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ON THE STABILITY REGIONS OF THE TROJAN ASTEROIDS

RUDOLF DVORAK and RICHARD SCHWARZ
University of Vienna, Institute for Astronomy, Türkenschanzstr. 17, A-1180 Wien, Austria, e-mails: {dvorak, schwarz}@astro.univie.ac.at

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Abstract. The orbits of fictitious bodies around Jupiter's stable equilibrium points L_4 and L_5 were integrated for a fine grid of initial conditions up to 100 million years. We checked the validity of three different dynamical models, namely the spatial, restricted three body problem, a model with Sun, Jupiter and Saturn and also the dynamical model with the Outer Solar System (Jupiter to Neptune). We determined the chaoticity of an orbit with the aid of the Lyapunov Characteristic Exponents (= LCE) and used also a method where the maximum eccentricity of an orbit achieved during the dynamical evolution was examined. The goal of this investigation was to determine the size of the regions of motion around the equilibrium points of Jupiter and to find out the dependance on the inclination of the Trojan's orbit. Whereas for small inclinations (up to $i = 20°$) the stable regions are almost equally large, for moderate inclinations the size shrinks quite rapidly and disappears completely for $i > 60^0$. Additionally, we found a difference in the dynamics of orbits around L_4 which – according to the LCE – seem to be more stable than the ones around L_5.

Key words: difference L_4 and L_5, stability regions, trojans

1. Introduction

The quite complicated dynamics of Trojans was discussed by several authors since the discovery of the first Trojan in 1906 (Achilles by Max Wolf in Heidelberg); especially in recent years numerical and analytical work has been accomplished. One of the first estimations of the stability regions around the equilibrium points was made by Rabe (1967) in the framework of the restricted three-body problem. Érdi in many papers (e.g. 1988, 1997) studied the motion of the Trojans also with analytical methods in the model of the spatial elliptic restricted three-body problem and even took into account partly Saturn's perturbations. Using numerical methods Milani (1993, 1994) could show that some of the real Trojans are on chaotic orbits. An extensive study via numerical simulations was undertaken by Levison et al. (1997) who found that the Trojans undergo a slow dispersion in Gigayears time scales. In simple dynamical models also analytical and semianalytical methods can be applied which lead to estimates of the stable regions too small compared to the real ones (e.g. Celletti and Giorgilli, 1991; Beaugé and

Roig, 2001; Skokos and Docoumetzidis, 2001; Efthymiopoulos, 2005). Also possible escapes from the Trojan cloud were discussed by different groups (e.g. Pilat-Lohinger et al., 1999; Dvorak and Tsiganis, 2000; Marzari and Scholl, 2002; Tsiganis et al., 2000) in connection with chaotic orbits. The goal of this investigation is twofold: first we wanted to find out how the largeness of the stable areas around the Lagrangian equilibrium points of Jupiter changes with the inclination of the asteroids (e.g. Schwarz and Gyergyovits, 2003; Brasser et al., 2004). Second we wanted to check why there is a different number of real Trojans around L_4 and L_5[1]. In fact this difference is significant ($N(L_4)/N(L_5) = 5/3$) and it is something which we do not understand up to now. In Figure 1 we show a histogram with respect to the inclinations and also one with respect to the eccentricities, where the difference in the distribution for L_4 and L_5 Trojans is well visible. Although we have some indications, our results cannot confirm that the reason for the actual difference is due to a different dynamical structure.

2. Dynamical Models and Numerical Methods

It is evident that the influence of the terrestrial planets is very small on asteroids around the stable Lagrangian equilibrium points. In numerical simulations the necessary time step – when we include inner planets – would be in the order of 1/10 (including Mars) down to 1/80 (including Mercury) of the time step needed for an integration of the OSS. Therefore any numerical work on the dynamics of Trojans is undertaken by adding the masses of the inner planets to the Sun, which is a way of taking into account their influence on the motion of the other bodies in the Solar system. In our case we tested three models, namely

1. SUN + JUPITER + massless asteroids (= SJA), which is the spatial elliptic restricted three-body problem
2. SUN + JUPITER + SATURN + massless asteroids (= SJS)
3. SUN + outerplanets (JUPITER to NEPTUNE) + massless asteroids (the outer Solar system = OSS).

As integration method for solving the equations of motion we used two different methods:

– On one hand we used the Lie-integrator with recurrence formulae for the Lie-terms which can also be utilized for high eccentric orbits due to the automatic step size (e.g. Hanslmeier and Dvorak, 1984; Lichtenegger,

[1] By September, 10th 2004 the numbers are for Jupiter Trojans L4: 1060, L5: 628.

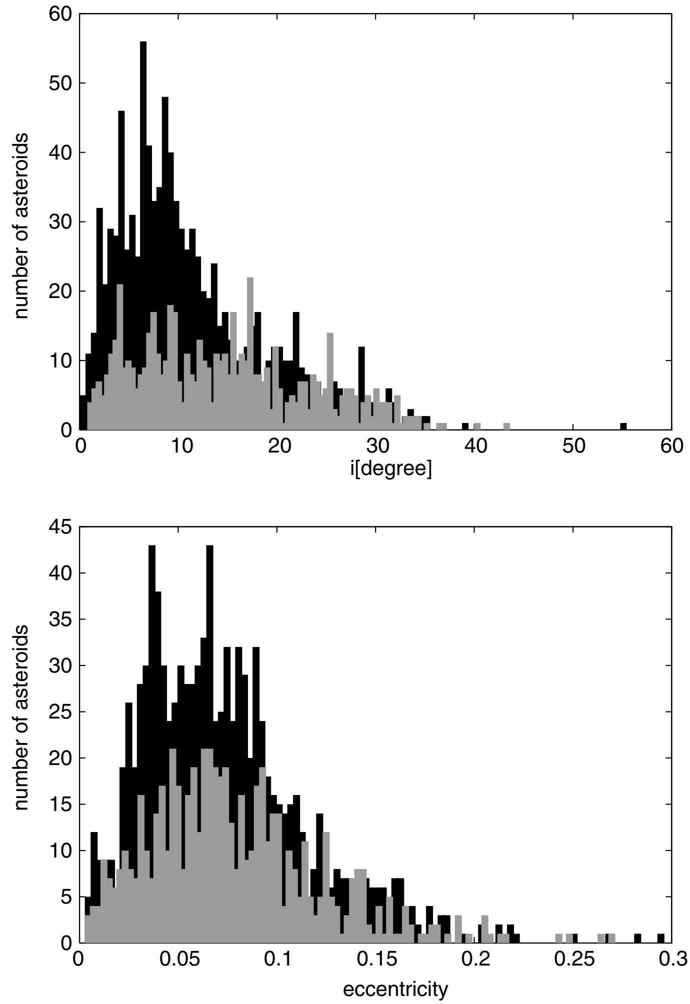

Figure 1. Distribution of all real L_4 and L_5 Trojans with respect to their inclination (top) and their eccentricity (bottom); L_5 Trojans are depicted in light grey.

1984). We already used this method for many numerical simulations and compared it to other methods (e.g. Tsiganis et al., 2000).

– On the other hand we used the program *orbit9*, developed by Milani[2] (1999), a high order Runge Kutta method. This software also computes the Lyapunov Characteristic Exponents (= LCE), respectively, the Lyapunovtime (= LT) which are essential for the determination of the chaoticity of an orbit (e.g. Froeschlé, 1984).

[2]For detailed information see: http://copernico.dm.unipi.it/~ milani/propel5/node3.html

Additionally to the LCEs we computed the maximum eccentricity during the evolution of an orbit to determine its stability and also the region of stable motion around the two equilibrium points. Our stability criterion for a Trojan was, that the eccentricity should not exceed $e = 0.5$; this is a good measure which we tested and compared to other definitions like crossing the line of syzygy.[3]

The integration times were slightly different for the different runs in the three models; we used always at least 10^7 years, but for the LCEs we used always 10^8 years.

We present the results of the numerical experiments as follows:

– the largeness of the stable region around the equilibrium points depending on the initial semimajor axes and the synodic longitudes with respect to the initial inclination in the model SJS (Figure 2);
– the state of chaoticity of the orbits close to L_4 and L_5 depending on initial eccentricity and inclination via the LT in the model OSS (Figures 3 and 4);
– the extension in the synodic longitude of the stable zones for a fixed value of the semimajor axis $a_{\mathrm{Trojan}} = a_{\mathrm{Jupiter}}$ depending on the inclination in the OSS (Figure 5);
– the comparison of L_4 and L_5 Trojans in all three models in an initial condition diagram eccentricity versus LT (Figure 6).

3. Stable Regions Around L_4 and L_5

We determined the extension of the stable regions with the Lie-integrator and the maximum eccentricity of an orbit as stability parameter. In the model of the SJS we integrated the equations of motion from 0° to 360° for a grid of $\Delta\lambda = 1°$ in the synodic longitude and semimajor axis of $4.9AU < a_{\mathrm{Trojan}} < 5.6AU$ with $\Delta a = 0.02$ AU for massless fictitious Trojans. We set the initial inclination to $0° < i_{\mathrm{Trojan}} < 60°$ with $\Delta i = 5°$ and the initial eccentricity to zero; Ω and ω of the Trojans were set to the respective orbital elements of Jupiter. In Fig. 2 we can see that the size of the stable region for small inclinations $0° < i < 15°$ is almost the same and it is also equally large for both Lagrangian points. There is only a small decrease in the size with decreasing inclination of the Trojan's orbit $20° \leqslant i \leqslant 35°$ (not shown here). For high inclinations $40° < i < 55°$ (see also Figure 2) the size shrinks very fast and the stable region disappears completely for $i = 60°$. The size of the stable region (number of stable orbits) was determined with a least square fit: $N(i) = a \cdot i^2 + b \cdot i + c$ with: $a = -0.0046 \pm 0.0007; b = 0.0800 \pm 0.0352;$

[3]Alignment of Sun, Jupiter and the Trojan.

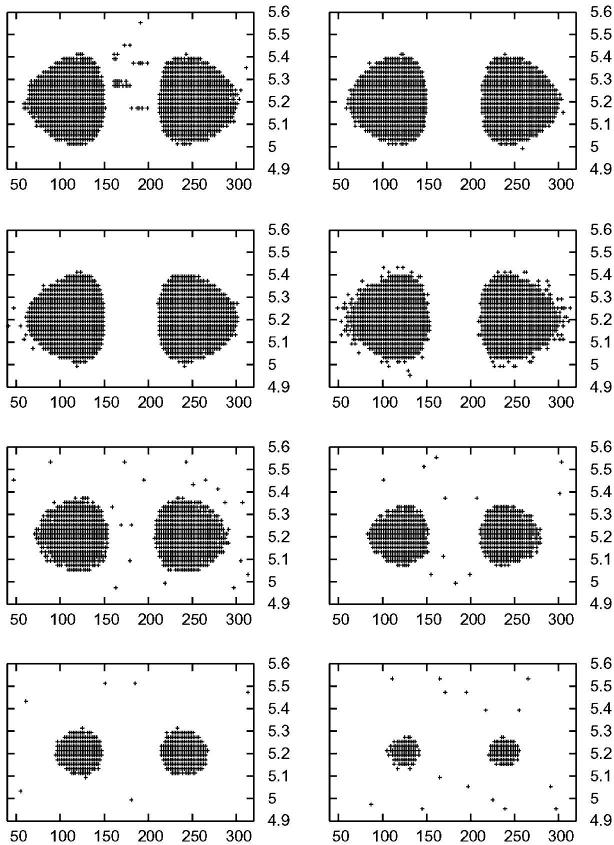

Figure 2. Largeness of the stable regions around L_4 and L_5 in the dynamical model SJS synodic longitude versus initial semimajor axis (in AU). Eight different initial inclinations of the fictitious Trojans are shown: $i = 0°$ and $i = 5°$ (first row), $i = 10°$ and $i = 15°$ (second row), $i = 40°$ and $i = 45°$(third row), $i = 50°$ and $i = 55°$ (fourth row); in the synodic coordinates the position of Jupiter is at $\lambda = 180°$ and a = 5.2 AU. Points indicate stable orbits.

$c = 9.13406 \pm 0.3782$. N corresponds to the percentage of the stable orbits out of the grid in initial conditions specified above.

4. Lyapunov-Times for Fictitious L_4 and L_5 Trojans Depending on i_{ini} and e_{ini}

With the forementioned program *orbit9* we integrated for a grid $\Delta e = 0.01$ between $0 < e < 0.2$ and $\Delta i = 1.°75$ between $0° < i < 34°$ fictitious Trojans around L_4 and L_5 with a semimajor axis $a = a_{\mathrm{Jupiter}}$.[4] In the dynamical model

[4]The angle λ was chosen such that for L_4 Trojans the actual position was 60° ahead of Jupiter's mean longitude and for L_5 Trojans 60° behind.

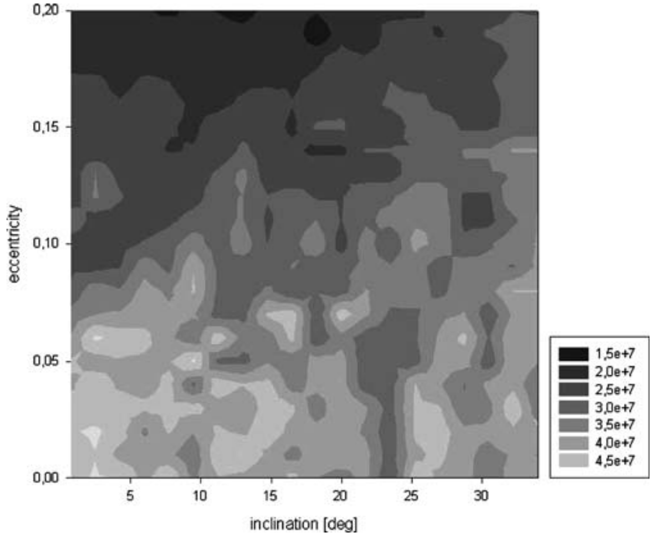

Figure 3. Initial condition diagram i_{ini} versus e_{ini} for fictitious L_4 Trojans showing LTs in the dynamical model OSS; the initial semimajor axes were set to the one of Jupiter.

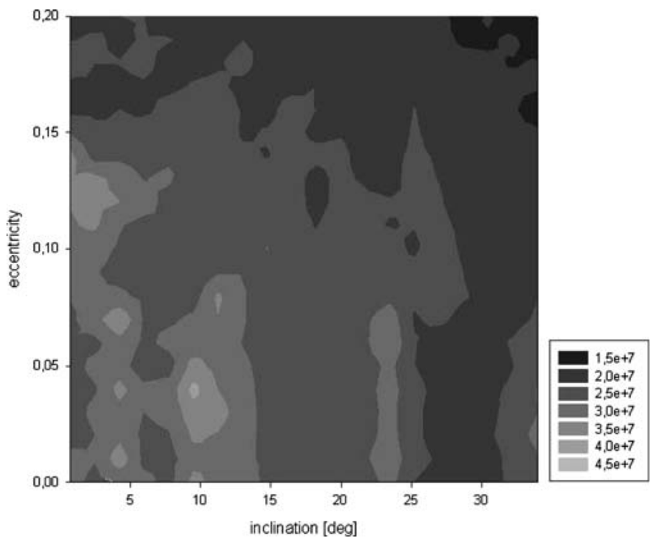

Figure 4. Caption like in Figure 4 but for L_5 Trojans.

OSS we fixed the integration time to 100 Million years, computed the LTs for each orbit and plotted it in the respective inclination versus eccentricity diagram. Globally there is the expected tendency to more chaotic orbits for larger eccentricities AND larger inclinations; this is true for both equilibrium

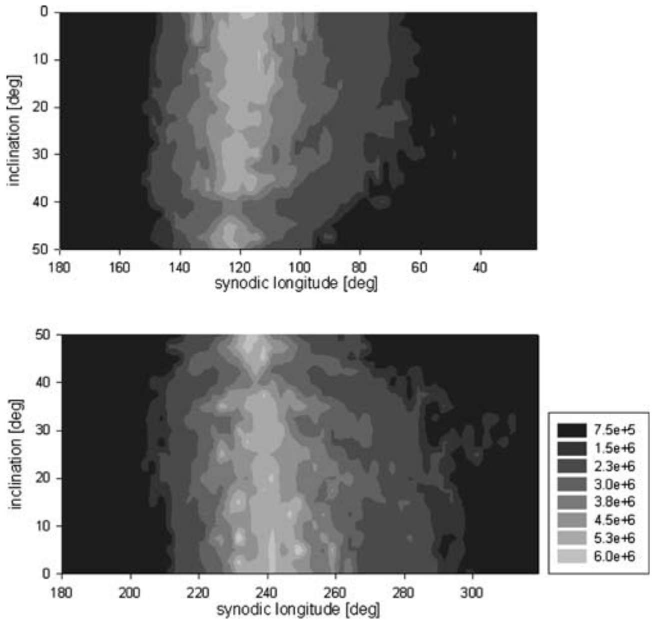

Figure 5. Initial condition diagram i_{ini} versus the synodic longitude λ for fictitious L_4 (lower graph) and L_5 (upper graph) Trojans showing the LTs in the dynamical model OSS. The initial semimajor a_{ini} of the fictitious Trojan with initially a circular orbit was set to the one of Jupiter, which is located at $\lambda = 180°$.

points. Furthermore we can see that in the L_4 diagram (Figure 3) even for large inclinations and small eccentricities the LT is relatively large; this is not the case for the L_5 Trojans (Figure 4). From a comparison of these two figures it seems that there is more chaos around L_5; this is the confirmation of results of a former study (Schwarz et al., 2004). In addition we checked the validity of these results by computations of the orbits of fictitious Trojans in the two simpler models SJA and SJS; this comparison will be discussed in the final chapter.

5. Extension of the Stable Zone Depending on the Synodic Longitude of the Trojan

In a further step of our investigation we checked the extension of the stable region with respect to the initial synodic longitude λ in the dynamical model OSS with *orbit9*. The initial conditions were the same as in the former runs ($a_{\text{ini}} = a_{\text{Jupiter}}$), but we varied λ from $\lambda = 0°$ to $\lambda = 360°$ with a grid of $\Delta\lambda = 1°$ and varied the inclination from $i = 0°$ to $i = 50°$ with $\Delta i = 2.°5$. The position

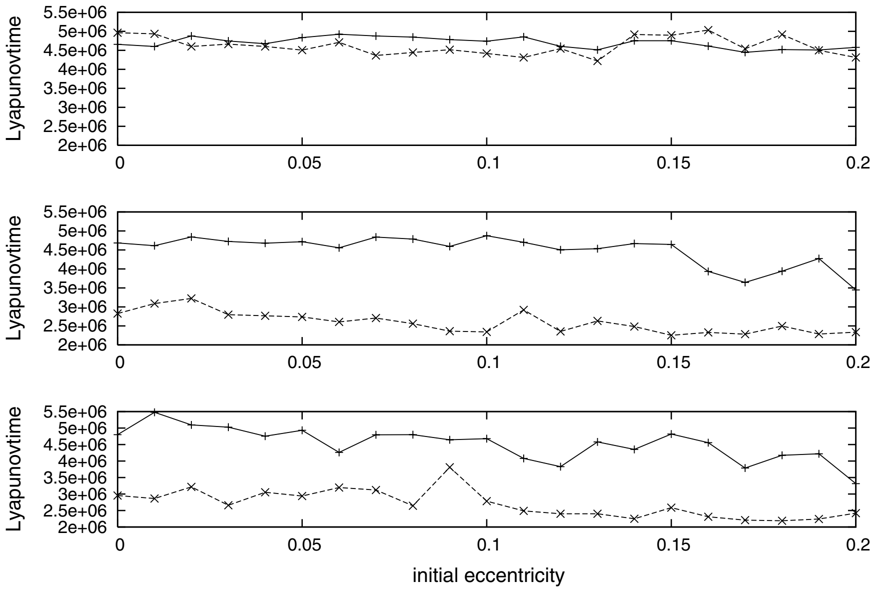

Figure 6. Comparison of L_4 and L_5 Trojans in the three dynamical models SJA (upper graph), SJS (middle graph) and OSS (lower graph) for $i_{\mathrm{ini}} = 30°$. LT is plotted versus the initial eccentricity for the L_4 (solid line) and the L_5 Trojans (dashed line).

of Jupiter (Figure 5) is at $\lambda = 180°$, the two Lagrangian points are at 120° and 240° for L_5 respectively, L_4. Again we can see a slightly different structure with more chaotic orbits in the L_5 region (upper graph).

6. Comparison of the Three Dynamical Models SJA, SJS and OSS

We directly compared the three dynamical models SJA, SJS and OSS[5] for a cut of the former Figures 3 and 4 with three different inclinations ($i = 10°, 20°$ and 30°) and a time interval of 10^7 years, where we varied the eccentricities from e = 0 to e = 0.2 with a grid of $\Delta e = 0.01$. Figure 6 shows the respective results for $i = 30°$; there we have plotted how the LT depends on the initial eccentricity of the fictitious body close to L_4 and to L_5. In the upper graph for model SJA we see that the two lines almost coincide and that the LT is around 5×10^6 for all different eccentricities. In the middle graph for model SJS we can see that the LT for all orbits around L_5 is lower than in the model SJA (dashed line) and differs significantly from L_4. The same effect is visible

[5]Note that taking a more realistic model we introduce additional degrees of freedom and thus new resonances appear (see Robutel et al., 2005).

in the lower graph when we compare the dynamical models OSS and SJS[6]. In the model SJA there is no decreasing LT visible and the lines are almost straight; in the two models SJS and OSS there is a small dependance on the eccentricity visible (slight decrease with e_{ini}). It should be emphasized that the cut for $i = 30°$ is representative and we got similar results for other inclinations. We conclude from this comparison that the effect of different chaoticity of the orbits for the Lagrangian points is an effect which already appears when we include Saturn in the model. But, the difference in the chaotic behaviour of the L_4 and the L_5 orbits – found with the aid of the LCE – could be due to a biased choice of initial conditions for Trojans around the two equilibrium points. We therefore can not yet claim that there is in fact a difference in the dynamics between the two stable Lagrangian points for more realistic models including other planets.

7. Conclusions

In this investigation we established the size of stable regions of fictitious Trojans around the equilibrium points of Jupiter for different dynamical models: the elliptic restricted problem, a model including also Saturn and one where the outer planets with their mutual perturbations and their gravitational force on the fictitious Trojans were fully taken into account. This goal was achieved with long term numerical experiments for fictitious Trojans around L_4 and L_5 for a chosen grid of initial conditions. We have tested different initial eccentricities and inclinations of these Trojans and determined their stability with a straightforward check of their maximum eccentricity (using results of the Lie-series integration). The major point of our investigation is that we can see how the stability regions shrink with larger inclinations and that they finally disappear completely for $i = 60°$. As proper tool for determining the dynamical state we also computed the LCE (respectively, the LT) which gave us an estimation of the chaoticity of an orbit depending on the inclination and the eccentricity of the Trojans. For that reason we used a programm provided by Milani et al. (loc.cit) for time scales of 10^8 years which gives a good estimate of the LCE. We also pointed out a possible difference in the stability between the two Lagrangian points, but this will be a topic of further investigations.

[6]We did not find this difference when we used the maximum eccentricity. The LCE is a more sensitive tool.

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