ANALYTICAL SOLUTIONS OF THE KDV-KZK EQUATION

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Abstract: The KdV-KZK equation for fluids developed by me was presented at the ICSV 11 in St. Petersburg in July 2004. In this paper, I made an attempt on the analytical solutions of this equation using the perturbation method. Some physical interpretation of the solutions is given. A brief introduction to KdV-KZK equation for solids is given

Key words: analytical solutions, perturbation method, dissipation, dispersion, nonlinearity, diffraction, shock wave, solitons, chaos theory

1. INTRODUCTION

We developed the KdV-KZK equation for fluids in a paper presented at the 11th ICSV in St. Petersburg in July 2004. The KdV-KZK equation covers all basic physical mechanisms of sound propagation in fluids: diffraction, nonlinearity, absorption, and dispersion. Mark Hamilton's group[1] in University of Texas has used a numerical method to solve an augmented Burgers equation using splitting procedure and incremental step approach by giving equal weights to all the four physical mechanisms. This does not reflect the actual phenomenon because dispersion may be predominant over

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diffraction etc. and so only gives approximation results. In this paper, we use a rigorous approach by using analytical method.

2. PERTURBATION METHOD USED

The KdV-KZK equation is given as

$$\frac{\partial}{\partial \tau}\left(\frac{\partial P}{\partial z}-\frac{\partial^2 P}{\partial c_0^3 \rho_0}\frac{\partial^2 P}{\partial \tau^2}-\frac{\varepsilon}{c_0^3 \rho_0}P\frac{\partial P}{\partial \tau}+\gamma\frac{\partial^3 P}{\partial \tau^3}\right)=\frac{c_0}{2}\Delta_\perp P \tag{1}$$

where P = acoustic pressure, z = direction of sound propagation, $\tau = t - z/c_0$, ε = parameters of nonlinearity, c_0 = sound velocity, $b = \zeta + 4\eta/3$ where ζ and η are the bulk and shear viscosity. ρ_0 = density of fluid and $\gamma = c_p/c_v$ = adiabatic index where c_p and c_v are the specific heats at constant pressure and constant volume.

With perturbation method and writing

$$P = P^{(1)} + P^{(2)} + P^{(3)} \tag{2}$$

First perturbation, we write

$$\frac{\partial}{\partial \tau}\left(\frac{\partial P^{(1)}}{\partial z}-\frac{b}{\partial c_0^3 \rho_0}\frac{\partial^2 P^{(1)}}{\partial \tau^2}\right)-\frac{c_0}{2}\Delta_\perp P^{(1)}=0 \tag{3}$$

where $\Delta_\perp = \partial^2/\partial x^2 + \partial^2/\partial y^2$ is the Lagrangian in transverse coordinates.

Here our solutions are given in the light of the simple modes of operation of a parametric radiator. The processes whereby low-frequency waves are formed in the field of nondiffracting plane and spherically diverging high-frequency beams are considered. We will consider waves of high intensity (high acoustic Reynolds number) and that the profile of the wave contains a discontinuity.

The solution of Eq. (3) for a biharmonic high-frequency pump can be written as

$$P^{(1)} = \frac{1}{2}A_1(r,z)e^{iw_1\tau} + \frac{1}{2}A_2(r,z)e^{iw_2\tau} + c.c. \tag{4}$$

The complex amplitudes $A_{1,2}$ satisfy the parabolic equation of diffraction theory:

$$\frac{\partial A}{\partial z} + \alpha A = \frac{1}{2ik}\left(\frac{\partial^2 A}{\partial r^2} + \frac{1}{r}\frac{\partial A}{\partial r}\right) \tag{5}$$

where αA describes the attenuation of the high-frequency waves.

For second perturbation, we have

$$\frac{\partial}{\partial \tau}\left(\frac{\partial P^{(2)}}{\partial z} - \frac{b}{2c_0^3\rho_0}\frac{\partial^2 P^{(2)}}{\partial \tau^2}\right) - \frac{c_0}{2}\Delta_\perp P^{(2)} = \frac{\varepsilon}{2c_0^3\rho_0}\frac{\partial^2 P^{(1)^2}}{\partial \tau^2} \tag{6}$$

Substituting in Eq. (4), we have

$$\frac{\varepsilon}{2c_0^3\rho_0}\frac{\partial^2 P^{(1)^2}}{\partial \tau^2} = -\frac{\varepsilon\Omega^2}{4c_0^3\rho_0}A_1(r,z)A_2^*(r,z)e^{i\Omega\tau} + c.c. \tag{7}$$

where Ω = difference angular frequency.

Substitute the expression Eq. (7) into the right hand side of Eq. (6) and seek its solution by inspection in the form

$$P^{(2)} = \frac{1}{2} P(r,z) e^{i\Omega r} + c.c. \tag{8}$$

We obtain the following equation for the complex amplitude of the difference frequency wave $P_{-}(r,z)$ in place of Eq. (6) :

$$\frac{\partial P_{-}}{\partial z} + \alpha_{-} P_{-} - \frac{1}{2ik} \Delta_{\perp} P_{-} = i \frac{\varepsilon \Omega}{2 c_0^3 \rho_0} A_1 A_2^{*} \tag{9}$$

Here the term that describes the damping of the difference frequency wave is eliminated with the help of the substitution $P_{-} \to P_{-} \exp(-\alpha_{-} z)$. Thus Eq. (9) becomes

$$\frac{\partial P_{-}}{\partial z} - \frac{1}{2ik} \Delta_{\perp} P_{-} = i \frac{\varepsilon K}{2 c_0^2 \rho_0} A_1 A_2^{*} e^{-\alpha z} = Q(r,z) \tag{10}$$

where α_{-} = damping coefficient of difference frequency wave, K = wave number of the difference frequency wave, $\alpha = \alpha_1 + \alpha_2 - \alpha_{-} = 2/l_a$ = effective attenuation coefficient, A_1, A_2^{*} = functions satisfying the parabolic equation Eq. (5) without account of attenuation.

To solve Eq. (10), we assume the beams to be cylindrically symmetric (circular). Using the Hankel transforms :

$$\begin{aligned} P_{-}(r,z) &= \int_0^{\infty} \widetilde{P}(v,z) J_0(vr) v dv, \\ \widetilde{P}(v,z) &= \int_0^{\infty} P_{-}(r,z) J_0(vr) r dr \end{aligned} \tag{11}$$

then Eq. (10) reduces to

$$\frac{d\widetilde{P}}{dz} = i\frac{v^2}{2k}\widetilde{P} = \widetilde{Q}(v,z) \tag{12}$$

where $\widetilde{Q} = \dfrac{i\varepsilon K}{2c_0^2\rho_0}e^{-\alpha z}\int_0^\infty A_1 A_2^{\,*} J_0(vr)rdr$ (13)

is the Hankel transform of the right hand side of Eq. (10). The solution of Eq. (12) with zero condition at the boundary z=0 has the form

$$\widetilde{P} = \int_0^z \widetilde{Q}(v,z')\exp(i\frac{v^2}{2k}(z-z'))dz' \tag{14}$$

Carrying out the inverse Hankel transformation, we find the desired solution of Eq. (10)

$$P_-(r,z) = \int_0^\infty J_0(vr)vdv\int_0^z \widetilde{Q}(v,z')\exp(\frac{iv^2}{2k}(z-z'))dz' \tag{15}$$

This solution is valid for any distributions of complex amplitudes of the high-frequency waves A_1 and A_2 on the surface of the pump transducer. Substituting any of the solutions of the parabolic equation in this form, the corresponding solution for the difference-frequency wave can be obtained.

For third perturbation, we have

$$\frac{\partial}{\partial \tau}\left(\frac{\partial P^{(3)}}{\partial z} - \frac{b}{2c_0^3\rho_0}\frac{\partial^2 P^{(3)}}{\partial \tau^2}\right) - \frac{c_0}{2}\Delta_\perp P^{(3)} = \frac{\varepsilon}{2c_0^3\rho_0}\frac{\partial^2 P^{(2)^2}}{\partial \tau^2} - \gamma\frac{\partial^3 P^{(2)}}{\partial \tau^3} \tag{16}$$

We know that the KdV-KZK equation is an extension of the KdVB equation to include diffraction effect. So the solution of the KdVB equation should throw some lights on the possible solution of the KdV-KZK equation. The solution of the KdVB equation describes a shock wave as a transition between two constant velocity values. This transition can have oscillations due to the dispersion. At low δ where $\delta = \frac{b}{2c_0^3\rho_0}$ the first oscillations are quite close to solitons (solitons correspond to a closed separatrix). In the latter case a nonstationary solution can be developed as well, describing a slowly decaying soliton.

Hence, let us seek a solution in the form of a soliton

$$P = \frac{P_0}{\cosh^2(\tau + bz/\Delta)} \tag{17}$$

where b = constant = $\frac{\varepsilon P_0}{c_0^3\rho_0 3}$ and $\Delta = \sqrt{\frac{12\gamma}{(\varepsilon/c_0^3\rho_0)P_0}}$.

Substituting Eq. (17) into Eq. (16), should give some light on the solution of the KdV-KZK equation.

3. ANALYTICAL SOLUTION OF THE KDV-KZK EQUATION USING CHAOTIC THEORY

The KdV-KZK equation is a nonlinear differential equation. There is a theorem stating that any nonlinear differential equation does possess a regime within which its solutions are chaotic. Using the above theorem, the KdV-KZK equation should have some chaotic solution subject to certain conditions on its parameters. The KdVB equation has some soliton solution. X. N. Chen and R. J. Wei[2] show that some solitons do possess chaotic characteristics. I feel that it is very promising to find analytical solutions of the KdV-KZK equation in the light of chaotic theory.

4. EXTENSION OF THE KDV-KZK EQUATION TO SOLIDS

There has been extension of the KZ equation from fluids to solids such as M. S. Cramer and M. F. Andrew's[3] work. In this paper, they consider a weakly nonlinear, weakly diffracting, two dimensional shear waves propagating in a prestrained hyper elastic solid. A modification of the classical KZ equation is derived using a systematic perturbation scheme. Both dissipative and non-dissipative materials are considered. The principle effect of the prestrain is seen to be the inclusion of a quadratic nonlinearity to the cubic nonlinearity found n the case of zero prestrain. Further results include the shock jump relations and the prediction of shocks having a speed which is identical to the nonlinear wave speed ahead of or behind the shock. The main difference of the KZ equation for fluids and for solids is the inclusion of elasticity and stress-strain relations in the equation. Also the Lagrangian coordinates instead of the Euler coordinates have to be used to account for the strain parameter.

5. CONCLUSION

The analytical solution for the KdV-KZK equation is possible. Analytical solutions for the KdVB equation are well established. Compared with the KdVB equation, the KdV-KZK equation only has one extra term and it will not introduce much complexity. It would be also useful to find numerical solutions of the KdV-KZK equation but would be different from that of the Lee-Hamilton method[1] which gives equal importance to each of the physical mechanism. KdV-KZK equation for fluids will find application in aerodynamics, underwater acoustics. Its extension to solids will find applications in medical imaging and in nonlinear nondestructive evaluation.

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