Chapter 4

First-Order Inference

In this chapter, three new layers of the logic, NAL-2, NAL-3, and NAL-4, will be defined, which introduce terms with internal structures and variants of the inheritance relation into NAL. Consequently, the expressive and inferential power of the logic will be increased, step by step. At the end of the chapter, we will get a complete First-Order Non-Axiomatic Logic.

4.1 Compound terms

In NAL-1, each term is "atomic," and named by a word, which is simply a unique identifier without internal structure. Obviously, Narsese-1 can only express simple statements.

To represent more complicated experience, "compound terms" are needed.

Definition 16 *A* compound term $(op\ c_1\ \cdots\ c_n)$ *is a term formed by one or more terms* $c_1, \cdots, c_n$*, called its* component(s)*, with a* term operator*,* op*. The order of the components usually matters.*

Sometimes we prefer the "infix" format of a compound term, that is, to write $(op\ c_1\ \cdots\ c_n)$ as $(c_1\ op\ \cdots\ op\ c_n)$. When introducing term operators with two or more components in the following, usually they are only defined with two components, and the general case (for both the above prefix representation and the infix representation) is translated into the two-component case by the following definition.

Definition 17 *If* $c_1 \cdots c_n$ $(n > 2)$ *are terms and op is a term operator defined as taking two arguments, both compound terms* $(op\ c_1 \cdots c_n)$ *and* $(c_1\ op \cdots op\ c_n)$ *are defined recursively as* $(op\ (op\ c_1 \cdots c_{n-1})\ c_n)$.

In NAL, the term operators are predefined as part of the grammar of Narsese, with determined (experience-independent) meaning. The meaning of a compound term has two parts, a *literal* part and an *empirical* part, where the former is determined by its definition and other literal truths about the term, while the latter comes from the system's experience when the compound term is used as a whole. In NAL, though empirical statements are all uncertain (i.e., with frequency in [0, 1] and confidence in (0, 1)), literal truths remain binary, so their truth values are omitted in the following description.

To indicate the syntactic complexity of a compound term, a notion of "level" is recursively defined as the following.

Definition 18 *Each term in NAL is on a certain* level *according to its syntactical complexity. If a term is atomic, then it is on level 1. If a term is a compound, then it is one level higher than the highest level of its components.*

Defined in this way, "level" is a syntactic concept, and it has nothing to do with semantics. Terms of different levels can have inheritance relations between each other.

All compound terms can be used by the inference rules defined in NAL-1. When doing so, their internal structures are ignored. In the following, three extensions of NAL-1 are defined, layer by layer, each of which processes some special types of compound term.

4.2 NAL-2: sets and variants of inheritance

NAL-1 is extended into NAL-2 by introducing new copulas to enrich the system's expressing capacity, so as to move Narsese closer to natural languages.

The copula in a term logic intuitively corresponds to the "to be" in English. However, even such a rough mapping cannot be simply established, because as a copula, "*to be*" has multiple usages, for example:

type: "Birds *are* animals."

element: "Tweety *is* a bird."

attribute: "Ravens *are* black."

identification: "The morning star *is* Venus."

The inheritance relation defined in NAL-1 can be used for the first case, but not for the others directly, though it is closely related to them. To introduce these new relations into NAL, the grammar, semantics, and inference rules all need to be extended.

4.2.1 Similarity

A symmetric inheritance relation *similarity* is written as "$\leftrightarrow$."

Definition 19 *The similarity statement "$S \leftrightarrow P$" is defined by two inheritance statements as $S \leftrightarrow P \equiv (S \rightarrow P) \wedge (P \rightarrow S)$.*

Therefore, the similarity relation is reflexive, symmetric, and transitive.

It follows that an inheritance relation between two terms is implied by a similarity relation between them.

Theorem 4 $(S \leftrightarrow P) \supset (S \rightarrow P)$

Here "$\supset$" is the *implication* operator defined in propositional logic. The expressions in the theorem are not statements in Narsese, but in its meta-language.

Two terms related by the similarity relation are in both the extension and the intension of each other. Here the extension and intension of a term are defined as before, that is, by the "$\rightarrow$" relation, not the new "$\leftrightarrow$" relation.

Theorem 5 $(S \leftrightarrow P) \equiv (S \in (P^E \cap P^I)) \equiv (P \in (S^E \cap S^I))$

Theorem 6 $(S \leftrightarrow P) \equiv (S^E = P^E) \equiv (S^I = P^I)$

That is, "$S \leftrightarrow P$" means "S and P have the same meaning." Or, we can say that the two terms are *identical*.

Two compounds terms are identical if they have the same term operator, and their corresponding components are identical pair by pair. Especially, if each of them has exactly one component, then the above "if" becomes "if and only if."

Definition 20 *The meaning of two compound terms are related in the following way:*

$$((c_1 \leftrightarrow d_1) \wedge \cdots \wedge (c_n \leftrightarrow d_n)) \supset ((op\ c_1\ \cdots\ c_n) \leftrightarrow (op\ d_1\ \cdots\ d_n))$$

$$(c \leftrightarrow d) \equiv ((op\ c) \leftrightarrow (op\ d))$$

To extend the relation to the situations of "incomplete similarity," the evidence of a similarity relation is defined like the evidence of an inheritance relation. For similarity statement "$S \leftrightarrow P$," its positive evidence is in $(S^E \cap P^E)$ and $(P^I \cap S^I)$, and its negative evidence is in $(S^E - P^E)$, $(P^E - S^E)$, $(P^I - S^I)$, and $(S^I - P^I)$. In this way, in general "similarity" is a matter of degree, measured by a truth value (defined in NAL-1, as a function of the amount of evidence). In the following, I reserve the word "identical" for the special situation where a similarity statement has truth value $<1, 1>$.

Corresponding to the syllogistic rules in NAL-1, in NAL-2 there are three combinations of inheritance and similarity, corresponding to *comparison*, *analogy*, and another form of *deduction*, respectively, as indicated by the names of truth-value functions in Table 4.1. To make the table (as well as the following inference tables) simpler, the truth values of the premises are omitted.

$J_2 \setminus J_1$	$M \rightarrow P$	$P \rightarrow M$	$M \leftrightarrow P$
$S \rightarrow M$		$S \leftrightarrow P < F_{com} >$	$S \rightarrow P < F'_{ana} >$
$M \rightarrow S$	$S \leftrightarrow P < F_{com} >$		$P \rightarrow S < F'_{ana} >$
$S \leftrightarrow M$	$S \rightarrow P < F_{ana} >$	$P \rightarrow S < F_{ana} >$	$S \leftrightarrow P < F_{dd2} >$

Table 4.1: The Syllogistic Rules of NAL-2

In the inference table, F'_x indicate the truth-value function obtained by exchanging $< f_1, c_1 >$ and $< f_2, c_2 >$ in function F_x, where x is the indicator of the inference rule (such as *ana* for analogy, *abd* for abduction, and *ind* for induction). Such a function is needed for every inference rule whose truth-value function is not symmetric with respect to the two premises.

In NARS, *comparison* refers to the inference rule by which a similarity judgment is obtained by comparing the inheritance relations of two terms to a third term. It is easy to see that the situation here is like the cases of abduction and induction, except that now the conclusion is symmetric. Using the same procedure as in NAL-1, we first build Boolean functions among the variables as the following:

$$w = and(or(f_1, f_2), c_1, c_2),\ w^+ = and(f_1, c_1, f_2, c_2)$$

which lead to the truth-value function

$$F_{com}:\ f = \frac{f_1 f_2}{f_1 + f_2 - f_1 f_2},\ c = \frac{c_1 c_2 (f_1 + f_2 - f_1 f_2)}{c_1 c_2 (f_1 + f_2 - f_1 f_2) + k}$$

When $f_1 = f_2 = 0$, we define f to be 0 for the sake of continuity.

From an inheritance judgment J_1 and a similarity judgment J_2, NAL does a certain type of *analogy* by replacing a term in J_1 by a similar term provided by J_2. The situation here is quite similar to that of deduction as defined in NAL-1. The difference is, when the similarity judgment goes to extreme to become an identity judgment, the conclusion should have the truth-value of the inheritance judgment. Therefore, the confidence should depend more on f_2 and c_2, and not on f_1 anymore. Under this consideration NAL uses the following truth-value function:

$$F_{ana}:\ f = f_1 f_2,\ c = c_1 f_2^2 c_2^2$$

If the two premises are both similarity relations, the inference here is based on the transitivity of the similarity relation. It can be treated as deduction going in both directions. However, in this case, if one similarity judgment goes to extreme to become an identity judgment, the conclusion should have the truth-value of the other similarity judgment. Therefore, the following truth-value function is used:

$$F_{dd2}:\ f = f_1 f_2,\ c = c_1 c_2 (f_1 + f_2 - f_1 f_2)$$

The above two truth-value functions are introduced as variants of the deduction function, rather than obtained by directly analyzing the truth-value relationship between the premises and the conclusion.

4.2.2 Sets, instance and property

With the inheritance relation defined in NAL-0, terms can form an inheritance hierarchy, with respect to their level of generality/specificity.

- If "$T_1 \to T_2$" is true, and "$T_2 \to T_1$" is false, then T_1 is more specific than T_2, and T_2 is more general than T_1.
- If both "$T_1 \to T_2$" and "$T_2 \to T_1$" are true (that is, "$T_1 \leftrightarrow T_2$" is true), then T_1 and T_2 are on the same level of generality/specificity.
- If both "$T_1 \to T_2$" and "$T_2 \to T_1$" are false, then T_1 and T_2 cannot be compared with respect to generality/specificity.

For various purposes, we often need to define the boundary of such a hierarchy, by treating certain terms as at the most specific or the most general level. In NAL-2, two kinds of such compound terms, "extensional set" and "intensional set," are introduced.

Definition 21 *If T is a term, the* extensional set *with T as the only component, $\{T\}$, is also a term, and its meaning is defined by*

$$(\forall x)((x \to \{T\}) \equiv (x \leftrightarrow \{T\})).$$

That is, a compound term with such a form is like a set defined by a sole element. The compound therefore has a special property: all terms in the extension of $\{T\}$ must be identical to it, and no term can be more specific than it (though it is possible for some terms to be more specific than T).

Theorem 7 *For any term T, $\{T\}^E \subseteq \{T\}^I$.*

On the other hand, $\{T\}^I$ is not necessarily included in $\{T\}^E$.

To name a term like this means to treat its extension as including an *individual*. In a natural language, T often corresponds to a proper name, and $\{T\}$ corresponds to a category with a single instance indicated by that proper name. For example, "Tweety is a bird" can be represented as "$\{Tweety\} \rightarrow bird$" (but not "$Tweety \rightarrow bird$," which means "Tweety is a kind of bird").

An *instance relation*, "$\circ\!\!\rightarrow$," is another way to represent the same information.

Definition 22 *The instance statement "$S \circ\!\!\rightarrow P$" is defined by the inheritance statement "$\{S\} \rightarrow P$."*

So "Tweety is a bird" can also be represented as "$Tweety \circ\!\!\rightarrow bird$."

The intuitive meaning of "$\circ\!\!\rightarrow$" is similar to the membership relation ("$\in$") in set theory, but in NAL this relation is no longer primary or necessary (since it is defined by other notions).[1]

Theorem 8 $((S \circ\!\!\rightarrow M) \wedge (M \rightarrow P)) \supset (S \circ\!\!\rightarrow P)$.

However, "$S \rightarrow M$" and "$M \circ\!\!\rightarrow P$" does not imply "$S \circ\!\!\rightarrow P$."

Theorem 9 $(S \circ\!\!\rightarrow \{P\}) \equiv (S \leftrightarrow P)$.

"$T \circ\!\!\rightarrow \{T\}$" follows as a special case. On the other hand, the statement "$T \circ\!\!\rightarrow T$" is not a literal truth, though may be an empirical one.

According to the duality between extension and intension, we can define another special compound term and the corresponding copula.

Definition 23 *If T is a term, the* intensional set *with T as the only component, $[T]$, is also a term, and its meaning is defined by*

$$(\forall x)(([T] \rightarrow x) \equiv ([T] \leftrightarrow x)).$$

That is, a compound term with such a form is like a set defined by a sole attribute. The compound therefore has a special property: all terms in the intension of $[T]$ must be identical to it, and no term can be more general than it (though it is possible for some terms to be more general than T).

Theorem 10 *For any term T, $[T]^I \subseteq [T]^E$.*

On the other hand, $[T]^E$ is not necessarily included in $[T]^I$.

[1]This issue will be discussed in detail in subsection 10.1.3.

To name a term like this means to treat its intension as having an *attribute*. In a natural language, T often corresponds to an adjective, and $[T]$ corresponds to a category with that adjective as the defining property. For example, "Ravens are black" can be represented as "$raven \rightarrow [black]$" (but not "$raven \rightarrow black$").

A *property relation*, "$\rightarrow\!\!\circ$," is another way to represent the same information.

Definition 24 *The property statement "$S \rightarrow\!\!\circ P$" is defined by the inheritance statement "$S \rightarrow [P]$."*

So "Ravens are black" can also be represented as "$raven \rightarrow\!\!\circ black$."

This relation can be used when we characterize terms by a set of primary properties. It can also be directly used in inference.

Theorem 11 $(S \rightarrow M) \wedge (M \rightarrow\!\!\circ P) \supset (S \rightarrow\!\!\circ P)$.

However, "$S \rightarrow\!\!\circ M$" and "$M \rightarrow P$" does not imply "$S \rightarrow\!\!\circ P$."

Theorem 12 $([S] \rightarrow\!\!\circ P) \equiv (S \leftrightarrow P)$.

"$[T] \rightarrow\!\!\circ T$" follows as a special case. On the other hand, the statement "$T \rightarrow\!\!\circ T$" is not a literal truth, though may be an empirical one.

An *instance-property* relation, "$\circ\!\!\rightarrow\!\!\circ$," is defined by combining "$\circ\!\!\rightarrow$" and "$\rightarrow\!\!\circ$."

Definition 25 *The instance-property statement "$S \circ\!\!\rightarrow\!\!\circ P$" is defined by the inheritance statement "$\{S\} \rightarrow [P]$."*

Intuitively, it states that an instance S has a property P. This relation is not really necessary, and it is just a way to simplify a statement.

So "Tweety is yellow" can also be represented as "$Tweety \circ\!\!\rightarrow\!\!\circ yellow$" (but not "$Tweety \rightarrow\!\!\circ yellow$," "$Tweety \circ\!\!\rightarrow yellow$," or "$Tweety \rightarrow yellow$").

Theorem 13 $(S \circ\!\!\rightarrow\!\!\circ P) \equiv (\{S\} \rightarrow\!\!\circ P) \equiv (S \circ\!\!\rightarrow [P])$

$<copula>$	::=	$\leftrightarrow \mid \circ\!\!\rightarrow \mid \rightarrow\!\!\circ \mid \circ\!\!\rightarrow\!\!\circ$
$<term>$	::=	$\{<term>\} \mid [<term>]$

Table 4.2: The New Grammar Rules of Narsese-2

$S \leftrightarrow P$	$\{S\} \leftrightarrow \{P\}$
$S \leftrightarrow P$	$[S] \leftrightarrow [P]$
$S \rightarrow \{P\}$	$S \leftrightarrow \{P\}$
$[S] \rightarrow P$	$[S] \leftrightarrow P$
$S \circ\!\!\rightarrow P$	$\{S\} \rightarrow P$
$S \rightarrow\!\!\circ P$	$S \rightarrow [P]$
$S \circ\!\!\rightarrow\!\!\circ P$	$\{S\} \rightarrow [P]$

Table 4.3: The Equivalence Rules of NAL-2

4.2.3 NAL-2 summary

In summary, while all the grammar rules of Narsese-1 are still valid in NAL-2, there are additional grammar rules of Narsese-2, as listed in Table 4.2.

Beside the syllogistic rules in Table 4.1, the previous definitions give the equivalence rules of NAL-2 in Table 4.3, where the two statements in the same row can replace each other. That is, a judgment with one statement can derive another judgment with the other statement, with the same truth value.

Since each new copula is defined in terms of the inheritance relation "$\rightarrow$," its semantics and relevant inference rules can be derived from those in NAL-1.

Please note that the extension and intension of a term are still defined by the inheritance relation, not by the new relations derived from it. Therefore,

- "$S \circ\!\!\rightarrow P$" says that the extension of P include $\{S\}$ (not S) as an element;

- "$S \rightarrow\!\!\circ\, P$" says that the intension of S include $[P]$ (not P) as an element.

To simplify the implementation of the system, relations "$\circ\!\!\rightarrow$," "$\rightarrow\!\!\circ$," and "$\circ\!\!\rightarrow\!\!\circ$" are only used in the input/output interface, and within the system they are translated into "$\rightarrow$." Therefore we do not really need to introduce inference rules for them. The same thing cannot be done to "$\leftrightarrow$." Though the "$\leftrightarrow$" relation is defined in terms of the "$\rightarrow$" relation, the system usually cannot translate a "$\leftrightarrow$" judgment into an equivalent "$\rightarrow$" judgment. Therefore, NAL-2 uses five copula in its interface language, but only keep two of them ("$\rightarrow$" and "$\leftrightarrow$") in its internal representation, without losing any expressive and inferential power.

As an example of inference in NAL-2, we start with the following judgments:

$$(1)\ Tweety \circ\!\!\rightarrow bird < 1, 0.9 >$$
$$(2)\ Tweety \circ\!\!\rightarrow\!\!\circ\ yellow < 1, 0.9 >$$
$$(3)\ Tweety \leftrightarrow Birdie < 1, 0.9 >$$

Using the equivalence rules, they derive the following judgments, respectively:

$$(4)\ \{Tweety\} \rightarrow bird < 1, 0.9 >$$
$$(5)\ \{Tweety\} \rightarrow [yellow] < 1, 0.9 >$$
$$(6)\ \{Tweety\} \leftrightarrow \{Birdie\} < 1, 0.9 >$$

From (4) and (5), by induction the system derives

$$(7)\ bird \rightarrow [yellow] < 1, 0.45 >$$

From (4) and (6), by analogy the system derives

$$(8)\ \{Birdie\} \rightarrow bird < 1, 0.73 >$$

which can be displayed as

$$(9)\ Birdie \circ\!\!\rightarrow bird < 1, 0.73 >$$

4.3 NAL-3: intersections and differences

In NAL-3, compound terms are composed by combining the extension or intension of existing terms in certain way.

4.3.1 Intersections

Definition 26 *Given terms* T_1 *and* T_2*, their* extensional intersection*,* $(T_1 \cap T_2)$*, is a compound term defined by*

$$(\forall x)((x \rightarrow (T_1 \cap T_2)) \equiv ((x \rightarrow T_1) \wedge (x \rightarrow T_2))).$$

From right to left, the equivalence expression defines the extension of the compound, i.e., "$(x \rightarrow T_1) \wedge (x \rightarrow T_2)$" implies "$x \rightarrow (T_1 \cap T_2)$"; from left to right, it defines the intension of the compound, i.e., "$(T_1 \cap T_2) \rightarrow (T_1 \cap T_2)$" implies "$(T_1 \cap T_2) \rightarrow T_1$" and "$(T_1 \cap T_2) \rightarrow T_2$."

As an example, "Ravens are black birds" can be represented as "$raven \rightarrow ([black] \cap bird)$," where the predicate term is an extensional intersection of the term $[black]$ and the term *bird*.

Theorem 14

$$(T_1 \cap T_2)^E = T_1^E \cap T_2^E, \; (T_1 \cap T_2)^I = T_1^I \cup T_2^I$$

In the above expressions, the "$\cap$" sign is used in two different senses. On the right-side of the first expression, it indicates the ordinary intersection of sets, but on the left-side of the two expressions, it is the new intersection operator of terms. Though these two senses are intuitively similar, they are not the same, because the term operator is related to both the extension and the intension of a term.

As the common extension of the two terms, the compound term $(T_1 \cap T_2)$ inherits properties of both T_1 and T_2. That is why its intension is the union of the intensions of its two components.

The above definition and theorem specify the *literal* meaning of the compound terms, which is formed when a compound is built from its components. Later, the meaning of the compound may become more or less different, as the result of new experience and inference activity.[2]

The intensional intersection of terms is defined symmetrically.

Definition 27 *Given terms* T_1 *and* T_2*, their* intensional intersection*,* $(T_1 \cup T_2)$*, is a compound term defined by*

$$(\forall x)(((T_1 \cup T_2) \rightarrow x) \equiv ((T_1 \rightarrow x) \wedge (T_2 \rightarrow x))).$$

[2]This issue is discussed in detail in subsection 11.1.5.

From right to left, the equivalence expression defines the intension of the compound, i.e., "$(T_1 \rightarrow x) \wedge (T_2 \rightarrow x)$" implies "$(T_1 \cup T_2) \rightarrow x$"; from left to right, it defines the extension of the compound, i.e., "$(T_1 \cup T_2) \rightarrow (T_1 \cup T_2)$" implies "$T_1 \rightarrow (T_1 \cup T_2)$" and "$T_2 \rightarrow (T_1 \cup T_2)$."

Intuitively, the intensional intersection of two terms is defined by the common properties of the terms.

Theorem 15

$$(T_1 \cup T_2)^I = T_1^I \cap T_2^I, \ (T_1 \cup T_2)^E = T_1^E \cup T_2^E$$

Now we can see that the duality of *extension* and *intension* in NAL corresponds to the duality of *intersection* and *union* in set theory — *intensional intersection* corresponds to *extensional union*, and *extensional intersection* corresponds to *intensional union*.[3]

From the definitions, it is obvious that both intersections are symmetric to their components:

Theorem 16

$$\begin{array}{c} (T_1 \cap T_2) \leftrightarrow (T_2 \cap T_1) \\ (T_1 \cup T_2) \leftrightarrow (T_2 \cup T_1) \end{array}$$

The relation between the compounds to their components is captured by the following theorem:

Theorem 17

$$\begin{array}{l} (T_1 \cap T_2) \rightarrow T_1 \\ T_1 \rightarrow (T_1 \cup T_2) \end{array}$$

Both operators can be extended to take more than two arguments. Since "$\cap$" and "$\cup$" are both associative and symmetric, the order of their components does not matter. A special situation for these two operators is when the components are the same.

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[3]Some people may think that it is more natural to call $(T_1 \cup T_2)$ "extensional union" than "intensional intersection," because the symbol used for the term operator is the union operator in set theory. I do it the other way to stress its relation with the extensional intersection, as well as with the "difference" terms to be introduced in the following.

Theorem 18

$$(T \cup T) \leftrightarrow T$$
$$(T \cap T) \leftrightarrow T$$

The following implications are derived from the definition of the compound terms.

Theorem 19

$$\begin{array}{rcllcr} T_1 \rightarrow M & \wedge & \neg((T_1 \cup T_2) \rightarrow M) & \supset & \neg(T_2 \rightarrow M) \\ \neg(T_1 \rightarrow M) & \wedge & (T_1 \cap T_2) \rightarrow M & \supset & T_2 \rightarrow M \\ M \rightarrow T_1 & \wedge & \neg(M \rightarrow (T_1 \cap T_2)) & \supset & \neg(M \rightarrow T_2) \\ \neg(M \rightarrow T_1) & \wedge & M \rightarrow (T_1 \cup T_2) & \supset & M \rightarrow T_2 \end{array}$$

These implications will be turned into inference rules in the next chapter, which can be used to "decompose" a compound term to get conclusions about its components.

The two intersection operators keep inheritance and similarity relations between terms:

Theorem 20

$$\begin{array}{rcl} S \rightarrow P & \supset & (S \cup M) \rightarrow (P \cup M) \\ S \rightarrow P & \supset & (S \cap M) \rightarrow (P \cap M) \\ S \leftrightarrow P & \supset & (S \cup M) \leftrightarrow (P \cup M) \\ S \leftrightarrow P & \supset & (S \cap M) \leftrightarrow (P \cap M) \end{array}$$

In these propositions, M can be any term in V_K. The same is assumed for the implications introduced later.

If a term is taken as a set, then all the above theorems can be proved in set theory. However, in NAL even if sets can be defined (as in NAL-2), not all terms are sets, and the above theorems should not be understood as equivalent to their counterparts in set theory, but as (partially) isomorphic to them.

4.3.2 Differences

The above two compound terms are defined by restricting the extension or intension of a term with an additional *positive* statement. In the following we do the same thing, but with a *negative* statement.

Definition 28 *If T_1 and T_2 are different terms, their* extensional difference*, $(T_1 - T_2)$, is a compound term defined by*

$$(\forall x)((x \rightarrow (T_1 - T_2)) \equiv ((x \rightarrow T_1) \wedge \neg(x \rightarrow T_2))).$$

From right to left, the equivalence expression defines the extension of the compound, i.e., "$(x \rightarrow T_1) \wedge \neg(x \rightarrow T_2)$" implies "$x \rightarrow (T_1 - T_2)$"; from left to right, it defines the intension of the compound, i.e., "$(T_1 - T_2) \rightarrow (T_1 - T_2)$" implies "$(T_1 - T_2) \rightarrow T_1$" and "$\neg((T_1 \cap T_2) \rightarrow T_2)$."

Given this definition, "Penguins are birds that cannot fly" can be represented as "$penguin \rightarrow (bird - [flying])$," where the predicate term is a extensional difference of the term *bird* and the term $[flying]$.

Obviously, $(T_2 - T_1)$ can also be defined, but it will be different from $(T_1 - T_2)$.

Theorem 21

$$(T_1 - T_2)^E = T_1^E - T_2^E, \ (T_1 - T_2)^I = T_1^I$$

Intuitively, $(T_1 - T_2)$ is $(T_1 \cap (non\text{-}T_2))$, where $(non\text{-}T_2)$ is a term whose extension includes all terms in V_K that are not in the extension of T_2. However, the intension of this term is empty, because such a definition specifies no common (affirmatively specified) property for the terms in its extension. Since no additional property can be assigned to the compound, the intension of $(T_1 - T_2)$ is the same as that of T_1. For the same reason, in Narsese there is no term defined as the negation of another term, though we can talk about $(non\text{-}T)$ in the meta-language.[4]

Symmetrically, intensional differences can be defined.

Definition 29 *If T_1 and T_2 are different terms, their* intensional difference*, $(T_1 \ominus T_2)$, is a compound term defined by*

$$(\forall x)(((T_1 \ominus T_2) \rightarrow x) \equiv ((T_1 \rightarrow x) \wedge \neg(T_2 \rightarrow x))).$$

From right to left, the equivalence expression defines the intension of the compound, i.e., "$(T_1 \rightarrow x) \wedge \neg(T_2 \rightarrow x)$" implies "$(T_1 \ominus T_2) \rightarrow x$"; from left to right, it defines the extension of the compound, i.e., "$(T_1 \ominus T_2) \rightarrow (T_1 \ominus T_2)$" implies "$T_1 \rightarrow (T_1 \ominus T_2)$" and "$\neg(T_2 \rightarrow (T_1 \ominus T_2))$."

[4]Though in Narsese there is no negated term, there are negated statements, which will be defined in the next chapter.

Theorem 22

$$(T_1 \ominus T_2)^I = T_1^I - T_2^I, \ (T_1 \ominus T_2)^E = T_1^E$$

Intuitively, intensional difference is used to relax the requirement of a category by removing some positive properties. As a result, the original instances remains.

The relation between the difference compounds to their first component is captured by the following theorem:

Theorem 23

$$\begin{array}{l} (T_1 - T_2) \rightarrow T_1 \\ T_1 \rightarrow (T_1 \ominus T_2) \end{array}$$

The relation between the difference compounds to their second component is captured by the following theorem:

Theorem 24

$$\begin{array}{l} M \rightarrow (T_1 - T_2) \supset \neg(M \rightarrow T_2) \\ (T_1 \ominus T_2) \rightarrow M \supset \neg(T_2 \rightarrow M) \end{array}$$

Please notice the difference between the above two theorems: while in the former the result can be used to address both the extension and the intension of the compound, in the latter it is only about one of the two aspects.

Unlike the intersection operators, the difference operators cannot take more than two arguments, though we can still use both prefix and infix formats to represent them, so as to be consistent with other compound terms. Also, neither $(T - T)$ nor $(T \ominus T)$ is a valid term.

According to the literal meaning of the compound terms, there are the following implications involving extensional/intensional differences:

Theorem 25

$$\begin{array}{rcrcr} T_1 \rightarrow M & \wedge & \neg((T_1 \ominus T_2) \rightarrow M) & \supset & T_2 \rightarrow M \\ \neg(T_1 \rightarrow M) & \wedge & \neg((T_2 \ominus T_1) \rightarrow M) & \supset & \neg(T_2 \rightarrow M) \\ M \rightarrow T_1 & \wedge & \neg(M \rightarrow (T_1 - T_2)) & \supset & M \rightarrow T_2 \\ \neg(M \rightarrow T_1) & \wedge & \neg(M \rightarrow (T_2 - T_1)) & \supset & \neg(M \rightarrow T_2) \end{array}$$

The two difference operators keep inheritance and similarity relations between terms, though may reverse the direction of the relation:

Theorem 26

$$\begin{array}{lcl}
S \rightarrow P & \supset & (S - M) \rightarrow (P - M) \\
S \rightarrow P & \supset & (M - P) \rightarrow (M - S) \\
S \rightarrow P & \supset & (S \ominus M) \rightarrow (P \ominus M) \\
S \rightarrow P & \supset & (M \ominus P) \rightarrow (M \ominus S) \\
S \leftrightarrow P & \supset & (S - M) \leftrightarrow (P - M) \\
S \leftrightarrow P & \supset & (M - P) \leftrightarrow (M - S) \\
S \leftrightarrow P & \supset & (S \ominus M) \leftrightarrow (P \ominus M) \\
S \leftrightarrow P & \supset & (M \ominus P) \leftrightarrow (M \ominus S)
\end{array}$$

4.3.3 Compound sets

To apply the set-theoretic operators defined above to the sets defined in NAL-2, we get the following definitions:

Definition 30 *If* $t_1, \cdots, t_n$ *(*$n \geq 2$*) are different terms, a* compound extensional set $\{t_1, \cdots, t_n\}$ *is defined as* $(\cup \ \{t_1\} \ \cdots \ \{t_n\})$*; a* compound intensional set $[t_1, \cdots, t_n]$ *is defined as* $(\cap \ [t_1] \ \cdots \ [t_n])$*.*

In this way, extensional sets and intensional sets can both have multiple components. Intuitively, the former defines a term by enumerating its instances, and the latter by enumerating its properties. Again, the order of the components does not matter.

Now we can see that a set is a kind of compound term, whose internal structure fully specifies its extension (for an extensional set) or intension (for an intensional set), in the sense of identical terms. For example, for "$\{x\} \rightarrow \{t_1, t_2, t_3\}$" to be true, x must be identical to t_1, t_2, or t_3.

These compound sets satisfy the following identity relations (where t_1, t_2, t_3 are different terms).

Theorem 27

$$\begin{array}{rcl} (\{t_1, t_2\} \cup \{t_2, t_3\}) & \leftrightarrow & \{t_1, t_2, t_3\} \\ (\{t_1, t_2\} \cap \{t_2, t_3\}) & \leftrightarrow & \{t_2\} \\ (\{t_1, t_2\} - \{t_2, t_3\}) & \leftrightarrow & \{t_1\} \\ ([t_1, t_2] \cap [t_2, t_3]) & \leftrightarrow & [t_1, t_2, t_3] \\ ([t_1, t_2] \cup [t_2, t_3]) & \leftrightarrow & [t_2] \\ ([t_1, t_2] \ominus [t_2, t_3]) & \leftrightarrow & [t_1] \end{array}$$

These relations can be extended to sets with more (or less) than two components. These results are similar to the ones in set theory, though in NAL not all terms can be treated as sets.

4.3.4 NAL-3 summary

The additional grammar rules of Narsese-3 are listed in Table 4.4.

$<term>$	::=	$\{<term>^+\} \mid [<term>^+]$
		$\mid (\cap <term><term>^+) \mid (\cup <term><term>^+)$
		$\mid (- <term><term>) \mid (\ominus <term><term>)$

Table 4.4: The New Grammar Rules of Narsese-3

The previous grammar rule for extensional set and intensional set becomes a special case of the new rule. For sets with multiple components, "," can be used to separate them. For an intersection or difference term, an "infix" format can also be used, as mentioned in the previous section.

Related to the new compound terms, the most important inference rules introduced in NAL-3 are those that combine two inheritance statements into a new one with a compound term as subject or predicate. These rules are listed in Table 4.5, which are applied only when T_1 and T_2 are different, and do not have each other as component.

$J_2 \setminus J_1$	$M \rightarrow T_1$	$T_1 \rightarrow M$
$T_2 \rightarrow M$		$(T_1 \cup T_2) \rightarrow M \ \ {<}F_{int}{>}$ $(T_1 \cap T_2) \rightarrow M \ \ {<}F_{uni}{>}$ $(T_1 \ominus T_2) \rightarrow M \ \ {<}F_{dif}{>}$
$M \rightarrow T_2$	$M \rightarrow (T_1 \cap T_2) \ \ {<}F_{int}{>}$ $M \rightarrow (T_1 \cup T_2) \ \ {<}F_{uni}{>}$ $M \rightarrow (T_1 - T_2) \ \ {<}F_{dif}{>}$	

Table 4.5: The Composition Rules of NAL-3

The truth-value functions in Table 4.5 are defined as the following:

$$
\begin{aligned}
F_{int}: \quad & f = and(f_1, f_2) \\
& c = or(and(not(f_1), c_1), and(not(f_2), c_2)) + and(f_1, f_2, c_1, c_2) \\
F_{uni}: \quad & f = or(f_1, f_2) \\
& c = or(and(f_1, c_1), and(f_2, c_2)) + and(not(f_1), not(f_2), c_1, c_2) \\
F_{dif}: \quad & f = and(f_1, not(f_2)) \\
& c = or(and(not(f_1), c_1), and(f_2, c_2)) + and(f_1, not(f_2), c_1, c_2)
\end{aligned}
$$

To understand the above functions, let us look at the extensional cases, where the two premises have a common subject.

The frequency functions directly come from the definitions of the compounds: "$M \rightarrow (T_1 \cap T_2)$" if "$M \rightarrow T_1$" *and* "$M \rightarrow T_2$"; "$M \rightarrow (T_1 \cup T_2)$" if "$M \rightarrow T_1$" *or* "$M \rightarrow T_2$"; "$M \rightarrow (T_1 - T_2)$" if "$M \rightarrow T_1$" *but not* "$M \rightarrow T_2$".

The confidence functions are determined by listing the various cases where the conclusion gets full evidence (in idealized situations). For example, for "$M \rightarrow (T_1 \cap T_2)$," it happens when one premise is absolutely false, or when both premises have full evidence. The other confidence functions can be obtained similarly.

The intensional cases are completely symmetric to the extensional ones, so the same set of truth functions is used, though for the (extensional and intensional) intersections, the functions are switched.

As an example, given the following judgments,

$$\begin{array}{l}(1)\ \{Tweety\} \rightarrow bird <1, 0.9> \\ (2)\ \{Tweety\} \rightarrow [yellow] <1, 0.9> \\ (3)\ \{Tweety\} \rightarrow canary <1, 0.9>\end{array}$$

from (1) and (2) the system composes a compound term "$([yellow] \cap bird)$" ("yellow bird"), with "$\{Tweety\}$" in its extension:

$$(4)\ \{Tweety\} \rightarrow ([yellow] \cap bird) <1, 0.81>$$

Then, from (3) and (4) by induction the following judgment is derived:

$$(5)\ canary \rightarrow ([yellow] \cap bird) <1, 0.41>$$

In the last step, the compound terms involved are treated just like atomic terms.

4.4 NAL-4: products, images, and ordinary relations

One common criticism to Aristotle's Syllogism, or to term logic in general, is that it cannot represent and process a relation that is not a copula [Bocheński, 1970]. In NAL-4, these relations are handled with the help of certain types of compound terms, using an idea borrowed from set theory.

4.4.1 Ordinary relations and products

In NAL-4, "ordinary relations" indicates the relations among terms that are not the inheritance relation or its variants (such as similarity, instance, property, and instance-property). These relations may be not reflexive, not transitive, not merely reflexive and transitive, or not defined on all terms. The copulas are defined in the meta-language of NAL, with fixed (built-in) meaning to the system. In contrary, the ordinary relations are described in Narsese, with experience-grounded meaning.

Definition 31 *For two terms T_1 and T_2, their* product $(T_1 \times T_2)$ *is a compound term defined by*

$$((S_1 \times S_2) \rightarrow (P_1 \times P_2)) \equiv ((S_1 \rightarrow P_1) \wedge (S_2 \rightarrow P_2)).$$

This definition can be extended as before to allow more than two components in a product. Also, the "prefix" format can be used for products.

Unlike the term operators introduced in NAL-3, the product operator allows the components to be the same. That is, $(T \times T)$ is a valid compound term. $(T_1 \times T_2)$ and $(T_2 \times T_1)$ are usually different, and so are $(T_1 \times (T_2 \times T_3))$ and $((T_1 \times T_2) \times T_3)$.

Theorem 28

$$(S \rightarrow P) \equiv ((M \times S) \rightarrow (M \times P)) \equiv ((S \times M) \rightarrow (P \times M))$$
$$(S \leftrightarrow P) \equiv ((M \times S) \leftrightarrow (M \times P)) \equiv ((S \times M) \leftrightarrow (P \times M))$$

Theorem 29

$$((S_1 \times S_2) \leftrightarrow (P_1 \times P_2)) \equiv ((S_1 \leftrightarrow P_1) \wedge (S_2 \leftrightarrow P_2))$$

That is, two products are identical if and only if and only if their corresponding components are identical.

Theorem 30

$$\{(x \times y) \,|\, x \in T_1^E, y \in T_2^E\} \subseteq (T_1 \times T_2)^E$$
$$\{(x \times y) \,|\, x \in T_1^I, y \in T_2^I\} \subseteq (T_1 \times T_2)^I$$

The "$\subseteq$" cannot be replaced by "$=$" in the above theorem, because $(T_1 \times T_2)^E$ and $(T_1 \times T_2)^I$ may contain other terms that are not products.

Definition 32 *A* relation *is a term R such that there is a product P satisfying "$P \rightarrow R$" or "$R \rightarrow P$".*

Therefore in NAL a relation is not a set, because it is not only defined extensionally. A product is a relation (because "$(T_1 \times T_2) \rightarrow (T_1 \times T_2)$"), but a relation is not necessarily a product. In NAL, a relation can be an atomic term.

For example, "Acid and base neutralize each other" can be represented as "$(acid \times base) \rightarrow neutralization$," and "Neutralization happens between acid and base" can be represented as "$neutralization \rightarrow (acid \times base)$."

4.4.2 Images

Given one component, the "image" operator identifies the other one in the extension or intension of a given relation with two components.

Definition 33 *For a relation R and a product $(\times\ T_1\ T_2)$, the* extensional image *operator, "$\perp$," and* intensional image *operator, "$\top$," of the relation on the product are defined as the following, respectively:*

$$((\times\ T_1\ T_2) \rightarrow R) \equiv (T_1 \rightarrow (\perp\ R\ \diamond\ T_2))) \equiv (T_2 \rightarrow (\perp\ R\ T_1\ \diamond)))$$

$$(R \rightarrow (\times\ T_1\ T_2)) \equiv ((\top\ R\ \diamond\ T_2)) \rightarrow T_1) \equiv ((\top\ R\ T_1\ \diamond)) \rightarrow T_2)$$

where "$\diamond$" is a special symbol indicating the location of T_1 or T_2 in the product, and it can appear in any place, except the first (which is the relation), in the component list. When it appears at the second place, the image can also be written in infix format as $(R \perp T_2)$ or $(R \top T_2)$ (in other cases, only the prefix format is used).

For example, "Acid corrodes metal" can be equivalently represented as "$(\times\ acid\ metal) \rightarrow corrosion$," "$acid \rightarrow (\perp\ corrosion\ \diamond\ metal)$," and "$metal \rightarrow (\perp\ corrosion\ acid\ \diamond)$."

The above definition can be extended to include products with more than two components, where the image can only be written in the prefix format.

In general, $(R \perp T_2)$ and $(R \top T_2)$ are different, but there are situations where they are the same.

Theorem 31

$$T_1 \leftrightarrow ((T_1 \times T_2) \perp T_2)$$

$$T_1 \leftrightarrow ((T_1 \times T_2) \top T_2)$$

Intuitively, the above theorem shows that starting from a term, a "product" followed by an "image" will go back to the same term. However, if the order of the two operators is switched, the result is different:

Theorem 32

$$((R \perp T) \times T) \rightarrow R$$

$$R \rightarrow ((R \top T) \times T)$$

The "$\rightarrow$" in the above theorem cannot be replaced by the "$\leftrightarrow$."

An image operator can be applied to both sides of an inheritance relation, though in the result, the subject and predicate may be switched:

Theorem 33

$$\begin{array}{lll} S \rightarrow P & \supset & (S \perp M) \rightarrow (P \perp M) \\ S \rightarrow P & \supset & (S \top M) \rightarrow (P \top M) \\ S \rightarrow P & \supset & (M \perp P) \rightarrow (M \perp S) \\ S \rightarrow P & \supset & (M \top P) \rightarrow (M \top S) \end{array}$$

4.4.3 NAL-4 summary

In summary, NAL-4 introduces the new grammar rules in Table 4.6.

$$\boxed{\begin{array}{lll} <term> & ::= & (\times <term><term>^{+}) \\ & & \mid (\perp <term><term>^{*} \diamond <term>^{*}) \\ & & \mid (\top <term><term>^{*} \diamond <term>^{*}) \end{array}}$$

Table 4.6: The New Grammar Rules of Narsese-4

There is no new inference rule directly defined in NAL-4, except the equivalence rules in Table 4.7, given by the previous definitions. The table does not include all variants of a rule obtained by changing the component list of the compound.

The following example shows the capability of NAL-4. Let's start with three judgments:

(1) $vinegar \rightarrow acid <1, 0.9>$
(2) $baking\text{-}soda \rightarrow base <1, 0.9>$
(3) $(\times\ vinegar\ baking\text{-}soda) \rightarrow neutralization <1, 0.9>$

$S \rightarrow P$	$(S \times M) \rightarrow (P \times M)$
$S \leftrightarrow P$	$(S \times M) \leftrightarrow (P \times M)$
$(\times\ T_1\ T_2) \rightarrow R$	$T_1 \rightarrow (\perp\ R\ \diamond\ T_2)$
$R \rightarrow (\times\ T_1\ T_2)$	$(\top\ R\ \diamond\ T_2) \rightarrow T_1)$

Table 4.7: The Equivalence Rules of NAL-4

Then the following judgment can be derived as an equivalent form of (3):

$$(4)\ \textit{vinegar} \rightarrow (\perp\ \textit{neutralization}\ \diamond\ \textit{baking-soda}) < 1, 0.9 >$$

From (1) and (4), by induction the system gets

$$(5)\ \textit{acid} \rightarrow (\perp\ \textit{neutralization}\ \diamond\ \textit{baking-soda}) < 1, 0.45 >$$

which can be equivalently transformed into

$$(6)\ \textit{baking-soda} \rightarrow (\perp\ \textit{neutralization}\ \textit{acid}\ \diamond) < 1, 0.45 >$$

From (2) and (6), by induction again:

$$(7)\ \textit{base} \rightarrow (\perp\ \textit{neutralization}\ \textit{acid}\ \diamond) < 1, 0.29 >$$

which can be rewritten as

$$(8)\ (\times\ \textit{acid}\ \textit{base}) \rightarrow \textit{neutralization} < 1, 0.29 >$$

This example shows that though the inference rules of NAL are defined on the inheritance relation and its variants, other relations (like "neutralization") can also be represented and processed properly.