numerical scheme and a comparison of analytical and numerical results.

4

Dressing chain equations

One very promising approach to solving integrable systems is based on the notion of a dressing chain [395, 438, 448]. This method covers soliton, finite-gap, rational, and other important solutions [423, 438] within the universal scheme, reducing the starting problem to a solution of closed sets of nonlinear ordinary equations with the bi-Hamiltonian structure [438]. Here we derive the dressing chain equations and study the above classes of solutions.

A scheme to reconstruct the potential entering an associated linear problem depends on a class to which solutions of the inverse problem belong [378]. The coefficients (potentials) of the linear equation are elements of some algebraic structure. Such a structure is generated by transformations that preserve the functional form of the equation. Hence, a general set of potentials splits into subsets invariant with respect to the action of the transformations. These transformations, for example, the Darboux (Schlesinger, Moutard) transformations, are generated by transformations of eigenfunctions ϕ of a given differential operator. As a general remark, note that it is possible to approximate locally solutions of the linear problem by a sequence of the Moutard and Ribacour transformations [170]. The Darboux transformation (DT) produces naturally an intermediate object $\sigma = (D\phi)\phi^{-1}$, where D is the differential operator, which is related to the potential by the generalized Miura transformation and satisfies the Riccati equation (compare with [164]). The main ideas of Darboux permit us, in principle, to determine the form of the transformation [102] by means of a factorization of the operator under consideration. The structure of the transformation depends on the ring to which the operator coefficients belong, as well as on the realization of an abstract differentiation operation [265, 321]. The DT and Miura transformations for non-Abelian entries (differential ring) are studied in [467]. This method allows us to operate effectively with the spectral data. Here we restrict ourselves to one-dimensional problems, but steps towards multidimensions have already been made [378].

If we substitute the potentials expressed in terms of σ in any iteration of the DT, we get the chain equations in the form of differential-difference

equations. They can be equally well treated as the transformed form of the spectral problem we start with. This fact was established for the simplest one-dimensional problem in [448] and extended in [395]. Section 4.1 contains simple but instructive examples of the appearance of dressing chains and their usefulness in applications. The dressing chain formalism opens new possibilities to produce explicit solutions as well as to study difficult questions of a uniform approximation of the potentials [355]. The technique of dressing chains is directly connected to the quantum inverse problem [84] and integration of soliton equations. In Sect. 4.2 we consider a general spectral problem, polynomial in differentiation. We start from an appropriate evolution equation and reduce the consideration to the stationary case that generates a spectral problem.

The solution of the chain equation can be analyzed from the point of view of the bi-Hamiltonian structure [53, 438]. Some important structures connected to the chain equations were studied in [165]. The symmetry of the system is naturally related to the DT and generates a finite group that we use to simplify the problem. In Sect. 4.3 we introduce projection operators for the irreducible subspaces of the symmetry group and the corresponding variables. Section 4.4 is devoted to symmetry (in particular, permutation symmetry) of the dressing chain equations. In Sect. 4.5 we concretize the results for the specific number of iterations in the dressing chain system.

In Sects. 4.6–4.8 we discuss a class of periodic or quasiperiodic potentials and associate with them a notion of the spectral curve, Dubrovin equations and general finite-gap potentials. We consider a transition to new variables in which solutions of the chain equations are expressed in quadratures. Note also that an important application of the dressing theory is concerned with the possibility to combine the finite-gap [45] and localized (solitonic) solutions (see the discussion in [324]). This idea, following the Shabat scheme [393], was implemented first by Kuznetsov and Mikhailov [258] using an example of dressing the cnoidal wave (stationary two-zone solution of the KdV equation) with N solitons. By means of the finite-gap integration theory, solutions of this type were also obtained in [24, 216, 251]. In Sect. 4.9 we formulate the DT for the non-Abelian Zakharov–Shabat (ZS) problem. Section 4.10 contains a derivation of the dressing chain equations produced by DT of operators polynomial in an automorphism of a ring. Taking this result, we build in Sect. 4.11 a dressing chain equation for the non-Abelian Hirota model. Section 4.12 contains some comments.

4.1 Instructive examples

As shown in Chap. 2, the operator of the classical DT has the universal form

$$L_\sigma = D - \sigma$$

and intertwines, for example, the operators of the equation

$$-\psi_{xx} + u_i\psi = \lambda\psi. \tag{4.1}$$

The potential u_i is linked to u_{i+1} by

$$u_{i+1} = u_i - 2\sigma_i', \tag{4.2}$$

with σ_i satisfying

$$\sigma_i' + \sigma_i^2 + \mu_i = u_i, \tag{4.3}$$

where μ_i is an eigenvalue. Equation (4.3) is solved by $\sigma_i = \phi_i'\phi_i^{-1}$, where ϕ_i is a solution of the problem (4.1) with the eigenvalue μ_i. Substitution of u_i from (4.3) into (4.2) yields the so-called *dressing chain*:

$$(\sigma_i + \sigma_{i+1})' = \sigma_i^2 - \sigma_{i+1}^2 + \mu_i - \mu_{i+1}. \tag{4.4}$$

The important problems of quantum mechanics can be solved by the dressing chain formalism. Let us look for solutions of (4.4) in the form [395]

$$\sigma_j = \xi_j a(x) + \eta_j, j = 0, \pm 1, \ldots . \tag{4.5}$$

Plugging (4.5) into (4.4) produces the Riccati equation for $a(x)$:

$$(\xi_{j+1}+\xi_j)a' = (\xi_j^2-\xi_{j+1}^2)a^2-2(\xi_{j+1}\eta_{j+1}-\xi_j\eta_j)a-\eta_{j+1}^2+\eta_j^2+\mu_j-\mu_{j+1}. \tag{4.6}$$

The solutions $a(x)$ via the relation (4.3) (known as the Miura transformation, see Chap. 2) produce the potentials

$$u_j = \xi_j a'(x) + (\xi_j a(x) + \eta_j)^2 + \mu_j.$$

The case $a = x$ means $\xi_{j+1} = \pm\xi_j$, $\eta_{j+1} = \pm\eta_j$, $j = 0, 1, \ldots$, and the relation (4.6) reads as the recurrence for the eigenvalues,

$$\mu_{j+1} = \mu_j + 2\xi_j, \tag{4.7}$$

and gives the equidistant spectrum for the choice $\xi_{j+1} = \xi_j = \xi$, $\eta_{j+1} = \eta_j = \eta$. The harmonic oscillator potential for $x \in (-\infty, \infty)$ is directly obtained in such a way:

$$u_i = -\xi + \xi^2 x^2 + 2\xi\eta x + \eta^2 + \mu_i.$$

The ground state ϕ_0 is annihilated by the DT:

$$(D - \sigma)\phi_0 = 0; \qquad \phi'\phi_0^{-1} = \xi x + \eta.$$

This fundamental property of the DT defines the explicit form of the function ϕ_0 and hence of all the dressed ones. Another choice $\xi_{j+1} = -\xi_j = -(-1)^j\xi$, $\eta_{j+1} = -\eta_j = -(-1)^j\eta$ does not produce essentially new potentials.

The radial Schrödinger equation in atomic units,

$$\left(-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} + \frac{l(l+1)}{2r^2} + u_l - E\right)\psi_l(r) = 0 \tag{4.8}$$

(for the scattering problem see Sect. 6.2.3), is transformed to (4.1) by $\psi_l = \psi/r$, $r \to x$ and hence the corresponding chain equation is equivalent to

(4.4). Equation (4.6) for the choice $a = 1/r$ yields two constraints for the constants:

$$(\xi_{j+1} + \xi_j)(\xi_{j+1} - \xi_j - 1) = 0, \quad \xi_{j+1}\eta_{j+1} - \xi_j\eta_j = 0.$$

The next choice

$$\xi_{j+1} = \xi_j + 1, \qquad \xi_0 = 0$$

yields for $j = 1, 2, \ldots$

$$\eta_{j+1} = \frac{\xi_j \eta_j}{\xi_{j+1}},$$

which results in

$$\eta_j = \frac{\eta}{j}.$$

The spectrum is determined by

$$\mu_{j+1} - \mu_j = \eta_j^2 - \eta_{j+1}^2$$

with the obvious solution

$$\mu_j = -\eta_j^2 = -\frac{\eta^2}{j^2}.$$

This means that (4.7) and

$$\psi_0(r) = \frac{C}{r} \exp\left(\int_1^r \sigma(x)\mathrm{d}x\right)$$

solve the Coulomb quantum problem for $l = j - 1$ and arbitrary principal number j. More details and two more potentials linked to the solutions of (4.6), namely, $\exp x$ and $\tan x$ are considered in Sect. 6.2.3.

4.2 Miura maps and dressing chain equations for differential operators

4.2.1 Linear problems

Reproducing for convenience the general conjectures from Chap. 2, let us take the differential operator

$$L = \sum_{n=0}^{N} a_n D^n \tag{4.9}$$

on a differential ring A with coefficients (potentials) $a_n \in A$, and the evolution equation

$$\psi_y = L\psi, \qquad \psi \in A. \tag{4.10}$$

Here the operator D is a differentiation with respect to some variable (maybe, the abstract one) and ψ_y is the derivative with respect to another variable (see [271, 321] for details and generalizations). Denote $D\psi = \psi'$. The transformation of the solutions is taken in the standard Darboux form

$$\psi[1] = D\psi - \sigma\psi,$$

where

$$\sigma = \phi'\phi^{-1} \tag{4.11}$$

with a linearly independent invertible solution ϕ of (4.10).

The transformation of the coefficients of the resulting operator

$$L[1] = \sum_{n=0}^{N} a_n[1]D^n$$

is determined by

$$a_N[1] = a_N \tag{4.12}$$

and for all the other $n = 0, \ldots, N-1$ by

$$a_n[1] = a_n + \sum_{k=n+1}^{N} [a_k B_{k,k-n} + (a_k' - \sigma a_k)B_{k-1,k-1-n}] \tag{4.13}$$

that yields the covariance principle. This means that the function $\psi[1]$ is a solution of the equation

$$\psi_y[1] = L[1]\psi[1].$$

This result is a compact reformulation of the Matveev theorem [321].

The functions $B_{m,n}$ were introduced in [467] and represent the generalized Bell polynomials defined in Sect. 2.2.

Proposition 4.1. *If the function σ satisfies the equation*

$$D_y\sigma \equiv \sigma_y = Dr + [r, \sigma], \tag{4.14}$$

where $r = \sum_{n=0}^{N} a_n B_n(\sigma)$, the operator $L_\sigma = D - \sigma$ intertwines the operators $D_y - L$ and $D_y - L[1]$.

For the derivation of the dressing chain equations we consider the stationary solution of the evolution equation (4.10):

$$D_y\phi = \phi\mu.$$

This gives $D_y\sigma = 0$. For example, in the matrix case $\mu = \mathrm{diag}\{\mu_1, \ldots, \mu_n\}$.

Note that (4.14) for σ taken from (4.11) becomes the identity. The parameters μ_i have the sense of eigenvalues of the operator L.

Hence, the consequence of (4.14),

$$Dr + [r, \sigma] = 0, \tag{4.15}$$

is the analog of the Riccati equation that we call the generalized Miura map. It connects the *potentials* [coefficients of the operator L (4.9)] and σ at every step i of the DT iterations. Further we will supply the functions with the upper index i to show the number of iterations made. In the scalar case the commutator is zero and (4.15) reads

$$\sum_{n=1}^{N} a_n^i \, B_n(\sigma_i) = c = \text{const}. \tag{4.16}$$

If we have the single potential a_0 with the other ones being invariant, $a_n^i = a_n^{i+1}$, $n \neq 0$, then the expression for the iterated potential is given by

$$a_0^i = -\sum_{n=1}^{N} a_n^i \, B_n(\sigma_i) + c_i. \tag{4.17}$$

In this case the derivation of the chain equation is made by the substitution of (4.17) into (4.13) for $n = 0$ and with the indices i and $i + 1$. Equation (4.17) for $N = 2$ gives the link between σ_i and the potential u_i that enters the second-order operator

$$L = -D^2 + u_i \, . \tag{4.18}$$

Supplying (4.16) for $N = 2$ with indices, we obtain

$$\sigma_{i,x} + \sigma_i^2 + c_i = u_i. \tag{4.19}$$

The next choice $N = 3$ leads to the generalized Miura transformation

$$\sigma_{i,xx} + \frac{3}{2}(\sigma_i^2)_x + \sigma_i^3 + u_i\sigma_i + w_i = \mu_i, \tag{4.20}$$

where $w_i = a_0^i$ and $u_i = a_1^i$. It connects the coefficients u_i and w_i with σ_i. For both cases the DT has a similar form

$$u_{i+1} = u_i - 2\sigma_{i,x} \tag{4.21}$$

(for $N = 3$ we have $a_3 = 1$; hence -2 in the DT should be changed to $+3$). Finally, if one starts from the second-order Sturm–Liouville equation (4.18), the associated dressing chain equation is written as

$$\sigma_{i+1,x} + \sigma_{i+1}^2 + \mu_{i+1} = -\sigma_{i,x} + \sigma_i^2 + \mu_i. \tag{4.22}$$

This dressing chain equation was studied in Sect. 4.1.

Further generalization is concerned with a nonisospectral DT. Such a DT has the same form as (4.21) but it links different λ-subspaces:

$$\psi_{i+1}(\lambda + \alpha_i) = (D - \sigma_i)\psi_i(\lambda). \tag{4.23}$$

It specifies the factorization condition as

$$u_{i+1} = u_i - 2\sigma_{i,x} + \alpha_i \tag{4.24}$$

and shifts the eigenvalue by α_i. Substitution of (4.19) into (4.24) yields

$$\sigma_{i+1,x} + \sigma_{i,x} = \sigma_i^2 - \sigma_{i+1}^2 + \alpha_i + c_i - c_{i+1}. \tag{4.25}$$

Note that c_i are arbitrary integration constants and could be fixed as, e.g., in the instructive example of Sect. 4.1, where $\alpha_i = 0$ and $c_i = \mu_i$, that corresponds to the choice of $\sigma_i = \phi'/\phi$ (check by direct substitution!). In Sects. 4.6–4.8, where the dressing chain equation is studied and used to build finite-gap solutions [422, 438], $c_i = 0$ is chosen to simplify the dressing chain equation Lax pair.

For the third-order operator, if $w_i = 0$, the dressing chain equation is written as

$$(\sigma_{i+1,xx} - \mu_{i+1})/\sigma_{i+1} + 3\sigma_{i+1,x} + \sigma_{i+1}^2 = (\sigma_{i,xx} - \mu_i)/\sigma_i + 6\sigma_{i,x} + \sigma_i^2. \tag{4.26}$$

The case of zero w_i is obviously a reduction for the space of solutions of the linear problem and some modification of the DT formula for the eigenfunctions is necessary [281]. Important connections with the Hamiltonian structures of some third-order problems (related to the Sawada–Kotera/Gibbon and Kaup–Kupershmidt equations) were studied in [164].

4.2.2 Lax pairs of differential operators

As regards a Lax pair of a nonlinear system, one should consider an additional operator, say,

$$A = \sum_{n=0}^{M} b_n D^n, \tag{4.27}$$

leading to the evolution equation

$$\psi_t = A\psi. \tag{4.28}$$

The coefficients of both equations (4.9) and (4.27) depend on a set of potentials $u_1, \ldots, u_n$, eventual solutions of nonlinear equations. In accordance with the joint covariance property [270, 271], the DTs of the coefficients induce the DTs of the potentials (Sect. 2.7). We can formulate a proposition similar to Proposition 4.1:

Proposition 4.2. *If a function σ satisfies (4.14) and the equation*

$$D_t\sigma \equiv \sigma_t = Dq + [q,\, \sigma], \tag{4.29}$$

where $q = \sum_{n=0}^{M} b_n\, B_n(\sigma)$, then the operator $L_\sigma = D - \sigma$ intertwines the pairs of operators $(D_y - L, D_t - A)$ and $(D_y - L[1], D_t - A[1])$. This means the integrability of the compatibility condition of (4.1) and (4.28) in the sense of the symmetry existence with respect to the DT $u_i \to u_i[1]$ of the potentials.

The compatibility condition of (4.14) and (4.29) yields the extra equation

$$Dq_y + [q_y,\, \sigma] + [q,\, Dr + [r,\, \sigma]] = Dr_t + [r_t,\, \sigma] + [r,\, Dq + [q,\, \sigma]], \tag{4.30}$$

which links the potentials and the element σ. In the case of the single potential u, it is possible to express it as a function of σ [273]. Considering the iterated potentials

$$u_i = f_i(\sigma_i)$$

(now the index is again a number of iterations) allows us to produce the dressing chain equation substituting the function into the DT formula $u_i[1] = u_{i+1}$.

The scalar case is much simpler. From (4.30) it follows that $D(q_y - r_t) = 0$, or

$$q_y - r_t = \sum_{n=0}^{M} [b_{ny}\, B_n(\sigma) + b_n\, B_n'(\sigma) Dq] - \sum_{n=0}^{N} [a_{nt}\, B_n(\sigma) + a_n\, B_n'(\sigma) Dr] = \text{const}. \tag{4.31}$$

The good example of this case is the Kadomtsev–Petviashvili equation and its dressing chain [273], whence the potential is extracted from (4.31). In the theory of solitons (4.26) generates the Sawada–Kotera equation, while (4.20) corresponds to the famous KdV equation (see also Sect. 8.7). From the point of view of the derivation of the dressing chain equation, other reductions are more complicated: it is necessary to express the potential from (4.20). For the chain equation associated with the reduction of the Boussinesq equation see [281].

4.3 Periodic closure and time evolution

The periodic closure of the dressing chain equation (4.22) for the KdV equation produces a finite system of equations that possesses the bi-Hamiltonian structure [438]. As Veselov and Shabat [438] wrote about the case $N = 3$ (one-gap potentials), "it is a useful exercise to derive explicit formulas for σ_i directly from equations of the chain." Below we briefly show how to do that and give the formula. There is an important question that occurs from this direct way: How does one extract the dependence of the potential on the additional parameter t from the Lax pair?

We propose to specify the "time" evolution via the t-dependence of x-conserved quantities [274]. In this section we begin to study this problem in terms of the same chain variables supplied with a time dependence. Let us start from the system for three functions σ_i for the simplest nontrivial closure

$$\begin{array}{rcl}(\mathrm{d}/\mathrm{d}x)\sigma_1(x) &=& \sigma_3^2(x) - \sigma_2^2(x) + \mu_3 - \mu_2,\\ (\mathrm{d}/\mathrm{d}x)\sigma_2(x) &=& \sigma_1^2(x) - \sigma_3^2(x) + \mu_1 - \mu_3,\\ (\mathrm{d}/\mathrm{d}x)\sigma_3(x) &=& \sigma_2^2(x) - \sigma_1^2(x) + \mu_2 - \mu_1.\end{array} \tag{4.32}$$

This direct way to produce the bi-Hamiltonian formalism was initiated in [438] by introducing a generating function and extracting conserved quantities (integrals). In this simplest example we take the integrals to solve the problem completely. If we express the third variable σ_3 as the linear combination of the other ones by means of the first integral (Casimir function) $c = \sigma_1+\sigma_2+\sigma_3$ and substitute it into the other equations of (4.32), then, after use of the second integral A, we arrive at the differential equation of the first order for elliptic functions. It is convenient to show this fact in terms of other variables [438]:

$$g_1(x) = \sigma_1(x) + \sigma_2(x), \quad g_2(x) = \sigma_2(x) + \sigma_3(x), \quad g_3(x) = \sigma_3(x) + \sigma_1(x).$$

Now we exclude $g_3(x)$ by the relation $g_3(x) = 2c - g_1(x) - g_2(x)$. Further, if we omit the argument x, the inverse transformations become

$$\sigma_1 = -g_2 + c, \qquad \sigma_2 = -c + g_1 + g_2, \qquad \sigma_3 = c - g_1.$$

Inserting the transforms into the system (4.26), we obtain two differential equations for new variables:

$$\frac{\mathrm{d}}{\mathrm{d}x}g_1 = \mu_1 - \mu_2 + 2cg_1 - g_1^2 - 2g_1g_2, \quad \frac{\mathrm{d}}{\mathrm{d}x}g_2 = -2cg_2 + 2g_1g_2 + g_2^2 + \mu_2 - \mu_3.$$

The second integral of motion in terms of g_i is more compact than in σ_is:

$$A = g_1g_2g_3 + \mu_2g_3 + \mu_1g_2 + \mu_3g_1. \tag{4.33}$$

It allows us to express g_2 as a function of g_1. Therefore,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}g_1(x) = -\mu_2 + \mu_1 + 2cg_1 - g_1^2 - 2\big[-g_1^2 + \mu_2 - \mu_1 + 2g_1c\\ +(g_1^4 - 4g_1^3c + 2g_1^2\mu_1 + 2c^2 - 2\mu_3 + \mu_2\\ +4A - \mu_2c - \mu_1cg_1 + \mu_1^2 + \mu_2^2 - 2\mu_2\mu_1)^{1/2}\big].\end{aligned} \tag{4.34}$$

The next problem is to solve (4.34) in elliptic functions. The Weierstrass or Legendre canonical form of the integral yields a solution of the problem after the Abel transformation [208] and use of the algebraic formulas that give $\sigma_{2,3}$. Finally, we have the explicit dependence of σ_i on x and parameters c and A [275]; see also Sect. 4.5.

Let us turn to the problem of a time evolution arising from the Lax representation for some nonlinear equation. The main instruction from the above example is to search for the time dependence of the x-independent entries c and A. Let us consider a DT-covariant "time" evolution. In the KdV case the second Lax operator has the form of (4.9). The account of a connection (4.19) between the potential and σ produces the modified KdV (MKdV) equation for the function σ. The substitution into the equation of the x-derivatives from the system (4.32) yields

$$\frac{\partial}{\partial t}\sigma_3(x,t) = \frac{1}{2}(\mu_1+\mu_2-5\mu_3-F)\frac{\partial}{\partial x}\sigma_3(x,t),$$
$$F = (\sigma_1+\sigma_2+\sigma_3)^2 = c^2. \tag{4.35}$$

The equations for $\sigma_{1,2}$ are written similarly if we use the cyclic symmetry in the indices. Such equations are general; we will call them the t-chain equations (for t-chains see also [313]). The form of the equations is typical for the so-called hydrodynamic-type equations. The system is diagonal and Hamiltonian, it can be integrated by the hodograph method [428]. The final step of this direct construction is the use of (4.35) and similar equations for $\sigma_{1,2}$. First we can check that

$$\frac{\partial}{\partial t}c = -3\sigma_2^2\mu_3+3\sigma_3^2\mu_2-3\sigma_3^2\mu_1+3\sigma_1^2\mu_3-3\sigma_1^2\mu_2+3\sigma_2^2\mu_1 = 3c_x,$$

which is zero if the x-dependence of σ_i is governed by (4.32). If one plugs the t-derivatives of σ_i from formulas like (4.35) into the derivative A_t, this gives

$$A_t = g_2^2g_1^2(3\mu_1-3\mu_3)+g_2^2g_3^2(3\mu_2-3\mu_1)+g_1^2g_3^2(3\mu_3-3\mu_2)$$
$$+g_2g_1(6\mu_3\mu_2-6\mu_1\mu_2)+g_2g_3(6\mu_3\mu_1-6\mu_3\mu_2)+g_1g_3(6\mu_1\mu_2-6\mu_3\mu_1)$$
$$+3\,(\mu_1-\mu_3)\,(\mu_2-\mu_3)\,(\mu_2-\mu_1)\,.$$

Further analysis leads to Proposition 4.3:

Proposition 4.3. *If a common period X exists for all g_i, the x-independent polynomial c does not depend on t, and A is a linear function of t.*

For the proof it is enough to notice that the second-order combination $g_2g_1(6\mu_3\mu_2-6\mu_1\mu_2)+g_2g_3(6\mu_3\mu_1-6\mu_3\mu_2)+g_1g_3(6\mu_1\mu_2-6\mu_3\mu_1)$ is a linear combination of the x-derivative and constant because, for example,

$$g_1\,(g_2-g_3) = \frac{\mathrm{d}}{\mathrm{d}x}g_2(x)+\frac{\mathrm{d}}{\mathrm{d}x}g_3(x)+\mu_2-\mu_1.$$

After integration of this equation in x over the period X and use of the conservation laws for the KdV and MKdV equations in combination with (4.26), we arrive at the linear dependence of AX on t. The coefficient may be recognized in the expression for A_t: it is a combination of the eigenvalues μ_i. For example, the $\int\sigma_i, \int\sigma_i^2, i=1,2,3$ and other more complicated conserved quantities [443] should be accounted for, and the integrals over X of the combinations are zero. The resulting formula for A should be substituted into the solution of (4.34).

4.4 Discrete symmetry

4.4.1 General remarks

One can easily check the invariance of every conserved quantity (e.g., the above c and A) against the permutations of the elements σ_i, as well as the covariance of the systems (4.32) and (4.35). The symmetry with respect to the cyclic permutation of the variables (or indices) is obvious if one recalls its DT origin; hence, this observation is general. For the illustration we again exploit the simplest KdV case starting from the representation of [438] (for the additional symmetry with respect to reflections see [165]). Let us consider the periodically closed dressing chain equation of the odd step $N = 2g + 1$ and introduce vectorial notations, namely

$$\Sigma = (\sigma_1, ..., \sigma_N)^{\mathrm{T}}, \qquad \mu = (\mu_1, ..., \mu_N)^{\mathrm{T}}$$

and

$$\Sigma^2 = (\sigma_1^2, ..., \sigma_N^2)^{\mathrm{T}}.$$

The closed chain equation (4.22) can be rewritten either in the form

$$(1 + S)\Sigma_x = (1 - S)(\Sigma^2 + \mu)$$

or in the form

$$\Sigma_x = \sum_{k=1}^{N-1} (-1)^k S^k (\Sigma^2 - \mu), \tag{4.36}$$

where the operator of permutation is represented as the matrix S [438]. Both forms are obviously invariant with respect to the S-transformation because the matrix S and operators in (4.36) commute. The same statement is valid for the equations of the time evolution.

Let us emphasize that the right-hand side of (4.36) is tensorial with respect to the components of Σ, so the action of the group transformation is the tensor (direct) product of the group representations in the corresponding vector spaces.

If we introduce the cyclic permutation operator T_s, its action determines the matrix S as

$$T_s \sigma_i = S_{ik} \sigma_k = \sigma_{i+1(\mathrm{mod}N)}.$$

The powers of the matrix of the previous section produce the group, $S^k \in C_n \subset S_n$ and give a basis for integration of the covariant equations. The technique uses the Poisson representation of the system (4.36),

$$\psi_x(\Sigma, \mu) = \{H(\Sigma, \mu), \psi(\Sigma, \mu)\},$$

where the operator

$$H(\Sigma, \mu) = \sum_{i=1}^{N} \left(\frac{1}{3} \sigma_i^3 + \mu_i \sigma_i \right)$$

is invariant with respect to the group transformations and defines a linear operator ad_H with respect to the Poisson bracket

$$\{\sigma_i, \sigma_k\} = (-1)^{k-i}(1 - \delta_{ik}), \qquad k \le i.$$

It is easy to check that

$$\{H(\Sigma), \sigma_j\} = \sum_{k=1}^{n} \left(\frac{1}{3}\{\sigma_k^3, \sigma_j\} + \mu_k\{\sigma_k, \sigma_j\} \right) = \sum_{k=1}^{n} (\sigma_k^2 + \mu_k)\{\sigma_k, \sigma_j\},$$

which yields the system (4.36).

The integration of the system can be understood in a "quantum-mechanical" language, introducing first the "commuting" functions $C_i(\Sigma)$ from the kernel $C_i \in K$:

$$\mathrm{ad}_H C_i = 0. \tag{4.37}$$

Next, the eigenvalue problem for $\psi(\Sigma, \mu)$ outside the kernel can be considered as a matrix one in some basis:

$$\mathrm{ad}_H \psi_i = \lambda_i \psi_i, \tag{4.38}$$

where $i = 1, \ldots, g$, $N = 2g + 1$. The symmetry of (4.37) and (4.38) with respect to the transformations T_s (4.18) follows from the obvious relation

$$H(T_s\Sigma, T_s\mu) = H(\Sigma, \mu).$$

This means that matrices of a representation of the symmetry group commute with the matrix of ad_H on the corresponding subspace with constant Casimir operator.

4.4.2 Irreducible subspaces

The symmetry and the tensor structure of the right-hand side of (4.36) show that the system may be simplified in the framework of the Wigner–Eckart theory [452]. In accordance with the Wigner theorem applied to equations (4.37) and (4.38), the operator ad_H has the quasidiagonal structure in the basis of irreducible representations. The projective operators p_i acting on the irreducible subspaces are defined in the subspaces produced by chains that appear as the sum over the transformation group action on some basic element. In the case of the commutative group, chosen for simplicity, the irreducible matrices are one-dimensional, and the basis is defined by the set of projectors

$$p_i = \sum_{s \in G} N_i D^i(s) T_s,$$

where N_i are normalizing constants, $D^i(s)$ are irreducible representations of the symmetry group, and T_s is the group transformation operator in the space under consideration.

In our case the operator T_s coincides with that introduced in the previous section and in the case of the cyclic permutation group C_n the irreducible matrices are one-dimensional. Namely, $D^j(e) = 1$, $D^j(s) = a_j$, $D^j(s^2) = a_j^2, \ldots$ with $a_j = \exp(j2\pi\mathrm{i}/N)$, where the integer N is the group order. Hence,

$$T_s s_j = a_j s_j. \tag{4.39}$$

For $N = 3$ we project the system (4.36) or, originally, (4.26) onto each subspace, having three equivalent equations. Let $a = \exp(2\pi\mathrm{i}/3)$; e.g., the second of the resulting equations gives

$$s_{2x} = n_1 + an_2 + a^2 n_3 \tag{4.40}$$

$$= \sigma_3^2 - \sigma_2^2 + \mu_3 - \mu_2 + a\left(\sigma_1^2 - \sigma_3^2 + \mu_1 - \mu_3\right) + a^2\left(\sigma_2^2 - \sigma_1^2 + \mu_2 - \mu_1\right).$$

Here n_i denotes the right-hand side of (4.26).

The inverse transform from original variables to the basis of irreducible representations reads

$$\sigma_1 = s_1 + s_2 + s_3,$$
$$\sigma_2 = s_1 + as_2 + a^2 s_3,$$
$$\sigma_3 = s_1 + a^2 s_2 + as_3.$$

Inserting them into (4.40) yields

$$\frac{\mathrm{d}}{\mathrm{d}x} s_1(x) = 0,$$

$$\frac{\mathrm{d}}{\mathrm{d}x} s_2(x) = -(a-1)\left[\, 3a\left(s_2^2 + 2s_1 s_3\right) - a\mu_2 + a\mu_1 - \mu_2 + \mu_3\right],$$

and

$$\frac{\mathrm{d}}{\mathrm{d}x} s_3(x) = (a-1)\left[\, 3a\left(s_3^2 + 2s_1 s_2\right) - a\mu_3 + a\mu_1 + \mu_2 - \mu_3\right].$$

The second conservation law (4.33) in terms of s_i allows us to express the Hamiltonian as a function of the only variable s_3. The conservation laws are obviously the combinations of the irreducible polynomials σ_i. Together with the Hamiltonian ($=\lambda$) conservation it leads to the spectral curve definition.

Returning to the general problem needs the tensor product space of the vectors Σ and μ. The problem of solution of (4.37) and (4.38) simplifies if we use the above symmetry written in terms of

$$s_i = \sum_{s\in G} N_i D^i(s) T_s \sigma_1 = N^{-1} \sum_{k=1}^{N} a_{i-1}^{k-1} \sigma_k.$$

Proposition 4.4. *By direct application of the operator T_s it can be checked that the tensor products of s_i [see (4.39)]*

$$s_i s_k \dots s_j \tag{4.41}$$

form a basis of irreducible tensors in the space of polynomials, and the result of the T_s operator action differs from (4.41) by a constant factor $a_i a_k \dots a_j$.

The further computations are conveniently made via the Poisson bracket

$$\{s_j, s_l\} = N^{-2} \sum_{ik} a_{j-1}^{i-1} a_{l-1}^{k-1} \{\sigma_i, \sigma_k\}. \tag{4.42}$$

Particularly it is shown that the C_i can be chosen as a combination of irreducible polynomials (one can check this statement for the examples in [438]), and the conservation laws presented are combinations of the irreducible polynomials of σ_i.

Next, the t-dependence may be introduced via the scheme of Sect. 4.3. Other nonlinear systems are treated similarly. The widened symmetry, e.g., from [165], includes reflections at the (Σ, μ) tensor product space and could give more information about solutions.

4.5 Explicit formulas for solutions of chain equations ($N = 3$)

Let us return to equations (4.32),

$$\begin{aligned} \sigma_1' &= \sigma_3^2 - \sigma_2^2 + \mu_3 - \mu_2, \\ \sigma_2' &= \sigma_1^2 - \sigma_3^2 + \mu_1 - \mu_3, \\ \sigma_3' &= \sigma_2^2 - \sigma_1^2 + \mu_2 - \mu_1, \end{aligned} \tag{4.43}$$

that are equivalent to the system (4.25) under the condition $\alpha = 0$ with $N = 3$. We use two integrals

$$\begin{aligned} c &= \sigma_1 + \sigma_2 + \sigma_3, \\ A &= g_1 g_2 g_3 + \mu_2 g_3 + \mu_1 g_2 + \mu_3 g_1 = \\ &\quad (\sigma_1 + \sigma_2)(\sigma_2 + \sigma_3)(\sigma_3 + \sigma_1) + \\ &\quad \mu_2(\sigma_3 + \sigma_1) + \mu_1(\sigma_2 + \sigma_3) + \mu_3(\sigma_1 + \sigma_2). \end{aligned} \tag{4.44}$$

From (4.43) we obtain

$$\sigma_1'^{\,2} = (\sigma_3^2 - \sigma_2^2 + \mu_3 - \mu_2)^2.$$

Using the first equation in (4.44) as $\sigma_3 = c - \sigma_1 - \sigma_2$, we obtain the equation containing σ_1' and σ_2. Then we eliminate the remaining variable σ_2 with the help of both integrals in (4.44). Thus, we get the equation

$$\begin{aligned} \sigma_1'^{\,2} &= \sigma_1^4 - 2(c^2 + \mu_3 + \mu_2 - 2\mu_1)\sigma_1^2 \\ &\quad - 4(2\mu c - A)\sigma_1 + c^4 + 2(\mu_2 + \mu_3 + 2\mu_1)c^2 - 4Ac + (\mu_3 - \mu_2)^2. \end{aligned} \tag{4.45}$$

This equation has the following structure:

$$\left(\frac{\mathrm{d}\sigma_1}{\mathrm{d}x}\right)^2 = \sigma_1^4 - 6a\sigma_1^2 + 4b\sigma_1 + d, \tag{4.46}$$

where a, b, and d are constants determined by (4.45). The extra multipliers 6 and 4 have been included for convenience. The relation (4.46) is an elliptic curve in variables (σ_1', σ_1) and therefore it is uniformized by elliptic functions. Let us build the invariants (capital letters are chosen to distinguish the invariants from variables g_j of the chain)

$$G_2 = d + a^2, \qquad G_3 = a^3 - b^2 - ad.$$

So, the pair (b, a) is a point on a curve

$$b^2 = 4a^3 - G_2 a - G_3.$$

Therefore, there exists a parameter ν such that the following equations occur:

$$b = \wp'(2\nu), \qquad a = \wp(2\nu),$$

where $\wp(2\nu)$ is the Weierstrass elliptic function. This means that we take three new parameters G_2, G_3, and ν instead of the old ones a, b, and d which in turn depend on five parameters of the chain: $(\mu_1, \mu_2, \mu_3, A, c)$; hence, we write

$$\left(\frac{\mathrm{d}\sigma_1}{\mathrm{d}x}\right)^2 = \sigma_1^4 - 6\wp(2\nu)\sigma_1^2 + 4\wp'(2\nu)\sigma_1 + [G_2 - 3\wp(2\nu)],$$

which yields

$$\sigma_1(x) = \zeta(x + \nu + x_0; G_2, G_3) - \zeta(x - \nu + x_0; G_2, G_3) - \zeta(2\nu; G_2, G_3).$$

Note that σ_1 is not the Weierstrass σ-function in the theory of elliptic functions, but ζ is the standard Weierstrass ζ-function. The solution $\sigma_1(x)$ depends on three arbitrary constants [according to the third order of equations (4.43)] G_2, G_3, and x_0 which are in turn defined by five constants μ_j, A, and c in explicit but transcendental way. Parameter ν is not exceptional,

$$\nu = \frac{1}{2}\wp^{-1}\left(\frac{\mu_3 + \mu_2 - 2\mu_1 + c^2}{3}\right),$$

where $\wp^{-1}$ denotes the elliptic integral of the first kind (inversion of the elliptic function $\wp$).

Remark 4.5. The solution obtained is exactly the logarithmic derivative of the Ψ-function for the one-gap Lamé potential,

$$\Psi'' - 2\wp(x)\Psi = \lambda\Psi, \qquad \Psi = \frac{\sigma(\alpha - x)}{\sigma(\alpha)\sigma(x)}\,\mathrm{e}^{\zeta(\alpha)x}, \qquad \lambda = \wp(\alpha),$$

and is distinguished from the solution

$$\frac{\Psi'}{\Psi} = \zeta(\alpha - x) - \zeta(\alpha) - \zeta(x)$$

by a shift of the spectral parameter α.

Remark 4.6. If one is interested in σ_i (4.46) in connection with the KdV equation theory, the dependence on time can be obtained using the t-chains [274] obtained by means of the MKdV equation for σ and conservation laws [443] (see the previous sections).

4.6 Towards the spectral curve

It is known that there exists a class of periodic or quasiperiodic potentials of the operator (4.18) for which the spectral problem in L_2 leads to the continuous spectrum with the finite number of gaps, i.e., intervals at which the values of the parameter λ do not belong to the spectrum [45]. Potentials of this kind exemplified in Sect. 4.5 correspond to solutions of the KdV equation, if time dependence is introduced by means of the Lax pair [215, 353]. The important explicit formulas for eigenfunctions of the operator (4.18) were obtained in [215]. As mentioned already, the chain equations closed (periodically) on an odd step $2g + 1$ produce the finite-gap solutions [438, 448]. We have demonstrated already that for $g = 1$ the potentials are expressed in the elliptic Weierstrass or Jacobi functions.

Further development allows us to include an additional evolution variable y [45, 55].

As shown in Sect. 4.2, especially the comments on Proposition 4.1, the value of the parameter α depends on the interpretation of σ. The case $\alpha = 0$ corresponds to $\sigma = \phi_x \phi^{-1}$, where ϕ is the eigenfunction of the operator (4.18) with the eigenvalue μ; it differs from that for $\alpha \neq 0$. For general statements and some applications see [284].

Let us consider the periodically closed ($\sigma_{n+N} = \sigma_n,\ \alpha_{n+N} = \alpha_n$) chain (4.25). Below we follow [17, 438] for the formalism of the 2×2 Lax pair (denoted as U and V) and [422, 423] for the link to the spectral curve and the Dubrovin equations concentrating mostly on the case of finite-gap solutions for $N = 2g + 1$.

As follows from (4.19), the link of the variable σ to the potential u_n connects the corresponding matrix operators. Let us start from the spectral equation

$$-\psi_n'' + u_n \psi_n = \lambda \psi_n. \tag{4.47}$$

With the first-order dressing chain equation in mind, it is useful to rewrite (4.47) in the matrix form. Speaking in physical language, we introduce the column of "state"

$$\Phi_n = \begin{pmatrix} \psi_n \\ \psi_n' \end{pmatrix},$$

which is a solution of

$$\Phi_n' = U_n(\mu)\Phi_n. \tag{4.48}$$

U_n is easily found from (4.47). Differentiation of the DT (4.23) gives

$$\psi_{n+1}'(\lambda+\alpha_n) = (u_n-\lambda)\psi_n(\lambda) - \sigma_n'\psi_n(\lambda) - \sigma_n\psi_n'(\lambda), \tag{4.49}$$

or, using (4.19) with the choice $c_i = 0$, gives

$$\psi_{n+1}'(\lambda+\alpha_n) = (\sigma_n^2-\lambda)\psi_n(\lambda) - \sigma_n\psi_n'(\lambda).$$

In terms of the column Φ_n the result may be written as

$$\Phi_{n+1}(\lambda+\alpha_n) = V_n(\lambda)\Phi_n. \tag{4.50}$$

The first row of this vector relation is simply (4.23) and the second line follows from (4.49) after use of the Miura link (4.19). Both (4.48) and (4.50) form the Adler Lax pair [17]

$$V_n(\lambda) = \begin{pmatrix} -\sigma_n & 1 \\ \sigma_n^2-\lambda & -\sigma_n \end{pmatrix}, \qquad U_n(\lambda) = \begin{pmatrix} 0 & 1 \\ u_n-\lambda & 0 \end{pmatrix}. \tag{4.51}$$

The operators satisfy the differential equation (Lax compatibility condition) with a shifted spectral parameter that is equivalent to the chain equation

$$V_n' = U_n(\lambda+\alpha_n)V_n(\lambda) - V_n(\lambda)U_n(\lambda). \tag{4.52}$$

Now we can reformulate the chain closure condition in terms of the operators V_n. The operator $V_1(\lambda)$ maps the state Φ_1 onto the space $\Phi_2(\lambda+\alpha_1)$, the operator $V_2(\lambda+\alpha_1)$ maps Φ_2 onto $\Phi_3(\lambda+\alpha_1+\alpha_2)$ and so on, till

$$\Phi_{N+1}(\lambda+\alpha) = (-1)^N T(\lambda)\Phi_1(\lambda),$$

where the transition operator

$$T(\lambda) = \prod_{n=1}^{N} V_n(\lambda+\beta_{n-1}) \tag{4.53}$$

is introduced, whence

$$\beta_n = \sum_{i=1}^{n}\alpha_i, \qquad \beta_0 = 0, \qquad \beta_N = \alpha.$$

Equation (4.52) yields the x-evolution equation for $T(\lambda)$:

$$T'(\lambda) = U_1(\lambda+\alpha)T(\lambda) - T(\lambda)U_1(\lambda).$$

Let the eigenvalue of the operator $T(\lambda)$ be denoted as z, then the condition

$$\det[zI - T(\lambda)] = z^2 - P(\lambda)z + Q(\lambda) = 0 \tag{4.54}$$

introduces the hyperelliptic spectral curve in the complex (λ, z)-plane. The spectral invariant

$$Q(\lambda) = \det T(\lambda) = (-1)^N \prod_{n=0}^{N-1} (\lambda + \beta_n) \tag{4.55}$$

is evaluated via (4.53). Note that $\det V(\lambda+\beta_n) = \lambda+\beta_n$. The coefficient $P(\lambda)$ is given by

$$P(\lambda) = \mathrm{tr}T(\lambda) = A(\lambda) + D(\lambda),$$

for

$$T(\lambda) = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{4.56}$$

where, for the case of $N = 2g + 1$,

$$A = (-1)^{g+1} c\lambda^g + a_1\lambda^{g-1} + \ldots, \quad B = b_0\lambda^g + b_1\lambda^{g-1} + \ldots,$$
$$C = (-1)^{g+1}\lambda^{g+1} + c_1\lambda^g + \ldots, \quad D = (-1)^{g+1} c\lambda^g + d_1\lambda^{g-1} + \ldots,$$

$b_0 = (-1)^g$, $c = \sum_{k=1}^N \sigma_k (-1)^g$, which may be proved by induction using the definitions (4.53) and (4.51) of the matrices T and V. One starts with $g = 1$,

$$V_1V_2V_3 = \begin{pmatrix} -\sigma_1 & 1 \\ \sigma_1^2 - \lambda & -\sigma_1 \end{pmatrix} \begin{pmatrix} -\sigma_2 & 1 \\ \sigma_2^2 - \lambda - \beta_1 & -\sigma_2 \end{pmatrix} \begin{pmatrix} -\sigma_3 & 1 \\ \sigma_2^2 - \lambda - \beta_2 & -\sigma_3 \end{pmatrix}$$

and applies the induction. This results in

$$P(\lambda) = \sum_{n=0}^{g} I_n \lambda^{g-n}. \tag{4.57}$$

Substitution of (4.56) into (4.52) yields

$$A' = C - (\lambda + u_1)B, \tag{4.58}$$
$$B' = D - A, \tag{4.59}$$
$$C' = (\lambda + \alpha + u_1)A - (\lambda + u_1)D, \tag{4.60}$$
$$D' = (\lambda + \alpha + u_1)B - C. \tag{4.61}$$

Using (4.58) and (4.61), one can check that

$$P'(\alpha) = \alpha B(\lambda);$$

hence, when $\alpha = 0$, the coefficients I_g of the polynomial $P(\lambda)$ are constants of motion. We can compare the results that follow from (4.42) and the conservation laws from Sect. 4.3 and (4.44).

The transformation of the spectral curve (4.54) to the normal form [208] is performed by the substitution

$$z = \frac{P(\lambda) + y}{2},$$

which yields

$$y^2 = P(\lambda)^2 - 4Q(\lambda). \tag{4.62}$$

4.7 Dubrovin equations. General finite-gap potentials

Let us introduce a system of moving points on the hyperelliptic spectral curve (4.54) and (4.62). Namely, note that at the points $\lambda = \lambda_j$ such that

$$B(\lambda) = \prod_{j=1}^{g} (\lambda_j - \lambda), \tag{4.63}$$

the transition matrix T is triangular

$$T(\lambda_j) = \begin{pmatrix} A(\lambda_j) & 0 \\ C(\lambda_j) & D(\lambda_j) \end{pmatrix}. \tag{4.64}$$

This allows us to choose the eigenvalue

$$z_j = A(\lambda_j),$$

which means the g-tuple of points (λ_j, z_j) of the (λ, z)-plane belongs to the spectral curve

$$z_j^2 - P(\lambda_j) z_j + Q(\lambda_j) = 0. \tag{4.65}$$

The Abel–Jacobi mapping AJ: $\Gamma^{(g)} \to \mathrm{Jac}(\Gamma)$ sends the degree g divisor

$$\mathrm{Di} = (\lambda_1, z_1) + \ldots + (\lambda_g, z_g)$$

of Γ, i. e., an element of the g-fold symmetric product $\Gamma^{(g)}$ of Γ, to a point of the Jacobi variety $\mathrm{Jac}(\Gamma)$.

Let us show that the x-motion of the g-tuple of points follows from (4.56), which means that a dependence of λ_j on x is linearized on $\mathrm{Jac}(\Gamma)$. The motion is extracted from the derivative of $B(\lambda_j) = 0$ [see (4.63)],

$$B'(\lambda_j) + B_\lambda(\lambda_j)\lambda_j' = 0.$$

Taking (4.59) for $B' = A(\lambda_j) - D(\lambda_j) = \sqrt{P(\lambda_j)^2 - 4Q(\lambda_j)}$ and using the identity

$$P(\lambda_j)^2 - 4Q(\lambda_j)^2 = [A^2 + 2AD + D^2 - 4AD]_{\lambda=\lambda_j}$$

yields from (4.64) and (4.57) the generalized Dubrovin equations

$$\lambda_j' = \frac{\sqrt{P(\lambda_j)^2 - 4Q(\lambda_j)}}{B_\lambda(\lambda_j)}. \tag{4.66}$$

Here the numerator is expressed as the difference of the roots of quadratic equation (4.65) (eigenvalues of the matrix T). The equations (4.66) appeared (perhaps for the first time) in the paper of Drach [130]. The comprehensive description of the finite-gap potentials in the context of algebrogeometric theory is given in [45]; see also [78] with the historical remarks on the theory.

The Lagrange interpolation formula reads

$$\frac{F(\lambda)}{B(\lambda)} = \sum_{j=1}^{g} \frac{F(\lambda_j)}{B_\lambda(\lambda_j)(\lambda - \lambda_j)} + G(\lambda), \tag{4.67}$$

where $F(\lambda)$ and $G(\lambda)$ are polynomials. In particular, if $F(\lambda) = \lambda^k, k = 0, ..., g-1$, the application of (4.67) gives

$$\frac{\lambda^l}{B(\lambda)} = \sum_{j=1}^{g} \frac{\lambda_j^l}{B_\lambda(\lambda_j)(\lambda - \lambda_j)}.$$

Let us take (4.63) for distinct λ_js and pick out the residue of both sides at $\lambda = \infty$:

$$\sum_{j=1}^{g} \frac{\lambda_j^k}{B_\lambda(\lambda_j)} = \delta_{k,g-1}, \quad k = 1, \ldots, g. \tag{4.68}$$

Substitution of the formula for $B_\lambda(\lambda_j)$ yields

$$\sum_{j=1}^{g} \frac{\lambda_j^k \lambda_j'}{\sqrt{P(\lambda_j)^2 - 4Q(\lambda_j)}} = \delta_{k,g-1}.$$

Integration of the equations is realized on a complex torus $\mathbf{C^g/L}$ that represents the Jacobi variety of the image AJ(Di) of the divisor Di:

$$\sum_{j=1}^{g} \int^{\lambda_j} \frac{\lambda^k \mathrm{d}\lambda}{\sqrt{P(\lambda)^2 - 4Q(\lambda)}} = \delta_{k,g-1} x + \text{const}. \tag{4.69}$$

Let us trace the technical details of integration of the dressing chain equations, starting from the Drach–Dubrovin equations (4.66) for $g = 2$. Namely, the system

$$\frac{\mathrm{d}\lambda_1}{\mathrm{d}x} = \frac{\nu_1}{\lambda_1 - \lambda_2}, \qquad \frac{\mathrm{d}\lambda_2}{\mathrm{d}x} = \frac{\nu_2}{\lambda_1 - \lambda_2},$$

with $\nu_i = \sqrt{P(\lambda_i) - 4Q(\lambda_i)}$ is equivalent to

$$\frac{\mathrm{d}\lambda_1}{\nu_1} = \frac{\mathrm{d}x}{\lambda_1 - \lambda_2}, \qquad \frac{\mathrm{d}\lambda_2}{\nu_2} = -\frac{\mathrm{d}x}{\lambda_1 - \lambda_2},$$

which yields (4.68)

$$\frac{\mathrm{d}\lambda_1}{\nu_1} + \frac{\mathrm{d}\lambda_2}{\nu_2} = 0, \qquad \frac{\lambda_1 \mathrm{d}\lambda_1}{\nu_1} + \frac{\lambda_2 \mathrm{d}\lambda_2}{\nu_2} = \mathrm{d}x.$$

Integration leads to quadratures:

$$\int^{\lambda_1} \frac{\mathrm{d}\lambda}{\nu} + \int^{\lambda_2} \frac{\mathrm{d}\lambda}{\nu} = a_1, \qquad \int^{\lambda_1} \frac{\mathrm{d}\lambda}{\nu} + \int^{\lambda_2} \frac{\mathrm{d}\lambda}{\nu} = x + a_2,$$

where $\nu = \pm\sqrt{P(\lambda) - 4Q(\lambda)}$; see again (4.55) and (4.57).

In the case $\alpha = 0$ the algebrogeometric method [45] applies the Lagrange interpolation formula to the system (4.66).

4.8 Darboux coordinates

We continue to study the case $N = 2g + 1$. Differentiation of the equation for the spectral curve (4.54) in x yields

$$z_j' = \frac{P_\lambda(\lambda_j)z_j - Q_\lambda(\lambda_j)}{B_\lambda(\lambda_j)}. \tag{4.70}$$

The general conservation law $P'(\lambda_j) = 0$ is taken into account. Equation (4.66) for λ_i may be rewritten as

$$\lambda_i' = \frac{z_i - Q(\lambda_i)z_i^{-1}}{B'(\lambda_i)}. \tag{4.71}$$

Let us solve the spectral curve equation (4.65) with the plugged $P(\lambda_j)$ for I_s:

$$I_s = \sum_{j=1}^{g} \frac{z_j + Q(\lambda_j)z_j^{-1} - I_0\lambda_j^g}{B_\lambda(\lambda_j)} \frac{\partial b_s}{\partial \lambda_j}. \tag{4.72}$$

This gives a new interpretation of the conserved quantities I_s as functions of the double set of variables $\lambda_1, \ldots, \lambda_g$ and $z_1, \ldots, z_g$. The Darboux coordinates [103] are λ_i and $\log z_i$, as follows from [423].

Proposition 4.7. *Equations (4.71) and (4.70) can be considered as the Hamiltonian system*

$$\lambda_j' = z_j \frac{\partial H}{\partial z_j}, \qquad z_j' = z_j \frac{\partial H}{\partial \lambda_j} \tag{4.73}$$

with the Hamiltonian

$$H = I_1/b_0 = \sum_{j=1}^{g} \frac{z_j + Q(\lambda_j)z_j^{-1} - I_0\lambda_j^g}{B_\lambda(\lambda_j)}. \tag{4.74}$$

For a proof, (4.71) is checked by the differentiation of (4.74) in z_j and λ_j. The first differentiation gives

$$z_j \frac{\partial H}{\partial z_j} = \frac{z_j - Q(\lambda_j)z_j^{-1}}{B_\lambda(\lambda_j)},$$

and the second one yields

$$z_j \frac{\partial H}{\partial \lambda_j} = -\frac{z_j}{b_0} \frac{\partial I_1}{\partial \lambda_j}.$$

The derivatives $(\partial I_1/\partial \lambda_j)$ can be extracted from the result of differentiating the spectral curve (4.54) in λ_j, if the variables z_j are considered as functions of I_l via (4.72).

The differentiation of I_1 from (4.72) and use of (4.65) gives

$$\frac{\partial I_1}{\partial \lambda_k} = \sum_{j=1}^{g} \frac{[-P(\lambda_k)z_j + Q'(\lambda_k)]\delta_{kj}}{z_k B_\lambda(\lambda_j)}(-b_0), \tag{4.75}$$

the link $b_1 = -b_0(\lambda_1 + ... + \lambda_g)$ is also used, while

$$\frac{\partial b_1}{\partial \lambda_j} = -b_0$$

and (4.75) yield the second part of (4.73).

4.9 Operator Zakharov–Shabat problem

In this section we discuss the non-Abelian ZS problem. We formulate the DT and the corresponding dressing chain equations. As an example, we consider the operator nonlinear Schrödinger (NLS) equation.

4.9.1 Sketch of a general algorithm

We consider a general equation with non-Abelian entries,

$$a_0\Psi + a_1 D\Psi = \Psi_t, \tag{4.76}$$

that has the (first order in D) form of the operator L with Ψ being treated as a matrix or, more generally, as an operator. The potential a_0 may be expressed in terms of s from (4.17),

$$Da_0^i + [a_0^i, s_i] = -Da_1 - [a_1, s_i]s_i,$$

or, in terms of $\mathrm{ad}_s = [s, .]$,

$$a_0^i = (D - \mathrm{ad}_{s_i})^{-1} \{-Da_1 - [a_1, s_i]s_i\}. \tag{4.77}$$

The existence of the inverse operator in (4.77) imposes some restriction for the expression in curly brackets. Namely, the expression should be outside the kernel of the operator $D + \mathrm{ad}_{s_i}$. The DT is simple in this case. Namely, a_1 is not transformed owing to (4.13) but

$$a_0^{i+1} = a_0^i + [a_1, s_i]. \tag{4.78}$$

Substituting (4.77) for i and $i + 1$ into (4.78) gives the chain equations. One could also express the matrix elements of a_0 in terms of the elements of the matrix s and plug them into the Darboux transform (4.78) separately. Similar tricks give results in the case of the alternative DT (see [324]).

4.9.2 Lie algebra realization

Let us reformulate the general scheme to derive the dressing chain in the non-Abelian case [270], starting again from the evolution (4.76),

$$u\Psi + JD\Psi = \Psi_t, \tag{4.79}$$

with the polynomial operator $L(D)$. This case provides the nontrivial example of a general equation (4.10) with operator entries J and u (y is changed to t). As a result, the form of the evolution operator (Hamiltonian) is fixed in the form $JD + u$. Within this scheme, the one-dimensional Dirac equation arises that can be applied for consideration of a multilevel system interacting with a quantum field [254]. We treat Ψ (and the other solution Φ necessary to construct the DT) as operators. The Dirac equation in the form of the ZS spectral problem enters the Lax pair of some integrable nonlinear equations as the NLS and Manakov equations.

The potential u is expressible in terms of σ (4.14) that we rewrite as

$$-\sigma_t + J\sigma_x + [J\sigma, \sigma] = [\sigma, u]. \tag{4.80}$$

The structure of this equation determines the algebraic properties of the admissible dressing construction.

For the x-independent version, when $u_x = \sigma_x = 0$, (4.80) yields

$$-\sigma_t + [J\sigma, \sigma] = [\sigma, u], \tag{4.81}$$

which means $[\mathrm{tr}(\sigma)]_t = 0$ and the choice of traceless σ. The structure of σ implies also the restriction

$$\det \sigma = \det M = \prod \mu_i. \tag{4.82}$$

Namely, introducing the iteration index i, we have the link

$$u^i = (\mathrm{ad}_{\sigma_i} - D)^{-1} (DJ + [J, \sigma_i]\sigma_i). \tag{4.83}$$

In the subspace $\ker(D - \mathrm{ad}_{\sigma_i}) = 0$, where the Lie product is zero, (4.83) trivializes. The DT (4.78) is reproduced as

$$u^{i+1} = u^i + [J, \sigma_i]. \tag{4.84}$$

Note that J is not changed under the DT; see (4.12). Substituting (4.83) for i and $i + 1$ into (4.84), we arrive at the chain equations. We can also express matrix elements of u in terms of the entries of σ and plug them into the Darboux transform (4.84) separately.

Let us give more details of the construction in the stationary case, restricting ourselves to $DJ = 0$; note that Ψ and Φ correspond to λ and μ, respectively. There are two possibilities for stationary equations that follow from the non-Abelian equation (4.79): either $\Psi_t = \lambda\Psi$ or $\Psi_t = \Psi\lambda$. The first of

the possibilities leads to the essentially trivial connection between solutions and potentials from the point of view of DT theory [321]. In the second case we write

$$\sigma = \Phi_x \Phi^{-1} = J^{-1}(\Phi\mu\Phi^{-1} - u)$$

and the DT takes the following form in terms of s_i:

$$u^{i+1} = J^{-1}u^i J - J^{-1}[s^i, J],$$

where $s = \Phi\mu\Phi^{-1}$; hereafter iteration number indices are omitted. The potentials u^i can be excluded from (4.14) for this case:

$$\sigma_t = Dr + [r, \sigma],$$

with

$$r = J\sigma + u = s. \tag{4.85}$$

The stationary case, after plugging u^i from (4.85) and reentering indices, gives

$$s^{i+1} = s^i + J\sigma^{i+1} - \sigma^i J, \tag{4.86}$$

while the relation

$$Ds^i + [s^i, \sigma^i] = 0 \tag{4.87}$$

links the derivative of s and the internal derivative of σ. The formal transformation that leads to the chain equations is similar to (4.83) and follows after substitution of $\sigma^i = -\mathrm{ad}_{s_i}^{-1} Ds^i$ into (4.86).

Further progress in the explicit realization of this program is connected with the choice of the additional algebraic structure over the differential ring we consider. For example, if the elements s^i and σ^i belong to a Lie algebra with structure constants $C^\gamma_{\alpha\beta}$, then we introduce the expansion (summation over the Greek indices is implied) in the basis elements e_α for $s^i = \xi^i_\alpha e_\alpha$ and $\sigma^i = \eta^i_\alpha e_\alpha$. Plugging into (4.87) gives the differential equation

$$D\xi^i_\alpha + C^\alpha_{\gamma\beta}\xi^i_\gamma \eta^i_\beta = 0.$$

In terms of the matrix B,

$$B_{\beta\alpha} = C^\alpha_{\gamma\beta}\xi^i_\gamma, \tag{4.88}$$

we have for vectors outside of the kernel of B

$$\eta^i_\beta = -B^{-1}_{\beta\alpha} D\xi^i_\alpha. \tag{4.89}$$

By the definition of the Cartan subalgebra C the corresponding subspace does not contribute to the Lie product of (4.87).

Proposition 4.8. *Let J belongs to a module over the Lie algebra, $Je_\alpha = J_{\beta\alpha}e_\beta$, and let there exist an external involutive automorphism τ [e. g., $(ab)^\tau = b^\tau a^\tau$]. Then the chain equation for the variables ξ^i_α takes the form*

$$\xi^{i+1}_\alpha = \xi^i_\alpha - B^{-1}_{\beta\gamma}(D\xi^{i+1}_\gamma)J_{\beta\alpha} + (J^\tau_{\alpha\beta}B^{-1}_{\beta\gamma}D\xi^i_\gamma)^\tau,$$

where the matrix B is defined by (4.88) and the components e_α lie outside the Cartan subalgebra C. Otherwise,

$$D\xi_\gamma^i = 0,$$

if $e_\gamma \in C$.

This proposition demonstrates in fact the DT in the form of (4.86) written in the basis of e_α, in which (4.89) is used. The subspace of C gives the second case. The system of differential equations is hence nonlinear as the matrix B depends on ξ_γ^i.

4.9.3 Examples of NLS equations

Split NLS dressing chain

The split NLS equation is associated with the 2×2 matrix spectral problem

$$\Psi_x = i \begin{pmatrix} \lambda & p \\ q & -\lambda \end{pmatrix} \Psi$$

under the reduction $p = \bar{q}$ and the choice $J = \sigma_3$. Let

$$\sigma = \eta_i \sigma_i, \qquad u = u_1\sigma_1 + u_2\sigma_2 \in sl(2, \mathbb{C})$$

with the Pauli matrices σ_i,

$$[\sigma_i, \sigma_k] = 2\mathrm{i}\varepsilon_{iks}\sigma_s,$$

as generators of the algebra $sl(2, \mathbb{C})$. The "Miura" connection (4.81) is specified by

$$J\sigma = i\sigma_2\eta_1 - i\sigma_1\eta_2 + \eta_3\sigma_3.$$

Inserting this result, as well as u and σ into (4.81), we arrive at the equations

$$\eta_1' + 2\eta_1\eta_3 = 2\imath\eta_3 u_2, \quad \eta_2' + 2\eta_2\eta_3 = -2\imath\eta_3 u_1, \tag{4.90}$$

$$\eta_3' - 2\eta_1^2 - 2\eta_2^2 = 2\imath\eta_2 u_1 - 2\imath\eta_1 u_2.$$

It follows from (4.82) that

$$\eta_3 = \sqrt{\mu_1\mu_2 - \eta_1^2 - \eta_2^2}, \tag{4.91}$$

in accordance with (4.90). Hence, the chain equations are

$$u_1^{i+1} = u_1^i - 2i\eta_2^i, \qquad u_2^{i+1} = u_2^i + 2i\eta_1^i, \tag{4.92}$$

where

$$u_1 = -(\eta_2'/\eta_3 + 2\eta_2)/2i, \qquad u_2 = (\eta_1'/\eta_3 + 2\eta_1)/2i,$$

and η_3 is the function (4.91). The structure of the matrix elements of the potential u is such that $q = u_1 + iu_2$, so the reduction to the NLS case means the reality of both u_i. The repulsive NLS equation corresponds to $p = -\bar{q}$.

The simplest chain closures and solutions

The simplest version of a periodic closure is

$$\eta_s^{i+1} = \alpha_s \eta_s^i, \qquad \eta_1^1 = x, \qquad \eta_2^1 = y, \qquad \eta_3^1 = z,$$

with the conditions

$$-\mu_1^1\mu_2^1 = x^2 + y^2 + z^2, \quad -\mu_1^2\mu_2^2 = \alpha_1^2 x^2 + \alpha_2^2 y^2 + \alpha_3^2 z^2$$

providing restrictions for the constants

$$-\mu_1^1\mu_2^1 = x^2 + y^2 - (\mu_1^2\mu_2^2 - \alpha_1^2 x^2 - \alpha_2^2 y^2)/\alpha_3^2$$

or

$$\alpha_3^2 = \alpha_1^2 = \alpha_2^2, \qquad \mu_1^1\mu_2^1\alpha_3^2 = \mu_1^2\mu_2^2.$$

Hence,

$$\alpha_1 = \pm\alpha_3, \qquad \alpha_2 = \pm\alpha_3. \tag{4.93}$$

Equations (4.92) yield

$$\left(\frac{\alpha_1}{\alpha_3} - 1\right)\frac{x_t}{z} = 2(\alpha_1 + 1)y, \quad \left(\frac{\alpha_1}{\alpha_3} - 1\right)\frac{x_t}{z} = 2(\alpha_1 + 1)x.$$

Excluding z,

$$\frac{\alpha_2 - \alpha_3}{1 + \alpha_2}(\ln y)_t = \frac{\alpha_3 - \alpha_1}{1 + \alpha_2}(\ln x)_t,$$

gives nontrivial conditions for constants (4.93),

$$\alpha_1 = -\alpha_3, \quad \alpha_2 = -\alpha_3.$$

In this case $(\ln y)_t = -(\ln x)_t$ or $y = c/x$,

$$x_t/x = -(1 - \alpha_3)\sqrt{-\mu_1^1\mu_2^1 - x^2 + y^2},$$

which is again solved in elliptic functions [208]. If the coefficients are chosen as

$$(x_t)^2 = -(1 - \alpha_3)(-\mu_1^1\mu_2^1 x^2 - x^4 - c^2) = (1 - x^2)(k^2x^2 + k'^2),$$

then $x = \mathrm{cn}(t, k)$ is the Jacobi elliptic function.

The t-chains are obtained in a way similar to that in Sect. 4.3 using the second Miura map.

4.10 General polynomial in T operator chains

4.10.1 Stationary equations as eigenvalue problems and chains

Let again A be an operator ring with the automorphism T. If for any two elements $f, g \in A$

$$T(fg) = T(f)T(g),$$

general formulas for the DT for polynomial in T operators exist [321]; see also [271]. Here we continue to study the versions of the ZS problem. We call the operator T a shift operator but it could be general as defined above.

The stationary equation that corresponds to the evolution generated by a polynomial in the automorphism T (see the previous chapter) appears when the solutions of the constraint equations $\psi_t(x,t) = \lambda\psi$ and $\psi_t(x,t) = \psi\lambda$ are considered, or, in the first case,

$$\sum_{m=-M}^{N} U_m T^m \psi = \lambda\psi \,. \tag{4.94}$$

Hence, there are two versions of the scheme depending on the position of λ. Let us rewrite the first version of the Miura equation from Sect. 4.2, omitting the index + which denotes the first version:

$$\sigma_t = \sum_{m=-M}^{N} \left[U_m \; B_m(\sigma)\,\sigma - \sigma T\left(U_m\right) B_{m+1}(\sigma)\,\sigma\right]. \tag{4.95}$$

The derivative is written as

$$\sigma_t = \varphi_t (T\varphi)^{-1} - \varphi (T\varphi)^{-1} (T\varphi)_t (T\varphi)^{-1} = \mu\sigma - \sigma\mu. \tag{4.96}$$

It is zero if σ and μ commute. Recall that $\varphi \in \{\psi_{\lambda=\mu}\}$.

The connection between potentials and σ follows as a consequence of (4.95) in the stationary case of (4.96). For a similar link in the case of differential operators see Sect. 4.2. When the reduction is such that all the coefficients in (4.94) are functions of some unique potential u, this connection allows us in principle to express the potential u as a function of σ. This connection has the same form for the dressed potential. If we take connections for both potentials with the corresponding elements σ_i and plug the result into the DT formulas, the chain equations result.

This algorithm is illustrated next for particular examples.

4.10.2 Nonlocal operators of the first order

Let us take the general equation (4.94) for $N = 1$:

$$\psi_t\,(x,t) = (J + UT)\,\psi. \tag{4.97}$$

There are two types of DT in this case [271], denoted by the indices $\pm$; see Chap. 3. The DT of the first kind $(+)$ leaves J unchanged. We rewrite the transform of U as

$$U^+ = \sigma^+\,(TU)\left(T\sigma^+\right)^{-1}, \tag{4.98}$$

where $\sigma^+ = \phi(T\phi)^{-1}$; further the superscript $+$ is omitted.

For the spectral problem corresponding to (4.97), the nontrivial transformations appear if in the stationary equation we introduce the constant element μ that does not commute with φ and σ:

$$(J + UT)\,\varphi = \varphi\mu. \tag{4.99}$$

The formula for the potential is then changed to

$$U = \varphi\mu(T\varphi)^{-1} - J\sigma. \tag{4.100}$$

Let us derive the identity that links the potential U and σ, doing so in a different manner from that in [271] or in Sect. 4.6. We start from

$$T(\sigma)T^2(\varphi) = T(\varphi),$$

and insert it into the shifted equation (4.99):

$$T(U)T^2(\varphi) = T(\varphi)\mu - JT(\varphi).$$

One has a Miura-like link

$$\sigma T(U)\sigma = U + [J,\sigma], \tag{4.101}$$

where $T(\sigma) = \sigma^{-1}$ is accounted for. Comparing with (4.98) yields a new form of the DT that coincides with

$$U + [J,\sigma] = U^+.$$

Direct use of (4.99) for expressing U in terms of $\tau = \varphi\mu\varphi^{-1}$ and σ gives

$$U = \tau\sigma - J\sigma. \tag{4.102}$$

The element τ is useful as well, for

$$T(U) = \sigma^{-1}\tau - J\sigma^{-1}. \tag{4.103}$$

Inserting (4.103) and (4.102) into (4.101), we arrive at the identity. The algorithm of the explicit derivation of the chain equations begins from (4.101)

solved with respect to U in an appropriate way. For matrix rings, it may be a system of equations for matrix elements, which could be effective in low matrix dimensions of the "Miura" link (4.101).

The role of σ^+ can be played also by the function $s = \varphi\mu(T\varphi)^{-1}$. Equation (4.100) connects U and σ. Let us rewrite (4.101) and the DT in terms of s, excluding U from (4.100) and denoting the number of iterations by index n:

$$U[n] = s_n - J\sigma_n.$$

Equation (4.101) reduces to

$$s_n = \sigma_n T(s_n)\sigma_n. \tag{4.104}$$

This result gives for the DT

$$s_{n+1} - s_n = J\sigma_{n+1} + \sigma_n J.$$

Then, solving (4.104) with respect to s, we obtain the chain system. This could be done similarly to the method in the previous section by means of the Lie algebra representation.

Let us mention that the chain equations for the classical ZS problem and two types of the DT were introduced in [396]. The closure of the chain equations specifies classes of solutions.

4.10.3 Alternative spectral evolution equation

Let us take (4.94) in the alternative version of the spectral problem (left position of the parameter μ):

$$(U_0 + U_1 T)\,\varphi = \mu\varphi. \tag{4.105}$$

The connection between the unique potential U_1 and σ^+ (recall that + denotes the first version considered in 4.10.1) is obtained from (4.95) and (4.96), or directly from (4.105),

$$U_1 = (\mu - U_0)\sigma^+ \ . \tag{4.106}$$

Introducing the number of iterations n for U_1 and $n+1$ for U_1^+ yields

$$U_1\,[n+1] = (\mu_n - U_0)\sigma^+[n+1] = \sigma^+\,[n]\,T\left\{(\mu_n - U_0)\sigma^+\ \ [n)\right\}\left(T\sigma^+\,[n]\right)^{-1}. \tag{4.107}$$

This should be the chain equation for the generalized ZS problem. We rewrite the chain equation (4.107) in a more compact form changing the notations as follows $U_0 \to J,\ U_1 \to U,\ \sigma^+\,[n] \to \sigma_n$ and supposing that $T(J) = J$. We arrive at

$$\sigma_{n+1} \ \ = \ \ (\mu_n - J)^{-1}\sigma_n\ (\mu_n - J).$$

This dressing is, however, almost trivial. Such a phenomenon is well known for the differential operators [324]. The alternative and effective transformations

appear if in the stationary equation (4.105) we introduce the element μ that does not commute with σ and changes the order of the elements μ and φ on the right-hand side. (It is the second case mentioned at the beginning of this section). We get

$$(J + UT)\varphi = \varphi\mu.$$

The formula for the potential is changed to

$$U = \varphi\mu(T\varphi)^{-1} - J\sigma \tag{4.108}$$

and the role of σ^+ is played by the function $s = \varphi\mu(T\varphi)^{-1}$. Equation (4.95) connects U and σ:

$$J\sigma + U - \sigma J\sigma + \sigma T(U)(T\sigma)^{-1} = [\mu, \sigma]. \tag{4.109}$$

The algorithm of the explicit derivation of the chain equations begins from (4.109) solved with respect to U in the appropriate way. For matrix rings, it may be a system of equations for matrix elements that could be effective in low matrix dimensions of (4.109), as in [396]. Otherwise it leads to a problem of the adequate choice of basis. Let us rewrite (4.109) and the DT (4.107) in terms of s, excluding U from (4.108) and denoting the number of iterations by the index n: $U[n] = s_n - J\sigma_n$. Equation (4.109) is transformed as

$$s - \sigma J\sigma + \sigma T(s)(T\sigma)^{-1} + \sigma J = [\mu, \sigma].$$

This result gives for the DT

$$s_{n+1} - s_n = J\sigma_{n+1} + \sigma_n J\sigma_n - [\mu, \sigma_n]. \tag{4.110}$$

Then, taking (4.110) for two indices (e. g., they could be n and $n+1$) we have the chain system. In the next section we give the explicit example for the bilinear Hirota equation.

4.11 Hirota equations

In this section we build dressing chain equations for the non-Abelian analog of the Hirota equations. Performing periodic closure, we obtain a solution of the equations.

4.11.1 Hirota equations chain

Let us return to the Hirota equations and to their non-Abelian analog from Sect. 2.10. Excluding the transformed potential from (2.127) and (2.128), we arrive at the equation that links the potential and the function $\sigma^-(r)$:

$$u\, T^{-1}\sigma^-(r) - \sigma^-(r-1)\, u = \left[\sigma^-(r-1) - \sigma^-(r)\right] T^{-1}\sigma^-(r). \tag{4.111}$$

Note that the expressions in (2.127), (2.128), and (4.111) are still general (non-Abelian) and may be used in the simplest (but sufficiently rich) closures (e. g., $\sigma^-_{n+1} = \sigma^-_n$). For the Lie-algebraic approach to the non-Abelian chains see [270].

In the scalar (Abelian) case we can easily solve (4.111) for the potential:

$$u = \frac{\left[\sigma^-(r-1) - \sigma^-(r)\right] T^{-1}\sigma^-(r)}{T^{-1}\sigma^-(r) - \sigma^-(r-1)}. \tag{4.112}$$

Supplying the entries of (4.112) with the index N (iteration number) and substituting into (2.127) and (2.128), we obtain two equivalent forms of the chain equations. For example,

$$u_{N+1} = u_N - \sigma^-_N(r-1) + \sigma^-_N(r)$$

yields

$$\frac{\left[\sigma^-_{N+1}(r-1) - \sigma^-_{N+1}(r)\right] T^{-1}\sigma^-_{N+1}(r)}{T^{-1}\sigma^-_{N+1}(r) - \sigma^-_{N+1}(r-1)} = \frac{\left[\sigma^-_N(r-1) - \sigma^-_N(r)\right] \sigma^-_N(r-1)}{T^{-1}\sigma^-_N(r) - \sigma^-_N(r-1)}. \tag{4.113}$$

Equation (4.113) generates the chain equation for the specific case of the system (2.123) by the choice $x \to n$, $Tf_n(j,r) = f_{n-1}(j,r)$. A solution of the resulting chain equation generates the solution of the system (2.126) by use of the connection formula (4.112) and the corresponding formula for v. The transition to $\tau_n(j,r)$ is made by (2.124).

4.11.2 Solution of chain equation

Let us denote

$$s_N = \frac{\left[\sigma^-_N(r-1) - \sigma^-_N(r)\right]}{T^{-1}\sigma^-_N(r) - \sigma^-_N(r-1)},$$

then the dressing chain equation (4.113) reads

$$s_{N+1} = s_N \frac{\sigma^-_N(r-1)}{T^{-1}\sigma^-_{N+1}(r)}.$$

Iterating this recurrence q times yields

$$s_{N+q} = s_N \frac{\prod_{s=0}^{q-1} \sigma^-_{N+s}(r-1)}{\prod_{s=1}^{q} T^{-1}\sigma^-_{N+s}(r)}.$$

In analogy with the continuous case, let us consider the periodic closure of the chain (4.113), starting from the simplest case $q = 0$, which means $\sigma_{N+1} = \sigma_N = \sigma$. As $s_{N+1} = s_N$,

$$T\sigma^-(r-1) = \sigma^-(r).$$

This means $\sigma^{-}(r+p) = T^{p}\sigma^{-}(r)$. If a boundary condition in the point $r = 0$ is given, then

$$\sigma^{-}(p) = T^{p}\sigma^{-}(0).$$

The equation for $\varphi(p, x)$ is then

$$T\varphi(p, x) = T\left[\sigma(p, x)\right]\varphi(p, x).$$

The solution depends on the choice of T. If $T\varphi(p, x) = \varphi(p, x + \delta)$, then

$$\varphi(p, x) = \exp(Ax),$$

with A from

$$A\delta = \ln\left[T^{p+1}\sigma^{-}(0)\right].$$

4.12 Comments

Let us mention that the KdV chain equations were introduced by Weiss [448] and for the classical ZS problem and two types of the DT were proposed by Shabat [396]. The closure of the chain equations specifies classes of solutions. The periodic closure of the chains produces integrable bi-Hamiltonian finite-dimensional systems and, in some special cases, the finite-gap potentials [438, 448]. We expect that the technical elements we develop are general enough. Derivation of the chain equations represents simply the result of substitution of a potential as the function of σ into the DT formulas, but the problem of the explicit form of the function could be nontrivial. The periodic closures of a chain for arbitrary N for the KdV and other equations are studied similarly and lead to the expressions for σ_i and, consequently, for the potentials in hyperelliptic functions by the algebraic construction [45]. Important dressing theory applications mentioned in [324] concern the possibility to combine the finite-gap [45] and localized (solitonic) configurations; see also [24]. The interpretation in terms of the finite-gap integration theory may be found in [216, 251]. We also believe that the finite closures for the chain equations may produce the solutions by accounting for a symmetry analysis, by means of the Wigner–Eckart theorem for both x and t evolutions. The development of the technique for infinite chains does not look impossible as well.

5

Dressing in 2+1 dimensions

In this chapter we speak again about the origin of the dressing technique, now in multidimensions. The important step was realized in the Moutard papers [340, 341] that the stabilization of the Laplace transformation chain can generate solutions. Notice again (see Chap. 1) that the net of points generated by the transform of the invariants of the *gauge transformations* has two possible symmetry reductions: the first reduction corresponds to the Moutard case and the second one was discovered by Goursat [192, 193]. The dressing procedure in two spatial dimensions opened a way to apply the Laplace equation in Lax pairs to solve some nonlinear 2+1 equations because their associated spectral problems are expressed in terms of the Laplace equation.

The celebrated 2+1 Kadomtsev–Petviashvili (KP) equation for surface water waves (there are lots of other applications [228]; see Chaps. 9, 10) and the corresponding dressing based on the direct extension of the Darboux theory (linear Schrödinger evolution as the first operator in the Lax pair) [313] have been the subject of intense studies [324]. The dressing methods for the Davey–Stewartson (DS) equation were introduced in [277], where, by means of eight Ablowitz–Kaup–Newell–Segur (AKNS) type pairs, ordinary and twofold elementary Darboux transformations (DTs) were studied and used for construction of multisoliton solutions of both types (DS I and DS II) of the DS equation. The dressed potentials were expressed in terms of quasideterminants studied previously in [176]. It was proved that nonlinear superposition formulas have a symmetry structure that gives a possibility to build networks of DTs that can be used to solve boundary problems via the construction proposed in [199]. An important class of solutions of a general Zakharov-Shabat (ZS) hierarchy that was not mentioned in [324] is generated by the dressing formulas from [313, 314]. In particular, solutions of the KP equations are given by the relation [313]

$$u = -2\partial^2 \ln W(\varphi_1, \ldots, \varphi_s),$$

where the Wronskian W is formed by the dressing functions φ_j depending on a parameter k and arbitrary function $g(k)$:

$$\varphi_j = [\partial_k + g(x)] \exp(kx + k^2 y + k^3 t)|_{k=k_j}.$$

This class of solutions contains the so-called general position solutions derived by Krichever [252] via the finite-gap formalism. Note also that these solutions generate the Calogero–Moser potentials

$$u = 2 \sum_j \frac{1}{x - x_j(y,t)},$$

which can be extracted from the dressing formulas. For the N-particle problems and polynomial solutions of the ZS hierarchy we refer to [315]. The 2+1 theory of generalized AKNS equations, including the DS, the Boiti–Leon–Manna–Pempinelli (BLMP1 and BLMP2) [58, 65], and some other equations, is studied in [140, 141, 143, 144, 142].

Here we concentrate on studying a general theory of dressing based on combinations of the following transformations: Laplace, Darboux (Sects. 5.1, 5.2), Goursat (Sect. 5.3), and Moutard (Sect. 5.4). Among other things, we derive a new integrable equation (5.19) which can be treated as the two-dimensional generalization of the sinh–Gordon equation. Sections 5.5 and 5.6 illustrate applications of this theory to the two-dimensional Korteweg–de Vries (KdV), two-dimensional modified KdV (MKdV), Nizhnik–Veselov–Novikov, and BLMP1 equations.

5.1 Combined Darboux–Laplace transformations

In this section we formulate constraints to coefficients of the Laplace equation which reduce it to the Moutard and Goursat equations. We show that a number of integrable nonlinear equations arise as a consequences of the reduction equations for the DTs. The content of this section is based on [287].

5.1.1 Definitions

For the Laplace equation

$$\psi_{xy} + a\psi_y + b\psi = 0 \tag{5.1}$$

the following were introduced:

1. The Laplace transformations (LTs) (Sect. 1.5)

$$a \to a_{-1} = a - \partial_x \ln(b - a_y), \quad b \to b_{-1} = b - a_y, \quad \psi \to \psi_{-1} = \psi_x + a\psi, \tag{5.2}$$

$$a \to a_1 = a + \partial_x \ln b, \qquad b \to b_1 = b + \partial_y \left(a + \partial_x \ln b\right), \qquad \psi \to \psi_1 = \frac{\psi_y}{b}. \tag{5.3}$$

2. The DTs

$$a \to a_1 = a - \partial_x \ln(a+\sigma), \quad b \to b_1 = b + \sigma_y, \quad \psi \to \psi_1 = \psi_x - \sigma\psi, \tag{5.4}$$

$$a \to {}_1a = -(\sigma + b\rho), \qquad b \to {}_1b = b - (b\rho)_y, \qquad \psi \to {}_1\psi = \rho\psi_y - \psi. \tag{5.5}$$

where $\sigma = \sigma(x,y) = \phi_x/\phi$, $\rho = \phi/\phi_y$, and ψ and ϕ are particular solutions of (5.1) with predetermined a and b. We refer to ϕ as *the support function* of the DT.

5.1.2 Reduction constraints and reduction equations

A constraint for the coefficients a and b of (5.1) fixes a particular class of equations which we are interesting in. Namely, the condition

$$a = 0, \qquad b = u \tag{5.6}$$

yields the *Moutard equation*

$$\psi_{xy} + u(x,y)\psi = 0, \tag{5.7}$$

while

$$a = -\frac{1}{2}\partial_x \ln\lambda, \qquad b = -\lambda \tag{5.8}$$

leads to the *Goursat equation*

$$\zeta_{xy} = 2\sqrt{\lambda\,\zeta_x\zeta_y}. \tag{5.9}$$

After the substitution $\psi = \sqrt{\zeta_x}$ and $\chi = \sqrt{\zeta_y}$ we get

$$\psi_y = \sqrt{\lambda}\chi, \qquad \chi_x = \sqrt{\lambda}\psi$$

or, in the form of the Laplace equation,

$$\psi_{xy} = \frac{1}{2}(\ln\lambda)_x\psi_y + \lambda\psi \tag{5.10}$$

and a similar equation for χ; see also Sect. 5.1.3. The functions u and λ are solutions of the special equations which we call the *reduction equations*. In this section we will derive these equations for the LT and the DT. We study mostly the example of the Goursat equation, but the approach is directly reformulated for the Moutard equation.

Let us consider the LTs (5.2). The invariance of the reduction constraint (5.8) means

$$\lambda_{-1} = \lambda - \frac{1}{2}\partial_x\partial_y \ln\lambda = \frac{C}{2\lambda}, \quad C = \text{const}. \tag{5.11}$$

It is obvious that (5.11) is valid for the LT (5.3) as well because the last one is inverse to (5.2).

The reduction equation for this transformation is the well-known sinh–Gordon equation

$$\partial_x \partial_y \ln \lambda = 2\lambda - \frac{C}{\lambda}, \tag{5.12}$$

and the new potential λ_{-1} is a solution of (5.12) too. In the case $C = 0$ we obtain $\lambda_{-1} = 0$ and the Liouville equation, instead of (5.12). The general integral for the Liouville equation is well known:

$$\lambda = \frac{f'g'}{(f+g)^2},$$

where $f = f(x)$ and $g = g(y)$ are arbitrary differentiable functions. The Goursat equation is integrated as

$$\zeta = -\frac{1}{C_1^2}\partial_y \ln(f+g) + V, \quad C_1 = \text{const}.$$

The function $V = V(y)$ is determined by the equation

$$V' = \left(\frac{1}{2C_1}(\ln g')'\right)^2 = \frac{1}{4C_1^2}\left(\frac{g''}{g'}\right)^2$$

and

$$\psi = \frac{\sqrt{f'g'}}{C_1(f+g)}, \qquad \chi = \frac{1}{2C_1}\partial_y \ln\left(-\partial_y \frac{1}{f+g}\right).$$

Proposition 5.1. *Let M and L be two Laplace invariants of (5.1). This means that*

$$M = \frac{1}{2}\partial_x \partial_y \ln \lambda - \lambda, \qquad L = -\lambda.$$

Using the reduction equation (5.12) yields

$$M = -\frac{C}{2\lambda}, \qquad L = -\lambda$$

and

$$M_{-1} = M_1 = L, \qquad L_{-1} = L_1 = M.$$

Now we take the DT (5.4). Inserting both transforms into the reduction condition (5.8), we get

$$\lambda_1 = \lambda - \sigma_y = \lambda\left(\sigma - \frac{\lambda_x}{2\lambda}\right). \tag{5.13}$$

Denote $\alpha = \ln \phi$ and $\Lambda = \ln \lambda$. Since

$$\lambda - \sigma_y = \left(-\frac{1}{2}\Lambda_x + \alpha_x\right)\alpha_y$$

and $\sigma = \alpha_x$, we obtain from the transform (5.13) the condition for Λ:

$$\left(\alpha_x - \frac{1}{2}\Lambda_x\right)\left[\alpha_y - \exp(\Lambda)\left(\alpha_x - \frac{1}{2}\Lambda_x\right)\right] = 0. \tag{5.14}$$

Equating to zero the first parentheses yields

$$\Lambda_{xy} = 2\exp(\Lambda)$$

and $\alpha = \Lambda/2 - c(y)$, where $c(y)$ is an arbitrary function. But in this case we get $\lambda_1 = 0$, and the Liouville equation is in the realm of the reduction equation.

Equating to zero the brackets in (5.14), we arrive at the equation

$$[\exp(-2\alpha)\lambda]_x = [\exp(-2\alpha)]_y\ ; \tag{5.15}$$

therefore,

$$\theta_x = \psi^2 = \frac{1}{F_x + C_2}, \qquad \lambda = \frac{F_y + C_1}{F_x + C_2},$$

where $F = F(x, y)$ is any differentiable function and $C_{1,2} = \text{const}$. Substituting (5.15) into (5.10) yields

$$\begin{aligned} &2(C_2 + F_x)C_1^2 + [(F_{yxx} + 4F_y)C_2 + F_xF_{yxx} + 4F_yF_x - F_{xx}F_{yx}]\,C_1 + 2F_y^2F_x \\ &+ \left(F_{yxx}F_y - \frac{1}{2}F_{yx}^2 + 2F_y^2\right)C_2 - \frac{1}{2}F_{yx}^2F_x - F_yF_{xx}F_{yx} + F_xF_yF_{yxx} = 0. \end{aligned} \tag{5.16}$$

Define new fields P and Q as

$$F_x = P - C_2, \qquad F_y = Q - C_1.$$

Then (5.16) can be split into the system

$$2Q_xQP_x - (2Q_{xx}Q - Q_x^2 + 4Q^2)P = 0, \qquad P_y = Q_x. \tag{5.17}$$

After integration of the first equation we get

$$P = \frac{C_3Q_x}{\sqrt{Q}}\exp G, \qquad G_x = 2\frac{Q}{Q_x},$$

where C_3 is the third constant of integration. It is necessary to obey the second equation in (5.17). Let

$$Q = n^2(x, y), \qquad G = \ln m(x, y).$$

Then the reduction equation is simplified:

$$\left(n^2\right)_x = 2C\left(mn_x\right)_y, \qquad m_xn_x = mn. \tag{5.18}$$

This system can be rewritten in more convenient form. Let

$$n_x = n\exp S, \qquad m_x = m\exp(-S),$$

$S = S(x, y)$. After substituting into (5.18) we get

$$S_y = \frac{1}{C}\frac{n}{m} - \partial_y \ln(mn);$$

therefore,

$$S_{xy} = 4(\sinh S)\,\partial_y \partial_x^{-1} \cosh S. \tag{5.19}$$

Equation (5.19) is the reduction equation for the DT (5.4). It looks like (5.12) and it is a generalization of the $d = 2$ sinh–Gordon equation. The Lax pair for (5.19) is introduced by means of the following:

Proposition 5.2. *The (L, A) pair for (5.19) is written as*

$$K\psi = 0, \qquad K_1 D\psi = 0,$$

where

$$D = \partial_x - \sigma, \quad K = \partial_x\partial_y - \frac{1}{2}\frac{\lambda_x}{\lambda}\partial_y - \lambda, \qquad K_1 = \partial_x\partial_y - \frac{1}{2}\frac{\lambda_{1,x}}{\lambda_1}\partial_y - \lambda_1,$$

and the variables λ and λ_1 are determined by

$$\lambda = \frac{(S_x + 2\cosh S)_y}{4\sinh S}\exp(-S), \qquad \lambda_1 = \frac{(S_x + 2\cosh S)_y}{4\sinh S}\exp S, \tag{5.20}$$

and $\sigma_y \equiv \lambda - \lambda_1$.

This statement is checked by direct substitution. Thus, the reduction equations for the DT (5.4) have either the form of (5.19) or the form of the Liouville equation.

The reduction equations for the DT (5.5) are obtained similarly. As a result, we get

$$\lambda = C_1\phi_y \exp F, \qquad {}_1\lambda = -\frac{C_1C_2\phi^2}{\phi_y}\exp F, \tag{5.21}$$

where ϕ is the support function of the DT (5.5) and the reduction equation can be written in the form of a system

$$\phi_{xy} = \phi_y[F_x + 2C_1\phi\exp F], \qquad F_y\phi_y = C_2\phi.$$

Proposition 5.3. *By the construction (5.20) for the DT (5.4) we get*

$$M = -\lambda_1, \qquad L = -\lambda$$

and

$$M_1 = M\exp(-2S), \qquad L_1 = L\exp(2S).$$

Similarly for the DT (5.5) the use of (5.21) gives

$$M = -\frac{C_2(-\phi_x + \phi F_x + C_1\phi^2 \exp F)}{\phi_y}, \qquad L = -C_1\phi_y \exp F$$

and

$${}_1M = -\frac{\phi_y^2}{C_2\phi^2}M, \qquad {}_1L = -\frac{C_2\phi^2}{\phi_y^2}L.$$

The product of the Laplace invariants ML is invariant in both cases. Combinations of LT and DT generate new equations and their Lax pairs.

5.1.3 Goursat equation, geometry, and two-dimensional MKdV equation

As shown in Sect. 5.1.2, the Goursat equation (5.9) is connected to the particular case of (5.1) with two potentials $a = a(x, y)$ and $b = b(x, y) = \lambda(x, y)$. We refer to λ as the potential function. The reduction (5.8) is valid only for special types of potentials if the form of the Laplace equation is maintained while transformations are performed. Our interest in the Goursat equation is caused by applications of this equation in geometry and in the soliton theory:

1. As regards geometry, let x be the complex coordinate, $y = -\overline{x}$, $\sqrt{\lambda}$ is the real-valued function, and ψ or χ as solutions of (5.10) are complex-valued functions. Then we define three real-valued functions X_i, $i = 1, 2, 3$ which are the coordinates of a surface in $\mathbb{R}^3$ [242]:

$$\begin{aligned} X_1 + \mathrm{i}X_2 &= 2\mathrm{i}\int_\Gamma \left(\overline{\psi^2}\mathrm{d}y' - \overline{\chi^2}\mathrm{d}x'\right), \\ X_1 - \mathrm{i}X_2 &= -2\mathrm{i}\int_\Gamma \left(\psi^2\mathrm{d}y' - \chi^2\mathrm{d}x'\right), \\ X_3 &= -2\int_\Gamma \left(\overline{\psi}\chi\mathrm{d}y' + \overline{\chi}\psi\mathrm{d}x'\right), \end{aligned} \tag{5.22}$$

   where Γ is an arbitrary path of integration in the complex plane. The corresponding first fundamental form, the Gaussian curvature K, and the mean curvature H yield:

$$\mathrm{d}s^2 = 4U^2\mathrm{d}x\mathrm{d}y, \qquad K = \frac{1}{U^2}\partial_x\partial_y \ln U, \qquad H = \frac{\sqrt{\lambda}}{U}.$$

   Here $U = \mid \psi \mid^2 + \mid \chi \mid^2$ and any analytic surface in $\mathbb{R}^3$ can be globally represented by (5.22) [244].

2. As an example of soliton equations, consider the system of the two-dimensional MKdV equations introduced by Boiti, Leon, Martina, and Pempinelli [58, 65]:

$$4\lambda^2(\lambda_t - A\lambda_x + B\lambda_y - \lambda_{xxx} - \lambda_{yyy}) + 4\lambda^3\left[(2\lambda + B)_y + (2\lambda - A)_x\right] +$$

$$+6\lambda(\lambda_y\lambda_{yy} + \lambda_x\lambda_{xx}) - 3(\lambda_x^3 + \lambda_y^3) = 0,$$

$$B_x = 3\lambda_y - \lambda_x, \qquad A_y = \lambda_y - 3\lambda_x. \tag{5.23}$$

Here $\lambda = \lambda(x, y, t)$, $A = A(x, y, t)$, and $B = B(x, y, t)$. If we introduce the function $u = \sqrt{\lambda}$, then we can rewrite (5.23) in the more customary form

$$u_t + 2u^2(u_x + u_y) + \tfrac{1}{2}\left(B_y - A_x\right)u + Bu_y - Au_x - u_{3y} - u_{3x} = 0,$$

$$B_x = (3\partial_y - \partial_x)\,u^2, \qquad A_y = (\partial_y - 3\partial_x)\,u^2. \tag{5.24}$$

The reduction conditions $A = -B = -2u^2$ and $u_y = u_x$ lead to the MKdV equation,

$$u_t + 12u^2u_x - 2u_{3x} = 0,$$

(here $u_{3x} \equiv u_{xxx}$) so we call (5.24) the two-dimensional MKdV equations. The two-dimensional MKdV equations (5.24) are the compatibility condition of the linear system comprising (5.10) and

$$\psi_t = \psi_{3x} + \psi_{3y} - \frac{3}{2}\frac{\lambda_y}{\lambda}\psi_{yy} + \left[\frac{3}{4}\left(\frac{\lambda_y}{\lambda}\right)^2 - \lambda - B\right]\psi_y + (A-\lambda)\psi_x + \frac{1}{2}(A_x - \lambda_x)\psi.$$

We will study (5.24) in Sect. 5.6.

Remark 5.4. Zenchuk [477] studied the chains of discrete transformations (5.2)–(5.5) of solutions and potentials in the general case of the linear second-order partial differential equation with two independent variables. Considering the simplest ($k = 2$) closed chains of these transformations, he obtained a novel *integrable* equation

$$\frac{1}{2}S_{xy} - \mathrm{e}^{S} - \mathrm{e}^{-S}\left[C_1 - C_2\partial_x^{-1}\left(\mathrm{e}^{-S}\right)_y\right] = 0,$$

where $C_2 > 0$.

In the present chapter we use the reduction restriction (5.8) as a (weak) condition of closure. In Sect. 5.1.2 we derived a new integrable equation (5.19), the two-dimensional generalization of the sinh–Gordon equation. In the next section we employ the Goursat transformation and the binary Goursat transformation to construct explicit solutions of the Goursat equation. These transformations allow us to obtain new solutions of the Goursat equation without solving the reduction equation. We also discuss the transformation for Laplace invariants.

5.2 Goursat and binary Goursat transformations

An analogy of the Moutard transformation for the Goursat equation was studied by Ganzha [169]. Such a Goursat transformation is valid without a reduction restriction and reduction equations. Many useful details can be found in the textbook of Ganzha and Tsarev [171], where the transformation is defined via two solutions of (5.9). The transformed function $\psi[1]$ and the potential $\lambda[1]$ are extracted by quadratures [169, 197].

Theorem 5.5. *Let the transform $\psi[1]$ be introduced by the relations*

$$(z_1\psi[1]/\psi_1)_x = z_1(\psi_2/\psi_1)_x, \tag{5.25}$$

$$(z_1\psi[1]/\psi_1)_y = [z_1 z_{1xy} - 2z_{1x}z_{1y}/z_{1xy}](\psi_2/\psi_1)_y,$$

where $z_{1,2}$ are solutions of (5.9) and $\psi_{1,2} = \sqrt{z_{1,2x}}$ solve (5.10). Then $\psi[1]$ is a solution of the (transformed) equation (5.10) with the potential

$$\lambda[1] = \lambda - (\ln z_1)_{xy}$$

and the transform $z[1]$ is found by a quadrature from

$$z[1]_x = \psi^2[1], \qquad z[1]_y = (\psi[1]_y)^2/\lambda[1]. \tag{5.26}$$

This transformation preserves the form of the Laplace–Goursat equation (5.10), e.g., possesses the covariance property. Below we introduce a twofold eDT for the Goursat equaton with the same property.

We introduce new variables $\xi = x + y$ and $\eta = x - y$ and rewrite (5.10) in matrix form,

$$\Psi_\eta = \sigma_3\Psi_\xi + U\Psi. \tag{5.27}$$

Here

$$\Psi = \begin{pmatrix} \psi_1 & \psi_2 \\ \chi_1 & \chi_2 \end{pmatrix}, \qquad U = \sqrt{\lambda}\sigma_1, \tag{5.28}$$

where $\psi_k = \psi_k(\xi, \eta)$ and $\chi_k = \chi_k(\xi, \eta)$, $k = 1, 2$ are particular solutions of (5.10) with some $\lambda(\xi, \eta)$, and $\sigma_{1,3}$ are the Pauli matrices. Let Ψ_1 be some solution of (5.27) and $\Psi \neq \Psi_1$. We define a matrix function $\sigma \equiv \Psi_{1,\xi}\Psi_1^{-1}$. Equation (5.27) is covariant with respect to the classical DT:

$$\Phi[1] = \Phi_\xi - \sigma\Phi, \qquad U[1] = U + [\sigma_3, \sigma]. \tag{5.29}$$

It is a particular case of the general classical non-Abelian formula from Chap. 2, the Matveev Theorem 2.19.

Remark 5.6. It is not difficult to check that the DT (5.29) is the superposition formula for two simpler DTs given by (5.4) and (5.5).

Remark 5.7. Equation (5.27) is the spectral problem for the DS equation [13, 277]. The LT produces explicitly invertible Bäcklund autotransformations for the DS equation. It is shown in [459] that these transformations permit solutions to the DS equation to be constructed that fall off in all directions in the plane according to exponential and algebraic laws.

Next we consider a closed 1-form

$$\mathrm{d}\Omega = \mathrm{d}\xi\, \Phi\Psi + \mathrm{d}\eta\, \Phi\sigma_3\Psi, \qquad \Omega = \int \mathrm{d}\Omega,$$

where a 2×2 matrix function Φ solves the equation

$$\Phi_\eta = \Phi_\xi \sigma_3 - \Phi U. \tag{5.30}$$

Let us apply the DT. It can be verified by immediate substitution that (5.30) is covariant with respect to the transformation

$$\Phi[+1] = \Omega(\Phi, \Psi_1)\Psi_1^{-1}.$$

We can alternatively affect U (5.28) by the following transformation:

$$U[+1,-1] = U + [\sigma_3, \Psi_1 \Omega^{-1} \Phi].$$

The particular solution of (5.30) has the form

$$\Phi_1 = \begin{pmatrix} s_1\psi_1 + s_2\psi_2 & -s_1\chi_1 - s_2\chi_2 \\ s_3\psi_1 + s_4\psi_2 & -s_3\chi_1 - s_4\chi_2 \end{pmatrix}, \tag{5.31}$$

where $s_k = \mathrm{const}$ $(k = 1, \ldots, 4)$. It is convenient to choose Φ_1 in the form

$$\Phi_1 = \Psi_1^{\mathrm{T}} \sigma_3, \tag{5.32}$$

where the superscript T stands for the transpose. Equation (5.32) is the particular case of (5.31). In this case

$$U[+1,-1] = U - 2A_F, \tag{5.33}$$

where A_F is the off-diagonal part of the matrix $A = \Psi_1 \Omega^{-1} \Psi_1^{\mathrm{T}}$, $\Omega = \Omega(\Phi_1, \Psi_1)$ and

$$A_F^{\mathrm{T}} = A_F = f\sigma_1. \tag{5.34}$$

Here $f = f(\xi, \eta)$ is some function. Using (5.29), (5.33), and (5.34), we see that $U[+1,-1]$ has the same form as for the initial matrix U,

$$U[+1,-1] \equiv \begin{pmatrix} 0 & \sqrt{\lambda[+1,-1]} \\ \sqrt{\lambda[+1,-1]} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \sqrt{\lambda} - 2f \\ \sqrt{\lambda} - 2f & 0 \end{pmatrix};$$

thus, the reduction restriction is valid without the reduction equations.

The new function $\Phi[+1,-1]$ has the form

$$\Phi[+1,-1] = \Phi - \Omega(\Phi,\Psi_1)\,[\Omega(\Phi_1,\Psi_1)]^{-1}\,\Phi_1, \tag{5.35}$$

where Φ is an arbitrary solution of (5.30).

Using the twofold DT (5.33) and (5.35), we can construct a new solution of the Goursat equaton by means of dressing a particular solution. As a result, we get the following theorem (returning to the former variables x and y):

Theorem 5.8. *Let*

$$\psi_{k,y} = \sqrt{\lambda}\chi_k, \qquad \chi_{k,x} = \sqrt{\lambda}\psi_k,$$

$$\alpha_{k,y} = -\sqrt{\lambda}\beta_k, \qquad \beta_{k,x} = -\sqrt{\lambda}\alpha_k,$$

where $k = 1, 2$. Then new functions

$$\alpha_1' = \alpha_1 - \frac{A_1\psi_1 + A_2\psi_2}{D}, \qquad \beta_1' = \beta_1 + \frac{A_1\chi_1 + A_2\chi_2}{D}$$

are solutions of the equations

$$\alpha'_{1,y} = \sqrt{\lambda'}\,\beta_1', \qquad \beta'_{1,x} = \sqrt{\lambda'}\,\alpha_1',$$

where

$$\sqrt{\lambda'} = -\sqrt{\lambda} + \frac{\psi_1\chi_1\Omega_{22} + \psi_2\chi_2\Omega_{11} - (\psi_1\chi_2 + \psi_2\chi_1)\Omega_{12}}{D}$$

and

$$\Omega_{11} = \int \mathrm{d}x\psi_1^2 + \mathrm{d}y\chi_1^2, \qquad \Omega_{12} = \Omega_{21} = \int \mathrm{d}x\psi_1\psi_2 + \mathrm{d}y\chi_1\chi_2,$$

$$\Omega_{22} = \int \mathrm{d}x\psi_2^2 + \mathrm{d}y\chi_2^2, \qquad D = \Omega_{11}\Omega_{22} - \Omega_{12}^2,$$

$$\Lambda_{11} = \int \mathrm{d}x\alpha_1\psi_1 + \mathrm{d}y\beta_1\chi_1, \qquad \Lambda_{12} = \int \mathrm{d}x\alpha_1\psi_2 + \mathrm{d}y\beta_1\chi_2,$$

$$\Lambda_{21} = \int \mathrm{d}x\alpha_2\psi_1 + \mathrm{d}y\beta_2\chi_1, \qquad \Lambda_{22} = \int \mathrm{d}x\alpha_2\psi_2 + \mathrm{d}y\beta_2\chi_2,$$

$$A_1 = \Lambda_{11}\Omega_{22} - \Lambda_{12}\Omega_{12}, \qquad A_2 = \Lambda_{12}\Omega_{11} - \Lambda_{11}\Omega_{12}.$$

Here $\int = \int_\Gamma$, where Γ is an arbitrary path of integration in the plane. The explicit expressions for the functions α_2' and β_2' are obtained by the direct picking up of the relations indicated.

Thus the twofold eDT allows us to construct explicit solutions of the Goursat equation without solving the reduction equation.

5.3 Moutard transformation

The Moutard transformation [340, 341] is a map of the DT type: it connects solutions and the coefficient $u(x, y)$ of the equation (5.7) so that if φ and ψ are different solutions of (5.7), then the solution of the twin equation with $\psi \to \psi[1]$ and $u(x, y) \to u[1](x, y)$ can be constructed by the solution of the system

$$(\psi[1]\varphi)_x = -\varphi^2(\psi\varphi^{-1})_x, \qquad (\psi[1]\varphi)_y = \varphi^2(\psi\varphi^{-1})_y.$$

In other terms,

$$\psi[1] = \psi - \varphi\Omega(\varphi, \psi)/\Omega(\varphi, \varphi), \tag{5.36}$$

where Ω is the integral of the exact differential form

$$\mathrm{d}\Omega = \varphi\psi_x \mathrm{d}x + \psi\varphi_y \mathrm{d}y. \tag{5.37}$$

The transformed coefficient (potential in mathematical physics) is given by

$$u[1] = u - 2(\log\varphi)_{xy} = -u + \varphi_x\varphi_y/\varphi^2.$$

The proof is straightforward; see [298] for details.

The important feature of the Moutard transformation is general for the DTs: the transform is parameterized by a pair of solutions of the equation and the transform vanishes if the solutions coincide. The Moutard equation is obviously transformed to the two-dimensional Schrödinger equation and studied in connection with the central problems of classical differential geometry [197].

In the soliton theory the Moutard equation enters the Lax pairs for nonlinear equations such as the KP equation [35, 168, 298, 430] (see Chaps. 9, 10 for more details).

5.4 Iterations of Moutard transformations

Analysis of the iteration sequences for the transformations of the form (5.36), where, in accordance with (5.37),

$$\Omega(\varphi, \varphi) = \int \Delta_2(\varphi, \varphi)\mathrm{d}x_i + c_\phi = \phi^2/2 \tag{5.38}$$

by the appropriate choice of the constant c_ϕ, is performed similarly to the algorithm given in [324] for the classical DT. Suppose the result of N iterations is a linear combination of the integrals $\Omega(\varphi_i, \psi)$ of (5.37):

$$\psi[N] = \psi + \sum_i s_i\Omega(\varphi_i, \psi). \tag{5.39}$$

This formula is proved by induction. The main property of the Mourtard transformation can be written as

$$\varphi_k + \sum_i s_i \Omega(\varphi_i, \varphi_k) = 0 \tag{5.40}$$

and gives

$$s_i = \Delta_i / \Delta \tag{5.41}$$

by Kramer's rule. Denoting

$$\Omega_i \equiv \Omega(\varphi_i, \psi) \quad \text{and} \quad \Omega_{ik} \equiv \Omega(\varphi_i, \varphi_k),$$

we get $\Delta = \det[\Omega_{ik}]$, and Δ_i is obtained from Δ by the known rule of action with the ith row. Hence, the results of the iterations can be presented in the compact determinant form as in the classical Crum case [324].

Differentiating (5.39) yields

$$\psi_{xy}[N] = \psi_{xy} + (s_i \Omega_i)_{xy} = -u[N]\psi[N] \tag{5.42}$$

$$= -u\psi + (s_{ix}\Omega_i + s_i \Omega_{ix})_y = -u[N](\psi + s_i \Omega_i),$$

and using the definition of the determinant Δ together with the properties $\Omega_{ix} = \varphi_i \psi_x$, $s_{ix} = -s_i \ln_x \varphi_i$ gives the DT for the iterated potential

$$u[N] = u + 6(\ln \Delta)_{xx}, \tag{5.43}$$

that is used for multikink (see the next section) and multidromions [145, 146] construction.

5.5 Two-dimensional KdV equation

Applications of the Moutard transformations for solution of the KP and DS equations are well known [324]; for the Nizhnik–Veselov–Novikov equation see [278]. Here we follow [145] concerning the equation

$$m_{ty} = (m_{xxy} + m_y m_x)_x, \tag{5.44}$$

which is the 2+1 version [281] of the KdV-like Hirota–Satsuma equation [211]. Equation (5.44) was integrated by inverse spectral transform in [58, 65]. Details of multisoliton (multikink) construction and asymptotic behavior are given in the next section. We also use this example in Sect. 7.3 to show how the singular manifold method generates the Moutard transformation.

5.5.1 Moutard transformations

Here we consider the asymptotic behavior of iterated solutions and the simplest example of repeated iterations from the zero seed potential that demonstrates the interaction of kinks. The formula for the N-times iterated solution is

$$m = 6(\ln \Delta)_x, \tag{5.45}$$

where, again, $\Delta = \det[\Delta_{ik}]$ and, like [277], the one-step transform was performed,

$$\Delta_{ik} = \int d\Omega(\phi_k, \phi_i) + C_{ik}, \quad C_{ik} + C_{ki} = \phi_k(0)\phi_i(0),$$

$$\Omega(\phi_k, \phi_i) = -2\int [\delta_1 \mathrm{d}x + \delta_2 \mathrm{d}y + \delta_3 \mathrm{d}t], \tag{5.46}$$

$$\delta_1 = \phi_k \phi_{ix}, \quad \delta_2 = \phi_{ky}\phi_i, \quad \delta_3 = \phi_k\phi_{it} - \phi_{kx}\phi_{ixx} + \phi_{kxx}\phi_{ix}.$$

This way we fix the constants of integration. A similar combination of solutions leads to multidromions [145], the localized solitons in two dimensions (first appeared in [62]).

5.5.2 Asymptotics of multikink solutions of two-dimensional KdV equation

To demonstrate the possibilities of the technique in 2+1 dimensions, we consider the example of kink interaction and choose the seed Lax pair solution as

$$\phi_k = A_k \exp(a_k x + a_k^3 t) + B_k \exp(b_k y). \tag{5.47}$$

Introducing the notations

$$\alpha_{ik} = \frac{a_i}{a_i + a_k}, \qquad \beta_{ik} = \frac{b_i}{b_i + b_k},$$

$$\xi_k = a_k x + a_k^3 t, \qquad \xi_{i0} = a_i x_0 + a_i^3 t_0, \qquad A_i/B_i = p_i,$$

we perform integration from x_0, y_0, t_0 to x, y, t and obtain

$$\Delta_{ik} = C_{ik} + \alpha_{ik} p_i p_k [\exp(\xi_i + \xi_k) - \exp(\xi_{i0} + \xi_{k0})] + \tag{5.48}$$

$$+ p_i \left[\exp(\xi_i + b_k y) - \exp(\xi_{i0} + b_k y_0)\right] + \beta_{ik} \left[\exp(b_i + b_k)y - \exp(b_i + b_k)y_0\right].$$

We would stop at kinks within the choice $a_i > 0$, $b_i > 0$ for $x_0, y_0, t_0 \to -\infty$; hence,

$$\Delta_{ik} = [\alpha_{ik} p_i p_k \exp(\chi_i + \chi_k) + p_i \exp(\chi_i) + \beta_{ik}] \exp[(b_i + b_k)y] + C_{ik}, \tag{5.49}$$

where $\chi_i = a_i x + a_i^3 t - b_i y$. Notice that it is impossible to represent Δ_{ik} as a sum of two exponents with the opposite powers like for the multisoliton determinant representation for the KP equation [324]. Hence, we should develop an asymptotic calculation technique.

Let us consider the case

$$0 < \mathrm{Re}(a_1^2) < \ldots < \mathrm{Re}(a_N^2)$$

and go to the reference frame of the sth kink that means fixing the phase χ_s. Running at the level y, we shall derive the asymptotic at $t \to \pm\infty$. We shall also put $C_{ik} = 0$ and $\Delta'_{ik} = \Delta_{ik} \exp(b_i + b_k)y$ and account for the relation $(\ln \Delta)'_x = (\ln \Delta)_x$. Finally, let us investigate

$$\Delta'_{ik} = \alpha_{ik} \exp(\chi'_i + \chi'_k) + \exp(\chi'_i) + \beta_{ik}, \tag{5.50}$$

where

$$x = -a_s^2 t + b_s y/a_s + \chi_s/a_s, \tag{5.51}$$

$$\chi'_k = a_k(a_k^2 - a_s^2)t + (a_k b_s/a_s - b_k)y + \chi_k/a_s + ln[p_k].$$

Therefore, at $t \to \infty$ and $\chi_s = \mathrm{const}$,

$$\chi_k = \begin{cases} -\infty, k < s \\ +\infty, k > s \end{cases}$$

and the elements of the determinant matrix have the following asymptotic values:

1. $\Delta_{ik} \to \beta_{ik}, \quad i, k < s$
2. $\Delta_{ik} \to \alpha_{ik} \exp(\chi'_i + \chi'_k), \quad i, k > s$
3. $\Delta_{ik} \to \exp \chi'_i, \quad i > s, \quad k < s$
4. $\Delta_{ik} \to \alpha_{ik} \exp(\chi'_i + \chi'_k) + \beta_{ik}, \quad i < s, \quad k > s$

It can be shown that only the first term contributes to the determinant asymptotic. We list below the special cases:

$i = s$
$k < s, \quad \Delta_{sk} = \exp(\chi_s) + \beta_{sk},$
$k = s, \quad \Delta_{ss} = [\alpha_{ss} \exp(\chi_s) + 1] \exp(\chi_s) + \beta_{sk},$
$k > s, \quad \Delta_{sk} = \alpha_{sk} \exp(\chi'_s + \chi'_k).$

$k = s$
$i < s, \quad \Delta_{is} = \beta_{is},$
$i > s, \quad \Delta_{is} = \alpha_{is}[\exp(\chi_s) + 1] \exp(\chi_i) \exp(\chi_i).$

It is convenient to present the explicit form of the determinant via the supermatrix

	$k<s$	$k=s$	$k>s$
$i<s$	β_{ik}	β_{ik}	$\alpha_{ik}\exp(\chi'_i+\chi'_k)$
$i=s$	$\Delta_{sk}=\exp(\chi_s)+\beta_{sk}$	Δ_{ss}	$\alpha_{sk}\exp(\chi_s+\chi_k)$
$i>s$	$\exp(\chi_i)$	$\exp(\chi_i)[\alpha_{is}\exp(\chi_s)+1]$	$\alpha_{ik}\exp(\chi_i+\chi_k)$

In this asymptotic determinant it is possible to extract $\exp\chi$ from rows $i>s$ and from columns $k>s$, i.e.,

$$\Delta=\exp\left(\sum_{i=1}^{n}\chi_i-\chi_s\right)\Delta_1. \tag{5.52}$$

Then

$$\Delta_1=\begin{vmatrix} \beta_{ik} & \beta_{is} & 0 \\ \exp(\chi_s)+\beta & \Delta_{ss} & \alpha_{sk}\chi_s \\ 1 & \alpha_{is}\exp(\chi_s)+1 & \alpha_{ik} \end{vmatrix},$$

where 0 and 1 are matrices with zero and unit elements. Obviously, it follows from (5.45) that

$$m=\left[\sum_{i=1}^{n}\chi_i-\chi_s\right]_x+(\ln\Delta_1)_x=\sum_{i=1}^{n}a_i-a_s+(\ln\Delta_1)_x. \tag{5.53}$$

A Lagrange expansion by the row of number s,

$$[0,\dots,0,\alpha_{ss}\exp(2\chi_s),0,\dots,0]+[1,\dots,1,\alpha_{s,s+1}\exp(\chi_{s+1}),1,\dots,1]+\dots$$

$$+(\beta_{s1},\dots,\beta_{ss},0,\dots,0)$$

allows us to present the result for the asymptotic Δ_1 in a "kink" form:

$$\begin{aligned}\Delta_1=\alpha_{ss}\exp(2\chi_s)&\begin{vmatrix} \beta_{ik} & \beta_{is} & 0 \\ 0 & 1 & 0 \\ 1 & \alpha_{is}\exp(\chi_s)+1 & \alpha_{ik} \end{vmatrix} \\ +\exp(\chi_s)&\begin{vmatrix} \beta_{ik} & \beta_{is} & 0 \\ 1 & 1 & \alpha_{sk} \\ 1 & \alpha_{is}\exp(\chi_s)+1 & \alpha_{ik} \end{vmatrix}+\begin{vmatrix} \beta_{ik} & \beta_{is} & 0 \\ \beta_{sk} & \beta_{ss} & 0 \\ 1 & \alpha_{is}\exp(\chi_s)+1 & \alpha_{ik} \end{vmatrix}.\end{aligned} \tag{5.54}$$

The first determinant is arranged via a sum of the columns with the number s terms:

$$\begin{vmatrix} \beta_{ik} \\ 1 \\ 1 \end{vmatrix}+\begin{vmatrix} 0 \\ 0 \\ \alpha_{is}\exp(\chi_s) \end{vmatrix}.$$

The second determinant is zero because it has a zero row. Finally,

$$\Delta^a=\exp(2\chi_s)(\alpha_{ss}\Delta_1+\Delta_2)+\exp[\chi_s](\Delta_3+\Delta_4)+\Delta_5, \tag{5.55}$$

where

$$\Delta_1 = \begin{vmatrix} \beta_{ik} & \beta_{is} & 0 \\ 0 & 1 & 0 \\ 1^0 & 1^0 & \alpha_{ik} \end{vmatrix}, \qquad \Delta_2 = \begin{vmatrix} \beta_{ik} & 0 & 0 \\ 1^0 & 1 & \alpha_{sk} \\ 1^0 & \alpha_{is} & \alpha_{ik} \end{vmatrix}, \tag{5.56}$$

$$\Delta_3 = \begin{vmatrix} \beta_{ik} & \beta_{is} & 0 \\ 1^0 & 1 & \alpha_{sk} \\ 1^0 & 1^0 & \alpha_{ik} \end{vmatrix}, \qquad \Delta_4 = \begin{vmatrix} \beta_{ik} & \beta_{is} & 0 \\ \beta_{sk} & \beta_{ss} & \alpha_{sk} \\ 1^0 & \alpha_{is} & \alpha_{ik} \end{vmatrix}, \tag{5.57}$$

$$\Delta_5 = \begin{vmatrix} \beta_{ik} & \beta_{is} & 0 \\ \beta_{sk} & \beta_{ss} & 0 \\ 1^0 & 1^0 & \alpha_{ik} \end{vmatrix}. \tag{5.58}$$

The determination of the phase of the sth kink is performed in the following way. If we introduce the phase χ and rewrite Δ^a as

$$\Delta^a = (\exp\chi + a)^2 + b, \tag{5.59}$$

then

$$m = (\ln\Delta^a)_x = \Delta^a_x/\Delta^a = 2[\exp\chi + a]\alpha/[(\exp\chi + a)^2 + b], \tag{5.60}$$

where $\alpha = \chi_x$. As a result,

$$m = 0, \qquad \chi \to \infty, \tag{5.61}$$

$$m = 2a\alpha/(a^2 + b), \qquad \chi \to -\infty. \tag{5.62}$$

Equating powers of exponential terms,

$$2\chi = 2\chi_k + \ln(\alpha_{ss}\Delta_1 + \Delta_2), \tag{5.63}$$

$$2a\exp\left[\frac{1}{2}\ln(\alpha_{ss}\Delta_1 + \Delta_2)\right] = \Delta_3 + \Delta_4, \tag{5.64}$$

$$a^2 + b = \Delta_s, \tag{5.65}$$

we immediately determine the phase χ and asymptotic value of the sth kink taking into account (5.53), (5.60), and (5.45).

Concluding, though this note is rather technical, it contains ideas about a development of asymptotic construction in the "dromionic" case of 2+1 equations, as well as symmetry reductions of explicit solutions or the two-step equation reduction. It follows from Sects. 5.1 and 5.2 that there exists a direct possibility to construct solutions of (5.10) or (5.7) via forms like (5.37). More general asymptotic behavior can be analyzed similarly. For example, equating the phases of (5.53) and (5.48) and linear combinations of ξ and η of the form (5.50) and (5.51) with $Y = \text{const}$ yields

$$a_i x + a_i^3 t - b_i y = A_i\xi + B_i\eta,$$

$$A_i = a_i c_2, \qquad B_i = a_i c_2 - a_i^3 T = Y b_i.$$

The three-phase solutions are possible with one determinant condition on the parameters a_i and b_i, and so on.

5.6 Generalized Moutard transformation for two-dimensional MKdV equations

In this section we generate solutions of the two-dimensional MKdV equations, giving one more example of efficient applications of the technique which exploits the generalized Moutard transformation.

5.6.1 Definition of generalized Moutard transformation and covariance statement

The Lax pair for the two-dimensional MKdV equations (5.23) has the form

$$\begin{aligned} \psi_{xy} &= \frac{u_x}{u}\psi_y + u^2\psi, \\ \psi_t &= \psi_{xxx} + \psi_{yyy} - 3\frac{u_y}{u}\psi_{yy} + \left[3\left(\frac{u_y}{u}\right)^2 - u^2 - B\right]\psi_y \\ &\quad + \left(A - u^2\right)\psi_x + \frac{1}{2}\left(A - u^2\right)_x\psi. \end{aligned} \tag{5.66}$$

Ganzha [169] studied one type of the Moutard transformation for the Goursat equation. To use this transformation for obtaining exact solutions of (5.23), we should complete the definition of the Moutard transformation. It is easy to do that. Let ϕ be the second solution of (5.66) (the support function). Then we have a closed 1-form

$$\mathrm{d}\theta = \mathrm{d}x\,\theta_1 + \mathrm{d}y\,\theta_2 + \mathrm{d}t\,\theta_3, \qquad \theta = \int \mathrm{d}\theta,$$

where

$$\theta_1 = \phi^2, \qquad \theta_2 = \left(\frac{\phi_y}{u}\right)^2, \qquad \theta_3 = (A - u^2)\phi^2 - \phi_y^2 - \phi_x^2 + 2\phi\phi_{xx} +$$

$$+u^{-4}\left[(2\phi_{3y}\phi_y - \phi_{yy}^2 - B\phi_y^2)u^2 - 2u\phi_y(u_y\phi_y)_y + 3\left(u_y\phi_y\right)^2\right].$$

We define the *generalized* Moutard transformation in the following way:

$$\begin{aligned} &u \to \tilde{u} = u - \sqrt{(\ln\theta)_x(\ln\theta)_y}, \qquad A \to \tilde{A} = A - (\partial_x\partial_y - 3\partial_x^2)\ln\theta, \\ &B \to \tilde{B} = B + (\partial_x\partial_y - 3\partial_y^2)\ln\theta, \qquad \psi \to \tilde{\psi} = \frac{\phi Q}{\theta}, \end{aligned} \tag{5.67}$$

where

$$Q \equiv \int \mathrm{d}Q, \qquad \mathrm{d}Q = dx\,Q_1 + \mathrm{d}y\,Q_2 + \mathrm{d}t\,Q_3,$$

and ($w = \psi/\phi$)

$$Q_1 = \theta w_x, \qquad Q_2 = -\frac{\theta^3 (1/\theta)_{xy} w_y}{\theta_{xy}},$$
$$Q_3 = \theta w_{xxx} + c_1 w_{yyy} + c_2 w_{xx} + c_3 w_{yy} + c_4 w_x + c_5 w_y$$

with

$$c_1 = -\frac{\theta_{xy}}{2u^2} + \theta, \qquad c_2 = \frac{3}{2}\theta (\ln \theta_x)_x - \theta_x,$$
$$c_3 = \frac{u_y \theta_{xy}}{2u^3} + \frac{\phi \phi_{yy}}{u^2} - \frac{3u_y \theta}{u} + 3\left(\frac{\theta}{2}(\ln \theta_x)_y - \theta_y\right),$$
$$c_4 = \left(\frac{3\phi_{xx}}{\phi} + A - u^2\right)\theta - \frac{\theta_{xx}}{2},$$
$$c_5 = -\frac{3u_y^2 \theta_{xy}}{2u^4} + + \frac{1}{u^3}\left(\theta_{xy} u_{yy} + u_y \phi \phi_{yy}\right)$$
$$+ \frac{1}{u^2}\left[3\theta u_y^2 - \phi \phi_{3y} + \frac{1}{2}\left(B - \frac{\phi_{yy}}{\phi}\right)\theta_{xy}\right]$$
$$+ \left(\frac{3\phi_{yy}}{\phi} - B\right)\theta + \frac{u_y}{u}\left(2\theta_y - \frac{3\theta \theta_{xy}}{\theta_x}\right) + \frac{\theta_{xy}}{2} - u^2 \theta.$$

The 1-form dQ is closed,

$$Q_{1,y} = Q_{2,x}, \qquad Q_{1,t} = Q_{3,x}, \qquad Q_{2,t} = Q_{3,y}.$$

It is easy to verify that the (L, A) pair (5.66) is covariant with respect to the generalized Mourtard transformation (5.67).

5.6.2 Solutions of two-dimensional MKdV (BLMP1) equations

Now we use these transformations to construct exact solutions of the two-dimensional MKdV equations (5.24). Let us choose $u = \text{const}$ and $A = B = 0$. We will consider two examples.

1. If we take the solution of (5.66) as $\phi = \sinh \xi$, where

$$\xi = ax + \frac{u^2}{a} y + \frac{(u^2 - a^2)(u^4 - a^4)}{a^3} t \tag{5.68}$$

with real $a = \text{const}$, then using (5.67) we get a new solution of the two-dimensional MKdV equations,

$$\tilde{u} = \frac{u\left[2\eta - a^3 \sinh(2\xi)\right]}{2\eta + a^3 \sinh(2\xi)}, \quad \tilde{A} = \frac{16a^3 \sinh \xi \left[3a^5 \sinh \xi - (u^2 - 3a^2)\eta \cosh \xi\right]}{\left[2\eta + a^3 \sinh(2\xi)\right]^2},$$

$$\tilde{B} = \frac{16av^2 \cosh \xi \left[3a^3 u^2 \cosh \xi - (3u^2 - a^2)\eta \sinh \xi\right]}{\left[2\eta + a^3 \sinh(2\xi)\right]^2},$$

where

$$\eta = a^2(u^2 y - a^2 x) + (u^2 - a^2)(3u^4 + 3a^4 + 2a^2u^2)t \ . \tag{5.69}$$

2. To construct the algebraic solutions of (5.44), we choose the solutions of (5.66) as

$$\phi = (-1)^n \int_\alpha^\beta \mathrm{d}k\zeta(k) \exp[\xi(k)] \frac{\mathrm{d}^n}{\mathrm{d}k^n}\delta(k - k_0),$$

with $\xi(k)$ from (5.68), $a = a(k)$, and $\beta > k_0 > \alpha > 0$, where $\zeta(k)$ is an arbitrary differentiable function. For $n = 1$, $\zeta = 1$ we get

$$\begin{aligned} \tilde{u} &= \frac{u(a^6 - 2\eta^2 - 2a^3\eta)}{2\eta^2 + 2a^3\eta + a^6}, \qquad \tilde{A} = -\frac{8a^6(u^2 + 3a^2)\eta(\eta + a^3)}{(2\eta^2 + 2a^3\eta + a^6)^2}, \\ \tilde{B} &= \frac{8u^2a^4(3u^2 + a^2)\eta(\eta + a^3)}{(2\eta^2 + 2a^3\eta + a^6)^2}, \end{aligned} \tag{5.70}$$

with η from (5.69) and $a = a(k_0)$. Equation (5.70) is a simple nonsingular algebraic solution of the two-dimensional MKdV equations.

There is a group of equations for which the dressing technique is directly applied. The BLMP2 equation is a generalization of the Nizhnik–Veselov–Novikov equation [58]. There is another new integrable equation that is usually called the Boiti–Leon–Pempinelli (BLP) equation. It was proposed and studied in [65]. An integrable generalization of the sine and sinh–Gordon equations in two spatial dimensions was proposed in [64].

6

Applications of dressing to linear problems

The dressing procedures, following our "extended" understanding of this ideology, have been used for years to solve linear problems. In this chapter we concentrate on some recent results obtained along these lines. We observe considerable reciprocal influence of the nonlinear theory on linear methods, in particular via a systematic application of the Lax representation developed previously when studying nonlinear systems.

We show how to dress a seed solution of a one-dimensional second-order linear differential equation when the corresponding operator allows explicit factorization. We also show how the Darboux transformation (DT) theory appears in this framework and produces the so-called integrable or solvable potentials entering linear differential equations. The important and far-reaching example of solvable potentials is represented by the famous regular shape-invariant potentials introduced by Gendenstein [179] (see also [412] and references therein). We could mention as well other classes of potentials, like those obtained by algebraic deformations [190], singular (pointlike of the Coulomb type or zero-range) potentials [284], and matrix potentials on the whole axis or half-axis.

There is an excellent book [367] on applications of the Lax representation to classical mechanics. The integrability is established and exploited by means of the Lie algebra technique. Here we point out some possibilities directly related to the dressing scheme we develop; see also the recent book [368]. Generally, an evolution equation

$$\dot{x_i} = F_i(x), \tag{6.1}$$

can be written in the form of the Lax representation (by means of the L–M pair [367]), so that (6.1) is equivalent to

$$\dot{L} = [M, L]. \tag{6.2}$$

Such an idea of Lax [263] is traced throughout the previous chapters of this book. He showed that the spectrum of the matrix operator L does not depend

on time, when L evolves as $L(t) = UL(0)U^{-1}$, $U^{+}U = \mathbb{1}$ (hence the name *isospectral representation* [338]). Such an approach constitutes the main algorithm of seeking for integrability: eigenvalues λ of the matrix L are conserved quantities ($\dot{\lambda} = 0$). Historically, the Lax representation was found "experimentally" [152].

There are also attempts to apply these ideas to 2+1 dimensions by means of the Moutard transformations [288, 369]. Specific results of the application are still rather poor [30, 290]. In higher dimensions the search was launched by [25] with increased efforts till now [28, 29].

In Sect. 6.1 we introduce the gauge–Darboux transformation (GDT) and the auto-gauge–Darboux transformation (auto-GDT) as a manifestation of the covariance property of the linear equation under consideration. These transformations permit us to derive recurrent relations between solutions of a given equation with different values of the set of parameters. Quantum-mechanical integrable potentials are discussed in Sect. 6.2 from the point of view of dressing. We consider shape-invariant nonsingular potentials of the Schrödinger equation and their algebraic deformations, as well as the Coulomb-like singular potentials and their shape-invariant iterations. A new approach to solve the Schrödinger equation with a zero-range potential (ZRP) is described in Sect. 6.3. We show that dressing of such a potential by means of a special DT improves the ZRP model, especially for low-energy scattering. Further development of this method is illustrated in Sects. 6.4 and 6.5 by solving the problem of multicenter scattering. We perform a detailed analysis of the electron–CH_4 scattering and clarify the nature of the Ramsauer–Townsend minimum in the cross-section spectrum. In Sect. 6.6 we use the dressing technique to construct Green functions for a wide class of multidimensional differential operators with reflectionless potentials. Finally, in Sect. 6.7 we demonstrate the possibility to construct supersymmetric quantum-mechanical potentials with a preassigned discrete spectrum by means of the DTs. We explicitly manage the spectrum by deleting or adding energy levels.

6.1 General statements

In Sect. 2.4 general dressing formulas for coefficients of operators polynomial in D were derived. In Sect. 3.1 the origin of the DT and gauge transformations (GT) were discussed. We outline now the algorithm of eigenfunction construction on the basis of these results. The theory goes back to the book [324], where the simplest case of a quantum harmonic oscillator is discussed from this point of view. The combined GDT was introduced in [466], where the covariance theorem (including dressing formulas for potentials) for a wide class of operators was proved. A development of this technique was given in [381]. Recall that in Sect. 2.11 a combination of GT and DT was applied to solve a linear differential-difference problem that enters the Lax pair for the Nahm equations.

6.1.1 Gauge–Darboux and auto-gauge–Darboux transformations

Following [466] and [380], we consider the linear equation

$$\psi_{xx}+p(x;\nu_1,...,\nu_n)\psi_x+q(x;\nu_1,...,\nu_n)\psi=\lambda h(x;\nu_1,...,\nu_n)\psi+f(x;\nu_1,...,\nu_n). \tag{6.3}$$

The covariance of this problem with respect to a substitution (we call it a GDT)

$$\psi\to\tilde{\psi}=s(\psi_x-\sigma\psi), \tag{6.4}$$

where s and σ depend on x, means a form invariance of (6.4) with the only change of the coefficients in accordance with

$$h\to\tilde{h}=h,\qquad p\to\tilde{p}=p-[\ln(hs^2)]_x,$$

$$q\to\tilde{q}=q+p_x-[p+\sigma-(\ln s)_x](\ln h)_x+2\sigma_x+(\ln s)_x[\ln(hs^2)]_x-s_{xx}/s,$$

$$f\to\tilde{f}=sf_x-s\left[\sigma+s(\ln h)_x\right]. \tag{6.5}$$

We call the GDT (6.4) and (6.5) as the auto-GDT if

$$\tilde{p}\,(x;\nu_1,\ldots,\nu_n)=p\,(x;\tilde{\nu}_1,\ldots,\tilde{\nu}_n), \tag{6.6}$$

$$\tilde{q}\,(x;\nu_1,\ldots,\nu_n)=q\,(x;\tilde{\nu}_1,\ldots,\tilde{\nu}_n), \tag{6.7}$$

$$\tilde{f}(x;\nu_1,\ldots,\nu_n)=f(x;\tilde{\nu}_1,\ldots,\tilde{\nu}_n), \tag{6.8}$$

i.e., the GDT action is equivalent to the transformation of the spectral parameters. This notion accumulates the shape-invariant potentials of quantum theory that are discussed in the next section. In [190] the function σ is called the *factorization dressing function*, while in [92] it is referred to as the *superpotential*. The GDT allows us to obtain recurrent relations between solutions with different values of the parameter set. In the particular case of special functions of mathematical physics, these relations are exactly the recurrent relations between them. Further discussion of this subject is given in Chap. 4.

The algorithm of working with the auto-GDT consists mainly of two steps:

1. We solve a differential equation (6.3) for $\lambda=0$ and then, choosing h, find a solution with $\lambda\neq 0$.
2. By means of the transformation (6.5) we go to a solution of the same equation (6.5) with $\lambda=0$ which is a solution of the initial equation with some other set of the parameters $\tilde{\nu}_1,\ldots,\tilde{\nu}_n$.

Consider an example of the Gegenbauer equation

$$G\psi\equiv\frac{d^2\psi}{dx^2}+\frac{(2\mu+1)x}{x^2-1}\frac{d\psi}{dx}-\frac{n(2\mu+n)}{x^2-1}\psi=0.$$

The choice of $h = (x^2-1)^2$ and of the fixed solutions of $G\psi = \lambda h\psi$ as $\phi_{1,2,3} = (x^2-1)^{\beta_{1,2,3}}$ with

$$\beta_1 = n/2, \qquad \beta_2 = -n/2 - \mu, \qquad \beta_3 = 1/2 - \mu,$$

$$\lambda_{1,2} = 4\beta_{1,2}(\beta_{1,2}-1) + 2\beta_{1,2}(2\mu+1), \quad \lambda_3 = 4\mu\beta_3 - n(2\mu+n)$$

forms the GDT operators

$$D_{1,2,3} = (x^2-1)\left(\frac{\mathrm{d}}{\mathrm{d}x} - \frac{2\beta_{1,2,3}x}{x^2-1}\right),$$

by means of (6.4).

The transformed parameters take the form

$$\begin{aligned} \tilde{n}_1 &= n-1, \quad & \mu_1 &= \mu, \\ \tilde{n}_2 &= n+1, \quad & \mu_2 &= \mu, \\ \tilde{n}_3 &= n+1, \quad & \mu_3 &= \mu-1. \end{aligned}$$

The first two GDTs correspond to the known operators that link the Gegenbauer functions.

6.1.2 Chains of shape-invariant superpotentials

The dressing chain equation (Chap. 4) is nothing more than the result of substitution of the Miura link in the DT formula. For convenience we reproduce it here:

$$(\sigma_i + \sigma_{i+1})' = \sigma_i^2 - \sigma_{i+1}^2 + \mu_i - \mu_{i+1}. \tag{6.9}$$

There is a class of potentials of the Schrödinger one-dimensional equation that are obtained by closure of the dressing chain (see Sect. 4.1), i.e., under the condition posed to the j-times iterated function (superpotential) σ_j:

$$\sigma_j = \xi_j a(x) + \eta_j, \qquad j = 0, \pm 1, \ldots. \tag{6.10}$$

It is a kind of the superpotential parameterization. The compatibility of this condition with the chain equation (6.9) yields the condition for the function $a(x)$ that fixes the shape of superpotentials and hence of the potential u_i (here we use notations from Sect. 3.1) by means of the Miura link

$$u_i = \sigma_i' + \sigma_i^2 + \mu_i. \tag{6.11}$$

This procedure introduces one more (spectral) parameter μ_j into the scheme. Now we can write the equation for $a(x)$ in the standard Riccati equation form:

$$a' + \psi a^2 + \phi a + \chi = 0, \tag{6.12}$$

where

$$\psi = \frac{\xi_{j+1}^2 - \xi_j^2}{\xi_{j+1} + \xi_j}, \qquad \phi = \frac{2(\xi_{j+1}\eta_{j+1} - \xi_j\eta_j)}{\xi_{j+1} + \xi_j},$$

$$\chi = \frac{\eta_{j+1}^2 - \eta_j^2 + \mu_{j+1} - \mu_j}{\xi_{j+1} + \xi_j}. \tag{6.13}$$

The differential equation (6.12) with constant coefficients is transformed to the second-order linear equation

$$y'' + Py' + Qy = 0 \tag{6.14}$$

with $P = \phi$ and $Q = \psi\chi$ by the standard substitution (for nonzero ψ)

$$a = \frac{(\ln y)_x}{\psi},$$

which can be used for a classification of the superpotentials. Indeed, the characteristic equation for (6.14) is the quadratic equation, whose solutions

$$\lambda_{1,2} = -\frac{P}{2} \pm \sqrt{P^2/4 - Q}$$

yield the first class superpotentials as solutions of (6.12) with $\lambda_1 \neq \lambda_2$:

$$a = \frac{A\lambda_1 \exp(\lambda_1 x) + B\lambda_2 \exp(\lambda_2 x)}{\psi[A\exp(\lambda_1 x) + B\exp(\lambda_2 x)]}. \tag{6.15}$$

The second class corresponds to the coincidence of the roots of $P^2/4 - Q = 0$, i.e., to the condition for the parameters of the superpotential:

$$4\psi\chi = \phi^2.$$

Inserting here the definitions (6.13) of the Riccati equation coefficients leads to

$$(\xi_{j+1}\eta_j - \xi_j\eta_{j+1})^2 = (\xi_{j+1}^2 - \xi_j^2)(\mu_{j+1} - \mu_j).$$

This relation links the neighbor eigenvalues μ_j. The shape of the potential in this case is given by

$$a = \frac{[\ln(Ax + B)\mathrm{e}^{\lambda x}]_x}{\psi} = \frac{(x + B/A)^{-1} + \lambda}{\psi}. \tag{6.16}$$

Generally the coefficients in (6.14) can be complex; one recognizes the elementary functions in the potential formula

$$u_j = \xi_j a(x)' + [\xi_j a(x) + \eta_j]^2 + \mu_j. \tag{6.17}$$

If the potential in the Schrödinger equation is real, it leads to some conditions for coefficients.

Remark 6.1. There are special cases of $\psi = 0$ and $\phi = 0$:

1. The condition $\psi = 0$ implies $\xi_{j+1} = \pm\xi_j$. We distinguish two possibilities:
   (a) $\xi_{j+1} = \xi_j$ ($\phi = \eta_{j+1} - \eta_j \neq 0$) gives a linear equation for $a(x)$:

$$a' + \phi a + \chi = 0\,;$$

   hence,

$$a = C\exp(-\phi x) - \chi/\phi.$$

   This form corresponds to the Pöschl–Teller potential

$$u = -\phi\xi_j C\exp(-\phi x) + [\xi_j C\exp(-\phi x) - \xi_j\chi/\phi + \eta_j]^2 + \mu_j. \quad (6.18)$$

   (b) $\xi_{j+1} = -\xi_j$ results in the algebraic equation

$$\xi_j a^2 + (\xi_{j+1}\eta_{j+1} - \xi_j\eta_j)a + \eta_{j+1}^2 - \eta_j^2 + \mu_{i+1} - \mu_i = 0$$

   with a constant solution.
2. The second case corresponds to both zero coefficients $\phi = 0$ and $\psi = 0$ and leads to the celebrated harmonic oscillator model $a = \chi x$; see the details in Sect. 4.1.

6.2 Integrable potentials in quantum mechanics

In this section we discuss exactly solvable shape-invariant potentials entering the Schrödinger equation and demonstrate the usefulness of their algebraic deformations. The Coulomb potential is treated as a typical example of the singular shape-invariant potential.

6.2.1 Peculiarities

Applications in quantum mechanics impose additional conditions on transformations of the potentials. The constraints are stipulated by the demand for the potentials to be real and to possess admissible singularities, as well as by the specific definition of a spectrum in a Hilbert space $\mathcal{H}$ of solutions of the Schrödinger equation. Consider first the case of the one-dimensional Schrödinger equation

$$-\frac{1}{2}\psi_{xx} + U(x)\psi = E\psi \quad (6.19)$$

on a line $x \in -\infty, +\infty$ with a potential $U(x)$ and

$$\psi(x, E_n) \in \mathcal{H}$$

for the points E_n of the discrete spectrum [324, 461], while

$$\int_E^{E+\Delta E} \psi(x, E')\mathrm{d}E' \in \mathcal{H}$$

for the continuum spectrum.

A *rigorous* treatment of the DT D_ϕ for (6.19) can be given within the assumption of the semibounded self-adjoint Hamiltonian. Let us also assume that the potential $U(x)$ is bounded from below [190]. As we demonstrated in Chap. 2, the unique factorization of the operator (6.19) leads to the dressed (partner) operator $U[1] = U + 2\sigma_x$, $\sigma = \phi_x/\phi$, $\phi = \phi(x, E_1)$, where $E_1 \neq E$ is an eigenvalue. The partner potential is nonsingular if ϕ is nonvanishing. The correspondence [165] $\psi \to \psi[1]$ maps spectral domains of the operators H and $H[1]$ with the following properties:

1. *Direct transformation*: the auxiliary (factorization [190]) function ϕ is square-integrable and the operator $\partial - \sigma$ transforms each point of the discrete spectrum to a point of the discrete spectrum excluding the lowest one that is removed: the corresponding function goes to zero. The continuum states rest on their places because of the same asymptotic behavior at infinities of the *eigendifferential* $\int_E^{E+\Delta E} \psi(x, E')\mathrm{d}E'$ [155] and its transform $\int_E^{E+\Delta E} \psi[1](x, E')\mathrm{d}E' \in \mathcal{H}$.
2. *Backward transformation*: the inverse *prop* function $\phi(x, E')^{-1}, E' < E_{inf}$, is square-integrable ($E_{inf}$ is the infinium of the spectral values of the Hamiltonian). The prop function is a combination of independent solutions $\phi_\pm$ of (6.19), growing at opposite infinities.
3. *Isospectral transformation*: both ϕ and ϕ^{-1} are not square-integrable; eigenvalues E' associated with eigenfunctions $\phi = \phi_\pm$ are out of the spectrum; the operator D_ϕ acts as an isomorphism.

Further development of the GDT consists in including transformations of the independent variables [190]. Following [190], we will say that H is exactly solvable by polynomials if it is equivalent to a second-order operator that preserves the infinite flag of finite-dimensional modules,

$$M_1 \subset M_2 \subset M_3 \subset \ldots,$$

because of the triangle form of the operator in such a basis. Integrability of this kind strictly corresponds to shape-invariant potentials.

6.2.2 Nonsingular potentials

Shape-invariant potentials

The direct DT

$$u[1] = u + 2(\ln \sigma)_x \tag{6.20}$$

shifts the potential in a constant value C if $(\ln \sigma)_x = C$, which yields

$$\sigma = \exp(Cx + C_1),$$

or the harmonic oscillator, Morse potential, and Pöschl–Teller potential. These cases were mentioned in Sect. 6.1.2; see also [324]. Moreover, there are a lot of papers whose titles begin with "On a shape-invariant ...."

The class of shape-invariant potentials connected with the operator

$$P(z)\frac{\partial^2}{\partial z^2} + Q(z)\frac{\partial}{\partial z}, \tag{6.21}$$

where P and Q are the first- and the second-order polynomials, corresponds to the exactly solvable Hamiltonians that preserve the infinite flag of polynomial modules

$$\mathcal{P}_\prime \subset \mathcal{P}_1 \subset \ldots \subset \mathcal{P}_n \subset \ldots, \qquad \mathcal{P}_n =< 1, z, \ldots, z^n > .$$

Rescaling x and shifting the spectrum leads to the canonical forms listed in Table 6.1.

	$P(x)$	$Q(x)$	$z(x)$	$U(x)$
Ia	-1	$2z$	x	x^2
Ib	$-4z$	$4z-2$	x^2	x^2
II	$-z^2$	$-(2A+3)z+1$	$\exp x$	$\exp(-2x)/4-(A+1/2)\exp(-x)$
III	$z(1-z)$	$(A-3/2)+1-A$	$\cosh^2(x/2)$	$(1/4-A^2)\mathrm{sech}^2(x/2)/4$

Table 6.1. Nonsingular shape-invariant potentials on the line

The Hamiltonian H is exactly solvable in polynomials if it is transformed by a change of variable

$$x = \int (-P)^{-1/2} \mathrm{d}z$$

and a GT

$$\phi = \exp(p) f_{z=z(x)} \tag{6.22}$$

with

$$p = \int^z \frac{1}{2P}\left(Q - \frac{P'}{2}\right)\mathrm{d}z \tag{6.23}$$

to a Hamiltonian with a potential among those listed in Table 6.1. The transformed potential goes to

$$U = \frac{(Q-P'/2)(Q-3P'/2)}{4P} + R|_{z=z(x)}.$$

Owing to the presence of the polynomial P in the denominator, the number and multiplicity of real roots ρ of P determine the singularity of U, if the numerator is nonzero. Otherwise the potential is nonsingular, i.e., we can formulate the following proposition:

Proposition 6.2. *Let the range of $z(x)$ be denoted as R. One of the following possibilities holds:*

1. *P has no real roots, $R = (-\infty, \infty)$, then U has no singularities.*
2. *There is a double root ρ. Then $\mathrm{R} = (-\infty, \rho)$ or $\mathrm{R} = (\rho, \infty)$, and $U(x)$ is nonsingular.*

3. *There are two roots $\rho_1 < \rho_2$; the case of the only simple root $\rho_1 = \rho_2$ is included as well. Then $R = (-\infty, \rho_1]$ or $R = [\rho_2, \infty)$, and $U(x)$ is nonsingular. Both $z(x)$ and $U(x)$ are even functions. The potential is nonsingular, iff*

$$Q(\rho) = P'(x)/2 \quad or \quad Q(\rho) = 3P'(x)/2.$$

4. *There are two roots $\rho_1 < \rho_2$ and $R = [\rho_1, \rho_2]$. Then both $z(x)$ and $U(x)$ are periodic functions and $U(x)$ is singular.*

Scattering amplitudes for the shape-invariant potentials are calculated by the dressing method; see also [236].

Algebraic deformations

Algebraic deformations of the shape-invariant potentials leave the potentials and eigenfunctions in the class of elementary functions. They are a backward DT which is generated by the factorization function ϕ that is a product of exponential and rational functions. We use the notion of a function f of the polynomial type, if $(\log f)'$ is a rational function.

The prop (factorization [190]) function is

$$\phi(x) = \exp\{\, p[z(x)]\}f[z(x)]. \tag{6.24}$$

Owing to (6.22) and possibility 3, the function p is polynomial and the function $f(z)$ is a hypergeometric function of the polynomial type.

As is seen from Table 6.1, in case Ib we obtain nontrivial deformation of the oscillator potential if the solutions $y(z) = \Phi(a, c, z)$ of the equation

$$zy'' + (c - z)y' - ay = 0$$

are expressed in terms of the generalized Laguerre polynomials $L_m^\alpha, m = 0, 1, 2, \ldots$ [206]. Here $a = -m$ and $c = \alpha + 1$. One of the solutions is used in the deformation of the harmonic oscillator potential [134, 135], namely, the function

$$y_3 = \exp(z)\Phi(-m, c, -z).$$

We will include a shift by x_0 in the transform (6.24), or $z = (x - x_0)^2$, which allows us to deform the potential in a nonsymmetric way.

The function

$$\phi(x; \lambda, A, B) = \left[A\Phi(1/4 - \lambda/4, 1/2; x^2) + Bx\Phi(1/4 - \lambda/4, 3/2; x^2)\right] \exp(-x^2/2)$$

produces a general solution of the equation

$$-\phi'' + x^2\phi = \lambda\phi.$$

When $\lambda = 2n + 1$, we have bound states proportional to $\exp(-x^2/2)$, and, by the nodeless solutions ($\lambda < 1$), the prop (factorization, superpotential) function is written as

$$\phi^{(m)} = \phi(x; -1 - 4m, 1, 0) = \exp(-x^2/2) y_3(x^2). \tag{6.25}$$

This produces a deformation of the harmonic oscillator potential (Fig. 6.1)

$$U^{(m)} = x^2 - 2\left[\ln \exp(-x^2/2)\right]_{xx} y_3(x^2).$$

Similar results for the Morse potential deformation are demonstrated by Fig. 6.2.

The regular solutions of the hypergeometric equation

$$z(1-z)f''(z) + [c - (a+b+1)z]f'(z) - abf(z) = 0$$

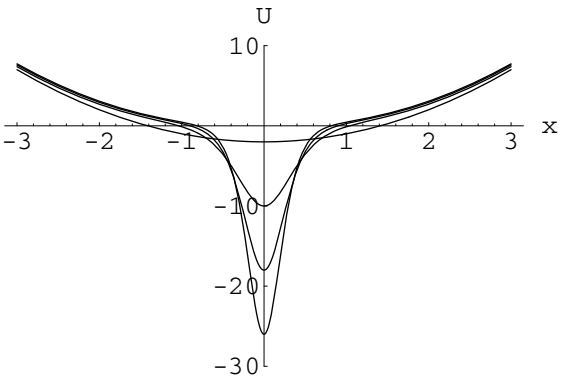

Fig. 6.1. The result of deformations in the case of the harmonic oscillator for $m = 0, 1, 2, 3$ [190]

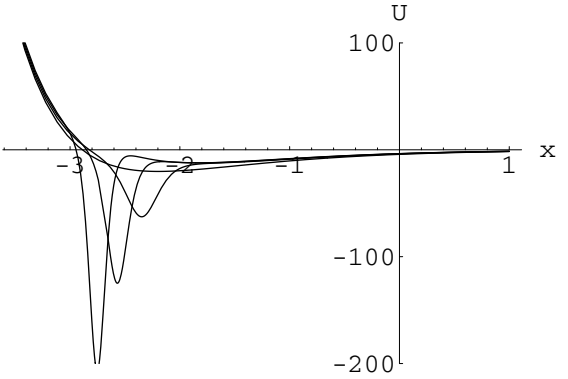

Fig. 6.2. Deformations of the Morse potential for $m = 0, 1, 2, 3$ [190]

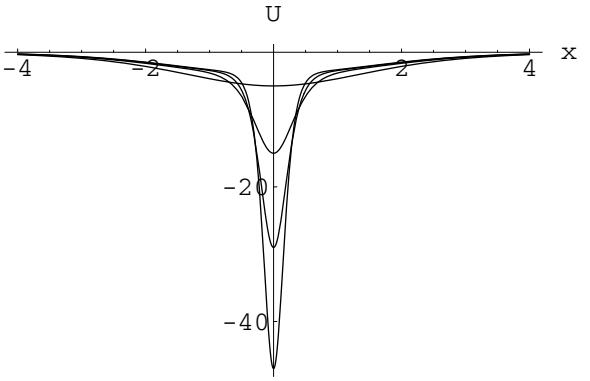

Fig. 6.3. Algebraic deformations of the hyperbolic Pöschl–Teller potential, $m = 0, 1, 2, 3$ [190].

are given by the Gauss hypergeometric function $f(z) = F(a, b, c; z)$. This function is of the polynomial type if $a = -m$ (or $b = -m$), where m is an integer [207]. In this case the solutions are given by the Jacobi polynomial $P_m^{(\alpha,\beta)}(z)$ which is used in deformations of the Pöschl–Teller potential (Fig. 6.3). Note that a bigger m corresponds to a deeper well.

6.2.3 Coulomb potential as a representative of singular potentials

The radial Schrödinger equation,

$$\left(-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}r^2} - \frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r} + \frac{l(l+1)}{2r^2} + u - E\right)\psi_l(r) = 0, \tag{6.26}$$

is transformed to (6.19) by the transformation $\psi_l = \phi_l/r$:

$$\left(-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}r^2} + \frac{l(l+1)}{2r^2} + u - E\right)\phi_l(r) = 0. \tag{6.27}$$

Hence, the corresponding chain equation is equivalent to that obtained in the one-dimensional case; see (6.9) and the instructive example in Sect. 4.1.

The general shape-invariant singular potential is given by (6.16). Equation (6.17) for the iterated potential yields

$$u_j = \frac{\xi_j(\xi_j/\psi - 1)}{(x + B/A)^2\psi} + 2\frac{\xi_j(\xi_j\lambda/\psi + \eta_j)}{(x + B/A)\psi} + (\xi_j\lambda/\psi + \eta_j)^2 + \mu_j, \tag{6.28}$$

where

$$\frac{\lambda}{\psi} = -\frac{(\xi_{j+1}\eta_{j+1} - \xi_j\eta_j)}{\xi_{j+1}^2 - \xi_j^2}, \qquad \psi = \xi_{j+1} - \xi_j.$$

The condition of existence of the representation (6.16) gives the following for the spectrum:

$$\mu_{j+1} = \mu_j + \frac{(\xi_{j+1}\eta_j - \xi_j\eta_{j+1})^2}{\xi_{j+1}^2 - \xi_j^2}.$$

The choice $B = 0$ that corresponds to the choice of the form function $a(r) = (\lambda r + 1)/r$ gives

$$u_j = \frac{\xi_j(\xi_j/\psi - 1)}{r^2\psi} - 2\frac{\xi_j(\eta_j - \xi_j\phi/2\psi)}{r\psi} + (\eta_j - \xi_j\phi/2\psi)^2 + \mu_j. \qquad (6.29)$$

Being the reduction of (6.28), this relation leads to

$$\phi = 2\frac{\xi_{j+1}\eta_{j+1} - \xi_j\eta_j}{\xi_{j+1} + \xi_j}.$$

Let us choose the potential u in the Coulomb form:

$$u = -Ze^2/r.$$

The coincidence between the r-dependent parts (6.29) and the double potential part of (6.27) yields two constraints for the constants:

$$\frac{\xi_j}{\psi}\left(\frac{\xi_j}{\psi} - 1\right) = l(l+1), \qquad \frac{\xi_j}{\psi}\frac{\xi_{j+1}}{\psi}\frac{\eta_{j+1}\xi_j - \eta_j\xi_{j+1}}{\xi_{j+1} + \xi_j} = -Ze^2.$$

The first relation gives ξ_j recursively as a function of l; the second one yields η_j via l, Z, and e. This means that (6.11) and the dressing of the basic state solve the Coulomb quantum problem, producing the spectrum of E in (6.27) and eigenfunctions via reproducing the Rodrig-like formulas.

On the other hand, the classification of shape-invariant potentials includes the Coulomb and related singular potentials [92, 100]; hence, the Coulomb potential deformations can be considered along the lines of Sect. 6.2.

Remark 6.3. An interesting extension of the above results is available. Let us add the so-called Manev potential [370] to the Coulomb field ($\alpha = -Ze^2$),

$$u = \frac{\alpha}{r} + \frac{\beta}{r^2},$$

the angular momentum term being incorporated in the last term. The conditions of coincidence of this potential with one of (6.29) yield constraints

$$\frac{\xi_j}{\psi}\left(\frac{\xi_j}{\psi} - 1\right) = \beta, \qquad \frac{\xi_j}{\psi}\frac{\xi_{j+1}}{\psi}\frac{\eta_{j+1}\xi_j - \eta_j\xi_{j+1}}{\xi_{j+1} + \xi_j} = \alpha.$$

Note also that [370] contains the paper of Todorov entitled "On Some Factorization Statements and Applications" that relates to the ladder (named staircase) operators in the case of infinite matrices with a finite number of nonzero matrix elements in each row/column.

6.2.4 Matrix shape-invariant potentials

For the case of the matrix Schrödinger equation we reproduce here in more detail the general formalism from Chap. 2. The DT for the one-dimensional matrix problem

$$a_2\frac{\partial^2\psi}{\partial^2 x}+a_1\frac{\partial\psi}{\partial x}+a_0\psi=E\psi \tag{6.30}$$

has the form

$$\psi^{(1)}=\psi_x-\phi_x\phi^{-1}\psi, \tag{6.31}$$

where $s=\phi_x\phi^{-1}$ is constructed by matrix solution ϕ of (6.30) for a different eigenvalue E'.

The transformation can be also defined by means of the covariance property of the Schrödinger equation with respect to a transformation of a wave function. The principal statement (on the covariance) formally yields ($\partial=\partial/\partial x$)

$$L^{(1)}\psi^{(1)}=E\psi^{(1)},\qquad L^{(1)}=\sum_{n=0}^{2}a_n^{(1)}\partial^n\ ; \tag{6.32}$$

explicit expressions for $a_n^{(1)}$ are given in [289] for an arbitrary-order operator. We cite here the matrix Darboux dressing formulas for the second-order operator. The coefficient a_2 does not transform, so it is chosen as $a_2=-1/2$, while the transform for a_1 generally contains the commutator

$$a_1^{(1)}=a_1+[a_2,s], \tag{6.33}$$

which for the given a_2 is zero. Finally, slightly changing the notations, the transforms are

$$\psi^{(1)}=\psi_x-s\psi, \tag{6.34}$$

$$a_0^{(1)}=a_0+a_1'+[a_1,s]+2a_2s'+a_2's+[a_2,s]s=a_0+a_1'-s',$$

where the commutator $[a_1,s]=0$ and the primed function is the shortcut for the derivatives in x. The functions $\psi^{(1)}$ and $u^{(1)}$ are again named "dressed" ones, and the auxiliary solutions of the problem (6.30) are referred to as the "prop" functions. The eigenfunction and the potential (ψ and u here) we start with are called the "seed" ones. Let us mention that the proof of the Darboux covariance relation includes a link that in the theory of solitons [324] is named the (general) Miura transformation (Chap. 3) and is solved identically by the substitution $s=\phi_x\phi^{-1}$. We, however, do not use this fact, and go in an alternative way. Notice that in the case of the radial Schrödinger equation $a_1=-1/r$, which also simplifies the transform (6.34).

6.3 Zero-range potentials, dressing, and electron–molecule scattering

Following Andrianov et al. [25], the Darboux formulas can be applied in multidimensional space in combination with those for the radial Schrödinger equation [324, 391]. This approach makes it possible to use the DT technique to work with an improved version of a ZRP model.

Our aim here is to dress the ZRP by means of a special choice of the DT in order to widen the possibilities of the ZRP model. The DT modifies the generalized ZRP boundary condition and creates a potential with arbitrarily arranged discrete spectrum levels for any angular momentum l.

6.3.1 ZRPs and Darboux transformations

Our statement consists in the fact that generalized ZRPs [38] appear as a result of the DTs applied to zero potential. In order to demonstrate this we consider a radial Schrödinger equation

$$\left(-\frac{1}{2}\frac{\mathrm{d}^2}{\mathrm{d}r^2}-\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}+\frac{l(l+1)}{2r^2}+u_l-E\right)\psi_l(r)=0 \tag{6.35}$$

for partial wave ψ_l with orbital momentum l. The atomic units $\hbar = m_\mathrm{e} = 1$ are used throughout the present section. Here u_l, $l = 0, 1, 2, \ldots$, are potentials for the partial waves with an asymptotic at infinity

$$\psi_l(r) \sim \frac{\sin(kr - \frac{l\pi}{2} + \delta_l)}{kr},$$

where δ_l are partial phase shifts. Equation (6.35) describes scattering of a particle with energy E and momentum $k = \sqrt{2E}$ by the rapidly decreasing potential u_l. In the absence of the potential, partial shifts $\delta_l = 0$ and partial waves can be expressed via spherical functions $\psi_l = j_l(kr)$.

It is known that (6.30) with

$$a_1 = -\frac{1}{r}, \qquad a_0 = \frac{l(l+1)}{2r^2} + u_l \tag{6.36}$$

is covariant with respect to the DT (6.33) and (6.34). The prop function ϕ plays an important role when applying the DT because it is used to calculate s. The function ϕ is a solution of (6.35) at a particular value of energy $E = -\kappa^2/2$, where we assume κ is a real number. If κ is a complex number, then the dressed potential will be a complex function in general.

Let us demonstrate how a generalized ZRP can be produced by the DT. It is convenient to use a sequence of DTs (Crum formulas [94] with the wave and prop functions multiplied by r), which for our equation look like

$$\psi_l \to \psi_l^{(1)} = \mathrm{const}\cdot\frac{W(r\psi_l, r\phi_1, \ldots, r\phi_{2l+1})}{rW(r\phi_1, \ldots, r\phi_{2l+1})}, \tag{6.37}$$

$$u_l \to u_l^{(1)} = u_l - [\ln W(r\phi_1, \ldots, r\phi_{2l+1})]''. \tag{6.38}$$

Here W is the Wronskian, ϕ_m are prop functions, and the double prime stands for $\partial^2/\partial r^2$. The transformation (6.37) combines the solution ψ_l and $2l+1$ solutions ϕ_m. The Crum formulas result from the replacement of a sequence of $2l+1$ first-order transformations by a single $(2l+1)$th-order transformation which happens to be more efficient in practical calculations. In order to obtain the ZRP, we start from zero potential and use prop functions

$$\phi_m = h_l^{(1)}(\kappa_m r), \tag{6.39}$$

where κ_m are solutions of the algebraical equation $\kappa_m^{2l+1} = \mathrm{i}\alpha_l$. Here we assume α_l is a real number. The explicit form of the spherical functions $h_l^{(1)}$ is

$$h_l^{(1)}(kr) = n_l(kr) + \mathrm{i}j_l(kr), \qquad h_l^{(2)}(kr) = n_l(kr) - \mathrm{i}j_l(kr). \tag{6.40}$$

The spherical functions j_l and n_l are related to usual Bessel functions with half-integer indices [30]. They obey the asymptotic at infinity $r \to \infty$ of the form

$$j_l(kr) \sim \frac{\sin(kr - \frac{l\pi}{2})}{kr}, \qquad n_l(kr) \sim \frac{\cos(kr - \frac{l\pi}{2})}{kr}. \tag{6.41}$$

For our purpose, it is important that in the vicinity of zero the spherical functions have asymptotic behavior

$$j_l(kr) \sim \frac{(kr)^l}{(2l+1)!!}, \qquad n_l(kr) \sim \frac{(2l-1)!!}{(kr)^{l+1}}. \tag{6.42}$$

Note the double factorial $(2l-1)!!$ satisfies the relation $(2l)! = 2^l l!(2l-1)!!$

The asymptotic for $h_l^{(1,2)}(kr)$ can be obtained by combining (6.42) in accordance with (6.40). For example, at infinity use of (6.41) leads to

$$h_l^{(1)}(kr) \sim (-\mathrm{i})^l \frac{\mathrm{e}^{\mathrm{i}kr}}{kr}, \qquad h_l^{(2)}(kr) \sim \mathrm{i}^l \frac{\mathrm{e}^{-\mathrm{i}kr}}{kr}.$$

Direct substitution of (6.39) into the Wronskian gives

$$W(r\phi_1, \ldots, r\phi_{2l+1}) = \text{const}.$$

This means that the dressed potential is again the ZRP, $u_l^{(1)}(r > 0) = 0$. The transformation (6.38) allows us to calculate potential for $r > 0$. We state that the DTs also yield a generalized ZRP at $r = 0$. In order to prove this, we perform the transformation (6.37) and show that $\psi_l^{(1)}$ is a solution for a generalized ZRP. Since the potential is zero in the region $r > 0$, it is sufficient to determine the asymptotic of the wave function. Substituting $\psi_l = j_l(kr)$ into the Crum formulas, we obtain

$$\psi_l^{(1)} = \text{const} \cdot \frac{W[rj_l(kr), r\phi_1, \ldots, r\phi_{2l+1}]}{rW(r\phi_1 \ldots, r\phi_{2l+1})} \sim$$

$$\frac{1}{2\mathrm{i}} \left((-\mathrm{i})^l \frac{\mathrm{e}^{\mathrm{i}kr}}{kr} \frac{\Delta(\mathrm{i}k, \kappa_1, \ldots, \kappa_{2l+1})}{\Delta(\kappa_1, \ldots, \kappa_{2l+1})} - \mathrm{i}^l \frac{\mathrm{e}^{-\mathrm{i}kr}}{kr} \frac{\Delta(-\mathrm{i}k, \kappa_1, \ldots, \kappa_{2l+1})}{\Delta(\kappa_1, \ldots, \kappa_{2l+1})} \right),$$

where Δ is the Vandermond determinant. Considering it as the product

$$\Delta(\mathrm{i}k, \kappa_1, \ldots, \kappa_{2l+1}) = \mathrm{const} \cdot \prod_{m=1}^{2l+1} (\kappa_m - \mathrm{i}k),$$

we obtain an asymptotic which coincides with the asymptotic of the solution:

$$\psi_l^{(1)} = \mathrm{const} \cdot [h_l^{(1)}(kr)\mathrm{e}^{2\mathrm{i}\delta_l} - h_l^{(2)}(kr)], \qquad \exp(2\mathrm{i}\delta_l) = \prod_{m=1}^{2l+1} \frac{\kappa_m - \mathrm{i}k}{\kappa_m + \mathrm{i}k}. \quad (6.43)$$

It is easy to show that the wave function (6.43) describes a scattering of the partial wave with orbital momentum l by the generalized ZRP . The potential is conventionally represented as the boundary condition at $r = 0$ for the wave function. This fact can be verified by direct substitution into the boundary condition for the generalized ZRP:

$$\left.\frac{(\mathrm{d}/\mathrm{d}r)^{2l+1}\, r^{l+1}\psi_l^{(1)}}{r^{l+1}\psi_l^{(1)}}\right|_{r=0} = -\frac{2^l l!\alpha_l}{(2l-1)!!}, \quad (6.44)$$

where α_l is the inverse scattering length for the partial wave with orbital momentum l. Recall that at low energies $\tan(\delta_l) \sim -a_l k^{2l+1}$ for a short-range potential, where a_l is the scattering length. In the special case of $l = 0$ we obtain $(\ln r\psi)' = -\alpha$. This generalized boundary condition can be obtained from the asymptotic of the wave function in the vicinity of zero, which was used in [38]. Let us consider the scattering matrix on the complex k-plane. Each element $\exp(2\mathrm{i}\delta_l)$ has $2l + 1$ poles at the points $k = \mathrm{i}\kappa_m$, which lie on a circle in the complex plane. Since the bound states correspond to the poles on the imaginary positive semi-axis in the complex k-plane, a bound state exists only if $\alpha_l > 0$ and l is an odd number or if $\alpha_l < 0$ and l is even. Otherwise the ZRP has an antibound state.

The ideas of the ZRP approach were recently developed [38, 116, 282] to extend the limits of the traditional treatment by Demkov and Ostrovsky [115] and Albeverio et al. [21]. The advantage of the theory is the possibility of obtaining an exact solution of the scattering problem. The ZRP is conventionally represented as the boundary condition on the wave function at some point. Alternatively, the ZRP can be represented as a pseudopotential [77, 116].

There is some "generalization" of the ZRP theory, when the inverse scattering length in the original boundary condition is replaced by $-k\cot\delta$ for $l = 0$. In such a model the ZRP may have two (or more) bound states with nonorthogonal wave functions. This problem does not appears in our model because our potential has only one bound state. However, we note that a generalized ZRP has another problem: the bound state with orbital momentum $l > 0$ does not belong to L^2 (the zero-range effect). But we think that this problem is not fatal because this model reasonably describes low-energy scattering.

Example 6.4. There is a simple example which proves our observation: the ZRP can be produced by the DT. Let us consider transformation of the regular solution $\psi = \sin(kr)/r$ with the prop function $\phi = \exp(\alpha r)/r$. Direct calculation yields the wave function

$$\psi^{(1)} = \text{const} \cdot (\psi' - s\psi) = \frac{\sin[kr - \arctan(k/\alpha)]}{kr}, \tag{6.45}$$

which satisfies the original ZRP condition with the inverse scattering parameter α. Repeating the transformation with $\phi = \exp(-\alpha r)/r$, we obtain $\psi^{(2)} = \psi$. This example shows a relation between the ZRP and the DT.

6.3.2 Dressing of ZRPs

We have shown the generalized ZRPs appear as a result of application of the DT to the seed solution. In this connection we can raise a problem of subsequent dressing of the ZRP. In the particular case of only one prop function ϕ the Crum formulas correspond to the usual DT:

$$\psi_l^{(1)} = \text{const} \cdot (\psi_l' - s\psi_l), \qquad s = (\ln\phi)',$$

$$u_l^{(1)} \equiv a_0^{(1)} - \frac{l(l+1)}{2r^2} = u_l + \frac{1}{r^2} - s', \tag{6.46}$$

where we suppose the potential u_l describes the ZRP. The functions ψ_l and ϕ are solutions of the Schrödinger equation (6.35). Since the potential $u_l(r > 0) = 0$, the solution ϕ can be written as a linear combination of spherical functions,

$$\phi = Cn_l(\mathrm{i}\kappa r) + C_1 j_l(\mathrm{i}\kappa r), \tag{6.47}$$

where C, C_1, and κ are parameters. Note the dressed potential $u_l^{(1)}$ is real for real prop function ϕ; hence, the parameters should be real. The direct application of (6.31) allows us to calculate the potential in the range $r > 0$, but not at $r = 0$! In order to solve this problem, we consider ϕ in the vicinity of zero. There are two different cases. The spherical function properties show that in the case $C = 0$ the leading term in ϕ is r^l and in the case $C = 1$ this term looks like r^{-l-1}. Therefore, the dressed coefficient $a_0^{(1)}$ has the following asymptotic at zero:

$$\frac{l(l+1)}{2r^2} + u_l^{(1)} \sim \begin{cases} \dfrac{(l+1)(l+2)}{2r^2}, & \text{when} \quad C = 0, \\ \dfrac{l(l-1)}{2r^2}, & \text{when} \quad C = 1. \end{cases}$$

As regards all the other possible cases, it is easy to see that they lead to the above results. According to (6.46), the dressed potential $u_l^{(1)}$ decreases as $\exp(-2|\kappa|r)$ at infinity. Thus, the DT produces a short-range core of the

centrifugal type (which depends on angular momentum l) in the potential. In this situation the boundary conditions on the dressed wave functions $\psi_l^{(1)}$ require some modification. We believe that in the general case the dressed ZRP is conventionally represented as the boundary condition

$$\left.\frac{(\mathrm{d}/\mathrm{d}r)^{2m+1}\, r^{m+1}\psi_l^{(1)}}{r^{m+1}\psi_l^{(1)}}\right|_{r=0} = \text{const},$$

where $m = l + 1$ for $C = 0$ and $m = l - 1$ for $C = 1$. However, repeating the DT for other values of κ and combining the results for $C = 0$ and 1, we can remove the short-range core. In the absence of the short-range core the boundary condition looks like (6.44). The sequence of N DTs leads to new poles of the S matrix which do not depend on C_1:

$$\exp\left(2\mathrm{i}\delta_l^{(N)}\right) = \frac{\alpha_l - \mathrm{i}k^{2l+1}}{\alpha_l + \mathrm{i}k^{2l+1}} \prod_{m=1}^{N} \frac{\kappa_m - \mathrm{i}k}{\kappa_m + \mathrm{i}k}. \tag{6.48}$$

Thus, we can use the DT in order to add (or remove) poles of the S matrix. The next step in the dressing procedure is the determination of the free parameters of the solutions ϕ. Changing parameter C_1, we obtain potentials with identical spectra, called the phase-equivalent potentials. The transformation of this kind is also known as the isospectral deformation.

Example 6.5. The simplest case $l = 0$ is instructive. Consider the original ZRP at $r = 0$ with the wave function (6.45). We can choose the solution ϕ as

$$\phi = \frac{\cosh(\kappa r)}{r}.$$

This choice corresponds to the parameters $C = 1$ and $C_1 = 0$. For brevity we omit the index $l = 0$. The DT (6.31) gives rise to the following property of the dressed wave function:

$$\left.\frac{(r\psi^{(1)})'}{r\psi^{(1)}}\right|_{r=0} = \frac{k^2 + \kappa^2}{\alpha},$$

which slightly differs from the usual boundary condition in ZRP theory $(\ln r\psi)' = -\alpha$. The dressed potential has the short-range tail:

$$u^{(1)}(r > 0) = -\frac{\kappa^2}{\cosh^2(\kappa r)}. \tag{6.49}$$

Our investigations show that some particular values of C_1 can give a long-range interaction which decreases like $\sim r^{-2}$.

The model we study describes the scattering of an electron on a compound particle. There were attempts to account for this important circumstance by means of matrix potentials to be applied not only to the well-known

multichannel problem, but to composite particles as well [406]. The matrix is a projection of a complicated basis that includes the orbital momenta; the only possible place for them to exist is the potential, if we restrict ourselves to the one-particle case. We will see in the next section that the DT-based approach provides a constructive method to describe general multiparticle systems, which is especially important in the multicenter problem.

6.4 Dressing in multicenter problem

The results of the previous section allow us to build a ZRP eigenfunction in the multicenter problem. In a more general situation we can also consider a system with a smooth potential plus a number of ZRPs. If the Green function for the smooth potential is known, we can provide a solution for the problem with the ZRPs added. This procedure was outlined in [115], where the case of a single ZRP was considered. Generalization to an arbitrary number of ZRPs is straightforward. In contrast, our general idea is to dress a multicenter system without using the Green function. This method gives simple formulas for partial phases and their corrections at low energies [284].

Let us consider a scattering problem for a nonspherical potential $\hat{U}$:

$$\left(-\frac{1}{2}\frac{\partial^2}{\partial r^2} - \frac{1}{r}\frac{\partial}{\partial r} + \frac{\hat{L}^2}{2r^2} + \hat{U} - E\right)\psi(\mathbf{r}) = 0, \tag{6.50}$$

where $\hat{L}^2$ is square of the angular momentum operator and E is the energy of a particle. The asymptotic of wave function $\psi(\mathbf{r})$ looks like

$$\psi(\mathbf{r}) \overset{r\to\infty}{\sim} \exp(\mathrm{i}\mathbf{k}\cdot\mathbf{r}) + f(\theta)\frac{\mathrm{e}^{\mathrm{i}kr}}{r}, \tag{6.51}$$

where $f(\theta)$ is the scattering amplitude which depends on scattering angle θ. The operator $\hat{L}^2$ commutes with all radial derivatives, in particular with $\partial = \partial/\partial r$. In the three-dimensional space the DT can be reduced to the one-dimensional Heisenberg matrix (or operator) problem (6.30) with appropriately chosen variable x and with a basis of (orthogonal) functions of the rest variables. In our case $x = r$, functions a_2 and a_1 are the same as in (6.36) and

$$a_0 = \frac{\hat{L}^2}{2r^2} + \hat{U}.$$

The radial DT for any solution of the Schrödinger equation is similar to that obtained in Sect. 6.1, but s should be treated as a function of the operator variable $\hat{L}^2$. The transformation of the potential is written as

$$\hat{U} \to \hat{U}^{(1)} = \hat{U} + \frac{1}{r^2} - s'. \tag{6.52}$$

In order to find the operator s, we use the covariance principle for (6.50). The covariance principle (6.32) formally yields an explicit constraint for s which looks like

$$a_0' + [a_0, s] + (a_1 s)' + [a_1, s]s + \{a_2(s' + s^2)\}' + [a_2, s](s' + s^2)$$
$$= a_0' + (a_1 s)' + \{a_2(s' + s^2)\}' = 0.$$

Integrating over r, we obtain the operator equation for s which in our case can be written as

$$s' + \frac{2}{r}s + s^2 = \frac{\hat{L}^2}{r^2} + 2\hat{U} + C(\hat{L}^2). \tag{6.53}$$

The integration "constant" $C(\hat{L}^2)$ is a function of the operator variable $\hat{L}^2$ which does not depend on r [289]. The sense of this "constant" can be understood from the asymptotic behavior of s at infinity $r \to \infty$, where (6.53) goes to

$$s' + s^2 = C(\hat{L}^2).$$

The general solution of this (Riccati) equation for the asymptotic in r at infinity gives either oscillations or $s(\infty) = K(\hat{L}^2)$, then $C(\hat{L}^2) = K(\hat{L}^2)^2$.

The operator s may be found as a series $\sum_{n=0}^{\infty} s_n \hat{L}^{2n}$, where coefficients s_n depend only on r. It is easy to show that (6.53) leads to the recursion relations for the coefficients s_n:

$$s_0' + \frac{2}{r}s_0 + s_0^2 = 2\hat{U} + C(0),$$

$$s_1' + \frac{2}{r}s_1 + s_0 s_1 + s_1 s_0 = \frac{\hat{L}^2}{r^2} + C'(0),$$

$$s_n' + \frac{2}{r}s_n + \sum_{k=0}^{n} s_k s_{n-k} = C^{(n)}(0), \quad n \geq 2.$$

For example, the first equation in the region where $U = 0$ looks like

$$s_0' + \frac{2}{r}s_0 + s_0^2 = K_0,$$

where K_0 is the zeroth coefficient in the expansion $K = \sum_{n=0}^{\infty} K_n \hat{L}^{2n}$. Equation (6.52) gives a nonlocal (with respect to angles) potential which depends on $\hat{L}^2$. Thus, we have the algorithm that determines the operator s and the dressed potential via the operator K. To evaluate the cross section, we need only partial phases or a scattering amplitude related to the operator K. In order to find the partial phases for the dressed potential, we should apply the DT to the wave function. However, a problem occurs: the DT in general modifies the plane wave $\exp(\mathrm{i}\mathbf{k} \cdot \mathbf{r})$. Thus, the DT applied to wave function $\psi(\mathbf{r})$ with the asymptotic (6.51) gives another asymptotic. In some particular

cases, special choice of the operator K allows us to avoid such a problem. We consider this choice as a condition in the formulation of a scattering problem.

Indeed, consider the partial wave asymptotic for a nonspherical potential [48]

$$\psi_J(\mathbf{r}) \sim \frac{1}{2\mathrm{i}kr}\left[\mathrm{e}^{\mathrm{i}kr+\mathrm{i}\delta_J}\Lambda_J(\mathbf{n}) - \mathrm{e}^{-\mathrm{i}kr-\mathrm{i}\delta_J}\Lambda_J(-\mathbf{n})\right], \tag{6.54}$$

where $\mathbf{n}$ is the unit vector directed as $\mathbf{r}$, δ_J denotes partial shifts, and $\Lambda_J(\mathbf{n})$ are normalized eigenvectors of the S matrix operator (partial harmonics). The simplest formulas for the shifts $\delta_J^{(1)}$ for the potential $\hat{U}^{(1)}$ result when partial harmonics Λ_J are also eigenvectors of the operator K. For example, suppose all partial harmonics Λ_J are eigenvectors of K but only Λ_0 has nonzero eigenvalue κ,

$$K\Lambda_0(\mathbf{n}) = \kappa\Lambda_0(\mathbf{n}).$$

The asymptotic dressing is reduced to the action of the operator $\partial - K$ on asymptotic (6.54). It is easy to show by using the expression

$$\ln\left(\frac{\kappa - \mathrm{i}k}{\kappa + \mathrm{i}k}\right) = -2\mathrm{i}\arctan(k/\kappa)$$

for real-valued variables k and κ that the DT changes only the partial shift δ_0:

$$\delta_0^{(1)} = \delta_0 - \arctan(k/\kappa). \tag{6.55}$$

In this special case we add only one additional parameter. In the region $k \gg |\kappa|$ the second term of (6.55) practically does not contribute to the partial cross section,

$$\sigma_J = \frac{4\pi}{k^2}\sin^2\delta_J. \tag{6.56}$$

It yields an essential contribution to the cross section when $k \approx |\kappa|$ and so it can be considered as a correction at low energies.

In a general case the DT modifies all partial harmonics and partial shifts. The DT allows us to construct new solvable models with additional parameters. One of the most important problems of solvable models is to fit them to some physically meaningful parameters. For example, in our case the parameter κ can be related to the effective radius of the interaction or the scattering length. It is well known that the scattering length is defined as derivative $A = -\delta'(k)$ at $k = 0$. Considering (6.55) at low energies, we obtain "renormalized" scattering length

$$A^{(1)} = A + \frac{1}{\kappa}. \tag{6.57}$$

6.5 Applications to X_n and YX_n structures

For the purpose of illustration we consider in this section the scattering problem for a dressed multicenter potential. The multicenter scattering within the framework of the ZRP model was investigated by Demkov and Rudakov

[114] (eight centers, cube), Szmytkowski and Szmytkowski [421] (four centers, regular tetrahedron), and others. We analyze here electron scattering by the X_n and YX_n structures within the framework of the ZRP model. At the end of this section we present a calculation of the electron–silane scattering cross section with the corrections caused by the dressing integral.

6.5.1 Electron–X_n scattering problem

Suppose a structure X_n contains n identical scatterers placed at the points $\mathbf{r}_m$, which involve only s waves. Let R denote the distance between any two scatterers. There are three such structures in the three-dimensional space: a line X_2, a regular triangle X_3, and a regular tetrahedron X_4. The partial waves $\psi_J(\mathbf{r})$ and phase shifts can be classified with respect to the symmetry group representation for the structures X_n ($n = 2, 3, 4$), degeneracy being defined by the dimension of the representation [48]. We use the partial waves for ordinary ZRPs in a general form,

$$\psi(\mathbf{r}) = \sum_{m=1}^{n} c_m \frac{\sin(k|\mathbf{r} - \mathbf{r}_m| + \delta)}{|\mathbf{r} - \mathbf{r}_m|}. \tag{6.58}$$

Our intention is to derive an algebraical equation for the partial phase. The s-wave boundary condition at the points $\mathbf{r}_m$ leads to an algebraical problem for a $n \times n$ matrix with compatibility condition

$$[p + (n-1)q](p-q)^{n-1} = 0,$$

where

$$p = akR + R\tan\delta, \quad q = a[\sin(kR) + \cos(kR)\tan\delta].$$

Then it is easy to show that the phases are determined by the following expressions:

$$\tan\delta_J = \begin{cases} -a\dfrac{kR + (n-1)\sin(kR)}{R + (n-1)a\cos(kR)}, & J = 0, \\ -a\dfrac{kR - \sin(kR)}{R - a\cos(kR)}, & J = 1, \ldots, n-1. \end{cases} \tag{6.59}$$

In the special case $n = 4$ we obtain the phases of the regular tetrahedron [421]. The integral cross section σ can be expressed as

$$\sigma = \sigma_0 + (n-1)\sigma_1,$$

where partial cross sections σ_J are given by (6.56). It is easy to relate a scattering length for a molecule X_n with the boundary parameter a. At large R the parameter a is reduced to the scattering length for an isolated atom. Starting from the equation

$$\delta(k) = -\arctan\left(a\frac{kR + (n-1)\sin(kR)}{R + (n-1)a\,\cos(kR)}\right),$$

we obtain

$$A_{\mathrm{X_n}} = -\delta'(0) = \frac{naR}{R + (n-1)a}. \tag{6.60}$$

Testing the result and taking $n = 1$ gives $A_{\mathrm{X}_n} = a$ for arbitrary R. The link (6.60) defines the monotonic function saturated at $a \to \infty$.

6.5.2 Electron–YX_n scattering problem

The structures YX_n can be used, for instance, to study a slow electron scattering by polyatomic molecules like H_2O, NH_3, and CH_4. For the sake of simplicity we suppose that the scatterers X are situated at the vertices of a regular structure X_n. Let D denotes the distance between scatterers Y–X and R denote the distance between scatterers X–X. In this case, the position of the scatterer Y is perfectly fixed only if $n = 4$ (geometric center of the tetrahedron), and we have the constraint $R = 2\sqrt{\frac{2}{3}}D$. The partial waves can be written as (6.58), where the summation should be performed from $m = 0$ to n. The partial phases can be derived analytically. The result is given by the expression

$$\tan\delta_J = -a_{\mathrm{x}}\,\frac{kR - \sin(kR)}{R - a_{\mathrm{x}}\cos(kR)}, \qquad J = 2, \ldots, n,$$

and $t = \tan\delta_J$, $J = 0, 1$, obeys the quadratic equation

$$(t + a_{\mathrm{y}}k)\left[\frac{t}{n-1} + a_{\mathrm{x}}\left(\frac{k}{n-1} + \frac{\sin(kR)}{R} + t\frac{\cos(kR)}{R}\right)\right]$$

$$= \frac{n}{n-1}a_{\mathrm{x}}a_{\mathrm{y}}\left(\frac{\sin(kD)}{D} + t\frac{\cos(kD)}{D}\right)^2, \tag{6.61}$$

where a_{x} and a_{y} denote boundary parameters. For large distances we can interpret these parameters as scattering lengths of isolated atoms. Thus, in the limiting case when the distance D is very large, the expression for $\tan\delta_0$ passes to the first equation of (6.59) and $\tan\delta_1 \sim -a_{\mathrm{y}}k$. This situation corresponds to independent scattering on a molecule X_n and atom Y. The substitution $a_{\mathrm{y}} = 0$ reduces $\tan\delta_0$ for the structure YX_n to $\tan\delta_0$ for the structure X_n.

Substitution $t = -Ak$, where A denotes the scattering length for a molecule YX_n, and passage to the limit $k \to 0$ in (6.61) gives the quadratic equation with the roots $A = 0$ and

$$A_{\mathrm{YX}n} = D\frac{(a_{\mathrm{y}} + na_{\mathrm{x}})RD + a_{\mathrm{x}}a_{\mathrm{y}}[(n-1)D - 2nR]}{[R + (n-1)a_{\mathrm{x}}]D^2 - na_{\mathrm{x}}a_{\mathrm{y}}R}. \tag{6.62}$$

The last root gives a monotonic function of the atomic length a_{x} with the same features as in the previous section. We believe that the scattering lengths for isolated atoms do not change much if the atoms form a polyatomic molecule.

6.5.3 Dressing and Ramsauer–Taunsend minimum

In the previous subsection we presented a detailed description of new solvable models for low-energy scattering in the electron–polyatomic molecule system. Now we compare the results of our model with other theoretical and experimental data. Among all possible applications we will discuss the scattering by the tetrahedral molecule SiH_4 (silane) because it has the most interesting point group, namely, the symmetry group of a tetrahedron. We focus our attention on one distinct feature of the integral cross section, namely, the Ramsauer–Townsend minimum around approximately 0.28 eV.

Jain et al. [219] classify the minimum as due to the s-wave scattering into ${}^2A_1$ symmetry and attribute the main contribution to the cross section at the minimum by the p-wave scattering. Also they state that the minimum is the result of a balance between the attractive long-range and repulsive short-range interactions.

The DT discussed in Sect. 6.4 allows us to correct cross sections at low energies; thus, using (6.57) and (6.62), we obtain the "renormalized" scattering length for a molecule YX_n. A model calculation with (6.55) and (6.61) has been performed to show that the Ramsauer–Townsend minimum appears as a consequence of balance between the attractive short-range and zero-range interactions.

The parameters used in calculation are as follows (in atomic units):

$$a_{\mathrm{x}} = 4.10, \quad R = 4.51, \quad \kappa = 0.185, \quad a_{\mathrm{y}} = 1.88, \quad D = 2.762,$$

which are regarded as constants in the range of interest. The equilibrium distances R and D were taken from ab initio calculation. The other parameters were chosen so as to reproduce the realistic low-energy asymptotic of σ and the position of the minimum. The result of our calculation is shown as the upper curve in Fig. 6.4. The circles show the numerical calculation [219]; the triangles and upper curve (least-squares fitting) describe the experimental data [446]. Our results show that dressing leads to additional finite-range attractive interaction which algebraically increases the partial phase ($\delta_0 < 0$) for partial wave A_1 for the YX_4 structure, and causes the deep minimum near 0.35 eV. Thus, our partial cross sections coincide well with the results obtained using other numerical data and coincide in shape with experimental data. The novel feature of our model of electron–molecule scattering is the dependence of the effective potential on electron momentum (spherical part of the Laplacian). This way we obtain a richer dependence of the scattering parameters on k, which improves the agreement with experiment in the low-energy region (Fig. 6.5). It could be considered as an alternative to the Demkov–Rudakov approach, with generalized partial waves introduced in each step of the dressing procedure. Let us also remind that the model uses only s waves; accounting for higher terms would extend the range of agreement.

We deal with the algebraical expressions for phases of the electron–X_n (and –YX_n) scattering problem. They are useful to study slow electron

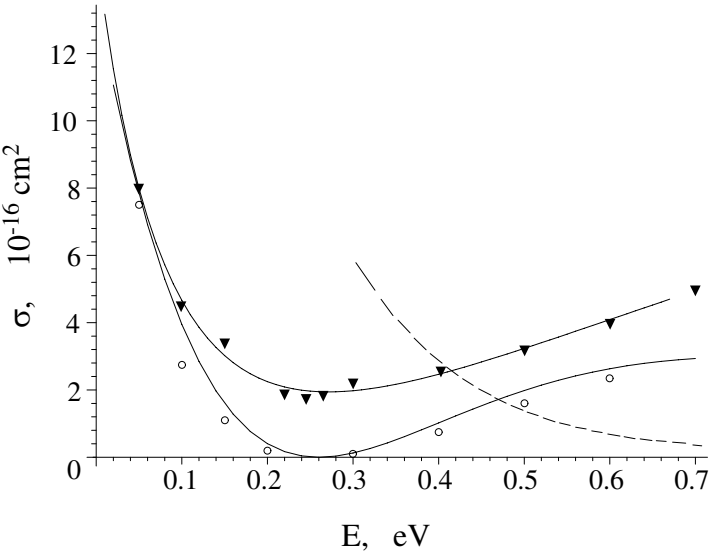

Fig. 6.4. Integral cross sections for electron–silane scattering around the Ramsauer–Townsend minimum. The *upper curve* gives the least-squares fitting to the experimental data [446] (*triangles*); the *lower curve* describes our model calculation for partial wave A_1; *circles* denote the calculation for partial wave A_1 [219]. The *dashed line* illustrates the calculation performed in [186]

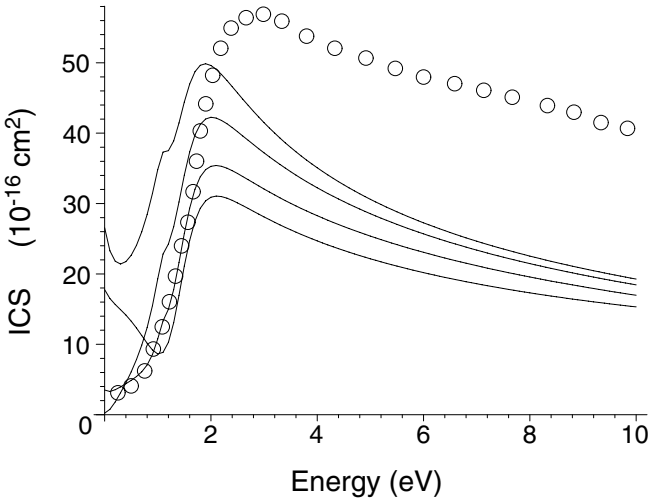

Fig. 6.5. Integral cross sections for electron$-SiH_4$ scattering on dressed zero-range potentials at a wider range of energies and different values of the matching parameter. Experimental results of Wan et al. [446] (*circles*)

scattering by a molecule. We also obtain expressions for scattering lengths which are helpful to fit the parameters (if the scattering length is known). Note that in our calculation we do not use scattering lengths of isolated atoms.

Among the most important aspects of the method is the demonstration of the power of the DT as applied to a multicenter scattering problem and ZRP theory. Namely, the ZRP is produced by the DT. Also, these transformations allow us to correct the ZRP model at low energies.

An alternative approach is formulated for the Calogero–Sutherland model in [172].

6.6 Green functions in multidimensions

This section is based on [290], where the dressing (factorization) ideas [412, 413, 414] are used to construct the simple poles expansion for the Jost function of the one-dimensional Schrödinger operator. The representation of the resolvent of the Schrödinger operator with a reflectionless potential does not contain an integral term. The expression of the potential has a free parameter that, in turn, allows us to build Green functions for a wide class of multidimensional differential operators. The explicit form of such a Green function is given for heat conductivity and wave equations in 2+1 dimensions with an arbitrary reflectionless potential.

6.6.1 Initial problem for heat equation with a reflectionless potential

Consider the nth-order inhomogeneous differential equation

$$(D_0 - L)\Psi = f(x). \tag{6.63}$$

The operators and the elements Ψ and f of a differential ring K are defined in Sect. 3.3. The following proposition is obtained as a consequence of the results of Sects. 3.1–3.3:

Proposition 6.6. *The functions $L_s\psi$, where $L_s = \partial - \phi_x\phi^{-1}$ is the dressing operator, are solutions of the equation*

$$(D_0 - \tilde{L})\Psi = L_s f(x),$$

where the DT-transformed operator $\tilde{L}$, the element s, and the equation for it are given in Sect. 3.3.

Following this statement, we can define a Green function of the dressed operator $\tilde{L}$. Indeed, take the function $f(x, x')$ in (6.63) as a solution of

$$L_s f(x, x') = \delta(x - x'),$$

and construct a solution of the same equation (6.63):

$$(D_0 - L)\Psi = \phi \int_{-\infty}^{x} \mathrm{d}x'' \phi^{-1}(x'')\delta(x'' - x') = \begin{cases} 0, & x < x', \\ \phi(x)\phi^{-1}(x'), & x > x'. \end{cases} \quad (6.64)$$

This results in a Green function that corresponds to the conditions which are used while the integration on the right-hand side of (6.64) is performed. The representation for the dressing operator $L_s = \partial - \phi_x \phi^{-1}$ is used here.

In a similar way we can perform a manipulation with an initial condition that is illustrated by the following example.

We proceed from the DT covariance of the heat equation for the function $\rho(\tau, x, y)$,

$$-\rho_\tau + \rho_{xx} + u(x)\rho = 0. \quad (6.65)$$

The covariance means the form invariance with respect to the iterated DT defined by the Wronskian $W[\phi_1, ..., \phi_N]$ of the solutions of (6.65):

$$\rho \to \rho[N] = \frac{W[\phi_1, \ldots, \phi_N, \rho]}{W[\phi_1, \ldots, \phi_N]}, \quad u \to u[N] = u + 2(\ln W[\phi_1, ..., \phi_N])_{xx}. \quad (6.66)$$

Consider now a Cauchy problem for (6.65), where $u(x)$ represents the reflectionless potential [354] and the initial condition is

$$\rho(0, x, y) = \delta(x - y). \quad (6.67)$$

The problem described by (6.67) is rather general and can be applied as a model of classical diffusion or heat conductivity. We, however, follow the application in the theory of quasiclassical quantization, where the function ρ is treated as a density matrix, whence τ stands for inverse temperature [104].

The algorithm to solve this problem consists in application of the dressing procedure organized by the sequence of DTs from (6.66),

$$\begin{aligned} \left\{ \frac{\partial}{\partial x} - (\ln \phi_1(x, y))_x \right\} \rho_0(0, x, y) &= g_1(x, y), \\ \left\{ \frac{\partial}{\partial x} - (\ln \phi_2[1](x, y))_x \right\} g_1(x, y) &= g_2(x, y), \\ \left\{ \frac{\partial}{\partial x} - (\ln \phi_k[k-1](x, y))_x \right\} g_{k-1}(x, y) &= g_k(x, y), \\ g_N(x, y) = \delta(x - y), \quad 2 \le k \le N, \end{aligned} \quad (6.68)$$

and the following theorem:

Theorem 6.7. *The function $\rho[N]$ being built by (6.66) will be a solution of the problem (6.65) and (6.67) with the potential $u[N]$, if $\rho(\tau, x, y)$ is a solution of (6.65) with the initial condition $\rho_0(0, x, y)$.*

This result is used when static solutions of the ϕ^4 model are quantized by means of the Riemann $\zeta(s)$-function [248] expressed via the Green functions of (6.65) [285]. The one-loop correction to the action is obtained directly as $S_q = -\zeta'(0)$.

Example 6.8. The most popular example of the kink is obtained in this scheme by means of the DT applied to zero seed solution $u = 0$. The solution ρ of (6.65) with ρ_0 as the initial condition for this case is a simple heat equation solution:

$$\rho(\tau, x, y) = \frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^{\infty} \rho_0(z, y) \exp\left[-(x-z)^2/4\tau\right] \mathrm{d}z.$$

The initial condition ρ_0 is evaluated by direct integration in (6.68):

$$\rho_0(x, y) = \phi_1(x) \begin{cases} \phi_1^{-1}(y), & x > y, \\ 0, & x < y. \end{cases} \tag{6.69}$$

The Green function $\rho[2]$ (density matrix) for the kink solution as the potential is built by the twofold DT in accordance with the Wronskian formula (6.66):

$$\rho[2](\tau, x, y) = \exp\left(-\frac{1}{2\sqrt{\pi\tau}} \frac{(x-y)^2}{4\nu(\tau)}\right) \tag{6.70}$$

$$+\frac{1}{2} \sum_{m=1}^{n} \rho_m \psi_m(x) \psi_m(x_0) \left[\operatorname{erf}\left(\frac{x-y+2b_m\tau}{2\sqrt{\tau}}\right) - \operatorname{erf}\left(\frac{x-y-2b_m\tau}{2\sqrt{\tau}}\right)\right],$$

where $\operatorname{erf}(x)$ is the probability integral [206].

6.6.2 Resolvent of Schrödinger equation with reflectionless potential and Green functions

Some analytic properties of the resolvent kernel from the previous subsection give the possibility to generalize the construction. Let the operator

$$L = -\frac{\mathrm{d}^2}{\mathrm{d}x^2} + u(x) \tag{6.71}$$

contain the function $u(x)$ which corresponds to the reflectionless potential [354]. The Jost function $\psi(x, k)$ satisfies

$$\lim_{x \to \infty} \psi(x, k) \exp(-\mathrm{i}kx) = 1.$$

It has n poles in the lower half k-plane that correspond to bound states (discrete spectrum), i.e., eigenvalues of the operator (6.71): $\lambda_m = -b_m^2$, $b_m > 0$, $m = 1, \ldots, n$. Eigenstates ψ_m are normalized as

$$\lim_{x \to \infty} \exp(b_m x) = 1.$$

These properties permit us to represent the Jost function $\psi(x,k)$ as [354]

$$\psi(x,k) = \exp(\mathrm{i}kx)R(x,k), \qquad R(x,k) = \frac{P_n(x,k)}{\prod_{m=1}^{n}(k+\mathrm{i}b_m)}. \tag{6.72}$$

The k-polynomial $P_n(x,k)$ has the leading term k^n, so

$$\lim_{|k|\to\infty} R(x,k) = 1.$$

The simple decimal expansion for the function $R(x,k)$ corresponds to the expansion from Sect. 6.6.1 obtained by the DT,

$$R(x,k) = 1 - \mathrm{i}\sum_{m=1}^{n} \frac{\rho_m \psi_m \exp(-b_m x)}{k+\mathrm{i}b_m}, \tag{6.73}$$

where $\rho_m^{-1} = \int_{-\infty}^{\infty} \psi_m^2(x)\mathrm{d}x$. Combining the Jost functions in a standard way,

$$2\mathrm{i}kG(x,x_0,k) = \begin{cases} \psi(x,-k)\psi(x_0,k), & x < x_0, \\ \psi(x,k)\psi(x_0,-k), & x > x_0, \end{cases} \tag{6.74}$$

we arrive at the resolvent kernel $G(x,x_0,k)$ that satisfies the equation

$$(L-k^2)G(x,x_0,k) = \delta(x-x_0). \tag{6.75}$$

Analyzing the representation (6.72) for the Jost function at both parts of the x-axis (6.74), we conclude that

$$G(x,x_0,k) = -\exp(\mathrm{i}k|x-x_0|)S(x,x_0,k)/2\mathrm{i}k,$$

where the factor $S(x,x_0,k)$ is symmetric with respect to x, x_0, and a rational function of k with the only simple poles at $k = \pm\mathrm{i}b_m$, $m = 1,\ldots,n$, owing to (6.72) and $\lim_{k\to\infty} S(x,x_0,k) = 1$. Hence,

$$\mathrm{Res}_{k=\pm\mathrm{i}b_m} S(x,x_0,k) = \pm\mathrm{i}\rho_m\psi_m(x)\psi_m(x_0)\exp(b_m|x-x_0|). \tag{6.76}$$

The resulting formula for the kernel takes the form

$$G(x,x_0,k) = -\frac{\exp(\mathrm{i}k|x-x_0|)}{2\mathrm{i}k} \tag{6.77}$$

$$-\sum_{m=1}^{n} \rho_m\psi_m(x)\psi_m(x_0)\left(\frac{\exp[(\mathrm{i}k+b_m)(x-x_0)]}{k-\mathrm{i}b_m} - \frac{\exp[(\mathrm{i}k-b_m)(x-x_0)]}{k+\mathrm{i}b_m}\right).$$

Inserting the formula for G into the determining equation (6.75), we see that the pole term disappears and the remaining one gives the expression for the potential

$$u(x) = -4\frac{\mathrm{d}}{\mathrm{d}x}\sum_{m=1}^{n} \rho_m\psi_m(x)\psi_m(x_0)\sinh[b_m(x-x_0)]. \tag{6.78}$$

This result gives the known representation of the reflectionless potential by squares of the eigenfunctions when $x_0 = 0$.

Equations (6.78) and (6.77) are used now to build Green functions for a wide class of multidimensional problems.

Theorem 6.9. *Let L_0 be a linear differential operator in an auxiliary variable y with constant coefficients and let $E(x, y)$ be the fundamental function of the operator $L_0 - (\partial^2/\partial x^2)$, i.e., $E(x, y)$ is the solution of the equation*

$$(L_0 - \frac{\partial^2}{\partial x^2})E(x, y) = \delta(x - y). \tag{6.79}$$

Then the fundamental function of the operator $L + L_0$, i.e., the solution of

$$(L_0 + L)G(x, y, x_0, y_0) = \delta(x - x_0, y - y_0), \tag{6.80}$$

is given by the following sum:

$$G(x, y, x_0, y_0) = E(x - x_0, y - y_0) + \sum_{m=1}^{n} \rho_m \psi_m(x)\psi_m(x_0)E_m(x - x_0, y - y_0), \tag{6.81}$$

where $E_m(x, y)$ is a solution of the equation

$$\frac{\partial}{\partial x}E_m(x, y) = -2\sinh(b_m x)E(x, y). \tag{6.82}$$

The verification of the representation (6.81) can be performed by the direct substitution into (6.80) and use of (6.77), (6.79), (6.80), and (6.82).

Let us build the Green functions for two natural examples of heat and wave equations.

1. The Green function for the operator $(\partial/\partial t) + \nu L$, $\nu > 0$ is written as

$$G(x, t, x_0, t_0) = \theta(t')\left\{\exp\left(-\frac{1}{2\sqrt{\nu t'}}\frac{(x - x_0)^2}{4\nu(t')}\right) + \sum_{m=1}^{n} \rho_m \psi_m(x)\psi_m(x_0)\right. \tag{6.83}$$
$$\left.\times\left[\operatorname{erf}\left(\frac{x - x_0 + 2\nu b_m t'}{2\sqrt{\nu t'}}\right) - \operatorname{erf}\left(\frac{x - x_0 - 2\nu b_m t'}{2\sqrt{\nu t'}}\right)\right]\right\}.$$

2. The Green function for the operator $(1/c^2)(\partial^2/\partial t^2) + L$ is

$$G(x, t, x_0, t_0) = c\theta(ct' - |x - x_0|) \tag{6.84}$$
$$\times\left(\frac{1}{2} + \sum_{m=1}^{n} \frac{\rho_m}{b_m}\psi_m(x)\psi_m(x_0)\left[\cosh(cb_m t') - \cosh(b_m t')\right]\right),$$

Here $t' = t - t_0$. These relations are obtained by means of classical fundamental solutions of the wave and heat equations for a homogeneous medium.

Problems of heat/mass diffusion or wave propagation in a medium with a model kink/soliton inhomogeneity may be solved by means of the Green functions (6.83) and (6.84). Such an inhomogeneity can be induced by a soliton propagation.

This method can be applied for any problem with a solvable operator related to some simple operator by the factorization.

6.6.3 Dirac equations

As shown in [460], the DT works as well to describe a fermion in an external field in two dimensions (r, t). The method of intertwining is used to construct the DT between one-dimensional electric potentials or one-dimensional external scalar fields for which the Dirac equation is exactly solvable. It is shown that a class of exactly solvable Dirac potentials corresponds to soliton solutions of the modified Korteweg–de Vries equation, just as certain Schrödinger potentials are solitons of the Korteweg–de Vries equation. It is also shown that the intertwining transformations are related to Bäcklund transformations for the modified Korteweg–de Vries equation. The structure of the intertwining relations is shown to be described by an $N = 4$ superalgebra, generalizing supersymmetric quantum mechanics to the Dirac case.

6.7 Remarks on $d = 1$ and $d = 2$ supersymmetry theory within the dressing scheme

Following [286], we review here the one-dimensional supersymmetric quantum mechanics and discuss the two-dimensional problems.

6.7.1 General remarks on supersymmetric Hamiltonian/quantum mechanics

Supersymmetric quantum mechanics realizes the quantum description of systems with double degeneracy of energy levels. When $d = 1$, the supersymmetry incorporates the one–dimensional factorization method which is intrinsically connected to the DT as shown in Chap. 3; see also [214, 324]. The DT groups together two Hamiltonians h_0 and h_1 with equivalent spectra:

$$h_0 = q^+ q + E_0, \qquad h_1 = q q^+ + E_0, \qquad q \equiv \frac{\mathrm{d}}{\mathrm{d}x} - (\ln \varphi)',$$

where q^+ is the Hermitian conjugate of q and φ is a solution of the equation $h_0 \varphi = E_0 \varphi$ (*support function*). It is easy to see that h_0 and h_1 are intertwined by q and q^+:

$$q\, h_0 = h_1\, q, \qquad h_0\, q^+ = q^+\, h_1$$

(see Sect. 1.1), and therefore

$$\psi^{(1)} = q\, \psi, \qquad \psi = q^+\, \psi^{(1)}, \tag{6.85}$$

if $h_0\psi = E_1\psi$, $h_1\psi^{(1)} = E_1\psi^{(1)}$, and $\varphi^{(1)} = \varphi^{-1}$. For brevity, the normalizing multipliers in (6.85) are omitted.

The DT is a tool to construct one-dimensional potentials with arbitrarily preassigned discrete spectra. For example, if the support function $\varphi(E_0; x)$ is the wave function of the ground state of h_0, then the discrete spectrum of h_1 coincides with the spectrum of h_0 without lower level E_0 [324]. In Sect. 6.2.1 it was explained how to add level E_0 to the spectrum of h_1. To this end, it is sufficient to exploit a solution of $h_0\phi = E_0\phi$ such that

$$\varphi \to +\infty, \qquad x \to \pm\infty, \tag{6.86}$$

and φ is a positively definite function for all values of x. It is convenient to choose φ as (some rigorous conditions for that are discussed in Sect. 6.2.1)

$$\varphi = \lambda\varphi_+ + (1-\lambda)\varphi_-,$$

where φ_+ and φ_- are positively definite functions with the following asymptotic behavior:

$$\varphi_+ \to \begin{cases} +\infty, \text{ for } & x \to +\infty, \\ 0, \text{ for } & x \to -\infty, \end{cases}, \quad \varphi_- \to \begin{cases} +\infty, \text{ for } & x \to -\infty, \\ 0, \text{ for } & x \to +\infty, \end{cases} \tag{6.87}$$

and λ is a real parameter lying in the interval $[0, 1]$. If $0 < \lambda < 1$, then level E_0 is the lower level of the spectrum of h_1. If $\lambda = 0$ or $\lambda = 1$, level E_0 is missing in both spectra of h_0 and h_1 and the spectra of these Hamiltonians coincide (isospectral case of Sect. 6.2.1).

All this relates to the one-dimensional supersymmetric quantum mechanics based on the following commutation relations:

$$[Q, H] = [Q^+, H] = 0, \qquad \{Q, Q^+\} = H, \tag{6.88}$$

where

$$Q = q\sigma_+, \qquad Q^+ = q^+\sigma_-, \qquad H = \operatorname{diag}(h_0 - E_0, h_1 - E_0),$$

and $\sigma_\pm = (\sigma_1 \pm i\sigma_2)/2$, $\sigma_{1,2}$ are Pauli matrices. This system can be interpreted in terms of a two-level atom interacting with a one-mode electromagnetic field.

Let us remember that q and q^+ are bosonic and σ_- and σ_+ are fermionic creation–annihilation operators.If the spectra of h_0 and h_1 are identical except for the single level, then (6.88) corresponds to the exact supersymmetry. If level E_0 is absent in the spectra of both operators, the supersymmetry is said to be broken. It is easy to see that level E_0 cannot *simultaneously* be present in the spectra of h_0 and h_1. This means that if the lowest level in the spectrum of the supersymmetric Hamiltonian is zero, it is degenerate.

Generally [454], the supersymmetry algebra contains N charges Q_i that commute with the Hamiltonian and their anticommutators make up a natural generalization of the anticommutator in (6.88):

$$\{Q_i, Q_j^+\} = \delta_{ij} H_S.$$

6.7.2 Symmetry and supersymmetry via dressing chains

Let us return again to the dressing chain equation for superpotentials σ_i introduced in Sect. 6.1.2; see (6.9). For an illustration let us close the dressing chain of the operators $L_i = \partial - \sigma_i$ at the third step:

$$\sigma_4 = \sigma_1, \qquad \alpha_4 = \alpha_1.$$

Multiplying the intertwining relation

$$LL_1 = L_1 L[1] \tag{6.89}$$

from the left by the operator L_2, and repeating the procedure with L_3, we arrive at

$$LL_1L_2L_3 = L_1L_2L_3L.$$

This means that $A = L_1L_2L_3$ is a symmetry [268].

The supersymmetry, as mentioned, is the direct consequence of the intertwining relations of the type (6.89). The Darboux operators L_s and L_s^+ are operators of supercharge and the super-Hamiltonian $H = L \oplus L[1]$ unifies L and the transformed operator [265]. Moreover, $H_2 = L \oplus L[1] \oplus L[2]$ fits the matrix supercharge constructed by the row $(0, L_1, L_1L_2)$, for the operator L_1L_2 intertwines L and $L[2]$; hence,

$$Q_1 = \begin{pmatrix} 0 & L_1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$Q_2 = \begin{pmatrix} 0 & 0 & L_1L_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$Q_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & L_2 \\ 0 & 0 & 0 \end{pmatrix}.$$

A similar observation holds for any $A = \prod_1^N L_i$ if $\alpha_{N+1} = \alpha_1$ and $s_{N+1} = s_1$. This construction is purely algebraic and looks to be general. If the chain concerns the one-dimensional Sturm–Liouville operator as in Sect. 4.5, the superpotentials s_i are one-gap ones.

6.7.3 $d = 2$ Supersymmetry example

In contrast to $d = 1$, for $d > 1$ a connection between spectra of h_0 and h_1 constitutes an open problem; see the discussions in [49, 50, 51]. To clarify the

assertion, let us stress that when $d > 1$, we have no formulas expressing wave functions of h_1 via those of h_0 that could be similar to the one-dimensional case.

However, the existence of Hamiltonians of a special form that allow the connections between spectra is not forbidden. Moreover, there could be expressions that connect wave functions of the corresponding Hamiltonians $h_{0,1}$ in a manner that does not relate to a physical spectrum. As we shall see, both possibilities have the corresponding realization. We consider an example of two-dimensional supersymmetric quantum mechanics [286] with such a property. The explicit form of operators that satisfy the algebraic relations (6.88) at $d = 2$ is determined by the expressions

$$Q = \begin{pmatrix} 0 & 0 & 0 & 0 \\ q_1 & 0 & 0 & 0 \\ q_2 & 0 & 0 & 0 \\ 0 & q_2 & -q_1 & 0 \end{pmatrix}, \qquad Q^+ = \begin{pmatrix} 0 & q_1^+ & q_2^+ & 0 \\ 0 & 0 & 0 & q_2^+ \\ 0 & 0 & 0 & -q_1^+ \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{6.90}$$

$$H = \mathrm{diag}(h_0 - E_0, \tilde{h}_{lm} - 2\delta_{lm}E_0, h_1 - E_0), \tag{6.91}$$

where

$$h_0 = q_m^+ q_m + E_0, \quad h_1 = q_m q_m^+ + E_0, \quad \tilde{h}_{lm} \equiv h_{lm} + H_{lm} - E_0\delta_{lm} \tag{6.92}$$

and

$$h_{lm} = q_l q_m^+ + E_0\delta_{lm}, \qquad H_{lm} = p_l p_m^+ + E_0\delta_{lm}. \tag{6.93}$$

Here $q_l = \partial_l - \partial_l(\ln\varphi)$, $p_l = \varepsilon_{lk}q_k^+$, ε_{lk} is the antisymmetric tensor, $\partial_l \equiv \partial/\partial x^l$ with indices $l = 1, 2$, and the summation in repeated indices is implied.

The general coupling between the spectra exists for pairs h_0, h_{lm} and h_1, H_{lm}. Really, taking into account that h_1 may be represented as $h_1 = p_m^+ p_m + E_0$, it is easy to verify the validity of the intertwining relations:

$$q_l h_0 = h_{lm} q_m, \quad p_l h_1 = H_{lm} p_m, \quad h_0 q_l^+ = q_m^+ h_{ml}, \quad h_1 p_l^+ = p_m^+ H_{ml}.$$

Similar relations appear in a two-directional (full) Jaynes–Cummings model (Sect. 1.2.3), in which two supercharges generate the Jaynes–Cummings Hamiltonian. The Hamiltonian is a combination of generators of the orthosymplectic superalgebra Osp(2,2,$\mathbb{R}$). By the same formulas the operator $\tilde{h}_{lm}$ is intertwined with h_0 and h_1. Its spectrum coincides with the spectra of the scalar Hamiltonians, excluding maybe level E_0.

In [25, 49, 50, 51] the supersymmetry defined by operators (6.90) and (6.91) was studied, with the assumption that φ is a wave function of the basic state of the Hamiltonian h_0. It was shown that such a choice of φ leads to the assertion that level E_0 is absent in the physical parts of spectra of $\tilde{h}_{lm}$ and h_1, or to unbroken supersymmetry. Here we study the inverse problem: the addition of level E_0, which is absent in the spectrum of h_0, to the spectra of both operators. We will show that the resulting supersymmetric Hamiltonian possesses a doubly degenerate level with $E = 0$. This situation cannot be realized for $d = 1$ in general and for $d = 2$ within the "level-deleting" case.

6.7.4 Level addition

Let the function $u = u(x, y)$ be an integrable potential, i.e., it is supposed that we are able to solve the Schrödinger equation $h_0\psi \equiv (-\Delta + u)\psi = E\psi$ explicitly for any spectral parameter value E. Unlike the one-dimensional case, the potential

$$u^{(1)} = u - 2\Delta \ln \varphi,$$

where φ is the *support function*, is not integrable. Suppose that the spectral parameter value E_0 lies below the ground-state energy of the Hamiltonian h_0. The following question is important: How does one choose the *support function* φ in order for level E_0 to appear in the physical part of the spectra of h_1 and $\tilde{h}_{lm}$?

For a scalar Hamiltonian the answer to this question is not difficult. Really, it is easy to verify that the function φ^{-1} satisfies the equation

$$h_1 \frac{1}{\varphi} = E_0 \frac{1}{\varphi}.$$

Therefore, it is sufficient to choose φ as a positive function for all x and y that grows exponentially in all directions in the (x, y)-plane. The situation coincides literally with the one-dimensional case of Sect. 6.2.1 (if such a solution exists, i.e., we do not consider the excited levels).

For the matrix Hamiltonian, a more advanced consideration is necessary. First of all, note that if the function ψ is the second solution of the Schrödinger equation with the eigenvalue E_0, then the function

$$\tilde{\psi}_m = q_m \psi \tag{6.94}$$

satisfies the equation

$$\tilde{h}_{lm}\tilde{\psi} = E_0\tilde{\psi}.$$

Show now that for rapidly decreasing $\tilde{\psi}$ the representation (6.94) is not only sufficient but necessary as well.

To start with, we prove the following:

Theorem 6.10. *Level E_0 belongs to the spectrum of $\tilde{h}_{lm}$, iff the corresponding* normalized *wave function $\tilde{\psi}_m$ satisfies the condition*

$$h_{lm}\tilde{\psi}_m = H_{lm}\tilde{\psi}_m = E_0\tilde{\psi}_m. \tag{6.95}$$

Proof. Let the function $\tilde{\psi}_m$ exist such that

$$\tilde{h}_{lm}\tilde{\psi}_m = E_0\tilde{\psi}_m, \qquad (\tilde{\psi}_m, \tilde{\psi}_m) = 1.$$

Define the functions ρ_m and σ_m by equalities

$$\rho_m \equiv h_{lm}\tilde{\psi}_m, \qquad \sigma_m \equiv H_{lm}\tilde{\psi}_m. \tag{6.96}$$

It follows from (6.92) and (6.93) that $\sigma + \rho = 2E_0\tilde{\psi}$ (indices omitted), i.e.,

$$(\rho + \sigma, \rho + \sigma) = 4E_0^2, \tag{6.97}$$

if ρ and σ are normalizable. Otherwise, we can check that

$$h_{mk}H_{kl} = H_{mk}h_{kl} = E_0\tilde{h}_{ml}.$$

It follows from (6.96) that

$$h_{lm}\sigma_m = H_{lm}\rho_m = E_0^2\tilde{\psi}_l.$$

Hence,

$$(\tilde{\psi}_m, h_{lm}\sigma_l) = (h_{lm}\tilde{\psi}_m, \sigma_l) = (\rho_m, \sigma_m) = E_0^2. \tag{6.98}$$

Combining (6.98) and (6.97), we obtain $(\rho - \sigma, \rho - \sigma) = 0$; therefore, $\sigma_m = \rho_m = E_0\tilde{\psi}_m$. Finally, we go from (6.96) to (6.95).

Thus, for level E_0 to lie in the physical spectrum $\tilde{h}_{lm}$, it is necessary to find a normalizable solution of (6.95). Let $\tilde{\psi}_m$ be such a function. Allowing $\tilde{h}_{lm}$ to act on it, we get the equation

$$q_m^+\tilde{\psi}_m = p_m^+\tilde{\psi}_m = 0.$$

This means that there exist two functions ψ and $\psi^{(1)}$ such that

$$\tilde{\psi}_m = q_m\psi = p_m\psi^{(1)} \tag{6.99}$$

and that satisfy the equations

$$h_0\psi = E_0\psi, \qquad h_1\psi^{(1)} = E_0\psi^{(1)}. \tag{6.100}$$

Solving (6.99) with respect to $\psi^{(1)}$, we get the important relation that couples ψ and $\psi^{(1)}$:

$$\psi^{(1)} = \frac{1}{\varphi}\int \mathrm{d}x_k\varepsilon_{km}(\varphi\partial_m\psi - \psi\partial_m\varphi), \tag{6.101}$$

which is known as the Moutard transformation [324]. It remains to note that from the established connections between $\tilde{\psi}_m$, ρ_m, and σ_m, the formula (6.94) obviously follows.

Thus, for the presence of level E_0 in the spectrum of $\tilde{h}_{lm}$ there should be two normalized solutions ψ, φ of the Schrödinger equation with a potential u and the spectral parameter $E = E_0$ for the function $\tilde{\psi}_m = q_m\psi$ to be normalizable. In the next section we will exemplify this procedure.

6.7.5 Potentials with cylindrical symmetry

Let ψ and $\varphi > 0$ be the solutions described at the end of the previous subsection. For the construction of matrix potentials with level E_0 it is convenient to introduce an auxiliary function $f = \psi/\phi$ that satisfies the equation

$$\partial_m(\varphi^2 \partial_m f) = 0. \tag{6.102}$$

Then

$$\tilde{\psi}_m = \varphi \partial_m f. \tag{6.103}$$

Consider the case when the seed potential possesses cylindrical symmetry, $u = u(r)$. Integrating (6.102) and substituting into (6.103), we get

$$\tilde{\psi}_m = \frac{x_m}{r^2 \varphi}. \tag{6.104}$$

The normalizing integral for (6.104) converges if φ grows at infinity as a polynomial function and in the vicinity of zero it behaves as r^{-k}, $k > 0$. If we require that the asymptotic behavior of φ is determined by the conditions

$$\varphi \to \begin{cases} r^a, \text{ for } & x^2 + y^2 \to \infty, \\ r^b, \text{ for } & x^2 + y^2 \to 0, \end{cases} \tag{6.105}$$

where $a > 1$ and $b < 1$, then the normalizing integral of φ^{-1} should converge as well. This means that level E_0 exists in spectra of both operators h_1 and $\tilde{h}_{lm}$ simultaneously.

It was shown in [49, 50, 51] that such a situation cannot take place for the Hamiltonians h_0 and $\tilde{h}_{lm}$. It is easy to see the difference between these couples. For example, using as the *support function* $1/\varphi$, it is possible to construct a new supersymmetric Hamiltonian

$$\widehat{H} = \operatorname{diag}(h_1 - E_0, \hat{h}_{lm} - 2\delta_{lm} E_0, h_0 - E_0). \tag{6.106}$$

The operator $\hat{h}_{lm}$ differs from $\tilde{h}_{lm}$ by the intertwining property. Namely, $\hat{h}_{lm}$ is intertwined with h_1 not by the operators p_m but by the dual ones q_m^+. Respectively, level E_0 does not exist in its spectrum, whereas for the rest of these operators the spectra coincide. Note that such "equivalent by spectrum" matrix operators were considered in [27].

The spectrum of the supersymmetric Hamiltonian (6.91) consists of the levels $\{E_i - E_0, E_i^{(1)} - E_0\}$, where E_i and $E_i^{(1)}$ are the levels of the discrete spectrum parts of h_0 and h_1, respectively. Now it is seen that if the condition (6.105) is satisfied, then in the spectrum of (6.91) there exists the doubly degenerate level $E = 0$ with the following eigenfunctions:

$$\Psi_1 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1/\varphi \end{pmatrix}, \qquad \Psi_2 = \begin{pmatrix} 0 \\ \varphi \partial_1 f \\ \varphi \partial_2 f \\ 0 \end{pmatrix}. \tag{6.107}$$

Using the explicit form of the odd supersymmetric operators (6.90) proves the validity of the relations for the wave functions for the zero level:

$$Q\Psi_{1,2} = Q^{+}\Psi_{1,2} = 0.$$

As an example, choose $\varphi = \exp(br)/r^k$, where $b > 0$ and $k > 0$. This function satisfies the necessary asymptotic (6.105). As a result, we obtain two scalar potentials of the Hamiltonians h_0 and h_1:

$$u = \frac{k^2}{r^2} - \frac{b(2k-1)}{r}, \qquad u^{(1)} = \frac{k^2}{r^2} - \frac{b(2k+1)}{r}. \tag{6.108}$$

The additional level corresponds to the energy $E_0 = -b^2$. It may be verified that the potentials are integrated by means of the confluent hypergeometric function. The discrete spectra are determined by

$$E_{_N} = -\frac{b^2(2k \mp 1)^2}{(1 + 2[N + \sqrt{m^2 + k^2}])^2}, \tag{6.109}$$

where the minus sign corresponds to u, the plus sign corresponds to $u^{(1)}$, N is the principal quantum number, and m stands for the magnetic quantum number.

The constructed potentials are interesting as an example that exhibits a difference between the DTs in multidimensions and their one-dimensional counterpart. Specifically, the comparison of the spectra of Hamiltonians h_0 and h_1 shows that the addition of the lowest level shifts all the spectrum. If we consider the potentials (6.108), it can be seen that when

$$k = \frac{(N+1)^2 - m^2}{2(N+1)},$$

the addition of the level $E_0 = -b^2$ does not move the excited level with the number N and fixed m. In general, the levels of the Hamiltonian h_1 go down in respect of the levels of h_0. This displacement is maximal in the lowest part of the well and decreases as $1/N^2$ in the higher part of the spectrum. In turn, the spectrum of the supersymmetric Hamiltonian (6.91) is doubly degenerate, including the level $E = 0$. Its normalizable vacuum wave functions are given by the expressions (6.107), and

$$\frac{1}{\varphi} = r^k \exp(-br), \qquad \partial_m f = \frac{x_m}{(r\varphi)^2}$$

yield their explicit form.

7

Important links

In this chapter we sketch some important links between ideas of the dressing Darboux transformation (DT), Bäcklund transformation (BT), etc. with related mathematical constructions. Firstly, it is the Hirota representation which originally produced many of the known families of multisoliton solutions, and these have often led to a disclosure of the underlying Lax systems and infinite sets of conserved quantities [209, 385]. In Sect. 7.1 we demonstrate a systematic derivation of the bilinear BTs from the so-called $\mathcal{Y}$-systems which are formulated in terms of the binary Bell polynomials. Taking as the example equations with the "sech^2" soliton solutions, we illustrate how to obtain the binary BTs for different weights of the $\mathcal{Y}$-polynomials. In Sect. 7.2 we represent the Darboux covariant Lax pairs in terms of the $\mathcal{Y}$-systems. In Sect. 7.3 we explain how to construct BTs from the explicit dressing formulas and, using the Noether theorem, how to derive discrete and continuous conservation laws. Next, in Sect. 7.4 the main formulas of the dressing theory are retrieved within the Weiss–Tabor–Carnevale procedure [449] of Painlevé analysis for partial differential equations (PDEs). In addition, we comment on a historical point connected with the appearance of the dressing method in the Zakharov–Shabat theory. Namely, we suggest in Sect. 7.5 an original revisiting of the technique of inverse scattering transform (IST) in terms of the Gel'fand–Levitan–Marchenko integral equation. Notice in connection with this that the search for perhaps the most general dressing scheme within the framework of the Zakharov and Shabat ideas is represented in [478].

7.1 Bilinear formalism. The Hirota method

A striking feature of the bilinear formalism is the ease with which direct insight can be gained into the nature of the eigenvalue problem associated with soliton equations (such as the KdV, Boussinesq, or Sawada–Kotera equations) derivable from the bilinear Hirota equation (representation) for a single Hirota function. The key element is the bilinear form of the BT which can be

straightforwardly obtained from the Hirota representation of these equations, through decoupling of a related "two-field condition" by means of an appropriate constraint of minimal weight [262]. The main point is that bilinear BTs are obtained systematically, without the need for tricky *exchange formulas* [209]. They arise in the form of "$\mathcal{Y}$-systems," each equation within such a system belonging to a linear space spanned by the basis of binary Bell polynomials ($\mathcal{Y}$-polynomials) [187].

An important element is the logarithmic linearizability of $\mathcal{Y}$-systems, which implies that each bilinear BT can be mapped onto a corresponding linear system of the Lax type. However, it turns out that these linear systems involve differential operators which, even in the simplest case, do *not* constitute a Darboux covariant [265, 324] Lax pair . This fact prevents us from obtaining large classes of solutions by direct application of the powerful Darboux machinery to the systems which arise by straightforward linearization of the $\mathcal{Y}$-systems. Here we present a simple scheme to resolve this difficulty for a variety of soliton equations which allow a bilinear BT that comprises a constraint of the lowest possible weight (weight 2). Darboux covariant Lax pairs for the KdV, Boussinesq, and Lax equations are obtained in a unified manner, by exploiting the relations between the coefficients of linear differential operators connected by the classical DT. Exponential Bell polynomials [44] and generalized "multipotential" $\mathcal{Y}$-systems are found to be useful for this purpose. This approach reveals deep connections between the (1+1)-dimensional equations and the underlying (higher-dimensional) Kadomtsev–Petviashvili (KP) hierarchy. We start our discussion by recalling the basic properties of the $\mathcal{Y}$-polynomials (derived in [187]) and by indicating how the use of the $\mathcal{Y}$-basis can lead systematically from the original nonlinear PDEs to the associated linear systems. The example of the Lax equation is instructive since this fifth-order equation has no single bilinear Hirota representation. The content of this section follows [260].

7.1.1 Binary Bell polynomials

The class of exponential Bell polynomials, originally defined for the Abelian entries as

$$Y_{mx}(v) = Y_m(v_x, v_{xx}, ..., v_{mx}) \equiv e^{-v}\frac{\partial^m}{\partial x^m}e^v, \quad m \in \mathbb{Z}, \tag{7.1}$$

was introduced in Sect. 2.1. It keeps a balance between linear and quadratic terms of the (generalized) Burgers equation, for

$$Y_{mx}(\ln\psi) = \psi_{mx}/\psi. \tag{7.2}$$

Examples are easily derived and are given in Sect. 2.1. The property of x-homogeneity,

$$Y_{m(\lambda x)}(v) = \lambda^{-m}(v)Y_{mx}(v), \tag{7.3}$$

introduces the weight m.

The binary polynomials that we shall use in this section are defined in terms of the exponential Bell polynomials

$$Y_{mx,nt}(f) = \mathrm{e}^{-f}\partial_x^m\partial_t^n\mathrm{e}^f \tag{7.4}$$

as follows:

$$\mathcal{Y}_{mx,nt}(v,w) \equiv Y_{mx,nt}(f)\Big|_{f_{px,qt}=\begin{cases} v_{px,qt} \text{ if } p+q=\text{odd}, \\ w_{px,qt} \text{ if } p+q=\text{even}, \end{cases}} \tag{7.5}$$

with the understanding that $f_{px,qt} \equiv \partial_x^p\partial_t^q f$. They inherit the easily recognizable partition structure of the Bell polynomials (for a recurrent definition see Sect. 2.2):

$$\begin{gathered} \mathcal{Y}_x(v) = v_x, \\ \mathcal{Y}_{2x}(v,w) = w_{2x} + v_x^2, \\ \mathcal{Y}_{x,t}(v,w) = w_{xt} + v_xv_t, \\ \mathcal{Y}_{3x}(v,w) = v_{3x} + 3v_xw_{2x} + v_x^3, \ \cdots \end{gathered} \tag{7.6}$$

The link between the $\mathcal{Y}$-polynomials and the standard Hirota expression

$$D_x^pD_t^qG'\cdot G \equiv (\partial_x - \partial_{x'})^p\,(\partial_t - \partial_{t'})^q\,G'(x,t)G(x',t')\Big|_{x'=x,t'=t} \tag{7.7}$$

is given by the identity

$$\mathcal{Y}_{mx,nt}(v=\ln G'/G,\ w=\ln G'G) \equiv (G'G)^{-1}D_x^mD_t^nG'\cdot G. \tag{7.8}$$

In the particular case $G'=G$, one has

$$G^{-2}D_x^mD_t^nG\cdot G \equiv \mathcal{Y}_{mx,nt}(0,\,Q=2\ln G) = \begin{cases} 0, & \text{if} \quad m+n=\text{odd}, \\ P_{mx,nt}(Q), & \text{if} \quad m+n=\text{even}, \end{cases} \tag{7.9}$$

the P-polynomials being characterized by an equally recognizable "even part" partition structure:

$$P_{2x}(Q) = Q_{2x}, \quad P_{x,t}(Q) = Q_{xt}, \quad P_{4x}(Q) = Q_{4x} + 3Q_{2x}^2,$$

$$P_{6x}(Q) = Q_{6x} + 15Q_{2x}Q_{4x} + 15Q_{2x}^3, \ldots. \tag{7.10}$$

A crucial property of the $\mathcal{Y}$-polynomials relates to the transformation $w = v + Q$, $v = \ln\psi$:

$$\mathcal{Y}_{px,qt}(v, w=v+Q)\Big|_{v=\ln\psi} \tag{7.11}$$

$$= \psi^{-1}\sum_{j=0}^{p}\sum_{\substack{k=0 \\ j+k=\text{even}}}^{q}\binom{p}{q}\binom{q}{k}P_{jx,kt}(Q)\psi_{(p-j)x,(q-k)t}$$

and originates from the addition formula for the polynomials $Y(v)$:

$$Y_{mx}(v_1+v_2)=\sum_{j=0}^{m}\binom{m}{j}Y_{(m-j)x}(v_1)Y_{jx}(v_2). \tag{7.12}$$

The proof is performed by use of the Newton–Leibnitz formula.

It should also be noticed that polynomials $\mathcal{Y}_{px,qt}(v,w)$, constructed with the derivatives of dimensionless variables v and w, are homogeneous expressions of the weight $p+qr$, if r stands for the dimension of t (the dimension of x is chosen equal to 1).

7.1.2 $\mathcal{Y}$-systems associated with "sech^2" soliton equations

We consider four examples of "sech^2" soliton equations with the order ranging from 3 to 5: the KdV, Boussinesq, Lax, and Sawada–Kotera equations.

KdV equation

The invariance of the KdV equation

$$\mathrm{KdV}(u)\equiv u_t+u_{3x}+6uu_x=0 \tag{7.13}$$

under the scale transformation

$$x\to\lambda x,\quad t\to\lambda^3 t,\quad u\to\lambda^{-2}u \tag{7.14}$$

shows that u has the dimension -2. A dimensionless field Q can be introduced by setting $u=cQ_{2x}$, with c being a dimensionless parameter to be determined. The resulting equation for Q can be derived from the *potential* equation

$$Q_{xt}+Q_{4x}+3cQ_{2x}^2=0, \tag{7.15}$$

which can be cast into the form

$$E(Q)\equiv P_{xt}(Q)+P_{4x}(Q)\equiv G^{-2}(D_xD_t+D_x^4)G\cdot G\Big|_{G=\exp(Q/2)}=0 \tag{7.16}$$

by setting $c=1$.

The well-known Hirota *two-field condition* on G and G', to be satisfied as a differential consequence of a bilinear BT (that we have to find), takes the form [209]

$$G'^{-2}(D_xD_t+D_x^4)G'\cdot G'-G^{-2}(D_xD_t+D_x^4)G\cdot G=0. \tag{7.17}$$

It corresponds to the following condition on $Q=2\ln G=w-v$ and $Q'=2\ln G'=w+v$:

$$E(w+v)-E(w-v)=2(v_{xt}+v_{4x}+6v_{2x}w_{2x})$$
$$\equiv 2\left\{\partial_x\left[\mathcal{Y}_t(v)+\mathcal{Y}_{3x}(v,w)\right]+6W[\mathcal{Y}_{2x}(v,w),\mathcal{Y}_x(v)]\right\}=0, \tag{7.18}$$

where $W(\mathcal{Y}_1, \mathcal{Y}_2)$ is the Wronskian. This condition can easily be decoupled into a pair of equations in the form of linear combinations of the $\mathcal{Y}$-polynomials. It suffices to impose such a constraint on v and w (p_j and q_j are integers or zero, c_j is a constant),

$$\sum_j c_j \mathcal{Y}_{p_j x, q_j t}(v, w) = 0, \tag{7.19}$$

of the lowest possible order (or weight). The simplest choice is a constraint of weight 2:

$$\mathcal{Y}_{2x}(v, w) \equiv w_{2x} + v_x^2 = 0. \tag{7.20}$$

In order to obtain a parameter-dependent decomposition, we should impose the condition

$$\mathcal{Y}_{2x}(v, w) = \lambda, \tag{7.21}$$

where λ is an arbitrary parameter of weight 2. This leads to the following $\mathcal{Y}$-system

$$\mathcal{Y}_{2x}(v, w) - \lambda = 0, \qquad \mathcal{Y}_t(v) + \mathcal{Y}_{3x}(v, w) + 3\lambda \mathcal{Y}_x(v) = 0, \tag{7.22}$$

the compatibility of which is guaranteed by the corresponding system for ψ [setting $w = v + Q$, $v = \ln \psi$ and using (7.10)]:

$$(L_2 - \lambda)\psi \equiv \psi_{2x} + (Q_{2x} - \lambda)\psi = 0, \tag{7.23}$$

$$(\partial_t + \mathcal{L}_3)\psi \equiv \psi_t + \psi_{3x} + 3(Q_{2x} + \lambda)\psi_x = 0,$$

i.e., to the (λ-independent) condition:

$$(Q_{xt} + Q_{4x} + 3Q_{2x})_x \equiv \partial_x E(Q) = 0. \tag{7.24}$$

The bilinear equivalent of the $\mathcal{Y}$-system (7.22) is obtained by means of (7.8):

$$D_x^2 G' \cdot G = \lambda G' G, \qquad (D_t + D_x^3 + 3\lambda D_x) G' \cdot G = 0. \tag{7.25}$$

It is the bilinear BT for the KdV proposed by Hirota [209].

Boussinesq equation

A similar analysis can be applied to the Boussinesq equation

$$\mathrm{Bq}(u) \equiv u_{2t} - u_{4x} + 3(u^2)_{2x} = 0. \tag{7.26}$$

This equation can be derived from a potential version obtained by setting $u = -Q_{2x}$:

$$E(Q) \equiv P_{2t}(q) - P_{4x}(Q) \equiv G^{-2}(D_t^2 - D_x^4)G \cdot G\Big|_{G=\exp(Q/2)} = 0. \tag{7.27}$$

The corresponding two-field condition

$$E(Q' = w + v) - E(Q = w - v) \equiv 2(v_{2t} - v_{4x} - 6v_2 w_{2x})$$

$$= -2\partial_x \mathcal{Y}_{3x}(v, w) + 2v_{2t} + 6W[\mathcal{Y}_{2x}(v, w), \mathcal{Y}_x(v)] = 0 \tag{7.28}$$

can still be decoupled into a pair of equations of the form (7.19) by means of the $\mathcal{Y}$-constraint of weight 2 (notice that in this case the dimension of $t = 2$, so we dispose of two $\mathcal{Y}$-polynomials of weight 2):

$$\mathcal{Y}_t(v) + a\mathcal{Y}_{2x}(v, w) = 0, \tag{7.29}$$

where a is a dimensionless constant to be determined.

The decoupling requires $a^2 = -3$ and produces the following parameter-dependent $\mathcal{Y}$-system (λ is an integration constant):

$$\mathcal{Y}_t + a\mathcal{Y}_{2x}(v, w) = 0, \quad a\mathcal{Y}_{x,t}(v, w) + \mathcal{Y}_{3x}(v, w) = \lambda. \tag{7.30}$$

The corresponding bilinear system

$$(D_t + aD_x^2)G' \cdot G = 0, \qquad (aD_x D_t + D_x^3 - \lambda)G' \cdot G = 0 \tag{7.31}$$

is exactly the bilinear BT for the Boussinesq equation obtained by Nimmo and Freeman [350]. Its compatibility is subject to that of the linear equivalent to the system (7.30):

$$\psi_t + a\psi_{2x} + aQ_{2x}\psi = 0, \tag{7.32}$$

$$a\psi_{xt} + \psi_{3x} + 3Q_{2x}\psi_x + (aQ_{xt} - \lambda)\psi = 0,$$

i.e., to the following potential version of the Boussinesq equation:

$$\mathrm{PBq}(Q) \equiv (Q_{2t} - Q_{4x} - 3Q_{2x}^2)_x = 0. \tag{7.33}$$

Lax equation

We now consider the Lax equation

$$\mathrm{Lax}(u) \equiv u_t + u_{5x} + 10uu_{3x} + 20u_x u_{2x} + 30u^2 u_x = 0. \tag{7.34}$$

Setting $u = cQ_{2x}$ brings it to the potential equation:

$$E_c(Q) \equiv Q_{xt} + Q_{6x} + 10cQ_{2x}Q_{4x} + 5cQ_{3x}^2 + 10c^2Q_{2x}^3 = 0. \tag{7.35}$$

The left-hand side of this equation is homogeneous with weight 6, but there is no value of c such that (7.35) can be expressed as a linear combination of the weight 6 polynomials $P_{6x}(Q)$ and $P_{xt}(Q)$. Setting $c = 1$, we may nevertheless consider the two-field condition

$$E_1(w+v) - E_1(w-v) \equiv 2\left\{\partial_x\left[\mathcal{Y}_t(v) + \mathcal{Y}_{5x}(v,w)\right] + R(v,w)\right\} = 0, \tag{7.36}$$

with

$$\begin{aligned} R(v,w) = -5\big(&v_xw_{5x} - v_{2x}w_{4x} + 6v_xw_{2x}w_{3x} + 2v_x^3w_{3x} - 3v_{2x}w_{2x}^2 \\ &+6v_x^2v_{2x}w_{2x} + 4v_xv_{2x}v_{3x} + 2v_x^2v_{4x} + v_x^4v_{2x} - 2v_{2x}^3\big). \end{aligned} \tag{7.37}$$

Eliminating w_{2x} and its derivatives by means of the weight 2 constraint (7.21), we find that the condition (7.36) can be decoupled into the following $\mathcal{Y}$-system:

$$\mathcal{Y}_{2x}(v,w) = \lambda, \quad \mathcal{Y}(v) + \mathcal{Y}_{5x}(v,w) + 15\lambda^2\mathcal{Y}_x(v) = 0. \tag{7.38}$$

Its compatibility is subjected to that of the corresponding linear system:

$$\begin{gathered} \psi_{2x} + (Q_{2x} - \lambda)\psi = 0, \qquad \psi_t + \mathcal{L}_5\psi = 0, \\ \mathcal{L}_5 = \partial_x^5 + 10Q_{2x}\partial_x^3 + 5(Q_{4x} + 3Q_{2x}^2 + 3\lambda^2)\partial_x, \end{gathered} \tag{7.39}$$

i.e., to the condition

$$(Q_{xt} + Q_{6x} + 10Q_{2x}Q_{4x} + 5Q_{3x}^2 + 10Q_{2x}^3)_x \equiv \partial_x E_1(Q) = 0. \tag{7.40}$$

Sawada–Kotera equation

We finally consider the Sawada–Kotera equation

$$\mathrm{SK}(u) \equiv u_t + u_{5x} + 15uu_{3x} + 15u_xu_{2x} + 45u^2u_x = 0, \tag{7.41}$$

which again can be derived from a potential equation by setting $u = Q_{2x}$, expressible in terms of $P_{6x}(Q)$ and $P_{xt}(Q)$:

$$E(Q) = P_{xt}(Q) + P_{6x}(Q) \equiv G^{-2}(D_xD_t + D_x^6)G\cdot G\Big|_{G=\exp(Q/2)} = 0. \tag{7.42}$$

It is easy to see that the corresponding two-field condition

$$E(w+v) - E(w-v) \equiv 2\partial_x\left[\mathcal{Y}_t(v) + \mathcal{Y}_{5x}(v,w)\right] + 10R(v,w) = 0, \tag{7.43}$$

with

$$\begin{aligned} R(v,w) = &-v_x w_{5x} + 2v_{2x} w_{4x} - 2v_{3x} w_{3x} + w_{2x} v_{4x} \\ &-2v_x^2 v_{4x} - 4v_x v_{2x} v_{3x} + 6v_{2x} w_{2x}^2 \\ &+3v_{2x}^3 - 6v_x w_{2x} w_{3x} - 2v_x^3 w_{3x} - 6v_x^2 v_{2x} w_{2x} - v_x^4 v_{2x}, \end{aligned} \tag{7.44}$$

can no longer be decoupled into a $\mathcal{Y}$-system by means of a weight 2 constraint of the form (7.20).

Yet, the weight 3 constraint

$$\mathcal{Y}_{3x}(v,w) \equiv v_{3x} + 3v_x w_{2x} + v_x^3 = \lambda \tag{7.45}$$

enables us to express $R(v,w)$ as follows:

$$R(v,w) = -\frac{1}{2}\partial_x \left[\mathcal{Y}_{5x}(v,w) + 3\lambda \mathcal{Y}_{2x}(v,w)\right]. \tag{7.46}$$

This means that the condition (7.43) can be decoupled into the following (λ-dependent) $\mathcal{Y}$-system:

$$\mathcal{Y}_{2x}(u,v) - \lambda = 0, \quad \mathcal{Y}_t(v) - \frac{3}{2}\mathcal{Y}_{5x}(v,w) - \frac{15}{2}\lambda \mathcal{Y}_{2x}(v,w) = 0. \tag{7.47}$$

Its compatibility is subjected to that of the corresponding ψ-system ($w = v + Q,\ v = \ln\psi$):

$$\begin{aligned} &\psi_{3x} + 3Q_{2x}\psi_x - \lambda\psi = 0, \\ &\psi_t - \frac{3}{2}\psi_{5x} - 15Q_{2x}\psi_{3x} - \frac{15}{2}P_{4x}(Q)\psi_x - \frac{15}{2}\lambda(\psi_{2x} + Q_{2x}\psi) = 0, \end{aligned} \tag{7.48}$$

i.e., to the condition:

$$\left(Q_{xt} + Q_{6x} + 15Q_{2x}Q_{4x} + 15Q_{2x}^3\right)_x \equiv \partial_x E(Q) = 0. \tag{7.49}$$

The bilinear equivalent of the system (7.47),

$$(D_x^3 - \lambda)G' \cdot G = 0, \qquad \left(D_t - \frac{3}{2}D_x^5 - \frac{15}{2}\lambda D_x^2\right) G' \cdot G = 0, \tag{7.50}$$

is the bilinear BT for the Sawada–Kotera equation reported in [386].

7.2 Darboux-covariant Lax pairs in terms of $\mathcal{Y}$-functions

In Sect. 3.7 a joint covariance property of operators was defined and investigated. It results in some necessary conditions, e.g., the joint covariance equations, whose solutions yield restriction on a form of solvable equations. Let us now go back to the KdV equation (7.13) and the associated linear system (7.23). It comprises the second-order eigenvalue equation considered by Lax

[263], with the covariance property we study throughout this book. According to this property, (nonvanishing) solutions ϕ to the spectral equation produce transformations

$$G_\phi = \phi\partial_x\phi^{-1} = \partial_x - \sigma, \quad \sigma = \partial_x \ln\phi, \tag{7.51}$$

which map $L_2 = \partial_x^2 + Q_{2x}$ onto the similar operator

$$\widetilde{L}_2 \equiv G_\phi L_2(Q_{2x})G_\phi^{-1} \equiv L_2(\widetilde{Q}_{2x}), \tag{7.52}$$

with $\widetilde{Q}_{2x} = Q_{2x} + 2\sigma_x$. With the second-order eigenvalue equation obtained from the constraint (7.21) through the map $v = \ln\psi$,

$$\mathcal{Y}_{2x}(v,\ v+Q) = \lambda, \tag{7.53}$$

we may try to associate a Darboux-covariant third-order evolution equation. Note that any equation of the form

$$\sum_n c_n \mathcal{Y}_{p_n x, q_n t}\left(v, v+Q^{(n)}\right) = 0 \tag{7.54}$$

corresponds to a linear equation for ψ. In particular, there is a correspondence between the evolution equation (c_2 and c_3 are constants)

$$\mathcal{Y}_t(v) + c_2\mathcal{Y}_{2x}\left(v, v+Q^{(2)}\right) + c_3\mathcal{Y}_{3x}\left(v, v+Q^{(3)}\right) = 0 \tag{7.55}$$

and its linear counterpart

$$\psi_t + L_3\psi = 0, \quad L_3 = c_3\partial_x^3 + c_2\partial_x^2 + b_1\partial_x + b_0, \tag{7.56}$$

with

$$b_1 = 3c_3Q_{2x}^{(3)}, \quad b_0 = c_2Q_{2x}^{(2)}. \tag{7.57}$$

Let G_ϕ be a transformation (7.51) generated by a (nonvanishing) solution ϕ of the system

$$\begin{aligned}&(L_2-\lambda)\phi \equiv (\partial_x^2 + Q_{2x} - \lambda)\phi = 0,\\ &(\partial_t + L_3)\phi \equiv (\partial_t + c_3\partial_x^3 + c_2\partial_x^2 + b_1\partial_x + b_0)\phi = 0.\end{aligned} \tag{7.58}$$

It maps the operators $L_2 - \lambda$ and $\partial_t + L_3$ onto the similar operators

$$G_\phi(L_2-\lambda)G_\phi^{-1} = L_2(\widetilde{Q}_{2x}) - \lambda, \quad G_\phi(\partial_t + L_3)G_\phi^{-1} = \partial_t + \widetilde{L}_3,$$

$$\widetilde{L}_3 \equiv c_3\partial_x^3 + c_2\partial_x^2 + \widetilde{b}_1\partial_x + \widetilde{b}_0, \tag{7.59}$$

where

$$\widetilde{b}_1 = b_1 + \Delta b_1, \qquad \Delta b_1 = 3c_3\sigma_x, \tag{7.60}$$

$$\widetilde{b}_0 = b_0 + \Delta b_0, \qquad \Delta b_0 = b_{1,x} + \sigma\Delta b_1 + 2c_2\sigma_x + 3c_3\sigma_{2x}, \tag{7.61}$$

and the following differential consequences of (7.58) have been taken into account:

$$\partial_x\left(\frac{\phi_{2x}}{\phi}+Q_{2x}-\lambda\right)=0 \quad\Longleftrightarrow\quad \partial_x Y_{2x}(\ln\phi)+Q_{3x}\equiv(\sigma_x+\sigma^2)_x+Q_{3x}=0,$$

$$\partial_x\left[Y_t(\ln\phi)+c_3Y_{3x}(\ln\phi)+c_2Y_{2x}(\ln\phi)+b_1Y_x(\ln\phi)+b_0\right]=0 \tag{7.62}$$

$$\Longleftrightarrow\quad \sigma_t+c_3(\sigma_{2x}+3\sigma\sigma_x+\sigma^3)_x+c_2(\sigma_x+\sigma^2)_x+(b_1\sigma)_x+b_{0,x}=0.$$

In order that ∂_t+L_3 be the Darboux-covariant with $L_2-\lambda$, we have to determine the coefficients b_i, $i=0,1$, as functions of Q_{2x} and its derivatives, in such a way that the covariance condition

$$\widetilde{L}_3(Q_{2x},\ Q_{3x},\ \ldots)=L_3(\widetilde{Q}_{2x},\ \widetilde{Q}_{3x},\ldots) \tag{7.63}$$

be satisfied with

$$\Delta Q_{(r+1)x}\equiv\widetilde{Q}_{(r+1)x}-Q_{(r+1)x}=2\sigma_{rx},\quad r=1,2,\ldots. \tag{7.64}$$

Hence, we should look for expressions $b_i=F_i(Q_{2x},Q_{3x},\ldots)$ such that the differences Δb_i which appear in (7.58) and (7.59) are expressible as

$$\Delta b_i=F_i(Q_{2x}+\Delta Q_{2x},Q_{3x}+\Delta Q_{3x},\ldots)-F_i(Q_{2x},Q_{3x},\ldots),\quad i=0,1. \tag{7.65}$$

Because

$$\Delta b_1=\frac{3}{2}c_3\Delta Q_{2x}, \tag{7.66}$$

it is clear that we can find an expression F_i, linear in Q_{2x}, which satisfies condition (7.64), yielding

$$b_1=\frac{3}{2}c_3Q_{2x}+c_1, \tag{7.67}$$

c_1 being an arbitrary constant. The difference Δb_0 is now given by the relation

$$\Delta b_0=\frac{3}{2}c_3Q_{3x}+3c_3\sigma\sigma_x+2c_2\sigma_x+3c_3\sigma_{2x}, \tag{7.68}$$

which, on account of (7.62), becomes

$$\Delta b_0=2c_2\sigma_x+\frac{3}{2}c_3\sigma_{2x}=c_2\Delta Q_{2x}+\frac{3}{4}c_3\Delta Q_{3x}. \tag{7.69}$$

It follows that we can find an expression F_0, linear in Q_{2x} and Q_{3x}, which satisfies condition (7.64), yielding

$$b_0=c_2Q_{2x}+\frac{3}{4}c_3Q_{3x}+c_0, \tag{7.70}$$

where c_0 is an arbitrary constant. Setting $c_0 = c_1 = 0$, we obtain

$$L_3 = c_3 \left(\partial_x^3 + \frac{3}{2} Q_{2x} \partial_x + \frac{3}{4} Q_{3x} \right) + c_2 L_2, \tag{7.71}$$

indicating that the simplest Darboux-covariant third-order evolution equation to be associated with (7.58) has the form (setting $c_2 = 0,\ c_3 = 4$)

$$(\psi_t + \bar{L}_3)\psi \equiv 0, \quad \bar{L}_3 = 4\partial_x^3 + 6Q_{2x}\partial_x + 3Q_{3x}. \tag{7.72}$$

Together with (7.54) it produces an equivalent version of our previous linear system (7.23) for the KdV equation, obtained by replacing the second equation by the combination

$$[\partial_t + \mathcal{L}_3 + 3\partial_x(L_2 - \lambda)]\psi = 0. \tag{7.73}$$

The operator $\bar{L}_3$ corresponds precisely to the third-order operator which gives rise to the KdV equation in the Lax formalism [350]:

$$[\partial_t + \bar{L}_3, L_2] = (Q_{xt} + Q_{4x} + 3Q_{2x}^2)_x = 0. \tag{7.74}$$

The full Darboux-covariant system obtained with expression (7.71) for L_3,

$$(L_2 - \lambda)\psi = 0, \qquad (\partial_t + L_3)\psi = 0, \tag{7.75}$$

corresponds, through the map $v = \ln \psi$, to the *multipotential* $\mathcal{Y}$-system

$$\mathcal{Y}_{2x}(v, v + Q) = \lambda, \tag{7.76}$$

$$\mathcal{Y}_t(v) + c_3 \mathcal{Y}_{3x}\left(v, v + Q^{(3)}\right) + c_2 \mathcal{Y}_{2x}\left(v, v + Q^{(2)}\right) = 0,$$

in which

$$Q_{2x}^{(3)} = \frac{1}{2} Q_{2x}, \quad Q_{2x}^{(2)} = Q_{2x} + \frac{3}{4}\frac{c_2}{c_3} Q_{3x}. \tag{7.77}$$

An interesting alternative to this system results from an interchange between $\mathcal{Y}_{2x}(v, v + Q)$ and $c_3 \mathcal{Y}_{3x}\left(v, v + Q^{(3)}\right) + c_2 \mathcal{Y}_{2x}\left(v, v + Q^{(2)}\right)$,

$$\begin{aligned} &c_3 \mathcal{Y}_{3x}\left(v, v + Q^{(3)}\right) + c_2 \mathcal{Y}_{2x}\left(v, v + Q^{(2)}\right) = \lambda, \\ &\mathcal{Y}_t(v) + \mathcal{Y}_{2x}(v, v + Q) = 0, \end{aligned} \tag{7.78}$$

which corresponds to an alternative Lax-like system with the third-order eigenvalue equation and second-order time evolution:

$$\begin{aligned} &L_3\psi \equiv (c_3\partial_x^3 + c_2\partial_x^2 + b_1\partial_x + b_0)\psi = \lambda\psi, \\ &(\partial_t + L_2)\psi \equiv \partial_t + \psi_{2x} + Q_{2x}\psi = 0, \end{aligned} \tag{7.79}$$

where the b_i, $i = 0, 1$, are given by (7.57).

Let G_ϕ be a transformation generated by a (nonvanishing) solution ϕ of the system (7.79). It still maps the operators $\partial_t + L_2$ and $L_3 - \lambda$ onto similar operators,

$$G_\phi(\partial_t + L_2)G_\phi^{-1} = \partial_t + \widetilde{L}_2, \quad \widetilde{L}_2 = L_2(\widetilde{Q}_{2x}), \tag{7.80}$$

$$G_\phi(L_3 - \lambda)G_\phi^{-1} = \widetilde{L}_3 - \lambda, \quad \widetilde{L}_3 = c_3\partial_x^3 + c_2\partial_x^2 + \widetilde{b}_1\partial_x + \widetilde{b}_0, \tag{7.81}$$

where the differences $\Delta b_i \equiv \widetilde{b}_i - b_i$ are given by (7.60) and (7.61) and where the following differential consequences of (7.79) have been taken into account:

$$\sigma_t + (\sigma_x + \sigma^2)_x + Q_{3x} = 0, \tag{7.82}$$

$$c_3(\sigma_{2x} + 3\sigma\sigma_x + \sigma^3)_x + c_2(\sigma_x + \sigma^2)_x + (b_1\sigma)_x + b_{0,x} = 0. \tag{7.83}$$

Extending the condition (7.64) to $r = 0$, we find by means of the above analysis that the covariance of $L_3 - \lambda$ with $\partial_t + L_2$ is guaranteed if

$$b_1 = \frac{3}{2}c_3Q_{2x} + c_1, \quad b_0 = c_2Q_{2x} + \frac{3}{4}c_3(Q_{3x} - Q_{xt}) + c_0, \tag{7.84}$$

where c_0 and c_1 are arbitrary constants. Setting $c_0 = c_1 = 0$, we find

$$L_3 = c_3\left(\partial_x^3 + \frac{3}{2}Q_{2x}\partial_x + \frac{3}{4}\left(Q_{3x} - Q_{xt}\right)\right) + c_2L_2, \tag{7.85}$$

yielding the simplest Darboux-covariant system of type (7.79):

$$\widehat{L}_3\psi = \lambda\psi, \quad (\partial_t + L_2)\psi = 0, \tag{7.86}$$

$$\widehat{L}_3 = 4\partial_x^3 + 6Q_{2x}\partial_x + 3\left(Q_{3x} - Q_{xt}\right).$$

The operators $\widehat{L}_3$ and $\partial_t + L_2$ are found to constitute the Lax pair for an equation which is nothing other than the potential version of the Boussinesq equation (7.26) in which t has been rescaled ($t = a\tau,\ a^2 = -3$):

$$[\partial_t + L_2, \widehat{L}_3] = -(3Q_{2t} + Q_{4x} + 3Q_{2x}^2)_x = (Q_{2\tau} - Q_{4x} - 3Q_{2x}^2)_x. \tag{7.87}$$

It is easy to verify that the system (7.86) taken with $t = a\tau$ and $a^2 = -3$ is the equivalent version of our previous linear system (7.32) for the Boussinesq equation which results from subtracting a times the x-derivative of the first equation from the second one. The full Darboux-covariant system obtained with expression (7.85),

$$(L_3 - \lambda)\psi = 0, \qquad (\partial_t + L_2)\psi = 0, \tag{7.88}$$

corresponds, through the map $v = \ln\psi$, to a "covariant" version of the system (7.78) in which

$$Q_{2x}^{(3)} = \frac{1}{2}Q_{2x} \quad \text{and} \quad Q_{2x}^{(2)} = Q_{2x} + \frac{3}{4}\frac{c_2}{c_3}(Q_{3x} - Q_{xt}). \tag{7.89}$$

The striking similarity between the *covariant* $\mathcal{Y}$-systems associated with the KdV and Boussinesq equations reveals a deep connection between both soliton systems. It suffices to consider the next step which leads us from the system (7.78) to an alternative version with two evolution equations corresponding to two t-variables (t_p has the dimension p):

$$\mathcal{Y}_{t_2}(v) + \mathcal{Y}_{2x}(v, v+Q) = 0, \tag{7.90}$$

$$\mathcal{Y}_{t_3}(v) + c_2\mathcal{Y}_{2x}\left(v, v+Q^{(2)}\right) + c_3\mathcal{Y}_{3x}\left(v, v+Q^{(3)}\right) = 0.$$

It is clear from the above analysis that the Darboux covariance of the corresponding linear system for $\psi = \exp v$,

$$(\partial_{t_2} + L_2)\psi = 0, \quad \left(\partial_{t_3} + c_3\partial_x^3 + c_2\partial_x^2 + 3c_3Q_{2x}^{(3)}\partial_x + c_2Q_{2x}^{(2)}\right)\psi = 0, \tag{7.91}$$

is still guaranteed by the conditions (7.89) on $Q_{2x}^{(3)}$ and $Q_{2x}^{(2)}$. In particular, it is found that the compatibility of the simplest covariant system (setting $c_2 = 0, c_3 = 4$),

$$\begin{aligned} &(\partial_t + L_2)\psi = 0, \quad (\partial_{t_3} + \widehat{L}_3)\psi = 0, \\ &\widehat{L}_3 = 4\partial_x^3 + 6Q_{2x}\partial_x + 3(Q_{3x} - Q_{xt_2}), \end{aligned} \tag{7.92}$$

is subjected to the condition

$$[\partial_{t_3} + \widehat{L}_3, \partial_{t_2} + L_2] = [P_{x,t_3}(Q) + 3P_{2t_2}(Q) + P_{4x}(Q)]_x = 0, \tag{7.93}$$

which is a potential version of the KP equation:

$$\mathrm{KP}(u) \equiv (u_{t_3} + u_{3x} + 6uu_x)_x + 3u_{2t_2} = 0, \tag{7.94}$$

obtained by setting $u = Q_{2x}$ and by integrating once with respect to x. We wish to stress that the above derivation of a covariant Lax pair for the KdV equation produced three closely related Darboux-covariant systems hinting in a direct manner at the (well-known) common origin of the KdV and Boussinesq equations as reductions of the KP equation.

We end our discussion with a direct derivation of a Darboux-covariant equivalent to the linear system (7.39) that we associated with the Lax equation (7.34). Our starting point is the multipotential $\mathcal{Y}$-system (c_i is a constant),

$$\mathcal{Y}_{2x}(v, v+Q) = \lambda, \quad \mathcal{Y}_t(v) + \sum_{i=2}^{5} c_i\mathcal{Y}_{ix}\left(v, v+Q^{(i)}\right) = 0, \tag{7.95}$$

or its linear version for $\psi = \exp v$,

$$(L_2 - \lambda)\psi = 0, \qquad (\partial_t + L_5)\psi = 0, \tag{7.96}$$

$$L_5 = c_5\partial_x^5 + c_4\partial_x^4 + b_3\partial_x^3 + b_2\partial_x^2 + b_1\partial_x + b_0,$$

with

$$\begin{aligned} b_3 &= 10c_5Q_{2x}^{(5)} + c_3, \quad b_2 = 6c_4Q_{2x}^{(4)} + c_2, \\ b_1 &= 3c_3Q_{2x}^{(3)} + 5c_5\left[Q_{4x}^{(5)} + 3\left(Q_{2x}^{(5)}\right)^2\right] + c_1, \\ b_0 &= c_2Q_{2x}^{(2)} + c_4\left[Q_{4x}^{(4)} + 3\left(Q_{2x}^{(4)}\right)^2\right] + c_0. \end{aligned} \tag{7.97}$$

Let G_ϕ be a transformation (7.51) generated by a (nonvanishing) solution ϕ of the system (7.96) and (7.97). It maps $L_2 - \lambda$ and $\partial_t + L_5$ onto the similar operators (7.60) and $\partial_t + \widetilde{L}_5$, with

$$\widetilde{L}_5 = c_5\partial_x^5 + c_4\partial_x^4 + \widetilde{b}_3\partial_x^3 + \widetilde{b}_2\partial_x^2 + \widetilde{b}_1\partial_x + \widetilde{b}_0, \tag{7.98}$$

where

$$\begin{aligned} \Delta b_3 \equiv \widetilde{b}_3 - b_3 &= 5c_5\sigma_x = \tfrac{5}{2}c_5\Delta Q_{2x}, \\ \Delta b_2 \equiv \widetilde{b}_2 - b_2 &= b_{3,x} + \sigma\Delta b_3 + 4c_4\sigma_x + 10c_5\sigma_{2x}, \\ \Delta b_1 \equiv \widetilde{b}_1 - b_1 &= b_{2,x} + \sigma\Delta b_2 + 3\sigma_x\widetilde{b}_3 + 6c_4\sigma_{2x} + 10c_5\sigma_{3x}, \\ \Delta b_0 \equiv \widetilde{b}_0 - b_0 &= b_{1,x} + \sigma\Delta b_1 + 2\sigma_x\widetilde{b}_2 + 3\sigma_{2x}\widetilde{b}_3 + 4c_4\sigma_{3x} + 5c_5\sigma_{4x}. \end{aligned} \tag{7.99}$$

In order to ensure the Darboux covariance of $\partial_t + L_5$ with $L_2 - \lambda$, we must again determine expressions F_i for b_i, $i = 0, 1, 2, 3$, in terms of Q_{2x} and its derivatives, which are such that condition (7.65) is satisfied at $i = 0, 1, 2, 3$, with (7.64). It is clear from (7.98) that F_3 can be chosen to be linear in Q_{2x}, so

$$b_3 = \frac{5}{2}c_5Q_{2x} + c_3, \tag{7.100}$$

where c_3 is an arbitrary constant. Equation (7.98) then becomes

$$\Delta b_2 = \frac{5}{2}c_sQ_{3x} + 5c_5\sigma\sigma_x + 4c_4\sigma_x + 10c_5\sigma_{2x}. \tag{7.101}$$

Using (7.62), we rewrite it as

$$\Delta b_2 = 4c_4\sigma_x + \frac{15}{2}c_5\sigma_{2x} = 2c_4\Delta Q_{2x} + \frac{15}{4}c_5\Delta Q_{3x}, \tag{7.102}$$

indicating that F_2 can be chosen to be linear in Q_{2x} and Q_{3x}, so

$$b_2 = 2c_4Q_{2x} + \frac{15}{2}c_5Q_{3x} + c_2, \tag{7.103}$$

where c_2 is an arbitrary constant. Hence, we obtain

$$\begin{aligned} \Delta b_1 = 2c_4Q_{3x} + \frac{15}{4}c_5Q_{4x} + 4c_4\sigma\sigma_x + \frac{15}{2}c_5\sigma\sigma_{2x} + \frac{15}{2}c_5\sigma_xQ_{2x} \\ +3c_3\sigma_x + 15c_5\sigma_x^2 + 6c_4\sigma_{2x} + 10c_5\sigma_{3x}, \end{aligned} \tag{7.104}$$

or, using (7.62),

$$\Delta b_1 = 2c_4\Delta Q_{3x} + \frac{25}{8}c_5\Delta Q_{4x} + \frac{15}{8}c_5\left(2Q_{2x}\Delta Q_{2x} + \Delta Q_{2x}\Delta Q_{2x}\right)$$
$$+\frac{3}{2}c_3\Delta Q_{2x} + 2c_4\Delta Q_{3x} + \frac{25}{8}c_5\Delta Q_{4x} + \frac{15}{8}\Delta\left(Q_{2x}^2\right). \tag{7.105}$$

It follows that F_1 can be chosen to be linear in $Q_{2x},\ Q_{3x},\ Q_{4x}$, and Q_{2x}^2, so

$$b_1 = \frac{3}{2}c_3Q_{2x} + 2c_4Q_{3x} + \frac{25}{8}c_5Q_{4x} + \frac{15}{8}c_5Q_{2x}^2 + c_1. \tag{7.106}$$

It is found from these results and (7.62) that Δb_0 becomes

$$\Delta b_0 = \frac{15}{16}c_5\left[\Delta Q_{5x} + 2\Delta(Q_{2x}Q_{3x})\right] \tag{7.107}$$
$$+\ c_4\left[\Delta Q_{4x} + \Delta\left(Q_{2x}^2\right)\right] + \frac{3}{4}c_3\Delta Q_{3x} + c_2\Delta Q_{2x},$$

indicating that the appropriate expression for b_0 is

$$b_0 = c_2Q_{2x} + \frac{3}{4}c_3Q_{3x} + c_4(Q_{4x} + Q_{2x}^2) + \frac{15}{16}c_5(Q_{5x} + 2Q_{2x}Q_{3x}) + c_0. \tag{7.108}$$

Setting $c_1 = c_0 = 0$, we obtain the following expression for L_5,

$$L_5 = c_4L_2^2 + c_3\left(\partial_x^3 + \frac{3}{2}Q_{2x}\partial_x + \frac{3}{4}Q_{3x}\right) + c_4L_2 + \widehat{L}_5, \tag{7.109}$$

with (choosing $c_5 = 16$)

$$\widehat{L}_5 = 16\partial_x^5 + 40Q_{2x}\partial_x^3 + 60Q_{3x}\partial_x^2 + (50Q_{4x} + 30Q_{2x}^2)\partial_x + 15(Q_{5x} + 2Q_{2x}Q_{3x}). \tag{7.110}$$

The relations between different potentials appearing in the *covariant* system (7.95) are determined by (7.97), (7.100), (7.103), (7.106), and (7.108). The simplest Darboux-covariant fifth-order evolution equation (7.97) to be associated with (7.96) has the form

$$\left(\partial_t + \widehat{L}_5\right)\psi = 0. \tag{7.111}$$

It is easy to see that the system (7.96) and (7.111) is equivalent to the original system (7.39):

$$\widehat{L}_5 = \mathcal{L}_5 + 15\left[\partial_x^3 + (Q_{2x} + \lambda)\partial_x + Q_{3x}\right](L_2 - \lambda). \tag{7.112}$$

Notice that the appearance of the third-order Darboux-covariant operator L_3 as a part of the general fifth-order covariant operator L_5 can be regarded as a direct confirmation of the close relationship between the KdV and Lax equations as the third- and fifth-order members of the same hierarchy.

7.3 Bäcklund transformations and Noether theorem

BTs naturally arise when the Darboux formalism is "projected" to solutions of nonlinear equations (the potentials of the corresponding Lax representation). The action is simple: "wave functions" of the Lax equations should be excluded [239].

7.3.1 BT and infinitesimal BT

In the previous section we showed that the bilinear BT is a DT covariant form of the equations of the Hirota method. For the KdV equation it is (7.25), which is obtained from (7.22). The second relation (7.22) is nothing more than the first equation of the classical BT, relating the fields $w = Q_x$ and $w' = Q'_x$:

$$(w + w')_x = (w - w')^2 - \kappa^2, \tag{7.113}$$

$$(w + w')_t = -2(w - w')(w - w')_{xx} + (w_x - w'_x)^2 + 3((w - w')^2 - \kappa^2)^2,$$

while the second equation (we take the form of [407], the appropriate change of notations is used) is derived from (7.18) in terms of the Q and Q' fields of the *potential* KdV equation (7.15); we denote $2\mu = -\kappa^2$. The form of this equation is not unique, because the first one can be used.

The famous consequence of the BT (7.113) is that both variables w and w' are solutions of the potential KdV equation

$$\Lambda w \equiv w_t - 6w_x^2 + w_{xxx} = 0. \tag{7.114}$$

Steudel [407, 408, 409] derived conservation laws for soliton equations by application of the Noether theorem, imposing the BT in a version of the extended interpretation of

$$w' = \mathbb{B}_\kappa w \equiv w + \kappa[1 + \kappa^{-2}(w'_x + w_x)]^{1/2}, \tag{7.115}$$

which is one of the solutions of the first relation in (7.113) with respect to $w' - w$. The real-valued w is in the realm of the extended BT transform, if $\inf[1 + \kappa^{-2}(w'_x + w_x)] \geq 0$, or $|w'_x + w_x| \leq M^2$, $|\kappa| \geq M$.

Theorem 7.1. *Let*

$$w_i = \mathbb{B}_{\kappa_i} w_0, \qquad w_3 = \mathbb{B}_{\kappa_2} w_1, \tag{7.116}$$

then

$$\mathbb{B}_{\kappa_1} \mathbb{B}_{\kappa_2} = \mathbb{B}_{\kappa_2} \mathbb{B}_{\kappa_1} \tag{7.117}$$

and

$$(w_3 - w_0)(w_2 - w_1) = \kappa_2^2 - \kappa_1^2. \tag{7.118}$$

The fundamental property of the extension basis is that (7.115) is valid not only for solutions of the potential KdV equation. In other words, the Laurent series

$$\delta w = \kappa + A_1\kappa^{-1} + A_2\kappa^{-2} + \dots \tag{7.119}$$

represents the infinitesimal transform at infinity on the κ-plane. Equating the x-derivative of the right-hand side of (7.119) and the right-hand side of the first relation in (7.113) yields

$$A_1 = w_x, \quad A_n = \frac{1}{2}A_{(n-1)x} - \frac{1}{2}\sum_{r=1}^{n-2} A_r A_{n-r-1}, \quad n = 2, 3, 4, \dots. \tag{7.120}$$

These formulas were first derived by Zakharov and Faddeev [469] in the context of the IST method; see also [385, 445]. Note also that the expansion (7.119) after differentiation in x gives an alternative representation of a DT as $\delta w_x \sim u[1] - u$. The recurrent relations (7.120) are solved explicitly:

$$A_2 = w_{xx}/2, \quad A_3 = w_{xxx} - w_x^2/2, \quad A_4 = w_{xxxx}/8 - w_x w_{xx}, \dots. \tag{7.121}$$

7.3.2 Noether identity and Noether theorem

A Lagrangian density for the KdV equation is chosen so that

$$\mathfrak{L} = \frac{1}{2}w_x w_t + \frac{1}{2}w_{xx}^2 - 2w_x^2 \tag{7.122}$$

gives the potential KdV equation (7.114) as the Euler equation. A variant of the Noether theorem for the dependence of $\mathfrak{L}$ on w_{xx} is based on the following form for the variation (the Frechét differential on the prolonged space):

$$\delta\mathfrak{L} \equiv \frac{\partial\mathfrak{L}}{\partial w_t}\delta w_t + \frac{\partial\mathfrak{L}}{\partial w_x}\delta w_x + \frac{\partial\mathfrak{L}}{\partial w_{xx}}\delta w_{xx}. \tag{7.123}$$

A decomposition of the right-hand side of (7.134) into a divergence and a term proportional δw gives the Noether identity

$$\delta\mathfrak{L} = A_t + B_x - \Lambda\delta w, \tag{7.124}$$

where

$$A = \frac{\partial\mathfrak{L}}{\partial w_t}\delta w = \frac{1}{2}w_x\delta w, \tag{7.125}$$

$$B = \left[\frac{\partial\mathfrak{L}}{\partial w_x} - \left(\frac{\partial\mathfrak{L}}{\partial w_{xx}}\right)_x\right]\delta w + \frac{\partial\mathfrak{L}}{\partial w_{xx}}\delta w_x = \left(\frac{1}{2}w_t + w_{xxx} - 6w_x^2\right)\delta w - w_{xx}\delta w_x. \tag{7.126}$$

The expression for Λ is given by (7.114). A proof of the theorem follows from the identity

$$\left(\frac{\partial\mathfrak{L}}{\partial w_t}\delta w\right)_t = \left(\frac{\partial\mathfrak{L}}{\partial w_t}\right)_t \delta w + \frac{\partial\mathfrak{L}}{\partial w_t}\delta w_t, \tag{7.127}$$

and similar ones for other derivatives and the Euler equation. The identity (7.124) proves the Noether theorem:

Theorem 7.2. *If the Lagrangian changes by a divergence*

$$\delta\mathfrak{L} = \epsilon(\Theta_t + \Xi_x) \tag{7.128}$$

under the infinitesimal transformations $w \to w + \epsilon f$, then, for all solutions of the potential KdV equation $\Lambda w = 0$, the conservation law

$$T_t + X_x = 0 \tag{7.129}$$

exists, with

$$T = \epsilon^{-1}A - \Theta, \tag{7.130}$$

$$X = \epsilon^{-1}B - \Xi. \tag{7.131}$$

The following lemma occurs:

Lemma 7.3. *Let $d_{ik} = w_i - w_k$ and*

$$\mathfrak{L}[w_1] - \mathfrak{L}[w_0] = \Theta_t^{01} + \Xi_x^{01}, \tag{7.132}$$

with

$$\Theta^{10} = -\frac{1}{12}d_{10}d_{10}^2 + \frac{1}{4}\kappa^2, \tag{7.133}$$

$$\Xi^{10} = d_{10}\left(-\frac{4}{5}d_{10}^4 - 2\kappa^2 + w_x(d_{10}^2 - \kappa^2) - 2w_x^2 + \frac{1}{4}(w_1 + w_0)\right).$$

Then the transformation $\mathbb{B}_\kappa$ is the Noether transformation.

This is proved by the definitions of the Lagrangian (7.122) and w_1 (7.116) on the basis of (7.113). The product $\mathbb{B}_{\kappa+\epsilon}\mathbb{B}_{-\kappa}$, being the Noether transformation, generates the vector (Θ, Ξ) such that

$$\delta\mathfrak{L} = \mathfrak{L}[w_2] - \mathfrak{L}[w_1] = \epsilon(\Theta_t + \Xi_x) \tag{7.134}$$

determines the variation about the fixed w_0. Finally, the part of the vector (Θ, Ξ),

$$T = -\frac{\kappa}{2}d_{40}, \qquad X = 2\kappa d_{40}(w_x - \kappa^2), \tag{7.135}$$

which is symmetric with respect to $\kappa \to -\kappa$ (the symmetry of the BT is accounted for), contributes indeed to the Noether conservation law:

$$\frac{1}{2}d_{40\,t} + 2d_{40}(\kappa^2 - w_x)_x. \tag{7.136}$$

The substitution of expansion (7.119) into (7.136) produces the conservation laws

$$(A_{2r-1})_t + 4(A_{2r+1} - w_x A_{2r-1})_x = 0, \qquad r = 1, 2, \ldots \tag{7.137}$$

in the form of Wadati et al. [445].

7.3.3 Comment on Miura map

The first relation in (7.113) for imaginary $\kappa = \mathrm{i}k$ in terms of $d_{01} = w_0 - w_1 = d_{41}^*$ is nothing more than the Miura link for $u = 2w_x$,

$$\sigma_x = \sigma^2 + k^2 - u,$$

or, in the notation of this section,

$$d_{01} = \sigma = \phi_x/\phi,$$

where ϕ is a solution of

$$-\phi_{xx} + u\phi = -k^2\phi.$$

This link immediately leads to the continuum conservation law from the celebrated paper of Miura et al. [335]

$$(\phi^*\phi)_t + (\phi^*\phi_{xx} + \phi\phi^*_{xx} - 4|\phi_x^2| - 6k^2\phi^*\phi)_x = 0$$

in the context of the Noether theorem.

Quite similarly the sine–Gordon equation is treated in [409].

7.4 From singular manifold method to Moutard transformation

Paper [10] contains the so-called Ablowitz–Ramani–Segur conjecture that incorporated the Painlevé property [360]. This result was extended by Weiss et al. [449] as the Weiss–Tabor–Carnevale theory to check the Painlevé property for a PDE.

Estévez and Leble [145, 146] developed a procedure to derive the Moutard transformation (and hence the DTs) in the framework of the singular manifold method. The generalization of these ideas for the case of two Painlevé branches was made in [143].

We will illustrate the idea using the example of the singular manifold method analysis of a version of the 2+1 KdV (Boiti–Leon–Manna–Pempinelli 1) equation ([59]). Let us write this equation in the form [145]

$$m_{ty} = (m_{xxy} + m_y m_x)_x. \tag{7.138}$$

It is proved that (7.138) has the standard Painlevé property, i.e., its solutions can be locally expanded in terms of four arbitrary functions. The truncated expansion produces the auto-BT

$$m[1] = m + 6\frac{\phi_x}{\phi}, \tag{7.139}$$

which links two solutions of (7.138) by the "singular manifold" function ϕ. The substitution of (7.139) into (7.138) and application of the generalized procedure [146] leads to the Lax pair

$$\phi_{xxx} - \phi_t + m_x\phi_x = 0, \qquad 3\phi_{xy} + m_y\phi = 0. \tag{7.140}$$

A consideration of (7.139) as a transformation $m \to m[1]$ and the truncated expansion for the transformed function $\psi[1]$,

$$\psi[1] = \frac{p}{\phi}, \tag{7.141}$$

which is the solution of the Lax pair (7.140) with the transform $m[1]$, yields the following equations for p:

$$p_x = -2\psi\phi_x, \quad p_y = -2\phi\psi_y, \quad p_t = 2\psi_x\phi_{xx} - 2\phi_x\psi_{xx} - 2\psi\phi_t. \tag{7.142}$$

It can be proved that the form

$$d\Omega = -\psi\phi_x \mathrm{d}x - \phi\psi_y \mathrm{d}y + (\psi_x\phi_{xx} - \phi_x\psi_{xx} - \psi\phi_t)\mathrm{d}t \tag{7.143}$$

is exact (i.e., $\mathrm{d}p = -2\mathrm{d}\Omega$) on solutions ψ and ϕ of the Lax equations and hence there exists

$$\psi[1] = \psi - 2\frac{\Omega(\psi,\phi)}{\phi}, \tag{7.144}$$

which coincides with the Moutard transformation [340, 341]. The method seems to be an effective tool to derive the Moutard transformation formalism in 2+1 dimensions [140]. It was further applied to generate the DTs for the Bogoyavlenskii equation in $2+1$ dimensions [144]. The constructive elements of the theory are presented in [141].

7.5 Zakharov–Shabat dressing method via operator factorization

7.5.1 Sketch of IST method

In the "new history" of the soliton theory, half a century after the Bäcklund–Moutard–Darboux transformations, the notion of dressing appeared within the inverse scattering problem, when solving the Cauchy problem for the KdV equation [474]. To begin with, let us sketch the IST method and introduce scattering data for the one-dimensional Sturm–Liouville problem

$$-\partial_x^2\psi + u(x)\psi = k^2\psi \tag{7.145}$$

with a localized potential $u(x)$ ($\epsilon > 0, |x| \to \infty \Rightarrow |u(x)x^{(1+\epsilon)}| \to 0$) and the spectral parameter k^2. The scattering data comprise eigenvalues $k_n = \mathrm{i}\kappa_n$,

normalization constants $a_n = \lim_{x\to\infty} \exp(\kappa_n)\psi_n$ for eigenfunctions ψ_n normalized as $\int_{-\infty}^{\infty} |\psi_n|^2 \mathrm{d}x = 1$, and the reflection coefficient $v(k)$. The last one is extracted from the asymptotic behavior of the continuum spectrum solutions

$$\psi(x,k) \simeq \begin{cases} \exp(-ikx) + v(k)\exp(ikx), & x \to \infty, \\ w(k)\exp(-ikx), & x \to -\infty. \end{cases} \tag{7.146}$$

Solving the scattering problem, we arrive at the function $F(x)$ [354]:

$$F(x) = \sum_m a_m \exp(-\kappa_m x) + \frac{1}{2\pi}\int_{-\infty}^{\infty} v(k)\exp(\mathrm{i}kx)\,\mathrm{d}k, \tag{7.147}$$

which determines the kernel of the Gel'fand–Levitan–Marchenko (GLM) integral equation

$$K(x,y) + F(x+y) + \int_x^{\infty} K(x,s)F(s+y)\,\mathrm{d}s = 0, \quad x \le y. \tag{7.148}$$

Then the potential $u(x)$ is retrieved from the solution $K(x,y)$ of (7.148) as

$$u(x) = 2\frac{\mathrm{d}}{\mathrm{d}x}K(x,x). \tag{7.149}$$

Equation (7.148) links K and F; it maps the scattering data to the potential and is referred to as the inverse scattering transformation. The Gardner–Green–Kruskal–Miura theory, using the second operator of the Lax pair (see Chap. 3), gives explicit dependence of the scattering data on time, $a_m(t)$ and $v(k,t)$, via the initial values of $a_m(0)$ and $v(k,0)$.

The GLM equation (7.148) is solved explicitly in some of the simplest cases [354]. The multisoliton solutions correspond to zero v (reflectionless potentials). The kernel of the integral operator factorizes in this case and has a finite number of terms, as is seen from (7.147).

7.5.2 Dressible operators

The idea of the dressing method in its original IST version [474] (we follow the modification given in [466]) uses the fact that each function F generates the function K and hence a potential. Let us write (7.147) symbolically as

$$K + F + K^*F = 0, \tag{7.150}$$

where the asterisk denotes the action of the integral operator and the function $F(x,y)$ goes to $F(x+y)$ for the standard GLM equation. Consider a pair of operators M and $\widehat{M}$ which obey the equation

$$\widehat{M}K + MF + (\widehat{M}K)^*F + K^*(MF) = 0. \tag{7.151}$$

The operator M is named the "bare" operator, and $\widehat{M}$ is the "dressed" operator. Suppose the function F obeys the equation

$$MF = 0. \tag{7.152}$$

Then we have

$$\widehat{M}K = 0, \tag{7.153}$$

if the operator $\widehat{M}$ exists. The set of pairs $(M,\ \widehat{M})$ forms a vector space.

As an example, consider the operator

$$M = \partial_x + \partial_y. \tag{7.154}$$

In this case (7.151) takes the form

$$MK(x,y)+MF(x,y)+\partial_x \int_x^\infty K(x,s)F(s,y)\,\mathrm{d}s+\int_x^\infty K(x,s)\partial_y F(s,y)\,\mathrm{d}s = 0. \tag{7.155}$$

Evidently,

$$\partial_x \int_x^\infty K(x,s)F(s,y)\,\mathrm{d}s = -K(x,x)F(x,y) + \int_x^\infty \left[(\partial_x K(x,s)\right] F(s,y)\,\mathrm{d}s. \tag{7.156}$$

Integration by parts gives

$$I = \int_x^\infty \left[\partial_s K(x,s)\right] F(s,y)\,\mathrm{d}s = -\int_x^\infty K(x,s)\partial_s F(s,y)\,\mathrm{d}s - K(x,x)F(x,y). \tag{7.157}$$

In (7.155) take into account (7.156) and introduce I as

$$\begin{aligned} MK(x,y) + MF(x,y) - K(x,x)F(x,y) + \int_x^\infty \left[\partial_x K(x,s)\right] F(s,y)\,\mathrm{d}s \\ + \int_x^\infty K(x,s)\partial_y F(s,y)\,\mathrm{d}s + I - I = 0. \end{aligned} \tag{7.158}$$

For $+I$ substitute the mid-positioned term in (7.157), and for $-I$ the right-hand side of (7.157) with the opposite sign:

$$\begin{aligned} MK(x,y) + MF(x,y) - K(x,x)F(x,y) + \int_x^\infty \left[\partial_x K(x,s)\right] F(s,y)\,\mathrm{d}s \\ + \int_x^\infty K(x,s)\partial_y F(s,y)\,\mathrm{d}s + \int_x^\infty \left[\partial_s K(x,s)\right] F(s,y)\,\mathrm{d}s \\ + \int_x^\infty K(x,s)\partial_s F(s,y)\,\mathrm{d}s + K(x,x)F(x,y) = 0. \end{aligned} \tag{7.159}$$

Ordering the terms, we get

$$(\partial_x + \partial_y)K(x,y) + MF(x,y) \tag{7.160}$$
$$+ \int_x^\infty \left[(\partial_x + \partial_s)\, K(x,s)\right] F(s,y)\,\mathrm{d}s + \int_x^\infty K(x,s)\,(\partial_y + \partial_s)\, F(s,y)\,\mathrm{d}s = 0.$$

Hence, the following operator arises:

$$\widehat{M} = M = \partial_x + \partial_y. \tag{7.161}$$

The operator M is called *dressible*. A set of dressible operators forms linear space.

A connection between scattering data and a potential $U = U(x,t)$ with the additional (time) parameter is used in integrable equations via the Lax representation [335] and, directly, in quantum evolution problems. The problems in which potentials are functions of time can be studied by the present method because the operator of the time derivative ∂_t is dressible. As before, from the equation

$$MF(x,y,t) = 0$$

we obtain

$$\widehat{M}K(x,y,t) = 0.$$

Let a function ψ be a solution of two equations

$$(\partial_t - L[U])\psi = 0, \tag{7.162}$$

$$(\partial_y - A[U])\psi = 0. \tag{7.163}$$

If derivatives with respect to t and y commute, then the Lax representation is

$$A_t - L_y = [A, L]. \tag{7.164}$$

Proposition 7.4. *If two operators M and $\widehat{M}$ are such that there exists a solution of (7.151) (operator M is dressible) and if the operator M forms the Lax pair with N, then the operators $\widehat{M}$ and $\widehat{N}$ also form a Lax pair. If a pair of operators M and N produces a nonlinear system, then the pair $\widehat{M}$ and $\widehat{N}$ produces the same system.*

The next example is

$$M = \alpha\frac{\partial}{\partial t} + \partial_x^2 - \partial_y^2. \tag{7.165}$$

We want to dress the operator M, applying it to the GLM equation (7.151). Integrating by parts yields the operator

$$\widehat{M} = \alpha\frac{\partial}{\partial t} + \partial_x^2 - \partial_y^2 + U(x), \tag{7.166}$$

where

$$U(x) = -2\frac{\mathrm{d}}{\mathrm{d}x}K(x,x). \tag{7.167}$$

Note that a function U has appeared in the dressed operator, while for the first-order operator (7.154) the dressed operator is the same as the bare one (7.161).

In general, we put

$$L_0 = l_0(x, t, \ldots)\partial_x^n. \tag{7.168}$$

Consider the operator D of the following structure:

$$DF = \alpha\partial_t F + L_0 F - F L_0^+, \tag{7.169}$$

where L_0^+ is the Hermitian conjugate to L and acts to the left.

Proposition 7.5. *The operator (7.169) is dressible. The dressed operator* $\widehat{D}$ *is*

$$\widehat{D}K = \alpha\partial_y K + LK - K L_0^+, \tag{7.170}$$

where

$$L = L_0 + \widetilde{L} \tag{7.171}$$

and

$$\widetilde{L} = \hat{l}_0\,\partial_x^{n-1} + \ldots, \qquad \hat{l}_0 \sim (\partial_x - \partial_y)^i K\big|_{y=x}. \tag{7.172}$$

7.5.3 Example

Let us take

$$L_0 = \partial_x^2 \quad \Rightarrow \quad L = \partial_x^2 + U.$$

Solving the equation $MF = 0$ yields $\widehat{M}K = 0$; hence, some class of solvable equations appears, with some linear space.

Let us consider operators D_1 and D_2,

$$D_1 F = \alpha_1\partial_{t_1} F + L_0^{(1)} - F L_0^{(1)+},$$

$$D_2 F = \alpha_2\partial_{t_2} F + L_0^{(2)} - F L_0^{(2)+}.$$

This class of operators contains the Lax representation

$$\alpha_1\partial_{t_1} L_0^{(2)} - \alpha_2\partial_{t_2} L_0^{(1)} + \left[L_0^{(1)}, L_0^{(2)}\right] = 0.$$

For relevant forms of the operators $L_0^{(1)}$ and $L_0^{(2)}$ and for $\alpha_1 = \alpha$, $\alpha_2 = -1$, $t_1 = y$, and $t_2 = t$ we obtain the KP equation

$$\partial_x\left(u_t + 6uu_x + u_{xxx}\right) + \alpha^2 u_{yy} = 0.$$

In the case of $\alpha^2 = -1$ we have the KP I equation; otherwise, if $\alpha^2 = 1$ we have the KP II equation. The KP equation is the two-dimensional equation that contains the KdV equation as a y-independent reduction:

$$u_t + 6uu_x + u_{xxx} = 0.$$

It was demonstrated in [324] that the triangular (Volterra) factorization of the operator

$$F = (1 + K^+)^{-1}(1 + K^-)$$

proved by Zakharov and Shabat [474] links the Zakharov–Shabat dressing scheme to the DT dressing.

8

Dressing via local Riemann–Hilbert problem

Beginning with this chapter, we proceed to a description of the second (mostly analytic) aspect of the dressing method. In this chapter we will show how to dress a seed solution of a (1+1)-dimensional nonlinear equation making use of the local Riemann–Hilbert (RH) problem. First we formulate in Sect. 8.1 a general approach to the RH problem based dressing method [354] in terms of the Lax representation associated with a given nonlinear equation. Then in the subsequent sections we will illustrate with examples of specific nonlinear equations the power of the RH problem method. Throughout this chapter we stress two basic facts concerning the applicability of the RH problem to solve nonlinear equations: (1) the RH problem naturally arises in the context of nonlinear equations and (2) this approach is substantially universal. In Sect. 8.2 we concretize the main ideas by means of the classic example of the nonlinear Schrödinger (NLS) equation. Sections 8.3 and 8.4 are devoted to mathematically more complicated equations: the modified NLS (MNLS) equation and the Ablowitz–Ladik (AL) equation. These two examples are particularly interesting from the point of view of the RH problem. Indeed, the reader will see that the formalisms we apply for solving the MNLS equation and the AL equation are practically the same though these equations are completely different: one of them is a partial differential equation (MNLS), while the other is a differential-difference equation (AL). Section 8.5 demonstrates some novel features of the RH problem formulation which arise in the case of higher-order matrix spectral problems. As an example, we consider in this section the three-wave resonant interaction equations. These equations are of interest in themselves because they represent the so-called dispersionless nonlinear equations. The non-Abelian version of this system was discussed in Sect. 3.7. In Sect. 8.6 we give one more argument in favor of the universality of the dressing method formalism developed. Namely, we will obtain the *homoclinic orbits* for the NLS and MNLS equations. Strictly speaking, this problem is not pertinent to the RH problem in the context of this chapter because we will dress the plane wave solution under periodic boundary conditions. Nevertheless, our formalism exhibits its effectiveness for solving nonsolitonic problems as well.

Finally, in the last section we briefly consider the well known Korteweg–de Vries (KdV) equation. This consideration is based on the method which allows a straightforward generalization to (2+1)-dimensional nonlinear equations and serves as a bridge to go in this direction.

8.1 RH problem and generation of new solutions

As indicated in previous chapters, the Lax representation [263] (or the zero-curvature representation) is of primary importance for the integration of nonlinear equations. In the framework of the Lax representation, a system of two linear matrix equations (sometimes these equations are scalar ones)

$$\psi_x = U\psi, \qquad \psi_t = V\psi \tag{8.1}$$

is associated with a given nonlinear equation. Here the matrices $U(x,t,k)$ and $V(x,t,k)$ depend on a solution of the nonlinear equation and on a complex spectral parameter k independent of the coordinates (x,t). These matrices are chosen in such a way that the compatibility condition [7]

$$U_t - V_x + [U,V] = 0 \tag{8.2}$$

resulting from the equality of mixed derivatives $\psi_{xt} = \psi_{tx}$ and providing the existence of a common solution for the system (8.1) would produce exactly the nonlinear equation we are considering. The matrices U and V have noncoinciding sets of poles (divisors) in some points of the extended complex k-plane $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. In fact, they are the divisors that determine all the essential features of the nonlinear equation with a given Lax representation.

Suppose we know some seed solution u_0 of the nonlinear equation (8.2). On frequent occasions, trivial solutions like zero can serve as the seed solution. We therefore know explicitly the matrices U_0 and V_0 which correspond to this solution. As a result, we can solve a system of linear equations

$$E_x = U_0 E, \qquad E_t = V_0 E \tag{8.3}$$

for the matrix function $E(x,t,k)$. Now we will demonstrate, following Zakharov and Shabat [475] (see also [148]), that there exists a possibility to build a class of new solutions of the nonlinear equation (8.2), this class being parameterized by a closed oriented contour L on the extended plane $\overline{\mathbb{C}}$ and by a nondegenerate bounded matrix function $G_0(k)$ defined on the contour. For this purpose we introduce first a matrix function $G(x,t,k)$,

$$G(x,t,k) = E(x,t,k)G_0(k)E^{-1}(x,t,k), \quad k \in L, \tag{8.4}$$

where $E(x,t,k)$ solves the system (8.3). Then we pose the RH problem [167] for the matrix $G(x,t,k)$ on the contour L as

$$\Phi_-^{-1}(x,t,k)\Phi_+(x,t,k) = G(x,t,k), \quad k \in L. \tag{8.5}$$

In other words, we want to factorize the matrix $G(x,t,k)$ into a product of the matrix functions Φ_+ and Φ_- in such a way that Φ_+ (Φ_-) is analytic inside (outside) the contour L and both of them satisfy equation (8.5) on the contour. Such an analytic factorization problem represents one of the formulations of the RH problem. If the contour L coincides with the real axis of the k-plane, then the matrix Φ_+ is analytically continuable to the upper half plane, while Φ_- is analytically continuable to the lower half plane.

The solution of the RH problem in the form formulated above is not unique. Evidently, matrices $\Phi'_\pm = M(x,t)\Phi_\pm$ with an arbitrary nondegenerate matrix M solve the same factorization problem. To provide the uniqueness of the solution, we should pose the normalization condition. This means that we should set a definite value of one of the matrices $\Phi_\pm$ in some predetermined point of the k-plane. Usually the infinite point $k=\infty$ is taken as the reference point. If $\Phi_-(\infty) = \mathbb{1}$, then this normalization is called canonical.

Let us differentiate (8.5) with respect to x, taking into account that $G_x = [U_0, G]$. Then we obtain

$$\Phi_{+x} = U\Phi_+ - \Phi_+ U_0,$$

where we introduce a matrix function $U(x,t,k)$,

$$U = \Phi_- U_0 \Phi_-^{-1} + \Phi_{-x}\Phi_-^{-1} = \Phi_+ U_0 \Phi_+^{-1} + \Phi_{+x}\Phi_+^{-1}.$$

It is clear that the matrix U has the same set of poles as U_0, if the poles do not lie on the contour. Moreover, if the contour contains the pole k_0 with multiplicity n_0, we should demand an additional property of the matrix $G_0(k)$, namely,

$$G_0(k) = \mathbb{1} + O\left(|k-k_0|^{n_0}\right),$$

near the point k_0; $\mathbb{1}$ is the identity matrix.

Similarly, the differentiation of (8.5) in t gives a matrix V,

$$V = \Phi_- V_0 \Phi_-^{-1} + \Phi_{-t}\Phi_-^{-1} = \Phi_+ V_0 \Phi_+^{-1} + \Phi_{+t}\Phi_+^{-1},$$

entering the equation

$$\Phi_{+t} = V\Phi_+ - \Phi_+ V_0.$$

The last step is to introduce functions $\psi_\pm = \Phi_\pm E$ which satisfy the compatible linear equations

$$\psi_{\pm x} = U\psi_\pm, \qquad \psi_{\pm t} = V\psi_\pm.$$

Hence, we constructed new matrices U and V which obey the compatibility condition (8.2), provided that we are able to solve the RH problem (8.5). It is important to stress that the matrices U and V have the same structure as U_0 and V_0. In other words, U and V depend on a new solution $u(x,t)$ in the same way as U_0 and V_0 depend on the seed solution $u_0(x,t)$. Therefore, we can restore purely algebraically the solution $u(x,t)$.

A distinction will be made between two types of the RH problem. The first one is the so-called regular RH problem when both Φ_+ and Φ_-^{-1} have no zeros in their domains of analyticity. In other words, $\det \Phi_+ \neq 0$ inside the contour L and $\det \Phi_-^{-1} \neq 0$ outside the contour. Otherwise we will deal with the RH problem with zeros. It is the RH problem with zeros that leads to soliton solutions of nonlinear equations.

The scheme described above gives general principles of dressing the seed solution. In the examples we give, we demonstrate that the RH problem arises naturally within the inverse spectral transform (IST) approach. Moreover, it will be clear how to adopt involutions that impose some restrictions on the general Lax representation (8.2) and reduce the matrices U and V to those belonging to some complex Lie algebras or symmetric spaces [331].

8.2 Nonlinear Schrödinger equation

Here we demonstrate the main stages of the application of the RH problem to obtain a soliton solution of the NLS equation

$$\mathrm{i}u_t + u_{xx} + 2|u|^2 u = 0. \tag{8.6}$$

A vast amount of literature exists about the NLS equation, the most important books are by Novikov et al. [354], Lamb [259], Dodd et al. [117], Ablowitz and Segur [13], Calogero and Degasperis [81], Newell [348], and Faddeev and Takhtajan [148]. Remember that solutions of the NLS equation in terms of the elliptic functions were given in Sects. 3.5 and 4.9.

8.2.1 Jost solutions

As is well known [473], the NLS equation (8.6) can be represented as the compatibility condition of the system (8.1) of two linear matrix equations with the 2×2 matrices U and V of the form

$$\begin{aligned} U &= -\mathrm{i}k\sigma_3 + Q, \quad Q = \begin{pmatrix} 0 & u \\ -\bar{u} & 0 \end{pmatrix}, \\ V &= -2\mathrm{i}k^2\sigma_3 + 2kQ + \mathrm{i}\sigma_3 Q_x - \mathrm{i}Q^2\sigma_3, \end{aligned} \tag{8.7}$$

this compatibility condition being fulfilled for arbitrary constant spectral parameter k. The matrix Q stands for the potential in the spectral equation $\psi_x = U\psi$. It will be more convenient for us to write the spectral equation in terms of the matrix $J = \psi E^{-1}$, where $E = \exp(-\mathrm{i}kx\sigma_3)$ is a solution of the spectral equation for zero potential. Hence, the spectral equation we shall deal with is written as

$$J_x = -\mathrm{i}k[\sigma_3, J] + QJ. \tag{8.8}$$

We consider the zero solution of the NLS equation as the seed solution to be dressed and are interested in deriving localized solutions.

First we introduce the so-called Jost solutions $J_\pm(x,k)$ of the spectral equation (8.8) obeying the asymptotic conditions $J_\pm \to \mathbb{1}$ at $x \to \pm\infty$. Since $\operatorname{tr} U = 0$, these boundary conditions guarantee that $\det J_\pm = 1$ for all x. In other words, the Jost solutions coincide asymptotically with the solution of the spectral equation with zero potential. It is clear now that going from the matrix ψ to the matrix J enables us to use the unit asymptotic for $J_\pm$, instead of the exponential asymptotic E.

Being solutions of the first-order differential equation, the Jost functions $J_\pm$ are not mutually independent. Indeed, they are interconnected by the scattering matrix $S(k)$,

$$J_- = J_+ E S E^{-1}, \quad S(k) = \begin{pmatrix} a(k) & -\bar{b}(k) \\ b(k) & \bar{a}(k) \end{pmatrix}, \quad \det\, S(k) = 1, \tag{8.9}$$

and the structure of $S(k)$ is dictated by the form of the potential Q. It follows directly from the spectral equation (8.8) that the Jost solutions obey the involutive condition

$$J_\pm^\dagger(x,\bar{k}) = J_\pm^{-1}(x,k), \tag{8.10}$$

where the dagger means the Hermitian conjugation. It is extremely important that the involution (8.10) manifests itself throughout all the other objects related to the spectral equation. For example, the scattering matrix $S(k)$ obeys the same involution $S^\dagger(\bar{k}) = S^{-1}(k)$.

8.2.2 Analytic solutions

What can we say about analytic properties of the Jost matrix functions with respect to the spectral parameter k? Let us rewrite the spectral equation (8.8) with the boundary conditions in the integral form. To take an example, we obtain the following integral equations

$$\begin{aligned} (J_-)_{11}(x,k) &= 1 + \int_{-\infty}^{x} \mathrm{d}\xi\; u(\xi)\; (J_-)_{21}(\xi,k), \\ (J_-)_{21}(x,k) &= -\int_{-\infty}^{x} \mathrm{d}\xi\; \bar{u}(\xi)\; (J_-)_{11}(\xi,k)\; \exp[2\mathrm{i}k(x-\xi)] \end{aligned} \tag{8.11}$$

for the first column entries of the Jost matrix J_-. We see that the exponent in the integrand (8.11) decreases for $\operatorname{Im} k > 0$. In other words, the first column $J_-^{[1]}$ of the matrix J_- is analytic in the upper half plane and continuous on the real axis $\operatorname{Im} k = 0$. In the same way we recognize that the second column $J_+^{[2]}$ of the matrix J_+ is analytic as well in the same domain. Therefore, we can define the matrix function $\Phi_+(x,k)$,

$$\Phi_+(x,k) = \left(J_-^{[1]}, J_+^{[2]}\right),$$

which is a solution of the spectral equation (8.8) and is analytic as a whole in the upper half plane.

The analytic solution $\Phi_+(x,k)$ can be expressed in terms of the Jost functions and some elements of the scattering matrix. Indeed, remembering (8.9), we have

$$\begin{aligned}\Phi_+ &= \left(J_-^{[1]}, J_+^{[2]}\right) = \left(a\, J_+^{[1]} + b\, \mathrm{e}^{2\mathrm{i}kx}\, J_+^{[2]}, J_+^{[2]}\right) \\ &= \left(J_+^{[1]}, J_+^{[2]}\right) \begin{pmatrix} a & 0 \\ b\, \mathrm{e}^{2\mathrm{i}kx} & 1 \end{pmatrix} = J_+ E S_+ E^{-1},\end{aligned}$$

where

$$S_+ = \begin{pmatrix} a & 0 \\ b & 1 \end{pmatrix}. \tag{8.12}$$

Similarly,

$$\Phi_+ = J_- E S_- E^{-1}, \quad S_- = \begin{pmatrix} 1 & \bar{b} \\ 0 & a \end{pmatrix}, \quad S_+ = S S_-. \tag{8.13}$$

It follows from the above formulas that

$$\det \Phi_+(x,k) = a(k). \tag{8.14}$$

Now, what about a matrix function analytic in the lower half plane? We can define such a function $\Phi_-^{-1}(x,k)$ by means of the involution (8.10), i.e.,

$$\Phi_-^{-1}(x,k) = \Phi_+^{\dagger}(x,\bar{k}).$$

It can be easily shown that $\Phi_-^{-1}(x,k)$ is expressed in terms of the rows of $J_\pm^{-1}$, namely,

$$\Phi_-^{-1} = \begin{pmatrix} (J_-)^{-1}_{[1]} \\ (J_+)^{-1}_{[2]} \end{pmatrix}.$$

Therefore, $\Phi_-^{-1}(x,k)$ is a solution of the adjoint spectral problem. On the real axis

$$\Phi_-^{-1}(x,k) = \Phi_+^{\dagger}(x,k) = E S_+^{\dagger} E^{-1} J_+^{-1} = E S_-^{\dagger} E^{-1} J_-^{-1}$$

and $\det \Phi_-^{-1}(x,k) = \bar{a}(k)$.

Let us write an asymptotic expansion for $\Phi_+(x,k)$,

$$\Phi_+(x,k) = \mathbb{1} + \frac{1}{k}\, \Phi_+^{(1)}(x) + O\left(\frac{1}{k^2}\right), \tag{8.15}$$

and substitute it into the spectral equation (8.8). Collecting terms with equal powers of k, we find a reconstruction formula for the potential:

$$Q = \mathrm{i}\left[\sigma_3, \Phi_+^{(1)}\right]. \tag{8.16}$$

Hence, in order to solve the NLS equation, we should find the analytic solution Φ_+ .

8.2.3 Matrix RH problem

Let us calculate a product $\Phi_-^{-1}(x,k)\Phi_+(x,k)$ for $\operatorname{Im} k = 0$. We easily find that this product depends essentially on k only, the x-dependence being given by the simple exponential function E. Indeed,

$$\Phi_-^{-1}(x,k)\Phi_+(x,k) = EG_0(k)E^{-1}, \qquad G_0 = S_+^\dagger S_+ = \begin{pmatrix} 1 & \bar{b} \\ b & 1 \end{pmatrix} \tag{8.17}$$

with account for $|a|^2 + |b|^2 = 1$. Hence, we arrive at the matrix RH problem! This problem arises naturally provided we operate with *analytic* solutions of the spectral problem. The contour L, being the real axis $\operatorname{Im} k = 0$, divides the complex k-plane into the domains $\mathbb{C}_+$, $\operatorname{Im} k > 0$, and $\mathbb{C}_-$, $\operatorname{Im} k < 0$. The normalization of the RH problem (8.17) is canonical,

$$\Phi_\pm(x,k) \longrightarrow \mathbb{1} \quad \text{for} \quad k \to \infty,$$

owing to (8.15).

The RH problem (8.17) is characterized by the so-called RH data which are categorized into discrete data (eigenvalues k_j and eigenvectors $|j\rangle$; see later) and continuous data [the matrix element $b(k)$]. Solitons correspond to the discrete data of the RH problem with zeros of the scattering coefficients $a(k)$ and $\bar{a}(k)$. Because we showed in the preceding subsection that the determinants of the matrices Φ_+ and Φ_-^{-1} are given by $a(k)$ and $\bar{a}(k)$, respectively, these matrices have zeros at the points k_j, $\bar{k}_l$ in their domains of analyticity, i.e., $\det \Phi_+(k_j) = 0$, $\operatorname{Im} k_j > 0$, $j = 1, 2, \ldots, \mathcal{N}$, and $\det \Phi_-^{-1}(\bar{k}_l) = 0$, $\operatorname{Im} \bar{k}_l < 0$, $l = 1, 2, \ldots, \mathcal{N}$. We suppose that all zeros are simple and of finite number. Besides, in virtue of the involution (8.10), we have an equal number $\mathcal{N}$ of zeros in both domains. The case of multiple points of the RH problem associated with the Zakharov–Shabat spectral problem has been studied by Shchesnovich and Yang [401].

We will solve the RH problem with zeros (8.17) by means of its regularization. This procedure consists in extracting rational factors from Φ_+ which are responsible for the existence of zeros. In fact, these rational factors represent specific Darboux transformations which produce simple zeros in the wave function Φ_+ (Chap. 3). Indeed, if $\det \Phi_+(k_j) = 0$, then at the point k_j there exists an eigenvector $|\chi_j\rangle$ with zero eigenvalue, $\Phi_+(k_j)|\chi_j\rangle = 0$. Let us introduce a rational matrix function

$$\Xi_j^{-1} = \mathbb{1} + \frac{k_j - \bar{k}_j}{k - k_j} P_j, \qquad P_j = \frac{|\chi_j\rangle\langle\chi_j|}{\langle\chi_j|\chi_j\rangle}.$$

Here P_j is the rank 1 projector, $P_j^2 = P_j$, and $\langle\chi_j| = |\chi_j\rangle^\dagger$ (cf. Chap. 3). In a relevant basis $P_j = \operatorname{diag}(1, 0)$; hence, $\det \Xi_j^{-1} = (k - \bar{k}_j)(k - k_j)^{-1}$. Because $\det \Phi_+(k) \sim (k - k_j)$ near the point k_j, we evidently have $\det(\Phi_+ \Xi_j^{-1}) \neq 0$ at the point k_j. Thereby we succeeded in regularizing the RH problem at the

point k_j. Zero $\bar{k}_l$ of the matrix function Φ_-^{-1} is regularized by the rational function

$$\Xi_l = \mathbb{1} - \frac{k_l - \bar{k}_l}{k - \bar{k}_l} P_l$$

and the matrix $\Xi_l \Phi_-^{-1}$ has no zero in $\bar{k}_l$. The regularization of all the other zeros is performed similarly and eventually we obtain the following representation for the analytic solutions:

$$\Phi_\pm = \phi_\pm \Gamma, \qquad \Gamma = \Xi_{\mathcal{N}} \Xi_{\mathcal{N}-1} \cdots \Xi_1, \tag{8.18}$$

where the rational matrix function $\Gamma(x,k)$ accumulates all zeros of the RH problem, while the matrix functions $\phi_\pm$ solve the regular RH problem (i.e., without zeros):

$$\phi_-^{-1}(x,k)\phi_+(x,k) = \Gamma(x,k) E G_0(k) E^{-1} \Gamma^{-1}(x,k). \tag{8.19}$$

If we restrict ourselves to obtaining the soliton solutions of the NLS equation, i.e., for $G_0 = \mathbb{1}$, we can pose without loss of generality $\phi_\pm = \mathbb{1}$.[1] As a result, $\Phi_+ = \Gamma$. The matrix Γ will be called the dressing factor. It follows from (8.18) that the asymptotic expansion for the dressing factor is written as

$$\Gamma(x,k) = \mathbb{1} + \frac{1}{k}\Gamma^{(1)}(x) + O\left(\frac{1}{k^2}\right). \tag{8.20}$$

For practical purposes, it is more convenient to decompose the product (8.18) into simple fractions [124, 235]. In general, the rational matrix function $\Gamma(k)$ and its inverse can be decomposed into terms of two sets of the vectors $|x_j\rangle$ and $|y_j\rangle$:

$$\begin{aligned}
\Gamma(k) &= \left(\mathbb{1} - \frac{k_{\mathcal{N}} - \bar{k}_{\mathcal{N}}}{k - \bar{k}_{\mathcal{N}}} P_{\mathcal{N}}\right) \cdots \left(\mathbb{1} - \frac{k_1 - \bar{k}_1}{k - \bar{k}_1} P_1\right) \\
&= \mathbb{1} - \sum_{l=1}^{\mathcal{N}} \frac{k_l - \bar{k}_l}{k - \bar{k}_l} |x_l\rangle\langle y_l|, \\
\Gamma^{-1}(k) &= \left(\mathbb{1} + \frac{k_1 - \bar{k}_1}{k - k_1} P_1\right) \cdots \left(\mathbb{1} + \frac{k_{\mathcal{N}} - \bar{k}_{\mathcal{N}}}{k - k_{\mathcal{N}}} P_{\mathcal{N}}\right) \\
&= \mathbb{1} + \sum_{j=1}^{\mathcal{N}} \frac{k_j - \bar{k}_j}{k - k_j} |y_j\rangle\langle x_j|.
\end{aligned} \tag{8.21}$$

Hence, instead of $\mathcal{N}$ vectors $|\chi_j\rangle$ we obtained $2\mathcal{N}$ vectors $|x_j\rangle$ and $|y_j\rangle$. The next problem is to express $|x_j\rangle$ in terms of $|y_j\rangle$. Consider the identity

[1] In the examples in the following sections such a simple choice will not be valid. Moreover, for the perturbed NLS equations the equality $\phi_\pm = \mathbb{1}$ is valid in the leading-order approximation only.

$\Gamma(k)\Gamma^{-1}(k) = \mathbb{1}$ at the point $k = k_j$. To avoid divergence at $k \to k_j$, we should pose $\Gamma(k_j)|y_j\rangle\langle x_j| = 0$, or

$$\left(\mathbb{1} - \sum_{l=1}^{\mathcal{N}} \frac{k_l - \bar{k}_l}{k_j - \bar{k}_l} |x_l\rangle\langle y_l|\right) |y_j\rangle\langle x_j| = 0\,.$$

Multiply it by $|y_j\rangle$ on the right. Because $\langle x_j|y_j\rangle \neq 0$, we obtain

$$|y_j\rangle = \sum_{l=1}^{\mathcal{N}} |x_l\rangle\langle y_l \left| \frac{k_l - \bar{k}_l}{k_j - \bar{k}_l} \right| y_j\rangle. \tag{8.22}$$

Let us introduce $\mathcal{N} \times \mathcal{N}$ matrices

$$X = (|x_1\rangle, |x_2\rangle, \ldots, |x_{\mathcal{N}}\rangle), \quad Y = (|y_1\rangle, |y_2\rangle, \ldots, |y_{\mathcal{N}}\rangle),$$

$$D = \{D_{lj}\} = \left\{ \langle y_l \left| \frac{1}{k_j - \bar{k}_l} \right| y_j\rangle \right\}, \quad F = \operatorname{diag}(\ldots, k_l - \bar{k}_l, \ldots).$$

Then (8.22) is written as $Y = XFS$, or $XF = YS^{-1}$. In components,

$$(k_l - \bar{k}_l)|x_l\rangle = \sum_{j=1}^{\mathcal{N}} \left(D^{-1}\right)_{jl} |y_j\rangle.$$

Substituting it into (8.21) and introducing more convenient notation $|j\rangle \equiv |y_j\rangle$, we obtain the desired formula for the dressing factor:

$$\Gamma(k) = \mathbb{1} - \sum_{j,l=1}^{\mathcal{N}} \frac{1}{k - \bar{k}_l} |j\rangle \left(D^{-1}\right)_{j\,l} \langle l|, \quad D_{lj} = \frac{\langle l|j\rangle}{k_j - \bar{k}_l}. \tag{8.23}$$

Similarly,

$$\Gamma^{-1}(k) = \mathbb{1} + \sum_{j,l=1}^{\mathcal{N}} \frac{1}{k - k_j} |j\rangle \left(D^{-1}\right)_{j\,l} \langle l|. \tag{8.24}$$

Let us remember the reconstruction formula (8.16), which is now written as

$$Q = \mathrm{i}\left[\sigma_3, \Gamma^{(1)}(x)\right]. \tag{8.25}$$

As a result, we will be able to find solutions of the NLS equation, provided we can calculate explicitly the matrix Γ or, more precisely, the vector $|j\rangle$. To this end, let us differentiate the equation $\Phi_+(k_j)|j\rangle = 0$ in x. Because Φ_+ is a solution of the spectral equation (8.8), we obtain

$$\partial_x \Phi_+(x,k)_{k_j} |j\rangle + \Phi_+(x,k_j)|j\rangle_x = \mathrm{i}k_j \Phi_+(x,k_j)\sigma_3|j\rangle + \Phi_+(x,k_j)|j\rangle_x = 0.$$

Therefore, the x-dependence of $|j\rangle$ is given by a simple linear equation

$$|j\rangle_x = -\mathrm{i}k_j\sigma_3|j\rangle. \tag{8.26}$$

In the same manner we find the evolutionary equation

$$|j\rangle_t = -2\mathrm{i}k_j^2\sigma_3|j\rangle. \tag{8.27}$$

Integrating them, we obtain explicitly the vector $|j\rangle$ as

$$|j\rangle = \exp\left[(-\mathrm{i}k_j x - 2\mathrm{i}k_j^2 t)\sigma_3\right]|j_0\rangle, \tag{8.28}$$

where $|j_0\rangle$ is a vector integration constant.

8.2.4 Soliton solution

With the above results in hand, we can now derive the soliton solution of the NLS equation. We have in this case $\mathcal{N} = 1$ and pose $k_1 = \xi + \mathrm{i}\eta$. Then the vector $|1\rangle$ (8.28) takes the form

$$|1\rangle = \begin{pmatrix} \exp\left\{\eta(x+4\xi t) - \mathrm{i}\left[\xi x + 2(\xi^2-\eta^2)t\right]\right\} p_1 \\ \exp\left\{-\eta(x+4\xi t) + \mathrm{i}\left[\xi x + 2(\xi^2-\eta^2)t\right]\right\} p_2 \end{pmatrix},$$

where p_1 and p_2 are components of the constant vector $|1_0\rangle$. The reconstruction formula (8.25) reduces to

$$u(x,t) = 2\mathrm{i}\Gamma_{12}^{(1)}(x,t). \tag{8.29}$$

It follows from

$$\Gamma = \mathbb{1} - \frac{k_1 - \bar{k}_1}{k - \bar{k}_1}P_1$$

that

$$\Gamma^{(1)} = -2\mathrm{i}\eta\frac{|1\rangle\langle 1|}{\langle 1|1\rangle}. \tag{8.30}$$

It is important that the vector $|1\rangle$ enters (8.30) both in the numerator and in the denominator. Therefore, we can divide both components of the vector by the same number, say p_2, without changing $\Gamma^{(1)}$. Denoting

$$\mathrm{e}^{a+\mathrm{i}\beta} = \frac{p_1}{p_2}, \quad z = 2\eta(x+4\xi t) + a, \quad \varphi = -2\xi x - 4(\xi^2-\eta^2)t + \beta,$$

we can represent the vector $|1\rangle$ in the very simple form:

$$|1\rangle = \mathrm{e}^{(1/2)(a+\mathrm{i}\beta)}\begin{pmatrix} \mathrm{e}^{(1/2)(z+\mathrm{i}\varphi)} \\ \mathrm{e}^{-(1/2)(z+\mathrm{i}\varphi)} \end{pmatrix}. \tag{8.31}$$

Substituting this vector into (8.30), we find

$$\Gamma^{(1)} = -\mathrm{i}\eta\begin{pmatrix} \mathrm{e}^{z} & \mathrm{e}^{\mathrm{i}\varphi} \\ \mathrm{e}^{-\mathrm{i}\varphi} & \mathrm{e}^{-z} \end{pmatrix}\mathrm{sech}z$$

and, in accordance with (8.29), we finally obtain the standard formula [473] for the NLS soliton:

$$u(x,t) = 2\eta\mathrm{e}^{\mathrm{i}\varphi}\mathrm{sech}z. \tag{8.32}$$

Here ξ and η determine the soliton velocity and amplitude, respectively, while a and β give the initial position and phase of the soliton.

8.2.5 NLS breather

In this subsection we will obtain the breather solution of the NLS equation. The breather is an oscillating "bound state" of two solitons centered at the same position and having equal velocities. Without loss of generality, we take the velocity of the solitons to be zero ($\xi_1 = \xi_2 = 0$). As a result, we have now four zeros of the RH problem, two of them lying on the positive imaginary axis, $k_1 = \mathrm{i}\eta_1$ and $k_2 = \mathrm{i}\eta_2$, and the other two zeros lying on the negative imaginary axis, $\bar{k}_1 = -\mathrm{i}\eta_1$ and $\bar{k}_2 = -\mathrm{i}\eta_2$. Therefore, the dressing factor Γ entering the reconstruction formula (8.25) is written in the form (8.23) for $\mathcal{N} = 2$ and gives after the asymptotic expansion

$$\begin{aligned}\Gamma^{(1)} &= -\sum_{j,l=1}^{2} |j\rangle (D^{-1})_{j\,l} \langle l| \qquad (8.33)\\ &= -(D^{-1})_{11}|1\rangle\langle 1| - (D^{-1})_{21}|2\rangle\langle 1| - (D^{-1})_{12}|1\rangle\langle 2| - (D^{-1})_{22}|2\rangle\langle 2|.\end{aligned}$$

The matrix elements D_{lj} are given by (8.23) and hence the matrix D takes the form

$$D = \begin{pmatrix} \dfrac{\langle 1|1\rangle}{k_1 - k_1} & \dfrac{\langle 1|2\rangle}{k_2 - k_1} \\ \dfrac{\langle 2|1\rangle}{k_1 - k_2} & \dfrac{\langle 2|2\rangle}{k_2 - k_2} \end{pmatrix}.$$

We can rewrite (8.33) immediately in terms of the matrix D:

$$\Gamma^{(1)} = (\det D)^{-1}[-D_{22}|1\rangle\langle 1| + D_{21}|2\rangle\langle 1| + D_{12}|1\rangle\langle 2| - D_{11}|2\rangle\langle 2|]. \qquad (8.34)$$

The vector $|j\rangle$, $j = 1, 2$, has the form [see (8.31)]

$$|j\rangle = \mathrm{e}^{(\mathrm{i}/2)\beta_j} \begin{pmatrix} \mathrm{e}^{(1/2)(z_j + \mathrm{i}\varphi_j)} \\ \mathrm{e}^{-(1/2)(z_j + \mathrm{i}\varphi_j)} \end{pmatrix}, \quad z_j = 2\eta_j x, \quad \varphi_j = 4\eta_j^2 t + \beta_j,$$

and we put $a_j = 0$ because the maxima of both solitons coincide. Let us first calculate matrix elements D_{lj}:

$$D_{12} = \frac{\mathrm{e}^{-(\mathrm{i}/2)(\beta_1 - \beta_2)}}{\mathrm{i}(\eta_1 + \eta_2)} \left(\exp \frac{1}{2}(z_1 + z_2 - \mathrm{i}\varphi_1 + \mathrm{i}\varphi_2) + \exp \frac{1}{2}(-z_1 - z_2 + \mathrm{i}\varphi_1 - \mathrm{i}\varphi_2) \right),$$

$$D_{21} = -D_{12}^*, \qquad D_{11} = \frac{\cosh z_1}{\mathrm{i}\eta_1}, \qquad D_{22} = \frac{\cosh z_2}{\mathrm{i}\eta_2}.$$

Then we obtain the determinant of D:

$$\det D = \frac{2}{(\eta_1 + \eta_2)^2} \left(\cosh(z_1 + z_2) + \cos(\varphi_1 - \varphi_2) - \frac{(\eta_1 + \eta_2)^2}{2\eta_1\eta_2} \cosh z_1 \cosh z_2 \right).$$

Remember that the two-soliton solution is given by (8.29). Calculating now $\Gamma_{12}^{(1)}$ by means of (8.32)–(8.34), we find

$$\Gamma_{12}^{(1)} = \frac{1}{\mathrm{i}\det D}\frac{\eta_1-\eta_2}{\eta_1+\eta_2}\left(\frac{\cosh z_1}{\eta_1}\mathrm{e}^{\mathrm{i}\varphi_2} - \frac{\cosh z_2}{\eta_2}\mathrm{e}^{\mathrm{i}\varphi_1}\right).$$

Hence,

$$\begin{aligned} u(x,t) = (\eta_1^2-\eta_2^2)\left(\frac{\cosh 2\eta_1 x}{\eta_1}\exp\left(4\mathrm{i}\eta_2^2 t+\mathrm{i}\beta_2\right) - \frac{\cosh 2\eta_2 x}{\eta_2}\exp\left(4\mathrm{i}\eta_1^2+\mathrm{i}\beta_1\right)\right) \\ \times\left(\cosh 2(\eta_1+\eta_2)x + \cos\left[4(\eta_1^2-\eta_2^2)t+\beta_1-\beta_2\right]\right. \\ \left. - \frac{(\eta_1+\eta_2)^2}{2\eta_1\eta_2}\cosh 2\eta_1 x\,\cosh 2\eta_2 x\right)^{-1}. \end{aligned} \tag{8.35}$$

This formula describes the two-soliton solution of the NLS equation but it is not yet a breather. The breather being a result of the evolution of the initial configuration $u(x,0) = 2\operatorname{sech} x$ is obtained under definite relations between η_1 and η_2 and between β_1 and β_2 [387]. Considering $u(x,t)$ (8.35) for $t=0$, we easily find that we should take $\eta_1 = 3/2$, $\eta_2 = 1/2$, $\beta_1 = 0$, and $\beta_2 = \pi$. As a result, the breather solution of the NLS equation is written as

$$u(x,t) = 4\mathrm{e}^{\mathrm{i}t}\frac{\cosh 3x + 3\mathrm{e}^{8\mathrm{i}t}\cosh x}{\cosh 4x + 4\cosh 2x + 3\cos 8t}. \tag{8.36}$$

The breather (8.36) oscillates with the frequency $\omega = 8$.

In conclusion, let us summarize the basic steps in the derivation of the soliton solution. First we built analytic solutions of the spectral problem from the components of the Jost solutions. Then we showed that the analytic solutions solve the RH problem with zeros. After regularization of the RH problem we extracted the rational dressing factor. The dressing factor is determined by the discrete RH data, i.e., eigenvalues and eigenvectors. The eigenvalues are constants of motion, while the eigenvectors are governed by simple linear equations. After integrating these equations we obtained the eigenvectors explicitly that enable us to calculate the dressing factor and finally to derive the soliton solution.

8.3 Modified nonlinear Schrödinger equation

In this section we will obtain soliton solutions of the MNLS equation taking as a seed solution the trivial one $u = 0$. The MNLS equation

$$\mathrm{i}u_t + u_{xx} + 2|u|^2u + \mathrm{i}\alpha\left(|u|^2u\right)_x = 0, \quad \alpha\in\mathrm{Re} \tag{8.37}$$

for a scalar complex function $u(x,t)$ has important applications in nonlinear optics [23, 121, 180, 432, 442] and plasma physics [213, 334]. In particular, this equation extends the famous NLS equation to the case of subpicosecond optical pulses. For definiteness we take hereafter the parameter $\alpha > 0$.

8.3.1 Jost solutions

Equation (8.37) allows the Lax representation (8.2) with the matrices U and V of the form

$$U = -\frac{\mathrm{i}}{\alpha}(k^2-1)\sigma_3 + \mathrm{i}kQ, \quad Q = \begin{pmatrix} 0 & u \\ \bar{u} & 0 \end{pmatrix}, \tag{8.38}$$

$$V = -\frac{2\mathrm{i}}{\alpha^2}(k^2-1)^2\sigma_3 + \frac{2\mathrm{i}}{\alpha}k(k^2-1)Q + \mathrm{i}k^2Q^2\sigma_3 - k\sigma_3 Q_x - \mathrm{i}\alpha k Q^3.$$

As for the NLS equation, we define a matrix $J = \psi E^{-1}$, where

$$E = \exp\left[-(\mathrm{i}/\alpha)(k^2-1)x\sigma_3\right]$$

is a solution of the equation $\psi_x = U\psi$ for zero potential $Q = 0$; hence, a spectral equation for the MNLS equation is written as

$$J_x = -\frac{\mathrm{i}}{\alpha}(k^2-1)\,[\sigma_3, J] + \mathrm{i}k\,QJ. \tag{8.39}$$

This spectral problem, being quadratic in the spectral parameter k, belongs to the Wadati–Konno–Ichikawa (WKI) class [444]. The Hamiltonian structure and squared solutions of equations solvable by the quadratic spectral problem have been studied by Gerdjikov and Ivanov [183, 184]. It should be noted that the MNLS equation can be transformed by a gauge transformation [330] to the so-called derivative NLS equation [233] which is also applicable in plasma physics [119, 337, 411].

Jost solutions $J_\pm(x,k)$ of (8.39) are determined by the asymptotics $J_\pm \to \mathbb{1}$ as $x \to \pm\infty$ and $\det J_\pm = 1$. The scattering matrix $S(k)$ is given by the equations of the form (8.9) for $\operatorname{Im} k^2 = 0$ and the involution (8.10) preserves its form for the MNLS Jost functions as well.

There exists another symmetry of the spectral equation (8.39). Indeed, it can be easily shown that a solution of the spectral equation satisfies a parity condition

$$J(k) = \sigma_3 J(-k)\sigma_3. \tag{8.40}$$

This condition means that the diagonal entries of the matrix J are even functions of k, while off-diagonal ones are odd functions. Evidently, the parity condition (8.40) is valid as well for the scattering matrix. In particular, we find that the scattering matrix elements obey the parity conditions $a(k) = a(-k)$ and $b(k) = -b(-k)$.

Let us consider now the asymptotic expansion of the Jost solutions at $k \to \infty$,

$$J_\pm(x,k) = J_\pm^{(0)}(x) + \frac{1}{k}\,J_\pm^{(1)}(x) + O\left(\frac{1}{k^2}\right). \tag{8.41}$$

As a consequence of the parity property (8.40), the expansion coefficients $J_\pm^{(2n)}$, $n = 0, 1, \ldots$, are diagonal matrices, while $J_\pm^{(2n+1)}$, $n = 0, 1, \ldots$, are

off-diagonal matrices. Substituting the series (8.41) into the spectral equation, we find the leading-order terms $J_{\pm}^{(0)}(x)$ as

$$J_{\pm}^{(0)}(x) = \exp\left(-\frac{\mathrm{i}\alpha}{2}\sigma_3 \int_{\pm\infty}^{x} |u(\xi)|^2 \mathrm{d}\xi\right).$$

Hence, we arrive at the important conclusion that the asymptotic expansion with the unit matrix as the leading-order term, like for the NLS equation, is incompatible with the spectral equation (8.39).

8.3.2 Analytic solutions

Rewriting equations for the first column of J_- in the integral form,

$$\begin{aligned} J_{-11} &= 1 + \mathrm{i}k \int_{-\infty}^{x} \mathrm{d}\xi\, u(\xi)\, J_{-21}(\xi), \\ J_{-21} &= \mathrm{i}k \int_{-\infty}^{x} \mathrm{d}\xi\, \bar{u}(\xi)\, J_{-11}(\xi) \exp\left(\frac{2\mathrm{i}}{\alpha}(k^2-1)(x-\xi)\right), \end{aligned} \tag{8.42}$$

we see that the exponent in the integrand of (8.42) decreases for $\operatorname{Im} k^2 > 0$, i.e., for k lying in the first and third quadrants of the k-plane. We denote this domain by $\mathbb{C}_+$. In other words, the first column $J_-^{[1]}$ is analytic in $\mathbb{C}_+$ and sectionally continuous on $\operatorname{Im} k^2 = 0$, i.e., on the real and imaginary axes, reaching them from $\mathbb{C}_+$. Similarly we reveal analyticity of the column $J_+^{[2]}$ in the same domain. Therefore, a matrix function

$$\Phi_+ = \left(J_-^{[1]}, J_+^{[2]}\right)$$

solves the spectral equation (8.39) and is analytic in $\mathbb{C}_+$. Similarly to the NLS equation, we can express the analytic solution Φ_+ in terms of the Jost functions:

$$\Phi_+ = J_+ E S_+ E^{-1} = J_- E S_- E^{-1},$$

with the same matrices $S_\pm$ as in (8.12) and (8.13). The asymptotic expansion for Φ_+ takes the form

$$\Phi_+(x,k) = \Phi_+^{(0)}(x) + \frac{1}{k}\Phi_+^{(1)}(x) + O\left(\frac{1}{k^2}\right). \tag{8.43}$$

Substituting this expansion into the spectral equation and equating terms with equal powers of k, we find

$$\Phi_+^{(1)} = \frac{\alpha}{2}\sigma_3 Q \Phi_+^{(0)}, \qquad \Phi_{+x}^{(0)} = \mathrm{i}Q\Phi_+^{(1)}.$$

Combining these relations, we get two important results: first, the equation for $\Phi_+^{(0)}$,

$$\Phi_{+x}^{(0)} = -\frac{\mathrm{i}\alpha}{2}\,\sigma_3\, Q^2 \Phi_+^{(0)}, \tag{8.44}$$

and, second, a formula for the reconstruction of the potential Q,

$$Q = \frac{2}{\alpha}\,\sigma_3\,\Phi_+^{(1)}\Phi_+^{(0)-1}\,. \tag{8.45}$$

In virtue of the involution of the type (8.10) we introduce a matrix function Φ_-^{-1},

$$\Phi_-^{-1}(x,k) = \Phi_+^{\dagger}(x,\bar{k}), \tag{8.46}$$

which is analytic for $\operatorname{Im} k^2 < 0$, i.e., in the second and fourth quadrants. This domain is denoted as $\mathbb{C}_-$. These formulas give the following relations for determinants:

$$\det\Phi_+(x,k) = a(k), \qquad \det\Phi_-^{-1}(x,k) = \bar{a}(k).$$

As follows from (8.45), we need to know explicitly the matrix Φ_+ to find solutions of the MNLS equation.

8.3.3 Matrix RH problem

Once again we calculate a product $\Phi_-^{-1}(x,k)\Phi_+(x,k)$ for $k \in \operatorname{Im} k^2 = 0$,

$$\Phi_-^{-1}(x,k)\Phi_+(x,k) = EG_0(k)E^{-1}\,, \tag{8.47}$$

where

$$G_0(k) = S_+^{\dagger}S_+ = \begin{pmatrix} 1 & \bar{b} \\ b & 1 \end{pmatrix}.$$

Hence, we obtain the RH problem (8.5). Figure 8.1 illustrates the contour L dividing the complex k-plane into two domains $\mathbb{C}_+$ and $\mathbb{C}_-$ with the Φ_+ function and the Φ_-^{-1} function, respectively. The positive direction of the contour

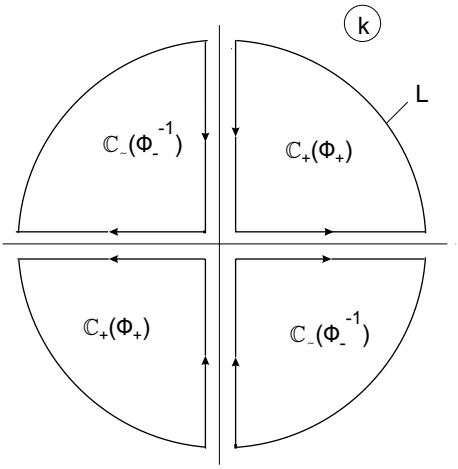

Fig. 8.1. Domains of analyticity and the contour L direction

corresponds to the rule that the $\mathbb{C}_+$ domain is on the left when traveling along the contour. The normalization of the RH problem (8.47) is noncanonical because, in accordance with (8.43),

$$\Phi_+(x,k) \to \Phi_+^{(0)}(x), \qquad k \to \infty\,.$$

In general we obtain the RH problem with zeros. Suppose that all zeros are simple. In virtue of the involution (8.46) we have an *equal* number $\mathcal{N}$ of zeros in $\mathbb{C}_+$ and $\mathbb{C}_-$. Moreover, because of the parity property (8.40), zeros appear in pairs as $\pm k_j$ and $\pm\bar{k}_l$. This means that the regularization of the RH problem at the points $\pm k_j$ is performed by *two* elementary rational multipliers, $\Phi_+\Xi_j^{-1}\Xi_{-j}^{-1}$, where

$$\Xi_j^{-1} = \mathbb{1} + \frac{k_j - \bar{k}_j}{k - \bar{k}_j} P_j, \quad \Xi_{-j}^{-1} = \mathbb{1} - \frac{k_j - \bar{k}_j}{k + \bar{k}_j} P_{-j}, \quad P_{\pm j} = \frac{|\chi_{\pm j}\rangle\langle\chi_{\pm j}|}{\langle\chi_{\pm j}|\chi_{\pm j}\rangle},$$

and $\Phi_+(\pm k_j)|\chi_{\pm j}\rangle = 0$. In virtue of the parity property, the vectors $|\chi_j\rangle$ and $|\chi_{-j}\rangle$ are interrelated, $|\chi_{-j}\rangle = \sigma_3|\chi_j\rangle$, and therefore $P_{-j} = \sigma_3 P_j \sigma_3$. After the complete regularization, we once again arrive at the factorizable representation of $\Phi_\pm$,

$$\Phi_\pm = \phi_\pm \Gamma, \quad \Gamma = \Xi_{\mathcal{N}}\Xi_{-\mathcal{N}}\cdots\Xi_1\Xi_{-1}\,, \tag{8.48}$$

where $\phi_\pm$ solve the regular RH problem

$$\phi_-^{-1}\phi_+ = \Gamma E G_0 E^{-1}\Gamma^{-1}. \tag{8.49}$$

Comparing the asymptotic expansion

$$\Gamma(x,k) = \mathbb{1} + k^{-1}\Gamma^{(1)}(x) + O(k^{-2})$$

with that for Φ_+ (8.43), we obtain from (8.48)

$$\Phi_+^{(0)} = \phi_+, \qquad \Phi_+^{(1)} = \Phi_+^{(0)}\Gamma^{(1)}.$$

Hence, we can take the leading-order term $\Phi_+^{(0)}(x)$ of the asymptotic expansion (8.43) as a k-independent solution of the regular RH problem. In turn, the reconstruction formula (8.45) now takes the form

$$Q = \frac{2}{\alpha}\,\Phi_+^{(0)}\Gamma^{(1)}\left(\Phi_+^{(0)}\right)^{-1}. \tag{8.50}$$

Note that because the RH problem for the NLS equation allows the standard normalization, we took a trivial solution ($\phi_+ = \mathbb{1}$) of the regular RH problem (8.19). It should be stressed once again that a choice of a k-independent solution of the regular RH problem is valid for solitons only. If we want to account for the nonsolitonic part of a solution, we should consider a nontrivial solution of the regular RH problem. As a rule, the regular RH problem does

not allow as complete an analytical investigation as the RH problem with $G_0 = \mathbb{1}$. In general, the regular RH problem can be formulated in terms of singular integral equations. Examples of a perturbative study of the regular RH problem to account for soliton radiation are given in [122, 398]. The description of the RH problem for the WKI spectral problem can be also found in the paper by Zabolotskii [463].

8.3.4 MNLS soliton

We can simplify the reconstruction formula (8.50) when soliton solutions are concerned. Indeed, equations (8.39), (8.44), and (8.48) yield the equation for the dressing factor:

$$\Gamma_x = \left(\Phi_+^{(0)-1}\Phi_+\right)_x = \frac{\mathrm{i}\alpha}{2}\,Q^2\sigma_3\Gamma - \frac{\mathrm{i}}{2}\,(k^2-1)\,[\sigma_3,\Gamma] + \mathrm{i}k\left(\Phi_+^{(0)}\right)^{-1} Q\,\Phi_+^{(0)}\,.$$

Therefore, $\Gamma^{-1}(x,k=0)$ obeys

$$\Gamma^{-1}(x,k=0)_x = -\frac{\mathrm{i}\alpha}{2}\,Q^2\sigma_3\Gamma^{-1}(x,k=0).$$

But exactly the same differential equation occurs for $\Phi_+^{(0)}$; see (8.44). Hence, we can identify $\Gamma^{-1}(x,k=0)$ and $\Phi_+^{(0)}$ and write the reconstruction formula (8.50) for solitons as

$$Q = \frac{2}{\alpha}\,\sigma_3\Gamma^{-1}(k=0)\Gamma^{(1)}\Gamma(k=0)\,. \tag{8.51}$$

As a result, it is the dressing factor Γ that completely determines soliton solutions of the MNLS equation.

Now we will derive the MNLS soliton. The discrete data of the RH problem comprise the eigenvalues k_1, $k_2 = -k_1$, $\bar{k}_1$, and $\bar{k}_2 = -\bar{k}_1$ (Fig. 8.2), as well as the eigenvectors $|1\rangle$ and $|2\rangle \equiv |-1\rangle = \sigma_3|1\rangle$. Following (8.23) and (8.24), we have

$$\begin{aligned}
\Gamma(k) = \mathbb{1} &- \frac{1}{k-\bar{k}_1}\left[\left(D^{-1}\right)_{11}|1\rangle\langle 1| + \left(D^{-1}\right)_{21}|2\rangle\langle 1|\right] \\
&- \frac{1}{k+\bar{k}_1}\left[\left(D^{-1}\right)_{12}|1\rangle\langle 2| + \left(D^{-1}\right)_{22}|2\rangle\langle 2|\right], \\
\Gamma^{-1}(k) = \mathbb{1} &+ \frac{1}{k-k_1}\left[\left(D^{-1}\right)_{11}|1\rangle\langle 1| + \left(D^{-1}\right)_{12}|1\rangle\langle 2|\right] \\
&+ \frac{1}{k+k_1}\left[\left(D^{-1}\right)_{21}|2\rangle\langle 1| + \left(D^{-1}\right)_{22}|2\rangle\langle 2|\right].
\end{aligned}$$

Hence, we need know the eigenvectors to obtain the matrix elements D_{lj} (8.23) and the dressing factor Γ. In the same way as was done for the NLS equation in (8.26) and (8.27), we have a system of linear equations for the vector $|1\rangle$:

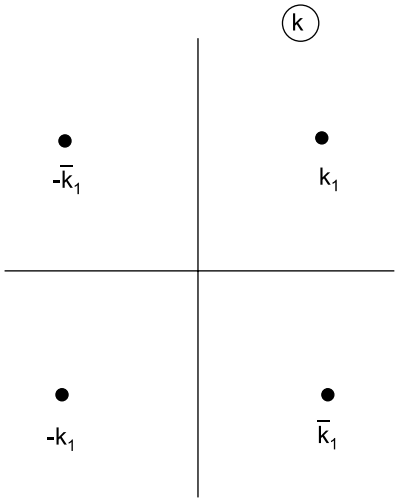

Fig. 8.2. Typical arrangement of zeros corresponding to the MNLS soliton

$$|1\rangle_x = -\frac{\mathrm{i}}{\alpha}\,(k_1^2-1)\sigma_3|1\rangle, \qquad |1\rangle_t = -\frac{2\mathrm{i}}{\alpha^2}\,(k_1^2-1)^2\sigma_3|1\rangle.$$

Solving it we obtain the vector $|1\rangle$ exactly in the form (8.31) but with different definitions of z and φ. Namely,

$$z = \frac{1}{2w}\,(x - Vt - x_0)\,, \quad x_0 = 2aw,$$

$$\varphi = Vwz + \frac{1}{4}(V^2 + w^{-2})t + \varphi_0, \quad \varphi_0 = aVw + \beta.$$

Here we have introduced real parameters V and w,

$$V = \frac{2}{\alpha}\,(2 - k_1^2 - \bar{k}_1^2), \qquad w = \frac{\mathrm{i}}{2}\frac{\alpha}{k_1^2 - \bar{k}_1^2},$$

which play the role of the soliton velocity and width, as will be seen later. Note once again that $|2\rangle = \sigma_3|1\rangle$. Therefore, we can now calculate the matrix D in accordance with (8.23) and represent Γ and Γ^{-1} as

$$\Gamma(k) = \mathbb{1} - \frac{\tilde{D}_-}{k - \bar{k}_1} - \frac{\tilde{D}_+}{k + \bar{k}_1}, \quad \Gamma^{-1}(k) = \mathbb{1} + \frac{D_-}{k - k_1} + \frac{D_+}{k + k_1}.$$

Here

$$D_- = \frac{k_1^2 - \bar{k}_1^2}{2}\begin{pmatrix} \dfrac{\mathrm{e}^{z}}{k_1\mathrm{e}^{z} + \bar{k}_1\mathrm{e}^{-z}} & \dfrac{\mathrm{e}^{\mathrm{i}\varphi}}{k_1\mathrm{e}^{-z} + \bar{k}_1\mathrm{e}^{z}} \\ \dfrac{\mathrm{e}^{-\mathrm{i}\varphi}}{k_1\mathrm{e}^{z} + \bar{k}_1\mathrm{e}^{-z}} & \dfrac{\mathrm{e}^{-z}}{k_1\mathrm{e}^{-z} + \bar{k}_1\mathrm{e}^{z}} \end{pmatrix}, \quad D_+ = -\sigma_3 D_- \sigma_3,$$

$$\tilde{D}_- = \frac{k_1^2 - \bar{k}_1^2}{2} \begin{pmatrix} \dfrac{\mathrm{e}^{z}}{k_1\mathrm{e}^{-z} + \bar{k}_1\mathrm{e}^{z}} & \dfrac{\mathrm{e}^{\mathrm{i}\varphi}}{k_1\mathrm{e}^{-z} + \bar{k}_1\mathrm{e}^{z}} \\ \dfrac{\mathrm{e}^{-\mathrm{i}\varphi}}{k_1\mathrm{e}^{z} + \bar{k}_1\mathrm{e}^{-z}} & \dfrac{\mathrm{e}^{-z}}{k_1\mathrm{e}^{z} + \bar{k}_1\mathrm{e}^{-z}} \end{pmatrix}, \quad \tilde{D}_+ = -\sigma_3 \tilde{D}_- \sigma_3.$$

This gives

$$\Gamma^{(1)} = -\left(\tilde{D}_+ + \tilde{D}_-\right) = (k_1^2 - \bar{k}_1^2) \begin{pmatrix} 0 & \dfrac{\mathrm{e}^{\mathrm{i}\varphi}}{k_1\mathrm{e}^{-z} + \bar{k}_1\mathrm{e}^{z}} \\ \dfrac{\mathrm{e}^{-\mathrm{i}\varphi}}{k_1\mathrm{e}^{z} + \bar{k}_1\mathrm{e}^{-z}} & 0 \end{pmatrix},$$

$$\Gamma(k=0) = \frac{k_1}{\bar{k}_1} \begin{pmatrix} \dfrac{k_1\mathrm{e}^{z} + \bar{k}_1\mathrm{e}^{-z}}{k_1\mathrm{e}^{-z} + \bar{k}_1\mathrm{e}^{z}} & 0 \\ 0 & \dfrac{k_1\mathrm{e}^{-z} + \bar{k}_1\mathrm{e}^{z}}{k_1\mathrm{e}^{z} + \bar{k}_1\mathrm{e}^{-z}} \end{pmatrix}.$$

Substituting these matrices into (8.51) eventually yields the soliton of the MNLS equation:

$$u_s(x,t) = \frac{\mathrm{i}}{w} \frac{k_1\mathrm{e}^{-z} + \bar{k}_1\mathrm{e}^{z}}{(k_1\mathrm{e}^{z} + \bar{k}_1\mathrm{e}^{-z})^2} \mathrm{e}^{\mathrm{i}\varphi}. \tag{8.52}$$

The most important feature of the soliton (8.52) consists in the fact that the parameter α enters the denominator of (8.52) through w. This means that the soliton (8.52) is nonperturbative with respect to α and cannot be obtained by considering the MNLS equation as an α-perturbed NLS equation. Despite a rather unaccustomed form, it is easy to check that the modulus of the soliton (8.52) behaves as prescribed for solitons, i.e., mainly in accordance with the hyperbolic secant rule:

$$|u_s| = \frac{\operatorname{sech} z}{2w|k_1|\cos\theta} \left(1 + \tan^2\theta \tanh^2 z\right)^{-1/2}.$$

Here

$$|k_1| = \left[\left(1 - \frac{\alpha}{4}V\right)^2 + \left(\frac{\alpha}{4w}\right)^2\right]^{1/4}, \quad \tan 2\theta = \frac{\alpha}{4w} \frac{1}{1 - (\alpha/4)V}.$$

The soliton (8.52) has a number of peculiarities which distinguish it from the standard NLS soliton. First, the soliton u_s has nonzero phase difference at its limits. Indeed,

$$\frac{k_1\mathrm{e}^{-z} + \bar{k}_1\mathrm{e}^{z}}{(k_1\mathrm{e}^{z} + \bar{k}_1\mathrm{e}^{-z})^2} \longrightarrow \begin{cases} (\bar{k}_1/k_1^2)\,\mathrm{e}^{-z}, & z \to \infty \\ (k_1/\bar{k}_1^2)\,\mathrm{e}^{z}, & z \to -\infty \end{cases}$$

and

$$\arg\left(u_s(z \to -\infty)\right) - \arg\left(u_s(z \to \infty)\right) = 6\arg(k_1) \neq 0.$$

Second, the important invariant of the MNLS equation, namely, the optical energy (or number of particles) $\int_{-\infty}^{\infty} |u_s|^2 \mathrm{d}x$, has the upper limit [399]

$$\int_{-\infty}^{\infty} |u_s|^2 \mathrm{d}x = \frac{8}{\alpha} \arg(k_1) < \frac{4\pi}{\alpha} .$$

These properties of the MNLS soliton resemble those of the dark NLS soliton which also has nonzero phase difference and relation between the optical energy and the phase difference.

At the same time, there exists a nontrivial limit transition from the MNLS soliton to the bright NLS one. To carry out this limit, we should take into account that the Lax pair (8.38) for the MNLS equation should produce in this limit the Lax pair (8.7) for the NLS equation. This condition implies that the spectral parameter k depends on α and gives the following prescription [399]:

$$\frac{1}{\alpha}(k^2 - 1) \longrightarrow k_{\mathrm{NLS}} \quad \text{at} \quad \alpha \to 0,$$

or

$$k = 1 + \frac{\alpha}{2} k_{\mathrm{NLS}} + O(\alpha^2) . \tag{8.53}$$

In the limit (8.53) the MNLS soliton (8.52) reproduces the NLS soliton (8.32).

Similarly to the NLS equation, we can construct a MNLS breather by means of the dressing factor Γ with $N = 4$ because of zeros $\pm k_1$ and $\pm k_2$ (as well as $\pm \bar{k}_1$ and $\pm \bar{k}_2$). Explicit calculation was performed by Doktorov [121]. Figures 8.3 and 8.4 demonstrate the temporal evolution of the MNLS breather. We see that the MNLS breather evolves as a whole object, without any decomposition into single solitons, as it should.

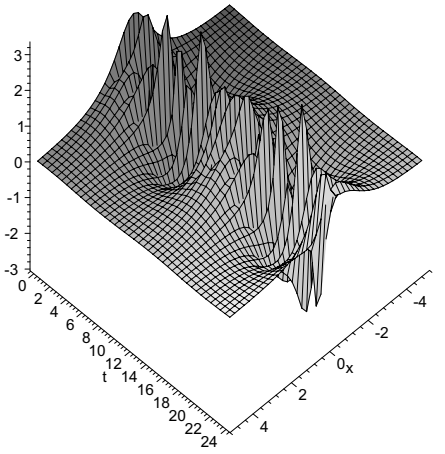

Fig. 8.3. Evolution of the real part of the MNLS breather

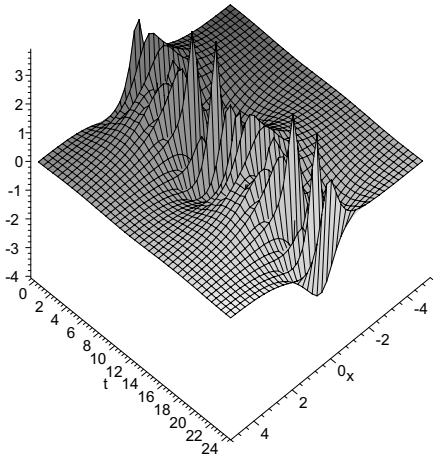

Fig. 8.4. Evolution of the imaginary part of the MNLS breather

8.4 Ablowitz–Ladik equation

This section is devoted to a consideration of a completely different example of nonlinear integrable equations, namely, the discrete nonlinear AL equation [8]. The propagation properties of waves arising as a result of the interplay of nonlinearity with the lattice discreteness can be quite distinct from those inherent in continuous nonlinear systems. For example, self-focusing and defocusing processes can be achieved in the same discrete medium, and wavelength diffraction management [9], the possibility forbidden in continuous systems, is possible. Our aim in this section is to derive the soliton solution of the AL equation. We will follow prescriptions developed in the preceding sections and will see that, despite some features peculiar to discrete equations, the main ideas of the dressing method based on the RH problem are valid for a wide class of nonlinear equations, no matter whether the coordinates are discrete or continuous. Moreover, a striking resemblance exists between solving the MNLS and AL equations, though it is the NLS equation that represents the continuous limit of the AL equation.

8.4.1 Jost solutions

The AL equation

$$\mathrm{i}u_{nt} + \frac{1}{h^2}(u_{n+1} + u_{n-1} - 2u_n) + |u_n|^2(u_{n+1} + u_{n-1}) = 0 \tag{8.54}$$

describes evolution of a scalar complex function $u_n(t)$ defined on an infinite one-dimensional lattice ($-\infty < n < \infty$) with the lattice spacing h. The terms

$u_{n+1} + u_{n-1} - 2u_n$ represent the discrete analog of the second derivative. To perform a limit $h \to 0$, we write $u_{n\pm1}$ as

$$u_{n\pm1} = u \pm hu_x + \frac{1}{2}\, h^2 u_{xx}$$

and substitute them into (8.54). This yields the NLS equation $\mathrm{i}u_t = u_{xx} + 2|u|^2 u$. In the following we put $h = 1$.

The AL equation allows the Lax representation with the AL spectral problem [8]

$$J(n+1) = (E + Q_n)J(n)E^{-1}, \tag{8.55}$$

$$Q_n = \begin{pmatrix} 0 & u_n \\ -u_n^* & 0 \end{pmatrix}, \qquad E = \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix},$$

and the evolutionary equation

$$\begin{aligned} J_t(n) &= V(n)J(n) - J(n)\Omega(z), \\ V(n) &= \mathrm{i}\begin{pmatrix} u_{n-1}^* u_n & zu_n - z^{-1}u_{n+1} \\ z^{-1}u_n^* - zu_{n-1}^* & -u_{n-1}u_n^* \end{pmatrix} + \Omega, \\ \Omega\,(z) &= \frac{\mathrm{i}}{2}(z - z^{-1})^2\sigma_3. \end{aligned} \tag{8.56}$$

It is customary to denote a spectral parameter for the AL spectral problem as z, instead of k in the preceding sections. Besides, for further convenience we use here an asterisk to denote complex conjugation.

Note that we encounter the first novel feature compared with continuous equations. Indeed, the spectral problem (8.55) is not differential but *algebraical*. Nevertheless, we can introduce matrix Jost functions $J_\pm(n, z)$ as solutions of the spectral equation (8.55) with the asymptotics $J_\pm \to \mathbb{1}$ as $n \to \pm\infty$. The scattering matrix $S(z)$ defined by

$$J_-(n, z) = J_+(n, z)E^n S(z)E^{-n} \tag{8.57}$$

has the structure

$$S(z) = \begin{pmatrix} a_+ & -b_- \\ b_+ & a_- \end{pmatrix}.$$

The AL spectral problem, like the WKI one, obeys the parity property

$$J(n, z) = \sigma_3 J(n, -z)\sigma_3. \tag{8.58}$$

Hence, we can suppose that the process of solving the AL equation will have much in common with that of the MNLS equation, and not with that of the NLS equation, as one might expect.

The second feature is the lack of a simple involution relation like (8.10). Instead we will now derive a discrete analog of the involution property. To this end, let us define a parameter $\bar{z}$ as $\bar{z} = 1/z^*$. Then it is easy to show that $(E(\bar{z}) + Q_n)^\dagger = E^{-1}(z) - Q_n$ and

$$[J_\pm(n+1, \bar{z})]^\dagger = \big[(E(\bar{z}) + Q_n)\, J_\pm(n, \bar{z})E^{-1}(\bar{z})\big]^\dagger = E(z)J_\pm(n, \bar{z})\big(E^{-1}(z) - Q_n\big).$$

Accounting for the relation $[E^{-1}(z) - Q_n](E + Q_n) = (1 + |u_n|^2)\,\mathbb{1}$, we obtain

$$J_\pm^\dagger(n+1,\bar z)\,J_\pm(n+1,z) = (1+|u_n|^2)E\,J_\pm^\dagger(n,\bar z)\,J_\pm(n,z)E^{-1}.$$

Iterating this relation as

$$J_\pm^\dagger(n,\bar z)\,J_\pm(n,z) = (1+|u_n|^2)E^{-1}J_\pm^\dagger(n+1,\bar z)\,J_\pm(n+1,z)E$$

$$= (1+|u_n|^2)^{-1}(1+|u_{n+1}|^2)^{-1}E^{-2}J_\pm^\dagger(n+2,\bar z)\,J_\pm(n+2,z)E^2 = \dots$$

and taking into account the asymptotic behavior $J_+ \to \mathbb{1}$ at $n \to +\infty$ eventually yields

$$J_+^\dagger(n,\bar z)J_+(n,z) = \prod_{l=n}^{\infty}\rho_l^{-1}\,\mathbb{1}\,,$$

where $\rho_l = 1 + |u_l|^2$. Denoting

$$\prod_{l=n}^{\infty}\rho_l^{-1} = v_+(n), \tag{8.59}$$

we can write the result of iterations as

$$J_+^\dagger(n,\bar z) = v_+(n)\,J_+^{-1}(n,z). \tag{8.60}$$

In the same way we obtain

$$J_-^\dagger(n,\bar z) = v_-(n)J_-^{-1}(n,z), \qquad v_-(n) = \prod_{l=-\infty}^{n-1}\rho_l. \tag{8.61}$$

Equations (8.60) and (8.61) determine the involution property for the AL spectral problem. We will call the relations (8.60) and (8.61) as *conjugation*. Now we show that the functions $v_\pm(n)$ are nothing more than determinants of $J_\pm$. Indeed, because $\det(E+Q_n) = \rho_n$, we have $\det J_+(n+1) = \rho_n \det J_+(n)$, or

$$\det J_+(n) = \rho_n^{-1}\det J_+(n+1) = \rho_n^{-1}\rho_{n+1}^{-1}\det J_+(n+2) = \dots = \prod_{l=n}^{\infty}\rho_l^{-1} = v_+,$$

and similarly for J_-. As a result, $\det J_\pm(n,z) = v_\pm(n)$. It should be noted in this connection that $\det J_-(n) = \det J_+(n)\det S$ and hence

$$\det S = v, \qquad v = v_+^{-1}v_- = \prod_{l=-\infty}^{\infty}\rho_l.$$

The next step in studying the Jost solutions is their asymptotic behavior. Let us write the spectral equation (8.55) in the explicit form

$$\begin{pmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{pmatrix}_{n+1} = \begin{pmatrix} J_{11} + z^{-1}u_nJ_{21} & z^2J_{12} + zu_nJ_{22} \\ z^{-2}J_{21} - z^{-1}u_n^*J_{11} & J_{22} - zu_n^*J_{12} \end{pmatrix}_n$$

and consider the limit $z \to \infty$, for which $J(n,z) = J^{(0)}(n) + z^{-1}J^{(1)}(n) + z^{-2}J^{(2)}(n) + \dots$. Then in the leading order of the asymptotic expansion we obtain

$$\begin{pmatrix} J_{11}^{(0)} & J_{12}^{(0)} \\ J_{21}^{(0)} & J_{22}^{(0)} \end{pmatrix}_{n+1} = \begin{pmatrix} J_{11}^{(0)} & z^2 J_{12}^{(0)} + z J_{12}^{(1)} + z u_n J_{22}^{(0)} \\ 0 & J_{22}^{(0)} - z u_n^* J_{12}^{(0)} - u_n^* J_{12}^{(1)} \end{pmatrix}_n .$$

Comparing the entries with the same power of z on both sides of this relation, we get

$$J^{(0)}(n+1) = \begin{pmatrix} 1 & 0 \\ 0 & \rho_n \end{pmatrix} J^{(0)}(n)\,, \tag{8.62}$$

while the potential u_n is retrieved as

$$u_n = -\frac{J_{12}^{(1)}}{J_{22}^{(0)}}. \tag{8.63}$$

In the limit $z \to 0$, for which $J(n,z) = J_{(0)}(n) + zJ_{(1)}(n) + z^2 J_{(2)}(n) + \dots$, we similarly obtain

$$J_{(0)}(n+1) = \begin{pmatrix} \rho_n & 0 \\ 0 & 1 \end{pmatrix} J_{(0)}(n).$$

8.4.2 Analytic solutions

To reveal analytic properties of the Jost solutions, we rewrote the spectral equation for the continuous nonlinear equations in the form of integral equations. Now it is natural to use infinite products. Indeed, we transform the spectral equation as

$$J(n,z) = (E + Q_n)^{-1} J(n+1,z)E = \frac{E^{-1} - Q_n}{\rho_n}\, J(n+1,z)\, E$$

$$= \lim_{N\to\infty} \prod_{l=n}^{N} \frac{E^{-1} - Q_l}{\rho_l}\, J(N+1)\, E^{N-n+1}$$

and for the first column we write

$$J_+^{[1]}(n,z) = [J_+(n,z)]_{\cdot 1} = \lim_{N\to\infty} \prod_{l=n}^{N} \left(\frac{E^{-1} - Q_l}{\rho_l} \right)_{\cdot\cdot} [J_+(N+1)]_{\cdot 1}\, z^{N-n+1}.$$

Since $J_+(N) \to \mathbb{1}$ at $N \to \infty$, the factor $[J_+(N+1)]_{\cdot 1}$ in the last equation can be treated as $[J_+(N+1)]_{11}$. Hence, the expression in parentheses does not contribute to the z-dependence of J_{+21} and therefore $[J_+(n,z)]_{21} \sim z^{N-n+1} \longrightarrow 0$ at $z \to 0$. As regards J_{+11}, we will gain $z^{-1} = (E^{-1})_{11}$ from every factor in the product, which results in $J_+(n,z)_{11} \sim z^{-(N-n+1)} z^{N-n+1}$

and $[J_+(n,z)]_{11}$ does not depend on z in this limit. Hence, the column $J_+^{[1]}$ is analytic for $|z|<1$, i.e., inside the unit circle at the z-plane. In the same way we can show that the columns $J_-^{[1]}$ and $J_+^{[2]}$ are analytic for $|z|>1$, i.e., outside the unit circle. Let us define a matrix function

$$\Phi_+(n,z)=\left(J_-^{[1]},J_+^{[2]}\right)$$

which is analytic for $|z|>1$ (the $\mathbb{C}_{\text{out}}$ domain) and solves the spectral equation (8.55). It follows from the conjugation formulas (8.60) and (8.61) that the rows $(J_-)^{-1}_{[1]}$ and $(J_+)^{-1}_{[2]}$ are analytic for $|z|<1$ (the $\mathbb{C}_{\text{in}}$ domain). As a result, the matrix function

$$\Phi_-^{-1}(n,z)=\begin{pmatrix}(J_-)^{-1}_{[1]}\\(J_+)^{-1}_{[2]}\end{pmatrix}(n,z) \tag{8.64}$$

is analytic as a whole in $\mathbb{C}_{\text{in}}$ and solves the adjoint spectral problem. From the definition (8.57) of the scattering matrix we have

$$\Phi_+=(J_-^{[1]},J_+^{[2]})=(a_+J_+^{[1]}+z^{-2n}J_+^{[2]},J_+^{[2]})=J_+E^nS_+E^{-n},$$

$$S_+=\begin{pmatrix}a_+ & 0\\ b_+ & 1\end{pmatrix},$$

and similarly

$$\Phi_+=J_-E^nS_-E^{-n},\qquad S_-=\begin{pmatrix}1 & b_-/v\\ 0 & a_+/v\end{pmatrix}.$$

Therefore,

$$\det\Phi_+=\det J_+\det S_+=\det J_-\det S_-=v_+(n)\,a_+(z). \tag{8.65}$$

Analogously,

$$\Phi_-^{-1}=E^nT_+E^{-n}J_+^{-1}=E^nT_-E^{-n}J_-^{-1},$$

$$T_+=\begin{pmatrix}a_-/v & b_-/v\\ 0 & 1\end{pmatrix},\quad T_-=\begin{pmatrix}1 & 0\\ b_+ & a_-\end{pmatrix},$$

and

$$\det\Phi_-^{-1}(n,z)=v_-^{-1}(n)a_-(z).$$

Asymptotic formulas for analytic solutions are derived directly from those for the Jost functions. In particular, at $z\to\infty$

$$\Phi_+=\begin{pmatrix}J_{-11} & J_{+12}\\ J_{-21} & J_{+22}\end{pmatrix}\longrightarrow\begin{pmatrix}\lim\limits_{z\to\infty}J_{-11} & 0\\ 0 & \lim\limits_{z\to\infty}J_{+22}\end{pmatrix}.$$

From (8.62) $J^{(0)}_{-11}(n+1)=J^{(0)}_{-11}(n)=J^{(0)}_{-11}(n-1)=\ldots=\mathbb{1}$ and

$$J^{(0)}_{+22}(n)=\rho_n^{-1}J^{(0)}_{+22}(n+1)=\rho_n^{-1}\rho_{n+1}^{-1}J^{(0)}_{+22}(n+2)=\ldots=\prod_{l=n}^{\infty}\rho_l^{-1}=v_+(n).$$

Therefore, at $z \to \infty$

$$\Phi_+(n,z) \longrightarrow \Phi_+^{(0)}(n) = \begin{pmatrix} 1 & 0 \\ 0 & v_+(n) \end{pmatrix}. \tag{8.66}$$

In the same way we derive at $z \to 0$

$$\Phi_-^{-1} \longrightarrow \Phi_{-(0)}^{-1} = \begin{pmatrix} v_-^{-1}(n) & 0 \\ 0 & 1 \end{pmatrix}.$$

Hence, $\det \Phi_+ \to v_+(n)$ at $z \to \infty$, which gives from (8.65) $a_+(z) \to 1$ in the same limit. Similarly, $a_-(z) \to 1$ at $z \to 0$.

The last thing we should do with the analytic solutions is to obtain a conjugation formula for them. Writing out (8.60) $J_\pm^{-1}(n,z) = v_\pm^{-1}(n)\, J_\pm^\dagger(n,\bar z)$ in components, we get

$$\begin{aligned}
&\left[J_-^{-1}(n,z)\right]_{11} = v_-^{-1}\left[J_-^\dagger(n,\bar z)\right]_{11} = v_-^{-1}\left[J_-^*(n,\bar z)\right]_{11},\\
&\left[J_-^{-1}(n,z)\right]_{12} = v_-^{-1}\left[J_-^\dagger(n,\bar z)\right]_{12} = v_-^{-1}\left[J_-^*(n,\bar z)\right]_{21},\\
&\left[J_+^{-1}(n,z)\right]_{21} = v_+^{-1}(n)\left[J_+^*(n,\bar z)\right]_{12}, \quad \left[J_+^{-1}(n,z)\right]_{22} = v_+^{-1}(n)\left[J_+^*(n,\bar z)\right]_{22}.
\end{aligned}$$

Inserting these relations into (8.64) yields

$$\Phi_-^{-1}(n,z) = \begin{pmatrix} J_{-[1]}^{-1}(n,z) \\ J_{+[2]}^{-1}(n,z) \end{pmatrix} = \begin{pmatrix} J_{-11}^{-1} & J_{-12}^{-1} \\ J_{+21}^{-1} & J_{+22}^{-1} \end{pmatrix}(n,z)$$

$$= \begin{pmatrix} v_-^{-1}J_{-11}^* & v_-^{-1}J_{-21}^* \\ v_+^{-1}J_{+12}^* & v_+^{-1}J_{+22}^* \end{pmatrix}(n,\bar z) = \begin{pmatrix} v_-^{-1} & 0 \\ 0 & v_+^{-1} \end{pmatrix}\left(J_-^{[1]}, J_+^{[2]}\right)^\dagger(n,\bar z)$$

$$= \begin{pmatrix} v_-^{-1} & 0 \\ 0 & v_+^{-1} \end{pmatrix}\Phi_+^\dagger(n,\bar z).$$

As a result, the conjugation formula for the analytic solutions takes the form

$$\Phi_+^\dagger(n,z) = B(n)\,\Phi_-^{-1}(n,\bar z), \quad B(n) = \begin{pmatrix} v_-(n) & 0 \\ 0 & v_+(n) \end{pmatrix}. \tag{8.67}$$

8.4.3 RH problem

As might be expected, the functions Φ_+ and Φ_-^{-1} enter the RH problem

$$\Phi_-^{-1}(n,z)\Phi_+(n,z) = E^n G(z) E^{-n}, \qquad |z| = 1, \tag{8.68}$$

$$G = T_+S_+ = T_-S_- = \begin{pmatrix} 1 & b_-/v \\ b_+ & 1 \end{pmatrix},$$

with the contour being the unit circle $|z| = 1$. The normalization of the RH problem is noncanonical and is given by (8.66). It should be noted that there

exists a possibility for the AL equation to formulate the RH problem with the standard normalization but at the cost of *nonlinear* dependence of the spectral problem on the potential Q_n [182].

As we know, solitons correspond to the RH problem with zeros. Suppose Φ_+ has zeros at some points $\pm z_j \in \mathbb{C}_{\text{out}}$, $j = 1, \dots, \mathcal{N}$, i.e., $\det \Phi_+(\pm z_j) = 0$, and $\det \Phi_-^{-1}(\pm \bar{z}_l) = 0$, $\pm \bar{z}_l \in \mathbb{C}_{\text{in}}$, $l = 1, \dots, \mathcal{N}$. Because of the modification of the involution property, we should match a definition of the projector P_j with the conjugation formula (8.67). In other words, we define the projector as

$$P_j = \frac{|j\rangle\langle j|B}{\langle j|B|j\rangle},$$

with the matrix B defined in (8.67). Owing to the parity property (8.58), we have $|-j\rangle = \sigma_3|j\rangle$.

Exactly as for the MNLS equation, we bring the matrix functions $\Phi_\pm$ to a factorizable representation $\Phi_\pm = \phi_\pm \Gamma$, with the dressing factor Γ written as

$$\Gamma(n,z) = \mathbb{1} - \sum_{j,l}^{\mathcal{N}} \frac{1}{z - \bar{z}_l} |j\rangle \left(D^{-1}\right)_{j\,l} \langle l|B, \tag{8.69}$$

$$\Gamma^{-1}(n,z) = \mathbb{1} + \sum_{j,l}^{\mathcal{N}} \frac{1}{z - z_j} |j\rangle \left(D^{-1}\right)_{j\,l} \langle l|B,$$

with the matrix elements

$$D_{lj} = \langle l| \frac{B}{z_j - \bar{z}_l} |j\rangle. \tag{8.70}$$

Now what about a coordinate dependence of the vector $|j\rangle$? As regards n-dependence, we have

$$\Phi_+(n+1,z)|j,n+1\rangle = 0 = \left[E(z_j) + Q_n\right] \Phi_+(n,z_j) E^{-1}(z_j)|j,n+1\rangle.$$

Comparing this with the spectral equation (8.55), we can take $E^{-1}(z_j)|j, n+1\rangle = |j,n\rangle$, or

$$|j,n\rangle = E^n(z_j)|\tilde{j}\rangle,$$

where $|\tilde{j}\rangle$ is an n-independent (but t-dependent) vector. It follows from (8.56) that $|j,n\rangle_t = \Omega(z_j)|j,n\rangle$. Therefore, the coordinate dependence of the vector $|j\rangle$ is given by

$$|j\rangle = E^n(z_j) \mathrm{e}^{\Omega(z_j)} |j\,0\rangle, \qquad |j\,0\rangle = \text{const}. \tag{8.71}$$

Finally, we find from the identity $\det \Phi_+(z_j) = 0$ that zeros do not depend on n and t. Zeros $\pm z_j$ and $\pm \bar{z}_l$ and vectors $|j\rangle$ comprise the discrete part of the RH problem. Once again, where solitons were concerned, i.e., $G(z) = \mathbb{1}$, we can choose the leading-order term $\Phi_+^{(0)}$ (8.66) of the asymptotic expansion as a solution of the regular RH problem $\phi_-^{-1}\phi_+ = \mathbb{1}$.

In accordance with (8.63), (8.66), and $\Phi_+ = \phi_+ \Gamma$, a solution $u_n(t)$ of the AL equation can be retrieved from the solution of the RH problem as

$$u_n(t) = -\lim_{z\to\infty} \frac{(z\Phi_+)_{12}}{\Phi_{+22}} = -\frac{\Phi^{(1)}_{+12}}{\Phi^{(0)}_{+22}} = -\frac{\Gamma^{(1)}_{12}}{v_+(n)}. \tag{8.72}$$

Hence, to obtain a soliton solution of the AL equation, we should find the dressing factor Γ.

8.4.4 Ablowitz–Ladik soliton

As we know, four zeros $\pm z_1$ and $\pm \bar{z}_1$ correspond to a single soliton solution of the AL equation (Fig. 8.5). To obtain the AL soliton, we will be guided by (8.72). For discrete RH data, it is possible to express $v_+(n)$ entering (8.72) in terms of $\Gamma(n, z = 0)$—just in the same way as we expressed $\Phi^{(0)}_+$ through $\Gamma(k = 0)$ in the case of the MNLS equation. Indeed, because for solitons $G(z) = \mathbb{1}$, we have $\Phi_+ = \Phi_-$. Hence, $\Phi_+ \to \mathrm{diag}(v_-(n), 1)$ as $z \to 0$, owing to (8.66). Now we obtain from $\Phi_+ = \Phi^{(0)}_+ \Gamma$

$$\Gamma(n,z) = \begin{pmatrix} 1 & 0 \\ 0 & v_+^{-1} \end{pmatrix} \Phi_+, \quad \Gamma(n,0) = \begin{pmatrix} 1 & 0 \\ 0 & v_+^{-1} \end{pmatrix} \begin{pmatrix} v_- & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} v_- & 0 \\ 0 & v_+^{-1} \end{pmatrix}.$$

Hence, $v_+^{-1}(n) = \Gamma_{22}(n, 0)$ and the reconstruction formula takes the form

$$u_n(t) = -\Gamma^{(1)}_{12}(n)\Gamma_{22}(n, 0). \tag{8.73}$$

Besides, we can express the matrix B (8.67) in terms of $\Gamma(n, 0)$:

$$B = \mathrm{diag}\left(\Gamma_{11}(n, 0), \Gamma^{-1}_{22}(n, 0)\right).$$

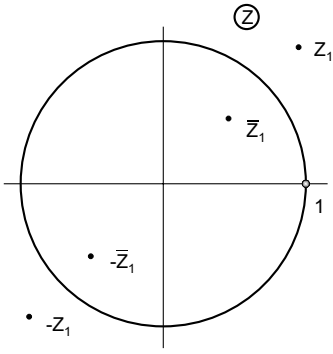

Fig. 8.5. Typical arrangement of zeros corresponding to the Ablowitz–Ladik soliton

Now we will calculate the dressing factor Γ. Denoting $z_1 = \exp[(1/2)(\mu + \mathrm{i}k)]$ and $(p_1/p_2) = \exp(a + \mathrm{i}\varphi)$, where p_1 and p_2 are components of the constant vector $|j_0\rangle$, we find from (8.71) the vector $|j\rangle$ explicitly:

$$|j\rangle = \mathrm{e}^{\frac{1}{2}(a+\mathrm{i}\beta)} \begin{pmatrix} \mathrm{e}^{\frac{1}{2}(x_n+\mathrm{i}\varphi_n)} \\ \mathrm{e}^{-\frac{1}{2}(x_n+\mathrm{i}\varphi_n)} \end{pmatrix}. \tag{8.74}$$

Here

$$x_n = \mu n - 2t \sinh\mu \sin k + a, \quad \varphi_n = kn + 2t(\cosh\mu \cos k - 1) + \varphi.$$

It should be noted that the vector $|j\rangle$ (8.74) has the same structure as the corresponding vector for the NLS and MNLS equations; see (8.31). As regards the dressing factor Γ, it follows from (8.69) with $\mathcal{N} = 1$, $z_2 = -z_1$, and $\bar{z}_2 = -\bar{z}_1$ that

$$\begin{aligned} \Gamma(n,z) = \mathbb{1} &- \frac{1}{z - \bar{z}_1} \left[|n\rangle (D^{-1})_{11} \langle n|B + \sigma_3 |n\rangle (D^{-1})_{21} \langle n|B \right] \\ &- \frac{1}{z + \bar{z}_1} \left[|n\rangle (D^{-1})_{12} \langle n|B\sigma_3 + \sigma_3 |n\rangle (D^{-1})_{22} \langle n|B\sigma_3 \right] . \end{aligned} \tag{8.75}$$

Calculating matrix elements D_{lj} (8.70) and taking into account that $\det \Gamma (n,0) = \exp(2\mu)$, we obtain from (8.75)

$$\Gamma(n,z) = \mathbb{1} - \frac{\sinh\mu}{2(z - \bar{z}_1)} \tilde{F}_-(n) - \frac{\sinh\mu}{2(z + \bar{z}_1)} \tilde{F}_+(n), \tag{8.76}$$

$$\Gamma^{-1}(n,z) = \mathbb{1} + \frac{\sinh\mu}{2(z - z_1)} F_-(n) + \frac{\sinh\mu}{2(z + z_1)} F_+(n),$$

where

$$\tilde{F}_-(n) = \begin{pmatrix} \dfrac{\exp\left[\mu(n - \frac{1}{2} - x) + \frac{\mathrm{i}}{2}k\right]}{\cosh\mu(n - 1 - x)} & \dfrac{\exp\left[\mathrm{i}k(n - x) + \mathrm{i}\alpha - \mu\right]}{\cosh\mu(n - 1 - x)} \\ \dfrac{\exp\left[-\mathrm{i}k(n - 1 - x) - \mathrm{i}\alpha + \mu\right]}{\cosh\mu(n - x)} & \dfrac{\exp\left[-\mu(n - \frac{1}{2} - x) + \frac{\mathrm{i}}{2}k\right]}{\cosh\mu(n - x)} \end{pmatrix},$$

$$F_-(n) = \begin{pmatrix} \dfrac{\exp\left[\mu(n - \frac{1}{2} - x) + \frac{\mathrm{i}}{2}k\right]}{\cosh\mu(n - x)} & \dfrac{\exp\left[\mathrm{i}k(n - x) + \mathrm{i}\alpha - \mu\right]}{\cosh\mu(n - 1 - x)} \\ \dfrac{\exp\left[-\mathrm{i}k(n - 1 - x) - \mathrm{i}\alpha + \mu\right]}{\cosh\mu(n - x)} & \dfrac{\exp\left[-\mu(n - \frac{1}{2} - x) + \frac{\mathrm{i}}{2}k\right]}{\cosh\mu(n - 1 - x)} \end{pmatrix},$$

$$\tilde{F}_+(n) = -\sigma_3 \tilde{F}_-(n) \sigma_3, \quad F_+(n) = -\sigma_3 F_-(n) \sigma_3.$$

Here

$$x(t) = 2t \frac{\sinh\mu}{\mu} \sin k + x_0, \quad x_0 = -\frac{a}{\mu} - \frac{3}{2},$$

$$\alpha(t) = 2t(\cosh\mu\,\cos k + \frac{k}{\mu}\sinh\mu\,\sin k - 1) + \alpha_0, \quad \alpha_0 = \beta - \frac{ak}{\mu} - k.$$

As a result, we obtain from (8.73) the AL soliton solution [8]:

$$u_n(t) = \exp[\mathrm{i}k(n-x) + \mathrm{i}\alpha]\frac{\sinh\mu}{\cosh\mu(n-x)}.$$

The AL soliton depends on four constant real parameters μ, k, x_0, and α_0 which determine the soliton mass 2μ, its group velocity $v_{\mathrm{gr}} = 2(\sinh\mu/\mu)\sin k$, soliton maximum position $x(t)$, and phase $\alpha(t)$.

8.5 Three-wave resonant interaction equations

In this section we shall deal with the problem of three-wave resonant interaction in nonlinear quadratic media, one of the classic examples of successful application of the IST to an actual physical problem. Three wave packets (envelopes) are involved in this process, the central frequencies and wave vectors obeying the resonance conditions

$$\omega_3 = \omega_1 + \omega_2, \qquad \mathbf{k}_3 = \mathbf{k}_1 + \mathbf{k}_2. \tag{8.77}$$

The physical nature of the wave packets can be arbitrary. The interaction of the type (8.77) describes, e.g., a decay of wave 3 (pumping wave) into waves 1 and 2 (secondary waves). Such a process occurring in a stable medium is exemplified by generation of harmonics in nonlinear optics [349] and decay instability in plasma [226]. In unstable media, e.g., in plasma, processes of the so-called explosive instability type are possible when the resonance conditions take a rather different form:

$$\omega_1 + \omega_2 + \omega_3 = 0, \qquad \mathbf{k}_1 + \mathbf{k}_2 + \mathbf{k}_3 = 0.$$

The (1+1)-dimensional form of the three-wave resonant interaction (in the following, the three-wave equations) for the case of the decay instability can be given as

$$\begin{aligned} u_{1t} + v_1 u_{1x} &= \mathrm{i}\gamma\bar{u}_2 u_3, \\ u_{2t} + v_2 u_{2x} &= \mathrm{i}\gamma\bar{u}_1 u_3, \\ u_{3t} + v_3 u_{3x} &= \mathrm{i}\gamma u_1 u_2, \end{aligned} \tag{8.78}$$

while for the explosive instability they are written slightly differently:

$$\begin{aligned} u_{1t} + v_1 u_{1x} &= \mathrm{i}\gamma\bar{u}_2\bar{u}_3, \\ u_{2t} + v_2 u_{2x} &= \mathrm{i}\gamma\bar{u}_1\bar{u}_3, \\ u_{3t} + v_3 u_{3x} &= \mathrm{i}\gamma\bar{u}_1\bar{u}_2. \end{aligned} \tag{8.79}$$

Here u_j and v_j are the scaled envelope and group velocity of the jth wave, and γ is the interaction constant. In what follows we restrict ourselves to (8.78). The case of the explosive instability (8.79) is treated in the same manner.

A possibility to solve the three-wave equations by means of the IST was discovered by Zakharov and Manakov [471, 472]. The detailed analysis of the three-wave equations can be found in [229, 234, 354]. From the point of view of the RH problem, we will see that because an associated spectral problem is realized with 3×3 matrices, some subtle details arise when determining analytic properties of Jost solutions.

Before proceeding to solving (8.78), we discuss some peculiarities inherent in these equations. The main one is the absence of dispersion in the model, i.e., the lack of second-order derivatives in x in (8.78). The reason is that the three-wave resonant interaction has a very short characteristic time scale compared with that for dispersive effects. In other words, the time needed for a three-wave interaction to exhibit a considerable effect on the system is much shorter that the time required for dispersion to manifest itself. Let us recall that in the case of the NLS model, it is the dispersion that leads to a separation between solitons and linear waves (radiation) because of the decay of the continuous spectrum in time. In contrast, for the three-wave equations the continuous spectrum is considered on an equal footing with the solitons: it remains with solitons for long times and mixes nonlinearly with them.

Nevertheless, as for the NLS solitons, the three-wave soliton solutions can be obtained in a closed form. Indeed, we can discriminate between solitons and radiation on the basis of the properties of the RH problem associated with the three-wave equations. Namely, there exists a subset of the RH problem data for which a system of singular integral equations reduces to the algebraic ones. Solutions of these algebraic equations are called solitons of the three-wave equations. At the same time, as a manifestation of the aforementioned mixing of solitons and radiation, different envelopes can in general exchange with solitons and radiation, provided that the total number of solitons is preserved. Moreover, as we showed in Sect. 3.7, the previous statements still hold for more general (non-Abelian) three-wave system.

8.5.1 Jost solutions

Let us start to solve (8.78). Those equations allow the Lax representation

$$\psi_x = \mathrm{i}(kJ + U)\psi, \qquad \psi_t = \mathrm{i}(kI + V)\psi, \tag{8.80}$$

where J and I are diagonal matrices with constant real entries,

$$J = \operatorname{diag}(a_1, a_2, a_3), \quad a_1 > a_2 > a_3, \quad I = \operatorname{diag}(b_1, b_2, b_3).$$

U and V are 3×3 matrices with zero diagonal. We suppose that the potential U falls fast enough for $|x| \to \infty$, $\int_{-\infty}^{\infty} |u_{ij}(x)|\,\mathrm{d}x < \infty \ \forall\, i,\ j$; as usual, k is the spectral parameter. The compatibility condition for (8.80) has the form

$$[J, V] = [I, U], \qquad U_t - V_x + \mathrm{i}[U, V] = 0. \tag{8.81}$$

Then equations (8.78) are obtained from (8.81) after the following identifications:

$$u_1 = \frac{u_{12}}{(a_1-a_2)^{1/2}}, \qquad u_2 = \frac{u_{23}}{(a_2-a_3)^{1/2}}, \qquad u_3 = \frac{u_{13}}{(a_1-a_3)^{1/2}},$$

$$\alpha_{ij} = \frac{b_i - b_j}{a_i - a_j}, \quad v_1 = -\alpha_{12}, \quad v_2 = -\alpha_{23}, \quad v_3 = -\alpha_{13}, \quad v_{ij} = \alpha_{ij}u_{ij},$$

$$\gamma = \frac{a_1b_3 - a_3b_1 + a_3b_2 - a_2b_3 + a_2b_1 - a_1b_2}{[(a_1-a_2)(a_1-a_3)(a_2-a_3)]^{1/2}},$$

as well as the reduction $U^\dagger = U$ and, as a consequence, $V^\dagger = V$. So, explicitly the matrices U and V are written as

$$U = \begin{pmatrix} 0 & u_{12} & u_{13} \\ \bar{u}_{12} & 0 & u_{23} \\ \bar{u}_{13} & \bar{u}_{23} & 0 \end{pmatrix}, \quad V = -\begin{pmatrix} 0 & v_1u_{12} & v_3u_{13} \\ v_1\bar{u}_{12} & 0 & v_2u_{13} \\ v_3\bar{u}_{13} & v_2\bar{u}_{23} & 0 \end{pmatrix}.$$

To formulate the RH problem with the canonical normalization, we introduce the matrix function $\chi = \psi E^{-1}$, where $E = \exp(\mathrm{i}kJx)$. Then the spectral problem is written as

$$\chi_x = \mathrm{i}k[J, \chi] + \mathrm{i}U\chi. \tag{8.82}$$

Let $J_\pm(k,x)$ be the Jost solutions to the spectral problem (8.82), $J_\pm \to \mathbb{1}$ at $x \to \pm\infty$. Then the scattering matrix $S(k)$ is defined as before: $J_-(k,x) = J_+(k,x)ES(k)E^{-1}$.

8.5.2 Analytic solutions

As usual, we write the spectral equation for the Jost solutions in the integral form:

$$J_\pm(k,x) = \mathbb{1} + \int_{\pm\infty}^{x} \mathrm{d}y\, \mathrm{e}^{\mathrm{i}kJ(x-y)}U(y)J_\pm(k,y)\mathrm{e}^{-\mathrm{i}kJ(x-y)}. \tag{8.83}$$

From (8.83) we can readily ascertain that the columns $J_+^{[1]}$ and $J_-^{[3]}$ are analytic in the upper half plane $\operatorname{Im} k > 0$, while $J_-^{[1]}$ and $J_+^{[3]}$ are analytic in the lower half plane $\operatorname{Im} k < 0$. As regards the second columns $J_\pm^{[2]}$, they do not allow an analytic continuation off the real axis $\operatorname{Im} k = 0$. The reason for such behavior is evident: while the first and the third columns have in the integrand (8.83) the exponential factors with the differences $a_i - a_j$ of the same sign [$(a_2 - a_1) < 0$ and $(a_3 - a_1) < 0$ for the first column, and $(a_1 - a_3) > 0$ and $(a_2 - a_3) > 0$ for the third column], the second column contains the differences of opposite signs [$(a_1 - a_2) > 0$ and $(a_3 - a_2) < 0$].

In order to apply the RH problem to solve (8.78), we should construct a matrix solution to (8.82) which will possess as a whole the definite analytic properties in the k-plane. To this end, we address ourselves to the adjoint spectral problem

$$\tilde{\psi}_x = -\mathrm{i}(kJ + U^{\mathrm{T}})\tilde{\psi}, \tag{8.84}$$

with transpose matrix U^{T}. It is easy to see that if ψ is a solution to (8.80), then $(\psi^{-1})^{\mathrm{T}}$ is a solution to the adjoint spectral problem (8.84). For the problem (8.84) we can also introduce the Jost solutions $\tilde{J}_{\pm} \to \mathbb{1}$ at $x \to \pm\infty$, which obey the equation

$$\tilde{J}_{\pm x} = -\mathrm{i}k[J,\, \tilde{J}_{\pm}] - \mathrm{i}\, U^{\mathrm{T}} \tilde{J}_{\pm},$$

and a scattering matrix $R(k)$ for the adjoint spectral problem is defined by means of $\tilde{J}_{-}(k,x) = \tilde{J}_{+}(k,x)ER(k)E^{-1}$, $R = (S^{-1})^{\mathrm{T}}$.

Now we can prove that the columns $\tilde{J}_{-}^{[1]}$ and $\tilde{J}_{+}^{[3]}$ are analytic for $\operatorname{Im} k > 0$, while the columns $\tilde{J}_{+}^{[1]}$ and $\tilde{J}_{-}^{[3]}$ are analytic for $\operatorname{Im} k < 0$. So, it is natural to construct a vector column being the vector product of the vector columns $\tilde{J}_{-}^{[1]}$ and $\tilde{J}_{+}^{[3]}$. Being by construction analytic in the upper half plane, it can be considered as the needed second column with the definite analytic behavior. More exactly, the vector column

$$J'^{[2]} = \left(\tilde{J}_{-}^{[1]} \times \tilde{J}_{+}^{[3]}\right) \exp[-\mathrm{i}k(a_1 + a_2 + a_3)x]$$

is a solution to (8.82) and the matrix function

$$\Phi_{+} = \left(J_{+}^{[1]},\, J'^{[2]},\, J_{-}^{[3]}\right) \tag{8.85}$$

is a solution to (8.82) and is analytic as a whole in the upper half plane $\operatorname{Im} k > 0$.

Just as in the preceding sections, we can represent Φ_{+} in the form $\Phi_{+} = J_{+}ES_{+}E^{-1}$, where

$$S_{-} = \begin{pmatrix} r_{11} & 0 & 0 \\ r_{12} & s_{33} & 0 \\ r_{13} & -s_{32} & 1 \end{pmatrix}, \quad S_{+} = \begin{pmatrix} 1 & -r_{21} & s_{13} \\ 0 & r_{11} & s_{23} \\ 0 & 0 & s_{33} \end{pmatrix}.$$

Here s_{jk} and r_{jk} are entries of the matrices S and R, respectively. Besides, we have $S_{+} = SS_{-}$.

8.5.3 RH problem

As in the case of the NLS equation, there is the involution $J_{\pm}^{\dagger}(\bar{k}) = J_{\pm}^{-1}(k)$ which permits us to construct the matrix function $\Phi_{-}^{-1}(k)$ analytic in the lower half plane, $\Phi_{-}^{-1}(k) = \Phi_{+}^{\dagger}(\bar{k})$. Thereby, we are in a position to formulate the RH problem,

$$\Phi_{-}^{-1}(k,x)\,\Phi_{+}(k,x) = E(k,x)G(k)E^{-1}(k,x), \quad \Phi_{\pm} \to \mathbb{1} \text{ at } k \to \infty, \tag{8.86}$$

with

$$G(k) = S_+^\dagger(k) S_+(k) = \begin{pmatrix} 1 & -r_{21} & s_{13} \\ -\bar{r}_{21} & 1-|r_{33}|^2 & \bar{s}_{32} \\ \bar{s}_{13} & s_{32} & 1 \end{pmatrix}.$$

The contour defining the RH problem coincides with the real axis $\operatorname{Im} k = 0$ and the normalization is canonical.

8.5.4 Solitons of three-wave equations

Our intention is to obtain a soliton solution of the three-wave equations; hence, we should consider the RH problem with zeros and for $G(k) = \mathbb{1}$. The discrete spectral data of the RH problem (8.86) are given by zeros $k_j \equiv \xi_j + \mathrm{i}\eta_j$ of the matrix function Φ_+ (8.85), $\det \Phi_+(k_j) = 0$, $\operatorname{Im} k_j > 0$, and vector $|j\rangle$. For the one-soliton solution the matrix Φ_+ reduces to the dressing factor Γ having the form

$$\Gamma = \mathbb{1} - \frac{k_1 - \bar{k}_1}{k - \bar{k}_1} P, \tag{8.87}$$

where the projective 3×3 matrix

$$P = \left(|p_1|^2 + |p_2|^2 + |p_3|^2\right)^{-1} \begin{pmatrix} |p_1|^2 & p_1\bar{p}_2 & p_1\bar{p}_3 \\ p_2\bar{p}_1 & |p_2|^2 & p_2\bar{p}_3 \\ p_3\bar{p}_1 & p_3\bar{p}_2 & |p_3|^2 \end{pmatrix} \tag{8.88}$$

is composed from the components of the vector $|1\rangle = (p_1, p_2, p_3)^{\mathrm{T}}$. Substituting the asymptotic decomposition $\Gamma = \mathbb{1} + k^{-1}\Gamma^{(1)} + \ldots$, $k \to \infty$ into (8.82), we reconstruct the potential U from the solution of the RH problem (8.86), $U = -[J, \Gamma^{(1)}]$, or, as follows from (8.87),

$$U = (k_1 - \bar{k}_1)[J, P]. \tag{8.89}$$

Therefore, the problem of finding the soliton solution reduces to finding the projective matrix P. Its coordinate dependence is determined by the evolution equations for the vector $|1\rangle$:

$$|1\rangle_x = -\mathrm{i}k_1 J|1\rangle, \qquad |1\rangle_t = -\mathrm{i}k_1 I|1\rangle.$$

Hence,

$$p_j = \exp[-\mathrm{i}(\xi_1 + \mathrm{i}\eta_1)(a_j x + b_j t)] p_j^{(0)},$$

where the complex parameters $p_j^{(0)}$ stand for the integration constants. These constants are determined from the initial conditions at $t = 0$. Thereby, we have the explicit expression for the matrix P (8.88). Substituting it into (8.89), we obtain the soliton solution to the three wave equations:

$$u_{ij} = 2\mathrm{i}\eta_1(a_i - a_j) p_i^{(0)} \bar{p}_j^{(0)} \exp\{-\mathrm{i}(\xi_1 + \mathrm{i}\eta_1)[(a_i - a_j)x + (b_i - b_j)t]\} \Delta_1^{-1}, \tag{8.90}$$

where

$$\Delta_1 = \sum_{l=1}^{3} |p_l^{(0)}|^2 \exp[2\eta_1(a_l x + b_l t)].$$

To gain greater insight into the solution (8.90), let us consider the case of only two nonzero constants $p_1^{(0)}$ and $p_2^{(0)}$; hence, the potential U contains only two nonzero entries u_{12} and $u_{21} = \bar{u}_{12}$. In this case the solution (8.90) transforms to

$$u_{12} = \mathrm{i}\eta_1(a_1 - a_2)\frac{\exp\{-\xi_1[(a_1 - a_2)x + (b_1 - b_2)t + \phi_{12}]\}}{\cosh[\eta_1(a_1 - a_2)(x - v_1 t - x_{12})]}, \qquad (8.91)$$

where

$$x_{12} = \frac{1}{\eta_1}\frac{\alpha_1 - \alpha_2}{a_1 - a_2}, \quad \phi_{12} = \frac{\phi_1 - \phi_2}{\xi_1}, \quad p_j^{(0)} \equiv \exp(\alpha_j + \mathrm{i}\phi_j).$$

This result is somewhat trivial because only one of the wave packets has a nonzero envelope, but the essential features of such a solution are inherent in the whole soliton (8.90). Indeed, let all $p_j^{(0)}$ be nonzero. Consider the solution (8.90) for $t \to \infty$, paying special attention to the wave u_{13}. We can put u_{13} in the following form:

$$u_{13} = 2\mathrm{i}\eta_1(a_1 - a_3)\exp\{-\mathrm{i}\xi_1[(a_1 - a_3)x + (b_1 - b_3)t + \phi_{13}]\}D_1^{-1}, \qquad (8.92)$$

where

$$D_1 = 2\cosh[\eta_1(a_1 - a_3)(x - v_3 t) + \alpha_1 - \alpha_3]$$
$$+ \exp[-\eta_1(a_1 - a_2)(x - v_1 t) - \alpha_1 + \alpha_2]\exp[\eta(a_1 - a_3)(x - v_2 t) + \alpha_2 - \alpha_3].$$

Let $v_1 > v_2$. Then we conclude from (8.92) that $u_{13} \to 0$ for $t \to \infty$, while for $t \to -\infty$ u_{13} tends to the expression of the type (8.91) with the change of indices $2 \to 3$, i.e., to the soliton of pumping wave. In contrast, the components u_{12} and u_{23} tend exponentially to zero for $t \to -\infty$ and to solitons of the type (8.91) for $t \to \infty$. Thereby, the solution (8.90) describes the decay of the composite soliton into two simple ones. For the case of $v_1 < v_2$ the solution (8.90) describes the reverse process of a fusion of two simple solitons into the composite one.

Hence, the three-wave resonant interaction process provides an example of the so-called nontrivial interaction of solitons, as distinct from the trivial interaction (scattering) of the NLS solitons.

So far we have considered the case of the simplest identity reduction $U^\dagger = U$. At the same time, the spectral problem (8.82) allows a reduction of a more general type (the so-called B-hermiticity [354]):

$$U^\dagger = BUB, \qquad B = \mathrm{diag}(r_1, r_2, r_3),$$

where $r_j = \pm 1$. Such a reduction provides the mutual conjugation of zeros of the RH problem which lie in different half planes of the k-plane. It is easy to see that the projective matrix P takes the form

$$P_{ij} = \frac{r_i p_i \bar{p}_j}{\sum_l r_l |p_l|^2}.$$

It is important that nonidentity reduction leads to a substantial modification of the above results. This conclusion follows from the fact that the denominator Δ_1 entering the soliton solution (8.90) takes the form

$$\Delta_1' = \sum_{l=1}^{3} r_l |p_l^{(0)}|^2 \exp\left[2\eta_l(a_l x + b_l t)\right]. \tag{8.93}$$

If some of r_l are equal to -1, a singularity occurs for some values of the coordinates. Let us consider the case $B = (-1, 1, 1)$. It is evident that the singularity is absent only for $p_1^{(0)} = 0$, which means that only the soliton of the simple wave u_{23} exists in the system with such a restriction. A similar conclusion follows for the reduction matrix $B = (1, 1, -1)$.

Nontrivial results take place for the reduction $B = (1, -1, 1)$. As for the previous cases, we have here the possibility of the existence of solitons of the wave u_{13} for two other zero waves. At the same time, a general solution for three envelopes can exist, but for a finite time interval. Indeed, let us write the denominator Δ_1' (8.93) in the form

$$\begin{aligned} \Delta_1' \exp\left[2\alpha_1 + 2\eta_1(a_1 x + b_1 t)\right] \\ = 1 - \exp\left[2(\beta_1 - \beta_2) - 2\eta_1(a_1 - a_2)x - 2\eta_1(b_1 - b_2)t\right] \\ + \exp\left[2(\beta_1 - \beta_3) - 2\eta_1(a_1 - a_3)x - 2\eta_1(b_1 - b_3)t\right] \end{aligned}$$

and analyze the expression on the right-hand side. This analysis shows that under the condition

$$\begin{aligned} &\frac{1}{a_1 - a_2}\left(|\beta_1 - \beta_2|\frac{a_2 - a_3}{a_1 - a_2} - (b_1 - b_2)t\right) \\ &< \frac{1}{a_1 - a_3}\left(|\beta_1 - \beta_3|\frac{a_2 - a_3}{a_1 - a_3} - (b_1 - b_3)t\right) \end{aligned}$$

Δ_1' does not take zero value. Excluding the exceptional case $v_1 = v_2$, we see that this inequality is broken for some positive or negative $t = t_0$. The case of the positive t_0 corresponds to the explosive instability, while for the negative t_0 we have the process of smoothing the initial singularity.

Note that the Darboux-dressing transformation was applied in [108] to construct a larger class of exact solutions of the three-wave equations with nontrivial seed solutions.

8.6 Homoclinic orbits via dressing method

In this section we will dress nonzero solutions of the NLS and MNLS equations. Along with solitons as stable solutions of nonlinear integrable equations with important applications in physics and mathematics, these equations allow unstable waves such as homoclinic orbits. The existence of homoclinic solutions serves as an indicator of chaotic behavior in a perturbed deterministic nonlinear dynamical system. The role of homoclinic solutions in the generation of chaos was revealed in the case of periodic boundary conditions for the damped-driven sine–Gordon equation [326, 327] and for the perturbed NLS equation [5, 11, 6, 201, 297]. Extended reviews of analytic and numerical methods in this topic are given by McLaughlin and Overman [328] and by Ablowitz et al. [6]. Different approaches have been proposed for derivation of homoclinic solutions for integrable partial differential equations: while the bilinear Hirota method [210] was used by Ablowitz and Herbst [5], the Bäcklund transformations were employed in [296, 326, 327, 455]. The problem of construction of the homoclinic orbits by means of the Darboux transformation method is discussed in the book of Matveev and Salle [324].

We will show in this section that the dressing method developed in the preceding sections is well suited to derive homoclinic solutions. In order to explain basic ideas, we first reproduce the known homoclinic solution of the NLS equation by means of the dressing method. Then we consider the MNLS equation.

8.6.1 Homoclinic orbit for NLS equation

The NLS equation

$$\mathrm{i}u_t = u_{xx} + 2(|u|^2 - \omega)u, \quad \omega \in \mathrm{Re} \tag{8.94}$$

with an additional real parameter ω has the Lax pair $\psi_x = U\psi$ and $\psi_t = V\psi$ with the matrices U and V of the form

$$U = \mathrm{i}k\sigma_3 + \mathrm{i}Q, \quad Q = \begin{pmatrix} 0 & u \\ \bar{u} & 0 \end{pmatrix},$$

$$V = \mathrm{i}(2k^2 - Q^2 + \omega)\sigma_3 + 2\mathrm{i}kQ + \sigma_3 Q_x.$$

We are interested in periodic solutions of (8.94) with a spatial period L, $u(x + L, t) = u(x, t)$. Hence, the Floquet theory should be applied to the spectral equation $\psi_x = U\psi$. The fundamental matrix $M(x, k)$ is defined as a solution of the spectral equation with the boundary condition $M(0, k) = \mathbb{1}$. The Floquet discriminant is defined as $\Delta(k) = \mathrm{tr}M(L, k)$, where $M(L, k)$ is the *transfer matrix*, and bounded eigenfunctions of the spectral problem correspond to $\Delta(k)$ satisfying the condition $-2 \le \Delta(k) \le 2$. The Floquet spectrum is characterized by the simple periodic points $\{k_j^{\mathrm{s}},\ \Delta(k_j^{\mathrm{s}}) = \pm 2,$

$(\mathrm{d}\Delta/\mathrm{d}k)_{k_j^{\mathrm{s}}} \neq 0\}$ and the double points $\{k_j^{\mathrm{d}},\ \Delta(k_j^{\mathrm{d}}) = \pm 2,\ (\mathrm{d}\Delta/\mathrm{d}k)_{k_j^{\mathrm{d}}} = 0,$ $(\mathrm{d}^2\Delta/\mathrm{d}k^2)_{k_j^{\mathrm{d}}} \neq 0\}$. We will deal with the complex double points indicating linearized instability of solutions of a nonlinear wave equation because these points label the orbits homoclinic to unstable solutions.

We are interested in orbits homoclinic to the periodic plane wave solution u_0 of the NLS equation (8.94) taken in the form

$$u_0 = c\exp[-2\mathrm{i}(c^2-\omega)t], \tag{8.95}$$

where c is a real amplitude. Simple calculation gives the fundamental matrix,

$$M(x,k) = \begin{pmatrix} \cos\mu x + \mathrm{i}(k/\mu)\sin\mu x & \mathrm{i}(c/\mu)\mathrm{e}^{-2\mathrm{i}(c^2-\omega)t}\sin\mu x \\ \mathrm{i}(c/\mu)\mathrm{e}^{2\mathrm{i}(c^2-\omega)t}\sin\mu x & \cos\mu x - \mathrm{i}(k/\mu)\sin\mu x \end{pmatrix}, \quad \mu^2 = c^2+k^2,$$

and hence $\Delta(k) = 2\cos\mu L$. Thereby, the Floquet spectrum comprises the real axis of the k-plane (the main spectrum) and a part of the imaginary axis lying between the simple periodic points $\pm\mathrm{i}c$. Besides, there exists an infinite sequence of real double points $k_n^{\mathrm{d}} = [(n\pi/L)^2 - c^2]^{1/2}$, where $c^2 \leq (n\pi/L)^2$ and n are integers, and a finite number of complex double points k_j^{d}, where j are integers, situated on the imaginary axis within the interval $(\mathrm{i}c, -\mathrm{i}c)$, $(j\pi/L)^2 < c^2$. In what follows we choose c and L in such a way to obtain a single pair of complex double points $k_1^{\mathrm{d}} = \pm\mathrm{i}[c^2 - (\pi/L)^2]^{1/2}$, which is a single unstable mode of the solution (Fig. 8.6). Hence, $j = 1$ and $n = 2, 3, \ldots$

After diagonalization of the transfer matrix $M(L,k)$, $R^{-1}M(L,k)R = \mathrm{diag}(\mathrm{e}^{\mathrm{i}\mu L}, \mathrm{e}^{-\mathrm{i}\mu L})$, we define the Blochsolution $\tilde{\chi} = M(x,k)R$ of the spectral

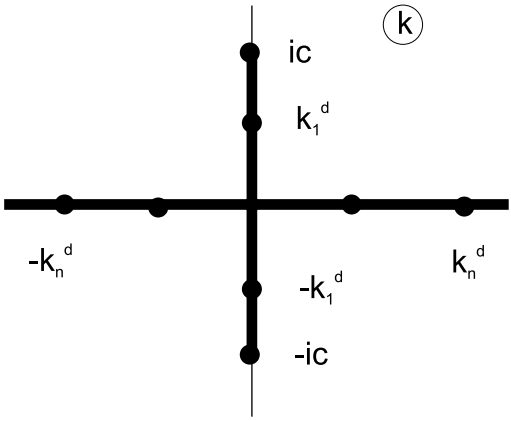

Fig. 8.6. The Floquet spectrum (*thick lines*), infinite sequence of the real double points $\pm k_n^{\mathrm{d}}$, simple periodic points $\pm ic$, and the single pair of the complex double points $\pm k_1^{\mathrm{d}}$

equation. Demanding the Bloch solution to satisfy both equations of the Lax pair, we obtain it explicitly as

$$\tilde{\chi} = \exp[-\mathrm{i}(c^2-\omega)t\sigma_3] \begin{pmatrix} 1 & -\dfrac{\mu-k}{c} \\ \dfrac{\mu-k}{c} & 1 \end{pmatrix} \exp[\mathrm{i}\mu(x+2kt)\sigma_3]\,.$$

In the following, it will be more convenient to work with a modified Bloch function $\chi = \tilde{\chi}\exp[-\mathrm{i}kx\sigma_3 - \mathrm{i}(2k^2+\omega)t\sigma_3]$ which satisfies the equations

$$\chi_x = U\chi - \mathrm{i}k\chi\sigma_3, \quad \chi_t = V\chi - \mathrm{i}(2k^2+\omega)\chi\sigma_3 \tag{8.96}$$

and allows the asymptotic expansion started with the unit matrix, $\chi = \mathbb{1} + k^{-1}\chi^{(1)} + O(k^{-2})$, while the potential Q is reconstructed via

$$Q = -[\sigma_3, \chi^{(1)}]. \tag{8.97}$$

Suppose now that a solution homoclinic to the plane wave (8.95) can be obtained from (8.97) with the Bloch function χ being a result of dressing the Bloch function χ_0 which satisfies (8.96) with $u = u_0$:

$$\chi = \Gamma\chi_0. \tag{8.98}$$

Here $\Gamma(k,x,t)$ is the dressing factor which is written in the form well known for us:

$$\Gamma = \mathbb{1} - \frac{k_1-\bar{k}_1}{k-\bar{k}_1}P, \quad \Gamma^{-1} = \mathbb{1} + \frac{k_1-\bar{k}_1}{k-k_1}P, \tag{8.99}$$

where P is a projector, $P = (|1\rangle\langle 1|)/\langle 1|1\rangle$, $\langle 1| = |1\rangle^\dagger$, and $|1\rangle = (p_1,p_2)^{\mathrm{T}}$ is a two-component vector. As regards the choice of the pole k_1 in (8.99), it is the point where we encounter a crucial difference from the standard applications of the dressing method. The positions of the poles in the dressing factors are usually taken quite arbitrarily, without reference to the seed solution u_0. In contrast, it is the seed solution u_0 which determines these poles in our case. Namely, we take the complex double points as the poles of the dressing factors; therefore, $k_1 = k_1^{\mathrm{d}}$.

Expanding (8.99) in the asymptotic series in k^{-1} gives a new solution in terms of the old one and the dressing factor:

$$Q = Q_0 - [\sigma_3, \Gamma^{(1)}],$$

where $\Gamma = \mathbb{1} + k^{-1}\Gamma^{(1)} + O(k^{-2})$. Hence, we need know the vector $|1\rangle$ to obtain new solution Q.

Differentiating (8.98) in x yields

$$U(k) = -\Gamma[\partial_x - U_0(k)]\Gamma^{-1}, \tag{8.100}$$

where $U_0 = U(u_0)$. Evidently, the left-hand side of (8.100) is regular at points k_1 and $\bar{k}_1$, while the right-hand side has simple poles at these points because of

the dressing factors. From the condition of a vanishing residue at the point k_1 we obtain $|1\rangle_x = U_0(k_1)|1\rangle$, and, similarly, $|1\rangle_t = V_0(k_1)|1\rangle$. These equations are easily integrated and we obtain

$$|1\rangle = \mathrm{e}^{-\mathrm{i}(c^2-\omega)t\sigma_3} \begin{pmatrix} A\exp(\mathrm{i}\mu_1 x - 2k_0\mu_1 t) - \dfrac{\mu_1 - \mathrm{i}k_0}{c}\exp(-\mathrm{i}\mu_1 x + 2k_0\mu_1 t) \\ A\dfrac{\mu_1 - \mathrm{i}k_0}{c}\exp(\mathrm{i}\mu_1 x - 2k_0\mu_1 t) + \exp(-\mathrm{i}\mu_1 x + 2k_0\mu_1 t) \end{pmatrix}.$$

Here $A = \mathrm{const}$, $\mu_1 = \mu(k_1)$, $k_1 = \mathrm{i}k_0$. Evidently, $\Gamma^{(1)} = -(k_1 - \bar{k}_1)P$ and hence $u = u_0 + 2(k_1 - \bar{k}_1)P_{12}$, with $P_{12} = (p_1\bar{p}_2)/(|p_1|^2 + |p_2|^2)$. Inserting here the vector $|1\rangle$ and introducing notations

$$A = \exp(\rho + \mathrm{i}\beta), \quad \tau = \sigma t - \rho, \quad \phi = \beta - \pi/2, \quad \sigma = 4k_0\mu_1, \quad \mu_1 + \mathrm{i}k_0 = c\mathrm{e}^{\mathrm{i}p},$$

we obtain the homoclinic solution in the form

$$u_h = \frac{\cos 2p - \sin p\ \operatorname{sech}\tau\ \cos(2\mu_1 x + \phi) - \mathrm{i}\sin 2p\ \tanh\tau}{1 + \sin p\ \operatorname{sech}\tau\ \cos(2\mu_1 x + \phi)}\, c\mathrm{e}^{-2\mathrm{i}(c^2-\omega)t}, \tag{8.101}$$

which coincides with the solution obtained by Li and McLaughlin [297] by means of the Bäcklund transformation.

It is easy to see that this solution is indeed homoclinic to the plane wave, reproducing this wave (up to a factor) at both infinities:

$$t \to \pm\infty: \quad u_h \to \exp(\pm 2\mathrm{i}p)c\exp[-2\mathrm{i}(c^2 - \omega)t]\,.$$

Solutions of the type (8.101) with the plane-wave asymptotic behavior were previously obtained in [20, 217]. In [20] the solution (8.101) was related with the long-time evolution of the modulational instability of the plane wave. The Darboux transformation was applied in [217] to dress the plane wave and the dressed solution was interpreted as describing a process of self-excitation and subsequent attenuation of periodic waves.

In the case of $\mathcal{N}$ unstable modes the above procedure can be iterated. However, a more efficient way to deal with multiple double points is described in the next subsection.

8.6.2 MNLS equation: Floquet spectrum and Bloch solutions

The MNLS equation

$$\mathrm{i}u_t = u_{xx} + \mathrm{i}\alpha(|u|^2 u)_x + 2(|u|^2 - \omega)u,$$

with a real constant ω, allows the Lax representation with the matrices U and V of the form (Sect. 8.3)

$$U = \mathrm{i}\Lambda\sigma_3 + \mathrm{i}kQ, \quad \Lambda(k) = \frac{1}{\alpha}(1 - k^2), \quad Q = \begin{pmatrix} 0 & u \\ \bar{u} & 0 \end{pmatrix},$$
$$V = \mathrm{i}\Omega\sigma_3 + 2\mathrm{i}k\Lambda Q - \mathrm{i}k^2Q^2\sigma_3 + k\sigma_3 Q_x + \mathrm{i}\alpha k Q^3, \quad \Omega(k) = 2\Lambda^2 + \omega.$$

Like the NLS equation, we take the plane wave solution of the MNLS equation

$$u_0 = c\exp[-2\mathrm{i}(c^2-\omega)t]$$

as a periodic solution with a spatial period L. The fundamental matrix $M(x,k)$ is obtained in the form

$$M(x,k) = \begin{pmatrix} \cos\mu x + \mathrm{i}\dfrac{\Lambda}{\mu}\sin\mu x & \mathrm{i}\dfrac{ck}{\mu}\mathrm{e}^{-2\mathrm{i}(c^2-\omega)t}\sin\mu x \\ \mathrm{i}\dfrac{ck}{\mu}\mathrm{e}^{2\mathrm{i}(c^2-\omega)t}\sin\mu x & \cos\mu x - \mathrm{i}\dfrac{\Lambda}{\mu}\sin\mu x \end{pmatrix},$$

where $\mu = (\Lambda^2 + c^2k^2)^{1/2}$. Hence, $\Delta(k) = 2\cos\mu L$ and four complex double points $k_j = \pm[1-(1/2)\alpha^2c^2 \pm \mathrm{i}\alpha c l_j]^{1/2}$, $l_j = [1-(1/4)\alpha^2c^2 - (j\pi/cL)^2]^{1/2}$, lying in four quadrants of the k-plane, correspond to each unstable mode. We choose c and L in such a way that only the single unstable mode exists, i.e., $l_1^2 > 0$ and $l_j^2 < 0$ for $j > 1$. The linearized stability analysis confirms that the above four complex double points $k_1 = [1-(1/2)\alpha^2c^2 - \mathrm{i}\alpha c l_1]^{1/2}$, $k_2 = -k_1$, $k_3 = \bar{k}_1$ and $k_4 = -\bar{k}_1$ (Fig. 8.7) are associated with the exponential instability.

The Bloch function which solves both Lax equations takes a surprisingly simple form:

$$\tilde{\chi}_0(k,x,t) = \exp[-\mathrm{i}(c^2-\omega)t\sigma_3] \begin{pmatrix} 1 & -\dfrac{\mu-\Lambda}{ck} \\ \dfrac{\mu-\Lambda}{ck} & 1 \end{pmatrix} \exp\left\{\mathrm{i}\mu\left[x + (2\Lambda + \alpha c^2)t\right]\sigma_3\right\}.$$

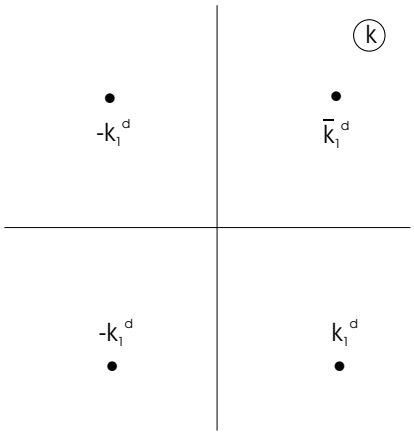

Fig. 8.7. Four complex double points for the single unstable mode

Now we define a modified Bloch function $\chi = \tilde{\chi}\exp(-\mathrm{i}\Lambda x - \mathrm{i}\Omega t)\sigma_3$ which satisfies the linear equations $\chi_x = U\chi - \mathrm{i}\Lambda\chi\sigma_3$, and $\chi_t = V\chi - \mathrm{i}\Omega\chi\sigma_3$ and allows the asymptotic expansion $\chi = \chi^{(0)} + k^{-1}\chi^{(1)} + O(k^{-2})$. Therefore, we obtain for the plane wave potential

$$\chi_0(k,x,t) = \exp[-\mathrm{i}(c^2-\omega)t\sigma_3]\begin{pmatrix} 1 & -\dfrac{\mu-\Lambda}{ck} \\ \dfrac{\mu-\Lambda}{ck} & 1 \end{pmatrix} \times \exp\left[\mathrm{i}(\mu-\Lambda)(x+2\Lambda t)\sigma_3\right]\exp\left[\mathrm{i}(\alpha\mu c^2-\omega)t\sigma_3\right]$$

and the leading term of the asymptotic series $\chi_0^{(0)} = \mathrm{e}^{-(\mathrm{i}/2)\alpha c^2[x+(3/2)\alpha c^2 t]\sigma_3}$. We see once again that this leading term is not a unit matrix, in contrast to the NLS equation. Once again this is a manifestation of the fact that the MNLS equation does not allow the canonical normalization of the associated RH problem. Therefore, we perform now one more transformation of the Bloch solution, $\phi = \chi^{(0)-1}\chi$, to have the unit matrix in the asymptotic expansion: $\phi = \mathbb{1} + k^{-1}\phi^{(1)} + O(k^{-2})$. ϕ satisfies the linear equations

$$\phi_x = U'\phi - \mathrm{i}\Lambda\phi\sigma_3, \quad \phi_t = V'\phi - \mathrm{i}\Omega\phi\sigma_3, \tag{8.102}$$

where

$$U' = \mathrm{i}\Lambda\sigma_3 + \mathrm{i}kQ' + \frac{i}{2}\alpha\sigma_3 Q'^2, \tag{8.103}$$

$$V' = \mathrm{i}\Omega\sigma_3 + 2\mathrm{i}k\Lambda Q' - \mathrm{i}k^2 Q'^2\sigma_3 + k\sigma_3 Q'_x - \frac{\alpha}{2}[Q', Q'_x] - \frac{\mathrm{i}}{4}\alpha^2\sigma_3 Q'^4.$$

Here the new potential Q' is related to the initial one Q as

$$Q' = \chi^{(0)-1}Q\chi^{(0)}. \tag{8.104}$$

Evidently, $Q'^2 = Q^2$. Besides, $\phi^{(1)}$ is expressed via the potential as follows:

$$\phi^{(1)} = \frac{\alpha}{2}\,\sigma_3\,Q'\,.$$

8.6.3 MNLS equation: dressing of plane wave

Suppose a new solution of the linear equations (8.102) follows from the known one ϕ_0 by dressing $\phi = \Gamma\phi_0$. As before, we take the complex double points as the poles of the dressing factor; therefore, we have four poles $k_1 = k_1^{\mathrm{d}}$, $\bar{k}_1 = \bar{k}_1^{\mathrm{d}}$, $k_2 = -k_1^{\mathrm{d}}$, and $\bar{k}_2 = -\bar{k}_1^{\mathrm{d}}$. Expanding the relation $\phi = \Gamma\phi_0$ in the asymptotic series gives in accordance with (8.104) and (8.103)

$$Q = \chi^{(0)}\left[\left(\chi_0^{(0)}\right)^{-1}Q_0\chi_0^{(0)} + \frac{2}{\alpha}\,\sigma_3\Gamma^{(1)}\right](\chi^{(0)})^{-1}.$$

Because $\phi(k=0)$ and $(\chi_0^{(0)})^{-1}$ obey the same equation $y_x = (\mathrm{i}/2)\,\alpha\,\sigma_3\,Q'^2\,y$ and $\phi(k=0) = \Gamma_0\,\phi_0(k=0)$, we get $\chi^{(0)} = \chi_0^{(0)}\Gamma_0^{-1}$, where $\Gamma_0 = \Gamma(k=0)$. Therefore, we obtain the connection between the new and old solutions of the MNLS equation:

$$Q = \Gamma_0^{-1}\left(Q_0 + \frac{2}{\alpha}\sigma_3\chi_0^{(0)}\Gamma^{(1)}\chi_0^{(0)-1}\right)\Gamma_0,$$

or in components

$$u = \frac{(\Gamma_0)_{22}}{(\Gamma_0)_{11}}\left[u_0 + \frac{2}{\alpha}\exp\left(-\mathrm{i}\alpha c^2 x - \mathrm{i}\frac{3}{2}\alpha^2 c^4 t\right)\right]. \tag{8.105}$$

In full agreement with the results of Sect. 8.2.3 we represent the dressing factor in the form

$$\Gamma(k) = \mathbb{1} - \sum_{j,l=1}^{2}\frac{1}{k-\bar{k}_l}|j\rangle(D^{-1})_{j\,l}\langle l|, \quad D_{lj} = \frac{\langle l|j\rangle}{k_j - \bar{k}_l}. \tag{8.106}$$

Differentiating $\phi = \Gamma\phi_0$ in x gives $U'(x,k) = -\Gamma\,[\,\partial_x - U_0'(x,k)]\,\Gamma^{-1}$. From the condition of vanishing residues at the points k_1 and k_2 we obtain the equations

$$|j\rangle_x = U_0'(k_j)|j\rangle, \qquad |j\rangle_t = V_0'(k_j)|j\rangle, \quad j = 1,2. \tag{8.107}$$

Note that the vector $|2\rangle$ is related to $|1\rangle$ as $|2\rangle = \sigma_3|1\rangle$, in virtue of the parity property $U_0'(k_2) = U_0'(-k_1) = \sigma_3 U_0'(k_1)\sigma_3$. Hence,

$$D_{11} = \frac{\langle 1|1\rangle}{k_1 - \bar{k}_1} = -D_{22}, \quad D_{21} = \frac{\langle 2|1\rangle}{k_1 - \bar{k}_2} = -D_{12}.$$

As a result, we obtain Γ_0 and $\Gamma^{(1)}$ entering (8.105) in the form

$$\Gamma_0 = \mathrm{diag}\left(1 + \frac{2}{\bar{k}_1}\frac{(|1\rangle\langle 1|)_{11}}{D_{11}-D_{21}}, 1 + \frac{2}{\bar{k}_1}\frac{(|1\rangle\langle 1|)_{22}}{D_{11}+D_{21}}\right) \equiv \mathrm{diag}\,(\Gamma_{01},\Gamma_{02}),$$

$$\Gamma^{(1)} = -2\begin{pmatrix} 0 & (|1\rangle\langle 1|)_{12}(D_{11}-D_{21})^{-1} \\ (|1\rangle\langle 1|)_{21}(D_{11}+D_{21})^{-1} & 0 \end{pmatrix}.$$

Because D_{ij} are expressed in terms of the vector $|1\rangle$ [see (8.106)], we have to obtain it explicitly. Next we will account for the explicit (x,t)-dependence of the vector $|1\rangle$ and justify the name "homoclinic" for the solution (8.105).

8.6.4 MNLS equation: homoclinic solution

Integrating linear equations (8.107), we obtain

$$|1\rangle = \exp\left(\frac{\mathrm{i}}{2}\alpha c^2 x\sigma_3\right)\exp\left[-\mathrm{i}\left(c^2 - \omega - \frac{3}{4}\alpha^2 c^4\right)t\sigma_3\right]\exp\left(\frac{1}{2}(\gamma + \mathrm{i}\beta)\right)$$
$$\times\begin{pmatrix} \mathrm{e}^{\mathrm{i}\xi-\tau} + \mathrm{e}^{-\mathrm{i}\xi+\tau} \\ \left[\mathrm{e}^{-(\tau+\Phi)}\mathrm{e}^{\mathrm{i}(\xi-\lambda_-)} - \mathrm{e}^{\tau+\Phi}\mathrm{e}^{-\mathrm{i}(\xi-\lambda_+)}\right]\mathrm{e}^{\mathrm{i}(\delta/2)} \end{pmatrix}.$$

Here γ and β are real integration constants, $\mu_1 = \mu(k_1) = \pi/L$, $\xi = \mu_1(x + 2\alpha c^2 t) + (1/2)\beta$, $\tau = 2c\mu_1 l_1 t - (1/2)\gamma$,

$$\Phi = \frac{1}{4}\log\frac{1+\alpha\mu_1}{1-\alpha\mu_1},\ \tan\lambda_\pm = \frac{l_1}{(\mu_1/c) + (1/2)\,\alpha c},\ \text{and}\ \tan\delta = \frac{\alpha c l_1}{1-(1/2)\,\alpha^2 c^2}.$$

Hence, the matrix elements are written as

$$D_{11} = 2\mathrm{e}^{\gamma}(A+B)(k_1-\bar{k}_1)^{-1},\quad D_{21} = 2\mathrm{e}^{\gamma}(A-B)(k_1+\bar{k}_1)^{-1},$$

where

$$A(\xi,\tau) = \cosh 2\tau + \cos 2\xi,\quad B(\xi,\tau) = \cosh 2(\tau+\Phi) - \cos(2\xi - \lambda_+ - \lambda_-),$$

and

$$\Gamma_{01} = \frac{k_1}{\bar{k}_1}\frac{k_1 A + \bar{k}_1 B}{\bar{k}_1 A + k_1 B} = \mathrm{e}^{\mathrm{i}(\Theta-\delta)},\quad \Gamma_{02} = \frac{k_1}{\bar{k}_1}\frac{\bar{k}_1 A + k_1 B}{k_1 A + \bar{k}_1 B} = \mathrm{e}^{-\mathrm{i}(\Theta+\delta)},$$

$$\tan\frac{\Theta}{2} = \frac{1}{\mathrm{i}}\frac{k_1-\bar{k}_1}{k_1+\bar{k}_1}\frac{A-B}{A+B},$$

$$\exp(-\mathrm{i}\alpha c^2 x)\exp\left(-\frac{3}{2}\mathrm{i}\alpha^2 c^4 t\right)\Gamma_{12}^{(1)}$$

$$= -\frac{\mathrm{i}\alpha l_1\exp(-\mathrm{i}\delta/2)}{\bar{k}_1 A + k_1 B}\left[\left(\mathrm{e}^{2\tau}+\mathrm{e}^{2\mathrm{i}\xi}\right)\mathrm{e}^{\Phi-\mathrm{i}\lambda_+} - \left(\mathrm{e}^{-2\tau}+\mathrm{e}^{-2\mathrm{i}\xi}\right)\mathrm{e}^{-\Phi+\mathrm{i}\lambda_-}\right]u_0\,.$$

Substituting the above formulas into (8.105), we obtain explicitly the homoclinic solution of the MNLS equation:

$$u = \left(1 - \frac{2\mathrm{i}\,l_1\mathrm{e}^{-\mathrm{i}(\delta/2)}}{\bar{k}_1 A + k_1 B}\left[\left(\mathrm{e}^{2\tau}+\mathrm{e}^{2\mathrm{i}\xi}\right)\mathrm{e}^{\Phi-\mathrm{i}\lambda_+} - \left(\mathrm{e}^{-2\tau}+\mathrm{e}^{-2\mathrm{i}\xi}\right)\mathrm{e}^{-\Phi+\mathrm{i}\lambda_-}\right]\right)u_0\mathrm{e}^{-2\mathrm{i}\Theta}. \tag{8.108}$$

The solution (8.108) is indeed homoclinic to the plane wave because

$$\tau\to\pm\infty:\quad u\to u_0\exp\left[-2\mathrm{i}(\Theta_\pm + \Phi_\pm)\right],$$

$$\Theta_\pm = \lim_{\tau\to\pm\infty}\Theta = \pm\arctan\frac{\alpha^2 c\,l_1\mu_1}{2\left(1-\frac{1}{2}\alpha^2c^2\right)\left(1-\frac{1}{4}\alpha^2c^2\right)+(\alpha c\,l_1)^2},$$

$$\Phi_\pm = \pm\frac{1}{2}\arctan\frac{2c\,l_1\mu_1}{\mu_1^2 - c^2 l_1^2}.$$

In the $\alpha\to 0$ limit, if the spectral parameter k is represented as $k = 1 - \frac{1}{2}\alpha\,k_{\mathrm{NLS}} + O(\alpha^2)$, the solution (8.108) reproduces the NLS homoclinic orbit (8.101).

8.7 KdV equation

At first sight it might appear as rather strange to devote the very last section in this chapter to one of the most popular and most studied nonlinear integrable equations. Our intention here is to perform an analysis of the KdV equation in the form most suitable for a generalization to (2+1)-dimensional equations. It is precisely this approach that will be realized in the next chapter, first using an example of the (1+1)-dimensional integrodifferential Benjamin–Ono equation and then for proper (2+1)-dimensional equations. We will not dwell on a detailed exposition of various results concerning the KdV equation and its solutions. There exists a vast amount of literature devoted to this topic (for example, [13, 81, 117, 348]). The traditional treatment of the KdV equation was mentioned in Sect. 7.5.1. Instead, our main goal in this section is to illustrate the basic steps of the approach, which goes back to Novikov et al. [354]. We recommend the reader to consult the content of this section when studying the next chapter in order to see strict parallels in the strategy of solving (1+1)- and (2+1)-dimensional equations.

8.7.1 Jost solutions

The KdV equation

$$u_t + 6uu_x + u_{xxx} = 0$$

for a smooth real function $u(x,t)$ is integrated by means of the spectral problem having the form of the time-independent Schrödinger equation [173]

$$\phi_{xx} + (u + k^2)\phi = 0. \tag{8.109}$$

To have linear dependence on the spectral parameter k, we perform a transformation $m(x,k) = \phi(x,k)\exp(\mathrm{i}kx)$. Hence, the spectral problem for KdV is written as

$$m_{xx} - 2\mathrm{i}km_x + um = 0. \tag{8.110}$$

The evolution part of the Lax pair has the form

$$m_t = (\alpha - 4\mathrm{i}k^3 + u_x + 2\mathrm{i}ku)m + (4k^2 - 2u)m_x, \quad \alpha = \text{const}. \tag{8.111}$$

Analysis of the spectral problem will be performed by means of the Green function method. The Green function $G(x,k)$ solves the equation

$$G_{xx} - 2ikG_x = -\delta(x)$$

and has the form

$$G(x,k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}p \frac{\mathrm{e}^{\mathrm{i}px}}{p\,(p-2k)}.$$

We see that the Green function has a discontinuity across the real axis of the complex k-plane. Accordingly, we can determine two Green functions $G_{\pm}(x,k)$ which are analytic in the half planes $\mathrm{Im}k \gtrless 0$:

$$G_{\pm}(x,k) = \pm\frac{1}{2\mathrm{i}k}\left(1 - \mathrm{e}^{2\mathrm{i}kx}\right)\theta(\pm x), \tag{8.112}$$

where $\theta(x)$ is the Heaviside step function, $\theta(x) = 1$ for $x > 0$ and $\theta(x) = 0$ for $x < 0$. Hence, the fundamental solutions (the Jost functions) $M_\pm(x,k)$ of the spectral equation (8.110) can be defined by means of the integral equations

$$M_\pm(x,k) = 1 + \int_{-\infty}^{\infty} \mathrm{d}x' G_\pm(x-x',k)u(x')M_\pm(x',k), \tag{8.113}$$

with the property $M_\pm(x,k) \to 1$ at $k \to \infty$. $M_\pm$ can be analytically extended in the upper and lower halves of the k-plane, respectively. Besides, these functions obey the boundary conditions $M_\pm(x,k) \to 1$ at $x \to \mp\infty$. Indeed, though the integral equations (8.113) have been written in the Fredholm form, they are in fact of the Volterra type because of the step function $\theta(\pm x)$.

The free term 1 in (8.113) represents a solution of the spectral equation with zero potential. This corresponds to dressing of the trivial solution. However, the second-order spectral equation with zero potential allows one more linearly independent solution $\mathrm{e}^{2\mathrm{i}kx}$. Accordingly, we can define two other fundamental solutions

$$N_\pm(x,k) = \mathrm{e}^{2\mathrm{i}kx} + \int_{-\infty}^{\infty} \mathrm{d}x' G_\pm(x-x',k)u(x')N_\pm(x',k), \tag{8.114}$$

with the limits $N_\pm(x,k) \to \mathrm{e}^{2\mathrm{i}kx}$ at $x \to \mp\infty$. As distinct from $M_\pm$, the eigenfunctions $N_\pm$, due to the exponential free term, cannot in general be analytically continued off the real axis $\mathrm{Im}k = 0$.

It follows from (8.112) that

$$M_\pm(x,k) \to a_\pm(k) \pm b_\pm(k)\mathrm{e}^{2\mathrm{i}kx} \tag{8.115}$$

for $x \to \pm\infty$, where

$$a_\pm(k) = 1 \pm \frac{1}{2\mathrm{i}k}\int_{-\infty}^{\infty} \mathrm{d}x u(x)M_\pm(x,k), \quad b_\pm(k) = -\frac{1}{2\mathrm{i}k}\int_{-\infty}^{\infty} \mathrm{d}x u(x)M_\pm(x,k)\mathrm{e}^{-2\mathrm{i}kx} \tag{8.116}$$

are the *scattering coefficients*. For real potentials $u(x)$ there are involution relations for scattering coefficients:

$$a_\pm^*(k) = a_\pm(-k), \quad b_\pm^*(k) = b_\pm(-k), \quad a_+^*(k) = a_-(k), \quad b_+^*(k) = b_-(k),$$

$$|a_+(k)|^2 = 1 + |b_+(k)|^2.$$

It is important that $a_\pm(k)$ are analytic functions in $\mathrm{Im}k \gtrless 0$, as is seen from (8.116). Therefore, $M_\pm/a_\pm$ are meromorphic functions in $\mathrm{Im}k \gtrless 0$ with a finite number of poles in zeros of $a_\pm(k)$.

8.7.2 Scattering equation and RH problem

In accordance with (8.115), for $x \to +\infty$

$$\frac{M_+(x,k)}{a_+(k)} \to 1 + \rho_+(k)\mathrm{e}^{2\mathrm{i}kx}, \qquad \rho_+ = \frac{b_+}{a_+}.$$

At the same time $M_-(x,k) \to 1$ at $x \to +\infty$. Let us now calculate a jump $\Delta(x,k) = M_+/a_+ - M_-$ of these functions across the real axis $\mathrm{Im}k = 0$. Calculation gives

$$\Delta(x,k) = (1/a_+) - 1$$
$$+ \int_{-\infty}^{\infty} \mathrm{d}x'(G_+ - G_-)(x-x',k)u(x')\frac{M_+(x',k)}{a_+(k)} + \int_{-\infty}^{\infty} \mathrm{d}x' G_-(x-x',k)\Delta(x',k).$$

Since $(G_+ - G_-)(x,k) = (1/2\mathrm{i}k)\left(1 - \mathrm{e}^{2\mathrm{i}kx}\right)$, in virtue of the definitions (8.116) we obtain

$$\Delta(x,k) = \rho_+(k)\mathrm{e}^{2\mathrm{i}kx} + \int_{-\infty}^{\infty} \mathrm{d}x' G_-(x-x',k)u(x')\Delta(x',k).$$

On the other hand [see (8.114)],

$$\rho_+(k)N_-(x,k) = \rho_+(k)\mathrm{e}^{2\mathrm{i}kx} + \int_{-\infty}^{\infty} \mathrm{d}x' G_-(x-x',k)u(x')\rho_+(k)N_-(x',k).$$

Comparing the last two equations, we conclude owing to the uniqueness of the solution of the above integral equations that $\Delta(x,k) = \rho_+(k)N_-(x,k)$. Therefore, we arrive at the *scattering equation*

$$\frac{M_+(x,k)}{a_+(k)} = M_-(x,k) + \rho_+(k)N_-(x,k). \tag{8.117}$$

In order to have reasons to treat (8.117) as the RH problem, we should express N_- in terms of M_-. To do that we note first of all that $G_\pm(x,k) = G_\pm(x,-k)\mathrm{e}^{2\mathrm{i}kx}$. Hence,

$$M_-(x,-k) = 1 + \int_{-\infty}^{\infty} \mathrm{d}x' G_-(x-x',k)u(x')M_-(x',-k)$$

$$= 1 + \int_{-\infty}^{\infty} \mathrm{d}x' \mathrm{e}^{-2\mathrm{i}k(x-x')} G_-(x-x',k)u(x')M_-(x',-k)$$

$$= \mathrm{e}^{-2\mathrm{i}kx}\left(\mathrm{e}^{2\mathrm{i}kx} + \int_{-\infty}^{\infty} \mathrm{d}x' G_-(x-x',k)u(x')M_-(x',-k)\mathrm{e}^{2\mathrm{i}kx'}\right).$$

Comparing this with the integral equation for N_- (8.114), we get the *symmetry relation*:

$$N_-(x,k) = M_-(x,-k)\mathrm{e}^{2\mathrm{i}kx}. \tag{8.118}$$

Therefore, we can write the scattering equation (8.117) in the form of the RH problem *with a shift*:

$$\frac{M_+(x,k)}{a_+(k)} = M_-(x,k) + \rho_+(k)\mathrm{e}^{2\mathrm{i}kx}M_-(x,-k), \quad \mathrm{Im}k = 0. \tag{8.119}$$

The shift is referred to the change of sign in k in the last term. The normalization of the RH problem (8.119) is canonical, $M_\pm \to 1$ as $k \to \infty$.

The form (8.119) of the RH problem corresponds to the *jump problem* for a piecewise analytic function $\Psi(x,k)$,

$$\Psi = \begin{cases} M_+/a_+, & \mathrm{Im}k > 0, \\ M_-, & \mathrm{Im}k < 0, \end{cases} \tag{8.120}$$

with discontinuity along the contour $\mathrm{Im}k = 0$. In previous sections we dealt with the factorization form of the RH problem. Evidently, both versions are equivalent. Indeed, the RH problem (8.17) for the NLS equation with the matrix $G = I + g$ takes the form $\Phi_+ = \Phi_- + g\Phi_-$ with the same structure as (8.119).

8.7.3 Inverse problem

To solve the inverse problem of determining the potential $u(x)$, we reconstruct the function $\Psi(x,k)$ (8.120). This function is analytic in the entire complex k-plane, except for a finite number $\mathcal{N}$ of simple poles at the points $k_j = \mathrm{i}\kappa_j$, $\kappa_j > 0, j = 1,\dots,\mathcal{N}$, and a discontinuity across the real axis. Accordingly, using the Cauchy formula, we obtain

$$\Psi(x,k) = 1 + \sum_{l=1}^{\mathcal{N}} \frac{\Phi_j(x)}{k-k_j} + \frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty} \frac{\mathrm{d}\ell}{\ell-k}\rho(\ell)\mathrm{e}^{2\mathrm{i}\ell x}M_-(x,-\ell). \tag{8.121}$$

Here $\Phi_j(x)$ is a residue of Ψ in k_j. A set of all $\Phi_j(x)$, $j = 1,\dots,\mathcal{N}$ comprises the set of eigenfunctions corresponding to the discrete spectrum $(k_j, j = 1,\dots,\mathcal{N})$ of the problem (8.110). For $\mathrm{Im}k > 0$ (8.121) and (8.118) give

$$\frac{M_+(x,k)}{a_+(k)} = 1 + \sum_{j=1}^{\mathcal{N}} \frac{\Phi_j(x)}{k-k_j} + \frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty} \frac{\mathrm{d}\ell}{\ell-k}\rho(\ell)N_-(x,\ell).$$

To determine the behavior of M_+ for $k \to k_j$, we define a regular part of M_+ as

$$M_+^{(j)}(x,k) = \frac{M_+(x,k)}{a_+(k)} - \frac{\Phi_j(x)}{k-k_j}.$$

With account for the integral equation (8.113) for M_+, we can write

$$M_+^{(j)}(x,k) - \int_{-\infty}^{\infty} \mathrm{d}x' G_+(x-x',k)u(x')M_+^{(j)}(x',k)$$
$$= \frac{1}{a_+(k)} - \frac{1}{k-k_j}\left(\Phi_j - \int_{-\infty}^{\infty} \mathrm{d}x' G_+(x-x',k)u(x')\Phi_j(x')\right).$$

The left-hand side of this equation is free of singularities for $k = k_j$; hence, the diverging terms on the right-hand side have to be canceled. This gives the integral equation for discrete eigenfunctions:

$$\Phi_j(x) = \mathrm{i}\gamma_j^{-1} + \int_{-\infty}^{\infty} \mathrm{d}x' G_+(x-x',k_j)u(x')\Phi_j(x'), \quad \gamma_j = \mathrm{i}(\mathrm{d}a_+/\mathrm{d}k)_{k_j}. \tag{8.122}$$

Comparing (8.122) with the integral equation for M_+, we find

$$\lim_{k\to k_j} M_+(x,k) = -\mathrm{i}\gamma_j\Phi_j(x). \tag{8.123}$$

In turn, (8.123) and asymptotics of M_+ at $x \to \pm\infty$ give

$$\Phi_j \to \begin{cases} \mathrm{i}\gamma_j^{-1}, & x \to -\infty \\ \mathrm{i}c_j\mathrm{e}^{-2\kappa_j x}, & x \to +\infty. \end{cases} \tag{8.124}$$

Here c_j is a normalization constant. Finally, it follows from (8.122) and (8.124) that it is possible to express eigenvalues as functionals of discrete eigenfunctions:

$$\kappa_j = -\frac{\mathrm{i}}{2}\gamma_j \int_{-\infty}^{\infty} \mathrm{d}x u(x)\Phi_j(x).$$

To reconstruct the potential, we at first derive a system of linear equations for eigenfunctions. Φ_j can be expressed in terms of M_-:

$$\Phi_j(x) = \frac{\mathrm{i}}{\gamma_j}M_+(x,k_j) = \mathrm{i}c_j N_-(x,k_j) = \mathrm{i}c_j\mathrm{e}^{-2\kappa_j x}M_-(x,-k_j).$$

On the other hand, we obtain from (8.121)

$$M_-(x,-k) = 1 - \sum_{j=1}^{\mathcal{N}} \frac{\Phi_j(x)}{k+k_j} + \frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty} \frac{\mathrm{d}\ell}{\ell+k+\mathrm{i}0}\rho(\ell)N_-(x,\ell) \tag{8.125}$$

and for $k = k_j$

$$M_-(x,-k_j) = 1 + \mathrm{i}\sum_{j=1}^{\mathcal{N}} \frac{\Phi_j(x)}{\kappa_j+\kappa_m} + \frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty} \frac{\mathrm{d}\ell}{\ell+\mathrm{i}\kappa_j}\rho(\ell)N_-(x,\ell).$$

Therefore, $\Phi_j(x)$ takes the form

$$\Phi_j(x) = \mathrm{i}c_j\mathrm{e}^{-2\kappa_j x}\left(1 + \mathrm{i}\sum_{j=1}^{\mathcal{N}} \frac{\Phi_j(x)}{\kappa_j+\kappa_m} + \frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty} \frac{\mathrm{d}\ell}{\ell+\mathrm{i}\kappa_j}\rho(\ell)N_-(x,\ell)\right). \tag{8.126}$$

Further, in virtue of (8.118) and (8.125),

$$N_-(x,k) = \mathrm{e}^{2\mathrm{i}kx}\left(1 - \sum_{j=1}^{\mathcal{N}} \frac{\Phi_j(x)}{k+\mathrm{i}\kappa_j} + \frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty} \frac{\mathrm{d}\ell}{\ell+k+\mathrm{i}0}\rho(\ell)N_-(x,\ell)\right). \tag{8.127}$$

As a result, we have a closed system of linear equations (8.126) and (8.127) to find $\Phi_j(x)$ and $N_-(x,k)$. This system is determined by the RH data $\{\rho_+(k), \kappa_j, c_j, j = 1,\dots,\mathcal{N}\}$. Then we expand the integral equation (8.114) for N_- in the asymptotic series in k^{-1} and compare it with (8.127) taken for $k \to \infty$. As a result, we get the reconstruction formula

$$u(x) = \partial_x\left(2\mathrm{i}\sum_{j=1}^{\mathcal{N}} \Phi_j(x) - \frac{1}{\pi}\int_{-\infty}^{\infty}\mathrm{d}k\rho(k)N_-(x,k)\right). \tag{8.128}$$

The first term on the right-hand side of (8.128) contributes from the discrete spectrum, while the second term is responsible for the continuous spectrum.

It should be stressed that it is the nonanalytic eigenfunction N_- that enters the complete set of eigenfunctions $\{N_-, \Phi_j, j = 1,\dots,\mathcal{N}\}$ [363]. It is not accidental because the spectral problem (8.110) is non-self-adjoint in contrast to the standard spectral problem (8.109).

8.7.4 Evolution of RH data

Substituting the asymptotics of M_+ at $x \to \pm\infty$ into the evolution equation (8.111), we obtain

$$\alpha = 4\mathrm{i}k^3, \quad a_+(k,t) = a_+(k,0), \quad \rho_+(k,t) = \rho_+(k,0)\mathrm{e}^{8\mathrm{i}k^3t}.$$

Because $a_+(k_j,t) = 0$, we get $\kappa_j = \text{const}$. Besides, owing to $M_+(x,k_j) \to b_j\mathrm{e}^{-2\kappa_j x}$ at $x \to +\infty$, we find

$$c_j(t) = c_j(0)\mathrm{e}^{8\kappa^3t}.$$

Hence, the time dependence of the RH data is extremely simple.

8.7.5 Soliton solution

Solitons correspond to reflectionless potentials, i.e., $\rho(k) = 0$. The simplest one-soliton solution is given by $u_1(x,t) = 2\mathrm{i}\Phi_{1x}(x,t)$. It follows from (8.126) that

$$\Phi_1(x,t) = 2\mathrm{i}\kappa_1\frac{\mathrm{e}^{-2\kappa_1(x-4\kappa_1^2t-x_0)}}{1+\mathrm{e}^{-2\kappa_1(x-4\kappa_1^2t-x_0)}}, \quad x_0 = \frac{1}{2\kappa_1}\ln\frac{c_1(0)}{2\kappa_1}.$$

Hence, the one-soliton solution to the KdV equation has the form

$$u_1(x,t) = 2\kappa_1^2\mathrm{sech}2\kappa_1(x-4\kappa_1^2t-x_0),$$

which goes back to Korteweg and de Vries [249]. An $\mathcal{N}$-soliton solution is also well known [354].

Let us remember that the aim of this chapter is purely methodological. We will see in the next chapter that the main ideas developed above will be naturally developed for (2+1)-dimensional equations.

9

Dressing via nonlocal Riemann–Hilbert problem

In the previous chapter we illustrated the efficiency of the dressing approach using the local Riemann–Hilbert (RH) problem for solution of the Cauchy problem for a number of (1+1)-dimensional nonlinear integrable equations. The essential progress in the development of the inverse spectral transform (IST) formalism has been achieved owing to the perception that the *nonlocal* RH problem can serve as a natural frame for solving nonlinear equations in 2+1 dimensions. Manakov [305] was the first to apply the nonlocal RH problem to treat the Kadomtsev–Petviashvili (KP) equation by means of the IST method. Besides, there exists an important class of (1+1)-dimensional nonlinear integrodifferential equations which cannot be solved by the methods discussed in Chap. 8.

This chapter contains an exposition of basic points related to the application of the nonlocal RH problem. We consider three featured examples. In Sect. 9.1 we consider the (1+1)-dimensional integrodifferential Benjamin–Ono (BO) equation . At the very beginning we work with real function $u(x,t)$ taking into account important constraints imposed on the spectral data by the reality condition, due to Kaup, Lakoba, and Matsuno [231]. Basic steps in application of the nonlocal RH problem developed for the BO equation are then used in Sect. 9.2 for the KP I equation. Along with the classical lumps, we discuss localized solutions associated with multiple-pole eigenfunctions found by Ablowitz and Villarroel [439]. Finally, Sect. 9.3 is devoted to solution of the initial boundary value problem for the Davey–Stewartson I (DS I) equation. It is this equation that allows true exponentially localized solitons in 2+1 dimensions discovered by Boiti, Leon, Martina, and Pempinelli [62]. In our consideration we follow the formalism adopted by Fokas and Santini [162].

9.1 Benjamin–Ono equation

The BO equation for a scalar function $u(x,t)$,

$$u_t + 2uu_x + Hu_{xx} = 0, \qquad Hu(x) = \frac{1}{\pi}\text{p.v.}\int_{-\infty}^{\infty} \mathrm{d}x' \, \frac{u(x')}{x'-x}, \tag{9.1}$$

where $Hu(x)$ is the Hilbert transform of $u(x)$ and p.v. means the principal value of the integral, describes the propagation of long internal waves in a stratified fluid [46, 106, 195, 357]. We require the function $u(x,t)$ to be real, in accordance with its physical meaning. In the general case of complex $u(x,t)$ Fokas and Ablowitz [158] developed an IST approach to solve the Cauchy problem for the BO equation within the class of initial conditions which have sufficient decay at infinity, i.e., $u(x,0) \equiv u_0(x) \to 0$ at $|x| \to \infty$. The consideration of real potentials $u(x)$[1] imposes nontrivial restrictions on the spectral data of the corresponding RH problem and, in addition, makes it possible to derive a body of practically important results, such as the prediction of a number of possible bound states (solitons) that can be produced by the initial data [232, 231]. Furthermore, we will restrict ourselves to the so-called nongeneric potentials which include, in particular, all N-soliton solutions and zero background [232, 362]. The nongeneric case has distinctly different features, compared with the general situation. The N-soliton solution of the BO equation was obtained for the first time in [83, 86] in the framework of the decomposition of $u(x,t)$ in a finite number of simple poles, as well as in [310] by the Hirota method . The IST method permits us to carry out the complete study of the Cauchy problem for the BO equation.

9.1.1 Jost solutions

The Lax pair for the BO equation (9.1) has the form [54, 346]

$$\mathrm{i}\Phi_x^+ + k(\Phi^+ - \Phi^-) + u\Phi^+ = 0, \tag{9.2}$$

$$\mathrm{i}\Phi_t^\pm - 2\mathrm{i}k\Phi_x^\pm + \Phi_{xx}^\pm - 2\mathrm{i}[u]_x^\pm + \nu\Phi^\pm = 0. \tag{9.3}$$

Here $\Phi^\pm(k,x)$ are limit values of the analytic function Φ at $y \to \pm 0$ in the upper and lower halves of the complex z-plane, where $z = x + \mathrm{i}y$ is a complexification of the physical variable x, k is a spectral parameter, and ν is an arbitrary constant. The functions $[u]^\pm(x)$ are defined as $[u]^\pm = P^\pm u$, where $P^\pm$ are projectors

$$\left(P^\pm u\right)(x) = \pm\frac{1}{2\pi\mathrm{i}}\int_{-\infty}^{\infty}\mathrm{d}x'\frac{u(x')}{x' - (x \pm \mathrm{i}0)}.$$

In other words, $[u]^+$ means that one takes the part of $u(x)$ that is analytic in the upper half z-plane. It is interesting that the spectral equation (9.2) can be considered as a differential RH problem.

Taking the (+) part of the spectral equation (i.e., acting on it by the projector P^+), we arrive at the integrodifferential equation

$$\mathrm{i}\Phi_x^+ + k\Phi^+ + [u\Phi]^+ = 0. \tag{9.4}$$

[1] On occasion, we will suppress for convenience time dependence in the potential and other quantities.

Now we introduce the Jost solutions $N_\pm(x,k)$ of (9.4) by means of their asymptotics:

$$N_+(x\to-\infty,k)=\mathrm{e}^{\mathrm{i}kx},\quad N_-(x\to\infty,k)=\mathrm{e}^{\mathrm{i}kx},\quad k>0. \tag{9.5}$$

There exist the other pair $M_\pm(x,k)$ of the Jost solutions of the spectral equation (9.2), which are defined by the asymptotics

$$M_+(x\to-\infty,k)=1,\qquad M_-(x\to\infty,k)=1 \tag{9.6}$$

and are solutions of the inhomogeneous integrodifferential equation

$$\mathrm{i}\Psi_x^+ + k\Psi^+ + [u\Psi]^+ - k = 0. \tag{9.7}$$

All the Jost functions are defined for positive values of k. These values of k comprise the continuous spectrum of the spectral problem (9.2).

The Jost functions introduced obey the following integral equations:

$$M_\pm(x,k)=1+\int_{-\infty}^{\infty}\mathrm{d}x' G_\pm(x-x',k)u(x')M_\pm(x',k), \tag{9.8}$$

$$N_\pm(x,k)=\mathrm{e}^{\mathrm{i}kx}+\int_{-\infty}^{\infty}\mathrm{d}x' G_\pm(x-x',k)u(x')N_\pm(x',k). \tag{9.9}$$

Here

$$G_\pm(x,k)=\frac{1}{2\pi}\int_0^\infty \mathrm{d}p\frac{\mathrm{e}^{\mathrm{i}px}}{p-(k\pm\mathrm{i}0)} \tag{9.10}$$

are limits of the Green function

$$G(x,k)=\frac{1}{2\pi}\int_0^\infty \mathrm{d}p\frac{\mathrm{e}^{\mathrm{i}px}}{p-k}$$

on the real k-axis which are analytic everywhere in the complex k-plane, except for the positive part of the real k-axis. Equation (9.10) can be obtained with account of the formula

$$\frac{1}{x'-(x+\mathrm{i}\epsilon)}=\mathrm{i}\int_0^\infty \mathrm{d}p\exp[-\mathrm{i}p\,(x'-x-\mathrm{i}\epsilon)].$$

It should be stressed that the integral equations (9.8) and (9.9) are of the Fredholm type, as distinct from the Volterra integral equations for the (1+1)-dimensional equations of the preceding chapter. This fact is of crucial importance because homogeneous versions of (9.8) and (9.9) can have nontrivial solutions $\Phi_j(x)$,

$$\Phi_j(x)=\int_{-\infty}^{\infty}\mathrm{d}x' G(x-x',k_j)u(x')\Phi_j(x'),\qquad k_j<0, \tag{9.11}$$

$$G(x,k_j)=\frac{1}{2\pi}\int_0^\infty \mathrm{d}p\frac{\mathrm{e}^{\mathrm{i}px}}{p-k_j}, \tag{9.12}$$

in some isolated points k_j, $j = 1, \ldots, \mathcal{N}$, lying on the negative part of the axis $\mathrm{Im}k = 0$. These solutions determine the discrete spectrum of the spectral problem and are associated with solitons (or lumps) of the BO equation. Each of these functions is analytic in the upper half z-plane.

Integrating (9.12) by parts, we obtain an asymptotic of $G(x, k_j)$:

$$G(x, k_j) = \frac{1}{2\pi \mathrm{i} k_j x} + \mathcal{O}(x^{-2}).$$

Therefore,

$$\Phi_j(x) \to \frac{1}{2\pi \mathrm{i} k_j x} \int_{-\infty}^{\infty} \mathrm{d}x' u(x') \Phi_j(x') \quad \text{at} \quad x \to \infty.$$

It is seen that it is natural to take the normalization of $\Phi_j(x)$ as

$$x\Phi_j(x) \to 1 \quad \text{at} \quad x \to \infty. \tag{9.13}$$

In this case, the eigenvalues k_j are given by the functional

$$k_j = \frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \mathrm{d}x u(x) \Phi_j(x), \qquad j = 1, \ldots, \mathcal{N}. \tag{9.14}$$

9.1.2 Scattering equation and symmetry relations

In the same way as for other spectral problems, the left and right Jost functions are interrelated by scattering equations. To obtain the scattering equations, we find at first a difference $\Delta(x, k) = M_+(x, k) - M_-(x, k)$ for $k > 0$. It follows from (9.8) and (9.9) that

$$\Delta(x,k) = \int_{-\infty}^{\infty} \mathrm{d}x' G_+(x-x', k) u(x') M_+(x', k) - \int_{-\infty}^{\infty} \mathrm{d}x' G_-(x-x', k) u(x') M_-(x', k).$$

As stated already, the Green function $G(x, k)$ has a discontinuity across the positive k-axis. We can calculate this jump:

$$G_+(x,k) - G_-(x,k)$$

$$= \frac{1}{2\pi} \int_0^{\infty} \mathrm{d}p \ \mathrm{e}^{ipx} \left(\frac{1}{p - k - \mathrm{i}0} - \frac{1}{p - k + \mathrm{i}0} \right) = \mathrm{i}\theta(k) \mathrm{e}^{\mathrm{i}kx},$$

where we have used

$$\frac{1}{k \mp \mathrm{i}0} = \pm \pi \mathrm{i} \delta(k) + \mathrm{p.v.} \left(\frac{1}{k} \right)$$

and $\theta(k)$ is the Heaviside step function. Hence,

$$\begin{aligned}\Delta(x,k) &= \int_{-\infty}^{\infty} \mathrm{d}x' G_-(x-x',k)u(x')\Delta(x',k) + \int_{-\infty}^{\infty} \mathrm{d}x'(G_+ - G_-)(x-x',k) \\ &= \beta(k)\mathrm{e}^{\mathrm{i}kx} + \int_{-\infty}^{\infty} \mathrm{d}x' G_-(x-x',k)u(x')\Delta(x',k), \end{aligned} \tag{9.15}$$

where the reflection coefficient $\beta(k)$ is defined as

$$\beta(k) = \mathrm{i}\theta(k) \int_{-\infty}^{\infty} \mathrm{d}x u(x) M_+(x,k)\mathrm{e}^{-\mathrm{i}kx}. \tag{9.16}$$

On the other hand, it follows from (9.9) that

$$\beta(k)N_-(x,k) = \beta(k)\mathrm{e}^{\mathrm{i}kx} + \int_{-\infty}^{\infty} \mathrm{d}x' G_-(x-x',k)u(x')\beta(k)N_-(x',k). \tag{9.17}$$

Comparing (9.15) and (9.17), we find $\Delta(x,k) = \beta(k)N_-(x,k)$. Thereby, we obtain the important relation between the Jost functions:

$$M_+(x,k) = M_-(x,k) + \beta(k)N_-(x,k). \tag{9.18}$$

There also exists a connection between the functions N_+ and N_- written as [383]

$$N_+(x,k) = \Gamma(k)N_-(x,k) \tag{9.19}$$

with some function $\Gamma(k)$. Let us differentiate $G_\pm$ in k and integrate the result by parts. This gives

$$\frac{\partial G_\pm}{\partial k} = -\frac{1}{2\pi k} \pm \mathrm{i}xG_\pm(x,k). \tag{9.20}$$

On the other hand, writing $N_-(x,k)$ from (9.9) as

$$N_-(x,k)\mathrm{e}^{-\mathrm{i}kx} = 1 + \int_{-\infty}^{\infty} \mathrm{d}x' G_-(x-x',k)\mathrm{e}^{-\mathrm{i}k(x-x')}u(x')N_-(x',k)\mathrm{e}^{-\mathrm{i}kx'}$$

and taking the k-derivative, we find with account of (9.20)

$$\begin{aligned}\frac{\partial}{\partial k}\left[N_-(x,k)\mathrm{e}^{-\mathrm{i}kx}\right] &= \int_{-\infty}^{\infty} \mathrm{d}x' \frac{\partial G_-}{\partial k}(x-x',k)\mathrm{e}^{-\mathrm{i}k(x-x')}u(x')N_-(x',k)\mathrm{e}^{-\mathrm{i}kx'} \\ &\quad -\mathrm{i}\int_{-\infty}^{\infty} \mathrm{d}x' G_-(x-x',k)\mathrm{e}^{-\mathrm{i}k(x-x')}(x-x')u(x')N_-\mathrm{e}^{-\mathrm{i}kx'} \\ &\quad + \int_{-\infty}^{\infty} \mathrm{d}x' G_-(x-x',k)\mathrm{e}^{-\mathrm{i}k(x-x')}u(x')\frac{\partial}{\partial k}(N_-\mathrm{e}^{-\mathrm{i}kx'}) \\ &= \mathrm{e}^{-\mathrm{i}kx}f_-(k) + \int_{-\infty}^{\infty} \mathrm{d}x' G_-(x-x',k)\mathrm{e}^{-\mathrm{i}k(x-x')}u(x')\frac{\partial}{\partial k}(N_-\mathrm{e}^{-\mathrm{i}kx'}), \end{aligned} \tag{9.21}$$

where

$$f_-(k) = -\frac{1}{2\pi k}\int_{-\infty}^{\infty} \mathrm{d}x u(x) N_-(x,k), \qquad k>0. \tag{9.22}$$

Multiplying $M_-(x,k)$ (9.8) by $f_-(k)$ and comparing the result with the right-hand side of (9.21), we arrive at the differential connection between N_- and M_- [158]:

$$\frac{\partial}{\partial k}\left[N_-(x,k)\mathrm{e}^{-\mathrm{i}kx}\right] = f_-(k)M_-(x,k)\mathrm{e}^{-\mathrm{i}kx}. \tag{9.23}$$

Similarly,

$$\frac{\partial}{\partial k}\left[N_+(x,k)\mathrm{e}^{-\mathrm{i}kx}\right] = f_+(k)M_+(x,k)\mathrm{e}^{-\mathrm{i}kx},$$

where

$$f_+(k) = -\frac{1}{2\pi k}\int_{-\infty}^{\infty} \mathrm{d}x u(x) N_+(x,k), \qquad k>0. \tag{9.24}$$

Equations (9.23) and (9.24) give the symmetry (or closure) relations for the Jost solutions. Multiplying both sides of (9.19) by $u(x)$ and integrating in x, we find a simple connection between $f_+(k)$ and $f_-(k)$:

$$f_+(k) = \Gamma(k) f_-(k). \tag{9.25}$$

Eventually, (9.23), (9.24), (9.19), and (9.25) give

$$\beta(k) f_+(k) = \frac{\partial}{\partial k}\Gamma(k). \tag{9.26}$$

Now we should find out how the function M_- behaves as $k \to k_j$. Define a function $M_-^{(j)}(x,k)$ as in [158] (we suppose here that M_- allows simple poles only; see Sect. 9.1.4)

$$M_-^{(j)}(x,k) = M_-(x,k) - \frac{c_j}{k-k_j}\Phi_j(x),$$

where c_j is a normalization constant. Then we can write, taking into account (9.8) and (9.11),

$$M_-^{(j)}(x,k) - \int_{-\infty}^{\infty} \mathrm{d}x' G_-(x-x',k)u(x')M_-^{(j)}(x',k) \tag{9.27}$$

$$= 1 - \frac{c_j}{k-k_j}\left(\Phi_j(x) - \int_{-\infty}^{\infty} \mathrm{d}x' G_-(x-x',k)u(x')\Phi_j(x')\right).$$

When $k \to k_j$, the last term in (9.27) develops an uncertainty $0/0$. The k-derivative of the numerator in this term gives with regard to (9.20)

$$\int_{-\infty}^{\infty} \mathrm{d}x' \left(\frac{\partial}{\partial k} G_-(x-x',k)\right) u(x')\Phi_j(x')$$

$$= -\frac{1}{2\pi k}\int_{-\infty}^{\infty} \mathrm{d}x u(x)\Phi_j(x) + \mathrm{i}\int_{-\infty}^{\infty} \mathrm{d}x'(x-x')G(x-x',k_j)u(x')\Phi_j(x').$$

Hence,

$$M_-^{(j)}(x,k) - \int_{-\infty}^{\infty} \mathrm{d}x' G(x-x',k_j)u(x')M_-(x',k_j) \tag{9.28}$$

$$= a_j + \mathrm{i}c_j \int_{-\infty}^{\infty} \mathrm{d}x'(x-x')G(x-x',k_j)u(x')\Phi_j(x'),$$

where

$$a_j = 1 - \frac{c_j}{2\pi k_j} \int_{-\infty}^{\infty} \mathrm{d}x u(x)\Phi_j(x).$$

Putting $c_j = -\mathrm{i}$ gives $a_j = 0$. It is easy to see that a particular solution to (9.28) has the form $M_-^{(j)}(x,k) = x\Phi_j(x)$. Then, applying the Fredholm alternative [425], we obtain a connection between Φ_j and M_-:

$$\lim_{k\to k_j} \left(M_-(x,k) + \frac{i}{k-k_j}\Phi_j(x) \right) = (x+\gamma_j)\Phi_j(x). \tag{9.29}$$

The normalization constant γ_j will be time-dependent if we account for the evolution of the potential $u(x,t)$.

9.1.3 Adjoint spectral problem and asymptotics

Following the paper by Kaup and Matsuno [232], we define now the adjoint spectral problem

$$\mathrm{i}\Phi_x^{\mathrm{A}} - k\Phi^{\mathrm{A}} + u\left[\Phi^{\mathrm{A}}\right]^+ = 0. \tag{9.30}$$

Applying the projector P^- to (9.30), we obtain the equation complex conjugate to (9.4) whose solution is $N_+^*(x,k)$. The $(+)$ projection of (9.30) gives a linear inhomogeneous equation. As a result, a solution N^{A} of the adjoint problem is given by

$$N^{\mathrm{A}}(x,k) = N_+^*(x,k) - \mathrm{i}\int_{-\infty}^{x} \mathrm{d}x' \left[uN_+^*\right]^+ (x',k)\mathrm{e}^{-\mathrm{i}k(x-x')}. \tag{9.31}$$

Similarly we can write a bound-state eigenfunction $N_j^{\mathrm{A}}(x)$:

$$N_j^{\mathrm{A}}(x) = \Phi_j^*(x) - \mathrm{i}\int_{-\infty}^{x} \mathrm{d}x' \left[u\Phi_j^*\right]^+ (x')\mathrm{e}^{-\mathrm{i}k_j(x-x')}.$$

In the following we will need asymptotic expressions for the eigenfunctions at $|x| \to \infty$. Taking into accountthe definitions (9.5) and (9.6) and equations

(9.18), (9.19), and (9.31), we have for $x \to -\infty$ and $k > 0$, in addition to (9.5) and (9.6)

$$N_-(x,k) \to \Gamma(k)\mathrm{e}^{\mathrm{i}kx}, \quad M_-(x,k) \to 1 + \beta(k)\mathrm{e}^{\mathrm{i}kx}, \qquad N^{\mathrm{A}}(x,k) \to \mathrm{e}^{\mathrm{i}kx} \tag{9.32}$$

and for $x \to \infty$ and $k > 0$

$$N_+(x,k) \to \Gamma(k)\mathrm{e}^{\mathrm{i}kx}, \quad M_+(x,k) \to 1 + \beta(k)\mathrm{e}^{\mathrm{i}kx}, \quad N^{\mathrm{A}}(x,k) \to \Gamma^*(k)\mathrm{e}^{-\mathrm{i}kx}, \tag{9.33}$$

while for the bound states we have for $x \to -\infty$ and $k_j < 0$

$$\Phi_j(x) \to \frac{1}{x}, \qquad N_j^{\mathrm{A}}(x) \to 0,$$

and for $x \to \infty$ and $k_j < 0$

$$\Phi_j(x) \to \frac{1}{x}, \qquad N_j^{\mathrm{A}}(x) \to -\mathrm{i}\mathrm{e}^{-\mathrm{i}k_j x} \int_{-\infty}^{\infty} \mathrm{d}x' u(x') \Phi_j^*(x') \mathrm{e}^{\mathrm{i}k_j x'}.$$

Now we can concretize the form of the function $\Gamma(k)$. Excluding the potential $u(x)$ from the direct (9.4) and adjoint (9.30) spectral problems , we obtain a relation (the Wronskian relation) between the left and right asymptotics:

$$\Phi^A(x,k')\Phi(x,k)|_{x=-\infty}^{+\infty} = \mathrm{i}(k-k') \int_{-\infty}^{\infty} \mathrm{d}x' \Phi^{\mathrm{A}}(x',k')\Phi(x',k). \tag{9.34}$$

Taking $k' = k$, $\Phi^{\mathrm{A}} = N^{\mathrm{A}}$, and $\Phi = N_-$ in (9.34) and using the above asymptotics, we easily get

$$\Gamma^*(k)\Gamma(k) = 1,$$

which allows us to introduce a real phase $\theta(k)$:

$$\Gamma(k) = \mathrm{e}^{-\mathrm{i}\theta(k)}.$$

To find the phase explicitly, we construct the Wronskian relation from the spectral problems (9.7) and (9.30):

$$N^{\mathrm{A}}(x,k') \left[M_-(x,k) - 1\right]_{x=-\infty}^{+\infty} \tag{9.35}$$

$$= \mathrm{i}(k-k') \int_{-\infty}^{\infty} \mathrm{d}x' N^{\mathrm{A}}(x',k') \left[M_-(x',k) - 1\right] - \mathrm{i} \int_{-\infty}^{\infty} \mathrm{d}x' u(x') \left[N^{\mathrm{A}}(x',k')\right]^- .$$

Taking $k' = k$ once again, we find

$$\beta(k) = -\mathrm{i}\Gamma(k) \int_{-\infty}^{\infty} \mathrm{d}x u(x) \left[N^{\mathrm{A}}(x,k)\right]^- . \tag{9.36}$$

Let us remember [see (9.31)] that $\left[N^{\mathrm{A}}\right]^- = N_+^*$. Then, with account for the connection (9.19) and the definition (9.22) of $f_-(k)$, we obtain from (9.36) $\beta(k) = -2\pi\mathrm{i}kf_-^*(k)$, i.e.,

$$f_-(k) = \frac{\beta^*(k)}{2\pi\mathrm{i}k}. \tag{9.37}$$

Because $(\partial/\partial k)\Gamma = -\mathrm{i}\Gamma(\partial/\partial k)\theta$, we find from (9.25), (9.26), and (9.37):

$$\frac{\partial\theta}{\partial k} = \frac{|\beta(k)|^2}{2\pi k}.$$

Hence, we represent explicitly the function $\Gamma(k)$ in terms of the reflection coefficient $\beta(k)$:

$$\Gamma(k) = \exp\left(\frac{1}{2\pi\mathrm{i}}\int_0^k \frac{\mathrm{d}k'}{k'}|\beta(k')|^2\right). \tag{9.38}$$

It should be stressed that (9.37) and (9.38) hold for real potentials only. In the case of complex potentials, the spectral functions $\beta(k)$ and $f_-(k)$ are independent [158].

We can make use of the Wronskian relation to derive integral expressions for "squared" eigenfunctions. Taking $\Phi^{\mathrm{A}} = N^{\mathrm{A}}(x,k')$ and $\Phi = N(x,k)$ in (9.34) and using the asymptotics (9.32) and (9.33), we obtain

$$\int_{-\infty}^{\infty}\mathrm{d}x N^{\mathrm{A}}(x,k')N(x,k) = -\mathrm{i}\Gamma^*(k')\lim_{x\to+\infty}\frac{\mathrm{e}^{\mathrm{i}(k-k')x}}{k-k'} + \frac{\mathrm{i}}{\Gamma(k)}\lim_{x\to-\infty}\frac{\mathrm{e}^{\mathrm{i}(k-k')x}}{k-k'}.$$

Invoking the formula

$$\lim_{x\to+\infty}\mathrm{p.v.}\frac{\mathrm{e}^{\pm\mathrm{i}kx}}{k} = \pm\mathrm{i}\pi\delta(k),$$

we find the following orthogonality relation for the continuous eigenfunctions:

$$\int_{-\infty}^{\infty}\mathrm{d}x N^{\mathrm{A}}(x,k')N_-(x,k) = \frac{2\pi}{\Gamma(k)}\delta(k-k'). \tag{9.39}$$

It is evident (because k and k_j belong to different parts of the real k-axis) that the continuous and bounded eigenfunctions are mutually orthogonal:

$$\int_{-\infty}^{\infty}\mathrm{d}x N_j^{\mathrm{A}}(x)N_-(x,k) = \int_{-\infty}^{\infty}\mathrm{d}x N^{\mathrm{A}}(x,k)\Phi_j(x) = 0, \tag{9.40}$$

as well as

$$\int_{-\infty}^{\infty}\mathrm{d}x N_j^{\mathrm{A}}(x)\Phi_n(x) = 0, \quad \text{if} \quad j\neq n. \tag{9.41}$$

Furthermore, it follows from (9.35) that

$$\int_{-\infty}^{\infty}\mathrm{d}x N_j^{\mathrm{A}}(x)\Phi_j(x) = \mathrm{i}\int_{-\infty}^{\infty}\mathrm{d}x u(x)\left[N_j^{\mathrm{A}}(x)\right]^-.$$

With account for $N_j^{\mathrm{A}} = \Phi_j^*$ and (9.14) we obtain

$$\int_{-\infty}^{\infty}\mathrm{d}x N_j^{\mathrm{A}}(x)\Phi_j(x) = -2\pi k_j. \tag{9.42}$$

Finally, the boundary condition (9.13) gives

$$\int_{-\infty}^{\infty} \mathrm{d}x \Phi_j(x) = -\pi \mathrm{i}. \tag{9.43}$$

9.1.4 RH problem

Now we have all ingredients to formulate the RH problem for the BO equation. Indeed, because the Green function G_+ is analytic in the upper half k-plane, it follows from the theory of Fredholm integral equations [425] that the solution $M_+(x,k)$ is analytic in the same region as well, except for possible poles at isolated points k_j, where nontrivial solutions of the homogeneous integral equation exist. The same analytic properties but in the lower half plane are inherent to the eigenfunction $M_-(x,k)$. As regards $N_\pm$, they cannot in general be continued off the real k-axis because of the exponent $\mathrm{e}^{\mathrm{i}kx}$. On the other hand, there is a differential connection (9.23) between the functions N_- and M_-. This fact enables us, by means of (9.18), (9.23), and (9.37), to pose the RH problem of the form

$$M_+(x,k) = M_-(x,k) + \frac{\beta(k)}{2\pi\mathrm{i}} \int_0^k \frac{\mathrm{d}k'}{k'} \beta^*(k') \mathrm{e}^{-\mathrm{i}k'x} M_-(x,k'). \tag{9.44}$$

The normalization of the RH problem (9.44) is given by

$$M_+(x,k) \to 1 \quad \text{at} \quad k \to \infty. \tag{9.45}$$

It should be noted that the RH problem (9.44) is essentially distinct from the RH problems we dealt with in the preceding chapter. Namely, the RH problem (9.44) is *nonlocal* in that stems from the nonlocality of the BO equation. The spectral data of the RH problem (9.44) are determined by the set $\{\beta(k), k > 0; k_j, \mathrm{Re}\gamma_j, j = 1, \ldots, \mathcal{N}\}$.[2] As usual, we can regularize the RH problem and represent its solutions M_+ and M_- as

$$M_+(x,k) = 1 - \mathrm{i} \sum_{j=1}^{\mathcal{N}} (k-k_j)^{-1} \Phi_j(x) + m_+(x,k), \;\; m_+(x, k \to \infty) = 0, \tag{9.46}$$

$$M_-(x,k) = 1 - \mathrm{i} \sum_{j=1}^{\mathcal{N}} (k-k_j)^{-1} \Phi_j(x) + m_-(x,k), \quad m_-(x, k \to \infty) = 0, \tag{9.47}$$

where the holomorphic functions $m_\pm(x,k)$ are $(+)$ and $(-)$ functions with respect to x, respectively. The fact that the spectral problem (9.2) allows only *simple* poles was proved in [362].

[2] It will be shown later that $\mathrm{Im}\gamma_j$ can be expressed in terms of k_j.

The solution M_-, $\mathrm{Im}k \le 0$ of the RH problem can be expressed in terms of $N_-(x,k)$ for positive k and of $\Phi_j(x)$ at the poles k_j. Let us write (9.18) as [see (9.46)]

$$1 - \mathrm{i}\sum_{j=1}^{\mathcal{N}}(k-k_j)^{-1}\Phi_j(x) + m_+(x,k) = M_-(x,k) + \beta(k)N_-(x,k)$$

and apply the projector P^- to it. This yields

$$M_-(x,k) = 1 - \mathrm{i}\sum_{j=1}^{\mathcal{N}}(k-k_j)^{-1}\Phi_j(x) + \frac{1}{2\pi\mathrm{i}}\int_0^\infty \mathrm{d}l \frac{\beta(\ell)}{\ell - (k - \mathrm{i}0)}N_-(x,\ell). \quad (9.48)$$

We see that the solution (9.48) clearly demonstrates the separation of contributions of the discrete and continuous spectra. It follows from (9.29) and (9.48) that in the limit $k \to k_j$

$$(x+\gamma_j)\Phi_j(x) + \mathrm{i}\sum_{n\ne j}(k-k_j)^{-1}\Phi_j(x) - \frac{1}{2\pi\mathrm{i}}\int_0^\infty \mathrm{d}\ell \frac{\beta(\ell)}{\ell - (k-\mathrm{i}0)}N_-(x,\ell) = 1. \quad (9.49)$$

Now we multiply (9.49) by $\Phi_j^*(x)$ and integrate in x with account for the orthogonality conditions (9.39)–(9.43). This yields

$$\int_{-\infty}^{\infty}\mathrm{d}x(x+\gamma_j)|\Phi_j(x)|^2 = \pi\mathrm{i}.$$

In accordance with (9.42) the last formula gives

$$\gamma_j = \frac{1}{2\pi k_j}\int_{-\infty}^{\infty}\mathrm{d}x x|\Phi_j(x)|^2 - \frac{\mathrm{i}}{2k_j}\ ; \quad (9.50)$$

therefore,

$$\mathrm{Im}\gamma_j = -\frac{1}{2k_j}. \quad (9.51)$$

Hence, only the $\mathrm{Re}\gamma_j$ should be included in the set of spectral data of the RH problem.

The reconstruction of the potential $u(x)$ from the solution of the RH problem can be performed as follows. Integrating by parts the integral in (9.9), we find

$$M_- \to 1 - \frac{[u]^+}{k}, \qquad k \to \infty.$$

As a result, we obtain from (9.48) in the same limit $k \to \infty$

$$[u]^+ = \frac{1}{2\pi\mathrm{i}}\int_0^\infty \mathrm{d}k\beta(k)N_-(x,k) + \mathrm{i}\sum_{j=1}^{\mathcal{N}}\Phi_j(x). \quad (9.52)$$

Therefore, the real potential $u(x)$ is given by

$$u(x) = [u(x)]^+ + [u(x)]^{+*}. \quad (9.53)$$

9.1.5 Evolution of spectral data

To determine the time evolution of the spectral data, we should consider the Lax equation (9.3):

$$\mathrm{i}M_{+t} - 2\mathrm{i}kM_{+x} + M_{+xx} - 2\mathrm{i}[u_x]^+M_+ + \nu M_+ = 0.$$

Since $M_+ \to 1$ for $x \to -\infty$, we have $\nu = 0$. At $x \to \infty$ this equation is reduced to

$$\mathrm{i}M_{+t} - 2\mathrm{i}kM_{+x} + M_{+xx} = 0. \tag{9.54}$$

Let us substitute (9.18) into (9.54):

$$\mathrm{i}(M_{-t} + \beta_t N_- + \beta N_{-t}) - 2\mathrm{i}k(M_{-x} + \beta N_{-x}) + M_{-xx} + \beta N_{-xx} = 0. \tag{9.55}$$

At $x \to +\infty$, $M_- \to 1$ and $N_- \to \mathrm{e}^{\mathrm{i}kx}$. As a result, (9.55) gives $\beta_t = \mathrm{i}k^2\beta$, i.e.,

$$\beta(k,t) = \beta(k,0)\mathrm{e}^{\mathrm{i}k^2t}. \tag{9.56}$$

Taking the time derivative of (9.29) yields

$$k_j = \text{const}, \qquad \gamma_j(t) = 2k_jt + \gamma_{j,0}, \qquad \gamma_{j,0} = \text{const}. \tag{9.57}$$

Therefore, (9.56) and (9.57) give the time dependence of the spectral data.

9.1.6 Solitons of BO equation

In the case of the pure soliton potential $u_\mathrm{s}(x,t)$ the reflection coefficient $\beta(k)$ vanishes; hence, the reconstruction of the potential is given by [see (9.52)]

$$[u_s(x,t)]^+ = \mathrm{i}\sum_{j=1}^{\mathcal{N}} \Phi_j(x,t).$$

Bound states Φ_j can be found from (9.49) as a solution of the algebraic system

$$(x + \gamma_j)\Phi_j + \mathrm{i}\sum_{n\neq j}(k_j - k_n)^{-1}\Phi_n = 1. \tag{9.58}$$

For $\mathcal{N} = 1$ we obtain from (9.58)

$$\Phi_1(x,t) = [x + \gamma_1(t)]^{-1}.$$

Putting $k_1 = -(1/2)v$, $v > 0$ and accounting for the relation (9.51), we can write $\gamma_{1,0} = -x_0 + \mathrm{i}/v$. Then $\gamma_1(t) = -x_0 - vt + \mathrm{i}/v$ and (9.52) and (9.53) eventually give

$$u_\mathrm{s}(x,t) = \frac{2v}{1 + v^2(x - vt - x_0)^2}. \tag{9.59}$$

We see that the BO soliton looks like a localized object moving with velocity v with the initial position of the maximum at point x_0. As for the Korteweg–de Vries (KdV) soliton, the same parameter v determines both the soliton amplitude and the soliton velocity. However, the BO soliton (9.59) decays algebraically only, as distinct from the KdV soliton, which decreases exponentially.

One further comment is in order concerning the Jost functions for the one-soliton potential (9.59). Though the function M_- can be easily recovered from (9.48), we have no recipe for similarly recovering the function N_- [remember that the solution (9.48) of the RH problem as well as the potential (9.52) are expressed in terms of $N_-(x,k)$ defined for positive k]. Instead we should solve the direct spectral problem (9.4) and (9.5) with the one-soliton potential (9.59). Because N_- has to be analytic in the upper half z-plane and the potential (9.59) is real, we can write

$$[uN_-]^+ = u(x)N_-(x,k) + \mathrm{i}\frac{N_-(x_0+\mathrm{i}/v,k)}{x-(x_0+\mathrm{i}/v)}.$$

Then the solution of the spectral problem has the form [232]

$$N_-(x,k)\mathrm{e}^{-\mathrm{i}kx} \tag{9.60}$$

$$=\frac{x-x_0-\mathrm{i}/v}{x-x_0+\mathrm{i}/v}\left\{1+N_-\left(x,k_0+\frac{\mathrm{i}}{v}\right)\left[\left(1+\frac{2k}{v}\right)\mathrm{e}^{k/v}E(\zeta)+\frac{2i}{v}\frac{e^{-\mathrm{i}kx}}{x-x_0-\mathrm{i}/v}\right]\right\},$$

where $\zeta=\mathrm{i}k(x-x_0-\mathrm{i}/v)$ and $E(\zeta)=\int_\zeta^\infty \mathrm{d}t\exp(-t)t^{-1}$, where $\arg(\zeta)<\pi$. In general, the function N_- (9.60) is not analytic in the upper half z-plane, in view of the fact that the integral $E(\zeta)$ has a logarithmic singularity at $\zeta=0$, $E(\zeta)\sim -\gamma-\ln\zeta+\mathcal{O}(z)$, where γ is the Euler constant. The only way to provide analyticity is to choose $N_-(x_0+\mathrm{i}/v,k)=0$, which results in

$$N_-(x,k)=\mathrm{e}^{\mathrm{i}kx}\frac{x-x_0-\mathrm{i}/v}{x-x_0+\mathrm{i}/v}. \tag{9.61}$$

This form of the Jost function $N_-(x,k)$ agrees with both asymptotics (9.32) and (9.33) [note that $\Gamma(k)=1$ for $\beta(k)=0$]. For the bound states, the problem of singularity is resolved automatically in view of $(1+2k/v)=0$ for $k=k_1$.

Now we can obtain from (9.59) and (9.61) that

$$\int_{-\infty}^{\infty}\mathrm{d}x u_s(x)N_-(x,k)\mathrm{e}^{-\mathrm{i}kx}=0. \tag{9.62}$$

The potentials obeying (9.62) are called *nongeneric*, as distinct from generic potentials for which the integral (9.62) is strictly nonzero [232, 362]. The $\mathcal{N}$-soliton solutions, including the trivial one $u=0$ which is considered here as the seed solution, belong to the class of nongeneric potentials. The IST approach

to the BO equation developed by Fokas and Ablowitz [158] is applicable to general complex generic potentials.

One of the main distinctions between the nongeneric and generic potentials is the limit $k \to 0$ for the Jost functions and the reflection coefficient $\beta(k)$. To determine whether one has the generic or the nongeneric case, it is sufficient to determine how fast the reflection coefficient vanishes for $k \to 0^+$. This problem is closely related to the problem of the edge of the continuous spectrum. It was shown by Pelinovsky and Sulem [362] that the point $k = 0$ belongs to the continuous spectrum for nongeneric potentials and does not belong to it for generic ones.

Now we briefly address a problem of estimating a number of bound states (solitons) generated by a smooth localized initial perturbation $u_0(x)$. Equation (9.48) for $k = 0$ gives

$$M_-(x,0) = 1 + \mathrm{i}\sum_{j=1}^{\mathcal{N}} k_j^{-1}\Phi_j(x) + \frac{1}{2\pi\mathrm{i}}\int_0^\infty \frac{\mathrm{d}k}{k}\beta(k)N_-(x,k). \tag{9.63}$$

Let us multiply this equation by $u(x)$ and integrate over the real line x. The integral $\int_{-\infty}^{\infty} \mathrm{d}x u(x) M_-(x,k)$ on the left-hand side of (9.63) gives zero for both generic and nongeneric potentials ($M_-(x,0) = 0$ for generic potentials [232]). Then it follows from (9.16) and (9.14) that the number of bound states is given in terms of the area $A[u]$ of $u_0(x)$, where $A[u] = \int_{-\infty}^{\infty} \mathrm{d}x u_0(x)$:

$$\mathcal{N} = \frac{1}{2\pi}\left(A[u] + \frac{1}{2\pi}\int_0^\infty \frac{\mathrm{d}k}{k}|\beta(k)|^2\right).$$

The last remark is concerned with soliton generation by a small initial perturbation. It was shown in [362] that there is a threshold in soliton generation for generic initial perturbation, while in the case of a perturbation of zero background or soliton state, a new eigenvalue emerges from the edge $k = 0$ of the continuous spectrum for an arbitrary small initial perturbation. This new eigenvalue is exponentially small.

9.2 Kadomtsev–Petviashvili I equation—lump solutions

The KP equation

$$(u_t + 6uu_x + u_{xxx})_x = \pm 3u_{yy} \tag{9.64}$$

appeared in plasma physics [227] and surface water waves [12] for the description of two-dimensional waves propagating in the x direction with slow variation in the y direction. This equation represents one of the possible generalizations of the KdV equation to $2+1$ dimensions. The properties of localized solutions of (9.64) depend crucially on the sign of the right-hand side of this equation. We refer to the KP I equation for the case of $(+)$ sign and to the

KP II equation for (–) sign. In this section we consider the KP I equation—it is the nonlocal RH problem that appears in studying the KP I equation by the inverse spectral method, as was shown by Manakov [305]. The KP II equation is integrated in the framework of the $\bar{\partial}$ method and is considered in the next chapter. Fokas and Ablowitz [157] succeeded in obtaining explicit formulas for scattering data by means of introducing nonanalytic eigenfunctions of the associated spectral problem and derived lump solution, previously found in [306] and in [324]. Fokas and Zakharov [163] generalized the dressing method to the case of nontrivial seed solutions. Boiti et al. [66] elaborated a spectral transform for the KP I equation based on analytic eigenfunctions and orthogonality relations. Boiti et al. [70, 71] applied a resolvent-based approach for obtaining solutions of the KP I equation on both zero and nonzero background. The classical results concerning various solutions of the KP I equation in the framework of the IST have been summarized by Ablowitz and Clarkson [3] and Konopelchenko [241]. Since then further important progress in the KP I theory has been achieved. In particular, Ablowitz and Villarroel [14, 439] discovered a class of solutions of the KP I equation associated with multiple poles of meromorphic eigenfunctions. Pelinovsky and Sulem [363] proved the completeness of a set of eigenfunctions which contains nonanalytic continuous eigenfunctions.

Note that we will be interested in solutions of the KP I equation that decrease as $x^2 + y^2 \to \infty$; therefore, no consideration will be given to the so-called line solitons of the KP I equation which do not decrease in some directions in the (x, y) plane and essentially give the KdV solitons directed at some angle relative to the x-axis. N-line solitons are discussed by Satsuma [385]. The IST theory for the line-soliton-type potentials has been considered by Boiti et al. [69].

9.2.1 Lax representation

Following the strategy of the dressing method, we first derive the Lax representation for the KP I equation. Let us introduce "long" derivatives

$$D_x = \partial_x + \mathrm{i}k, \quad D_y = \partial_y + \mathrm{i}k^2, \quad D_t = \partial_t + \mathrm{i}k^3.$$

Evidently, these operators have poles at $k = \infty$ and are mutually commuting. Our aim is to make up "balance equations" in order to eliminate poles. It is clear that the difference $\mathrm{i}D_y m - D_x^2 m \sim \mathcal{O}(1)$ has no poles in k for the function $m(x, y, t, k)$ which allows the asymptotic expansion $m = 1 + m_1/k + m_2/k^2 + \mathcal{O}(k^{-3})$. Therefore, we can write this difference as $\mathrm{i}D_y m - D_x^2 m = um$ with some function $u(x, y, t)$. Taking into account the above expansion, we can show that $u = -2\mathrm{i}m_{1x}$ and hence

$$im_y = m_{xx} + 2\mathrm{i}km_x + um. \tag{9.65}$$

In the same way we can form the balance equation with the operator D_t. Indeed, the third-order pole is eliminated if we take the sum

$$D_t m + D_x^3 m = m_t + m_{xxx} + 3\mathrm{i}km_{xx} - 3k^2 m_x.$$

In order to eliminate the second-order and first-order poles, we should write

$$m_t + m_{xxx} + 3\mathrm{i}km_{xx} - 3k^2 m_x = \mu D_x^2 m + \nu D_x m + \rho m \tag{9.66}$$

with some yet unknown functions μ, ν, and ρ that do not depend on k. Inserting the series expansion of m in (9.66) and equating terms with equal powers of k^{-1}, we find

$$\mu = 0, \quad \nu = 3\mathrm{i}m_{1x} = -(3/2)u, \quad \rho = 3\mathrm{i}m_{1xx} - 3m_{2x} - \mathrm{i}\nu m_1.$$

Terms with k^{-2} in (9.65) give $\mathrm{i}m_{1y} = m_{1xx} + 2\mathrm{i}m_{2x} + um_1$; therefore, we can write ρ as

$$\rho = \frac{3}{2}\mathrm{i}m_{1xx} - \frac{3}{2}m_{1y} = -\frac{3}{4}u_x - \frac{3}{4}\mathrm{i}\partial_x^{-1}u_y.$$

Hence,

$$m_t + m_{xxx} + 3\mathrm{i}km_{xx} - 3k^2 m_x + \frac{3}{2}um_x + \frac{3}{2}\mathrm{i}kum + \frac{3}{4}u_x m + \frac{3}{4}\mathrm{i}\partial_x^{-1}u_y m = 0. \tag{9.67}$$

Performing a scaling transformation $y \to -y$, $t \to 4t$, equations (9.65) and (9.67) take the form

$$\mathrm{i}m_y + m_{xx} + 2\mathrm{i}km_x + um = 0, \tag{9.68}$$

$$m_t + 4m_{xxx} + 12\mathrm{i}km_{xx} - 12k^2 m_x + 6um_x + 6\mathrm{i}kum + 3u_x m - 3\mathrm{i}\left(\partial_x^{-1}u_y\right)m = 0. \tag{9.69}$$

It can be shown that the compatibility condition for this system of linear equations gives precisely the KP I equation:

$$(u_t + 6uu_x + u_{xxx})_x = 3u_{yy}. \tag{9.70}$$

Hence, equations (9.68) and (9.69) constitute the Lax representation for the KP I equation with the spectral parameter k [133]. Moreover, the KP I equation follows immediately from (9.69) if we express terms with k by means of (9.68), expand (9.69) in k^{-1}, and account for the relation $m_{1x} = (\mathrm{i}/2)u$. An alternative derivation of the Lax representation for the KP equation is given in Sect. 7.5. In what follows we consider the function $u(x, y, t)$ to be real, nonsingular, and decaying rationally at infinity.

9.2.2 Eigenfunctions and eigenvalues

As for the BO equation, we start with the determination of the Green function for the spectral problem (9.68). By the standard Fourier analysis we get

$$G(x, y, k) = \frac{1}{4\pi^2}\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty} \mathrm{d}\xi\mathrm{d}\eta \frac{\mathrm{e}^{\mathrm{i}(\xi x + \eta y)}}{\eta + \xi(\xi + 2k)}. \tag{9.71}$$

It is seen from (9.71) that $G(x,y,k)$ has a discontinuity across the real axis $\mathrm{Im}k=0$ of the complex k-plane. Taking $k=k_{\mathrm{R}}\pm\mathrm{i}0$, we will have two functions

$$G_{\pm}(x,y,k)=\frac{\mathrm{i}}{2\pi}\int_{-\infty}^{\infty}\mathrm{d}\xi\exp\left[\mathrm{i}\xi x-\mathrm{i}\xi(\xi+2k)y\right]\left[\theta(y)\theta(\mp\xi)-\theta(-y)\theta(\pm\xi)\right], \tag{9.72}$$

which allow analytic continuation in the half planes $\mathrm{Im}k\gtrless 0$. As before, $\theta(\xi)$ stands for the Heaviside step function.

According to the fact that the spectral equation (9.68) with zero potential has two linearly independent solutions,

$$m_{01}=1,\quad m_{02}=\exp[\mathrm{i}\beta(x,y,k,\ell)],\quad \beta(x,y,k,\ell)=(\ell-k)x-(\ell^2-k^2)y, \tag{9.73}$$

where ℓ is an additional parameter, we can build two pairs of eigenfunctions of the spectral equation, $M_{\pm}(x,y,k)$ and $N_{\pm}(x,y,k,\ell)$ [157]. With account for (9.73), these eigenfunctions obey the following inhomogeneous integral equations:

$$(\mathcal{G}_{\pm}M_{\pm})(x,y,k)=1, \tag{9.74}$$

$$(\mathcal{G}_{\pm}N_{\pm})(x,y,k,\ell)=\mathrm{e}^{\mathrm{i}\beta(x,y,k,\ell)}, \tag{9.75}$$

where the operator $\mathcal{G}_{\pm}$ acts as

$$(\mathcal{G}_{\pm}F)(x,y,k,\ell)=F(x,y,k,\ell) \tag{9.76}$$

$$-\iint_D\mathrm{d}x'\mathrm{d}y'G_{\pm}(x-x',y-y',k)u(x',y')F(x',y',k,\ell).$$

Integration in (9.76) is performed over the two-dimensional region D from $-\infty$ to $+\infty$ with respect to both variables. It should be remarked that the order of integration is important when the potential $u(x,y)$ is not absolutely integrable. We adopt the rule that the first integration is in x, i.e.,

$$\iint_D\mathrm{d}x\mathrm{d}y=\int_{-\infty}^{\infty}\mathrm{d}y\int_{-\infty}^{\infty}\mathrm{d}x.$$

Eigenfunctions $M_{\pm}$ allow analytic continuation in the half planes $\mathrm{Im}k\gtrless 0$, while eigenfunctions $N_{\pm}$ are in general nonanalytic for $\mathrm{Im}k\neq 0$.

Bound states of the spectral equation (9.68) are given by solutions $\Phi_j^{\pm}(x,y)$, decaying at infinity, of the homogeneous Fredholm equations

$$(\mathcal{G}_{\pm}\Phi_j^{\pm})(x,y,k_j^{\pm})=0,\quad G_{\pm}(x,y,k_j^{\pm})=\frac{1}{4\pi^2}\int\limits_{-\infty-\infty}^{\infty}\int\limits^{\infty}\mathrm{d}\xi\mathrm{d}\eta\frac{\mathrm{e}^{\mathrm{i}(\xi x+\eta y)}}{\eta+\xi(\xi+2k_j^{\pm})} \tag{9.77}$$

for a set of isolated complex values $k_j^{\pm}$, $j=1,\ldots,\mathcal{N}$. Eigenvalues $k_j^{\pm}$ comprise the discrete spectrum of (9.68). For real potentials $k_j^{\pm}$ appear in

pairs, $k_j^- = \bar{k}_j^+$. The asymptotic behavior of the Green function for $r = (x^2+y^2)^{1/2} \to \infty$ follows from (9.72) and has the form

$$G_\pm(x,y,k) \to \pm\frac{1}{x-2(k\pm \mathrm{i}0)y} + \mathcal{O}(r^{-2}). \tag{9.78}$$

Inserting (9.78) into (9.77) and putting $k \to k_j^\pm$, we get asymptotics of the discrete eigenfunctions:

$$\Phi_j^\pm(x,y) \to \pm\frac{\mathrm{i}}{x-2k_j^\pm y}\frac{1}{2\pi\mathrm{i}}\iint_D \mathrm{d}x'\mathrm{d}y' u(x',y')\Phi_j^\pm(x',y').$$

Let us introduce the notation

$$Q(k_j^\pm) = \pm\frac{1}{2\pi\mathrm{i}}\iint_D \mathrm{d}x\mathrm{d}y u(x,y)\Phi_j^\pm(x,y). \tag{9.79}$$

Then it is natural to normalize $\Phi_j^\pm$ by the condition

$$-\mathrm{i}(x-2k_j^\pm y)\Phi_j^\pm(x,y) \to Q(k_j^\pm) \quad \text{at} \quad r\to\infty. \tag{9.80}$$

A particular value of Q depends on the pole structure of the meromorphic functions $M_\pm$. With this in mind we will call $Q(k_j^\pm)$ the index of pole $k_j^\pm$ [14]. Besides, suppose that (9.77) has a unique solution, i.e., $\dim\ker \mathcal{G}_\pm(k_j^\pm) = 1$.

Now we introduce the adjoint operator $\mathcal{G}^\dagger$ and the adjoint integral equation

$$(\mathcal{G}^\dagger\chi)(x,y,k) \equiv \chi(x,y) - \iint_D \mathrm{d}x'\mathrm{d}y'\bar{G}(x'-x,y'-y,k)u(x,y)\chi(x',y') = 1. \tag{9.81}$$

In the differential form the adjoint spectral problem is written as

$$(-\mathrm{i}\partial_y + \partial_{xx} + u - 2\mathrm{i}\bar{k}\partial_x)\chi = 0. \tag{9.82}$$

In virtue of the symmetry $\bar{G}(x,y,\bar{k}) = G(-x,-y,k)$, it follows from (9.81) that

$$\ker\mathcal{G}^\dagger(k_j^\pm) = u(x,y)\ker\mathcal{G}(k_j^\mp). \tag{9.83}$$

This means that $\Phi_j^\pm$ will be solutions of the homogeneous equations (9.77) at the points $k_j^\pm$ if $u(x,y)\Phi_j^\pm$ are solutions of the adjoint homogeneous equation at the points $k_j^\mp$. In particular, we obtain

$$\dim\ker\mathcal{G}(k_j^\pm) = \dim\ker\mathcal{G}^\dagger(k_j^\pm) = \dim\ker\mathcal{G}(k_j^\mp).$$

A crucial difference between the spectral problem for KP I and that for the BO equation lies in the fact that (9.68) allows multiple eigenvalues. First we will consider the case of simple eigenvalues.

It follows from analytic properties of $G_\pm$ that eigenfunctions $M_\pm$ are represented in terms of meromorphic functions with simple poles at the points $k_j^\pm$:

$$M_\pm(x,y,k) = 1 + \sum_{j=1}^{\mathcal{N}}(k-k_j^\pm)^{-1}\Phi_j^\pm(x,y) + m_\pm(x,y,k), \tag{9.84}$$

where bound states $\Phi_j^{\pm}$ are residues of $M_{\pm}$ at the points $k_j^{\pm}$ and $m_{\pm}$ are holomorphic functions in $\mathrm{Im}k \gtrless 0$, $m_{\pm} \to 0$ at $|k| \to \infty$. In order to determine residues $\Phi_j^{\pm}$, we need to investigate the behavior of $M_{\pm}$ in the limit $k \to k_j^{\pm}$. Around $k = k_j^-$ we evidently get

$$M_-(x,y,k) = \nu_-(x,y,k) + \frac{\Phi_j^-}{k - k_j^-}, \tag{9.85}$$

where ν_- is regular in k_j^- and tends to 1 at $|k| \to \infty$. Inserting (9.85) into (9.74) gives

$$(\mathcal{G}_-\nu_-)(x,y,k) + (k - k_j^-)^{-1}(\mathcal{G}_-\Phi_j^-)(x,y,k) = 1.$$

Expanding $\mathcal{G}_-$ in the Taylor series around k_j^-, we obtain in the limit $k \to k_j^-$ two integral equations

$$(\mathcal{G}_-\Phi_j^-)(x,y,k_j^-) = 0, \quad (\mathcal{G}_-\nu_-)(x,y,k_j^-) + \left(\frac{\partial \mathcal{G}_-}{\partial k}\Phi_j^-\right)(x,y,k_j^-) = 1. \tag{9.86}$$

The first equation is satisfied owing to (9.77). The derivative of the Green function follows from (9.72):

$$\frac{\partial G_-}{\partial k}(x,y,k_j^-) = -\mathrm{i}(x - 2k_j^- y)G_-(x,y,k_j^-) + \frac{1}{2\pi\mathrm{i}}. \tag{9.87}$$

Substituting it into the second equation in (9.86) transforms this equation to

$$\left\{\mathcal{G}_-\left[\nu_- + \mathrm{i}(x - 2k_j^- y)\Phi_j^-\right]\right\} = 1 - Q(k_j^-). \tag{9.88}$$

If $Q(k_j^-) = 1$ (the so-called *normalization constraint*), it follows from (9.86) and (9.88) that in virtue of $\dim\ker\mathcal{G} = 1$ we have

$$\nu_- + \mathrm{i}(x - 2k_j^- y)\Phi_j^- = -\mathrm{i}\gamma_j^-\Phi_j^-,$$

where γ_j^- is a proportionality constant. As a result, we get [compare the similar formula (9.29) for the BO equation]

$$\lim_{k\to k_j^-}\left(M_- - \frac{\Phi_j^-}{k - k_j^-}\right) = -\mathrm{i}\xi_j^-\Phi_j^-, \qquad \xi_j^- = x - 2k_j^- y + \gamma_j^-. \tag{9.89}$$

An analogous formula in terms of $\xi_j^+ = x - 2k_j^+ + \gamma_j^+$ exists for $k \to k_j^+$.

9.2.3 Scattering equation and closure relations

To formulate the RH problem, we derive here the scattering equation and closure (symmetry) relations. For this purpose we first calculate a jump $\Delta = M_+ - M_-$ of eigenfunctions across the real k-axis:

$$\Delta(x,y,k) = \iint_D \mathrm{d}x'\mathrm{d}y'(G_+ - G_-)(x-x',y-y',k)u(x',y')M_+(x',y',k)$$

$$+ \iint_D \mathrm{d}x'\mathrm{d}y'G_-(x-x',y-y',k)u(x',y')\Delta(x',y',k), \quad \operatorname{Im} k = 0.$$

Equation (9.72) gives

$$(G_+ - G_-)(x,y,k) = \frac{1}{2\pi\mathrm{i}} \int_{-\infty}^{\infty} \mathrm{d}\ell \operatorname{sign}(\ell - k)\mathrm{e}^{\mathrm{i}\beta(x,y,k,\ell)},$$

where we put $\xi + k = \ell$. Then

$$\Delta(x,y,k) = \int_{-\infty}^{\infty} \mathrm{d}\ell \operatorname{sign}(\ell-k)T(k,\ell)\mathrm{e}^{\mathrm{i}\beta(x,y,k,\ell)} \tag{9.90}$$

$$+ \iint_D \mathrm{d}x'\mathrm{d}y'G_-(x-x',y-y',k)u(x',y')\Delta(x',y',k),$$

where

$$T(k,\ell) = \frac{1}{2\pi\mathrm{i}} \iint_D \mathrm{d}x\mathrm{d}y\mathrm{e}^{-\mathrm{i}\beta(x,y,k,\ell)}u(x,y)M_+(x,y,k). \tag{9.91}$$

Let us multiply N_- (9.75) by $\operatorname{sign}(\ell-k)T(k,\ell)$ and integrate in ℓ:

$$\int_{-\infty}^{\infty} \mathrm{d}\ell \operatorname{sign}(\ell-k)T(k,\ell)N_-(x,y,k,\ell) = \int_{-\infty}^{\infty} \mathrm{d}\ell \operatorname{sign}(\ell-k)T(k,\ell)\mathrm{e}^{\mathrm{i}\beta(x,y,k,\ell)}$$

$$+ \int_{-\infty}^{\infty} \mathrm{d}\ell \operatorname{sign}(\ell-k)T(k,\ell) \iint_D \mathrm{d}x'\mathrm{d}y'G_-(x-x',y-y',k)u(x',y')N_-(x',y',k,\ell). \tag{9.92}$$

Comparing (9.90) and (9.92) and taking into account $\dim\ker\mathcal{G}_- = 1$, we obtain the scattering equation

$$M_+(x,y,k) - M_-(x,y,k) = \int_{-\infty}^{\infty} \mathrm{d}\ell \operatorname{sign}(\ell-k)T(k,\ell)N_-(x,y,k,\ell), \quad \operatorname{Im} k = 0. \tag{9.93}$$

We will be closer to the formulation of the RH problem for the KP I equation if we find a relation between the eigenfunctions M_- and N_- (the closure relation). It follows from (9.75) that

$$\left(\hat{\mathcal{G}}_- \hat{N}_-\right)(x,y,k,\ell) = \mathrm{e}^{\mathrm{i}\ell x - \mathrm{i}\ell^2 y}, \tag{9.94}$$

where

$$\hat{G}_- = G_- \mathrm{e}^{\mathrm{i}kx - \mathrm{i}k^2 y}, \qquad \hat{N}_- = N_- \mathrm{e}^{\mathrm{i}kx - \mathrm{i}k^2 y}.$$

Differentiation of (9.94) in k with account for (9.87) gives [157]

$$\left(\hat{\mathcal{G}} \frac{\partial}{\partial k} \hat{N}_-\right)(x, y, k, \ell) = -F(k, \ell) \mathrm{e}^{\mathrm{i}kx - \mathrm{i}k^2 y}, \tag{9.95}$$

where

$$F(k, \ell) = \frac{i}{2\pi} \iint_D \mathrm{d}x\mathrm{d}y u(x, y) N_-(x, y, k, \ell). \tag{9.96}$$

Multiply now (9.74) for M_- by $-F(k, \ell)\mathrm{e}^{\mathrm{i}kx - \mathrm{i}k^2 y}$ and compare it with (9.95). As a result, we get the closure relation

$$\frac{\partial N_-}{\partial k}(x, y, k, \ell) + \mathrm{i}(x - 2ky) N_-(x, y, k, \ell) = -F(k, \ell) M_-(x, y, k). \tag{9.97}$$

In the integral form, taking into account the boundary condition $N_-(x, y, k, k) = M_-(x, y, k)$, relation (9.97) has the form [3]

$$N_-(x, y, k, \ell) = M_-(x, y, \ell)\mathrm{e}^{\mathrm{i}\beta(x,y,k,\ell)} - \int_\ell^k \mathrm{d}p F(p, \ell) M_-(x, y, p) \mathrm{e}^{\mathrm{i}\beta(x,y,k,p)}. \tag{9.98}$$

The spectral transforms $T(k, \ell)$ (9.91) and $F(k, \ell)$ (9.96) are not independent. Indeed, let us multiply $N_-(x, y, k, \ell)$ (9.75) by $u(x, y)\bar{M}_+(x, y, k)$ and integrate in x and y:

$$\iint_D \mathrm{d}x\mathrm{d}y u(x, y) \bar{M}_+(x, y, k) N_-(x, y, k, \ell) = \iint_D \mathrm{d}x\mathrm{d}y u(x, y) \bar{M}_+(x, y, k) \mathrm{e}^{\mathrm{i}\beta(x,y,k,\ell)}$$

$$+ \iint_D \mathrm{d}x\mathrm{d}y u(x,y) \bar{M}_+(x,y,k) \iint_D \mathrm{d}x'\mathrm{d}y' G_-(x - x', y - y', k) u(x', y') N_-(x', y', k, \ell).$$

Inserting $\bar{M}_+$ (9.74) into the second line of this formula, accounting for the symmetry $G_\pm(x, y, k) = \bar{G}_\mp(-x, -y, k)$ for real k and the definitions (9.91) and (9.96), we get

$$T(k, \ell) = \bar{F}(k, \ell). \tag{9.99}$$

9.2.4 RH problem

Formulas (9.89), (9.93), and (9.98) make it possible to formulate the RH problem for the KP I equation. Indeed, substituting N_- (9.98) into the right-hand side of (9.93), we obtain after some manipulations the nonlocal RH problem [157]

$$M_+(x, y, k) - M_-(x, y, k) = \int_{-\infty}^{\infty} \mathrm{d}\ell f(k, \ell) M_-(x, y, \ell) \mathrm{e}^{\mathrm{i}\beta(x,y,k,\ell)}, \tag{9.100}$$

$$f(k, \ell) = \operatorname{sign}(k - \ell) F(k, \ell)$$

with the standard normalization condition $M_\pm \to 1$ at $|k| \to \infty$. Acting on (9.100) with the projector P_-,

$$(P_- g)(k) = -\frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \frac{\mathrm{d}\ell}{\ell - k + \mathrm{i}0} g(\ell),$$

we obtain a linear integral equation

$$M_-(x,y,k) = 1 + \sum_{j=1}^{\mathcal{N}} \left[(k - k_j^+)^{-1} \Phi_j^+(x,y) + (k - k_j^-)^{-1} \Phi_j^-(x,y) \right] \quad (9.101)$$

$$+ \frac{1}{2\pi \mathrm{i}} \iint_D \frac{\mathrm{d}\ell \mathrm{d}p}{p - k + \mathrm{i}0} f(p,\ell) M_-(x,y,\ell) \mathrm{e}^{\mathrm{i}\beta(x,y,p,\ell)}, \qquad \operatorname{Im} k = 0.$$

The limit $k \to k_j^\pm$ gives with account for (9.89)

$$-\mathrm{i}\xi_j^\pm \Phi_j^\pm(x,y) = 1 + {\sum_{i=1}^{\mathcal{N}}}' \left[(k_j^\pm - k_i^+)^{-1} \Phi_i^+(x,y) + (k_j^\pm - k_i^-)^{-1} \Phi_i^-(x,y) \right]$$
$$+ \frac{1}{2\pi \mathrm{i}} \iint_D \frac{\mathrm{d}\ell \mathrm{d}p}{p - k_j^\pm} f(p,\ell) M_-(x,y,\ell) \mathrm{e}^{\mathrm{i}\beta(x,y,p,\ell)}. \quad (9.102)$$

The prime near the sign symbol means that terms with zero denominators are excluded. Equations (9.101) and (9.102) comprise a complete system of linear equations for finding eigenfunctions M_- and $\Phi_j^\pm$ in terms of the RH data $[f(k,\ell);\ k_j^\pm, \gamma_j^\pm, j = 1,\dots,\mathcal{N}]$ [remember that $\gamma_j^\pm$ enter $\xi_j^\pm$ (9.89)]. Hence, the potential $u(x,y)$ is reconstructed as

$$u(x,y) = \partial_x \left(-2\mathrm{i} \sum_{j=1}^{\mathcal{N}} (\Phi_j^+ + \Phi_j^-) + \frac{1}{\pi} \iint_D \mathrm{d}k \mathrm{d}\ell f(k,\ell) M_-(x,y,\ell) \mathrm{e}^{\mathrm{i}\beta(x,y,k,\ell)} \right). \quad (9.103)$$

A distinctive role of the nonanalytic eigenfunctions $N_\pm$ should be especially emphasized. As shown in [363], they are the functions $N_\pm$ having the additional parameter ℓ that comprise, along with the bound states $\Phi_j^\pm$, a complete set of functions. In particular, the potential is expressed in terms of the complete set of eigenfunctions as

$$u(x,y) = \partial_x \left(-2\mathrm{i} \sum_{j=1}^{\mathcal{N}} (\Phi_j^+ + \Phi_j^-) + \frac{1}{\pi} \iint_D \mathrm{d}k \mathrm{d}\ell \operatorname{sign}(k - \ell) T(k,\ell) N_-(x,y,k,\ell) \right).$$

9.2.5 Evolution of RH data

Evolution equations for the RH data are found, as usual, from (9.69). Substituting (9.100) into (9.69), we obtain the evolution equation for $f(k,\ell)$:

$f_t(k,\ell) = 4\mathrm{i}(\ell^3 - k^3) f(k,\ell)$. Therefore, the time dependence of the continuous RH data is given by the simple formula

$$f(k,\ell,t) = f(k,\ell,0)\exp\left[4\mathrm{i}(\ell^3 - k^3)t\right]. \tag{9.104}$$

To get discrete data evolution, we insert (9.84) with $m_\pm = 0$ into (9.69). It follows in the limit $k \to k_j^\pm$ that

$$\partial_t k_j^\pm = 0. \tag{9.105}$$

Finally, in the same limit, equation (9.89) gives

$$\gamma_j^\pm(t) = \gamma_j^\pm(0) + 12(k_j^\pm)^2 t. \tag{9.106}$$

9.2.6 Soliton solution

Algebraic solitons (lumps) of the KP I equation correspond to "reflectionless" potentials when $f(k,\ell) = T(k,\ell) = 0$. The $\mathcal{N}$-soliton solution is reconstructed from (9.103) as

$$u_{\mathcal{N}}(x,y) = -2\mathrm{i}\partial_x \sum_{j=1}^{\mathcal{N}} \left[\Phi_j^+(x,y,t) + \Phi_j^-(x,y,t)\right], \tag{9.107}$$

where $\Phi_j^\pm$ are found from the system of linear algebraic equations [see (9.102)]

$$1 + \mathrm{i}\xi_j^\pm \Phi_j^\pm + \sum_{i=1}^{\mathcal{N}}{}' \left[(k_j^\pm - k_i^+)^{-1}\Phi_i^+ + (k_j^\pm - k_i^-)^{-1}\Phi_i^-\right] = 0. \tag{9.108}$$

In particular, equation (9.108) gives for $\mathcal{N} = 1$

$$\Phi_1^\pm = \frac{\mathrm{i}}{\Delta}\left(x - 2k_1^\mp y + 12(k_1^\mp)^2 t + \gamma_1^\mp(0) \pm \frac{\mathrm{i}}{k_1^+ - k_1^-}\right),$$

$$\Delta = (X - 2k_\mathrm{R}Y)^2 + 4k_\mathrm{I}^2 Y^2 + \frac{1}{4k_\mathrm{I}^2}.$$

The notations are

$$X = x - 12(k_\mathrm{R}^2 + k_\mathrm{I}^2)t - x_0, \quad Y = y - 12k_\mathrm{R}t - y_0, \quad k_1^\pm = k_\mathrm{R} \pm \mathrm{i}k_\mathrm{I}, \tag{9.109}$$

$$x_0 = \frac{k_\mathrm{R}\gamma_\mathrm{I} - \gamma_\mathrm{R}k_\mathrm{I}}{k_\mathrm{I}}, \quad y_0 = \frac{\gamma_\mathrm{I}}{2k_\mathrm{I}}, \quad \gamma_1^\pm(0) = \gamma_\mathrm{R} \pm \mathrm{i}\gamma_\mathrm{I}.$$

With $\Phi_1^\pm$ found, we obtain from (9.107) the one-lump solution:

$$u_1(x,y,t) = \partial_x^2 \ln\Delta = 4\frac{-(X - 2k_\mathrm{R}Y)^2 + 4k_\mathrm{I}^2 Y^2 + 1/4k_\mathrm{I}^2}{[(X - 2k_\mathrm{R}Y)^2 + 4k_\mathrm{I}^2 Y^2 + 1/4k_\mathrm{I}^2]^2}. \tag{9.110}$$

This solution describes a smooth weakly localized (decaying as r^{-2}) configuration which moves uniformly with velocity $\mathbf{v} = (v_x, v_y) = [12(k_{\mathrm{R}}^2 + k_{\mathrm{I}}^2), 12k_{\mathrm{R}}]$. For $2\mathcal{N}$ noncoinciding eigenvalues $k_j^{\pm}$, $j = 1, \ldots, \mathcal{N}$, the $\mathcal{N}$-lump solution can be compactly written in the form [354]

$$u_{\mathcal{N}}(x, y, t) = 2\partial_x^2 \ln \det\, B, \tag{9.111}$$

where entries B_{mn} of the $2\mathcal{N} \times 2\mathcal{N}$ matrix B are given by

$$B_{mn} = (x - 2k_m y + \gamma_m)\delta_{mn} - \mathrm{i}(1 - \delta_{mn})(k_m - k_n)^{-1}$$

and the eigenvalues and normalization factors are arranged as

$$(k_1^+, \ldots, k_{\mathcal{N}}^+, k_1^-, \ldots, k_{\mathcal{N}}^-) \quad \text{and} \quad (\gamma_1^+, \ldots, \gamma_{\mathcal{N}}^+, \gamma_1^-, \ldots, \gamma_{\mathcal{N}}^-).$$

The solution (9.111) describes a process of collision of $\mathcal{N}$ lumps. It can be shown [306] that (9.111) is decomposed into a sum of $\mathcal{N}$ one-lump solutions for $t \to \pm\infty$. It is important that phase shifts of lumps stemming from their mutual interaction are zero. This means that lump interaction is trivial.

9.2.7 KP I equation—multiple poles

Following Ablowitz and Villarroel [14, 439], consider here the case of a purely discrete spectrum but assume that eigenfunctions can have multiple poles. When a continuous spectrum is absent, the solutions $M_{\pm}$ of the RH problem are the same, $M_+ = M_- = M$. Suppose $M(x, y, k)$ corresponds to the purely discrete spectrum of the problem (9.68) and has $2\mathcal{N}$ poles $k_j^{\pm}$, $j = 1, \ldots, \mathcal{N}$ with multiplicities $r_j^{\pm}$:

$$M(x, y, k) = 1 + \sum_{m=1}^{2\mathcal{N}} \left[\frac{\Phi_m(x, y)}{k - k_m} + \sum_{r=2}^{r_m} \frac{\Psi_{m,r}(x, y)}{(k - k_m)^r} \right]. \tag{9.112}$$

Here $\Phi_m = \Phi_m^+$, $m = 1, \ldots, \mathcal{N}$ and $\Phi_m = \Phi_m^-$ for $m = \mathcal{N} + 1, \ldots, 2\mathcal{N}$. To have explicitly the function M, we need know the Laurent coefficients Φ_m and $\Psi_{m,r}$. Therefore, we derive first of all equations for them. Around the pole k_m we have

$$M(x, y, k) = \nu(x, y, k) + \frac{\Phi_m}{k - k_m} + \sum_{r=2}^{r_m} \frac{\Psi_{m,r}}{(k - k_m)^r}, \tag{9.113}$$

where $\nu(x, y, k)$ is a regular part of M, and $\nu \to 1$ at $|k| \to \infty$. Now we insert (9.113) into the equation $(\mathcal{G}M)(x, y, k) = 1$. Expanding the Green function up to the r_mth order around k_m and equating terms with equal powers of $(k - k_m)^{-r}$, we obtain a system of integral equations

$$
\begin{aligned}
&(\mathcal{G}\Psi_{m,r_m})(x,y,k_m)=0\,,\\
&\left(\mathcal{G}\Psi_{m,r_m-1}+\frac{\partial\mathcal{G}}{\partial k}\Psi_{m,r_m}\right)(x,y,k_m)=0,\\
&\vdots\\
&\left(\mathcal{G}\Phi_m+\frac{\partial\mathcal{G}}{\partial k}\Psi_{m,2}+\ldots+\frac{1}{(m-1)!}\frac{\partial^{r_m-1}\mathcal{G}}{\partial k^{r_m-1}}\Psi_{m,r_m}\right)(x,y,k_m)=0,\\
&\left(\mathcal{G}\nu+\frac{\partial\mathcal{G}}{\partial k}\Phi_m+\frac{1}{2}\frac{\partial^2\mathcal{G}}{\partial k^2}\Psi_{m,2}+\ldots+\frac{1}{r_m!}\frac{\partial^{r_m}\mathcal{G}}{\partial k^{r_m}}\right)(x,y,k_m)=1.
\end{aligned}
\tag{9.114}
$$

The first equation in this system shows that the poles k_m are indeed the discrete data of the RH problem , while Ψ_{m,r_m} are eigenfunctions corresponding to the eigenvalues k_m. It should be noted that equations (9.86) make up a particular case of the system (9.114).

As in Sect. 9.2.1, we define the indices of poles

$$
Q(k_m)=\frac{1}{2\pi\mathrm{i}}\operatorname{sign}(\operatorname{Im}k_m)\iint_D \mathrm{d}x\mathrm{d}y\,u\Phi_m.
$$

For bound-state eigenfunctions Φ_m the orthogonality conditions are fulfilled:

$$
\langle\Phi_\ell^-,\Phi_j^+\rangle=0,\qquad j\neq\ell,
\tag{9.115}
$$

where the scalar product is defined as

$$
\langle f,g\rangle=\frac{1}{\pi}\iint_D \mathrm{d}x\mathrm{d}y\,\bar{f}_x g.
\tag{9.116}
$$

The orthogonality conditions are easily obtained from the spectral equation (9.68) for Φ_j^+ by multiplying it by $\bar{\Phi}_\ell^-$ and integrating by parts. The indices of the poles can be expressed by means of this scalar product:

$$
\begin{aligned}
Q(k_j^+)&=\frac{1}{2\pi\mathrm{i}}\iint_D \mathrm{d}x\mathrm{d}y\,u\Phi_j^+=-\frac{1}{\pi}\iint_D \mathrm{d}x\mathrm{d}y\sum_{m=1}^{2\mathcal{N}}(\partial_x\Phi_m)\Phi_j^+\\
&=\frac{1}{\pi}\iint_D \mathrm{d}x\mathrm{d}y\sum_{m=1}^{2\mathcal{N}}(\partial_x\bar{\Phi}_m)\Phi_j^+=\sum_{m=1}^{2\mathcal{N}}\langle\Phi_m,\Phi_j^+\rangle=\langle\Phi_j^-,\Phi_j^+\rangle.
\end{aligned}
$$

Because $\langle f,g\rangle=-\langle g,f\rangle$, we get the important equality

$$
Q(k_j^+)=Q(k_j^-).
\tag{9.117}
$$

In what follows we shall restrict our consideration to the simplest example of a double pole. In other words, we assume that the eigenfunction $M(x,y)$ has the following structure:

$$
M(x,y,k)=1+\frac{\Phi_1^+}{k-k_1^+}+\frac{\Phi_1^-}{k-k_1^-}+\frac{\Psi_2}{(k-k_1^+)^2}.
\tag{9.118}
$$

For determination of the Laurent coefficients $\Phi_1^{\pm}$ and Ψ_2 we consider first the function M near $k = k_1^+$:

$$M(x,y,k) = \nu_+(x,y,k) + \frac{\Phi_1^+(x,y)}{k-k_1^+} + \frac{\Psi_2}{(k-k_1^+)^2}, \tag{9.119}$$

where ν_+ is regular in k_1^+ and tends to 1 at $|k| \to \infty$. In this case the system (9.114) is reduced to three equations

$$(\mathcal{G}\Psi_2)(x,y,k_1^+) = 0, \tag{9.120}$$

$$\left(\mathcal{G}\Phi_1^+ + \frac{\partial \mathcal{G}}{\partial k}\Psi_2\right)(x,y,k_1^+) = 0, \tag{9.121}$$

$$\left(\mathcal{G}\nu_+ + \frac{\partial \mathcal{G}}{\partial k}\Phi_1^+ + \frac{1}{2}\frac{\partial^2 \mathcal{G}}{\partial k^2}\Psi_2\right)(x,y,k_1^+) = 1. \tag{9.122}$$

In virtue of the similarity with equations (9.86), we can invoke the result (9.89) and write by analogy

$$\Phi_1^+ + \mathrm{i}\xi_1^+\Psi_2 = 0. \tag{9.123}$$

As regards (9.122), we need the second derivative of the Green function that is obtained from (9.87):

$$\frac{\partial^2 G_+}{\partial k^2} = \left[2y - (x-2ky)^2\right] G_+ + \frac{x-2ky}{2\pi}. \tag{9.124}$$

Substituting (9.87) and (9.124) into (9.122), we get

$$\left(\mathcal{G}\left[\nu_+ + \mathrm{i}(x-2k_1^+y)\Phi_1^+ - \left(2\mathrm{i}y + (x-2k_1^+y)^2\right)\Psi_2\right]\right)(x,y,k_1^+)$$

$$= 1 - Q(k_1^+) - \frac{1}{2}q_2 + \frac{1}{2}(x-2k_1^+y)Q_2, \tag{9.125}$$

where

$$Q_2 = \frac{1}{2\pi\mathrm{i}}\iint_D \mathrm{d}x\mathrm{d}y\, u(x,y)\Psi_2(x,y),\ q_2 = \frac{1}{2\pi}\iint_D \mathrm{d}x\mathrm{d}y(x-2k_1^+y)u(x,y)\Psi_2(x,y). \tag{9.126}$$

Now we prove that $Q_2 = 0$ and $q_2 = -Q(k_1^+)$. Indeed, if $\Phi_1^+ \sim r^{-\alpha_1}$ and $\Psi_2 \sim r^{-\alpha_2}$, then it follows from (9.121) written in a differential form as

$$(\mathrm{i}\partial_y + \partial_x^2 + 2\mathrm{i}k_1^+\partial_x + u)\Phi_1^+ + 2\mathrm{i}\partial_x\Psi_2 = 0$$

that $\alpha_2 = \alpha_1 + 1$. Further, by analogy with the one-lump solution (9.110) we suppose that $u \sim \mathcal{O}(r^{-2})$ at $r \to \infty$. Then $\alpha_1 = 1$. Hence, performing integration for Q_2 in (9.126), we obtain in virtue of the decay rate of Ψ_2

$$Q_2 = -\frac{1}{2\pi\mathrm{i}}\iint_D \mathrm{d}x\mathrm{d}y(\mathrm{i}\partial_y + \partial_x^2 + 2\mathrm{i}k_1^+\partial_x)\Psi_2 = 0.$$

Taking then $(x - 2k_1^+ y)\Psi_2$ from (9.123) and inserting it into (9.125) gives $q_2 = -Q$. Eventually, eliminating Φ_1^+ in (9.125) by means of (9.123), we can write (9.122) as

$$\left[\mathcal{G}\left(\nu^+ + \frac{1}{2}(\xi_1^{+2} - 2\mathrm{i}y + \gamma_2^+)\Psi_2\right)\right](x, y, k_1^+) = 1 - \frac{1}{2}Q(k_1^+), \quad \gamma_2^+ = \gamma_2^+(t). \tag{9.127}$$

The inhomogeneous integral equation (9.127) will have a solution if the right-hand side is orthogonal to the complex conjugated solution $\bar{\chi}$ of the adjoint integral equation (9.81) (the Fredholm condition [425]). In the case of the second-order pole in k_1^+ we get $\bar{\chi}(x, y, k_1^+) = u(x, y)\Phi_1^+(x, y)$, in accordance with (9.83). As a result, the Fredholm condition provides

$$0 = \left(1 - \frac{1}{2}Q(k_1^+)\right) \iint_D \mathrm{d}x\mathrm{d}y\bar{\chi}$$

$$= \left(1 - \frac{1}{2}Q(k_1^+)\right) \iint_D \mathrm{d}x\mathrm{d}y u(x, y)\Phi_1^+(x, y) \sim \left(1 - \frac{1}{2}Q(k_1^+)\right) Q(k_1^+).$$

Because we assume $Q(k_1^+) \neq 0$, this gives $Q(k_1^+) = 2$.

Now we will treat the simple pole k_1^-, taking into account that $Q(k_1^-) = Q(k_1^+) \equiv Q = 2$. Near k_1^- we write

$$M(x, y, k) = \nu_-(x, y, k) + \frac{\Phi_1^-}{k - k_1^-}, \qquad (\mathcal{G}M)(x, y, k) = 1, \tag{9.128}$$

and ν_- is regular in k_1^- and tends to 1 as $|k| \to \infty$. Let us remember that for simple pole k_1^- equations (9.86) exist, the second one of them being transformed to (9.88). Because $Q \neq 1$, equation (9.89) is not valid and cannot be used to determine Φ_1^-.

In order to have analog of (9.89) for $Q \neq 1$, we differentiate $(\mathcal{G}M) = 1$ (9.128) in k:

$$\left(\frac{\partial \mathcal{G}}{\partial k}\nu_-\right) + \left(\mathcal{G}\frac{\partial \nu_-}{\partial k}\right) + \frac{1}{k - k_1^-}\left(\frac{\partial \mathcal{G}}{\partial k}\Phi_1^-\right) - \frac{1}{(k - k_1^-)^2}\left(\mathcal{G}\Phi_1^-\right) = 0.$$

Expanding $\mathcal{G}$ and $(\partial\mathcal{G}/\partial k)$ near k_1^- and collecting terms with equal powers of $(k - k_1^-)^{-1}$, we obtain the following integral equations:

$$(\mathcal{G}\Phi_1^-)(x, y, k_1^-) = 0, \quad \left(\frac{\partial \mathcal{G}}{\partial k}\nu_- + \mathcal{G}\frac{\partial \nu_-}{\partial k} + \frac{1}{2}\frac{\partial^2 \mathcal{G}}{\partial k^2}\Phi_1^-\right)_{k_1^-} = 0.$$

Taking into account explicit formulas (9.87) and (9.124) for derivatives of the Green function, we get

$$\left[\mathcal{G}\left(\frac{\partial \nu_-}{\partial k} + \mathrm{i}(x - 2k_1^- y)\nu_- - \frac{1}{2}(2\mathrm{i}y + (x - 2k_1^-)^2)\Phi_1^-\right)\right](x, y, k_1^-)$$

$$= \mathrm{i}(x - 2k_1^- y)\left(1 - \frac{1}{2}Q\right) - q_0 - \frac{1}{2}\tilde{\eta}, \tag{9.129}$$

where

$$q_0 = \frac{-1}{2\pi \mathrm{i}} \iint_D \mathrm{d}x\mathrm{d}y u(x,y)\nu_-(x,y),\ \tilde{\eta} = \frac{-1}{2\pi} \iint_D \mathrm{d}x\mathrm{d}y (x - 2k_1^- y)u(x,y)\Phi_1^-(x,y).$$

On the other hand, equation (9.88) gives for $Q = 2$

$$\left\{\mathcal{G}\left[\nu_- + \mathrm{i}(x - 2k_1^- y)\Phi_1^-\right]\right\}(x, y, k_1^-) = -1. \tag{9.130}$$

Let us multiply (9.130) by $-[q_0 + (1/2)\tilde{\eta}]$ and add it to (9.129). As a result, we obtain an analog of (9.89) for the case $Q = 2$:

$$\left(\frac{\partial \nu_-}{\partial k} + \mathrm{i}\xi_1^- \nu_- - \frac{1}{2}(2\mathrm{i}y + (\xi_1^-)^2 + \delta^-)\Phi_1^-\right)_{k_1^-} = 0, \quad \delta^- = \delta^-(t). \tag{9.131}$$

Now we have all that is necessary to determine $\Phi_1^{\pm}$ and Ψ_2. Near $k = k_1^+$

$$\nu_+(k) = 1 + \frac{\Phi_1^-}{k - k_1^-}.$$

and (9.123) and (9.127) give the following system of linear algebraic equations:

$$\Phi_1^+ + \mathrm{i}\xi_1^+ \Psi_2 = 0, \quad 1 + (k_1^+ - k_1^-)^{-1}\Phi_1^- + \frac{1}{2}(\xi_1^{+2} - 2\mathrm{i}y + \gamma_2^+)\Psi_2 = 0. \tag{9.132}$$

Eliminating Ψ_2, we obtain a relation between Φ_1^+ and Φ_1^-:

$$1 + \frac{\mathrm{i}}{2\xi_1^+}(\xi_1^{+2} - 2\mathrm{i}y + \gamma_2^+)\Phi_1^+ + \frac{\Phi_1^-}{k_1^+ - k_1^-} = 0. \tag{9.133}$$

Near $k = k_1^-$

$$\nu_-(k) = 1 + \frac{\Phi_1^+}{k - k_1^+} + \frac{\Psi_2}{(k - k_1^+)^2}$$

and (9.131) gives

$$1 + \left(\frac{1}{k_1^- - k_1^+} + \frac{\mathrm{i}}{\xi_1^-}\frac{1}{(k_1^- - k_1^+)^2}\right)\Phi_1^+ + \left(\frac{1}{(k_1^- - k_1^+)^2} + \frac{2\mathrm{i}}{(k_1^- - k_1^+)^3}\right)\Psi_2$$
$$+ \frac{\mathrm{i}}{2\xi_1^-}[(\xi_1^-)^2 + 2\mathrm{i}y + \delta^-]\Phi_1^- = 0. \tag{9.134}$$

Eliminating once again Ψ_2, we obtain the second equation for Φ_1^+ and Φ_1^-:

$$1 + \frac{1}{k_1^- - k_1^+}\left[1 + \mathrm{i}\left(\frac{1}{\xi_1^+} + \frac{1}{\bar{\xi}_1^+}\right)\frac{1}{k_1^- - k_1^+} - \frac{2}{|\xi_1^+|^2}\frac{1}{(k_1^- - k_1^+)^2}\right]\Phi_1^+$$
$$+ \frac{\mathrm{i}}{2\bar{\xi}_1^+}[(\bar{\xi}_1^+)^2 + 2\mathrm{i}y + \delta^-]\Phi_1^- = 0. \tag{9.135}$$

Hence, we have a system of two algebraic equations (9.133) and (9.135) to determine $\Phi_1^{\pm}$ and hence Ψ_2 from (9.132). Expanding M in the asymptotic series in k^{-1} leads to the reconstruction formula

$$u(x,y) = -2\mathrm{i}\partial_x(\Phi_1^+ + \Phi_1^-) = 2\partial_x^2 \ln \Delta. \tag{9.136}$$

To obtain evolution of the parameters $\gamma_1^{\pm}(t)$ and $\delta^-(t)$, we substitute M found above into (9.69) taken at $r \to \infty$. This gives

$$\gamma_1^{\pm}(t) = \gamma + 12k_1^{\pm 2}t, \qquad \delta^-(t) = \delta - 24\mathrm{i}k_1^- t, \qquad \gamma,\ \delta = \text{const}.$$

After rather lengthy but transparent calculations we obtain the function Δ in the form [439]

$$\begin{aligned}
&\Delta(x,y,t) \qquad (9.137)\\
&= \left\{[X - 2k_{\mathrm{R}}Y - 12(k_{\mathrm{R}}^2 - k_{\mathrm{I}}^2)t]^2 - 4k_{\mathrm{I}}^2(Y + 12k_{\mathrm{R}}t)^2 + \delta_{\mathrm{R}}^-(t)\right\}^2\\
&+ \left(2(Y + 12k_{\mathrm{R}}t)\left\{1 + 2k_{\mathrm{I}}[X - 2k_{\mathrm{R}}Y - 12(k_{\mathrm{R}}^2 - k_{\mathrm{I}}^2)t]\right\} + \frac{\gamma_{\mathrm{I}}^+(t)}{k_{\mathrm{I}}} - \delta_{\mathrm{I}}^-(t)\right)^2\\
&+ \frac{1}{k_{\mathrm{I}}^2}\left[\left(X - 2k_{\mathrm{R}}Y - 12(k_{\mathrm{R}}^2 - k_{\mathrm{I}}^2)t - \frac{1}{2k_{\mathrm{I}}}\right)^2 + 4k_{\mathrm{I}}^2(Y + 12k_{\mathrm{R}}t)^2 + \frac{1}{4k_{\mathrm{I}}^2}\right],
\end{aligned}$$

where X and Y were introduced in (9.109). The lump solution of the KP I equation corresponding to the multiple poles (9.118) is given by

$$u_\ell = 2\partial_x^2 \ln \Delta = 2\left[\frac{\Delta_{XX}}{\Delta} - \left(\frac{\Delta_X}{\Delta}\right)^2\right].$$

Note that the existence of indices (topological charges) is stipulated by the fact that the potential decays sufficiently slowly (algebraically) at infinity.

Villarroel and Ablowitz [439] investigated this solution for $t \to \pm\infty$. It was shown that the solution decomposes in this limit into two humps each having its own velocity. Figures 9.1–9.3 illustrate a typical scattering process described by the solution (9.137). The interaction of the lumps can be treated in terms of two-particle dynamics under the action of attractive force. Mutual attraction is not strong enough to form a bound state. Hence, as distinct from the N-lump configuration, interaction between multiple-pole humps is nontrivial. These authors also considered more complicated versions of the multiple-pole structure.

Note that a class of (in general, singular) solutions of the KP I equation (and some other equations) with multiple poles was obtained by Dubrovsky [136].

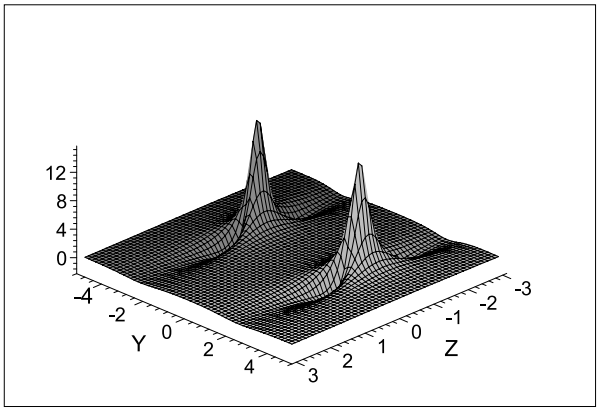

Fig. 9.1. Lumps of the solution (9.137) before interaction. $Z = X - 2k_{\mathrm{R}}Y - 12(k_{\mathrm{R}}^2 - k_{\mathrm{I}}^2)t$, $k_{\mathrm{R}} = 1/2$, $k_{\mathrm{I}} = 1$, $\gamma = \delta = 0$ [439]

9.3 Davey–Stewartson I equation

In the context of shallow water waves, the DS I equation

$$\begin{aligned} &\mathrm{i}q_t + \frac{1}{2}(q_{xx} + q_{yy}) + \epsilon|q|^2 q = \phi_x q, \\ &\phi_{xx} - \phi_{yy} = 2\epsilon\left(|q|^2\right)_x, \qquad \epsilon = \pm 1 \end{aligned} \tag{9.138}$$

describes the (2+1)-dimensional evolution of a small-amplitude, slowly modulated packet of surface waves with dominant surface tension [105, 470]. Here

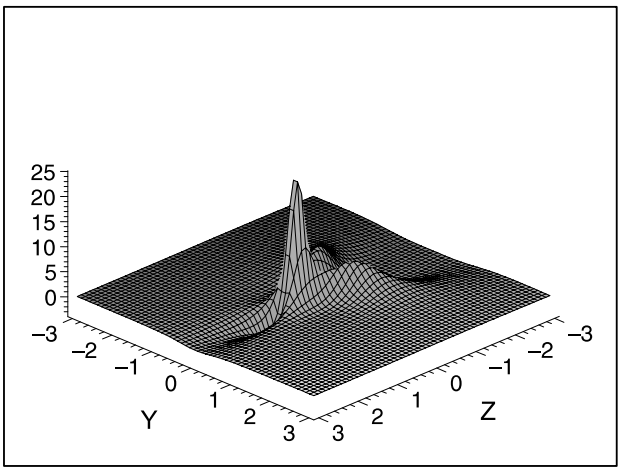

Fig. 9.2. Interaction of lumps described by the solution (9.137) [439]

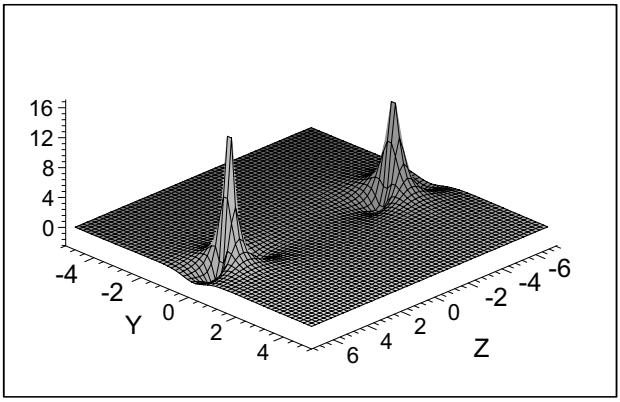

Fig. 9.3. Lumps of the solution (9.137) after interaction [439]

$q(x, y, t)$ is the dimensionless envelope of the wave packet and $\phi(x, y, t)$ is the dimensionless amplitude of the mean fluid flow. The initial-value problem for DS I was addressed by Fokas and Ablowitz [156, 159]. They formulated the RH problem for the eigenfunctions of the spectral problem but localized solutions were not been found. In a somewhat more general (in fact, nonintegrable) form the DS I type equation arises in nonlinear optics when studying propagation of a single quasimonochromatic optical pulse in a nonresonant quadratic medium [2].

A breakthrough in finding true solitons in 2+1 dimensions was caused by the remarkable discovery by Boiti et al. [62, 63, 365]. They demonstrated by means of the Bäcklund gauge transformation that exponentially localized solitons of the DS I equation exist if specific boundary conditions are properly taken into account. This new situation can be explained in physical language. Indeed, in 1+1 dimensions, where solitons are the result of the balance between counter-acting nonlinearity and dispersion, both of these effects are of the same order of magnitude and are able to compensate each other. In contrast, in 2+1 dimensions dispersion is, as a rule, much stronger than nonlinearity; hence, additional sources are needed to stop dispersive broadening. Just the boundaries serve as these sources.

There are two versions of the IST formalism to find solitons of the DS I equation. Fokas and Santini [162] used the unit normalization of analytic eigenfunctions of the spectral problem and modified the second (evolutionary) Lax equation to incorporate nontrivial boundary conditions, while Boiti et al. [67, 68] normalized eigenfunctions by the boundary conditions, retaining the second Lax equation to be explicitly integrable. We will follow in this section the approach of Fokas and Santini as it is technically simpler, though the method by Boiti et al. seems perhaps more natural from the viewpoint of the

IST ideology. Note that the so-called dromion solutions of the DS I equation were derived by the $\bar{\partial}$ formalism in the book by Konopelchenko [241].

9.3.1 Spectral problem and analytic eigenfunctions

The DS I equation (9.138) arises as the compatibility condition of the system of linear equations (the Lax pair)

$$\psi_x + \sigma_3\psi_y + Q\psi = 0, \qquad Q = \begin{pmatrix} 0 & q \\ \epsilon\bar{q} & 0 \end{pmatrix}, \tag{9.139}$$

$$\mathrm{i}\psi_t + \sigma_3\psi_{yy} + Q\psi_y + A\psi = 0. \tag{9.140}$$

Here $\psi(x, y, t)$ is a 2×2 matrix function and the 2×2 matrix A will be specified later. Because the spectral problem (9.139) is hyperbolic, it is reasonable to use the coordinates $\xi = x + y$ and $\eta = x - y$. In new coordinates the second equation in (9.138) takes the form

$$\phi_{\xi\eta} = \frac{\epsilon}{2}(\partial_\xi + \partial_\eta)|q|^2. \tag{9.141}$$

Integrating (9.141) in turn in ξ and η, we obtain

$$\phi_\xi = -U_1 + \frac{\epsilon}{2}|q|^2, \quad U_1(\xi,\eta) = -\frac{\epsilon}{2}\int_{-\infty}^{\eta} \mathrm{d}\eta' \left(|q|^2\right)_\xi + u_1(\xi, t), \tag{9.142}$$

$$\phi_\eta = -U_2 + \frac{\epsilon}{2}|q|^2, \quad U_2(\xi,\eta) = -\frac{\epsilon}{2}\int_{-\infty}^{\xi} \mathrm{d}\xi' \left(|q|^2\right)_\eta + u_2(\eta, t).$$

Here the real functions $u_1(\xi, t)$ and $u_2(\eta, t)$ represent the boundary values of U_1 and U_2:

$$u_1(\xi, t) = \lim_{\eta\to-\infty} U_1(\xi, \eta, t), \qquad u_2(\eta, t) = \lim_{\xi\to-\infty} U_2(\xi, \eta, t). \tag{9.143}$$

Then the DS I equation is written as

$$\mathrm{i}q_t + q_{\xi\xi} + q_{\eta\eta} + (U_1 + U_2)q = 0. \tag{9.144}$$

Hence, we can consider the DS I equation as the integrodifferential equation for complex function $q(x, y, t)$ with boundary conditions (9.143) and definite dependence (9.142) of U_1 and U_2 on q. Just the real functions U_1 and U_2 enter the matrix A in the evolutionary part (9.140) of the Lax pair:

$$A = \begin{pmatrix} U_2 & -q_\eta \\ \epsilon\bar{q} & -U_1 \end{pmatrix}. \tag{9.145}$$

We assume that $q(\xi, \eta, 0)$, $u_1(\eta, t)$, and $u_2(\xi, t)$ decay for large ξ and η. Our aim is to solve the initial boundary value problem for the DS I equation.

Now we introduce a spectral parameter k by means of the transformation

$$\psi(\xi,\eta) = M(\xi,\eta,k)E_k(\xi,\eta), \qquad E_k(\xi,\eta) = \operatorname{diag}\left(\mathrm{e}^{\mathrm{i}k\eta}, \mathrm{e}^{-\mathrm{i}k\xi}\right).$$

In terms of the matrix M, the spectral equation (9.139) takes the form

$$DM - \frac{\mathrm{i}}{2}k[\sigma_3, M] + \frac{1}{2}QM = 0, \qquad D = \operatorname{diag}(\partial_\xi,\ \partial_\eta). \tag{9.146}$$

We will seek solutions $M(\xi,\eta,k)$ of the spectral equation that are bounded in the $(\xi,\ \eta)$-plane for any k and allow the asymptotic expansion

$$M(\xi,\eta,k) = \mathbb{1} + k^{-1}M^{(1)}(\xi,\eta) + \mathcal{O}(k^{-2}).$$

Substituting this expansion into (9.146), we can reconstruct the potential:

$$Q(\xi,\eta) = \mathrm{i}[\sigma_3, M^{(1)}(\xi,\eta)], \quad \text{or} \quad q(\xi,\eta) = 2\mathrm{i}M^{(1)}_{12}(\xi,\eta). \tag{9.147}$$

As before, we will work with solutions of the spectral problem which are written in the integral form. It can be shown that there exist eigenfunctions with definite analytic properties in the k-plane. Namely, the matrix functions $M^{\pm}(\xi,\eta,k)$ determined by the Green functions $G^{\pm}$,

$$M^{\pm}(\xi,\eta,k) = \mathbb{1} - \left(G^{\pm}(\cdot,k)\frac{1}{2}Q(\cdot)M^{\pm}(\cdot,k)\right)(\xi,\eta), \tag{9.148}$$

where

$$\left[G^{+}(\cdot,k)\Phi(\cdot,k)\right](\xi,\eta) \tag{9.149}$$

$$= \begin{pmatrix} \int\limits_{-\infty}^{\xi} \mathrm{d}\xi'\Phi_{11}(\xi',\eta,k) & \int\limits_{-\infty}^{\xi} \mathrm{d}\xi'\Phi_{12}(\xi',\eta,k)\mathrm{e}^{\mathrm{i}k(\xi-\xi')} \\ -\int\limits_{\eta}^{\infty} \mathrm{d}\eta'\Phi_{21}(\xi,\eta',k)\mathrm{e}^{-\mathrm{i}k(\eta-\eta')} & \int\limits_{-\infty}^{\eta} \mathrm{d}\eta'\Phi_{22}(\xi,\eta',k) \end{pmatrix}$$

and

$$\left[G^{-}(\cdot,k)\Phi(\cdot,k)\right](\xi,\eta) \tag{9.150}$$

$$= \begin{pmatrix} \int\limits_{-\infty}^{\xi} \mathrm{d}\xi'\Phi_{11}(\xi',\eta,k) & -\int\limits_{\xi}^{\infty} \mathrm{d}\xi'\Phi_{12}(\xi',\eta,k)\mathrm{e}^{\mathrm{i}k(\xi-\xi')} \\ \int\limits_{-\infty}^{\eta} \mathrm{d}\eta'\Phi_{21}(\xi,\eta',k)\mathrm{e}^{-\mathrm{i}k(\eta-\eta')} & \int\limits_{-\infty}^{\eta} \mathrm{d}\eta'\Phi_{22}(\xi,\eta',k) \end{pmatrix}$$

are analytic in the upper and lower half planes of the k-plane, respectively.

9.3.2 Spectral data and RH problem

To determine spectral data for the spectral problem (9.146) with the potential Q, we should calculate a jump $\Delta = M^+ - M^-$ of eigenfunctions across the real axis $\mathrm{Im}k = 0$. After straightforward calculation making use of (9.148)–(9.150) we obtain

$$\Delta(\xi,\eta,k) = \Gamma(\xi,\eta,k) - \left(\widetilde{G}(\cdot,k)\frac{1}{2}Q(\cdot)\Delta(\cdot,k)\right)(\xi,\eta), \quad \mathrm{Im}k = 0, \tag{9.151}$$

where

$$\Gamma = \frac{1}{2}\begin{pmatrix} 0 & -\int_{-\infty}^{\infty} \mathrm{d}\xi'(qM_{22}^-)(\xi',\eta,k)\mathrm{e}^{\mathrm{i}k(\xi-\xi')} \\ \epsilon\int_{-\infty}^{\infty} \mathrm{d}\eta'(\bar{q}M_{11}^+)(\xi,\eta',k)\mathrm{e}^{-\mathrm{i}k(\eta-\eta')} & 0 \end{pmatrix}$$

and the Green function $\widetilde{G}$ is defined as

$$\left[\widetilde{G}(\cdot,k)\Phi(\cdot,k)\right](\xi,\eta)$$

$$= \begin{pmatrix} \int\limits_{-\infty}^{\xi} \mathrm{d}\xi'\Phi_{11}(\xi',\eta,k) & \int\limits_{-\infty}^{\xi} \mathrm{d}\xi'\Phi_{12}(\xi',\eta,k)\mathrm{e}^{\mathrm{i}k(\xi-\xi')} \\ \int\limits_{-\infty}^{\eta} \mathrm{d}\eta'\Phi_{21}(\xi,\eta',k)\mathrm{e}^{-\mathrm{i}k(\eta-\eta')} & \int\limits_{-\infty}^{\eta} \mathrm{d}\eta'\Phi_{22}(\xi,\eta',k) \end{pmatrix}.$$

Hereafter we will not specify the integration limits in the case of integration along the whole line.

Let us seek the jump in the form

$$\Delta(\xi,\eta,k) = \int \mathrm{d}\ell M^-(\xi,\eta,\ell)E_\ell(\xi,\eta)f(k,\ell)E_k^{-1}(\xi,\eta) \tag{9.152}$$

with a 2×2 matrix function $f(k,l)$ that determines the spectral data. Substituting (9.152) into (9.151), we have

$$\Delta(\xi,\eta,k) = \Gamma(\xi,\eta,\ell) - \int \mathrm{d}\ell\left(\widetilde{G}(\cdot,\ell)\frac{1}{2}Q(\cdot)M^-(\cdot,\ell)E_\ell(\cdot)f(k,\ell)E_k^{-1}(\cdot)\right)(\xi,\eta). \tag{9.153}$$

Comparing (9.153) and (9.151), we obtain after simple calculation

$$\int \mathrm{d}\ell E_\ell(\xi,\eta)f(k,\ell) = -\frac{1}{2}\int \mathrm{d}\ell\begin{pmatrix} 0 & \int \mathrm{d}\xi'(qM_{22}^-)(\xi',\eta,k)\mathrm{e}^{-\mathrm{i}\ell\xi'} \\ 0 & 0 \end{pmatrix} f(k,\ell)$$

$$+\frac{1}{2}\begin{pmatrix} 0 & -\int \mathrm{d}\xi'(qM_{22}^-)(\xi',\eta,k) \\ \epsilon\int \mathrm{d}\eta'(\bar{q}M_{11}^+)(\xi,\eta',k)\mathrm{e}^{\mathrm{i}k\eta'} & 0 \end{pmatrix}. \tag{9.154}$$

Let us act on (9.154) with the matrix operator

$$\frac{1}{2\pi}\,\mathrm{diag}\left(\int \mathrm{d}\eta'\mathrm{e}^{-\mathrm{i}p\eta'}, \int \mathrm{d}\xi'\mathrm{e}^{\mathrm{i}p\xi'}\right).$$

This results in

$$f(k,\ell)=\begin{pmatrix}0 & -S(k,\ell)\\ T(k,\ell) & 0\end{pmatrix}-\int \mathrm{d}p\begin{pmatrix}0 & S(p,\ell)\\ 0 & 0\end{pmatrix}f(k,\ell), \tag{9.155}$$

where the spectral functions are defined as

$$S(k,\ell)=\frac{1}{4\pi}\iint \mathrm{d}\xi\mathrm{d}\eta\, qM_{22}^{-}\mathrm{e}^{-\mathrm{i}k\xi-\mathrm{i}\ell\eta},\quad T(k,\ell)=\frac{\epsilon}{4\pi}\iint \mathrm{d}\xi\mathrm{d}\eta\,\bar{q}M_{11}^{+}\mathrm{e}^{\mathrm{i}k\eta+\mathrm{i}l\xi} \tag{9.156}$$

and $T(k,\ell)=\epsilon\bar{S}(\ell,k)$ [162]. It immediately follows from (9.155) that $f_{22}=0$; hence,

$$f(k,\ell)=\begin{pmatrix}-\int \mathrm{d}pT(k,p)S(p,\ell) & -S(k,\ell)\\ T(k,\ell) & 0\end{pmatrix}. \tag{9.157}$$

Inserting (9.157) into (9.152), we find that the second column of the jump satisfies the equation

$$\begin{pmatrix}M_{12}^{+}\\ M_{22}^{+}\end{pmatrix}(k)-\begin{pmatrix}M_{12}^{-}\\ M_{22}^{-}\end{pmatrix}(k)=-\int \mathrm{d}\ell S(k,\ell)\mathrm{e}^{\mathrm{i}k\xi+\mathrm{i}\ell\eta}\begin{pmatrix}M_{11}^{-}\\ M_{21}^{-}\end{pmatrix}(\ell). \tag{9.158}$$

Similarly, for the first column we have

$$\begin{pmatrix}M_{11}^{+}\\ M_{21}^{+}\end{pmatrix}(k)-\begin{pmatrix}M_{11}^{-}\\ M_{21}^{-}\end{pmatrix}(k)=-\int \mathrm{d}\ell T(k,\ell)\mathrm{e}^{-\mathrm{i}k\eta-\mathrm{i}\ell\xi}\begin{pmatrix}M_{12}^{+}\\ M_{22}^{+}\end{pmatrix}(\ell). \tag{9.159}$$

Equations (9.158) and (9.159) determine the nonlocal RH problem. In accordance with the Cauchy–Green formula (1.98) the solution of the RH problem is given by the integral equations of the form

$$\begin{pmatrix}M_{11}^{\pm}\\ M_{21}^{\pm}\end{pmatrix}(k)=\begin{pmatrix}1\\ 0\end{pmatrix}+\frac{1}{2\pi}\int\frac{\mathrm{d}p}{p-(k\pm\mathrm{i}0)}\int\mathrm{d}\ell T(p,\ell)\mathrm{e}^{-\mathrm{i}p\eta-\mathrm{i}\ell\xi}\begin{pmatrix}M_{12}^{+}\\ M_{22}^{+}\end{pmatrix}(\ell),$$

$$\begin{pmatrix}M_{12}^{\pm}\\ M_{22}^{\pm}\end{pmatrix}(k)=\begin{pmatrix}0\\ 1\end{pmatrix}-\frac{1}{2\pi}\int\frac{\mathrm{d}p}{p-(k\pm\mathrm{i}0)}\int\mathrm{d}\ell S(p,\ell)\mathrm{e}^{\mathrm{i}p\xi+\mathrm{i}\ell\eta}\begin{pmatrix}M_{11}^{-}\\ M_{21}^{-}\end{pmatrix}(\ell). \tag{9.160}$$

Hence, the reconstruction of the potential $q(\xi,\eta)$ from the solution of the RH problem is given with account for (9.147) by the formula

$$q(\xi,\eta)=\frac{1}{\pi}\iint \mathrm{d}k\mathrm{d}\ell S(k,\ell)\mathrm{e}^{\mathrm{i}\ell\eta+\mathrm{i}k\xi}M_{11}^{-}(\xi,\eta,\ell). \tag{9.161}$$

9.3.3 Time evolution of spectral data and boundaries

To find solutions of the DS I equation using the IST method, we should determine the time evolution of the spectral function $S(k,\ell)$. As usual, the second Lax equation is exploited for this sake. In the case of the initial boundary value problem we, however, cannot naively invoke (9.140) in its

present form because of the nontrivial boundary conditions. In order to reconcile boundary conditions with the evolutionary Lax equation, note that we can add a term $\int \mathrm{d}\ell\psi(k,\ell)\gamma(k-\ell)$ with some matrix function γ to the right-hand side of (9.140), without breaking the compatibility condition. Now we invoke such a freedom to incorporate boundary values u_1 and u_2 into the Lax pair. Assume that the right-hand side of (9.140) contains an additional term $W(\xi,\eta,k)$ and consider a matrix function ψ^- related to M^- by $\psi^-(\xi,\eta,k) = M^-(\xi,\eta,k)E_k(\xi,\eta)$. From the integral formulas (9.148) and (9.150) for M^- it immediately follows that matrix elements of ψ^- obey simple asymptotics in some directions $\xi \to \pm\infty$ and/or $\eta \to \pm\infty$. As an example, $\psi^-_{12} \to 0$ at $\xi \to +\infty$ and $\psi^-_{22} \to \mathrm{e}^{-\mathrm{i}k\xi}$ at $\eta \to -\infty$; hence, we can write

$$\psi^-_{12t} = -\int_\xi^\infty \mathrm{d}\xi' \left(\psi^-_{12t}\right)_{\xi'} \quad \text{and} \quad \psi^-_{22t} = \int_{-\infty}^\eta \mathrm{d}\eta' \left(\psi^-_{22t}\right)_{\eta'}.$$

Inserting into the integrands evolution equations for the entries of ψ^- which follow from

$$\mathrm{i}\psi^-_t + \sigma_3\psi^-_{yy} + Q\psi^-_y + A\psi^- + W = 0,$$

yields, in particular,

$$\begin{pmatrix} W_{12} \\ W_{22} \end{pmatrix} = -k^2 \begin{pmatrix} \psi^-_{12} \\ \psi^-_{22} \end{pmatrix}(k) + \int \mathrm{d}\ell \begin{pmatrix} \psi^-_{12} \\ \psi^-_{22} \end{pmatrix}(\ell)\gamma_1(k-\ell).$$

Here $\gamma_1(k-\ell)$ is related to the boundary value $u_1(\xi,t)$ by means of

$$u_1(\xi,t) = \int \mathrm{d}\ell\gamma_1(k-\ell)\mathrm{e}^{\mathrm{i}(k-\ell)\xi}. \tag{9.162}$$

Similarly,

$$\begin{pmatrix} W_{12} \\ W_{22} \end{pmatrix}(k) = k^2 \begin{pmatrix} \psi^-_{11} \\ \psi^-_{21} \end{pmatrix}(k) - \int \mathrm{d}\ell \begin{pmatrix} \psi^-_{11} \\ \psi^-_{21} \end{pmatrix}(\ell)\gamma_2(k-\ell)$$

and

$$u_2(\eta,t) = \int \mathrm{d}\ell\gamma_2(k-\ell)\mathrm{e}^{-\mathrm{i}(k-\ell)\eta}.$$

As a result, the improved evolutionary Lax equation has the form

$$\mathrm{i}\psi^-_t + \sigma_3\psi^-_{yy} + Q\psi^-_y + A\psi^- + k^2\psi^-\sigma_3 - \int \mathrm{d}\ell\psi^-(\ell)\gamma(k-\ell)\sigma_3 = 0, \tag{9.163}$$

$$\gamma(k-\ell) = \mathrm{diag}\left[\gamma_2(k-\ell), \gamma_1(k-\ell)\right].$$

Subsequent actions are rather standard. In terms of ψ^-, the spectral function $S(k,\ell)$ is written as

$$S(k,\ell) = \frac{1}{4\pi}\iint \mathrm{d}\xi\mathrm{d}\eta q(\xi,\eta)\psi^-_{22}(\xi,\eta,k)\mathrm{e}^{-\mathrm{i}\ell\eta}.$$

Hence, it is natural to consider ψ_{12}^- for $\xi \to -\infty$:

$$\lim_{\xi\to-\infty} \psi_{12}^-(\xi,\eta,k) \equiv \chi(k,\eta) = \frac{1}{2}\int \mathrm{d}\xi\mathrm{d}\eta\; q(\xi,\eta)\psi_{22}^-(\xi,\eta,k).$$

Moreover, it is easy to show that $\chi(k,\eta) = \int \mathrm{d}\ell \mathrm{e}^{\mathrm{i}\ell\eta} S(k,\ell)$.

We consider (9.163) for ψ_{12}^- at $\xi \to -\infty$. In this limit

$$\mathrm{i}\chi_t + \chi_{\eta\eta} + u_2\chi - k^2\chi + \int \mathrm{d}\ell\chi(\ell,\eta)\gamma_1(k-\ell) = 0. \tag{9.164}$$

Let us analyze the term with the integral. Expressing $\gamma_1(k-\ell)$ from (9.162) as

$$\gamma_1(k-\ell) = \frac{1}{2\pi}\int \mathrm{d}\xi' \mathrm{e}^{-\mathrm{i}(k-\ell)\xi'} u_1(\xi',t)$$

and substituting it in (9.164) yields the purely exponential k-dependence of the integrand. This means that it is reasonable to multiply this term by $(2\pi)^{-1}\int \mathrm{d}k\mathrm{e}^{\mathrm{i}k\xi}$ and integrate in k to have the delta-function $\delta(\xi-\xi')$. As a result, we obtain

$$\frac{1}{2\pi}\int \mathrm{d}k\mathrm{e}^{\mathrm{i}k\xi}\int \mathrm{d}\ell\chi(\ell,\eta)\gamma_2(k-\ell) = u_1(\xi,t)\widehat{S}(\xi,\eta).$$

Here $\widehat{S}(\xi,\eta) = (1/2\pi)\int \mathrm{d}\ell\mathrm{e}^{\mathrm{i}\ell\xi}\chi(\ell,\eta)$, i.e., $\widehat{S}(\xi,\eta)$ represents in fact the Fourier transform of the spectral function $S(k,\ell)$:

$$\widehat{S}(\xi,\eta) = \frac{1}{2\pi}\iint \mathrm{d}\xi\mathrm{d}\eta\,\mathrm{e}^{\mathrm{i}k\xi+\mathrm{i}\ell\eta} S(k,\ell).$$

Transforming the rest of the terms in (9.164) after multiplying by the above integral operator, we obtain the linear evolution equation for $\widehat{S}(\xi,\eta)$ [162]:

$$\mathrm{i}\widehat{S}_t + \widehat{S}_{\xi\xi} + \widehat{S}_{\eta\eta} + (u_1+u_2)\widehat{S} = 0. \tag{9.165}$$

We should solve this equation with the initial value $\widehat{S}(\xi,\eta,0)$, which in turn is determined by the initial value $q(\xi,\eta,0)$, and the known boundary functions u_1 and u_2.

Equation (9.165) allows the separation of variables of the form $\widehat{S}(\xi,\eta,t) = X(\xi,t)Y(\eta,t)$. This leads to the appearance of the nonstationary linear Schrödinger equations with the boundary functions as potentials:

$$\mathrm{i}X_t + X_{\xi\xi} + u_1(\xi,t)X = 0, \qquad \mathrm{i}Y_t + Y_{\eta\eta} + u_2(\eta,t)Y = 0. \tag{9.166}$$

In what follows we will be interested in the purely discrete spectra of (9.166) (the so-called *reflectionless boundaries*). It is easy to verify directly that the orthonormal eigenfunctions $X_n(\xi,t)$ and $Y_n(\eta,t)$ of the discrete spectrum of (9.166) can be written in a closed form as solutions of the algebraic equations

$$X_n + \sum_{j=1}^{N_1} \frac{m_n \bar{m}_j}{\mu_n + \bar{\mu}_j} \exp\left[-(\mu_n + \bar{\mu}_j)\xi + \mathrm{i}(\mu_n^2 - \bar{\mu}_j^2)t\right] X_j = m_n \mathrm{e}^{-\mu_n(\xi - \mathrm{i}\mu_n t)}, \tag{9.167}$$

$$Y_n + \sum_{j=1}^{N_2} \frac{\ell_n \bar{\ell}_j}{\lambda_n + \bar{\lambda}_j} \exp\left[-(\lambda_n + \bar{\lambda}_j)\eta + \mathrm{i}(\lambda_n^2 - \bar{\lambda}_j^2)t\right] Y_j = \ell_n \mathrm{e}^{-\lambda_n(\eta - \mathrm{i}\lambda_n t)}, \tag{9.168}$$

while the potentials are expressed in terms of X_n and Y_n as

$$u_1(\xi, t) = -2\partial_\xi \sum_{i=1}^{N_1} \bar{m}_i \exp\left[-\bar{\mu}_i(\xi + \mathrm{i}\bar{\mu}_i t)\right] X_i(\xi, t),$$
$$u_2(\eta, t) = -2\partial_\eta \sum_{j=1}^{N_2} \bar{\ell}_j \exp\left[-\bar{\lambda}_j(\eta + \mathrm{i}\bar{\lambda}_j t)\right] Y_j(\eta, t). \tag{9.169}$$

Evidently, the solutions of (9.167) and (9.168) take the form

$$X_n(\xi, t) = \sum_{i=1}^{N_1} \left[(\mathbb{1} + C_\xi)^{-1}\right]_{nj} m_j \mathrm{e}^{-\mu_j(\xi - \mathrm{i}\mu_j t)},$$
$$Y_n(\eta, t) = \sum_{j=1}^{N_2} \left[(\mathbb{1} + C_\eta)^{-1}\right]_{nj} \ell_j \mathrm{e}^{-\lambda_j(\eta - \mathrm{i}\lambda_j t)}, \tag{9.170}$$

where the Hermitian matrices C_ξ and C_η are defined as follows:

$$(C_\xi)_{ni} = \frac{m_n \bar{m}_i}{\mu_n + \bar{\mu}_i} \exp\left[-(\mu_n + \bar{\mu}_i)\xi + \mathrm{i}(\mu_n^2 - \mu_i^2)t\right],$$
$$(C_\eta)_{nj} = \frac{\ell_n \bar{\ell}_j}{\lambda_n + \bar{\lambda}_j} \exp\left[-(\lambda_n + \bar{\lambda}_j)\eta + \mathrm{i}(\lambda_n^2 - \bar{\lambda}_j^2)t\right].$$

Therefore, the spectral function $\widehat{S}(\xi, \eta, t)$ is written as

$$\widehat{S}(\xi, \eta, t) = \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} \rho_{ij} X_i(\xi, t) Y_j(\eta, t) \tag{9.171}$$

and the complex parameters ρ_{ij} are determined from the *initial* conditions:

$$\rho_{ij} = \iint \mathrm{d}\xi \mathrm{d}\eta \widehat{S}(\xi, \eta, 0) \bar{X}_i(\xi, 0) \bar{Y}_j(\eta, 0). \tag{9.172}$$

Orthonormality of X_n and Y_n gives rise to

$$X_n \bar{X}_i = \partial_\xi \left[(\mathbb{1} + C_\xi)^{-1}\right]_{ni}, \qquad Y_n \bar{Y}_j = \partial_\eta \left[(\mathbb{1} + C_\eta)^{-1}\right]_{nj}. \tag{9.173}$$

9.3.4 Reconstruction of potential $q(\xi, \eta, t)$

The factorized representation (9.171) of $\widehat{S}(\xi, \eta, t)$ corresponds to the degeneracy of the kernel $S(k, \ell)$ of the integral equations (9.160) that determines the solution of the nonlocal RH problem. Namely,

$$S(k, \ell) = \sum_{i=1}^{N} S_i(k)\widetilde{S}_i(\ell). \tag{9.174}$$

As a result, the second column $\left(M_{12}^{+}, M_{22}^{+}\right)^{\mathrm{T}}(k)$ in (9.160) can be written as

$$\begin{pmatrix} M_{12}^{+} \\ M_{22}^{+} \end{pmatrix}(k) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - \frac{\sqrt{2\pi}}{2\pi} \int \frac{\mathrm{d}p}{p - (k + \mathrm{i}0)} \mathrm{e}^{\mathrm{i}p\xi} \sum_{i=1}^{N} S_i(p) F_i(\xi, \eta), \tag{9.175}$$

where

$$F_i(\xi, \eta) = \frac{1}{\sqrt{2\pi}} \int \mathrm{d}\ell \mathrm{e}^{\mathrm{i}\ell\eta} \widetilde{S}_i(\ell) \begin{pmatrix} M_{11}^{-} \\ M_{21}^{-} \end{pmatrix}(\ell).$$

Let us multiply (9.175) by $(2\pi)^{-1/2} \int \mathrm{d}k \mathrm{e}^{-\mathrm{i}k\xi} \bar{S}_r(k)$ and integrate in k. This gives

$$G_r(\xi, \eta) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \bar{\sigma}_r(\xi) \tag{9.176}$$

$$- \int \mathrm{d}k \bar{S}_r(k) \mathrm{e}^{-\mathrm{i}k\xi} \sum_i \frac{1}{\sqrt{2\pi}} \int \mathrm{d}\ell \sigma_i(\ell) F_i(\xi, \eta) \frac{1}{2\pi\mathrm{i}} \int \frac{\mathrm{d}p}{p - (k + \mathrm{i}0)} \mathrm{e}^{\mathrm{i}(\xi - \ell)p}.$$

Here

$$G_r(\xi, \eta) = \frac{1}{\sqrt{2\pi}} \int \mathrm{d}k \mathrm{e}^{-\mathrm{i}k\xi} \bar{S}_r(k) \begin{pmatrix} M_{12}^{+} \\ M_{22}^{+} \end{pmatrix}(k)$$

and $\sigma_i(\xi)$ determines the Fourier transform of $S_i(k)$:

$$\sigma_i(\xi) = \frac{1}{\sqrt{2\pi}} \int \mathrm{d}k \mathrm{e}^{\mathrm{i}k\xi} S_i(k).$$

The last integral in (9.176) gives $2\pi\mathrm{i}\theta(\xi - \ell) \exp[\mathrm{i}k(\xi - \ell)]$. Then after simple manipulations with (9.176) we eventually obtain the algebraic equation

$$G_r(\xi, \eta) + \sum_{i=1}^{N} \alpha_{ri}(\xi) F_i(\xi, \eta) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \bar{\sigma}_i(\xi), \tag{9.177}$$

where

$$\alpha_{ri} = \int_{-\infty}^{\xi} \mathrm{d}\ell \bar{\sigma}_r(\ell) \sigma_i(\ell). \tag{9.178}$$

Performing a similar calculation with the column $(M_{11}^{-}, M_{21}^{-})^{\mathrm{T}}$, we obtain

$$F_r(\xi,\eta)+\epsilon\sum_{j=1}^{N}\beta_{rj}G_j(\xi,\eta)=\begin{pmatrix}1\\0\end{pmatrix}\widetilde{\sigma}_r(\eta). \tag{9.179}$$

Here

$$\beta_{rj}(\eta)=\int_{-\infty}^{\eta}\mathrm{d}\ell\widetilde{\sigma}_r(\ell)\bar{\widetilde{\sigma}}_j(\ell) \tag{9.180}$$

and $\widetilde{\sigma}_j(\ell)$ is the Fourier transform of $\widetilde{S}_j$:

$$\widetilde{\sigma}_j=\frac{1}{\sqrt{2\pi}}\int\mathrm{d}\ell\mathrm{e}^{\mathrm{i}\ell\eta}\widetilde{S}_j(\ell). \tag{9.181}$$

Then from (9.177) and (9.179) we find a system of algebraic equations for the first component f_r of F_r:

$$f_r-\epsilon\sum_{j=1}^{N}\left(\sum_{j=1}^{N}\beta_{rj}\alpha_{ji}\right)f_i=\widetilde{\sigma}_r. \tag{9.182}$$

Inserting into (9.161) the factorized form (9.174) of $S(k,\ell)$, we arrive at the closed formula for the potential q:

$$q(\xi,\eta)=2\sum_{i=1}^{N}\sigma_i(\xi)f_i(\xi,\eta). \tag{9.183}$$

Hence, we have two representation of the Fourier transform of $S(k,\ell,t)$. One of them is written in terms of σ_i and $\widetilde{\sigma}_i$, $S(\xi,\eta,t)=\sum_i\sigma_i(\xi,t)\widetilde{\sigma}_i(\eta,t)$, and the other one is given by (9.171). Therefore, we can take

$$\sigma_i(\xi,t)=X_i(\xi,t),\qquad \widetilde{\sigma}_i(\eta,t)=\sum_{j=1}^{N_2}\rho_{ij}Y_j(\eta,t). \tag{9.184}$$

Recall that we should calculate $f_i(\xi,\eta)$ which enters (9.183). Taking advantage of the representation (9.184) and the orthonormality property (9.173), we can calculate integrals in (9.178) and (9.180). Then after rather lengthy but straightforward calculation we obtain the formula for the potential q [162]:

$$q(\xi,\eta,t)=\sum_{i=1}^{N_1}\sum_{j=1}^{N_2}X_i(\xi,t)Y_j(\eta,t)Z_{ij}(\xi,\eta,t), \tag{9.185}$$

where Z_{ij} obeys the matrix algebraic equation $Z-\epsilon HZ=\rho$. Here the matrix H is given by $H=\rho(\mathbb{1}+C_\eta)^{-1}\bar{\rho}^{\dagger}(\mathbb{1}+C_\xi)^{-1\mathrm{T}}$ and the matrix ρ is defined in (9.172). Solutions of the DS I equation which correspond to the reflectionless boundaries are called (N_1,N_2) dromions [162].

9.3.5 (1, 1) Dromion solution

It is instructive to derive explicitly the simplest $(1,1)$ dromion solution. It corresponds to $\epsilon = N_1 = N_2 = 1$. In what follows we will omit the summation subscript 1. In this case equations (9.167) and (9.170) give

$$\begin{aligned}
X(\xi,t) &= \frac{1}{2} m \mathrm{e}^{-\mu_{\mathrm{R}}\xi_0} \mathrm{e}^{\mathrm{i}\mu_{\mathrm{I}}\xi + \mathrm{i}(\mu_{\mathrm{R}}^2 - \mu_{\mathrm{I}}^2)t} \operatorname{sech}\mu_{\mathrm{R}}(\xi + 2\mu_i t - \xi_0),\\
Y(\eta,t) &= \frac{1}{2} \ell \mathrm{e}^{-\lambda_{\mathrm{R}}\eta_0} \mathrm{e}^{\mathrm{i}\lambda_{\mathrm{I}}\eta + \mathrm{i}(\lambda_{\mathrm{R}}^2 - \lambda_{\mathrm{I}}^2)t} \operatorname{sech}\lambda_{\mathrm{R}}(\eta + 2\lambda_{\mathrm{I}} t - \eta_0),\\
u_1(\xi,t) &= 2\mu_{\mathrm{R}}^2 \operatorname{sech}^2 \mu_{\mathrm{R}}(\xi + 2\mu_{\mathrm{R}} t - \xi_0),\\
u_2(\eta,t) &= 2\lambda_{\mathrm{R}}^2 \operatorname{sech}^2 \lambda_{\mathrm{R}}(\eta + 2\lambda_{\mathrm{I}} t - \eta_0),
\end{aligned}$$

where $\mu = \mu_{\mathrm{R}} + \mathrm{i}\mu_{\mathrm{I}}$, $\lambda = \lambda_{\mathrm{R}} + \mathrm{i}\lambda_{\mathrm{I}}$, and real parameters ξ_0 and η_0 are defined as

$$\xi_0 = \frac{1}{\mu_{\mathrm{R}}} \ln \frac{|m|}{\sqrt{2\mu_{\mathrm{R}}}}, \qquad \eta_0 = \frac{1}{\lambda_{\mathrm{R}}} \ln \frac{|\ell|}{\sqrt{2\lambda_{\mathrm{R}}}}.$$

It is convenient to introduce running coordinates $z_\xi = \mu_{\mathrm{R}}(\xi + 2\mu_{\mathrm{I}} t - \xi_0)$ and $z_\eta = \lambda_{\mathrm{R}}(\eta + 2\lambda_{\mathrm{I}} t - \lambda_0)$. Then

$$Z = \rho \frac{\exp(-z_\xi - z_\eta)\cosh z_\xi \cosh z_\eta}{\exp(-z_\xi - z_\eta)\cosh z_\xi \cosh z_\eta + |\rho|^2/4}$$

and the $(1,1)$ dromion solution takes the form

$$q(\xi,\eta,t) = 4\rho(\mu_{\mathrm{R}}\lambda_{\mathrm{R}})^{1/2} \frac{\exp(-\mathrm{i}\Phi)}{4\cosh z_\xi \cosh z_\eta + |\rho|^2 \exp(z_\xi + z_\eta)}$$

with the phase

$$\Phi = \mu_{\mathrm{I}}\xi + \lambda_{\mathrm{I}}\eta - (\mu_{\mathrm{R}}^2 + \lambda_{\mathrm{R}}^2 - \mu_{\mathrm{I}}^2 - \lambda_{\mathrm{I}}^2)t - \arg(m\ell).$$

This solution describes a localized object in the (ξ,η)-plane which decays exponentially in all directions in the plane and moves with velocity $(-2\mu_{\mathrm{I}}, -2\lambda_{\mathrm{I}})$. Therefore, the dromion velocity is completely determined by the boundaries, while the initial value ρ influences the direction of motion in the (ξ,η)-plane.

Explicit calculation of the $(2,2)$ dromion solution is too cumbersome to be reproduced here. Fokas and Santini [162] performed the analysis of the asymptotic behavior of the $(2,2)$ dromion. They showed that this solution decays asymptotically into four single-hump constituents. Though the total energy of the $(2,2)$ dromion is conserved, the constituents can exchange energy among themselves. Besides, dromions do not in general preserve their form upon interaction.

10

Generating solutions via $\bar{\partial}$ problem

This chapter is devoted to a brief exposition of the $\bar{\partial}$ formalism, as applied to nonlinear equations. The first three sections deal with the so-called nonlinear equations with *self-consistent sources* (or with *nonanalytic dispersion relations*). This class of nonlinear equations is physically interesting because nonanalytic dispersion relations are directly associated with the resonant interaction of radiation with matter. In Sects. 10.1 and 10.2 we consider the (1+1)-dimensional nonlinear Schrödinger (NLS) and modified NLS equations with self-consistent sources, respectively, along with their gauge equivalents, while Sect. 10.3 is devoted to the Davey–Stewartson I equation with a nonanalytic dispersion relation. We analyze these equations by means of the $\bar{\partial}$ approach. It should be noted that the Riemann–Hilbert (RH) problem could be applied as well for this aim but, in our opinion, the $\bar{\partial}$ approach is frequently the most transparent and leads directly to the final results. In the first three sections, the $\bar{\partial}$ formalism is outlined in a rather unusual setting, but we prove its usefulness for practical calculations.

The last two sections comprise examples of nonlinear equations where using the $\bar{\partial}$ problem is necessary. Namely, in Sect. 10.4 we consider the Kadomtsev–Petviashvili II (KP II) equation and Sect. 10.5 is concerned with the Davey–Stewartson II (DS II) equations. The exposition in Sect. 10.4 is fairly standard. Section 10.5 contains some recent results pertaining to multiple poles of discrete eigenfunctions.

10.1 Nonlinear equations with singular dispersion relations: 1+1 dimensions

One of the ways to generalize nonlinear equations integrable by the IST is to add a source (the so-called self-consistent source) to a given equation. Needless to say, this operation has to preserve the integrability of the equation. Mel'nikov [329] represented the source as a Fourier transform of eigenfunctions of the recursion operator associated with the spectral problem.

Claude et al. [90, 295, 293, 291] related the source to the singular (nonanalytic) component of the dispersion law which follows from the evolution part of the Lax pair. Besides, (1+1)-dimensional nonlinear equations with a source can arise as a result of reductions via the symmetry constraints of (2+1)-dimensional equations [247, 479]. The importance of the singular dispersion relations (SDR) stems from the fact that adding a source transforms, as a rule, the initial-value problem to the initial boundary value problem. As is well known, boundaries play a vital part in many physical applications.

We start the study of nonlinear equations with SDR from the NLS equation with a source. A general approach to the solution of the Cauchy problem for nonlinear equations with SDR associated with the Zakharov–Shabat spectral problem was discussed by Leon [292] in terms of the RH problem. Our aim in this section is mainly to demonstrate the basic rules of working within the framework of the $\bar\partial$ formalism. We will closely follow the formalism used by Beals and Coifman [42] in their review article. A different approach to the $\bar\partial$ problem can be found in [220].

10.1.1 Spectral transform and Lax pair

We start from the matrix $\bar\partial$ problem in the complex k-plane,

$$\bar\partial\psi = \psi R, \tag{10.1}$$

where $R(x,t,k)$ is a spectral transform matrix which will be associated with a nonlinear equation. For simplicity we omit $\bar k$ in arguments of $R(x,t,k)$ and $\psi(x,t,k)$, so the quantities like ψ and R are, in general, nonanalytic in some domains in the k-plane (this may be everywhere in the k-plane). It is the operator $\bar\partial$ that measures the "departure from analyticity," when $\bar\partial\psi \neq 0$. As shown in Sect. 1.11, a solution of the $\bar\partial$ problem (10.1) with the canonical normalization is written as

$$\psi(k) = \mathbb{1} + \frac{1}{2\pi\mathrm{i}} \iint \frac{\mathrm{d}\ell \wedge \mathrm{d}\bar\ell}{\ell - k} \psi(\ell) R(\ell) \equiv \mathbb{1} + \psi R C_k. \tag{10.2}$$

Here C_k is the Cauchy–Green integral operator acting *on the left*. It transforms the argument k to ℓ in the function in front of it and integrates the result with the weight $(2\pi\mathrm{i})^{-1}(\ell - k)^{-1}$ over the whole complex plane. The representation (10.2) enables us to write formally a solution of the $\bar\partial$ problem (10.1) in terms of the matrix R:

$$\psi(k) = \mathbb{1} \cdot (\mathbb{1} - RC_k)^{-1}. \tag{10.3}$$

We will see later that though (10.3) looks rather symbolic, we can do with it all the manipulations we need.

Define a pairing

$$\langle f, g\rangle = \frac{1}{2\pi\mathrm{i}} \iint \mathrm{d}k \wedge \mathrm{d}\bar k f(k) g^{\mathrm{T}}(k), \qquad \langle f, g\rangle^{\mathrm{T}} = \langle g, f\rangle, \tag{10.4}$$

where the superscript T means transposition. The pairing (10.4) possesses easily verified properties:

$$\langle fR, g\rangle = \langle f, gR^{\mathrm{T}}\rangle, \qquad \langle fC_k, g\rangle = -\langle f, gC_k\rangle. \tag{10.5}$$

It is important that the space–time dependence of the matrix $R(x,t,k)$ dictates completely the form of the Lax pair of a given equation. Let the x-dependence be given by a simple linear equation

$$R_x = \mathrm{i}k[R, \sigma_3]. \tag{10.6}$$

Then we can perform with account of (10.3) the following calculation:

$$\begin{aligned}\psi_x(k) &= \mathbb{1}\cdot(\mathbb{1} - RC_k)^{-1}R_xC_k(\mathbb{1} - RC_k)^{-1} = \mathrm{i}k\psi(R\sigma_3 - \sigma_3R)C_k(\mathbb{1} - RC_k)^{-1}\\ &= \mathrm{i}k\psi R\sigma_3C_k(\mathbb{1} - RC_k)^{-1} - \mathrm{i}k\psi\sigma_3RC_k(\mathbb{1} - RC_k)^{-1}.\end{aligned}$$

The first term on the right-hand side is transformed in this way:

$$\mathrm{i}k\psi RC_k = \frac{\mathrm{i}}{2\pi\mathrm{i}}\iint\frac{\mathrm{d}\ell\wedge\mathrm{d}\bar{\ell}}{\ell - k}\ell\psi(\ell)R(\ell) = \frac{\mathrm{i}}{2\pi\mathrm{i}}\iint\mathrm{d}\ell\wedge\mathrm{d}\bar{\ell}\left(1 + \frac{k}{\ell - k}\right)\psi(\ell)R(\ell)$$

$$= \mathrm{i}\langle\psi R, \mathbb{1}\rangle + \mathrm{i}k(\psi RC_k) = \mathrm{i}\langle\psi R, \mathbb{1}\rangle + \mathrm{i}k(\psi - \mathbb{1}) \equiv \mathrm{i}\langle\psi R\rangle + \mathrm{i}k(\psi - \mathbb{1}) \tag{10.7}$$

(sometimes we will omit $\mathbb{1}$ in $\langle\cdot, \mathbb{1}\rangle$ unless its presence is important). As regards the second term, we write $RC_k(\mathbb{1} - RC_k)^{-1} = (\mathbb{1} - RC_k)^{-1} - \mathbb{1}$; hence,

$$\psi_x(k) = \mathrm{i}\langle\psi R\rangle - \mathrm{i}k\sigma_3(\mathbb{1} - RC_k)^{-1} + \mathrm{i}k\psi\sigma_3.$$

Now the only problem is concerned with the term $k(\mathbb{1} - RC_k)^{-1}$. We have from (10.7) $k\psi RC_k = \langle\psi R\rangle + k\psi - k$; hence, $k = \langle\psi R\rangle + k\psi(\mathbb{1} - RC_k)$ and finally

$$k(\mathbb{1} - RC_k)^{-1} = \langle\psi R\rangle\cdot(\mathbb{1} - RC_k)^{-1} + k\psi = (\langle\psi R\rangle + k)\psi. \tag{10.8}$$

As a result, $\psi_x = -\mathrm{i}[\sigma_3, \langle\psi R\rangle]\psi - \mathrm{i}k[\sigma_3, \psi]$. Introducing a potential

$$Q = \begin{pmatrix} 0 & q\\ r & 0\end{pmatrix} = -\mathrm{i}[\sigma_3, \langle\psi R\rangle], \tag{10.9}$$

we arrive at the Zakharov–Shabat spectral problem

$$\psi_x + \mathrm{i}k[\sigma_3, \psi] - Q\psi = 0. \tag{10.10}$$

Hence, the x-equation for R (10.6) with linear dependence on k leads to the ZS spectral problem.

For the time dependence of R we choose a *linear* equation as well:

$$R_t = [R, \Omega], \tag{10.11}$$

where $\Omega(k)$ is a dispersion relation. Suppose $\Omega(k)$ comprises both a polynomial part $\Omega_{\rm p}(k)$ and a singular (nonanalytic) part $\Omega_{\rm s}(k)$. We put

$$\Omega = \Omega_{\rm p} + \Omega_{\rm s} = \alpha_n k^n \sigma_3 + \frac{1}{2\pi {\rm i}} \iint \frac{{\rm d}\ell \wedge {\rm d}\bar\ell}{\ell - k}\, \omega(\ell)\sigma_3. \tag{10.12}$$

Here α_n is a constant and $\omega(k)$ is some scalar function. Note that $\bar\partial \Omega_{\rm s}(k) = \omega(k)\sigma_3$, in accordance with (1.100). To derive the evolution Lax equation, we proceed once again from the representation (10.3). First we consider the polynomial dispersion relation only, $\Omega = 2{\rm i}k^2\sigma_3$. Hence,

$$\begin{aligned}\psi_t &= \psi R_t C_k (\mathbb{1} - RC_k)^{-1} = \psi R \Omega C_k (\mathbb{1} - RC_k)^{-1} - \psi \Omega R C_k (\mathbb{1} - RC_k)^{-1} \\ &= \psi R \Omega C_k (\mathbb{1} - RC_k)^{-1} - \psi \Omega (\mathbb{1} - RC_k)^{-1} + \psi\Omega \\ &= 2{\rm i}\left[k^2 \psi R C_k \sigma_3 (\mathbb{1} - RC_k)^{-1} - k^2 \psi \sigma_3 (\mathbb{1} - RC_k)^{-1}\right] + \psi\Omega.\end{aligned}$$

Note that we cannot factor out k^2 from the brackets because different operators $C_k\sigma_3(\mathbb{1} - RC_k)^{-1}$ and $(\mathbb{1} - RC_k)^{-1}$ act on k^2. The term $k^2\psi RC_k$ is transformed as follows:

$$\begin{aligned}k^2 \psi R C_k &= \frac{1}{2\pi{\rm i}} \iint \frac{{\rm d}\ell \wedge {\rm d}\bar\ell}{\ell - k} \ell^2 \psi(\ell) R(\ell) = \frac{1}{2\pi{\rm i}} \iint {\rm d}\ell \wedge {\rm d}\bar\ell \left(\ell + k + \frac{k^2}{\ell - k}\right) \psi(\ell) R(\ell) \\ &= \langle k\psi R\rangle + k\langle \psi R\rangle + k^2\psi - k^2. \end{aligned} \tag{10.13}$$

This yields

$$\psi_t = 2{\rm i}\left[\langle k\psi R\rangle \sigma_3 \psi + \langle \psi R\rangle \sigma_3 k (\mathbb{1} - RC_k)^{-1} - k^2 \sigma_3 (\mathbb{1} - RC_k)^{-1}\right] + \psi\Omega. \tag{10.14}$$

The expression for $k(\mathbb{1} - RC_k)^{-1}$ was obtained in (10.8). As regards $k^2 (\mathbb{1} - RC_k)^{-1}$, it follows from (10.13) that

$$k^2 = \langle k\psi R\rangle + k\langle \psi R\rangle + k^2 \psi (\mathbb{1} - RC_k)$$

and hence

$$k^2 (\mathbb{1} - RC_k)^{-1} = \left(\langle k\psi R\rangle + \langle \psi R\rangle^2 + k\langle \psi R\rangle + k^2\right)\psi.$$

Then we obtain from (10.14)

$$\psi_t = -2{\rm i}\left([\sigma_3, \langle k\psi R\rangle] + {\rm i}Q\langle \psi R\rangle + {\rm i}kQ + k^2\sigma_3\right)\psi + \psi\Omega. \tag{10.15}$$

We can further simplify this equation owing to the relation between $\langle k\psi R\rangle$ and Q_x. Indeed,

$$Q_x = -{\rm i}[\sigma_3, \langle \psi R\rangle_x] = -2{\rm i}\langle \psi R\rangle_x^{\rm a}.$$

Here the superscript a means the off-diagonal part of a matrix (the superscript d will denote the diagonal part). Further,

$$(\psi R)_x = -{\rm i}[\sigma_3, \langle \psi R, \mathbb{1}\rangle]\psi R - {\rm i}[\sigma_3, k\psi R].$$

Hence, $\langle\psi R\rangle_x = Q\langle\psi R\rangle - \mathrm{i}[\sigma_3, \langle k\psi R\rangle]$ and $\langle\psi R\rangle_x^{\mathrm{a}} = Q\langle\psi R\rangle^{\mathrm{d}} - 2\mathrm{i}\sigma_3\langle k\psi R\rangle^{\mathrm{a}}$. Therefore, we derive the following expression for Q_x:

$$Q_x = -2\mathrm{i}\sigma_3\left(Q\langle\psi R\rangle^{\mathrm{d}} - 2\mathrm{i}\sigma_3\langle k\psi R\rangle^{\mathrm{a}}\right),$$

which enables us to express $\langle k\psi R\rangle^{\mathrm{a}}$ as

$$\langle k\psi R\rangle^{\mathrm{a}} = -\frac{1}{4}Q_x - \frac{\mathrm{i}}{2}\sigma_3 Q\langle\psi R\rangle^{\mathrm{d}}.$$

Substituting this formula into (10.15), we obtain the well -known second Lax equation for the NLS equation:

$$\psi_t = 2\mathrm{i}(-k^2\sigma_3 - \mathrm{i}kQ + \frac{1}{2}\sigma_3 Q_x - \frac{1}{2}\sigma_3 Q^2)\psi + \psi\Omega_{\mathrm{p}}.$$

Now we account for the SDR Ω_{s} (10.12). Again we calculate ψ_t: $\psi_t = (\psi R\Omega_{\mathrm{s}}C_k - \psi\Omega_{\mathrm{s}})(\mathbb{1} - RC_k)^{-1} + \psi\Omega_{\mathrm{s}}$. The first term gives

$$\psi R\Omega_{\mathrm{s}}C_k = \frac{1}{2\pi\mathrm{i}}\iint\frac{\mathrm{d}\ell\wedge\mathrm{d}\bar{\ell}}{\ell-k}\psi(\ell)R(\ell)\frac{1}{2\pi\mathrm{i}}\iint\frac{\mathrm{d}m\wedge\mathrm{d}\overline{m}}{m-\ell}\,\omega(m)\sigma_3.$$

The denominator is transformed as

$$\frac{1}{\ell-k}\frac{1}{m-\ell} = \frac{1}{m-k}\left(\frac{1}{\ell-k} - \frac{1}{\ell-m}\right).$$

Hence,

$$\begin{aligned}\psi R\Omega_{\mathrm{s}}C_k = {} & \frac{1}{2\pi\mathrm{i}}\iint\frac{\mathrm{d}\ell\wedge\mathrm{d}\bar{\ell}}{\ell-k}\psi(\ell)R(\ell)\frac{1}{2\pi\mathrm{i}}\iint\frac{\mathrm{d}m\wedge\mathrm{d}\overline{m}}{m-k}\,\omega(m)\sigma_3 \\ & - \frac{1}{2\pi\mathrm{i}}\iint\frac{\mathrm{d}m\wedge\mathrm{d}\overline{m}}{m-k}\,\omega(m)\frac{1}{2\pi\mathrm{i}}\iint\frac{\mathrm{d}\ell\wedge\mathrm{d}\bar{\ell}}{\ell-m}\psi(\ell)R(\ell)\sigma_3.\end{aligned}$$

The first term of this formula gives $[\psi(k) - \mathbb{1}]\Omega(k)$, while the second one produces

$$-\iint\frac{\mathrm{d}m\wedge\mathrm{d}\overline{m}}{m-k}\,\psi(m)\omega(m)\sigma_3 + \Omega(k),$$

because the integral over ℓ gives $\psi RC_m = \psi(m) - \mathbb{1}$. As a result,

$$\psi_t = \frac{1}{2\pi\mathrm{i}}\iint\frac{\mathrm{d}m\wedge\mathrm{d}\overline{m}}{k-m}\,\psi(m)\omega(m)\sigma_3(\mathbb{1} - RC_k)^{-1} + \psi\Omega_{\mathrm{s}}.$$

How does one calculate $(k-m)^{-1}(\mathbb{1} - RC_k)^{-1}$ in this integral? Following our above experience, we consider first the term $(k-m)^{-1}\psi RC_k$:

$$\frac{1}{k-m}\psi RC_k = \frac{1}{2\pi\mathrm{i}}\iint\frac{\mathrm{d}\ell\wedge\mathrm{d}\bar{\ell}}{\ell-k}\frac{1}{\ell-m}\psi(\ell)R(\ell)$$

$$= \frac{1}{k-m}\frac{1}{2\pi\mathrm{i}}\iint \frac{\mathrm{d}\ell\wedge\mathrm{d}\bar\ell}{\ell-k}\psi(\ell)R(\ell) - \frac{1}{k-m}\frac{1}{2\pi\mathrm{i}}\iint \frac{\mathrm{d}\ell\wedge\mathrm{d}\bar\ell}{\ell-m}\psi(\ell)R(\ell)$$

$$= \frac{1}{k-m}\left[(\psi(k)-\mathbb{1}) - (\psi(m)-\mathbb{1})\right] = \frac{1}{k-m}\left[\psi(k)-\psi(m)\right].$$

Hence,

$$\frac{1}{k-m}\psi(k)(\mathbb{1}-RC_k) = \frac{1}{k-m}\psi(m).$$

Taking the inverse of the last formula yields

$$\frac{1}{k-m}(\mathbb{1}-RC_k)^{-1} = \frac{1}{k-m}\psi^{-1}(m)\psi(k).$$

Therefore,

$$\psi_t = -\frac{1}{2\pi\mathrm{i}}\iint \frac{\mathrm{d}\ell\wedge\mathrm{d}\bar\ell}{\ell-k}\,\omega(\ell)\psi(\ell)\sigma_3\psi^{-1}(\ell)\psi(k) + \psi\Omega_\mathrm{s} = V_\mathrm{s}\psi + \psi\Omega_\mathrm{s}. \quad (10.16)$$

Hence, we derived the Lax operator V_s with distinctive squared eigenfunction structure $\psi\sigma_3\psi^{-1}$.

In fact, working with the $\bar\partial$ method, we need neither the spectral problem (10.10) nor the evolution part (10.15) and (10.16) of the Lax pair. All we need is the $\bar\partial$ problem (10.1) together with the linear equations (10.6) and (10.11) governing the space–time dependence of the spectral transform $R(x,t,k)$. Indeed, we will establish later a gauge equivalence of the NLS and Heisenberg spin chain equations with SDR, derive the recursion operators for them, and find their soliton solutions without any resort to the Lax pair. It should be also noted that when searching for the time evolution of spectral data, we use only the asymptotic of the second Lax operator, i.e., the dispersion relation. Just this necessary information is contained in the evolution equation (10.11) for the spectral transform matrix.

10.1.2 Recursion operator

By means of (10.9) and (10.11) we will find the time evolution of the potential Q:

$$Q_t = -\mathrm{i}[\sigma_3, \langle\psi R\rangle_t].$$

Calculation gives

$$(\psi R)_t = \bar\partial\psi_t = \bar\partial\left\{\mathbb{1}\cdot(\mathbb{1}-RC_k)^{-1}R_iC_k(\mathbb{1}-RC_k)^{-1}\right\}$$

$$= \bar\partial\left\{\psi R_tC_k(\mathbb{1}-RC_k)^{-1}\right\} = \bar\partial\left\{\psi R_t(\mathbb{1}-RC_k)^{-1}\right\}C_k = \psi R_t(\mathbb{1}-RC_k)^{-1}.$$

In performing the last step we made use of the evident relation $\bar\partial f(k)C_k = f(k)$. Therefore, in virtue of the properties (10.5), we can write

$$Q_t = -\mathrm{i}\left[\sigma_3\langle\psi R_t(\mathbb{1}-RC_k)^{-1},\mathbb{1}\rangle\right] = -\mathrm{i}\left[\sigma_3,\langle\psi R_t,\mathbb{1}\cdot(\mathbb{1}+R^\mathrm{T}C_k)^{-1}\rangle\right]. \quad (10.17)$$

It can be shown, owing to $\bar\partial(\psi^{-1})^{\mathrm{T}} = -(\psi^{-1})^{\mathrm{T}}R^{\mathrm{T}}$, that

$$\mathbb{1}\cdot(\mathbb{1}+R^{\mathrm{T}}C_k)^{-1} = (\psi^{-1})^{\mathrm{T}}. \tag{10.18}$$

Hence,

$$Q_t = -\mathrm{i}\left[\sigma_3, \langle\psi(R\Omega-\Omega R),(\psi^{-1})^{\mathrm{T}}\rangle\right] = -\mathrm{i}\left[\sigma_3, \langle(\bar\partial\psi)\Omega\psi^{-1}, \mathbb{1}\rangle\right]$$
$$+\mathrm{i}\left[\sigma_3, \langle\psi\Omega, \bar\partial(\psi^{-1})^{\mathrm{T}}\rangle\right] = -\mathrm{i}\left[\sigma_3, \langle(\bar\partial\psi)\Omega\psi^{-1}\rangle + \langle\psi\Omega\bar\partial\psi^{-1}\rangle\right]$$
$$= -\mathrm{i}\left[\sigma_3, \langle\bar\partial(\psi\Omega_{\mathrm{p}}\psi^{-1})\rangle - \langle\psi(\bar\partial\Omega_{\mathrm{s}})\psi^{-1}\rangle\right],$$

where we have used $\Omega_{\mathrm{s}} \to 0$ at $k\to\infty$. Taking into account (10.12), we obtain

$$Q_t = -\mathrm{i}\alpha_n[\sigma_3, \langle\bar\partial(k^n\psi\sigma_3\psi^{-1})\rangle] + \mathrm{i}[\sigma_3, \langle\omega(k)\psi\sigma_3\psi^{-1}\rangle]. \tag{10.19}$$

Denote $M(k) = \psi\sigma_3\psi^{-1}$. This function satisfies the equation

$$M_x + \mathrm{i}k[\sigma_3, M] - [Q, M] = 0. \tag{10.20}$$

The next steps in deriving the recursion operator are rather standard [148]. Let us write the 2×2 matrix M as a sum of diagonal and off-diagonal parts,

$$M = M^{\mathrm{d}} + M^{\mathrm{a}} = \frac{1}{2}\sigma_3\mathrm{tr}(M\sigma_3) + \frac{1}{4}\left[\sigma_3, [\sigma_3, M]\right]. \tag{10.21}$$

Then (10.20) gives two equations

$$M^{\mathrm{d}}_x = [Q, M^{\mathrm{a}}], \qquad M^{\mathrm{a}}_x + 2\mathrm{i}k\sigma_3 M^{\mathrm{a}} = [Q, M^{\mathrm{d}}]. \tag{10.22}$$

According to the asymptotic condition $\psi\to\mathbb{1}$ at $x\to\infty$, we obtain from the first equation in (10.22) $M^{\mathrm{d}} = \sigma_3 + \partial^{-1}[Q, M^{\mathrm{a}}]$, which enables us to write the second equation as

$$M^{\mathrm{a}}_x + 2\mathrm{i}k\sigma_3 M^{\mathrm{a}} = \left[Q, \sigma_3 + \partial^{-1}[Q, M^{\mathrm{a}}]\right]. \tag{10.23}$$

Hence, it is natural to introduce a recursion operator in the form

$$\Lambda\cdot = \frac{\mathrm{i}}{2}\sigma_3\left(\partial_x - \left[Q, \partial^{-1}[Q, \cdot]\right]\right), \tag{10.24}$$

which evidently does not depend on k. Then (10.23) gives $M^{\mathrm{a}} = -\mathrm{i}(\Lambda - k)^{-1}Q$ and

$$Q_t = -\alpha_n\left[\sigma_3, \langle\bar\partial\left(k^n(\Lambda-k)^{-1}\right)Q\rangle\right] + \mathrm{i}[\sigma_3, \langle\omega(k)M(k)\rangle].$$

Now we expand $(\Lambda-k)^{-1}$ in a series, $(\Lambda-k)^{-1} = -\sum_{j=1}^{\infty}k^{-j}\Lambda^{j-1}$, and make use of $\bar\partial k^{n-j} = \pi\delta(k)\delta_{j,n+1}$. Then

$$\left[\sigma_3, \sum_{j=1}^{\infty}\langle\bar\partial k^{n-j}\rangle\Lambda^{j-1}Q\right] = -[\sigma_3, \Lambda^n Q] = -2\sigma_3\Lambda^n Q.$$

As a result, we obtain the hierarchy of equations with SDR associated with the particular x-dependence (10.6) of the spectral transform (or, in other words, with the ZS spectral problem):

$$\begin{aligned} &Q_t + 2\alpha_n \sigma_3 \Lambda^n Q = \mathrm{i}[\sigma_3, \langle \omega(k) M(k) \rangle], \\ &M_x = [-\mathrm{i}k\sigma_3 + Q, M]. \end{aligned} \tag{10.25}$$

Let us consider two examples of the initial boundary value problem described by the system (10.25).

1. $n = 1$, $\alpha_1 = -\mathrm{i}$, $\omega(k) = \pi g(k_\mathrm{R})\delta(k_\mathrm{I})$, $k = k_\mathrm{R} + \mathrm{i}k_\mathrm{I}$,

$$Q = \frac{1}{2}\begin{pmatrix} 0 & E \\ -E^* & 0 \end{pmatrix}, \qquad M = \frac{\mathrm{i}}{2}\begin{pmatrix} -n & p \\ p^* & n \end{pmatrix}.$$

   In this case $\Lambda Q = (\mathrm{i}/2)\sigma_3 Q_x$. Therefore,

$$E_t + E_x = \langle\langle p \rangle\rangle, \quad p_x + 2\mathrm{i}k_\mathrm{R} p = En, \quad n_x = -\frac{1}{2}(E\bar p + \bar E p). \tag{10.26}$$

   Here the double brackets $\langle\langle p \rangle\rangle = \int_{-\infty}^{\infty} \mathrm{d}k_\mathrm{R} g(k_\mathrm{R}) p(k_\mathrm{R})$ stand for the average over the inhomogeneous broadening with the distribution function $g(k_\mathrm{R})$, when referring to the model of the radiation–matter interaction. Equations (10.26) are solved with the initial condition for E at $t = 0$ and boundary conditions for p and N at $x \to -\infty$ or $x \to +\infty$. Equations (10.26) are of the Maxwell–Bloch equation type and describe a number of phenomena like self-induced transparency [259, 325] and stimulated Raman scattering [91, 230, 410]. Leon [294] has shown that the system (10.26) is integrable for arbitrary boundary values.
2. $n = 2$, $\Lambda^2 Q = -(1/4)Q_{xx} + (1/2)Q^3$,

$$Q = \begin{pmatrix} 0 & E \\ -E^* & 0 \end{pmatrix}, \qquad M = \mathrm{i}\begin{pmatrix} -n & p \\ p^* & n \end{pmatrix}.$$

   Equations (10.25) give

$$\mathrm{i}E_t + E_{xx} + 2|E|^2 E = 2\mathrm{i}\langle\langle p \rangle\rangle, \quad p_x + 2\mathrm{i}k_\mathrm{R} p = -2En, \quad n_x = -(E\bar p + \bar E p). \tag{10.27}$$

   These equations (without inhomogeneous broadening) were derived from physical motivations by Doktorov and Vlasov [127, 441] to describe the dynamics of picosecond optical pulses in a combined resonant-cubic medium. This system can also be applied to nonlinear interaction of the electrostatic high-frequency wave with an ion-acoustic wave in two-component homogeneous plasma [90].

10.1.3 NLS–Maxwell–Bloch soliton

Here we will obtain the soliton solution of equations (10.27) within the $\bar\partial$ method. The soliton corresponds to the spectral transform matrix located at

the points k_1 and $\bar{k}_1$ of the complex plane where a solution ψ of the $\bar{\partial}$ problem has simple poles. Namely,

$$R = 2\pi \mathrm{i} \mathrm{e}^{-\mathrm{i}k\sigma_3 x} \begin{pmatrix} 0 & c\delta(k-k_1) \\ \bar{c}\delta(k-\bar{k}_1) & 0 \end{pmatrix} \mathrm{e}^{\mathrm{i}k\sigma_3 x}, \tag{10.28}$$

where $c = c(t)$ and this time dependence should be found from (10.11). As follows from (10.9), the soliton solution is given by

$$q_\mathrm{s} = -2\mathrm{i}\langle \psi R \rangle_{12} = -\frac{1}{\pi} \iint \mathrm{d}k \wedge \mathrm{d}\bar{k} \psi_{11}(k) R_{12}(k). \tag{10.29}$$

Substituting the explicit form of R (10.28) into (10.29), we derive a linear algebraic system

$$\psi_{11}(k) = 1 + \frac{2\mathrm{i}\bar{c}}{k-k_1} \psi_{12}(\bar{k}_1) \mathrm{e}^{2\mathrm{i}\bar{k}_1 x}, \quad \psi_{12} = \frac{2\mathrm{i}c}{k-k_1} \psi_{11}(k_1) \mathrm{e}^{-2\mathrm{i}k_1 x}.$$

Solving this system with respect to $\psi_{11}(k)$ and substituting the result into (10.29) yields

$$q_\mathrm{s} = 2\eta \mathrm{e}^{-2\mathrm{i}\xi x + \mathrm{i}\phi} \mathrm{sech} 2\eta(x - x_0). \tag{10.30}$$

Here $k_1 = \xi + \mathrm{i}\eta$, $c = -\eta \exp(-2\eta x_0 + \mathrm{i}\phi)$, and x_0 and ϕ are time-dependent. The time dependence is found from (10.11). Let us denote

$$\Omega_\mathrm{s}(k_1) = \frac{1}{2\pi\mathrm{i}} \iint \frac{\mathrm{d}\ell \wedge \mathrm{d}\bar{\ell}}{\ell - k_1} \omega(\ell)\sigma_3 \equiv (\omega_1 - \mathrm{i}\omega_2)\sigma_3.$$

Then $c_t = -2c(2\mathrm{i}k_1^2 + \omega_1 - \mathrm{i}\omega_2)$. On the other hand, $c_t = c(-2\eta x_{0t} + \mathrm{i}\phi_t)$. Comparing these two equations, we find

$$x_0 = \left(-4\xi + \frac{\omega_1}{\eta} \right) t + \xi_0, \quad \phi = -4(\xi^2 - \eta^2)t + 2\omega_2 t + \phi_0, \tag{10.31}$$

where ξ_0 and ϕ_0 are constants. As a result, the soliton solution of the NLS–Maxwell–Bloch system has the form of the standard NLS soliton (8.32) but with modulated velocity and phase. This modulation is caused by the resonant component of the combined medium and manifests itself through the frequencies ω_1 and ω_2. Other quantities p and n are easily calculated with the known soliton solution (10.30) by means of (10.26).

10.1.4 Gauge equivalence

Up to now we considered the $\bar{\partial}$ problem with the canonical normalization $\psi(k) \to \mathbb{1} + \mathcal{O}(1/k)$. To what extent is the demand of the canonical normalization critical ? To answer this question, let us multiply (10.2) by a nondegenerate matrix $g^{-1}(x,t)$ from the left [matrix function $g(x,t)$ does not depend on k] and put $\varphi(k) = g^{-1}\psi(k)$. Then evidently we get

$$\varphi(k) = g^{-1}(\mathbb{1} - RC_k)^{-1}. \tag{10.32}$$

Calculation of the x-derivative of φ yields

$$\varphi_x = \left(-g^{-1}g_x + \mathrm{i}\langle\varphi R\rangle\sigma_3 g - \mathrm{i}g^{-1}\sigma_3 g\langle\varphi R\rangle g\right)\varphi - \mathrm{i}kg^{-1}\sigma_3 g\varphi + \mathrm{i}k\varphi\sigma_3. \quad (10.33)$$

Now we choose $g(x,t)$ in such a way that the expression in brackets in (10.33) vanishes. This gives $g_x + \mathrm{i}[\sigma_3, \langle\psi R\rangle]g = 0$ or $g_x = Qg$. It means that we can put $g = \psi(k=0)$, with ψ being a solution of the $\bar\partial$ problem. Denoting

$$S(x,t) = g^{-1}\sigma_3 g, \quad (10.34)$$

we obtain from (10.32)

$$\varphi_x + \mathrm{i}kS\varphi - \mathrm{i}k\varphi\sigma_3 = 0. \quad (10.35)$$

This equation is nothing more than the spectral problem for the Heisenberg spin chain model [424] which is described by the equation

$$S_t = \frac{1}{2\mathrm{i}}[S, S_{xx}], \qquad S^2 = \mathbb{1}. \quad (10.36)$$

In the same way we could derive the evolution part of the Lax pair for (10.36) and reproduce the well-known gauge equivalence between the Lax operators for the NLS and Heisenberg equations [476]. However, because we work with the $\bar\partial$ problem, we do not need the Lax operators. Within the $\bar\partial$ method, the gauge equivalence is realized as a change of the normalization condition by means of the function $\psi(k=0)$.

10.1.5 Recursion operator for Heisenberg spin chain equation with SDR

To derive the recursion operator for the Heisenberg spin chain equation with SDR, we start, as in Sect. 10.1.2, from the calculation of the time derivative of S: $S_t = [S, g^{-1}g_t]$. Because

$$g = \psi(k=0) = \mathbb{1} + \frac{1}{2\pi\mathrm{i}}\iint \frac{\mathrm{d}\ell\wedge\mathrm{d}\bar\ell}{\ell}\psi(\ell)R(\ell) = \mathbb{1} + \langle\frac{1}{k}\psi R, \mathbb{1}\rangle,$$

we obtain

$$g_t = \langle\frac{1}{k}\psi R_t(\mathbb{1} - C_k R)^{-1}, \mathbb{1}\rangle = \langle\psi R_t, \frac{1}{k}(\mathbb{1} + R^{\mathrm{T}}C_k)^{-1}\rangle. \quad (10.37)$$

We have shown already [see (10.18)] that the operator $\mathbb{1} + R^{\mathrm{T}}C_k$ is intimately related to the function $(\psi^{-1})^{\mathrm{T}}$. Therefore, to calculate $(1/k)(\mathbb{1} + R^{\mathrm{T}}C_k)^{-1}$ in (10.37), we begin with

$$\frac{1}{k}(\psi^{-1})^{\mathrm{T}}R^{\mathrm{T}}C_k = \frac{1}{2\pi\mathrm{i}}\iint \frac{\mathrm{d}\ell\wedge\mathrm{d}\bar\ell}{\ell - k}\frac{1}{\ell}(\psi^{-1})^{\mathrm{T}}(\ell)R^{\mathrm{T}}(\ell)$$

$$= \frac{1}{k}\left(\frac{1}{2\pi\mathrm{i}}\iint\frac{\mathrm{d}\ell\wedge\mathrm{d}\bar{\ell}}{\ell-k}(\psi^{-1})^{\mathrm{T}}(\ell)R^{\mathrm{T}}(\ell) - \frac{1}{2\pi\mathrm{i}}\iint\frac{\mathrm{d}\ell\wedge\mathrm{d}\bar{\ell}}{\ell}(\psi^{-1})^{\mathrm{T}}(\ell)R^{\mathrm{T}}(\ell)\right)$$

$$= \frac{1}{k}\left[(\psi^{-1})^{\mathrm{T}}R^{\mathrm{T}}C_k - (\psi^{-1})^{\mathrm{T}}R^{\mathrm{T}}C_0\right] = \frac{1}{k}\left[(\psi^{-1})^{\mathrm{T}}(0) - (\psi^{-1})^{\mathrm{T}}(k)\right],$$

because $(\psi^{-1})^{\mathrm{T}}R^{\mathrm{T}}C_k = \mathbb{1} - (\psi^{-1})^{\mathrm{T}}$. Therefore, $(1/k)(\psi^{-1})^{\mathrm{T}}(\mathbb{1}+R^{\mathrm{T}}C_k)^{-1} = k^{-1}(\psi^{-1})^{\mathrm{T}}(0)$, or

$$\frac{1}{k}(\mathbb{1}+R^{\mathrm{T}}C_k)^{-1} = \frac{1}{k}\psi^{\mathrm{T}}(0)(\psi^{-1})^{\mathrm{T}}(k). \tag{10.38}$$

Continuing the calculation in (10.37), we get

$$g_t = \langle\psi R_t, \frac{1}{k}\psi^{\mathrm{T}}(0)(\psi^{-1})^{\mathrm{T}}(k)\rangle = \langle\frac{1}{k}\psi R\Omega\psi^{-1}, \mathbb{1}\rangle g - \langle\frac{1}{k}\psi\Omega R, (\psi^{-1})^{\mathrm{T}}\rangle g$$

$$= \langle\frac{1}{k}(\bar{\partial}\psi)\Omega\psi^{-1}, \mathbb{1}\rangle g + \langle\frac{1}{k}\psi\Omega, \bar{\partial}(\psi^{-1})^{\mathrm{T}}\rangle g = \langle\frac{1}{k}\bar{\partial}(\psi\Omega_p\psi^{-1})\rangle g - \langle\frac{1}{k}\omega_s\psi\sigma_3\psi^{-1}\rangle g.$$

Consider first the polynomial dispersion relation $\Omega_{\mathrm{p}} = \alpha_n k^n\sigma_3$. Because $g(x,t)$ does not depend on k, we can insert it into the brackets $\langle\cdots\rangle$. Then $g^{-1}g_t = \alpha_n\langle(1/k)\bar{\partial}(k^n\varphi\sigma_3\varphi^{-1})\rangle$. As a result,

$$S_t = \alpha_n\left[S, \langle\frac{1}{k}\bar{\partial}(k^n\varphi\sigma_3\varphi^{-1})\rangle\right]. \tag{10.39}$$

Denote $\varphi\sigma_3\varphi^{-1} = \mathcal{M}$, then $\mathcal{M}_x = -\mathrm{i}k[S,\mathcal{M}]$. Let us introduce a moving trihedral element $\tilde{\sigma}_\pm = g^{-1}\sigma_\pm g$ and $\tilde{\sigma}_3 = g^{-1}\sigma_3 g = S$, where $\sigma_\pm = \sigma_1 \pm \mathrm{i}\sigma_2$ are Pauli matrices, and a covariant derivative

$$\nabla_x\mathcal{M} = \mathcal{M}_x + [g^{-1}g_x, \mathcal{M}].$$

The trihedral elements are covariantly constant, $\nabla_x\tilde{\sigma}_\alpha = 0$; therefore, we can write $\nabla_x^{-1}\sum_\alpha\tilde{\sigma}_\alpha Q_\alpha = \sum_\alpha\tilde{\sigma}_\alpha\int^x\mathrm{d}yQ_\alpha(y)$. It can be easily shown that $g^{-1}g_x$ is written in terms of S: $g^{-1}g_x = (1/4)[S,S_x]$; hence,

$$\nabla_x\mathcal{M} = -\mathrm{i}k[S,\mathcal{M}] + \frac{1}{4}[S,[S,\mathcal{M}]]. \tag{10.40}$$

Now we decompose the 2×2 matrix $\mathcal{M}$ into "diagonal" and "off-diagonal" parts,

$$\mathcal{M} = \mathcal{M}^{\mathrm{d}} + \mathcal{M}^{\mathrm{a}} = \frac{1}{2}S\,\mathrm{tr}(\mathcal{M}S) + \frac{1}{4}[S,[S,\mathcal{M}]]\ ;$$

cf. (10.21). The quotation marks are relevant because diagonality and off-diagonality are used here with respect to the moving trihedral elements. In other words, $S = S^{\mathrm{d}}$, $[S^d,\mathcal{M}^{\mathrm{d}}] = 0$, $[S,\mathcal{M}^{\mathrm{a}}] = 2S\mathcal{M}^{\mathrm{a}}$, etc., but the upper indices d and a do not mean the usual diagonality and off-diagonality.

Decomposing (10.40) yields

$$\nabla_x \mathcal{M}^{\mathrm{d}} = \frac{1}{4}\left[[S, S_x], \mathcal{M}^{\mathrm{a}}\right], \quad \nabla_x \mathcal{M}^{\mathrm{a}} + 2\mathrm{i}kS\mathcal{M}^{\mathrm{a}} = \frac{1}{4}\left[[S, S_x], \mathcal{M}^{\mathrm{d}}\right]. \quad (10.41)$$

Hence,

$$\mathcal{M}^{\mathrm{d}} = S + \frac{1}{4}\int^x \mathrm{d}y\,[[S, S_y], \mathcal{M}^{\mathrm{a}}(y)]\,.$$

Inserting this $\mathcal{M}^{\mathrm{d}}$ into the second equation in (10.41), we get

$$\nabla_x \mathcal{M}^{\mathrm{a}} + 2\mathrm{i}kS\mathcal{M}^{\mathrm{a}} = \frac{1}{4}\left[[S, S_x], S\right] + \frac{1}{16}\left[[S, S_x], \int^x \mathrm{d}y\,[[S, S_y], \mathcal{M}^{\mathrm{a}}]\right].$$

The first term on the right-hand side gives $-S_x$. Introduce now the recursion operator Λ':

$$\Lambda'\cdot = \frac{\mathrm{i}}{2}\left(\nabla_x - \frac{1}{16}\left[[S, S_x], \int^x \mathrm{d}y\,[[S, S_y], \cdot]\right]\right).$$

We have $(\Lambda' - k)\mathcal{M}^{\mathrm{a}} = -(\mathrm{i}/4)[S, S_x]$ and therefore

$$\mathcal{M}^{\mathrm{a}} = -\frac{\mathrm{i}}{4}\left(\Lambda' - k\right)^{-1}[S, S_x].$$

As a result, equation (10.39) yields ($n \geq 1$)

$$S_t = 2\alpha_n S\langle\bar{\partial}(k^{n-1}\mathcal{M}^{\mathrm{a}})\rangle = \frac{\mathrm{i}}{2}\alpha_n S\langle\bar{\partial}\left(k^{n-1}\sum_{m=1}^{\infty} k^{-m}\right)\rangle \Lambda'^{m-1}[S, S_x]$$

$$= \frac{\mathrm{i}}{2}\alpha_n \pi S\langle\delta(k)\rangle \sum_{m=1}^{\infty}\delta_{mn}\Lambda'^{m-1}[S, S_x] = -\frac{\mathrm{i}}{2}\alpha_n S\Lambda'^{n-1}[S, S_x],$$

because $\bar{\partial}(k^{-n}) = \pi\delta(k)\delta_{n1}$. Hence, the Heisenberg spin chain hierarchy with SDR is given by

$$S_t + \frac{\mathrm{i}}{2}\,\alpha_n S\Lambda'^{n-1}[S, S_x] = -2S\langle\frac{1}{k}\,\omega_s(k)\mathcal{M}^{\mathrm{a}}\rangle, \quad (10.42)$$

$$\nabla_x \mathcal{M}^{\mathrm{d}} = \frac{1}{4}\left[[S, S_x], \mathcal{M}^{\mathrm{a}}\right], \quad \nabla_x \mathcal{M}^{\mathrm{a}} + 2\mathrm{i}kS\mathcal{M}^{\mathrm{a}} = \frac{1}{4}\left[[S, S_x], \mathcal{M}^{\mathrm{d}}\right].$$

The Heisenberg spin chain equation with SDR corresponds to $n = 2$ and $\alpha_2 = -2\mathrm{i}$.

Let us compare the recursion operators for the NLS and Heisenberg equations:

$$(\Lambda - k)M^{\mathrm{a}} = -\mathrm{i}Q \quad \text{and} \quad (\Lambda' - k)\mathcal{M}^{\mathrm{a}} = -\frac{\mathrm{i}}{4}[S, S_x].$$

It is trivial to show that the right-hand sides are interconnected by the gauge transformation, $(1/4)[S, S_x] = g^{-1}Qg$. Therefore, the same connection should exist for the recursion operators, $\Lambda' = g^{-1}\Lambda g$. We omit rather technical calculation that proves this fact.

10.2 Nonlinear evolutions with singular dispersion relation for quadratic bundle

In this section we extend the approach developed in Sect. 10.1 to the case of the spectral transform matrix whose x-evolution is dictated by the quadratic-in-k equation [126]. We saw in the previous section that the linear-in-k equation $R_x = \mathrm{i}k[R, \sigma_3]$ leads to the ZS spectral problem. Hence, we can expect that the present case of k^2 will lead us to equations intimately related to the modified NLS equation which was analyzed in Chap. 8. We know from this analysis that the modified NLS equation is noncanonical (in the sense of the RH normalization condition) in the class of equations integrable by the quadratic bundle. It will be seen that the same noncanonicity occurs for the $\bar\partial$ problem.

10.2.1 $\bar\partial$ Problem and recursion operator

As in Sect. 10.1, we begin with the $\bar\partial$ problem

$$\bar\partial\psi(k) = \psi(k)R(k), \quad \psi(k) = \mathbb{1} + \mathcal{O}(1/k), \quad k \to \infty. \tag{10.43}$$

Consider the spectral transform as an off-diagonal matrix even in k, $R(-k) = R(k)$. This condition resembles the parity property of the Jost solutions. The spatial and temporal dependences of $R(k)$ are given by

$$R_x = \frac{\mathrm{i}}{\alpha}(k^2 + \beta)[R, \sigma_3], \qquad R_t = [R, \Omega], \tag{10.44}$$

where α and β are real parameters, and the dispersion relation $\Omega = \Omega_\mathrm{p} + \Omega_\mathrm{s}$ includes both the polynomial Ω_p and singular Ω_s parts,

$$\Omega_\mathrm{p} = \omega_\mathrm{p}\sigma_3 = \sum_{j=0}^{J} \gamma_{2j} k^{2j}\sigma_3, \quad \Omega_\mathrm{s} = \frac{1}{2\pi\mathrm{i}} \iint \frac{\mathrm{d}\ell \wedge \mathrm{d}\bar\ell}{\ell^2 - k^2} \ell^2 \omega(\ell^2)\sigma_3, \quad \bar\partial\Omega_\mathrm{s} = \omega_s(k^2). \tag{10.45}$$

As before, the solution of the $\bar\partial$ problem (10.43) with the canonical normalization is given by the Cauchy–Green integral formula

$$\psi(k) = \mathbb{1} + \frac{1}{2\pi\mathrm{i}} \iint \frac{\mathrm{d}\ell \wedge \mathrm{d}\bar\ell}{\ell - k} \psi(\ell)R(\ell) \equiv \mathbb{1} + \psi R C_k, \tag{10.46}$$

or

$$\psi(k) = \mathbb{1} \cdot (\mathbb{1} - RC_k)^{-1} \tag{10.47}$$

with the diagonal and off-diagonal parts of ψ obeying the parity properties

$$\psi^\mathrm{d}(-k) = \psi^\mathrm{d}(k), \qquad \psi^\mathrm{a}(-k) = -\psi^\mathrm{a}(k). \tag{10.48}$$

In view of (10.44), differentiation of (10.47) in x gives

$$\psi_x = -\frac{\mathrm{i}}{\alpha}(k^2+\beta)[\sigma_3,\psi]-\frac{\mathrm{i}}{\alpha}k[\sigma_3,\langle\psi R\rangle]\psi-\frac{\mathrm{i}}{\alpha}[\sigma_3,\langle\psi R\rangle]\langle\psi R\rangle\psi-\frac{\mathrm{i}}{\alpha}[\sigma_3,\langle k\psi R\rangle]\psi. \tag{10.49}$$

In virtue of the parity property, $\langle\psi R\rangle$ is an off-diagonal matrix, while $\langle k\psi R\rangle$ is the diagonal one leading to $[\sigma_3,\langle k\psi R\rangle]=0$. Define a potential Q:

$$Q(x,t)=\begin{pmatrix} 0 & q\\ -\bar q & 0\end{pmatrix}=-\frac{\mathrm{i}}{\alpha}[\sigma_3,\langle\psi R\rangle]. \tag{10.50}$$

Then (10.49) gives the linear spectral problem of the Wadati–Konno–Ichikawa (WKI) type:

$$\psi_x=-\frac{\mathrm{i}}{\alpha}(k^2+\beta)[\sigma_3,\psi]+kQ\psi-\frac{\mathrm{i}\alpha}{2}Q^2\sigma_3\psi. \tag{10.51}$$

Now we obtain a hierarchy of evolutions Q_t. From (10.50) we get $Q_t=-\mathrm{i}\alpha^{-1}[\sigma_3,\langle\psi R\rangle_t]$. The right-hand side is transformed as

$$(\psi R)_t=\bar\partial\psi_t=\bar\partial\left[\mathbb{1}\cdot(\mathbb{1}-RC_k)_t^{-1}\right]=\bar\partial\left[\psi R_tC_k(\mathbb{1}-RC_k)^{-1}\right]$$

$$=\bar\partial\left[\psi R_t(\mathbb{1}-C_kR)^{-1}\right]C_k=\psi R_t(\mathbb{1}-C_kR)^{-1}.$$

Continuing, we get

$$Q_t=-\frac{\mathrm{i}}{\alpha}[\sigma_3,\langle\psi R_t(\mathbb{1}-C_kR)^{-1},\mathbb{1}\rangle]=-\frac{\mathrm{i}}{\alpha}[\sigma_3,\langle\psi R_t,\mathbb{1}\cdot(\mathbb{1}+R^{\mathrm{T}}C_k)^{-1}\rangle].$$

Here we take into account once again that $\mathbb{1}\cdot(\mathbb{1}+R^{\mathrm{T}}C_k)^{-1}=(\psi^{-1})^{\mathrm{T}}(k)$. Consequently, we arrive at

$$Q_t=-\frac{\mathrm{i}}{\alpha}\left[\sigma_3,\langle\psi R\Omega\psi^{-1},\mathbb{1}\rangle-\langle\psi\Omega,(\psi^{-1})^{\mathrm{T}}R^{\mathrm{T}}\rangle\right]$$

$$=-\frac{\mathrm{i}}{\alpha}\left[\sigma_3,\langle\bar\partial(\psi\Omega_p\psi^{-1})\rangle-\langle(\bar\partial\Omega_s)\psi^{-1}\rangle\right]$$

$$=-\frac{\mathrm{i}}{\alpha}\left[\sigma_3,\sum_{j=0}^{n}\gamma_{2j}\langle\bar\partial(k^{2j}M)\rangle\right]+\frac{\mathrm{i}}{\alpha}\left[\sigma_3,\langle k\omega_s(k^2)M\rangle\right],$$

where $M(k)=\psi(k)\sigma_3\psi^{-1}(k)$. This function satisfies the associated spectral equation

$$M_x+\frac{\mathrm{i}}{\alpha}\left(k^2+\beta+\frac{1}{2}\alpha^2Q^2\right)[\sigma_3,M]-k[Q,M]=0 \tag{10.52}$$

and can be expanded in the asymptotic series

$$M = \sigma_3 + \sum_{\ell=1}^{\infty} \frac{M^{(\ell)}}{k^\ell},$$

with $M^{(2\ell+1)}$ being off-diagonal and $M^{(2\ell)}$ being diagonal matrices. They satisfy the following equations:

$$M_x^{(2\ell)} = [Q, M^{(2\ell+1)}],$$
$$M_x^{(2\ell-1)} = -\frac{\mathrm{i}}{\alpha}[\sigma_3, M^{(2\ell+1)}] - \frac{\mathrm{i}}{\alpha}\left(\beta + \frac{1}{2}\alpha^2 Q^2\right)[\sigma_3, M^{(2\ell-1)}] + [Q, M^{(2\ell)}].$$

Hence,

$$M^{(2\ell)} = \sigma_3 \delta_{\ell 0} + \int^x \mathrm{d}x [Q, M^{(2\ell+1)}]$$

and

$$M^{(2\ell+1)} = (\mathbb{1} + L)\left(\frac{\mathrm{i}}{2}\alpha\sigma_3\partial_x - \beta - \frac{1}{2}\alpha^2 Q^2\right) M^{(2\ell-1)}, \quad \ell \geq 1, \quad M^{(1)} = \mathrm{i}\alpha Q.$$

Here

$$L\cdot = -\frac{\mathrm{i}\alpha}{2}\sigma_3 \left[Q, \int^x \mathrm{d}x [Q, \cdot]\right].$$

Now we define the recursion operator

$$\Lambda = (\mathbb{1} + L)\left(\frac{\mathrm{i}}{2}\alpha\sigma_3\partial_x - \beta - \frac{1}{2}\alpha^2 Q^2\right).$$

Then it follows that $M^{(2\ell+1)} = \mathrm{i}\alpha\Lambda^\ell Q$ and $M^{\mathrm{a}} = -\mathrm{i}\alpha k(\Lambda - k^2)^{-1}Q$. Hence, the hierarchy of nonlinear evolution equations with SDR associated with the quadratic equation (10.44) is represented by

$$Q_t = -2\sigma_3 \sum_{j=0}^{n} \gamma_{2j} \Lambda^j Q + \frac{2\mathrm{i}}{\alpha}\sigma_3 \langle k\omega(k^2) M^{\mathrm{a}}(k)\rangle$$

together with the spectral equation (10.52).

Putting $n = 2$, $\gamma_0 = \beta^2\gamma_4$, $\gamma_2 = 2\beta\gamma_4$, and $\gamma_4 = 2\mathrm{i}/\alpha^2$, we obtain the nonlinear equation corresponding to the WKI spectral problem with the canonical normalization and with SDR:

$$\mathrm{i}Q_t + \sigma_3 Q_{xx} + \mathrm{i}\alpha Q Q_x Q + 2\beta\sigma_3 Q^3 + \frac{1}{2}\alpha^2\sigma_3 Q^5 = -\frac{2}{\alpha}\langle k\omega(k^2) M^{\mathrm{a}}\rangle. \quad (10.53)$$

In order to find the soliton solution to equations (10.53) and (10.52), we take the spectral transform R in the form

$$R(k) = 2\pi\mathrm{i}E^{-1}\begin{pmatrix} 0 & c\left[\delta(k-k_1) + \delta(k+k_1)\right] \\ \bar{c}\left[\delta(k-\bar{k}_1) + \delta(k+\bar{k}_1)\right] & 0 \end{pmatrix} E, \quad (10.54)$$

where $E = \exp\left[(\mathrm{i}/\alpha)(k^2+\beta)\sigma_3 x\right]$ and $c = c(t)$. This choice of R is compatible with the parity property. Substituting (10.54) into (10.46), we find explicitly the matrix ψ:

$$\psi = \begin{pmatrix} \dfrac{k^2-k_1^2}{k^2-\bar k_1^2} + \dfrac{k_1^2-\bar k_1^2}{k^2-\bar k_1^2}\Delta^{-1} & \dfrac{4\mathrm{i}ck}{k^2-k_1^2}\Delta^{-1}\exp\left(-\dfrac{2\mathrm{i}}{\alpha}(k_1^2+\beta)x\right) \\ \dfrac{4\mathrm{i}\bar c k}{k^2-k_1^2}\bar\Delta^{-1}\exp\left(\dfrac{2\mathrm{i}}{\alpha}(\bar k_1^2+\beta)x\right) & \dfrac{k^2-\bar k_1^2}{k^2-k_1^2} + \dfrac{\bar k_1^2-k_1^2}{k^2-k_1^2}\bar\Delta^{-1} \end{pmatrix}, \tag{10.55}$$

with

$$\Delta(x,t) = 1 + \frac{|c|^2\bar k_1^2}{\xi^2\eta^2}\mathrm{e}^{(8/\alpha)\xi\eta x}, \qquad k_1 = \xi + \mathrm{i}\eta.$$

The time dependence $c(t)$ is found from the second equation in (10.44) and gives $c(t) = c_0\exp\left\{4[\nu_{\mathrm{p}}' + \nu_{\mathrm{s}}' - \mathrm{i}(\nu_{\mathrm{p}}'' + \nu_{\mathrm{s}}'')]t\right\}$, $c_0 = \mathrm{const}$. Here

$$\nu_{\mathrm{p}}' = \frac{4\xi\eta}{\alpha^2}(\xi^2-\eta^2+\beta), \quad \nu_{\mathrm{p}}'' = \frac{1}{\alpha^2}\left[(\xi^2-\eta^2+\beta)^2 - 4\xi^2\eta^2\right],$$

$$\frac{1}{4\pi}\iint\frac{\mathrm{d}\ell\wedge\mathrm{d}\bar\ell}{\ell^2-k_1^2}\ell^2\omega_{\mathrm{s}}(\ell^2) = \nu_{\mathrm{s}}' - \mathrm{i}\nu_{\mathrm{s}}''.$$

Inserting ψ (10.55) and R (10.54) into (10.50), we immediately obtain the soliton solution ($k_1 = |k_1|\mathrm{e}^{\mathrm{i}\mu}$, $|c_0||k_1| = \xi\eta\ \mathrm{e}^{\rho}$, $\xi\eta > 0$)

$$q_{\mathrm{s}}(x,t) = -\frac{4\xi\eta}{\alpha|k_1|}\exp\left[-\mathrm{i}\left(\frac{\alpha\nu_{\mathrm{p}}'}{2\xi\eta}x + 4(\nu_{\mathrm{p}}'' + \nu_{\mathrm{s}}'')t + \mu\right)\right] \tag{10.56}$$

$$\times\operatorname{sech}\left[\frac{4\xi\eta}{\alpha}\left(x + \alpha\frac{\nu_{\mathrm{p}}' + \nu_{\mathrm{s}}'}{\xi\eta}t\right) + \rho - \mathrm{i}\mu\right].$$

Equation (10.53) being canonical within the class of equations integrable via the quadratic bundle has no physical applications. We will show next that the physically important modified NLS equation follows from (10.53) by means of a gauge transformation. In fact, this transformation is reduced to a change of the normalization of the $\bar\partial$ problem.

10.2.2 Gauge transformation

To obtain the integrable modified NLS equation with a source, we perform a gauge transformation $\psi = g\psi'$ with a function $g(x,t) = \psi(k=0)$; hence, we arrive at the $\bar\partial$ problem with the normalization $g^{-1}(x,t)$, $\psi' = g^{-1}(x,t) + \mathcal{O}(1/k)$, and with the same matrix R. It is not difficult

to see that at the same time $\Lambda' = g^{-1}\Lambda g$ and the system of equations (10.52) and (10.53) transforms to

$$\begin{aligned} &\mathrm{i}Q'_t + \sigma_3 Q'_{xx} - \mathrm{i}\alpha(Q'^3)_x + 2\beta\sigma_3 Q'^3 \\ &= -\left\langle \omega_{\mathrm{s}}(k^2)\left(\frac{k}{\alpha}[\sigma_3, M'^{\mathrm{a}}(k)] + \mathrm{i}[Q', M'^{\mathrm{d}}(k)]\right)\right\rangle, \qquad (10.57) \\ &M'_x + \left[\frac{\mathrm{i}}{\alpha}(k^2+\beta)\sigma_3 - kQ', M'\right] = 0. \end{aligned}$$

Here $Q' = g^{-1}Qg$ and $M' = g^{-1}Mg$. Let $\omega_{\mathrm{s}} = \mathrm{i}a\pi\delta(\operatorname{Im} k_1)\delta\left[(\operatorname{Re} k_1)^2 - \gamma^2\right]$, $a, \gamma \in \mathrm{Re}$, and

$$Q' = \begin{pmatrix} 0 & \mathcal{E} \\ -\bar{\mathcal{E}} & 0 \end{pmatrix}, \qquad M'(\gamma) = \begin{pmatrix} -n & p \\ \bar{p} & n \end{pmatrix}(\gamma).$$

Then we obtain the following physically interesting system (the modified NLS equation with SDR),

$$\mathrm{i}\mathcal{E}_t + \mathcal{E}_{xx} + \mathrm{i}\alpha(|\mathcal{E}|^2\mathcal{E})_x - 2\beta|\mathcal{E}|^2\mathcal{E} = \frac{\mathrm{i}a}{\alpha}p - \frac{a}{\gamma}\mathcal{E}n, \qquad (10.58)$$

$$p_x + \frac{2\mathrm{i}}{\alpha}(\gamma^2+\beta)p = 2\gamma\mathcal{E}n, \quad n_x = -\gamma(\mathcal{E}\bar{p} + \bar{\mathcal{E}}p),$$

which describes a propagation of a subpicosecond optical pulse with the complex envelope $\mathcal{E}$ in a nonlinear fiber containing resonant two-level impurities; p and n are polarization and population difference,s respectively (a, α, β, and γ are real parameters). The soliton solution to (10.58) follows immediately from (10.56) as $\mathcal{E}_{\mathrm{s}} = (g_{22}/g_{11})q_{\mathrm{s}}$. Hence, we get

$$\mathcal{E}_{\mathrm{s}}(x,t) = \left(\frac{\Delta(x,t)}{\bar{\Delta}(x,t)}\right)^2 q_{\mathrm{s}}(x,t).$$

The N-soliton solution can be found in the same way with the use of an evident generalization of the spectral transform matrix R. Equations (10.58) were derived from physical arguments in [462].

10.3 Nonlinear equations with singular dispersion relation: 2+1 dimensions

In the previous sections we elaborated a method to construct nonlinear SDR equations in 1+1 dimensions. Now we demonstrate how to generalize this approach to 2+1 dimensions. As shown by Boiti et al. [61], the SDR equations in 2+1 dimensions possess a number of peculiarities, the main one being the absence of an explicit expression for the second Lax operator $T_2 = \partial_t - V$.

These authors proposed a (2+1)-dimensional generalization of the Maxwell–Bloch equations in the form of a rather complicated system of four equations. The approach of [61] was essentially based on the function V given implicitly. On the other hand, we know that the $\bar\partial$ formalism does not rely on the Lax representation. Therefore, it is seems reasonable to use the $\bar\partial$ method to derive the above class of equations, without making direct use of the function V.

This program realized below relies on the bilocal approach initiated by Konopelchenko and Dubrovsky [243] and elaborated to a full extent by Fokas and Santini [161, 384]. It is precisely the bilocal formalism that allows us to generate in a natural manner (2+1)-dimensional counterparts of many structures which successfully work in 1+1 dimensions.

10.3.1 Nonlocal $\bar\partial$ problem

Our starting point is the nonlocal $\bar\partial$ problem

$$\bar\partial\psi(k) = \iint \mathrm{d}\ell\wedge\mathrm{d}\bar\ell\psi(\ell)R(k,\ell), \quad k,\ell\in\mathbb{C}, \quad \psi(k) = \mathbb{1} + \mathcal{O}(1/k), \quad k\to\infty, \tag{10.59}$$

where $R(k,\ell)$ is a distribution in $\mathbb{C}^2$. We denote the integral in (10.59) as $\psi(k)R_kF$, where F is an integral operator acting on the left in accordance with (10.59); hence,

$$\bar\partial\psi(k) = \psi(k)R_kF. \tag{10.60}$$

A solution of the $\bar\partial$ problem is given, as usual, by the Cauchy–Green integral:

$$\psi(k) = \mathbb{1} + \frac{1}{2\pi\mathrm{i}}\iint\frac{\mathrm{d}\ell\wedge\mathrm{d}\bar\ell}{\ell-k}\iint\mathrm{d}m\wedge\mathrm{d}\bar m\psi(m)R(\ell,m)$$

$$= \mathbb{1} + \frac{1}{2\pi\mathrm{i}}\iint\frac{\mathrm{d}\ell\wedge\mathrm{d}\bar\ell}{\ell-k}\,(\psi(\ell)R_\ell F) = \mathbb{1} + \psi(k)R_kFC_k. \tag{10.61}$$

Therefore, a solution of the $\bar\partial$ problem is compactly written as

$$\psi(k) = \mathbb{1}\cdot(\mathbb{1} - R_kFC_k)^{-1}. \tag{10.62}$$

The pairing is defined as for 1+1 dimensions (10.4), except for the property $\langle\psi R_kF,\phi\rangle = \langle\psi,\phi\hat R_kF\rangle$, where $\hat R(k,\ell) = R^{\mathrm{T}}(\ell,k)$.

Assume a linear parametric dependence of $R(k,\ell)$ on spatial variables of the form

$$\partial_xR(k,\ell) = \mathrm{i}\ell\sigma_3R(k,\ell) - \mathrm{i}kR(k,\ell)\sigma_3, \qquad \partial_yR(k,\ell) = \mathrm{i}(k-\ell)R(k,\ell). \tag{10.63}$$

Now we show that this choice of the dependence of $R(k,\ell)$ on spatial variables leads to the ZS spectral problem on the plane. Differentiating (10.62) in x, we obtain $\partial_x\psi = \psi(\partial_x R_k)FC_k(\mathbb{1}-R_kFC_k)^{-1}$. By means of the definitions of the integral operators F and C_k and (10.63) we perform the following calculation:

$$\psi(\partial_x R_k)FC_k = \frac{1}{2\pi\mathrm{i}}\iint\frac{\mathrm{d}\ell\wedge\mathrm{d}\bar{\ell}}{\ell-k}\iint\mathrm{d}m\wedge\mathrm{d}\bar{m}\psi(m)\partial_x R(\ell,m)$$

$$= \frac{1}{2\pi\mathrm{i}}\iint\frac{\mathrm{d}\ell\wedge\mathrm{d}\bar{\ell}}{\ell-k}\iint\mathrm{d}m\wedge\mathrm{d}\bar{m}\psi(m)\left[\sigma_3 R(\ell,m)-R(\ell,m)\sigma_3\right]$$

$$= \frac{1}{2\pi\mathrm{i}}\iint\frac{\mathrm{d}\ell\wedge\mathrm{d}\bar{\ell}}{\ell-k}(\mathrm{i}\ell\psi\sigma_3 R_\ell F) - \frac{1}{2\pi\mathrm{i}}\iint\mathrm{d}\ell\wedge\mathrm{d}\bar{\ell}\,\mathrm{i}\left(1+\frac{k}{\ell-k}\right)(\psi R_\ell F)\sigma_3.$$

Since we have from (10.61) $\psi R_k FC_k = \psi - \mathbb{1}$, then (10.63) and the evident relation $R_kFC_k(\mathbb{1}-R_kFC_k)^{-1} = (\mathbb{1}-R_kFC_k)^{-1} - \mathbb{1}$ yield

$$\partial_x\psi = -\mathrm{i}k\psi\sigma_3 - \mathrm{i}\langle\psi R_kF\rangle\sigma_3\psi + \mathrm{i}k\sigma_3(\mathbb{1}-R_kFC_k)^{-1}. \tag{10.64}$$

Similarly,

$$\partial_y\psi = \mathrm{i}k\psi + \mathrm{i}\langle\psi R_kF\rangle\psi - \mathrm{i}k(\mathbb{1}-R_kFC_k)^{-1}. \tag{10.65}$$

Adding (10.64) and (10.65) yields

$$\partial_x\psi + \sigma_3\partial_y\psi - \mathrm{i}k[\sigma_3,\psi] - \mathrm{i}[\sigma_3,\langle\psi R_kF\rangle]\psi = 0. \tag{10.66}$$

Hence, if we identify

$$-\mathrm{i}[\sigma_3,\langle\psi R_kF\rangle] = Q(x,y) \tag{10.67}$$

with the potential, then (10.66) gives the ZS spectral problem on the plane:

$$(\partial_x + \sigma_3\partial_y + Q)\psi - \mathrm{i}k[\sigma_3,\psi] = 0. \tag{10.68}$$

Note that our formalism works much easier in 2+1 dimension than in 1+1 dimensions. Indeed, we do not need to transform terms like $k(\mathbb{1}-R_kFC_k)^{-1}$ in (10.64) and (10.65) because they cancel each other in the combination $(\partial_x+\sigma_3\partial_y)\psi$.

The time dependence of $R(k,\ell)$ is given, as usual, by a linear equation

$$\partial_t R(k,\ell) = R(k,\ell)\Omega(k) - \Omega(\ell)R(k,\ell). \tag{10.69}$$

Here $\Omega(k)$ is a matrix-valued dispersion relation. It consists of a holomorphic (polynomial) part $\Omega_{\mathrm{p}}(k)$ and a nonanalytic (singular) part $\Omega_{\mathrm{s}}(k)$. It is instructive to derive the evolution linear problem $\partial_t\psi = V\psi + \psi\Omega$. It follows from (10.60) and (10.69) that

$$\partial_t\psi = \psi(\partial_t R)FC_k(\mathbb{1}-R_kFC_k)^{-1} = (\psi R_kF\Omega C_k - \psi\Omega R_kFC_k)(\mathbb{1}-R_kFC_k)^{-1}$$

$$= (\psi R_kF\Omega C_k - \psi\Omega)(\mathbb{1}-R_kFC_k)^{-1} + \psi\Omega, \tag{10.70}$$

which gives

$$V\psi = (\psi R_k F \Omega C_k - \psi\Omega)(\mathbb{1} - R_k F C_k)^{-1}.$$

In order to reveal peculiarities of the Lax operator V, it is sufficient to restrict ourselves to consideration of the singular part of the dispersion relation. Let

$$\Omega_{\rm s}(k) = \frac{1}{2\pi{\rm i}} \iint \frac{{\rm d}\ell \wedge {\rm d}\bar\ell}{\ell - k}\,\omega(\ell)\sigma_3, \qquad \bar\partial\Omega(k) = \omega(k)\sigma_3.$$

The calculation yields

$$V(k)\psi(k)(\mathbb{1} - R_k F C_k) = \frac{1}{2\pi{\rm i}} \iint \frac{{\rm d}\ell \wedge {\rm d}\bar\ell}{\ell - k}[\psi(\ell) R_\ell F]\Omega(\ell) - \psi\Omega$$

$$= \frac{1}{2\pi{\rm i}} \iint \frac{{\rm d}\ell \wedge {\rm d}\bar\ell}{\ell - k} \iint {\rm d}m \wedge {\rm d}\bar m \psi(m) R(\ell, m) \frac{1}{2\pi{\rm i}} \iint \frac{{\rm d}s \wedge {\rm d}\bar s}{s - \ell}\omega(s)\sigma_3 - \psi\Omega.$$

The denominator is written as

$$\frac{1}{(\ell - k)(s - \ell)} = \frac{1}{s - k}\left(\frac{1}{\ell - k} - \frac{1}{\ell - s}\right).$$

Then we have

$$V(k)\psi(k)(\mathbb{1} - R_k F C_k)$$

$$= \frac{1}{2\pi{\rm i}} \iint \frac{{\rm d}\ell \wedge {\rm d}\bar\ell}{\ell - k} \iint {\rm d}m \wedge {\rm d}\bar m \psi(m) R(\ell, m) \frac{1}{2\pi{\rm i}} \iint \frac{{\rm d}s \wedge {\rm d}\bar s}{s - k}\omega(s)\sigma_3$$

$$- \frac{1}{2\pi{\rm i}} \iint \frac{{\rm d}s \wedge {\rm d}\bar s}{s - k}\omega(s) \frac{1}{2\pi{\rm i}} \iint \frac{{\rm d}\ell \wedge {\rm d}\bar\ell}{\ell - s} \iint {\rm d}m \wedge {\rm d}\bar m \psi(m) R(\ell, m)\sigma_3 - \psi\Omega$$

$$= \psi R_k F C_k \Omega(k) - \frac{1}{2\pi{\rm i}} \iint \frac{{\rm d}s \wedge {\rm d}\bar s}{s - k}\omega(s)(\psi R_s F C_s)\sigma_3 - \psi\Omega = -\omega(k)\psi(k)\sigma_3 C_k,$$

where we have used $\psi R_s F C_s = \psi(s) - \mathbb{1}$. Hence,

$$V(k)\psi(k) = -\omega(k)\sigma_3 C_k (\mathbb{1} - R_k F C_k)^{-1}.$$

Multiplying this relation by $(\mathbb{1} - R_k F C_k)$ and applying the $\bar\partial$ operator, we obtain

$$(\bar\partial V)\psi + V(\psi R_k F) - V\psi R_k F = -\psi\bar\partial\Omega_{\rm s},$$

which gives the integral equation for the Lax function V [61]:

$$\bar\partial V(k) = -\psi\omega_{\rm s}(k)\psi^{-1} + \iint {\rm d}\ell \wedge {\rm d}\bar\ell\,[V(\ell) - V(k)]\,\psi(\ell) R(k, \ell)\psi^{-1}(k). \quad (10.71)$$

Hence, the function V is known only within a solution of the integral equation.

Note that (10.71) includes the inverse function ψ^{-1}. However, in 2+1 dimensions (in contrast to 1+1 dimensions), there does not exist a simple equation [like (10.68)] for ψ^{-1}. Hence, a problem arises of finding a natural (2+1)-dimensional counterpart of the inverse function well defined in 1+1 dimensions. We will see next that such a function does exist and, moreover, that our formalism unambiguously suggests a true choice of this function.

10.3.2 Dual function

Let us calculate the time evolution of Q given explicitly by (10.67):

$$\partial_t Q = -\mathrm{i}[\sigma_3, \langle \partial_t(\psi R_k F)\rangle].$$

The right-hand side can be transformed as follows:

$$\partial_t(\psi R_k F) = V\psi R_k F + \psi R_k F\Omega.$$

Further calculation, owing to (10.70), yields

$$\partial_t(\psi R_k F) = \psi R_k F\Omega C_k(\mathbb{1} - R_k F C_k)^{-1} R_k F - \psi\Omega(\mathbb{1} - R_k F C_k)^{-1} R_k F$$

$$+\psi R_k F\Omega = \psi R_k F\Omega(\mathbb{1} - C_k R_k F)^{-1} - \psi\Omega R_k F(\mathbb{1} - C_k R_k F)^{-1}.$$

Hence,

$$\partial_t Q = -\mathrm{i}[\sigma_3, \langle \psi R_k F\Omega(\mathbb{1} - C_k R_k F)^{-1}, \mathbb{1}\rangle - \langle \psi\Omega R_k F(\mathbb{1} - C_k R_k F)^{-1}, \mathbb{1}\rangle].$$

Taking into account the pairing properties, we get

$$\partial_t Q = -\mathrm{i}[\sigma_3, \langle \psi R_k F\Omega, \mathbb{1}\cdot(\mathbb{1} + \hat{R}_k F C_k)^{-1}\rangle - \langle \psi\Omega, \mathbb{1}\cdot(\mathbb{1} + \hat{R}_k F C_k)^{-1}\hat{R}_k F\rangle]. \tag{10.72}$$

Now we introduce a dual function $\tilde{\psi}(k)$ by means of the relation [120]

$$\tilde{\psi}^{\mathrm{T}} = \mathbb{1}\cdot(\mathbb{1} + \hat{R}_k F C_k)^{-1}. \tag{10.73}$$

The $\bar{\partial}$ problem for the dual function has the form

$$\bar{\partial}\tilde{\psi}(k) = -\iint \mathrm{d}\ell \wedge \mathrm{d}\bar{\ell} R(\ell, k)\tilde{\psi}(\ell), \quad \text{or} \quad \bar{\partial}\tilde{\psi}^{\mathrm{T}} = -\tilde{\psi}^{\mathrm{T}}(k) R_k F \tag{10.74}$$

and $\tilde{\psi}(k)$ satisfies the dual spectral problem

$$\partial_x\tilde{\psi} + \partial_y\tilde{\psi}\sigma_3 - \tilde{\psi}Q - \mathrm{i}k[\sigma_3, \tilde{\psi}] = 0. \tag{10.75}$$

In order to derive (10.74), we proceed from $\tilde{\psi}^{\mathrm{T}} = \mathbb{1} - \tilde{\psi}^{\mathrm{T}}\hat{R}_k F C_k$ and take into account the identity $\bar{\partial} f(k) C_k = f(k)$. Then

$$\bar{\partial}\tilde{\psi}^{\mathrm{T}} = -\tilde{\psi}^{\mathrm{T}}\hat{R}_k F = -\iint \mathrm{d}\ell \wedge \mathrm{d}\bar{\ell}\tilde{\psi}^{\mathrm{T}}(\ell)\hat{R}(k, \ell)$$

$$= -\iint \mathrm{d}\ell \wedge \mathrm{d}\bar{\ell}\tilde{\psi}^{\mathrm{T}}(\ell) R^{\mathrm{T}}(\ell, k) = -\iint \mathrm{d}\ell \wedge \mathrm{d}\bar{\ell}[R(\ell, k)\tilde{\psi}(\ell)]^{\mathrm{T}}$$

and (10.74) follows. The derivation of (10.75) is slightly more cumbersome. Differentiating (10.73) in x, we find $\partial_x\tilde{\psi}^{\mathrm{T}} = -\tilde{\psi}^{\mathrm{T}}\partial_x\hat{R}_k F C_k(\mathbb{1} + \hat{R}_k F C_k)^{-1}$. Accounting for $\hat{R}(k, \ell) = R^{\mathrm{T}}(\ell, k)$, we obtain from (10.63)

$$\partial_x\hat{R}(k, \ell) = \mathrm{i}k\hat{R}(k, \ell)\sigma_3 - \mathrm{i}\ell\sigma_3\hat{R}(k, \ell), \quad \partial_y\hat{R}(k, \ell) = -\mathrm{i}(k - \ell)\hat{R}(k, \ell).$$

Then following the derivation of the Zakharov–Shabat spectral problem, we obtain

$$\partial_x \tilde{\psi}^{\mathrm{T}} = \mathrm{i}k\tilde{\psi}^{\mathrm{T}}\sigma_3 - \mathrm{i}\langle\tilde{\psi}^{\mathrm{T}}\hat{R}_k F, \mathbb{1}\rangle\sigma_3\tilde{\psi}^{\mathrm{T}} - \mathrm{i}k\sigma_3(\mathbb{1} + \hat{R}_k F C_k)^{-1}$$

and

$$\partial_x \tilde{\psi} = \mathrm{i}k\sigma_3\tilde{\psi} - \mathrm{i}\tilde{\psi}\sigma_3\langle\mathbb{1}, \tilde{\psi}^{\mathrm{T}}\hat{R}_k F\rangle - \mathrm{i}k\left((\mathbb{1} + \hat{R}_k F C_k)^{-1}\right)^{\mathrm{T}}\sigma_3.$$

Similarly,

$$\partial_y \tilde{\psi} = -\mathrm{i}k\tilde{\psi} + \mathrm{i}\tilde{\psi}\langle\mathbb{1}, \tilde{\psi}^{\mathrm{T}}\hat{R}_k F\rangle + \mathrm{i}k\left((\mathbb{1} + \hat{R}_k F C_k)^{-1}\right)^{\mathrm{T}}.$$

Hence,

$$\partial_x\tilde{\psi} + \partial_y\tilde{\psi}\sigma_3 - \mathrm{i}k[\sigma_3, \tilde{\psi}] + \mathrm{i}\tilde{\psi}[\sigma_3, \langle\mathbb{1}, \tilde{\psi}^{\mathrm{T}}\hat{R}_k F\rangle] = 0.$$

Now we need a connection between $\langle\mathbb{1}, \tilde{\psi}^{\mathrm{T}}\hat{R}_k F\rangle$ and Q. It can be found as follows:

$$\langle\psi R_k F, \mathbb{1}\rangle = \langle\mathbb{1}\cdot(\mathbb{1} - R_k F C_k)^{-1} R_k F, \mathbb{1}\rangle = \langle\mathbb{1}\cdot(\mathbb{1} - R_k F C_k)^{-1}, \hat{R}_k F\rangle$$

$$= \langle\mathbb{1}, \hat{R}_k F(\mathbb{1} + \hat{R}_k F C_k)^{-1}\rangle = \langle\mathbb{1}, \mathbb{1}\cdot(\mathbb{1} + \hat{R}_k F C_k)^{-1}\hat{R}_k F\rangle = \langle\mathbb{1}, \tilde{\psi}^{\mathrm{T}}\hat{R}_k F\rangle.$$

Hence,

$$Q = -\mathrm{i}[\sigma_3, \langle\psi R_k F, \mathbb{1}\rangle] = -\mathrm{i}[\sigma_3, \langle\mathbb{1}, \tilde{\psi}^{\mathrm{T}}\hat{R}_k F\rangle]$$

and we arrive at the spectral equation (10.75).

Taking into account the above relations concerning the dual function, we write the evolution (10.72) in the form

$$\partial_t Q = -\mathrm{i}[\sigma_3, \langle\psi R_k F\Omega, \tilde{\psi}\rangle - \langle\psi\Omega, \tilde{\psi}^{\mathrm{T}}\hat{R}_k F\rangle] = -\mathrm{i}[\sigma_3, \langle(\bar{\partial}\psi)\Omega\tilde{\psi}\rangle + \langle\psi\Omega\bar{\partial}\tilde{\psi}\rangle].$$

Finally, dividing the dispersion relation into the regular and singular parts, we obtain under the condition $\Omega_{\mathrm{s}}(k) \to 0$ for $k \to \infty$

$$\partial_t Q = -\mathrm{i}[\sigma_3, \langle\bar{\partial}(\psi\Omega_p\tilde{\psi}\rangle - \langle\omega_{\mathrm{s}}\psi\sigma_3\tilde{\psi}\rangle]. \tag{10.76}$$

As a result, it is the dual function $\tilde{\psi}$ that is a true (2+1)-dimensional generalization of inverse functions. It should be stressed that the definition (10.73) of the dual function arises naturally within the framework of the formalism based on the representation (10.62).

10.3.3 Recursion operator

To derive the recursion operator, we introduce a bilocal object [243]

$$M_{12}(x, y_1, y_2, k) = \psi(x, y_1, k)\sigma_3\tilde{\psi}(x, y_2, k) \equiv \psi_1\sigma_3\tilde{\psi}_2.$$

The function M_{12} satisfies the equation

$$\partial_x M_{12} + \sigma_3 \partial_{y_1} M_{12} + \partial_{y_2} M_{12} \sigma_3 - \mathrm{i}k[\sigma_3, M_{12}] + Q_1 M_{12} - M_{12} Q_2 = 0, \quad (10.77)$$

where $Q_i \equiv Q(x, y_i)$; hence, (10.76) takes the form

$$\delta_{12} \partial_t Q_2 = -\mathrm{i}\alpha_n \delta_{12} [\sigma_3, \langle \bar{\partial}(k^n M_{12}) \rangle] + \mathrm{i}\delta_{12} [\sigma_3, \langle \omega(k) M_{12} \rangle], \quad (10.78)$$

where $\delta_{12} = \delta(y_1 - y_2)$. Following [384], we introduce the notations

$$P_{12} M_{12} = \partial_x M_{12} + \sigma_3 \partial_{y_1} M_{12} + \partial_{y_2} M_{12} \sigma_3, \quad Q_{12}^{\pm} M_{12} = Q_1 M_{12} \pm M_{12} Q_2. \quad (10.79)$$

Let M_{12}^{d} and M_{12}^{a} be the diagonal and off-diagonal parts of the matrix M. Then (10.77) and (10.79) yield

$$P_{12} M_{12}^{\mathrm{d}} + Q_{12}^{-} M_{12}^{\mathrm{a}} = 0, \quad P_{12} M_{12}^{\mathrm{a}} - 2\mathrm{i}k\sigma_3 M_{12}^{\mathrm{a}} + Q_{12}^{-} M_{12}^{\mathrm{d}} = 0. \quad (10.80)$$

Because $M_{12}^{\mathrm{d}} = \sigma_3 - P_{12}^{-1} Q_{12}^{-} M_{12}^{\mathrm{a}}$, the second equation in (10.80) is written in the form $(\Lambda - k) M_{12}^{\mathrm{a}} = (2\mathrm{i})^{-1} Q_{12}^{+} \cdot \mathbb{1}$ [$Q_{12}^{+} \cdot \mathbb{1} = Q_1 + Q_2$ in accordance with the definition (10.79)], where the operator Λ is defined as

$$\Lambda = \frac{1}{2\mathrm{i}} \sigma_3 (P_{12} - Q_{12}^{-} P_{12}^{-1} Q_{12}^{-}).$$

Then $M_{12}^{\mathrm{a}} = (2\mathrm{i})^{-1} (\Lambda - k)^{-1} Q_{12}^{+} \cdot \mathbb{1}$ and after the expansion $(\Lambda - k)^{-1} = -\sum_{m=1}^{\infty} k^{-m} \Lambda^{m-1}$ we can write the polynomial contribution to $\partial_t Q$ in (10.78) as

$$-\mathrm{i}\alpha_n \delta_{12} [\sigma_3, \langle \bar{\partial}(k^n M_{12}) \rangle] = \alpha_n \sigma_3 \delta_{12} \sum_{m=1}^{\infty} \langle \bar{\partial} k^{n-m} \rangle \Lambda^{m-1} Q_{12}^{+} \cdot \mathbb{1}$$

$$= -\frac{\mathrm{i}}{2} \alpha_n \sigma_3 \delta_{12} \Lambda^n Q_{12}^{+} \cdot \mathbb{1}.$$

Now we have all we need to formulate a closed system of equations describing the evolution of the potential Q with account for both parts of the dispersion relations:

$$\delta_{12} \partial_t Q_2 = -\frac{\mathrm{i}}{2} \alpha_n \sigma_3 \delta_{12} \Lambda^n Q_{12}^{+} \cdot \mathbb{1} + \mathrm{i}\delta_{12} [\sigma_3, \langle \omega_s(k) M_{12} \rangle],$$

$$P_{12} M_{12} - \mathrm{i}k[\sigma_3, M_{12}] + Q_{12}^{-} M_{12} = 0. \quad (10.81)$$

Here the operator Λ plays the role of a recursion operator (more precisely, Λ is related to the true recursion operator by means of σ_3). If $M_{12} = \sigma_3$ and $\omega_s(k) = 0$, we get from (10.81) the hierarchy including the Davey–Stewartson I equation derived by Santini and Fokas [384] on the basis of an integral representation for the Lax evolution operator V.

For $\Omega_{\mathrm{p}} = 0$ the system (10.81) takes the form

$$\delta_{12} \partial_t Q_2 = \mathrm{i}\delta_{12} [\sigma_3, \langle \omega_s(k) M_{12} \rangle], \quad (10.82)$$

$$P_{12} M_{12} - \mathrm{i}k[\sigma_3, M_{12}] + Q_{12}^{-} M_{12} = 0.$$

It is seen that the structure of this system is similar to that (10.25) with $\alpha_n = 0$; hence, we can treat (10.82) as the Maxwell–Bloch equation in 2+1 dimensions. Its soliton solution can be found in [61].

10.4 Kadomtsev–Petviashvili II equation

In the previous sections we applied the $\bar{\partial}$ formalism to solve some problems for nonlinear equations in 1+1 and 2+1 dimensions. Though this approach has proved its efficiency, the $\bar{\partial}$ formalism was not absolutely necessary to solve these problems. In particular, the RH problem could be applied equally well for this aim.

The present section is devoted to analysis of the KP II equation. The KP II equation plays a distinctive role in the theory of nonlinear equations. It is the KP II equation that demonstrated for the first time the nonuniversality of the nonlocal RH problem for solving nonlinear equations.

Ablowitz et al. [1] showed in a beautiful paper that the inverse problem for the KP II equation can be successfully solved by means of the $\bar{\partial}$ problem. Following this paper, we demonstrate in this section the main steps in realizing the program for solution of the KP II equation in the framework of the $\bar{\partial}$ method.

10.4.1 Eigenfunctions and scattering equation

The KP II equation

$$(u_t + 6uu_x + u_{xxx})_x + 3u_{yy} = 0 \tag{10.83}$$

describes the evolution of weakly nonlinear, weakly dispersive, and weakly two-dimensional water waves (all these effects are of the same order) when gravity dominates surface tension. The physical derivation of the KP II equation can be found in [3]. The KP II equation represents the compatibility condition of two linear Lax equations

$$\begin{aligned} &-\ \psi_y + \psi_{xx} + u\psi = 0, \\ &\quad \psi_t + 4\psi_{xxx} + 6u\psi_x + 3u_x\psi + 3(\partial_x^{-1}u_y)\psi = 0, \end{aligned} \tag{10.84}$$

where $\partial_x^{-1} f = (1/2)\left(\int_{-\infty}^{x} - \int_{x}^{\infty}\right) \mathrm{d}x' f(x')$. In order to introduce a spectral parameter, we transform the Lax pair (10.84) to the function $m(x, y, k) = \psi(x, y)\exp(-\mathrm{i}kx + k^2 y)$. As a result, we will work with the Lax pair of the form

$$-\ m_y + m_{xx} + 2\mathrm{i}km_x + um = 0, \tag{10.85}$$

$$\begin{aligned} &m_t + 4m_{xxx} + 12\mathrm{i}km_{xx} - 12k^2 m_x + 6um_x + 6\mathrm{i}kum + 3u_x m \\ &+3(\partial_x^{-1}u_y)m + (\alpha(k) - 4\mathrm{i}k^3)m = 0, \end{aligned} \tag{10.86}$$

where $\alpha(k)$ is an arbitrary function. As usual, we choose two linearly independent solutions of (10.85) with zero potential:

$$M_0 = 1, \qquad N_0 = \exp\left[-\mathrm{i}(k + \bar{k})x + (k^2 - \bar{k}^2)y\right].$$

One of them, M_0, provides the canonical normalization of the solution M of the full spectral equation (10.85):

$$M(x,y,k) = 1 + \iint_{-\infty}^{\infty} \mathrm{d}x'\mathrm{d}y' G(x-x', y-y', k) u(x',y') M(x',y',k), \quad (10.87)$$

which is bounded for all complex k. The other solution reads

$$N(x,y,k) = \exp[-\mathrm{i}(k+\bar{k})x + (k^2 - \bar{k}^2)y] \quad (10.88)$$
$$+ \iint_{-\infty}^{\infty} \mathrm{d}x'\mathrm{d}y' G(x-x', y-y', k) u(x',y') N(x',y',k).$$

Here $G(x,y,k)$ is the Green function which obeys the equation

$$-G_y + G_{xx} + 2\mathrm{i}kG_x = -\delta(x)\delta(y)$$

and can be written as

$$G(x,y,k) = \frac{1}{4\pi^2} \iint_{-\infty}^{\infty} \mathrm{d}\xi \mathrm{d}\eta \frac{\mathrm{e}^{\mathrm{i}(\xi x + \eta y)}}{\xi^2 + 2k\xi + \mathrm{i}\eta}. \quad (10.89)$$

As distinct from the KP I equation, the Green function (10.89) has no jump across the real axis. Moreover, this function is analytic nowhere in the k-plane, as can be explicitly seen after integration in η. Indeed, calculating by residues the integral (10.89) with respect to η, we obtain the following formula for the Green function [1]:

$$G(x,y,k) = \frac{1}{2\pi} \Bigg\{ \theta(k_{\mathrm{R}}) \left[-\theta(-y) \int_{-2k_{\mathrm{R}}}^{0} \mathrm{d}\xi + \theta(y) \left(\int_{0}^{\infty} \mathrm{d}\xi + \int_{-\infty}^{-2k_{\mathrm{R}}} \mathrm{d}\xi \right) \right]$$
$$+ \theta(-k_{\mathrm{R}}) \left[-\theta(-y) \int_{0}^{-2k_{\mathrm{R}}} \mathrm{d}\xi + \theta(y) \left(\int_{-\infty}^{0} \mathrm{d}\xi + \int_{-2k_{\mathrm{R}}}^{\infty} \mathrm{d}\xi \right) \right] \Bigg\} \mathrm{e}^{\mathrm{i}\xi x - (\xi^2 + 2k\xi)y}. \quad (10.90)$$

Formula (10.90) contains explicitly k_{R} and hence the Green function is non-analytic. As a result, the eigenfunctions M and N are analytic nowhere in the k-plane. We will stress this fact denoting the eigenfunctions as $M(x,y,k,\bar{k})$ and $N(x,y,k,\bar{k})$.

Nonanalyticity of eigenfunctions prevents us from making use of the RH problem. It was found by Ablowitz et al. [1] that it is the $\bar{\partial}$ problem that should be employed to formulate scattering equations and to solve the inverse scattering problem. Hence, we calculate first the $\bar{\partial}$ derivative $\bar{\partial}M = (1/2)(\partial_{k_{\mathrm{R}}} + \mathrm{i}\partial_{k_{\mathrm{I}}})M(x,y,k,\bar{k})$:

$$\bar{\partial}M(x,y,k,\bar{k}) = \iint_{-\infty}^{\infty} \mathrm{d}x'\mathrm{d}y' \left[\bar{\partial} G(x-x', y-y', k, \bar{k}) \right] u(x',y') M(x',y',k,\bar{k})$$
$$+ \iint_{-\infty}^{\infty} \mathrm{d}x'\mathrm{d}y' G(x-x', y-y', k, \bar{k}) u(x',y') \bar{\partial} M(x',y',k,\bar{k}).$$

It is easily obtained from (10.90) that

$$\bar\partial G(x,y,k,\bar k)=\frac{1}{2\pi}\mathrm{sign}(-k_{\mathrm R})\exp\left[-\mathrm{i}(k+\bar k)x+(k^2-\bar k^2)y\right].$$

Therefore,

$$\bar\partial M(x,y,k,\bar k)=F(k,\bar k)\mathrm{e}^{-\mathrm{i}(k+\bar k)x+(k^2-\bar k^2)y} \tag{10.91}$$
$$+\iint_{-\infty}^{\infty}\mathrm{d}x'\mathrm{d}y'G(x-x',y-y',k,\bar k)u(x',y')\bar\partial M(x',y',k,\bar k),$$

where the spectral data are given by the function

$$F(k,\bar k)=\frac{1}{2\pi}\mathrm{sign}(-k_{\mathrm R})\iint_{-\infty}^{\infty}\mathrm{d}x\mathrm{d}yu(x,y)M(x,y,k,\bar k)\mathrm{e}^{\mathrm{i}(k+\bar k)x-(k^2-\bar k^2)y}. \tag{10.92}$$

Comparing (10.91) with the integral equation (10.88), we obtain the scattering equation in the form of the linear $\bar\partial$ problem:

$$\bar\partial M(x,y,k,\bar k)=F(k,\bar k)N(x,y,k,\bar k). \tag{10.93}$$

The next step consists in finding a symmetry (closure) relation, in order to express N in terms of M. The Green function obeys the symmetry property

$$G(x,y,-\bar k,-k)=G(x,y,k,\bar k)\exp\left[\mathrm{i}(k+\bar k)x-(k^2-\bar k^2)y\right].$$

Now from comparison of (10.87), where substitutions $k\to-\bar k$ and $\bar k\to-k$ have been performed, with (10.88) the discrete closure relation follows:

$$N(x,y,k,\bar k)=M(x,y,-\bar k,-k)\exp\left[-\mathrm{i}(k+\bar k)x+(k^2-\bar k^2)y\right].$$

Thereby, the $\bar\partial$ problem for the eigenfunction M is written in terms of the scattering data:

$$\bar\partial M(x,y,k,\bar k)=F(k,\bar k)M(x,y,-\bar k,-k)\exp\left[-\mathrm{i}(k+\bar k)x+(k^2-\bar k^2)y\right]. \tag{10.94}$$

It should be stressed that the above calculation is valid under the assumption that there are no nontrivial solutions of the homogeneous integral equation obtained from (10.88).

10.4.2 Inverse spectral problem

The inverse spectral problem is solved by means of the Cauchy–Green formula [see (1.99)]

$$M(x,y,k,\bar k)=1+\frac{1}{2\pi\mathrm{i}}\iint\frac{\mathrm{d}\ell\wedge\mathrm{d}\bar\ell}{\ell-k}\,\bar\partial M(x,y,\ell,\bar\ell)$$
$$=1+\frac{1}{2\pi\mathrm{i}}\iint\frac{\mathrm{d}\ell\wedge\mathrm{d}\bar\ell}{\ell-k}F(\ell,\bar\ell)M(x,y-\bar\ell,-\ell)\mathrm{e}^{-\mathrm{i}(\ell+\bar\ell)x+(\ell^2-\bar\ell^2)y}.$$

To reconstruct the potential $u(x,y)$, note that we have two representation for $M-1$:

$$M(x,y,k,\bar{k})-1=\begin{cases}\displaystyle\iint \mathrm{d}x'\mathrm{d}y' G(x-x',y-y',k,\bar{k})u(x',y')M(x',y',k,\bar{k})\\ \displaystyle\frac{1}{2\pi\mathrm{i}}\iint\frac{\mathrm{d}\ell\wedge\mathrm{d}\bar{\ell}}{\ell-k}\,F(\ell,\bar{\ell})M(x,y,-\bar{\ell},-\ell)\mathrm{e}^{-\mathrm{i}(\ell+\bar{\ell})x+(\ell^2-\bar{\ell}^2)y}\end{cases}.$$

Now we compare them in the order of $\mathcal{O}(k^{-1})$. From (10.89) for $|k|\to\infty$ we obtain

$$G=\frac{1}{8\pi^2 k}\int\limits_{-\infty}^{\infty}\mathrm{d}\eta\ \mathrm{v.p.}\int\limits_{-\infty}^{\infty}\frac{\mathrm{d}\xi}{\xi}\,\mathrm{e}^{\mathrm{i}(\xi x+\eta y)}+\mathcal{O}(k^{-2})=\frac{\mathrm{i}}{4k}\mathrm{sign}(x)\delta(y)+\mathcal{O}(k^{-2}).$$

Besides, $M=1+\mathcal{O}(k^{-1})$. Hence, the Green function representation of $M-1$ yields

$$M-1=\frac{\mathrm{i}}{4k}\left(\int\limits_{-\infty}^{x}\mathrm{d}x'u(x',y)-\int\limits_{x}^{\infty}\mathrm{d}x'u(x',y)\right)+\mathcal{O}(k^{-2}).\tag{10.95}$$

From the $\bar{\partial}$ representation we obtain

$$M-1=\frac{\mathrm{i}}{2\pi k}\iint\mathrm{d}\ell\wedge\mathrm{d}\bar{\ell}F(\ell,\bar{\ell})M(x,y,-\bar{\ell},-\ell)\mathrm{e}^{-\mathrm{i}(\ell+\bar{\ell})x+(\ell^2-\bar{\ell}^2)y}+\mathcal{O}(k^{-2}).\tag{10.96}$$

Comparing (10.95) and (10.96), we get the reconstruction formula

$$u(x,y)=\frac{1}{\pi}\,\partial_x\iint\mathrm{d}k\wedge\mathrm{d}\bar{k}\,F(k,\bar{k})M(x,y,-\bar{k},-k)\mathrm{e}^{-\mathrm{i}(k+\bar{k})x+(k^2-\bar{k}^2)y}.$$

We will be able to completely solve the KP II equation if the time evolution of the spectral data is determined. By the standard manipulations with (10.86) considered at the asymptotic $x^2+y^2\to\infty$ we obtain $\alpha(k)=4\mathrm{i}k^3$ and

$$F(k,\bar{k},t)=F(k,\bar{k},0)\mathrm{e}^{-4\mathrm{i}(k^3+\bar{k}^3)t}.$$

The uniqueness of a solution of the KP II equation for small initial data $u(x,y,0)$ was proved by Wickerhauser [451].

10.5 Davey–Stewartson II equation

The DS II equation in the focused case

$$\mathrm{i}u_t=\frac{1}{2}(u_{xx}-u_{yy})+(\phi+|u|^2)u,\qquad \phi_{xx}+\phi_{yy}=2|u|^2_{xx}\tag{10.97}$$

describes an evolution of quasimonochromatic wave packets with slowly varying amplitude $u(x,y,t)$ on a two-dimensional water surface under gravity

[12, 105], where $\phi(x, y, t)$ is the velocity potential. Besides, the DS II equation found use in plasma physics [352]. The IST method for (10.97) was realized in terms of the $\bar{\partial}$ problem by Fokas and Ablowitz [159]. Rational nonsingular localized solutions (lumps) of the DS II equation decaying at infinity as $(x^2 + y^2)^{-1}$ have been derived by Arkadiev et al. [31]. Various aspects of the IST approach for solving the DS II equation have been discussed by Beals and Coifman [41, 42] and Konopelchenko and Matkarimov [246]. The Darboux method was used in [308] to obtain soliton solutions which demonstrate nontrivial dynamics under interaction. The completeness of the eigenfunction system of the elliptic spectral problem associated with the DS II equation was established in [364].

In the papers cited above, solitons of the DS II equation correspond to simple poles of the solutions of the spectral equation. As we know from the example of the KP I equation, a novel class of solutions with more diverse properties arises if the eigenfunctions allow multiple poles. The same situation exists for the DS II equation. Villarroel and Ablowitz [440] found a variety of rationally decaying, regular, localized solutions of the DS II equation which stem from meromorphic eigenfunctions with multiple poles in the spectral parameter.

10.5.1 Eigenfunctions and scattering equation

The DS II equation (10.97) allows the matrix-valued Lax representation

$$\psi_x + \mathrm{i}\sigma_3\psi_y - Q\psi = 0, \qquad Q = \begin{pmatrix} 0 & u \\ -\bar{u} & 0 \end{pmatrix} \tag{10.98}$$

$$\psi_t = A\psi - Q\psi_y + \mathrm{i}\sigma_3\psi_{yy}. \tag{10.99}$$

Here ψ and A are 2×2 matrices. The compatibility condition for (10.98) and (10.99) gives the DS II equation (10.97) provided the entries of A are given by

$$(\partial_x + \mathrm{i}\partial_y)A_{11} = \frac{1}{2\mathrm{i}}(\partial_x - \mathrm{i}\partial_y)|u|^2, \qquad A_{12} = \frac{1}{2\mathrm{i}}(\partial_x - \mathrm{i}\partial_y)u, \tag{10.100}$$

$$(\partial_x - \mathrm{i}\partial_y)A_{21} = -\frac{1}{2\mathrm{i}}(\partial_x + \mathrm{i}\partial_y)|u|^2, \qquad A_{21} = \frac{1}{2\mathrm{i}}(\partial_x + \mathrm{i}\partial_y)\bar{u},$$

while the potential ϕ is expressed in terms of A_{ij} as $\phi = \mathrm{i}(A_{11} - A_{22}) - |u|^2$. Because the spectral problem (10.98) is elliptic, it is reasonable to introduce complex coordinates $z = x + \mathrm{i}y$ and $\bar{z} = x - \mathrm{i}y$. Then the DS II equation takes the form

$$\mathrm{i}u_t = u_{zz} + u_{\bar{z}\bar{z}} + (g + \bar{g})u, \qquad g_{\bar{z}} = \frac{1}{2}\left(|u|^2\right)_z, \tag{10.101}$$

where $g = \mathrm{i}A_{11}$. The Lax representation for (10.101) is written as

$$D\psi = \frac{1}{2}Q\psi, \qquad \psi_t = A\psi - \mathrm{i}Q(\partial_z - \partial_{\bar{z}})\psi - \mathrm{i}\sigma_3(\partial_z - \partial_{\bar{z}})^2\psi. \tag{10.102}$$

Here

$$D = \begin{pmatrix} \partial_{\bar z} & 0 \\ 0 & \partial_z \end{pmatrix}, \quad \text{and} \quad \text{now} \quad A = -\mathrm{i} \begin{pmatrix} g & u_z \\ u_{\bar z} & -\bar g \end{pmatrix}.$$

Let us choose a solution of the free equation $D\psi = 0$ in the form

$$E = \begin{pmatrix} \mathrm{e}^{\mathrm{i}kz} & 0 \\ 0 & \mathrm{e}^{-\mathrm{i}k\bar z} \end{pmatrix} \tag{10.103}$$

with a spectral parameter k and transform ψ as $\psi = ME$. Then the Lax pair (10.102) takes the form

$$DM - \frac{\mathrm{i}}{2}k[\sigma_3, M] = \frac{1}{2}QM, \tag{10.104}$$

$$M_t = AM - \mathrm{i}Q(\partial_z - \partial_{\bar z})M + kQM - \mathrm{i}\sigma_3(\partial_z - \partial_{\bar z} + \mathrm{i}k)^2 M. \tag{10.105}$$

Taking the asymptotic expansion

$$M = \mathbb{1} + \frac{1}{\mathrm{i}k}m + \mathcal{O}(k^{-2})$$

in (10.104), we obtain the potential reconstruction formula

$$Q = -[\sigma_3, m], \quad \text{or} \quad u(x, y, t) = -2\overline{m}_{21}(x, y, t). \tag{10.106}$$

In virtue of the specific structure of the potential matrix Q, there is a symmetry relation between the entries of the matrix M. Namely,

$$M = \begin{pmatrix} M_{11}(k) & -\overline{M}_{21}(\bar k) \\ M_{21}(k) & \overline{M}_{11}(\bar k) \end{pmatrix}. \tag{10.107}$$

Hence, it is sufficient to study only the first column $\mathbf{M}_1$ of the matrix M. The components of the column $\mathbf{M}_1 = (M_1, M_2)^{\mathrm{T}}$ obey the following (spectral) equations

$$\partial_{\bar z} M_1 = \frac{1}{2}uM_2, \qquad \partial_z M_2 = -\mathrm{i}kM_2 - \frac{1}{2}\bar u M_1, \tag{10.108}$$

with the boundary condition

$$\lim_{|k|\to\infty} \mathbf{M}_1(z, \bar z, k, \bar k) \equiv \mathbf{M}_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{10.109}$$

As the second linearly independent solution of the free equations (10.108) with $u = 0$ we can choose [159]

$$\mathbf{N}_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mathrm{e}^{-\mathrm{i}(kz + \bar k \bar z)}. \tag{10.110}$$

Note that equations (10.108) can be treated as $\bar\partial$ (∂) problems in the coordinate space. Therefore, in accordance with the Cauchy–Green formula (1.98) and boundary condition (10.109) we can write a solution of (10.108) in the integral form:

$$M_1(z,\bar z,k,\bar k) = 1 + \frac{1}{2\pi\mathrm{i}}\iint \frac{\mathrm{d}z'\wedge\mathrm{d}\bar z'}{z'-z}\frac{1}{2}u(z',\bar z')M_2(z',\bar z',k,\bar k), \tag{10.111}$$

$$M_2(z,\bar z,k,\bar k) = -\frac{1}{2\pi\mathrm{i}}\iint \frac{\mathrm{d}z'\wedge\mathrm{d}\bar z'}{\bar z'-\bar z}\frac{1}{2}\bar u(z',\bar z')M_1(z',\bar z',k,\bar k)\mathrm{e}^{-\mathrm{i}k(z-z')-\mathrm{i}\bar k(\bar z-\bar z')}.$$

Similarly,

$$N_1(z,\bar z,k,\bar k) = \frac{1}{2\pi\mathrm{i}}\iint \frac{\mathrm{d}z'\wedge\mathrm{d}\bar z'}{z'-z}\frac{1}{2}u(z',\bar z')N_2(z',\bar z',k,\bar k), \tag{10.112}$$

$$N_2(z,\bar z,k,\bar k) = \mathrm{e}^{-\mathrm{i}(kz+\bar k\bar z)} - \frac{1}{2\pi\mathrm{i}}\iint \frac{\mathrm{d}z'\wedge\mathrm{d}\bar z'}{\bar z'-\bar z}\frac{1}{2}\bar u(z',\bar z')N_1(z',\bar z',k,\bar k)\mathrm{e}^{-\mathrm{i}k(z-z')-\mathrm{i}\bar k(\bar z-\bar z')}.$$

It is easily seen that equations (10.111) can be written in the standard form with the Green function,

$$(\mathcal{G}\mathbf{M})(z,\bar z,k,\bar k) = \begin{pmatrix}1\\0\end{pmatrix}, \tag{10.113}$$

where

$$(\mathcal{G}\mathbf{M})(z,\bar z,k,\bar k) \equiv \mathbf{M} - \iint \mathrm{d}z'_{\mathrm{R}}\mathrm{d}z'_{\mathrm{I}}G(z_{\mathrm{R}}-z'_{\mathrm{R}},z_{\mathrm{I}}-z'_{\mathrm{I}},k,\bar k)\,(Q\mathbf{M})(z',\bar z',k,\bar k)$$

and

$$G(z,\bar z,k,\bar k) = \begin{pmatrix} G_1 & 0\\ 0 & G_2\end{pmatrix},\quad G_1 = \frac{1}{2\pi z},\quad G_2 = \frac{1}{2\pi\bar z}\mathrm{e}^{-\mathrm{i}(kz+\bar k\bar z)}. \tag{10.114}$$

The presence of $\exp[-\mathrm{i}(kz+\bar k\bar z)] = \exp[-2\mathrm{i}(k_{\mathrm{R}}x - k_I y)]$ in (10.114) means that the Green function is nowhere analytic. In turn, the eigenvectors $\mathbf{M}$ and $\mathbf{N}$ are nowhere analytic as well. "Departure from analyticity" $\bar\partial\mathbf{M} \equiv \partial\mathbf{M}/\partial\bar k$ determines the continuous spectrum and can be calculated directly from (10.111). Indeed,

$$\bar\partial\mathbf{M} = \frac{1}{2\pi\mathrm{i}}\iint \frac{\mathrm{d}z'\wedge\mathrm{d}\bar z'}{z'-z}\frac{1}{2}u(z',\bar z')\bar\partial M_2(z',\bar z',k,\bar k),$$

$$\bar\partial\mathbf{M}_2 = b(k,\bar k)\mathrm{e}^{-\mathrm{i}(kz+\bar k\bar z)} \tag{10.115}$$

$$-\frac{1}{2\pi\mathrm{i}}\iint \frac{\mathrm{d}z'\wedge\mathrm{d}\bar z'}{\bar z'-\bar z}\frac{1}{2}\bar u(z',\bar z')\bar\partial\mathbf{M}_1(z',\bar z',k,\bar k)\mathrm{e}^{-\mathrm{i}k(z-z')-\mathrm{i}\bar k(\bar z-\bar z')},$$

where

$$b(k,\bar{k}) = \frac{\mathrm{i}}{4\pi}\iint \mathrm{d}z_R \mathrm{d}z_I \bar{u}(z,\bar{z})\mathbf{M}_1(z,\bar{z},k,\bar{k})\mathrm{e}^{\mathrm{i}(kz+\bar{k}\bar{z})}.$$

Comparing (10.115) with (10.112), we conclude that the $\bar{\partial}$ equation for $\mathbf{M}$ takes the form

$$\bar{\partial}\mathbf{M} = b(k,\bar{k})\mathbf{N}. \tag{10.116}$$

To obtain the closure relation which connects $\mathbf{N}$ and $\mathbf{M}$, we write the second equation in (10.112) as

$$N_2 = \mathrm{e}^{-\mathrm{i}(kz+\bar{k}\bar{z})}\left(1 + \frac{1}{2\pi\mathrm{i}}\iint \frac{\mathrm{d}z'\wedge\mathrm{d}\bar{z}'}{\bar{z}'-\bar{z}}\frac{1}{2}(Q\mathbf{N})_2(z',\bar{z}',k,\bar{k})\mathrm{e}^{\mathrm{i}(kz'+\bar{k}\bar{z}')}\right)$$

and compare it with the equation (10.111) for $\overline{M}_1$. As a result, we get

$$\mathbf{N} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\overline{\mathbf{M}}\mathrm{e}^{-\mathrm{i}(kz+\bar{k}\bar{z})}. \tag{10.117}$$

10.5.2 Discrete spectrum and inverse problem solution

The discrete spectrum for the spectral problem (10.108) is given by a set of isolated eigenvalues $k_j, j = 1,\dots,\mathcal{N}$, for which homogeneous integral equations allow nontrivial solutions. As distinct from the KP I equation, the eigenvalue k_j for the DS II equation has double degeneracy with eigenstates $\Phi_j(z,\bar{z})$ and $\chi_j(z,\bar{z})$ [31]. The bound state vector $\mathbf{\Phi}^{(j)} = (\Phi_1^{(j)}, \Phi_2^{(j)})^{\mathrm{T}}$ is a solution of the equations

$$\Phi_1^{(j)}(z,\bar{z}) = \frac{1}{2\pi\mathrm{i}}\iint \frac{\mathrm{d}z'\wedge\mathrm{d}\bar{z}'}{z'-z}\frac{1}{2}u(z',\bar{z}')\Phi_2^{(j)}(z',\bar{z}'), \tag{10.118}$$

$$\Phi_2^{(j)}(z,\bar{z}) = -\frac{1}{2\pi\mathrm{i}}\iint \frac{\mathrm{d}z'\wedge\mathrm{d}\bar{z}'}{\bar{z}'-\bar{z}}\frac{1}{2}\bar{u}(z',\bar{z}')\Phi_1^{(j)}(z',\bar{z}')\mathrm{e}^{-\mathrm{i}k_j(z-z')-\mathrm{i}\bar{k}_j(\bar{z}-\bar{z}')}$$

and $\boldsymbol{\chi}^{(j)}$ is expressed through $\mathbf{\Phi}^{(j)}$ as

$$\boldsymbol{\chi}^{(j)} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\overline{\mathbf{\Phi}}^{(j)}\mathrm{e}^{-\mathrm{i}(k_j z+\bar{k}_j\bar{z})}. \tag{10.119}$$

It is seen from (10.118) that it is natural to normalize bound eigenfunctions by the condition

$$\mathbf{\Phi}^{(j)} \to \frac{1}{z}\begin{pmatrix} Q_j \\ 0 \end{pmatrix}, \quad |z| \to \infty$$

under the constraint

$$\iint \mathrm{d}z\wedge\mathrm{d}\bar{z}\bar{u}(z,\bar{z})\Phi_1^{(j)}(z,\bar{z})\mathrm{e}^{-\mathrm{i}(k_j z+\bar{k}_j\bar{z})} = 0.$$

Here the index Q_j is defined as

$$Q_j = \frac{1}{2\pi} \iint \mathrm{d}x\mathrm{d}y u(x,y) \Phi_2^{(j)}(x,y). \tag{10.120}$$

We will see that the index Q_j plays the same role as that for the KP I equation.

The existence of the discrete spectrum indicates that the eigenvector $\mathbf{M}$ is singular and has (in general, multiple) poles in the points k_j. First we consider the case of simple poles:

$$\mathbf{M} = \sum_{j=1}^{\mathcal{N}} (k-k_j)^{-1} \mathbf{\Phi}^{(j)}(z,\bar{z}) + \mathbf{M}_{\mathrm{reg}},$$

where $\mathbf{M}_{\mathrm{reg}}$ is regular in k_j. In order to determine residues $\mathbf{\Phi}^{(j)}$, we need to find the limit of $\mathbf{M}$ when $k \to k_j$. We have around $k = k_j$

$$\mathbf{M} = \frac{\mathbf{\Phi}^{(j)}}{k-k_j} + \boldsymbol{\mu}, \tag{10.121}$$

where $\boldsymbol{\mu}$ is regular in k_j and tends to $(1,0)^{\mathrm{T}}$ at $|k| \to \infty$. Taking first $Q_j = 1$, we substitute (10.121) into (10.113). Following the procedure described in Sect. 9.2.2 and accounting for the double degeneracy of k_j yields [31]

$$\lim_{k \to k_j} \left(\mathbf{M}(z,\bar{z},k,\bar{k}) - \frac{1}{k-k_j} \mathbf{\Phi}^{(j)}(z,\bar{z}) \right) = -\mathrm{i}(z+z_j)\mathbf{\Phi}^{(j)}(z,\bar{z}) - \mathrm{i}c_j \boldsymbol{\chi}^{(j)}(z,\bar{z}), \tag{10.122}$$

where z_j and c_j are constants and $\mathbf{\Phi}^{(j)}$ and $\boldsymbol{\chi}^{(j)}$ are related by (10.119).

Combining now the contributions of both the continuous and the discrete spectra, we can write the eigenvector $\mathbf{M}$ as

$$\mathbf{M}(z,\bar{z},k,\bar{k}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sum_{j=1}^{\mathcal{N}} \frac{\mathbf{\Phi}^{(j)}(z,\bar{z})}{k-k_j} + \frac{1}{2\pi\mathrm{i}} \iint \frac{\mathrm{d}\ell \wedge \mathrm{d}\bar{\ell}}{\ell - k} \, b(\ell,\bar{\ell}) \mathbf{N}(z,\bar{z},\ell,\bar{\ell}). \tag{10.123}$$

The limit $k \to k_j$ gives in accordance with (10.122) and for $Q_j = 1$

$$-\mathrm{i}(z+z_j)\mathbf{\Phi}^{(j)}(z,\bar{z}) - \mathrm{i}c_j \boldsymbol{\chi}^{(j)} \tag{10.124}$$

$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \sum_{i \neq j}^{\mathcal{N}} \frac{\mathbf{\Phi}^{(i)}(z,\bar{z})}{k_j - k_i} + \frac{1}{2\pi\mathrm{i}} \iint \frac{\mathrm{d}\ell \wedge \mathrm{d}\bar{\ell}}{\ell - k_j} \, b(\ell,\bar{\ell}) \mathbf{N}(z,\bar{z},\ell,\bar{\ell}).$$

Equations (10.123) and (10.124) comprise the closed system of equations which enable us to solve the inverse problem. Indeed, owing to (10.106), the potential $u(z,\bar{z})$ is retrieved as

$$u(z,\bar{z}) = 2\mathrm{i} \sum_{j=1}^{\mathcal{N}} \overline{\Phi}_2^{(j)}(z,\bar{z}) - \frac{1}{\pi} \iint \mathrm{d}k \wedge \mathrm{d}\bar{k} \bar{b}(k,\bar{k}) N_2(z,\bar{z},k,\bar{k}). \tag{10.125}$$

10.5.3 Lump solutions

As an example, we derive the one-lump solution to the DS II equation. In this case $b(k,\bar{k})=0$, $\mathcal{N}=1$ and the eigenvector takes the form

$$\mathbf{M}=\begin{pmatrix}1\\0\end{pmatrix}+\frac{\mathbf{\Phi}^{(1)}}{k-k_1}. \tag{10.126}$$

In virtue of (10.119), equation (10.124) reduces to

$$(z+z_1)\mathbf{\Phi}^{(1)}+c_1\begin{pmatrix}0&-1\\1&0\end{pmatrix}\mathrm{e}^{-\mathrm{i}(k_1z+\bar{k}_1\bar{z})}\overline{\mathbf{\Phi}}^{(1)}=\mathrm{i}\begin{pmatrix}1\\0\end{pmatrix}.$$

Solving this algebraic equation yields

$$\mathbf{\Phi}^{(1)}=\frac{\mathrm{i}}{|z+z_1|^2+|c_1|^2}\begin{pmatrix}\bar{z}+\bar{z}_1\\c_1\exp\left[-\mathrm{i}(k_1z+\bar{k}_1\bar{z})\right]\end{pmatrix}. \tag{10.127}$$

Hence, from (10.125) we obtain the lump potential

$$u_1(x,y)=\frac{2\bar{c}_1}{|z+z_1|^2+|c_1|^2}\mathrm{e}^{\mathrm{i}(k_1z+\bar{k}_1\bar{z})}. \tag{10.128}$$

As usual, in order to have a lump solution of the DS II equation, we need to know the temporal evolution of the parameters k_1, z_1, and c_1. Substituting the vector function $\mathbf{M}$ (10.126) with the residue (10.127) into (10.105) and taking into account (10.100), we obtain

$$k_1\equiv\xi+\mathrm{i}\eta=\mathrm{const},\quad z_1(t)=z_1(0)+2k_1t,\quad c_1(t)=c_1(0)\exp\left[-2\mathrm{i}(\xi^2-\eta^2)t\right]. \tag{10.129}$$

Therefore, the lump solution of the DS II equation is written as [31]

$$u_1(x,y,t)=2\bar{c}_1(0)\frac{\exp\left[2\mathrm{i}(\xi x-\eta y+(\xi^2-\eta^2)t)\right]}{[x+2\xi t+z_{1\mathrm{R}}(0)]^2+[y+2\eta t+z_{1\mathrm{I}}(0)]^2+|c_1(0)|^2}. \tag{10.130}$$

It describes a localized nonsingular object which moves on the (x,y)-plane with constant velocity $(-2\xi,-2\eta)$ and decays as $(x^2+y^2)^{-1}$.

The solution (10.130) corresponds to $Q=1$. A novel situation arises for the DS II equation, as distinct from the KP I equation. Namely, there exist meromorphic functions of the type (10.126) related to higher indices. For example, for $Q=2$ and the simple pole k_1 the analog of the limit relation (10.122) takes a more complicated form [440]:

$$\frac{1}{2}\left[(z+z_1)^2-\delta\right]\mathbf{\Phi}^{(1)}+\bar{c}_1(\bar{z}+\bar{z}_1+\bar{\epsilon})\mathrm{e}^{-\mathrm{i}(k_1z+\bar{k}_1\bar{z})}\begin{pmatrix}0&-1\\1&0\end{pmatrix}\overline{\mathbf{\Phi}}^{(1)}$$

$$=\mathrm{i}\begin{pmatrix}z+z_1\\c_1\mathrm{e}^{-\mathrm{i}(k_1z+\bar{k}_1\bar{z})}\end{pmatrix}.$$

Solving this equation with respect to $\overline{\Phi}_2^{(1)}$, we obtain from (10.125) the lump solution with $Q = 2$:

$$u_2(x,y,t) = \Delta^{-1}\left[\bar c_1(z+z_1)^2 - 2c_1(z+z_1)(z+z_1+\epsilon) - \bar c_1\delta\right] \mathrm{e}^{\mathrm{i}(k_1 z+\bar k_1\bar z)}, \tag{10.131}$$

with additional complex parameters ϵ and δ. Here

$$\Delta = \frac{1}{4}|(z+z_1)^2 - \delta|^2 + |c_1|^2|z+z_1+\epsilon|^2 \tag{10.132}$$

and the complex parameters k_1, z_1, c_1, δ, and ϵ evolve in accordance with (10.129) and $\delta_t = 2\mathrm{i}$, $\epsilon_t = 0$. The solution (10.131) is nonsingular and decays as $(x^2+y^2)^{-1}$. Analysis of the denominator in (10.131) shows that asymptotically this solution decomposes into two separate lumps [440]. These lumps attract each other but the attractive force is not strong enough to form a bound state. After collision the lumps scatter at the angle $\pi/2$ (Figs. 10.1–10.3).

The simplest example of a meromorphic column function with a double pole is given by

$$\mathbf{M}(k) = \begin{pmatrix}1\\0\end{pmatrix} + \frac{\mathbf{\Phi}}{k-k_1} + \frac{\mathbf{\Psi}}{(k-k_1)^2}. \tag{10.133}$$

In the same way as for the KP I equation, we substitute (10.133) into (10.113) and expand the Green function (10.114) up to the second order in $(k-k_1)$. As a result, the limit relation takes the form for $Q_1 = 2$

$$\mathbf{\Phi} = -\mathrm{i}(z+z_1)\mathbf{\Psi} + \mathrm{i}\bar c_1 \mathrm{e}^{-\mathrm{i}(k_1 z+\bar k_1\bar z)} \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} \overline{\mathbf{\Psi}},$$

$$\begin{pmatrix}1\\0\end{pmatrix} = \frac{1}{2}\left[-(z+z_1)^2+\delta\right]\mathbf{\Psi} + c_1(z+z_1+\epsilon)\mathrm{e}^{-\mathrm{i}(k_1 z+\bar k_1\bar z)} \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} \overline{\mathbf{\Psi}}.$$

Solving this algebraic system, we obtain a solution to the DS II equation for the double pole with $Q = 2$. It turns out that this solution coincides with the solution (10.130) after letting $t \to -t$ and taking complex conjugation

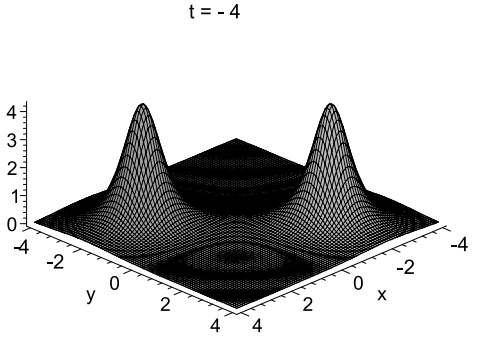

Fig. 10.1. Two lumps described by the solution (10.131) before interaction [440]

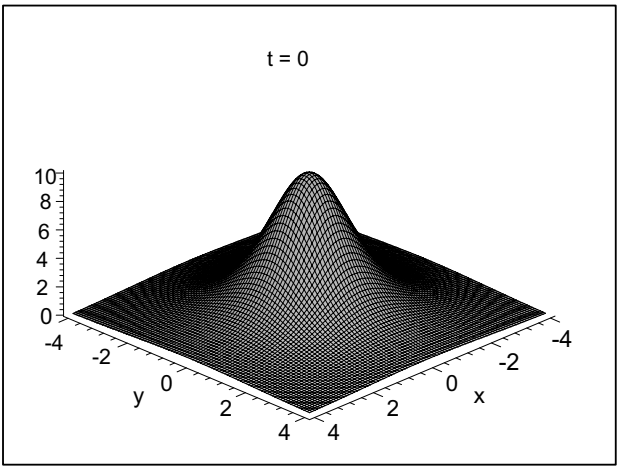

Fig. 10.2. Interaction of lumps described by the solution (10.131). Two lumps are merged into a single rotationally symmetric object with an amplitude equal to the sum of those of the separate lumps [440]

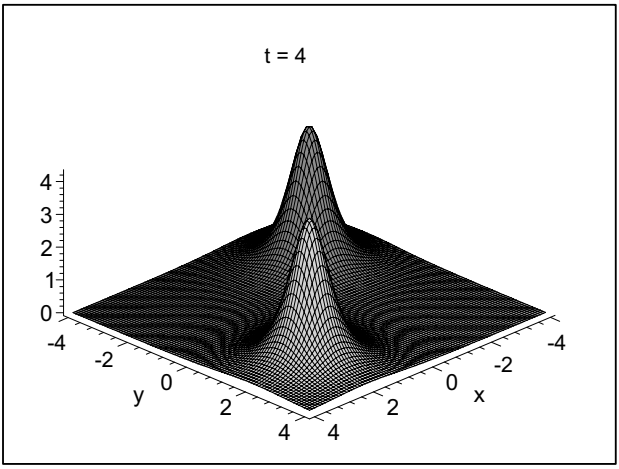

Fig. 10.3. Two lumps after interaction. The positions of lumps after scattering differ from those before interaction at the angle $\pi/2$ [440]

[440]. This result implies that the physically related solutions $u(x, y, t)$ and $\bar{u}(x, y, -t)$ for $Q = 2$ have different pole structures. Examples of meromorphic vector functions with a more complicated combination of poles can be found in the paper by Villarroel and Ablowitz [440].

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