TOPOLOGICAL-SHAPE SENSITIVITY METHOD: THEORY AND APPLICATIONS

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Abstract: The topological derivative allow us to quantify the sensitivity of a given cost function when the domain of definition of the problem is perturbed by introducing a hole or an inclusion. This concept has been successfully applied in the context of topology design and inverse problems. In order to find close expressions for the topological derivative several methods can be achieved in the literature. In particular, we have proposed the Topological-Shape Sensitivity Method, whose main feature is that all mathematical framework (and results), already developed for shape sensitivity analysis, can be used in the calculation of the topological derivative. In this paper we present the Topological-Shape Sensitivity Method and use it as a systematic methodology for computing the topological derivative for holes and inclusions in problems governed by Poisson's and Navier's equations.

Keywords: Topological derivative, shape sensitivity analysis, asymptotic expansion.

1. INTRODUCTION

The topological derivative [3, 5, 15] has been recognized as a promising tool to solve topology optimization problems (see [4] and references therein). In addition, extension of the topological sensitivity in order to include arbitrary shaped holes and its applications to Laplace, Poisson, Helmoltz, Navier, Stokes and Navier–Stokes equations were developed by Masmoudi and his co-workers and by Sokolowsky and his co-workers (see, for instance, [11]).

Although the topological sensitivity is extremely general, this concept may become restrictive due to mathematical difficulties involved in its calculation. However, several approaches to compute the topological derivative may be found in the literature [3, 14, 15]. In particular, we have proposed an alternative approach, called Topological-Shape Sensitivity Method [12], which is

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based on classical shape sensitivity analysis. On the other hand, the topological derivative (TD) concept is wider. In fact, this same idea can also be used to calculate the sensitivity of the problem when, instead of a hole, a small inclusion is introduced at a point in the domain. In this last case, no topology change occurs, then we have called it as configurational derivative (CD). Despite the conceptual difference between TD and CD, we will show that this last one can also be computed using the Topological-Shape Sensitivity Method.

In this paper, we firstly present a brief description of the Topological-Shape Sensitivity Method. Next we apply this approach to obtain the TD for Poisson's (considering both homogeneous and non-homogeneous Neumann and Dirichlet and also Robin boundary conditions on the hole) and Navier's (plane-stress, plane-strain and three-dimensional linear elasticity problems) equations . Furthermore, we compute the CD for steady-state heat conduction and plane-stress linear elasticity. Finally, it is also shown that in general the CD cannot be used to obtain the TD for homogeneous Neumann boundary condition on the hole simply taking the limit when the material property associated to the inclusion vanishes.

2. TOPOLOGICAL-SHAPE SENSITIVITY METHOD

Let us consider an open bounded domain $\Omega \subset \mathbb{R}^N$ $(N = 2, 3)$ with a smooth boundary $\partial\Omega$. If the domain Ω is perturbed by introducing a small hole B_ε of radius ε at an arbitrary point $\hat{\mathbf{x}} \in \Omega$, we have a new domain $\Omega_\varepsilon = \Omega - \overline{B}_\varepsilon$, whose boundary is denoted by $\partial\Omega_\varepsilon = \partial\Omega \cup \partial B_\varepsilon$. Then, considering a cost function ψ defined in both domains, its topological derivative for holes is given in [3], for $f(\varepsilon) > 0$, such that $f(\varepsilon) \to 0$ with $\varepsilon \to 0^+$, as

$$D_T\left(\hat{\mathbf{x}}\right) = \lim_{\varepsilon \to 0} \frac{\psi\left(\Omega_\varepsilon\right) - \psi\left(\Omega\right)}{f\left(\varepsilon\right)} . \tag{1}$$

We have proposed in [12] an alternative procedure to compute the topological derivative called Topological-Shape Sensitivity Method. This approach makes use of the whole mathematical framework (and results) developed for shape sensitivity analysis (see, for instance, the pioneer work of Murat and Simon [10]). The main result obtained in [12] is given by the following theorem:

THEOREM 1 *Let $f\left(\varepsilon\right)$ be a function chosen in order to $0 < \left|D_T\left(\hat{\mathbf{x}}\right)\right| < \infty$, then the topological derivative given by Equation (1) can be written as*

$$D_T\left(\hat{\mathbf{x}}\right) = \lim_{\varepsilon \to 0} \frac{1}{f'\left(\varepsilon\right)} \frac{d}{d\varepsilon} \psi\left(\Omega_\varepsilon\right) , \tag{2}$$

where the derivative of the cost function with respect to the parameter ε may be seen as its classical shape sensitivity analysis.

In general the cost function $\psi(\Omega) := \mathcal{J}_\Omega(u)$ may depends explicitly and implicitly on the domain Ω. This last dependence comes from the solution of a variational problem associated to Ω: find $u \in \mathcal{U}(\Omega)$, such that

$$a\,(u,\eta) = l\,(\eta) \quad \forall \eta \in \mathcal{V}(\Omega)\ , \tag{3}$$

where $\mathcal{U}(\Omega)$ and $\mathcal{V}(\Omega)$ respectively are the sets of admissible functions and admissible variations defined on Ω and $a\,(\cdot,\cdot) : \mathcal{U} \times \mathcal{V} \to \mathbb{R}$ is a bilinear form and $l\,(\cdot) : \mathcal{V} \to \mathbb{R}$ is a linear functional, which will be characterized later according to the problem under analysis. Likewise, the state equation written in the original configuration Ω (without hole) may also be satisfied in the perturbed configuration Ω_ε (with the introduction of a hole at point $\hat{\mathbf{x}} \in \Omega$). Therefore, we have the following variational problem associated to function Ω_ε: find $u_\varepsilon \in \mathcal{U}_\varepsilon(\Omega_\varepsilon)$, such that

$$a_\varepsilon\,(u_\varepsilon,\eta) = l_\varepsilon\,(\eta) \quad \forall \eta \in \mathcal{V}_\varepsilon(\Omega_\varepsilon)\ , \tag{4}$$

where $a_\varepsilon\,(\cdot,\cdot) : \mathcal{U}_\varepsilon \times \mathcal{V}_\varepsilon \to \mathbb{R}$, $l_\varepsilon\,(\cdot) : \mathcal{V}_\varepsilon \to \mathbb{R}$ and $\mathcal{U}_\varepsilon(\Omega_\varepsilon)$ and $\mathcal{V}_\varepsilon(\Omega_\varepsilon)$ respectively are the sets of admissible functions and admissible variations defined on Ω_ε, which will be also defined later according to the problem under analysis and the boundary condition on the hole.

Formally, the shape derivative of the cost function $\psi(\Omega_\varepsilon) := \mathcal{J}_{\Omega_\varepsilon}(u_\varepsilon)$ in relation to the parameter ε reads

$$\begin{cases} \textbf{Calculate :} & \dfrac{d}{d\varepsilon}\mathcal{J}_{\Omega_\varepsilon}(u_\varepsilon) \\ \textbf{Subject to :} & a_\varepsilon\,(u_\varepsilon,\eta) = l_\varepsilon(\eta) \quad \forall\, \eta \in \mathcal{V}_\varepsilon(\Omega_\varepsilon) \end{cases} . \tag{5}$$

Let us relax the constraint of the above problem, given by the state equation (Equation 5), by Lagrangian multipliers. Therefore, the Lagrangian is written as

$$\mathcal{L}_\varepsilon\,(v,\mu) = \mathcal{J}_{\Omega_\varepsilon}\,(v) + a_\varepsilon\,(v,\mu) - l_\varepsilon\,(\mu) \quad \forall \mu \in \mathcal{V}_\varepsilon(\Omega_\varepsilon) \text{ and } v \in \mathcal{U}_\varepsilon(\Omega_\varepsilon)\ . \tag{6}$$

Then, we have the following well-known result

$$\frac{d}{d\varepsilon}\mathcal{J}_{\Omega_\varepsilon}(u_\varepsilon) = \left.\frac{\partial}{\partial\varepsilon}\mathcal{L}_\varepsilon(v,\mu)\right|_{\substack{v=u_\varepsilon \\ \mu=\lambda_\varepsilon}}\ , \tag{7}$$

where u_ε is the solution of the state equation (Equation 4) and λ_ε is the solution of the *adjoint equation* given by: find $\lambda_\varepsilon \in \mathcal{V}_\varepsilon(\Omega_\varepsilon)$, such that

$$a_\varepsilon\,(\lambda_\varepsilon,\eta) = -\left\langle \frac{\partial}{\partial u_\varepsilon}\mathcal{J}_{\Omega_\varepsilon}(u_\varepsilon), \eta \right\rangle \qquad \forall \eta \in \mathcal{V}_\varepsilon(\Omega_\varepsilon)\ . \tag{8}$$

It should be observed that only the part of the boundary $\partial\Omega_\varepsilon$ associated to ∂B_ε is submitted to a perturbation (a uniform expansion of the ball B_ε in this case). Therefore, the shape derivative of the cost function results in an integral on the boundary ∂B_ε. In addition, considering the result of Theorem 1 (Equation 2), the topological derivative becomes

$$D_T\left(\hat{\mathbf{x}}\right) = -\lim_{\varepsilon \to 0} \frac{1}{f'\left(\varepsilon\right)} \int_{\partial B_\varepsilon} \mathbf{\Sigma}_\varepsilon \mathbf{n} \cdot \mathbf{n} \,, \tag{9}$$

where tensor $\mathbf{\Sigma}_\varepsilon$, that depends on u_ε and λ_ε, can be interpreted as a generalization of the Eshelby energy-momentum tensor [6]. As a consequence, this tensor plays a central role in the Topological-Shape Sensitivity Method and should be clearly identified according to the problem under consideration.

Finally, we need to calculate the limit $\varepsilon \to 0$ in Equation (9). Thus, we should know the behavior of the solutions u_ε and λ_ε when $\varepsilon \to 0$, which may be obtained from an asymptotic analysis around the neighborhood of the hole. For that, we can define a new function w_ε such as $u_\varepsilon = u + w_\varepsilon$ and, after making $\mathbf{y} = \mathbf{x}/\varepsilon$, we need to solve an exterior boundary value problem (define in $\mathbb{R}^N - \overline{B}_1$, where B_1 is a unit ball) associated to w_ε. At least for linear cases, this problem may be solved using separation of variables. From this result, we can choose a function $f(\varepsilon)$ in order to take the limit $\varepsilon \to 0$, obtaining the final expression of the topological derivative. Therefore, the Topological-Shape Sensitivity Method may be summarized in the following steps:

1. choose the cost function $\psi(\Omega) := \mathcal{J}_\Omega(u)$, where u is the solution of the state equation associated to the original domain Ω;
2. define the associated cost function $\psi(\Omega_\varepsilon) := \mathcal{J}_{\Omega_\varepsilon}(u_\varepsilon)$, where u_ε is the solution of the state equation defined in the perturbed domain Ω_ε;
3. compute the shape derivative of the cost function $\mathcal{J}_{\Omega_\varepsilon}(u_\varepsilon)$ using the Lagrangian Method, identifying tensor $\mathbf{\Sigma}_\varepsilon$ and writing the sensitivity expression as a boundary integral only defined on ∂B_ε;
4. use the result of Theorem 1;
5. make an asymptotic analysis around the neighborhood of the hole B_ε in order to know the behavior of the solutions u_ε and λ_ε when $\varepsilon \to 0$;
6. finally, choose function $f(\varepsilon)$ and compute the final expression of the topological derivative taking the limit $\varepsilon \to 0$.

Now, let us apply the above method to compute the topological (holes) and configurational (inclusions) sensitivities for some classical problems.

REMARK 1 *For the sake of simplicity, we will choose a cost function that depends only implicitly on the domain of definition of the problem through the solution of the state equation. Therefore* $\psi\left(\Omega\right) := \mathcal{J}\left(u\right)$ *and* $\psi\left(\Omega_\varepsilon\right) := \mathcal{J}\left(u_\varepsilon\right)$*, where* u *is the solution of Equation (3), associated to* Ω*, and* u_ε *is the solution of Equation (4), associated to* Ω_ε*.*

3. TOPOLOGICAL DERIVATIVE (HOLES)

In this section we will compute the topological derivative for steady-state heat conduction (considering both homogeneous and non-homogeneous Neumann and Dirichlet and also Robin boundary conditions on the hole) and linear elasticity (plane-stress, plane-strain and three-dimensional problems).

3.1 Steady-State Heat Conduction

Let us state the following variational problem associated to the original domain Ω: given a constant excitation b in Ω and a Dirichlet data $\bar{u}$ on $\partial\Omega$, find the temperature field $u \in \mathcal{U}(\Omega)$, such that

$$\int_{\Omega} k\nabla u \cdot \nabla\eta = \int_{\Omega} b\eta \quad \forall\eta \in \mathcal{V}(\Omega) \ , \tag{10}$$

where k is a material property and $\mathcal{U}(\Omega)$ and $\mathcal{V}(\Omega)$ are given, respectively, by

$$\mathcal{U} = \left\{u \in H^1(\Omega) : u|_{\partial\Omega} = \bar{u}\right\}, \ \mathcal{V} = \{\eta \in H_0^1(\Omega)\} \ . \tag{11}$$

Now, let us state a new variational problem associated to the perturbed domain Ω_ε: considering that on ∂B_ε we have Dirichlet, Neumann or Robin boundary conditions, find the temperature field $u_\varepsilon \in \mathcal{U}_\varepsilon(\Omega_\varepsilon)$, such that

$$\int_{\Omega_\varepsilon} k\nabla u_\varepsilon \cdot \nabla\eta + \gamma\int_{\partial B_\varepsilon} u_\varepsilon\eta = \int_{\Omega_\varepsilon} b\eta + (\beta+\gamma)\int_{\partial B_\varepsilon} h\eta \quad \forall\eta \in \mathcal{V}_\varepsilon(\Omega_\varepsilon) \ , \tag{12}$$

where $\mathcal{U}_\varepsilon(\Omega_\varepsilon)$ and $\mathcal{V}_\varepsilon(\Omega_\varepsilon)$ are given, respectively, by

$$\mathcal{U}_\varepsilon = \left\{u_\varepsilon \in \mathcal{U}(\Omega_\varepsilon) : \alpha(u_\varepsilon|_{\partial B_\varepsilon} - h) = 0\right\} \ , \tag{13}$$

$$\mathcal{V}_\varepsilon = \left\{\eta \in \mathcal{V}(\Omega_\varepsilon) : \alpha\ \eta|_{\partial B_\varepsilon} = 0\right\} \ , \tag{14}$$

and $\alpha, \beta, \gamma \in \{0, 1\}$, with $\alpha + \beta + \gamma = 1$. This notation should be interpreted as follows: when $\alpha = 1$, $u_\varepsilon = h$ and $\eta = 0$ on ∂B_ε, and when $\alpha = 0$, u_ε and η are free on ∂B_ε, where h is a data. Considering Remark 1, the shape derivative of the cost function becomes

$$\frac{d}{d\varepsilon}\mathcal{J}(u_\varepsilon) = -\int_{\partial B_\varepsilon}\left(\boldsymbol{\Sigma}_\varepsilon\mathbf{n}\cdot\mathbf{n} - \frac{1}{\varepsilon}\left(\gamma(u_\varepsilon - h) - \beta h\right)\lambda_\varepsilon\right) \ , \quad \text{where} \tag{15}$$

$$\boldsymbol{\Sigma}_\varepsilon = (k\nabla u_\varepsilon\cdot\nabla\lambda_\varepsilon - b\lambda_\varepsilon)\,\mathbf{I} - k\left(\nabla u_\varepsilon\otimes\nabla\lambda_\varepsilon + \nabla\lambda_\varepsilon\otimes\nabla u_\varepsilon\right) \ . \tag{16}$$

Finally, from an asymptotic analysis of u_ε and λ_ε, we can choose function $f(\varepsilon)$ depending on each type of boundary condition on ∂B_ε, which allow us to compute the limit $\varepsilon \to 0$ in Equation (15). This procedure leads to the results presented in Table 1, where u and λ are the solutions of the state and adjoint equations, respectively, both defined in the original domain Ω (without hole).

See [12] for applications of the results shown in Table 1. We also observe that the exceptional case $h = h^*$ appears in the Saint-Venant theory of torsion of elastic shafts.

Table 1. Topological derivatives for Poisson's problem in 2D domains.

Boundary conditions	$f(\varepsilon)$	$D_T(\hat{\mathbf{x}})$
$\beta = 1, \alpha = \gamma = 0$ and $h = 0$	$\pi\varepsilon^2$	$-2k\nabla u \cdot \nabla\lambda + b\lambda$
$\beta = 1, \alpha = \gamma = 0$ and $h \neq 0$	$2\pi\varepsilon$	$-h\lambda$
$\gamma = 1, \alpha = \beta = 0$	$2\pi\varepsilon$	$(u - h)\lambda$
$\alpha = 1, \beta = \gamma = 0$ and $h = h^*$	$\pi\varepsilon^2$	$2k\nabla u \cdot \nabla\lambda$
$\alpha = 1, \beta = \gamma = 0$ and $h \neq h^*$	$-\frac{2\pi}{\log(\varepsilon)}$	$(u - h)\lambda$

3.2 Linear Elasticity

The mechanical model associated to linear elasticity problem can be stated in its variational formulation as following: find the displacement vector field $\mathbf{u} \in \mathcal{U}(\Omega)$, such that

$$\int_\Omega \mathbf{T}(\mathbf{u}) \cdot \mathbf{E}(\boldsymbol{\eta}) = \int_{\Gamma_N} \bar{\mathbf{q}} \cdot \boldsymbol{\eta} \quad \forall \boldsymbol{\eta} \in \mathcal{V}(\Omega) \, , \tag{17}$$

where $\mathcal{U}(\Omega)$ and $\mathcal{V}(\Omega)$ are given by

$$\mathcal{U} = \{\mathbf{u} \in H^1(\Omega) : \mathbf{u}|_{\Gamma_D} = \bar{\mathbf{u}}\}, \ \mathcal{V} = \{\boldsymbol{\eta} \in H^1(\Omega) : \boldsymbol{\eta}|_{\Gamma_D} = \mathbf{0}\} \tag{18}$$

and Ω represents a deformable body submitted to a set of surface forces $\bar{\mathbf{q}}$ on the Neumann boundary Γ_N and displacement constraints $\bar{\mathbf{u}}$ on the Dirichlet boundary Γ_D. In addition, $\mathbf{E}(\mathbf{u})$ is the linearized Green deformation tensor and $\mathbf{T}(\mathbf{u})$ is the Cauchy stress tensor respectively given by

$$\mathbf{E}(\mathbf{u}) = \frac{1}{2}\left(\nabla\mathbf{u} + \nabla\mathbf{u}^T\right) := \nabla\mathbf{u}^s \qquad \text{and} \qquad \mathbf{T}(\mathbf{u}) = \mathbf{C}\mathbf{E}(\mathbf{u}) \, , \tag{19}$$

where $\mathbf{C} = \mathbf{C}^T$ is the elasticity tensor for linear elastic isotropic material. The problem stated in the original domain Ω can also be written in the domain Ω_ε with a hole B_ε. Therefore, assuming null forces on the hole, we have the following variational problem: find the displacement vector field $\mathbf{u}_\varepsilon \in \mathcal{U}_\varepsilon(\Omega_\varepsilon)$, such that

$$\int_{\Omega_\varepsilon} \mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon) \cdot \mathbf{E}_\varepsilon(\boldsymbol{\eta}) = \int_{\Gamma_N} \bar{\mathbf{q}} \cdot \boldsymbol{\eta} \quad \forall \boldsymbol{\eta} \in \mathcal{V}_\varepsilon(\Omega_\varepsilon) \ . \tag{20}$$

where $\mathcal{U}_\varepsilon(\Omega_\varepsilon) = \mathcal{U}(\Omega_\varepsilon)$ and $\mathcal{V}_\varepsilon(\Omega_\varepsilon) = \mathcal{V}(\Omega_\varepsilon)$. Observe that in accordance with the variational problem given by Equation (20), the natural boundary condition on ∂B_ε is $\mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon)\mathbf{n} = \mathbf{0}$ (homogeneous Neumann condition). Considering Remark 1, the shape derivative of the cost function becomes

$$\frac{d}{d\varepsilon}\mathcal{J}(\mathbf{u}_\varepsilon) = -\int_{\partial B_\varepsilon} \boldsymbol{\Sigma}_\varepsilon \mathbf{n} \cdot \mathbf{n} \, , \quad \text{where} \tag{21}$$

$$\boldsymbol{\Sigma}_\varepsilon = \left(\mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon) \cdot \mathbf{E}_\varepsilon(\boldsymbol{\lambda}_\varepsilon)\right) \mathbf{I} - (\nabla \boldsymbol{\lambda}_\varepsilon)^T \, \mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon) - (\nabla \mathbf{u}_\varepsilon)^T \, \mathbf{T}_\varepsilon(\boldsymbol{\lambda}_\varepsilon) \; . \tag{22}$$

Finally, taking into account homogeneous Neumann boundary condition on the hole and considering a classical stress distribution around the void, we can choose function $f(\varepsilon)$ and take the limit $\varepsilon \to 0$ in Equation (21) to obtain the final expression for the topological derivative. Thus, for $\mathbf{u}$ and $\boldsymbol{\lambda}$ solutions of the direct and adjoint problems, respectively, both associated to the original domain Ω (without hole) and with ν being the Poisson's ratio, we have the following results (see also [8] and [9]):

- plane-stress linear elasticity, $f(\varepsilon) = \pi \varepsilon^2$

$$D_T(\hat{\mathbf{x}}) = -\frac{4}{1+\nu} \mathbf{T}(\mathbf{u}) \cdot \mathbf{E}(\boldsymbol{\lambda}) + \frac{1-3\nu}{1-\nu^2} \mathrm{tr}\mathbf{T}(\mathbf{u}) \, \mathrm{tr}\mathbf{E}(\boldsymbol{\lambda}); \tag{23}$$

- plane-strain linear elasticity, $f(\varepsilon) = \pi \varepsilon^2$

$$D_T(\hat{\mathbf{x}}) = -4(1-\nu) \mathbf{T}(\mathbf{u}) \cdot \mathbf{E}(\boldsymbol{\lambda}) + \frac{(1-4\nu)(1-\nu)}{1-2\nu} \mathrm{tr}\mathbf{T}(\mathbf{u}) \, \mathrm{tr}\mathbf{E}(\boldsymbol{\lambda}); \tag{24}$$

- three-dimensional linear elasticity, $f(\varepsilon) = (4/3)\pi \varepsilon^3$

$$D_T(\hat{\mathbf{x}}) = -\frac{3}{2} \frac{1-\nu}{7-5\nu} \left[10 \mathbf{T}(\mathbf{u}) \cdot \mathbf{E}(\boldsymbol{\lambda}) - \frac{1-5\nu}{1-2\nu} \mathrm{tr}\mathbf{T}(\mathbf{u}) \, \mathrm{tr}\mathbf{E}\,(\boldsymbol{\lambda}) \right]. \tag{25}$$

For applications of these results, see [7] for 2D and [13] for 3D problems.

4. CONFIGURATIONAL DERIVATIVE (INCLUSIONS)

In this section we will compute the configurational derivative in steady-state heat conduction and plane-stress linear elasticity. Therefore, let us consider that the domain Ω is now perturbed by introducing, instead a hole, a small inclusion represented by B_ε. Therefore, we have a perturbed domain $\Omega_\varepsilon \cup B_\varepsilon$, where $\Omega_\varepsilon = \Omega - B_\varepsilon$. Thus, considering a cost function ψ defined in both domains Ω and $\Omega_\varepsilon \cup B_\varepsilon$, its configurational derivative is defined as

$$D_C\left(\hat{\mathbf{x}}\right) = \lim_{\varepsilon \to 0} \frac{\psi\left(\Omega_\varepsilon \cup B_\varepsilon\right) - \psi\left(\Omega\right)}{|B_\varepsilon|} \; , \tag{26}$$

where $|B_\varepsilon|$ is the Lebesgue measure of the inclusion. It is important to observe that all the mathematical framework introduced in section 2 can also be applied in this context. See also [1] and references therein.

4.1 Steady-State Heat Conduction

Let us consider again the Poisson's equation. Therefore, the problem formulation associated to the original domain Ω is given by Equation (10) and the state equation associated to the domain $\Omega_\varepsilon \cup B_\varepsilon$ is given by the following variational problem: find the temperature field $u_\varepsilon \in \mathcal{U}_\varepsilon(\Omega_\varepsilon \cup B_\varepsilon)$, such that

$$\int_{\Omega_\varepsilon \cup B_\varepsilon} k_\delta \nabla u_\varepsilon \cdot \nabla \eta = \int_{\Omega_\varepsilon \cup B_\varepsilon} b\eta \quad \forall \eta \in \mathcal{V}_\varepsilon(\Omega_\varepsilon \cup B_\varepsilon) \ , \tag{27}$$

where, according to definitions of the set $\mathcal{U}(\Omega)$ and the space $\mathcal{V}(\Omega)$ given by Equation (11), we have $\mathcal{U}_\varepsilon(\Omega_\varepsilon \cup B_\varepsilon) = \mathcal{U}(\Omega_\varepsilon \cup B_\varepsilon)$ and $\mathcal{V}_\varepsilon(\Omega_\varepsilon \cup B_\varepsilon) = \mathcal{V}(\Omega_\varepsilon \cup B_\varepsilon)$. In addition, the material property k_δ is defined, for $\delta \in \mathbb{R}^+$, as

$$k_\delta = k \quad \forall \mathbf{x} \in \Omega_\varepsilon \quad \text{and} \quad k_\delta = \delta k \quad \forall \mathbf{x} \in B_\varepsilon \ . \tag{28}$$

Introducing the notation $[\![\cdot]\!] := (\cdot)|_e - (\cdot)|_i$, where $(\cdot)|_e$ is associated to the bulk material e, represented by Ω_ε, and $(\cdot)|_i$ is associated to the inclusion i, represented by B_ε. Then the shape derivative of the cost function, according to Remark 1, results in

$$\frac{d}{d\varepsilon}\mathcal{J}(u_\varepsilon) = -\int_{\partial B_\varepsilon} [\![\boldsymbol{\Sigma}_\varepsilon \mathbf{n}]\!] \cdot \mathbf{n} \ , \tag{29}$$

where tensor $\boldsymbol{\Sigma}_\varepsilon$ is given by Equation (16) for $k = k_\delta$. Using the jump conditions associated to the normal derivatives of solutions u_ε and λ_ε, we can compute the limit $\varepsilon \to 0$ to get the configurational derivative, that is

$$D_C\left(\hat{\mathbf{x}}\right) = -2k\frac{1-\delta}{1+\delta}\nabla u \cdot \nabla \lambda \ , \tag{30}$$

where u and λ are the solutions of the state and adjoint equations, respectively, both defined in the original domain Ω (without inclusion).

4.2 Plane-Stress Linear Elasticity

In this section we compute the configurational derivative in plane-stress linear elasticity problem, whose variational formulation, associated to the original domain Ω, is given by Equation (17). On the other hand, the mechanical model associated to the domain $\Omega_\varepsilon \cup B_\varepsilon$ is given by the following variational problem: find the displacement vector field $\mathbf{u}_\varepsilon \in \mathcal{U}_\varepsilon(\Omega_\varepsilon \cup B_\varepsilon)$, such that

$$\int_{\Omega_\varepsilon \cup B_\varepsilon} \mathbf{T}_\varepsilon(\mathbf{u}_\varepsilon) \cdot \mathbf{E}_\varepsilon(\boldsymbol{\eta}) = \int_{\Gamma_N} \bar{\mathbf{q}} \cdot \boldsymbol{\eta} \quad \forall \boldsymbol{\eta} \in \mathcal{V}_\varepsilon(\Omega_\varepsilon \cup B_\varepsilon) \ . \tag{31}$$

where, according to definitions of the set $\mathcal{U}(\Omega)$ and the space $\mathcal{V}(\Omega)$ given by Equation (18), we have $\mathcal{U}_\varepsilon(\Omega_\varepsilon \cup B_\varepsilon) = \mathcal{U}(\Omega_\varepsilon \cup B_\varepsilon)$ and $\mathcal{V}_\varepsilon(\Omega_\varepsilon \cup B_\varepsilon) =$

$\mathcal{V}(\Omega_\varepsilon \cup B_\varepsilon)$. In addition, the elasticity tensor is now defined, for $\delta \in \mathbb{R}^+$, as

$$\mathbf{C}_\delta = \mathbf{C} \quad \forall \mathbf{x} \in \Omega_\varepsilon \quad \text{and} \quad \mathbf{C}_\delta = \delta \mathbf{C} \quad \forall \mathbf{x} \in B_\varepsilon \ . \tag{32}$$

From Remark 1, the shape derivative of the cost function results in

$$\frac{d}{d\varepsilon}\mathcal{J}_\varepsilon(\mathbf{u}_\varepsilon) = -\int_{\partial B_\varepsilon} [\![\boldsymbol{\Sigma}_\varepsilon \mathbf{n}]\!] \cdot \mathbf{n} \ , \tag{33}$$

remembering that $[\![\cdot]\!] := (\cdot)|_e - (\cdot)|_i$ and that $\boldsymbol{\Sigma}_\varepsilon$ is the generalized Eshelby tensor, given in by Equation (22) for $\mathbf{C} = \mathbf{C}_\delta$. Finally, taking into account the jump condition on the boundary of the inclusion, we find $f(\varepsilon) = \pi\varepsilon^2$ and the configurational derivative, for $\alpha = (3-\nu)/(1+\nu)$, becomes

$$D_T\left(\hat{\mathbf{x}}\right) = -\frac{1-\delta}{2}\frac{1+\alpha}{1+\delta\alpha}\left[2\mathbf{T}\left(\mathbf{u}\right)\cdot\mathbf{E}(\boldsymbol{\lambda}) - \frac{(1-\delta)(\alpha-2)}{2\delta+\alpha-1}\mathrm{tr}\mathbf{T}\left(\mathbf{u}\right)\,\mathrm{tr}\mathbf{E}(\boldsymbol{\lambda})\right], \tag{34}$$

where $\mathbf{u}$ and $\boldsymbol{\lambda}$ are solutions of the direct and adjoint problems, respectively, both associated to the original domain Ω (without inclusion).

5. FINAL REMARKS

In this paper, we have applied the Topological-Shape Sensitivity Method as a systematic procedure to compute the topological (holes) and configurational (inclusions) sensitivities for some classical problems in continuum mechanics.

We have observed that the CD in general doesn't converge in the limit case (for $\delta = 0$) to the TD for homogeneous Neumann boundary condition on the holes. In order to illustrate this issue, let us consider a cost function that also depends explicitly on the domain Ω as follows

$$\psi(\Omega) := \mathcal{J}_\Omega(u) = \int_\Omega w(u-u^*)^2 \ , \tag{35}$$

where u is the solution of Equation (10), u^* is a target temperature and w is a weighting factor defined, for a given subset $\varpi \subset \Omega$, as $w = 1$ if $\mathbf{x} \in \varpi$ and $w = 0$ if $\mathbf{x} \in \Omega - \overline{\varpi}$. From the Topological-Shape Sensitivity Method, we have respectively obtained the following results for holes and inclusions:

$$D_T\left(\hat{\mathbf{x}}\right) = -w(u-u^*)^2 - 2k\nabla u \cdot \nabla\lambda + b\lambda \ , \tag{36}$$

$$D_C\left(\hat{\mathbf{x}}\right) = -2k\frac{1-\delta}{1+\delta}\nabla u \cdot \nabla\lambda \ . \tag{37}$$

Observe that the result given by Equation (36) cannot be obtained taking the limit $\delta \to 0$ in Equation (37). Therefore, this fact suggests that in general the CD cannot be used to compute the TD for homogeneous Neumann boundary condition on the hole simply taking the limit when the material property associated to the inclusion vanishes.

Finally, we would like to point out that this paper only deals with linear problems. In addition, only linear problems or when the nonlinear term is a compact perturbation of the principal part of the operator have been considered in the current literature [2]. Therefore, we are now interested in the applications of the topological sensitivity for the cases in that the nonlinear term involves the principal part of the operator, like the p-Poisson's equation, plasticity, finite deformations and so on.

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