CHAPTER 7

ON THE QUANTUM POTENTIAL

1. RESUMÉ

We have seen already how the quantum potential arises in many contexts as a fundamental ingredient connected with quantum matter, and how it provides linkage between e.g. statistics and uncertainty, Fisher information and entropy, Weyl geometry, and quantum Kähler geometry. We expand further on certain aspects of the quantum potential in this Chapter aafter the resumé from Chapters 1-6 Moreover we have seen how the trajectory representation à la deBroglie-Bohm (dBB) can be used to develop meaningful insight and results in quantum field theory (QFT) and cosmology. This is achieved mainly without the elaborate machinery of Fock spaces, Feynman diagrams, operator algebras, etc. in a straightforward manner. The conclusion seems inevitable that dBB theory is essentially all pervasive and represents perhaps the most powerful tool available for understanding not only QM but the universe itself. There are of course many papers and opinions concerning such conclusions (some already discussed) and we will make further comments along these lines in this Chapter. We remark that there is some hesitation in postulating an ensemble interpretation when using dBB theory in cosmology with a wave function of the universe for example but we see no obstacle here, once an ensemble of universes is admitted. This is surely as reasonable as dealing with many string theories as is now fashionable. In any event we begin with a resumé of highlights from Chapters 1-6 and gather some material here (see also [**1026**] for information on the map $SE \to Q$).

1.1. THE SCHRÖDINGER EQUATION. We list some examples primarily concerned with QM.

(1) The SE in 1-dimension with $\psi = Rexp(iS/\hbar)$ and associated HJ and continuity equations are

$$-\frac{\hbar^2}{2m}\psi_{xx} + V\psi = i\hbar\psi_t;\ S_t + \frac{S_x^2}{2m} + V + Q = 0; \tag{1.1}$$

$$Q = -\frac{\hbar^2 R''}{2mR};\ \partial_t(R^2) + \frac{1}{m}(R^2 S_x)_x = 0$$

Here $P = R^2 = |\psi|^2$ is a probability density, $p = m\dot{x} = S_x$ (with S not constant!) is the momentum, $\rho = mP$ is a mass density, and Q is the quantum potential of Bohm. Classical mechanics involves a HJ equation with $Q = 0$ and can be derived as follows (cf. [**304**]). Consider a Lagrangian $L = p\dot{x} - H$ with Hamiltonian $H = (p^2/2m) + V$ and action

$S = \int_{t_1}^{t_2} dtL(x, \dot{x}, t)$. One computes

$$\delta S = \int_{t_1}^{t_2} dt \left[p\frac{d}{dt}\delta x + \dot{x}\delta p - \delta H - H\frac{d}{dt}\delta t \right] \quad (1.2)$$

where $\delta H = H_x\delta x + H_p\delta p + H_t\delta t = V_x\delta x + (p/m)\delta p + H_t\delta t$. Then

$$\delta S = \int_{t_1}^{t_2} dt \left[p\frac{d}{dt}\delta x + \dot{x}\delta p - V_x\delta x - \frac{p}{m}\delta p - H_t\delta t - H\frac{d}{dt}\delta t \right] = \quad (1.3)$$

$$= \int_{t_1}^{t_2} dt \left\{ \frac{d}{dt}[p\delta x - H\delta t] + \delta p\left(\dot{x} - \frac{p}{m}\right) - \delta x\left(\frac{dp}{dt} + V_x\right) + \delta t[\dot{H} - H_t] \right\}$$

Consequently, writing $\delta S = (S_x\delta x + S_t\delta t)|_{t_1}^{t_2}$ one arrives at

$$\dot{x} = \frac{p}{m};\ \dot{p} = -V_x;\ \dot{H} = H_t;\ p = S_x;\ S_t + H = 0 \quad (1.4)$$

Note here the "surface" term (from integration) is $G = p\delta x - H\delta t$ and $\delta S = G_2 - G_1$ which should equal $\delta S = (\partial S/\partial x_1)\delta x_1 + (\partial S/\partial x_2)\delta x_2 + (\partial S/\partial t_1)\delta t_1 + (\partial S/\partial t_2)\delta t_2$ where $x_1 = x(t_1)$ and $x_2 = x(t_2)$ One sees then directly how the addition of Q to a classical HJ equation produces a quantum situation.

(2) Another classical connection comes via hydrodynamics (cf. Section 1.1) where (1.1) can be put in the form

$$\partial_t(\rho v) + \partial(\rho v^2) + \frac{\rho}{m}(\partial V + \partial Q) = 0 \quad (1.5)$$

which is like an Euler equation in fluid mechanics moduo a pressure term $-\rho^{-1}\partial\mathfrak{P}$ on the right. If we identify $(\rho/m)\partial Q = \rho^{-1}\partial\mathfrak{P} \equiv \mathfrak{P} = \partial^{-1}(\rho^2/m)\partial Q$ (with some definition of ∂^{-1} - cf. [**205**]) then the quantum term could be thought of as providing a pressure with Q corresponding e.g. to a stress tensor of a quantum fluid. We refer also to Remark 1.1.2 and work of Kaniadakis et al where the quantum state coressponds to a subquantum statistical ensemble whose time evolution is governed by classical kinetics in phase space.

(3) The Fisher information connection à la Remarks 1.1.4 - 1.1.5 involves a classical ensemble with particle mass m moving under a potential V

$$S_t + \frac{1}{2m}(S')^2 + V = 0;\ P_t + \frac{1}{m}\partial(PS')' = 0 \quad (1.6)$$

where S is a momentum potential; note that no quantum potential is present but this will be added on in the form of a term $(1/2m)\int dt(\Delta N)^2$ in the Lagrangian which measures the strength of fluctuations. This can then be specified in terms of the probability density P as indicated in Remark 1.1.4 leading to a SE where by Theorem 1.1.1 $(\Delta N)^2 \sim c\int[(P')^2/P]dx$. A "neater" approach is given in Remark 1.1.5 leading in 1-D to

$$S_t + \frac{1}{2m}(S')^2 + V + \frac{\lambda}{m}\left(\frac{(P')^2}{P^2} - \frac{2P''}{P}\right) = 0 \quad (1.7)$$

Note that $Q = -(\hbar^2/2m)(R''/R)$ becomes for $R = P^{1/2}$ an equation $Q = -(\hbar^2/8m)[(2P''/P) - (P'/P)^2]$. Thus the addition of the Fisher information serves to quantize the classical system via a quantum potential and this gives a direct connection of the quantum potential with fluctuations.

(4) The Nagasawa theory (based in part on Nelson's work) is very revealing and fascinating. The essense of Theorem 1.1.2 is that $\psi = exp(R+iS)$ satisfies the SE $i\psi_t + (1/2)\psi'' + ia\psi' - V\psi = 0$ if and only if

$$V = -S_t + \frac{1}{2}R'' + \frac{1}{2}(R')^2 - \frac{1}{2}(S')^2 - aS';\ 0 = R_t + \frac{1}{2}S'' + S'R' + aR' \tag{1.8}$$

Changing variables ($X = (\hbar/\sqrt{m})x$ and $T = \hbar t$) one arrives at $i\hbar\psi_T = -(\hbar^2/2m)\psi_{XX} - iA\psi_X + V\psi$ where $A = a\hbar/\sqrt{m}$ and

$$i\hbar R_T + (\hbar^2/m^2)R_X S_X + (\hbar^2/2m^2)S_{XX} + AR_X = 0; \tag{1.9}$$

$$V = -i\hbar S_T + (\hbar^2/2m)R_{XX} + (\hbar^2/2m^2)R_X^2 - (\hbar^2/2m^2)S_X^2 - AS_X$$

The diffusion equations then take the form

$$\hbar\phi_T + \frac{\hbar^2}{2m}\phi_{XX} + A\phi_X + \tilde{c}\phi = 0;\ -\hbar\hat{\phi}_T + \frac{\hbar^2}{2m}\hat{\phi}_{XX} - A\hat{\phi}_X + \tilde{c}\hat{\phi} = 0; \tag{1.10}$$

$$\tilde{c} = -\tilde{V}(X,T) - 2\hbar S_T - \frac{\hbar^2}{m}S_X^2 - 2AS_X$$

It is now possible to introduce a role for the quantum potential in this theory. Thus from $\psi = exp(R+iS)$ (with $\hbar = m = 1$ say) we have $\psi = \rho^{1/2}exp(iS)$ with $\rho^{1/2} = exp(R)$ or $R = (1/2)log(\rho)$. Hence $(1/2)(\rho'/\rho) = R'$ and $R'' = (1/2)[(\rho''/\rho) - (\rho'/\rho)^2]$ while the quantum potential is $Q = (1/2)(\partial^2\rho^{1/2}/\rho^{1/2}) = -(1/8)[(2\rho''/\rho) - (\rho'/\rho)^2]$. Equation (1.8) becomes then

$$V = -S_t + \frac{1}{8}\left(\frac{2\rho''}{\rho} - \frac{(\rho')^2}{\rho^2}\right) - \frac{1}{2}(S')^2 - aS' \equiv \tag{1.11}$$

$$\equiv S_t + \frac{1}{2}(S')^2 + V + Q + aS' = 0;\ \rho_t + \rho S'' + S'\rho' + a\rho' = 0 \equiv \rho_t + (\rho S')' + a\rho' = 0$$

Thus $-2S_t - (S')^2 = 2V + 2Q + 2aS'$ and one has

PROPOSITION 1.1. *The creation-annihilation term c in the diffusion equations (cf. Theorem 1.1.2) becomes*

$$c = -V - 2S_t - (S')^2 - 2aS' = V + 2Q \tag{1.12}$$

where Q is the quantum potential.

(5) Going to Remarks 1.1.6-1.1.8 we set

$$\tilde{Q} = \frac{\hbar^2}{2m^2}\frac{\partial^2\sqrt{\rho}}{\sqrt{\rho}} = -\frac{1}{m}Q;\ D = \frac{\hbar}{2m};\ u = D\partial(log(\rho)) = \frac{h}{2m}\frac{\rho'}{\rho} \tag{1.13}$$

Then u is called an osmotic velocity field and Brownian motion involves $v = -u$ for the diffusion current. In particular

$$\tilde{Q} = \frac{1}{2}u^2 + D\partial u \tag{1.14}$$

One defines an entropy term $\mathfrak{S} = -\int \rho log(\rho)dx$ leading, for suitable regions of integration and behavior of ρ at infinity, and using $\rho_t = -\partial(v\rho)$ from (1.1), to

$$\frac{\partial\mathfrak{S}}{\partial t} = -\int \rho_t(1+log(\rho)) = \int (1+log(\rho))\partial(v\rho) = \tag{1.15}$$

$$= -\int v\rho' = \int u\rho' = D\int \frac{(\rho')^2}{\rho}$$

Note also

$$\tilde{Q} = \frac{D^2}{2}\left(\frac{2\rho''}{\rho} - \left(\frac{\rho'}{\rho}\right)^2\right) \Rightarrow \int \rho\tilde{Q} = -\frac{D^2}{2}\int \frac{(\rho')^2}{\rho} \tag{1.16}$$

Thus generally $\partial\mathfrak{S}/\partial t \geq 0$ and $\mathfrak{F} = -(2/D^2)\int \rho\tilde{Q}$ is a functional form of Fisher information with $\mathfrak{S}_t = D\mathfrak{F}$.

(6) The development of the SE by Nottale, Cresson, et al in Section 1.2 is basically QM and is peripheral to scale relativity as such. The idea is roughly to to imagine e.g. continuous nondifferentiable quantum paths and to describe the velocity in terms of an average $V = (1/2)(b_+ + b_-)$ and a discrepancy $U = (1/2)(b_+ - b_-)$ where $b_\pm$ are given by (2.1). Nottale's derivation is heuristic but revealing and working with a complex velocity he captures the complex nature of QM. In particular the quantum potential can be written as $Q = -(m/2)U^2 - (\hbar/2)\partial U$ corresponding to (1.14) via $\tilde{Q} = -(1/m)Q$. As indicated in Proposition 1.2.1 this reveals the quantum potential as a manifestation of the "fractal" nature of quantum paths - smooth paths correspond to $Q = 0$ which seems to preclude smooth trajectories for quantum particles. In such a case the standard formula $\dot{x} = (\hbar/2m)\Im[\psi^*\partial\psi/|\psi|^2]$ requires a discontinuous $\dot{x}$ wich places some constraints on ψ and the whole guidance idea. This whole matter should be addressed further along with considerations of osmotic velocity, etc. Another approach to quantum fractals is given in Section 1.5.

(7) In section 2.3.1 we sketched some of the Bertoldi-Faraggi-Matone (BFM) version of Bohmian mechanics and in particular for the stationary quantum HJ equation (QHJE) $(1/2m)S_x^2 + W = E$ $(W = V + Q)$, arising from $-(\hbar^2/2m)\psi'' + V\psi = E\psi$, one can extract from [**347**] the formulas for trajectories (using Floydian time $t \sim \partial S/\partial E$). Thus $(\partial_E W = \partial_E Q)$

$$t \sim \partial_E \int S_x dx = \partial_E \int [E-W]^{1/2}dx = \left(\frac{m}{2}\right)^{1/2}\int \frac{(1-\partial_E Q}{\sqrt{E-W}}dx \tag{1.17}$$

Hence

$$\frac{dt}{dx} = \left(\frac{m}{2}\right)^{1/2}\frac{1-\partial_E Q}{\sqrt{E-W}} \Rightarrow \dot{x} = \frac{S_x}{m}\frac{1}{1-\partial_E Q} \tag{1.18}$$

Thus $m(1-\partial_E Q)\dot{x} = m_Q\dot{x} = S_x$ and this is defined as p with m_Q representing a quantum mass. Note $\dot{x} \neq p/m$ and we refer to [**194, 191, 347, 373, 374**] for discussion of all this. Further via $p' = m_Q'\dot{x} +$

$m_Q(\ddot{x}/\dot{x})$, etc., one can rewrite the QSHJ as a third order trajectory (or microstate) equation (see also Remark 7.4)

$$\frac{m_Q^2}{2m}\dot{x}^2 + V - E + \frac{\hbar^2}{4m}\left(\frac{m_Q''}{m_Q} - \frac{3}{2}\left(\frac{m_Q'}{m_Q}\right)^2 - \frac{m_Q'}{m_Q}\frac{\ddot{x}}{\dot{x}^2} + \frac{\dddot{x}}{\dot{x}^3} - \frac{5}{2}\frac{\ddot{x}^2}{\dot{x}^4}\right) = 0 \quad (1.19)$$

In Remark 2.2.2 with Theorem 2.1 we observed how the uncertainty principle of QM can be envisioned as due to incomplete information about microstates when working in the Hilbert space formulation of QM based on the SE. It was shown how $\Delta q \Delta x = O(\hbar)$ arises automatically from a BFM perspective. Thus the canonical QM in Hilbert space cannot see a single trajectory and hence is obliged to operate in terms of ensembles and probability. We have also seen how a probabilistic ensemble picture with quantum fluctuations comes about with the fluctuations corresponding to the quantum potential (see Item 3 above). This suggests that a background motivation for the Hilbert space may really exist (beyond its pragmatic black magic) since these fluctuations represent a form of information (and uncertainty). The hydrodynamic and diffusion models are also directly connected to this and produce as in Item 5 above a connection to entropy.

1.2. DEBROGLIE-BOHM. There are many approaches to dBB theory and in fact much of the book is concerned with this. David Bohm wrote extensively about the subject but we have omitted much of the philosophy (implicate order, etc.). The book by Holland [**471**] is excellent and a modern theory is being constructed by Dürr, Goldstein, Zanghi, et al (cf. also the work of Bertoldi, Farragi, Matone, and Floyd). Some new directions in QFT, Weyl geometry, and cosmology are also covered in the book, due to Barbosa, Pinto-Neto, Nikolić, A. and F. Shojai, et al, and we will try to summarize some of that here.

(1) The BFM theory is quite novel (and profound) in that it is based entirely on an equivalence principle (EP) stating that all physical systems can be connected by a coordinate transformation to the free situation with vanishing energy. One bases the stationary situation of energy E in the nonrelativistic case with the SE as in Item 7 above. In the relativistic case (Remark 2.2.3) one can work in the same spirit directly with a Minkowski metric to obtain the Klein-Gordon (KG) equation with a relativistic quantum potential

$$Q_{rel} = -\frac{\hbar^2}{2m}\frac{\Box R}{R} \quad (1.20)$$

Note that the probability aspects concerning R appear to be absent in the relativistic theory. It is interesting to note (cf. Remark 2.2.1) that the EP implies that all mass can be generated by a coordinate transformation and since mass can be expressed in terms of the quantum potential Q this provides yet another role for the quantum potential.

(2) Some quantum field theory (QFT) aspects of the Bohm theory are developed in [**471**] and sketched here in Example 2.1.1. One arrives at a

formula, namely

(1.21) $$\Box\psi = -\left.\frac{\delta Q[\psi(x),t]}{\delta\psi(x)}\right|_{\psi(x)=\psi(x,t)};\ Q[\psi,t] = -\frac{1}{2R}\int d^3x\frac{\delta^2 R}{\delta\psi^2}$$

More recently there have been some impressive papers by Nikolić involving Bohmian theory and QFT. First there are papers on bosonic and fermionic Bohmian QFT sketched in Sections 3.2 and 3.3. These are lovely but even more attractive are two newer papers [**708, 713**] by Nikolić which we displayed in Sections 2.4 and 2.6. In [**708**] one utilizes the deDonder-Weyl formulation of QFT (reviewed in Appendix A) and a Bohmian formulation is not postulated but derived from the technical requirements of covariance and consistency with standard QM. One introduces a preferred foliation of spacetime with R^μ normal to the leaf Σ and writes $\mathfrak{R}([\phi],\Sigma) = \int_\Sigma d\Sigma_\mu R^\mu$ with $\mathfrak{S}([\phi],x) = \int_\Sigma d\Sigma_\mu S^\mu$. This produces a covariant version of Bohmian mechanics with $\Psi = \mathfrak{R}exp(i\mathfrak{S}/\hbar)$ via

(1.22) $$\frac{1}{2}\frac{dS_\mu}{d\phi}\frac{dS^\mu}{d\phi} + V + Q + \partial_\mu S^\mu = 0;\ \frac{dR^\mu}{d\phi}\frac{dS^\mu}{d\phi} + J + \partial_\mu R^\mu = 0$$

(1.23) $$Q = -\frac{\hbar^2}{2\mathfrak{R}}\frac{\delta^2\mathfrak{R}}{\delta_\Sigma\phi^2(x)};\ J = \frac{\mathfrak{R}}{2}\frac{\delta^2\mathfrak{S}}{\delta_\Sigma\phi^2(x)}$$

In [**713**] one uses the many fingered time (MF) Tomonaga-Schwinger (TS) equation where a Cauchy hypersurface Σ is defined via $x^0 = T(\mathbf{x})$ with $\mathbf{x}$ corresponding to coordinates on Σ. The TS equation is

(1.24) $$i\frac{\delta\Psi[\phi,T]}{\delta T(\mathbf{x})} = \hat{\mathfrak{H}}\Psi[\phi,T]$$

Take a free scalar field for convenience with

(1.25) $$\hat{\mathfrak{H}}(\mathbf{x}) = -\frac{1}{2}\frac{\delta^2}{\delta\phi^2(\mathbf{x})} + \frac{1}{2}[(\nabla\phi(\mathbf{x}))^2 + m^2\phi^2(\mathbf{x})]$$

Then for a manifestly covariant theory one introduces parameters $\mathbf{s} = (s^1,s^2,s^3)$ to serve as coordinates on a 3-dimensional manifold Σ in spacetime with $x^\mu = X^\mu(\mathbf{s})$ the embedding coordinates. The induced metric on Σ is

(1.26) $$q_{ij}(\mathbf{s}) = g_{\mu\nu}(X(\mathbf{s}))\frac{\partial X^\mu(\mathbf{s})}{\partial s^i}\frac{\partial X^\nu(\mathbf{s})}{\partial s^j}$$

Similarly a normal (resp. unit normal - transforming as as a spacetime vector) to the surface are

(1.27) $$\tilde{n}(\mathbf{s}) = \epsilon_{\mu\alpha\beta\gamma}\frac{\partial X^\alpha}{\partial s^1}\frac{\partial X^\beta}{\partial s^2}\frac{\partial X^\gamma}{\partial s^3};\ n^\mu(\mathbf{s}) = \frac{g^{\mu\nu}\tilde{n}_\nu}{\sqrt{|g^{\alpha\beta}\tilde{n}_\alpha\tilde{n}_\beta|}}$$

Then for $\mathbf{x}\to\mathbf{s}$ and $\frac{\delta}{\delta T(\mathbf{x})}\to n^\mu(\mathbf{s})\frac{\delta}{\delta X^\mu(\mathbf{s})}$ the TS equation becomes

(1.28) $$\hat{\mathfrak{H}}(\mathbf{s})\Psi[\phi,X] = in^\mu(\mathbf{s})\frac{\delta\Psi[\phi,X]}{\delta X^\mu(\mathbf{s})}$$

and the Bohmian equations of motion are ($\Psi = Rexp(iS)$)

(1.29) $$\frac{\partial\Phi(\mathbf{s},T]}{\partial\tau(\mathbf{s})} = \frac{1}{|q(\mathbf{s})|^{1/2}}\left.\frac{\delta S}{\delta\phi(\mathbf{s})}\right|_{\phi=\Phi};\ \frac{\partial}{\partial\tau(\mathbf{s})} \equiv lim_{\sigma_x\to 0}\int_{\sigma_x} d^3 s n^{\mu}(\mathbf{s})\frac{\delta}{\delta X^{\mu}(\mathbf{s})}$$

In the same spirit the quantum MFT KG equation is

(1.30) $$\left[\left(\frac{\partial}{\partial\tau(\mathbf{s})}\right)^2 + \nabla^i\nabla_i + m^2\right]\Phi(\mathbf{s},X] = -\frac{1}{|q(\mathbf{s})|^{1/2}}\left.\frac{\partial\mathfrak{Q}(\mathbf{s},\phi,X]}{\partial\phi\mathbf{s})}\right|_{\phi=\Phi}$$

where ∇_i is the covariant derivative with respect to s^i and

(1.31) $$\mathfrak{Q}(\mathbf{s},\phi,X] = -\frac{1}{|q(\mathbf{s})|^{1/2}}\frac{1}{2R}\frac{\delta^2 R}{\delta\phi^2(\mathbf{s})}$$

(3) The QFT model in Section 2.5 involving stochastic jumps is quite technical and should be read in conjunction with Nagasawa's book [**674**]. The idea (cf. [**326**]) is that for the Hamiltonian of a QFT there is associated a $|\psi|^2$ distributed Markov process, typically a jump process (to accound for creation and annihilation processes) on the configuration space of a variable number of particles. One treats this via functional analysis, operator thery, and probability, which leads to mountains of detail, only a small portion of which is sketched in this book.

(4) In Section 3.2 we give a sketch of dBB in Weyl geometry following A. and F. Shojai [**873**]. This is a lovely approach and using Dirac-Weyl methods one is led comfortably into general relativity (GR), cosmology, and quantum gravity, in a Bohmian context. Such theories dominate Chapters 3 and 4. First one looks at the relativistic energy equation $\eta_{\mu\nu}p^{\mu}p^{\nu} = m^2c^2$ generalized to

(1.32) $$\eta_{\mu\nu}P^{\mu}P^{\nu} = m^2c^2(1+\mathcal{Q}) = \mathcal{M}^2c^2;\ \mathcal{Q} = (\hbar^2/m^2c^2)(\Box|\Psi|/|\Psi|)$$

(1.33) $$\mathcal{M}^2 = m^2\left(1+\alpha\frac{\Box|\Psi|}{|\Psi|}\right);\ \alpha = \frac{\hbar^2}{m^2c^2}$$

(obtained e.g. by setting $\psi = \sqrt{\rho}exp(iS/\hbar)$ in the KG equation). Here $\mathcal{M}^2$ is not positive definite and in fact (1.32) is the wrong equation! Some interesting arguments involving Lorentz invariance lead to better equations and for a particle in a curved background the natural quantum HJ equation is most comfortably phrased as

(1.34) $$\nabla_{\mu}(\rho\nabla^{\mu}S) = 0;\ g^{\mu\nu}\nabla_{\mu}S\nabla_{\nu}S = \mathfrak{M}^2c^2;\ \mathfrak{M}^2 = m^2e^{\mathfrak{Q}};\ \mathfrak{Q} = \frac{\hbar^2}{m^2c^2}\frac{\Box_g|\Psi|}{|\Psi|}$$

This is equivalent to

(1.35) $$\left(\frac{m^2}{\mathcal{M}^2}\right)g^{\mu\nu}\nabla_{\mu}S\nabla_{\nu}S = m^2c^2$$

showing that the quantum effects correspond to a change in spacetime metric $g_{\mu\mu} \to \tilde{g}_{\mu\nu} = (\mathfrak{M}^2/m^2)g_{\mu\nu}$. This is a conformal transformation and leads to Weyl geometry where (1.35) takes the form $\tilde{g}^{\mu\nu}\tilde{\nabla}_{\mu}S\tilde{\nabla}_{\nu}S =$

m^2c^2 with $\tilde{\nabla}_\mu$ the covariant derivative in the metric $\tilde{g}_{\mu\nu}$. The particle motion is then

$$\mathfrak{M}\frac{d^2x^\mu}{d\tau^2} + \mathfrak{M}\Gamma^\mu_{\nu\kappa}u^\nu u^\kappa = (c^2g^{\mu\nu} - u^\mu u^\nu)\nabla_\nu\mathfrak{M} \tag{1.36}$$

and the introduction of a quantum potential is equivalent to introducing a conformal factor $\Omega^2 = \mathcal{M}^2/m^2$ in the metric (i.e. QM corresponds to Weyl geometry). One considers then a general relativistic system containing gravity and matter (no quantum effects) and links it to quantum matter by the conformal factor Ω^2 (using an approximation $1 + Q \sim exp(Q)$ for simplicity); then the appropriate Einstein equations are written out. Here the conformal factor and the quantum potential are made into dynamical fields to create a scalar-tensor theory with two scalar fields. Examples are developed and we refer to Section 3.2 and [**873**] for more details. Back reaction effects of the quantum factor on the background metric are indicated in the modified Einstein equations. Thus the conformal factor is a function of the quantum potential and the mass of a relativistic particle is a field produced by quantum corrections to the classical mass. In general frames both the spacetime metric and the mass field have quantum properties.

(5) The Dirac-Weyl theory is developed also in [**873**] via the action

$$\mathfrak{A} = \int d^4x\sqrt{-g}(F_{\mu\nu}F^{\mu\nu} - \beta^2\ {}^W\mathcal{R} + (\sigma+6)\beta_{;\mu}\beta^{;\mu} + \mathfrak{L}_{matter} \tag{1.37}$$

The gravitational field $g_{\mu\nu}$ and Weyl field ϕ_μ plus β determine the spacetime geometry and one finds a Bohmian theory with $\beta \sim \mathcal{M}$ (Bohmian quantum mass field). We will say much more about Dirac-Weyl theory below.

(6) There is an interesting approach by Santamato in [**840, 841**] dealing with the SE and KG equation in Weyl geometry (cf. Section 3.3 and [**189, 203**]). In the first paper on the SE one assumes particle motion given by a random process $q^i(t,\omega)$ with probability density $\rho(q,t)$, $\dot{q}^i(t,\omega) = v^i(q(t,\omega),t)$, and random initial conditions $q_0^i(\omega)$ $(i = 1,\cdots,n)$. One begins with a stochastic construction of (averaged) classical type Lagrange equations in generalized coordinates for a differentiable manifold M in which a notion of scalar curvature R is meaningful (this is where statistics enters the geometry). It is then shown that a theory equivalent to QM (via a SE) can be constructed where the "quantum force" (arising from a quantum potential Q) can be related to (or described by) geometric properties of space. To do this one assumes that a (quantum) Lagrangian can be constructed in the form $L(q,\dot{q},t) = L_C(q,\dot{q},t) + \gamma(\hbar^2/m)R(q,t)$ where $\gamma = (1/6)(n-2)/(n-1)$ with $n = dim(M)$ and R is a curvature scalar. Now for a Riemannian geometry $ds^2 = g_{ik}(q)dq^idq^k$ it is standard that in a transplantation $q^i \to q^i + \delta q^i$ one has $\delta A^i = \Gamma^i_{k\ell}A^\ell dq^k$, and here it is assumed that for $\ell = (g_{ik}A^iA^k)^{1/2}$ one has $\delta\ell = \ell\phi_k dq^k$ where the ϕ_k are covariant components of an arbitrary vector of M (Weyl geometry). Thus the actual affine connections $\Gamma^i_{k\ell}$ can be found by comparing this

with $\delta\ell^2 = \delta(g_{ik}A^iA^k)$ and one finds

$$\Gamma^i_{k\ell} = -\left\{ \begin{array}{c} i \\ k\ \ell \end{array} \right\} + g^{im}(g_{mk}\phi_\ell + g_{m\ell}\phi_k - g_{k\ell}\phi_m) \tag{1.38}$$

Thus we may prescribe the metric tensor g_{ik} and ϕ_i and determine via (1.38) the connection coefficients. Covariant derivatives are defined via commas and the curvature tensor $R^i_{k\ell m}$ in Weyl geometry is introduced via $A^i_{,k,\ell} - A^i_{,\ell,k} = F^i_{mk\ell}A^m$ from which arises the standard formula of Riemannian geometry $R^i_{mk\ell} = -\partial_\ell\Gamma^i_{mk} + \partial_k\Gamma^i_{m\ell} + \Gamma^i_{n\ell}\Gamma^n_{mk} - \Gamma^i_{nk}\Gamma^n_{m\ell}$ where (1.38) must be used in place of the Riemannian Christoffel symbols. The Ricci symmetric tensor R_{ik} and the scalar curvature R are defined via $R_{ik} = R^\ell_{i\ell k}$ and $R = g^{ik}R_{ik}$, while

$$R = \dot{R} + (n-1)[(n-2)\phi_i\phi^i - 2(1/\sqrt{g})\partial_i(\sqrt{g}\phi^i)] \tag{1.39}$$

where $\dot{R}$ is the Riemannian curvature built by the Christoffel symbols. Now the geometry is to be derived from physical principles so the ϕ_i cannot be arbitrary but are obtained by the same (averaged) least action principle giving the motion of the particle (statistical determination of geometry) and when $n \geq 3$ the minimization involves only (1.39). One shows that $\hat{\rho}(q,t) = \rho(q,t)/\sqrt{g}$ transforms as a scalar in a coordinate change and this will be called the scalar probability density of the random motion of the particle. Starting from $\partial_t\rho + \partial_i(\rho v^i) = 0$ a manifestly covariant equation for $\hat{\rho}$ is found to be

$$\partial_t\hat{\rho} + (1/\sqrt{g})\partial_i(\sqrt{g}v^i\hat{\rho}) = 0 \tag{1.40}$$

Some calculation then yields a minimum over R when

$$\phi_i(q,t) = -[1/(n-2)]\partial_i[log(\hat{\rho})(q,t)] \tag{1.41}$$

This shows that the geometric properties of space are indeed affected by the presence of the particle and in turn the alteration of geometry acts on the particle through the quantum force $f_i = \gamma(\hbar^2/m)\partial_i R$ which according to (1.39) depends on the gauge vector and its derivatives. It is this peculiar feedback between the geometry of space and the motion of the particle which produces quantum effects. In this spirit one goes next to a geometrical derivation of the SE. Thus inserting (1.41) into (1.39) one gets

$$R = \dot{R} + (1/2\gamma\sqrt{\hat{\rho}})[1/\sqrt{g})\partial_i(\sqrt{g}g^{ik}\partial_k\sqrt{\hat{\rho}})] \tag{1.42}$$

where the value $\gamma = (1/6)[(n-2)/(n-1)]$ has been used. On the other hand the HJ equation can be written as

$$\partial_t S + H_C(q, \nabla S, t) - \gamma(\hbar^2/m)R = 0 \tag{1.43}$$

When (1.42) is introduced into (1.43) the HJ equation and the continuity equation (1.40), with velocity field given by $v^i = (\partial H/\partial p_i)(q, \nabla S, t)$, form a set of two nonlinear PDE which are coupled by the curvature of space. Therefore self consistent random motions of the "particle" are obtained by solving (1.40) and (1.43) simultaneously. For every pair of solutions

$S(q,t,\hat{\rho}(q,t))$ one gets a possible random motion for the particle whose invariant probability density is $\hat{\rho}$. The present approach is so different from traditional QM that a proof of equivalence is needed and this is only done for Hamiltonians of the form $H_C(q,p,t) = (1/2m)g^{ik}(p_i - A_i)(p_k - A_k) + V$ (which is not very restrictive) leading to

$$\partial_t S + \frac{1}{2m} g^{ik}(\partial_i S - A_i)(\partial_k S - A_k) + V - \gamma\frac{\hbar^2}{m}R = 0 \tag{1.44}$$

(R in (1.43)). The continuity equation (1.40) is

$$\partial_t \hat{\rho} + (1/m\sqrt{g})\partial_i[\hat{\rho}\sqrt{g}g^{ik}(\partial_k S - A_k)] = 0 \tag{1.45}$$

Owing to (1.42), (1.44) and (1.45) form a set of two nonlinear PDE which must be solved for the unknown functions S and $\hat{\rho}$. Then a straightforward calculation shows that, setting $\psi(q,t) = \sqrt{\hat{\rho}(q,t)exp}](i/\hbar)S(q,t)]$ the quantity ψ obeys a linear SE

$$i\hbar\partial_t\psi = \frac{1}{2m}\left\{\left[\frac{i\hbar\partial_i\sqrt{g}}{\sqrt{g}} + A_i\right] g^{ik}(i\hbar\partial_k + A_k)\right\}\psi + \left[V - \gamma\frac{\hbar^2}{m}\dot{R}\right]\psi \tag{1.46}$$

where only the Riemannian curvature $\dot{R}$ is present (any explicit reference to the gauge vector ϕ_i having disappeared).

We recall that in the nonrelativistic context the quantum potential has the form $Q = -(\hbar^2/2m)(\partial^2\sqrt{\rho}/\sqrt{\rho})$ ($\rho \sim \hat{\rho}$ here) and in more dimensions this corresponds to $Q = -(\hbar^2/2m)(\Delta\sqrt{\rho}/\sqrt{\rho})$. The continuity equation in (1.45) corresponds to $\partial_t\rho + (1/m\sqrt{g})\partial_i[\rho\sqrt{g}g^{ik}(\partial_k S)] = 0$ ($\rho \sim \hat{\rho}$ here). For $A_k = 0$ (1.44) becomes $\partial_t S + (1/2m)g^{ik}\partial_i S\partial_k S + V - \gamma(\hbar^2/m)R = 0$. This leads to an identification $Q \sim -\gamma(\hbar^2/m)R$ where R is the Ricci scalar in the Weyl geometry (related to the Riemannian curvature built on standard Christoffel symbols via (1.39). Here $\gamma = (1/6)[(n-2)/(n-1)]$ which for $n = 3$ becomes $\gamma = 1/12$; further by (1.41) the Weyl field is $\phi_i = -\partial_i log(\rho)$. Consequently for the SE (1.46) in Weyl space the quantum potential is $Q = -(\hbar^2/12m)R$ where R is the Weyl-Ricci scalar curvature. For Riemannian flat space $\dot{R} = 0$ this becomes via (1.42)

$$R = \frac{1}{2\gamma\sqrt{\rho}}\partial_i g^{ik}\partial_k\sqrt{\rho} \sim \frac{1}{2\gamma}\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}} \Rightarrow Q = -\frac{\hbar^2}{2m}\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}} \tag{1.47}$$

as desired; the SE (1.46) reduces to the standard SE $i\hbar\partial_t\psi = -(\hbar^2/2m)\Delta\psi + V\psi$ ($A_k = 0$). Moreover (1.39) provides an interaction between gravity (involving $\dot{R}$ and g) and QM (which generates ϕ_i via ρ and R via Q).

(7) In [**841**] the KG equation is also derived via an average action principle with the restriction of a priori Weyl geometry removed. The spacetime geometry is then obtained from the average action principle to be of Weyl type with a gauge field $\phi_\mu = \partial_\mu log(\rho)$. One has a kind of "moral" equivalence between QM in Riemannian spaces and classical statistical mechanics in a Weyl space. Traditional QM based on wave equations and ad hoc probability calculus is merely a convenient tool to overcome the complications arisin from a nontrivial spacetime geometrical structure.

In the KG situation there is a relation $m^2 - (R/6) \sim \mathcal{M}^2 \sim m^2(1+Q)$ (approximating $m^2 exp(Q)$) and $Q \sim \Box\sqrt{\rho}/m^2\sqrt{\rho}$ which implies $R/6 \sim -\Box\sqrt{\rho}/\sqrt{\rho}$.

(8) Referring to Item 3 in Section 6.1.1 we note that for $\phi_\mu \sim A_\mu = \partial_\mu log(P)$ ($P \sim \rho$) one can envision a complex velocity $p_\mu + i\lambda A_\mu$ leading to

$$|p_\mu + i\sqrt{\lambda}A_\mu|^2 = p_\mu^2 + \lambda A_\mu^2 \sim g^{\mu\nu}\left(\frac{\partial S}{\partial x^\mu}\frac{\partial S}{\partial x^\nu} + \frac{\lambda}{P^2}\frac{\partial P}{\partial x^\mu}\frac{\partial P}{\partial x^\nu}\right) \tag{1.48}$$

This is exactly the term arising in a Fisher information Lagrangian

$$L_{QM} = L_{CL} + \lambda I = \int P\left\{\frac{\partial S}{\partial t} + \frac{1}{2}g^{\mu\nu}\left[\frac{\partial S}{\partial x^\mu}\frac{\partial S}{\partial x^\nu} + \frac{\lambda}{P^2}\frac{\partial P}{\partial x^\mu}\frac{\partial P}{\partial x^\nu}\right] + V\right\} dtd^n x \tag{1.49}$$

where I is the information term (see Section 3.1)

$$I = g^{ik}I_{ik} = \frac{g^{ik}}{2}\int \frac{1}{P}\frac{\partial P}{\partial y^i}\frac{\partial P}{\partial y^k}d^n y \tag{1.50}$$

known from ϕ_μ. Hence we have a direct connection between Fisher information and the Weyl field ϕ_μ along with a motivation for a complex velocity (cf. [**223**]). Further we note, via [**189**] and quantum geometry in the form $ds^2 \sim \sum dp_j^2/p_j$ on a space of probability distributions, that (1.50) can be defined as a Fisher information metric (positive definite via its connection to $(\Delta N)^2$) and

$$Q \sim -2\hbar^2 g^{\mu\nu}\left[\frac{1}{P^2}\frac{\partial P}{\partial x^\mu}\frac{\partial P}{\partial x^\nu} - \frac{2}{P}\frac{\partial^2 P}{\partial x^\mu \partial x^\nu}\right] \tag{1.51}$$

(corresponding to $-(\hbar^2/2m)(\partial^2\sqrt{\rho}/\sqrt{\rho}) = -(\hbar^2/8m)[(2\rho''/\rho) - (\rho'/\rho)^2]$). Further from $\mathbf{u} = -D\vec{\phi}$ with $Q = D^2((1/2)|\mathbf{u}|^2 - \nabla\cdot\vec{\phi})$, one expresses Q directly in terms of the Weyl vector. This enforces the idea that QM is built into Weyl geometry and moreover that fluctuations generate Weyl geometry.

(9) In Section 3.3 the WDW equation is treated following [**876, 870**] from a Bohmian point of view. One builds up a Lagrangian and Hamiltonian in terms of lapse and shift functions with a quantum potential

$$Q = \int d^3x\mathfrak{Q};\ \mathfrak{Q} = \hbar^2 NqG_{ijk\ell}\frac{1}{|\psi|}\frac{\delta^2|\psi|}{\delta q_{ij}\delta_{k\ell}} \tag{1.52}$$

The quantum potential changes the Hamiltonian constraint algebra to require weak closure (i.e. closure modulo the equations of motion); regularization and ordering are not considered here but will not affect the constraint algebra. The quantum Einstein equations are derived in the form

$$\mathfrak{G}^{ij} = -\frac{1}{N}\frac{\delta\mathfrak{Q}}{\delta q_{ij}};\ \mathfrak{G}^{0\mu} = \frac{\mathfrak{Q}}{2\sqrt{-g}}g^{0\mu} \tag{1.53}$$

The Bohmian HJ equation is

$$G_{ijk\ell}\frac{\delta S}{\delta q_{ij}}\frac{\delta S}{\delta q_{k\ell}} - \sqrt{q}\,({}^3\mathfrak{R} - \mathfrak{Q}) = 0 \tag{1.54}$$

where S is the phase of the WDW wave function and this leads to the same equations of motion (1.53). The modified Einstein equations are given in Bohmian form via

$$\mathfrak{G}^{ij} = -\kappa\mathfrak{T}^{ij} - \frac{1}{N}\frac{\delta(\mathfrak{Q}_G + \mathfrak{Q}_m)}{\delta g_{ij}};\ \mathfrak{G}^{0\mu} = -\kappa\mathfrak{T}^{0\mu} + \frac{\mathfrak{Q}_G + \mathfrak{Q}_m}{2\sqrt{-g}}g^{0\mu} \tag{1.55}$$

$$\mathfrak{Q}_m = \hbar^2\frac{N\sqrt{q}}{2}\frac{1}{|\psi|}\frac{\delta^2|\psi|}{\delta\phi^2};\ \mathfrak{Q}_G = \hbar^2 NqG_{ijk\ell}\frac{1}{|\psi|}\frac{\delta^2|\psi|}{\delta q_{ij};\delta q_{k\ell}} \tag{1.56}$$

$$Q_G = \int d^3x\mathfrak{Q}_G;\ Q_m = \int d^3x\mathfrak{Q}_m$$

In the third paper of [**876**] the Ashtekar variables are employed and it is shown that the Poisson bracket of the Hamiltonian with itself changes with respect to its classical counterpart but is still weakly equal to zero (modulo regularization, etc.); we refer to [**876**] and Section 4.3.1 for details.

1.3. GEOMETRY, GRAVITY, AND QM. We have already indicated some interaction of QM and geometry via Bohmian mechanics and remark here upon other aspects.

(1) It is known that one can develop a quantum geometry via Kähler geometry on a preHilbert space $P(H)$ (see e.g. [**54, 153, 188, 189, 203, 244, 245, 246, 247, 248**]). Thus $P(H)$ is a Kähler manifold with a Fubini-Study metric based on $(|d\psi_\perp >= |d\psi > -|\psi >< \psi|d\psi >)$

$$\frac{1}{4}ds^2_{PS} = [cos^{-1}(| < \tilde{\psi}|\psi > |)]^2 \sim 1 - | < \tilde{\psi}|\psi > |^2 =< d\psi_\perp|d\psi_\perp > \tag{1.57}$$

where $ds^2_{PS} = \sum dp_j^2/p_j = \sum p_j(d\,log(p_j))^2$ gives the connection to probability distributions. We have already seen in Item 8 of Section 6.1.2 how this probability metric is related to Fisher information, fluctuations, and the quantum potential.

(2) There is a fascinating series of papers by Arias, Bonal, Cardenas, Gonzalez, Leyva, Martin, and Quiros dealing with general relativity (GR) and conformal variations (cf. Section 3.2.2). We omit details here but simply remark that conformal GR with $\hat{g}_{ab} = \Omega^2 g_{ab}$ is shown to be the only consistent formulation of gravity. Here consistent refers to invariance under the group of transformations of units of length, time, and mass.

(3) In Section 4.5.1 one goes into the Bohmian interpretation of quantum cosmology à la [**770, 772, 774, 961**] for example (cf. also [**123, 124, 571, 572, 573**]). Thus write $H = \int d^3x(N\mathfrak{H} + N^j\mathfrak{H}_j)$ where for GR with a scalar field

$$\mathfrak{H}_j = -2D_i\pi^i_j\pi_\phi\partial_j\phi;\ \mathfrak{H} = \kappa G_{ijk\ell}\pi^{ij}\pi^{k\ell} + \frac{1}{2}h^{-1/2}\pi^2_\phi + \tag{1.58}$$

$$+h^{1/2}\left[-\kappa^{-1}(R^{(3)} - 2\Lambda) + \frac{1}{2}h^{ij}\partial_i\phi\partial_j\phi + U(\phi)\right]$$

The canonical momentum is

$$\pi^{ij} = -h^{1/2}(K^{ij} - h^{ij}K) = G^{ijk\ell}(\dot{h}_{k\ell} - D_kN_\ell - D_\ell N_k); \tag{1.59}$$

$$K_{ij} = -\frac{1}{2N}(\dot{h}_{ij} - D_i N_j - D_j N_i)$$

K is the extrinsic curvature of the 3-D hypersurface Σ in question with indices lowered and raised via the surface metric h_{ij} and its inverse) and $\pi_\phi = (h^{1/2}/N)(\dot{\phi} - N^j \partial_j \phi)$ is the momentum of the scalar field (D_i is the covariant derivative on Σ). Recall also the deWitt metric

$$G_{ijk\ell} = \frac{1}{2} h^{-1/2}(h_{ik}h_{j\ell} + h_{i\ell}h_{jk} - h_{ij}h_{k\ell}) \tag{1.60}$$

The classical 4-metric and scalar field which satisfy the Einstein equations can be obtained from the Hamiltonian equations

$$ds^2 = -(N^2 - N^i N_i)dt^2 + 2N_i dx^i dt + h_{ij} dx^i dx^j; \tag{1.61}$$

$$\dot{h}_{ij} = \{h_{ij}, H\};\ \dot{\pi}^{ij} = \{\pi^{ij}, H\};\ \dot{\phi} = \{\phi, H\};\ \dot{\pi}_\phi = \{\pi_\phi, H\}$$

One has the standard constraint equations which when put in Bohmian form with $\psi = Aexp(iS/\hbar)$ become

$$-2h_{\ell i} D_j \frac{\delta S(h_{ij}\phi)}{\delta h_{\ell j}} + \frac{\delta S(h_{ij}, \phi)}{\delta \phi}\partial_i \phi = 0;\ -2h_{\ell i} D_j \frac{\delta A(h_{ij}, \phi)}{\delta h_{\ell j}} + \frac{\delta A(h_{ij}, \phi)}{\delta \phi} = 0 \tag{1.62}$$

These depend on the factor ordering but in any case will have the form

$$\kappa G_{ijk\ell} \frac{\delta S}{\delta h_{ij}} \frac{\delta S}{\delta h_{k\ell}} + \frac{1}{2} h^{-1/2} \left(\frac{\delta S}{\delta \phi}\right)^2 + V + Q = 0 \tag{1.63}$$

$$Q = -\frac{\hbar^2}{A}\left(\kappa G_{ijk\ell} \frac{\delta^2 A}{\delta h_{ij} \delta h_{k\ell}} + \frac{h^{-1/2}}{2} \frac{\delta^2 A}{\delta \phi^2}\right)$$

$$\kappa G_{ijk\ell} \frac{\delta}{\delta h_{ij}} \left(A^2 \frac{\delta S}{\delta h_{k\ell}}\right) + \frac{1}{2} h^{-1/2} \frac{\delta}{\delta \phi} \left(A^2 \frac{\delta S}{\delta \phi}\right) = 0 \tag{1.64}$$

Now in the dBB interpretation one has guidance relations

$$\pi^{ij} = \frac{\delta S(h_{ab}, \phi)}{\delta h_{ij}};\ \pi_\phi = \frac{\delta S(h_{ij}, \phi)}{\delta \phi} \tag{1.65}$$

One then develops the Bohmian theory and from (1.65) results

$$\dot{h}_{ij} = 2N G_{ijk\ell} \frac{\delta S}{\delta h_{k\ell}} + D_i N_j + D_j N_i;\ \dot{\phi} = N h^{-1/2} \frac{\delta S}{\delta \phi} + N^i \partial_i \phi \tag{1.66}$$

The question posed now is to find what kind of structure arises from (1.66). The Hamiltonian is evidently $H_Q = \int d^3x \left[N(\mathfrak{H} + Q) + N^i \mathfrak{H}_i\right]$; $\mathfrak{H}_Q = \mathfrak{H} + Q$ and the first question is whether the evolution of the fields driven by H_Q forms a 4-geometry as in classical gravitational dynamics. Various situations are examined and (for Q of a specific form) sometimes the quantum geometry is consistent (i.e. independent of the choice of lapse and shift functions) and forms a nondegenerate 4-geometry (of Euclidean type). However it can also be consistent and not form a nondegenerate 4-geometry. In general, and always when the quantum potential is nonlocal, spacetime is broken and the evolving 3-geometries do not stick together to form a nondegenerate 4-geometry. These are very interesting results and mandate further study.

(4) Next (cf. Section 4.6) one goes to noncommutative (NC) theories folowing [**77, 78, 79, 772**]. First from [**77, 78**] one considers canonical comutation relations $[\hat{X}^\mu, \hat{X}^\nu] = i\theta^{\mu\nu}$ and develops a Bohmian theory for a noncommutative QM (NCQM) via a Moyal product

$$(f * g) = \frac{1}{(2\pi)^n}\int d^m k d^n p e^{i(k_\mu + p_\mu)x^\mu - (1/2)k_\mu\theta^{\mu\nu}p_\nu} f(k)g(p) = \tag{1.67}$$

For $\theta^{0i} = 0$ one has a Hilbert space as in commutative QM with a NC SE

$$i\hbar\frac{\partial\psi(x^i,t)}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi(x^i,t) + V(x^i) * \psi(x^i,t) = \tag{1.68}$$

$$= \frac{\hbar^2}{2m}\nabla^2\psi(x^i,t) + V\left(x^j + i\frac{\theta^{jk}}{2}\partial_k\right)\psi(x^i,t)$$

The operators $\hat{X}^j = x^j + \frac{i\theta^{jk}\partial_k}{2}$ are the observables with canonical coordinates x^i and $\rho d^3x = |\psi|^2 d^3x$ is intepreted as the porbability that the system is in a region of volume d^3x around x at time t. One writes $\psi = Rexp(iS/\hbar)$ and there results

$$\frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2m} + V + V_{nc} + Q_K + Q_I = 0;\ V_{nc} = V\left(x^i - \frac{\theta^{ij}}{2\hbar}\partial_j S\right) - V(x^i); \tag{1.69}$$

$$Q_K = \Re\left(-\frac{\hbar^2}{2m}\frac{\nabla^2\psi}{\psi}\right) - \left(\frac{\hbar^2}{2m}(\nabla S)^2\right) = -\frac{\hbar^2}{2m}\frac{\nabla^2 R}{R};$$

$$Q_I = \Re\left(\frac{V[x^j + (i\theta^{jk}/2)\partial_k]\psi}{\psi}\right) - V\left(x^i - \frac{\theta^{ij}}{2\hbar}\partial_j S\right)$$

One arrives at a formal structure involving

$$\hat{X}^j = x^i + i\theta^{jk}\partial_k/2;\ X^i(t) = x^i(t) - (\theta^{ij}/2\hbar)\partial_j S(x^i(t),t); \tag{1.70}$$

$$\frac{dx^i(t)}{dt} = \left[\frac{\partial^i S(\vec{x},t)}{m} + \frac{\theta^{ij}}{2\hbar}\frac{\partial V(X^k)}{\partial X^j} + \frac{\mathfrak{Q}^i}{2}\right]\Bigg|_{x^i = x^i(t)}$$

One finds then

$$\frac{\partial\rho}{\partial t} + \frac{\partial(\rho\dot{x}^i}{\partial x^i} - \frac{\partial}{\partial x^i}\left[\rho\left(\frac{\theta^{ij}}{2\hbar}\frac{\partial V(X^k)}{\partial X^j} + \frac{\mathfrak{Q}^i}{2}\right)\right] + \Sigma_\theta = 0 \tag{1.71}$$

so for equivariance ($\rho = |\psi|^2$) it is necessary that the sum of the last two terms in (1.71) vanish and when $V(X^i)$ is linear or quadratic this holds. In [**77, 78**] one also looks at NC theory in Kantowski-Sachs (KS) universes and at Friedman-Robertson-Walker (FRW) universes with a conformally coupled scalar field. Many specific situations are examined, especially in a minisuperspace context.

(5) Next (cf. Item 3 in Section 6.1.1 and Item 8 in Section 6.1.2) we consider [**449, 444**]. One gives a new derivation of the SE via the exact uncertainty principle and a formula

$$\tilde{H}_q[P,S] = \tilde{H}_c[P,S] + C\int dx \frac{\nabla P\cdot\nabla P}{2mP} \tag{1.72}$$

for the quantum situation. Consider then the gravitational framework

$$ds^2 = -(N^2 - h^{ij}N_iN_j)dt^2 + 2N_i dx^i dt + h_{ij}dx^i dx^j \tag{1.73}$$

One introduces fluctuations via $\pi^{ij} = (\delta S/\delta h_{ij}) + f^{ij}$ and arrives at a WDW equation

$$\left[-\frac{\hbar^2}{2}\frac{\delta}{\delta h_{ij}}G_{ijk\ell}\frac{\delta}{\delta h_{k\ell}} + V\right]\Psi = 0 \tag{1.74}$$

Note that an operator ordering is implicit and thus ordering ambiguities do not arise (similarly for quantum particle motion). The work here in [**449, 444**] is significant and very interesting; it is developed in some detail in Section 4.7.

(6) In Section 4.1 we followed work of M. Israelit and N. Rosen on Dirac-Weyl geometry (see in particular [**498, 499, 817**]). Recall that in Weyl geometry $g_{\mu\nu} \to \tilde{g}_{\mu\nu} = exp(2\lambda)g_{\mu\nu}$ is a gauge transformation and for a vector $\vec{B}$ of length B one has $dB = Bw_\nu dx^\nu$ where $w_\nu \sim \phi_\nu$ is the Weyl vector. The Weyl connection coefficients are

$$\Delta B^\lambda = B^\sigma K^\lambda_{\sigma\mu\nu}dx^\mu\delta x^\nu;\ \Delta B = BW_{\mu\nu}dx^\mu\delta x^\nu \tag{1.75}$$

and under a gauge transformation $w_\mu \to \bar{w}_\mu = w_\mu + \partial_\mu\lambda$. One writes $W_{\mu\nu} = w_{\mu,\nu} - w_{\nu,\mu}$ (where commas denote partial derivatives). The Dirac-Weyl action here is given via a field β ($\beta \to \bar{\beta} = exp(-\lambda)\beta$ under a gauge transformation) in the form

$$I = \int [W^{\lambda\sigma}W_{\lambda\sigma} - \beta^2 R + \beta^2(k-6)w^\sigma w_\sigma + 2(k-6)\beta w^\sigma\beta_{,\sigma} + \tag{1.76}$$
$$+k\beta_{,\underline{\sigma}}\beta_{,\sigma} + 2\Lambda\beta^4 + L_M]\sqrt{-g}d^4x$$

Note here the difference in appearance from (1.37) or the Dirac form in Appendix D; these are all equivalent after suitable adjustment (cf. Remark 6.1). Under parallel transport $\Delta B = BW_{\mu\nu}dx^\mu\delta x^\nu$ so one takes $W_{\mu\nu} = 0$ via $w_\nu = \partial_\nu w$ and we have what is called an integrable Weyl geometry with generating elements $(g_{\mu\nu}, w, \beta)$. Further set $b_\mu = \partial_\mu(log(\beta)) = \beta_\mu/\beta$ and use a modified Weyl connection vector $W_\mu = w_\mu + b_\mu$. Then varying (1.76) in w and $g_{\mu\nu}$ gives

$$2(\kappa\beta^2 W^\nu)_{;\nu} = S;\ G^\nu_\mu = -8\pi\frac{T^\nu_\mu}{\beta^2} + \tag{1.77}$$
$$+16\pi\kappa\left(W^\nu W_\mu - \frac{1}{2}\delta^\nu_\mu W^\sigma W_\sigma\right) + 2(\delta^\nu_\mu b^\sigma_{;\sigma} - b^\nu_{;\mu}) + 2b^\nu b_\mu + \delta^\nu_\mu b^\sigma_\sigma - \delta^\nu_\mu\beta^2\Lambda$$

where S is the Weyl scalar charge $16\pi S = \delta L_M/\delta w$, G^ν_μ is the Einstein tensor, and the energy momentum tensor of ordinary matter is

$$8\pi\sqrt{-g}T^{\mu\nu} = \delta(\sqrt{-g}L_M)/\delta g_{\mu\nu} \tag{1.78}$$

Finally variation in β gives an equation for he β field

$$R + k(b^\sigma_{;\sigma} + b^\sigma b_\sigma) = 16\pi\kappa(w^\sigma w_\sigma - w^\sigma_{;\sigma}) + 4\beta^2\Lambda + 8\pi\beta^{-1}B \tag{1.79}$$

(here $16\pi B = \delta L_M/\delta\beta$ is the Dirac charge conjugate to β). Note

$$\delta I_M = 8\pi\int(T^{\mu\nu}\delta g_{\mu\nu} + 2S\delta w + 2B\delta\beta)\sqrt{-g}d^4x \tag{1.80}$$

yielding the energy momentum relation $T^\lambda_{\mu;\lambda} - Sw_\mu - \beta Bb_\mu = 0$. Actually via (1.77) with $S + T = \beta B$ one obtains again (1.79) which is seen therefore as a corollary and not an independent equation. One derives now conservation laws etc. and following [**817**] produces an equation of motion for a test particle. Thus consider matter made up on identical particles of rest mass m and Weyl scalar charge q_s, being in the stage of a pressureless gas so $T^{\mu\nu} = \rho U^\mu U^\nu$ where U^ν is the 4-velocity and note also $T^\lambda_{\mu;\lambda} - Tb_\mu = SW_\mu$. Then one arrives at

$$\frac{dU^\mu}{ds} + \left\{\begin{array}{c}\mu \\ \lambda\ \sigma\end{array}\right\}U^\lambda U^\sigma = \left(b_\lambda + \frac{q_s}{m}W_\lambda\right)(g^{\mu\lambda} - U^\mu U^\lambda) \tag{1.81}$$

Further a number of illustrations are worked out involving the creation of mass like objects from Weyl-Dirac geometry, in a FRW universe for example (i.e. an external observer sitting in Riemannian spacetime would recognize the object as massive). Cosmological models are also constructed with the Weyl field serving to create matter. The treatment is extensive and profound.

REMARK 7.1.1. We have encountered Dirac-Weyl-Bohm (DWB) in Section 2.1 (Section 7.1.2, Item 5) and Dirac-Weyl geometry in Section 4.1 and Appendix D (Section 7.1.3, Item 6). The formulations are somewhat different and we try now to compare certain features. In Section 7.1.2 one has

$$\mathfrak{A} = \int d^4x\sqrt{-g}(F_{\mu\nu}F^{\mu\nu} - \beta^2\ {}^W\mathcal{R} + (\sigma+6)\beta_{;\mu}\beta^{;\mu} + \mathfrak{L}_{matter} \tag{1.82}$$

for the action; in Section 7.1.3 the action is

$$I = \int[W^{\lambda\sigma}W_{\lambda\sigma} - \beta^2 R + \beta^2(k-6)w^\sigma w_\sigma + 2(k-6)\beta w^\sigma\beta_{,\sigma} + \tag{1.83}$$
$$+k\beta_{,\underline{\sigma}}\beta_{,\sigma} + 2\Lambda\beta^4 + L_M]\sqrt{-g}d^4x$$

and in Appendix D we have (for the simplest vacuum equations and $\mathfrak{g} = \sqrt{-g}$)

$$I = \int[(1/4)F_{\mu\nu}F^{\mu\nu} - \beta^2 R + 6\beta^\mu\beta_\mu + c\beta^4]\mathfrak{g}d^4x \tag{1.84}$$

We recall the idea of co-covariant derivative from Appendix D where for a scalar S of Weyl power n one has $S_{*\mu} = S_{;\mu} = nw_\mu S \equiv S_\mu - nw_\mu S$ ($S_\mu = \partial_\mu S$) and I in

(1.84) is originally

$$I = \int [(1/4)F_{\mu\nu}F^{\mu\nu} - \beta^{2*}R + k\beta^{*\mu}\beta_{*\mu} + c\beta^4]\mathfrak{g}d^4x \tag{1.85}$$

However β is a co-scalar of weight -1 and $\beta^{*\mu}\beta_{*\mu} = (\beta^\mu + \beta w^\mu)(\beta_\mu + \beta w_\mu)$ so using

$$-\beta^{2*}R + k\beta^{*\mu}\beta_{*\mu} = -\beta^2 R + k\beta^\mu\beta_\mu + (k-6)\beta^2\kappa^\mu\kappa_\mu + \tag{1.86}$$

$$+6(\beta^2\kappa^\mu)_{:\mu} + (2k-12)\beta\kappa^\mu\beta_\mu$$

one obtains (1.84) for $k = 6$. Further recall that $W_\mu = w_\mu + \partial_\mu log(\beta)$ so $W_{\mu\nu} = F_{\mu\nu}$ and c in (1.84) corresponds to Λ in (1.83). The notation ${}^W\mathcal{R}$ in (1.82) is the same as R and $\beta_{;\mu} \sim \beta_\mu = \partial_\mu\beta$ so $(\sigma + 6) = 6$ provides a complete identification (modulo matter terms to be added in (1.84)); note $\sigma = k - 6$ from [**817**]. Now in [**872, 873**] one takes a Dirac-Weyl action of the form (1.82) and relates it to a Bohmian theory as in Section 3.2.1. The same arguments hold also for $\sigma = 0$ here with

$$\beta \sim \mathfrak{M};\ \frac{8\pi\mathfrak{T}}{R} \sim m^2;\ \alpha = \frac{\hbar^2}{m^2c^2} \sim -\frac{6}{R};\ \nabla_\nu\mathfrak{T}^{\mu\nu} - \mathfrak{T}\frac{\nabla^\mu\beta}{\beta} = 0 \tag{1.87}$$

Note also $16\pi\mathfrak{T} = \beta\psi$ where $\psi = \delta L_M/\delta\beta \sim 16\pi B$ so $B \sim \mathfrak{T}/\beta$. One assumes here that $16\pi J_\mu = \delta L_M/\delta w_\mu = 0$ where $w_\mu \sim \phi_\mu$.

REMARK 7.1.2. We recall from Section 3.2 that $\beta \sim \mathfrak{M}$ and $\mathfrak{M}^2/m^2 = exp(\mathfrak{Q})$ is a conformal factor. Further for $\beta_0 \to \beta_0 exp(-\Xi(x))$ one has $w_\mu \to w_\mu + \partial_\mu\Xi$ where $-\Xi = log(\beta/\beta_0)$ showing an interplay between mass and geometry. Recall also the relation $\nabla_\mu(\beta w^\mu + \beta\nabla^\mu\beta) = 0$. This indicates a number of connections between the quantum potential, geometry, and mass. Hence virtually any results in Dirac-Weyl theory models will involve the quantum potential. This is made explicit in Section 3.2 from [**873**] and could be developed for the examples and theory from [**499**] once wave functions and Bohmian ideas are inserted.

1.4. GEOMETRIC PHASES. We go now to [**283**] for some remarks on geometric phase and the quantum potential. One refers back here to geometric phases of Berry [**108**] and Levy-Leblond [**603**] for example where the latter shows that when a quanton propagates through a tube, within which it is confined by impenetrable walls, it acquires a phase when it comes out of the tube. Thus consider a tube with square section of side a and length L. Before entering the tube the quanton's wave function is $\phi = exp(ipx/\hbar)$ where p is the initial momentum. In the tube the wave function has the form

$$\psi = Sin\left(n_x\pi\frac{x}{a}\right)Sin\left(n_y\pi\frac{y}{a}\right)exp(ip'x/\hbar) \tag{1.88}$$

with appropriate transverse boundary conditions. After entering the tube the energy E of the quanton is unchanged but satisfies

$$E = \frac{(p')^2}{2m}(n_x^2 + n_y^2)\frac{\pi^2}{2ma} \tag{1.89}$$

For the simplest case $n_x = n_y = 1$ it was found that after the quanton left the tube there was an additional phase

$$\Delta\Phi = \frac{\pi^2\hbar^2}{pa^2}L \tag{1.90}$$

Subsequently Kastner [**539**] related this to the quantum potential that arises in the tube. Thus let the wave function in the tube be $Rexp[(iS/\hbar)+(ipx/\hbar)]$ in polar form. The eventual changes in the phase of the wave function, due to the tube, are now concentrated in S. In order to single out the influence of the tube on the wave function write $\psi_1 = \psi exp(ipx/\hbar)$ and the quantum potential corresponding to ψ is then

$$Q = -\frac{\hbar^2}{2m}\frac{\Delta R}{R} = \frac{\pi^2\hbar^2}{ma^2} \tag{1.91}$$

Now turn to the laws of parallel transport where for the Berry phase the law of parallel transport for the wave function is (cf. [**892**])

$$\Im < \psi|\dot{\psi} >= 0 \tag{1.92}$$

For the Levy-Leblond phase the law of parallel transport is given by

$$\Im < \psi|\dot{\psi} >= -\frac{1}{\hbar}Q|\psi|^2 \tag{1.93}$$

In this approach the wave function acquires an additional phase after the quanton has left the tube in the form

$$\psi(t+\Delta t) = exp(-iQ\Delta t/\hbar)\psi(t) \tag{1.94}$$

which after expansion in Δt leads to the law of parallel transport in (1.93). Indeed

$$Q\Delta t = \frac{\pi^2\hbar^2}{ma^2}\Delta t = \frac{\pi^2\hbar^2}{ma^2}\frac{mL}{p} = \frac{\pi^2\hbar^2}{pa^2}L = \Delta\Phi \tag{1.95}$$

If we use the polar form for the wave function (1.93) gives $(\partial S/\partial t) = -Q$ and this means that this new law of parallel transport eliminates the quantum potential from the quantum HJ equation. The whole quantum information is now carried by the phase of the wave function. One can see that the nature of this phase is quite different from Berry's phase; it is related to the presence or not of constraints in the system (in this case the tube).

Now consider a quite different type of constrained system where again a new geometric phase will arise. Look at a quantum particle constrained to move on a circle. The wave function has the form $\psi \sim sin(ns/\rho_0)$ where ρ_0 is the radius of the circle and s is the arc length with origin at a tangent point. Then the wave function will have a node at this tangent point. For a circle the value $n = 1/2$ is also allowed (cf. [**467, 570**]) and the corresponding quantum potential for $n = 1/2$ is

$$Q = \frac{h^2}{8m\rho_0^2} \tag{1.96}$$

which is exactly equal to the constant E_0 appearing in the Hamiltonian for a particle on a circle with radius ρ_0 following the Dirac quantization procedure for constrained systems (cf. [**846**]). The phase which a quanton would acquire traveling along the circle is then

$$Q\frac{2\pi\rho_0 m}{p} = \frac{\pi\hbar^2}{4\rho_0 p} \tag{1.97}$$

Note that if the circle becomes very small then the geometrical phase can not get bigger than $\sim (\hbar/m)$. This limit is imposed by the Heisenberg uncertainty relation $\rho_0 p \sim \hbar$. This is not the case for the Levy-Leblond phase which can get very large provided $L >> a$.

1.5. ENTROPY AND CHAOS. Connections of the quantum potential to Fisher information have already been recalled in e.g. Section 7.1.1 and we recall here from Chapter 6 a few matters.

(1) An extensive discussion relating Fisher information (as a "mother" information) to various forms of entropy is developed in Chapter 6 and this gives implicitly at least many relations between entropy, kinetic theory, uncertainty, and the quantum potential.
(2) A particular result of interest in Section 6.2.1 shows how the quantum potential acts as a constraining force to prevent deterministic chaos.

2. HYDRODYNAMICS AND GEOMETRY

We mentioned briefly some hydrodynamical aspects of the SE in Sections 1.1 and 1.3.2 and return to that now following [**294**]. Here one wants to limit the role of statistics and measurement to unveil some geometric features of the so called Madelung approach. Thus, with some repetition from Section 1.1, consider a SE $(\hbar/i)\psi_t + H(x,(\hbar/i)\nabla)\psi = 0$ with $\psi = Rexp(iS/\hbar)$ to arrive at

$$\frac{\partial S}{\partial t} + H(x,\nabla S) - \frac{\hbar^2}{2m}\frac{\Delta R}{R} = 0;\ \frac{\partial P}{\partial t} + \frac{\partial}{\partial x^i}(P\dot{x}^i) = 0;\ \dot{x}^i = \left[\frac{\partial H}{\partial p_i}\right]_{p=\nabla S} \tag{2.1}$$

(where $P = R^2$), and Madelung equations of the form (cf. (1.5))

$$\frac{\partial S}{\partial t} + H(x,\nabla S) - \frac{\hbar^2}{2m}\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}} = 0;\ \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x^i}(\rho\dot{x}^i) = 0 \tag{2.2}$$

where, in a continuum picture, $\rho = mP$ is the mass density of an extended particle whose shape is dictated by P. Setting $v^i = \dot{x}^i$ one has then an Euler equation of the form

$$\frac{\partial}{\partial t}(\rho v^i) + \frac{\partial}{\partial x^k}(\rho v^i v^k) = -\frac{\rho}{m}\frac{\partial V}{\partial x^i} + \frac{\partial}{\partial x^k}\tau^{ik} \tag{2.3}$$

Following [**924**] one has expressed the quantum force term here as the divergence of a symmetric "quantum stress" tensor

$$\tau_{ij} = \left(\frac{\hbar}{2m}\right)^2 \rho\frac{\partial^2(log(\rho))}{\partial x^i\partial x^j};\ p_i = \frac{\hbar^2}{2m^2}\rho\frac{\partial^2\sigma}{\partial(x^i)^2} \tag{2.4}$$

where p_i denotes diagonal elements or principal stresses expressed in normal coordinates (with $\rho = exp(2\sigma)$). The stress p_i is tension like (resp. pressure like) if

$p_i > 0$ (resp. $p_i < 0$) and the mean pressure is

$$\bar{p} = -\frac{1}{3}Tr(\tau_{ij}) = -\frac{\hbar^2}{6m^2}\rho\Delta\sigma \tag{2.5}$$

In classical hydrodynamics negative pressures are often associated with cavitation which involves the formation of topological defects in the form of bubbles. For an ideal fluid one would need $\tau_{ij} = -\bar{p}\delta_{oj}$ and this occurs if and only if the mass density is Gaussian $\sigma \propto -x^i x_i$ in which case $\bar{p} \propto (\hbar^2/2m)\rho$. Generally the stress tensor will not be isotropic, and not an ideal fluid; moreover if one had a viscous fluid one would expect τ_{ij} to be coupled to the rate of deformation tensor (derived from $D\mathbf{v}$). Since this does not occur one does not call this form of matter a fluid but rather a Madelung continuum, corresponding to something like an inviscid fluid which also supports shear stresses, whereas the Gaussian wave packet of QM corresponds to an ideal compressible irrotational fluid medium.

If now one adds time as the zeroth coordinate and extends the velocity vector by $v^0 = 1$ then, defining the energy momentum tensor as

$$\mathfrak{T}^{\mu\nu} = \rho\left[v^\mu v^\nu - \left(\frac{\hbar}{2m}\right)^2 \frac{\partial^2(log(\rho))}{\partial x^\mu \partial x^\nu}\right] \tag{2.6}$$

then the Euler and continuity equations can be combined in the form $\partial\mathfrak{T}^{\mu\nu}/\partial x^\mu = -(\rho/m)\partial^\nu V$; this is somewhat misleading since it is based on a nonrelativistic approach but it leads now to the relativistic theory. First start with the KG equation

$$\left[-\hbar^2\eta^{\mu\nu}\frac{\partial^2}{\partial x^\mu \partial x^\nu} + m_0^2c^2\right]\psi = 0 \tag{2.7}$$

For $\psi = Rexp(iS/\hbar)$ one gets now

$$\eta^{\mu\nu}\frac{\partial S}{\partial x^\mu}\frac{\partial S}{\partial x^\nu} + m_0^2c^2 - \hbar^2\frac{\Box R}{R} = 0;\ \eta^{\mu\nu}\frac{\partial}{\partial x^\mu}\left(P\frac{\partial S}{\partial x^\nu}\right) = 0 \tag{2.8}$$

Define now the 4-velocity, rest mass energy, energy momentum, and stress tensor via

$$u_\mu = \frac{1}{m_0}\frac{\partial S}{\partial x^\mu};\ \rho = m_0 P;\ p_\mu = \rho u_\mu; \tag{2.9}$$

$$\tau_{\mu\nu} = \left(\frac{\hbar}{2m_0}\right)^2\frac{\partial^2(log(\rho))}{\partial x^\mu \partial x^\nu};\ \mathfrak{T}_{\mu\nu} = \rho[u_\mu u_\nu + \tau_{\mu\nu}]$$

to arrive at relativistic equations for the medium described by ρ and u_μ in the form (♣) $\partial^\mu\mathfrak{T}_{\mu\nu} = 0$ and $\partial^\mu p_\mu = 0$ (the second equation is an incompressibility equation and this does not contradict the nonrelativistic compressibility of the medium since in relativity incompressibility in fluid media is equivalant to an infinite speed of light corresponding to rigidity in solid media).

One then erects an elegant mathematical framework involving spacetime foliations related to the complex character of ψ (see also the second paper in [**294**] on foliated cobordism, etc.). This is lovely but rather too abstract for the style of this book so we will not try to reproduce it here; we can however skip to some calculations involving the geometric origin of the quantum potential. Thus consider the consequences of choosing a scale of unit norm via the function $\sqrt{\rho}$. One takes a conformally related metric $\bar{g} = \Omega^2 g$ on a manifold M (where $\Omega^2 > 0$) and, writing $\Omega = exp(\sigma)$, one obtains the following formulas for the Levi-Civita connection, Ricci curvature, and scalar curvature

$$\bar{\Gamma}^i_{jk} = \Gamma^i_{jk} = \delta^i_j \partial_k \sigma + \delta^i_k \partial_j \sigma - g_{jk} g^{i\ell} \partial_\ell \sigma; \tag{2.10}$$

$$\bar{R}_{ij} = R_{ij} - (n-2)\sigma_{ij} - [\Delta\sigma + (n-2)(\partial^k \sigma \partial_k \sigma)] g_{ij}$$

$$\bar{R} = e^{-2\sigma}[R - 2(n-1)\Delta\sigma - (n-1)(n-2)(\partial^i \sigma \partial_i \sigma)]$$

where $\sigma_{ij} = \partial_i \partial_j \sigma - \partial_j \sigma \partial_i \sigma$. Now if the constant m is replaced by the function ρ then one must contend with the derivatives $\partial_i \rho$ and for $\rho = exp(2\sigma)$ the Minkowski metric will be deformed from $g = \eta$, $\Gamma^i_{jk} = 0$, $R_{ij} = R = 0$ to

$$\bar{\Gamma}^i_{jk} = \delta^i_j \partial_k \sigma + \delta^i_k \partial_j \sigma - \eta_{jk} \eta^{i\ell} \partial_\ell \sigma; \ \bar{R}_{ij} = -2\sigma_{ij} - [\Box\sigma + 2(\partial^k \sigma \partial_k \sigma)]\eta_{ij} \tag{2.11}$$

$$\bar{R} = -6e^{-2\sigma}[\Box\sigma + (\partial^i \sigma \partial_i \sigma]$$

Putting Ω back into the equation for scalar curvature one obtains

$$\bar{R} = -\frac{6}{\Omega^2}\left(\frac{\Box\Omega}{\Omega}\right); \ \frac{\Box\sqrt{\rho}}{\sqrt{\rho}} = \frac{\Box\Omega}{\Omega} = -\frac{1}{6}\bar{R}\Omega^2 = -\frac{1}{6}\rho\bar{R} \tag{2.12}$$

This identifies the quantum potential as a mass density times a scalar curvature and resembles some results obtained earlier from [**840, 841**] for example (cf. Section 3.3 and 3.3.2). One has also

$$\partial^i \bar{R}_{ij} = \partial^i \left(\frac{1}{2} g_{ij} \bar{R}\right) = \frac{1}{2}\partial^j \bar{R} \Rightarrow \partial^i \bar{R}_{ij} = -3\partial^i \tau_{ij} \tag{2.13}$$

so the Takabayashi stress tensor differs from the Ricci curvature only by a term of vanishing divergence. Hence there is no loss of generality in using the Ricci curvature of $g = \rho\eta$ as the stress tensor since both define the same force field; this means in particular that one is dealing with principal curvatures instead of principal stresses. To extend all this to a more general Lorentz manifold one notes that under a conformal change of spacetime metric to the energy metric the Einstein tensor becomes

$$\bar{G}_{\mu\nu} = \bar{R}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{R} = G_{\mu\nu} - 2\sigma_{\mu\nu} + [2\Box\sigma + \partial^\lambda \sigma \partial_\lambda \sigma] g_{\mu\nu} \tag{2.14}$$

Thus it seems to assume that this implies a quantum correction to the Einstein equation

$$G_{\mu\nu} + \left(\Box\sigma + \frac{1}{2}\partial^\lambda \sigma \partial_\lambda \sigma\right) g_{\mu\nu} = 8\pi G T_{\mu\nu} + 2\sigma_{\mu\nu} \tag{2.15}$$

2.1. PARTICLE AND WAVE PICTURES. An interesting discussion of hydrodynamic features of QM, electrodynamics, and Bohmian mechanics appears in [**474**] and we will sketch more or less thoroughly a few ideas here. Thus a hydrodynamic model of QM provides an interpretation of two pictures, wave mechanical (Eulerian) and particle (Lagrangian), and the two versions of QM have associated Hamiltonian formulations that are connected by a canonical transformation. This gives a new and precise meaning to the notion of wave-particle duality. However it is necessary to distinguish the dBB corpuscle from a fluid particle. Consider a fluid as a continuum of particles with history encoded in the position variables $q(a,t)$ where each particle is distinguished by a continuous vector label a. The motion is continuous in that the mapping from a-space to q-space is single valued and differentiable with inverse $a(q,t)$. Let $\rho_0(a)$ be the initial quantum probability density with $\int \rho_0(a)d^3a = 1$. Introduce a mass parameter m so that the mass of an elementary volume d^3a attached to the point a is given by $m\rho_0(a)d^3a$. Note $\int m\rho_0(a)d^3a = m$ so this is a total mass of the system. The conservation of mass of a fluid element in the course of its motion is

$$m\rho(q(a,t))d^3q(a,t) = m\rho_0(a)d^3a;\ \rho(a,t) = J^{-1}(a,t)\rho_0(a); \tag{2.16}$$

$$J = \frac{1}{3!}\epsilon_{ijk}\epsilon_{\ell mn}\frac{\partial q_i}{\partial a_\ell}\frac{\partial q_j}{\partial a_m}\frac{\partial q_k}{\partial a_n}$$

J is the Jacobian of the transformation between the two sets of coordinates and ϵ_{ijk} is the completely antisymmetric tensor with $\epsilon_{123} = 1$. Let V be the potential of an external classical body force and U the internal potential energy of the fluid due to interparticle interactions. Here assume U depends on $\rho(q)$ and its first derivatives and hence via (2.16) on the second order derivatives of q with respect to a. The Lagrangian is then (with integrand $\ell = [\cdots]$)

$$L[q,q_t,t] = \int \left[\frac{1}{2}m\rho_0(a)\left(\frac{\partial q(a,t)}{\partial t}\right)^2 - \rho_0(a)U(\rho) - \rho_0(a)V(q(a))\right] d^3a \tag{2.17}$$

(one substitutes for ρ from (2.16)). It is the action of the conservative force derived from U on the trajectories that represents the quantum effects here. They are characterized by the following choice for U, motivated by the known Eulerian expression for internal energy,

$$U = \frac{\hbar^2}{8m}\frac{1}{\rho^2}\frac{\partial\rho}{\partial q_i}\frac{\partial\rho}{\partial q_i} = \frac{\hbar^2}{8m}\frac{1}{\rho_0^2}J_{ij}J_{ik}\frac{\partial}{\partial a_j}\left(\frac{\rho_0}{J}\right)\frac{\partial}{\partial a_k}\left(\frac{\rho_0}{J}\right); \tag{2.18}$$

$$\frac{\partial}{\partial q_i} = J^{-1}J_{ij}\frac{\partial}{\partial a_j};\ J_{i\ell} = \frac{\partial J}{\partial(\partial q_i/\partial a_\ell)} = \frac{1}{2}\epsilon_{ijk}\epsilon_{\ell mn}\frac{\partial q_j}{\partial a_m}\frac{\partial q_k}{\partial a_n};\ \frac{\partial q_k}{\partial a_j}J_{ki} = J\delta_{ij}$$

Thus $J_{i\ell}$ is the cofactor of $\partial q_i/\partial a_\ell$. The interaction in the quantum case is not conceptually different from classical fluid dynamics but differs in that the order of derivative coupling of the particles is higher than in a classical equation of state. The Euler-Lagrange equations for the coordinates are

$$\frac{\partial}{\partial t}\frac{\partial L}{\partial(\partial q_i(a,t)/\partial t)} - \frac{\delta L}{\delta q_i(a)} = 0; \tag{2.19}$$

$$\frac{\delta L}{\delta q_i} = \frac{\partial \ell}{\partial q_i} - \frac{\partial}{\partial a_j}\frac{\partial \ell}{\partial(\partial q_i/\partial a_j)} + \frac{\partial^2}{\partial a_j \partial a_k}\frac{\partial \ell}{\partial(\partial^2 q_i/\partial a_j \partial a_k)}$$

which yield

$$m\rho_0(a)\frac{\partial^2 q_i(a)}{\partial t^2} = -\rho_0(a)\frac{\partial V}{\partial q_i} - \frac{\partial W_{ij}}{\partial a_j}; \tag{2.20}$$

$$W_{ij} = -\rho_0(a)\frac{\partial U}{\partial(\partial q_i/\partial a_j)} + \frac{\partial}{\partial a_k}\left(\rho_0(a)\frac{\partial U}{\partial(\partial^2 q_i/\partial a_j \partial a_k)}\right)$$

This has the form of Newton's second law and instead of giving an explicit form for W_{ij} one uses a more useful tensor σ_{ij} defined via $W_{ik} = J_{jk}\sigma_{ij}$ where σ_{ij} is the analogue of the classical pressure tensor $p\delta_{ij}$. Using (2.18) one can invert to obtain

(2.21)

$$\sigma_{ij} = J^{-1}W_{ik}\frac{\partial q_j}{\partial a_k} = \frac{\hbar^2}{4mJ^3}J_{ik}\left[\rho_0^{-1}J_{j\ell}\frac{\partial \rho_0}{\partial a_k}\frac{\partial \rho_0}{\partial a_\ell} + (J^{-1}J_{j\ell}J_{mn} - J_{jm}J_{\ell n})\times\right.$$

$$\times\frac{\partial \rho_0}{\partial a_\ell}\frac{\partial^2 q_m}{\partial a_k \partial a_n} - J_{j\ell}\frac{\partial^2 \rho_0}{\partial a_k \partial a_\ell} + \rho_0(J^{-1}J_{mn}J_{jr\ell s} + J^{-1}J_{j\ell}J_{mrns} - 2J^{-2}J_{j\ell}J_{mn}J_{rs})\times$$

$$\left.\times\frac{\partial^2 q_r}{\partial a_k \partial a_s}\frac{\partial^2 q_m}{\partial a_\ell \partial a_n} + \rho_0 J^{-1}J_{j\ell}J_{mn}\frac{\partial^3 q_m}{\partial a_k \partial a_\ell \partial a_n}\right];$$

$$J_{jm\ell n} = \frac{\partial J_{j\ell}}{\partial(\partial q_m/\partial a_n)} = \epsilon_{jmk}\epsilon_{\ell nr}\frac{\partial q_k}{\partial a_r}$$

One checks that σ_{ij} is symmetric and the equation of motion of the a^{th} particle moving in the field of the other particles and the external force is then

$$m\rho_0(a)\frac{\partial^2 q_i(a)}{\partial t^2} = -\rho_0(a)\frac{\partial V}{\partial q_i} - J_{kj}\frac{\partial \sigma_{ik}}{\partial a_j};\ \frac{\partial J_{ij}}{\partial a_j} = 0 \tag{2.22}$$

(the latter equation is an identity used in the calculation). The result in (2.22) is the principal equation for the quantum Lagrangian method; its solutions, subject to specification of $\partial q_{i0}/\partial t$, lead to solutions of the SE. Multiplying by $\partial q_i/\partial a_k$ one obtains the Lagrangian form

$$m\rho_0(a)\frac{\partial^2 q_i(a)}{\partial t^2}\frac{\partial q_i}{\partial a_k} = -\rho_0(a)\frac{\partial V}{\partial a_k} - \frac{\partial q_i}{\partial a_k}J_{kj}\frac{\partial \sigma_{ik}}{\partial a_j}; \tag{2.23}$$

$$p_i(a) = \frac{\partial L}{\partial(\partial q_i(a)/\partial t)} = m\rho_0(a)\frac{\partial q_i(a)}{\partial t}$$

A Hamiltonian form can also be obtained via the canonical field momenta $p_i(a) = \partial L/\partial(\partial q_i(a)/\partial t) = m\rho_0(a)(\partial q_i(a)/\partial t)$ with

(2.24)

$$H = \int p_i(a)\frac{\partial q_i(a)}{\partial t}d^3a - L = \int\left[\frac{p(a)^3}{2m\rho_0(a)} + \rho_0(a)U(J^{-1}\rho_0) + \rho_0(a)V(q(a))\right]d^3a$$

Hamilton's equations via Poisson brackets $\{q_i(a), q_j(a')\} = \{p_i(a), p_j(a')\} = 0$ and $\{q_i(a), p_j(a')\} = \delta_{ij}(a - a')$ are $\partial_t q_i(a) = \delta H/\delta p_i(a)$ and $\partial_t p_i(a) = -\delta H/\delta q_i(a)$ which when combined reproduce (2.22). Now to obtain a flow that is representative of QM one restricts the initial conditions for (2.22) to something corresponding to quasi-potential flow which means (♦) $\partial q_{i0}/\partial t = (1/m)(\partial S_0(a)/\partial a_i)$. However the flow is not irrotational everywhere because the potential $S_0(a)$ obeys the quantization condition

$$\oint_C \frac{\partial q_{i0}(a)}{\partial t} da_i = \oint_C \frac{1}{m}\frac{\partial S_0(a)}{\partial a_i} da_i = \frac{n\hbar}{m} \ \ (n \in \mathbf{Z}) \tag{2.25}$$

where C is a closed curve composed of material paricles. If it exists vorticity occurs in nodal regions (where the density vanishes) and it is assumed that C passes through a region of good fluid where $\rho_0 \neq 0$. To show that these assumptions imply motion characteristic of QM one demonstrates that they are preserved by the dynamical system. One first puts (2.23) into a more convenient form. Thus, using (2.16), the stress tensor (2.21) takes a simpler form

$$\sigma_{ij} = \frac{\hbar^2}{4m}\left(\frac{1}{\rho}\frac{\partial\rho}{\partial q_i}\frac{\partial\rho}{\partial q_j} - \frac{\partial^2\rho}{\partial q_i \partial q_j}\right) \tag{2.26}$$

Using

$$\frac{1}{\rho}\frac{\partial\sigma_{ij}}{\partial q_j} = \frac{\partial V_Q}{\partial q_i};\ V_Q = \frac{\hbar^2}{4m\rho}\left(\frac{1}{2\rho}\frac{\partial\rho}{\partial q_i}\frac{\partial\rho}{\partial q_i} - \frac{\partial^2\rho}{\partial q_i\partial q_i}\right) \tag{2.27}$$

(note V_Q is the dBB quantum potential) one sees that (2.23) can also be simplified as

$$m\frac{\partial^2 q_i}{\partial t^2}\frac{\partial q_i}{\partial a_k} = \frac{\partial}{\partial a_k}(V + V_Q) \tag{2.28}$$

Now integrate this equation between limits $(0,t)$ to get

$$m\frac{\partial q_i}{\partial t}\frac{\partial q_i}{\partial a_k} = m\frac{\partial q_{i0}}{\partial t} + \frac{\partial\chi(a,t)}{\partial a_k};\ \chi = \int_0^t \left(\frac{1}{2}m\left(\frac{\partial q}{\partial t}\right)^2 - V - V_Q\right) dt \tag{2.29}$$

Then using (♦) one has

$$\frac{\partial q_i}{\partial t}\frac{\partial q_i}{\partial a_k} = \frac{1}{m}\frac{\partial S}{\partial a_k};\ S(a,t) = S_0(a) + \chi(a,t) \tag{2.30}$$

with initial conditions $q = a$, $\chi_0 = 0$. To obtain the q-components multiply by $J^{-1}J_{ik}$ and use (2.18) to get (♠) $\partial q_i/\partial t = (1/m)(\partial S/\partial q_i)$ where $S = S(a(q,t),t)$. Thus the velocity is a gradient for all time and (♠) is a form of the law of motion. Correspondingly one can write (2.28) as

$$m\frac{\partial^2 q_i}{\partial t^2} = -\frac{\partial}{\partial q_i}(V + V_Q) \tag{2.31}$$

This puts the fluid dynamical law of motion (2.22) in a form of Newton's law for a particle of mass m. Note that the motion is quasi potential since the value (2.25) is preserved, i.e. (•) $\partial_t \oint_{C(t)} (\partial q_i/\partial t) dq_i = 0$ where $C(t)$ is the evolute of the material particles conposing C (cf. [**113**]). To obtain the equation governing S use the chain rule $F_t|_a = \partial_t F|_q + (\partial_t q_i)(\partial F/\partial q_i)$ and since $\chi = S - S_0$ from (2.30) one has

$$\left.\frac{\partial\chi}{\partial t}\right|_a = \left.\frac{\partial S}{\partial t}\right|_q + \frac{\partial q_i}{\partial t}\frac{\partial S}{\partial q_i} - \left(\left.\frac{\partial S_0}{\partial t}\right|_q + \frac{\partial q_i}{\partial t}\frac{\partial S_0}{\partial q_i}\right) \tag{2.32}$$

The two terms in the bracket sum to $\partial S_0(a)/\partial t = 0$ and using (♠) one obtains $(\partial\chi/\partial t)|_a = (\partial S/\partial t)|_q + m(\partial q/\partial t)^2$. Hence from (2.29) and (♠) one has

$$\frac{\partial\chi}{\partial t}\bigg|_a = \frac{1}{2}m\left(\frac{\partial q}{\partial t}\right)^2 - V - V_Q \Rightarrow \frac{\partial S}{\partial t} + \frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 + V + V_Q = 0 \tag{2.33}$$

This is the quantum HJ equation and one has shown that the equations (2.16), (2.22), and (♦) are equivalent to the 5 equations (2.16), (2.30), and (2.33); they determine the functions (q_i, ρ, S). Note that although the particle velocity is orthogonal to a moving surface $S = c$ the surface does not keep step with the particles that initially compose it and hence is not a material surface. There is also some interesting discussion about vortex lines for which we refer to [**474**].

The fundamental link between the particle (Lagrangian) and wave mechanical (Eulerian) pictures is defined by the following expression for the Eulerian density

$$\rho(x,t) = \int \delta(x - q(a,t))\rho_0(a)d^3a \tag{2.34}$$

The corresponding formula for the Eulerian velocity is contained in the expression for the current

$$\rho(x,t)v_i(x,t) = \int \frac{\partial q_i(a,t)}{\partial t}\delta(x - q(a,t))\rho_0(a)d^3a \tag{2.35}$$

These relations play an analogous role in the approach to the Huygen's formula

$$\psi(x,t) = \int G(x,t;a,0)\psi_0(a)d^3a \tag{2.36}$$

in the Feynman theory; thus one refers to $\delta(x - q9a, t))$ as a propagator. Unlike the many to one mapping embodied in (2.36) the quantum evolution here is described by a local point to point development. Using the result

$$\delta(x - q(a,t)) = J^{-1}\big|_{a(x,t)}\,\delta(a - a_0(x,t));\ x - q(a_0,t) = 0 \tag{2.37}$$

and evaluating the integrals (2.34) and (2.22) are equivalent to

$$\rho(x,t) = J^{-1}\big|_{a(x,t)}\,\rho_0(a(x,t));\ \rho(x(a,t),t) = J^{-1}(a,t)\rho_0(a) \tag{2.38}$$

$$v_i(x,t) = \frac{\partial q_i(a,t)}{\partial t}\bigg|_{a(x,t)};\ v_i(x,t)|_{a(x,t)} = \frac{\partial q_i(a,t)}{\partial t}$$

These restate the conservation equation (2.16) and give the relations between the velocities in the two pictures; J^{-1} could be called a local propagator. Now from (2.38) one can relate the accelerations in the two pictures via

$$\frac{\partial v_i}{\partial t} + v_j\frac{\partial v_i}{\partial x_j} = \frac{\partial^2 q_i(a,t)}{\partial t^2}\bigg|_{a(x,t)};\ \left(\frac{\partial v_i}{\partial t} + v_j\frac{\partial v_i}{\partial x_j}\right)\bigg|_{a(x,t)} = \frac{\partial^2 q_i(a,t)}{\partial t^2} \tag{2.39}$$

One can now translate the Lagrangian flow equations into Lagrangian language. Differentiating (2.34) in t and using (2.35) one finds the continuity equation

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho v_i) = 0 \tag{2.40}$$

Next differentiating (2.35) and using (2.31) and (2.40) one obtains the quantum analogue of Euler's equation

$$\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} = -\frac{1}{m}\frac{\partial}{\partial x_i}(V + V_Q) \tag{2.41}$$

Finally the quasi potential condition (♠) becomes (★) $v_i = (1/m)(\partial S(x,t)/\partial x_i)$. (2.38) gives the general solutions of the continuity equation (2.40) and Euler's equation (2.41) in terms of the paths and initial density. To establish the connection between the Eulerian equations and the SE note that (2.41) and (★) can be written

$$\frac{\partial}{\partial x_i}\left(\frac{\partial S}{\partial t} + \frac{1}{2m}\frac{\partial S}{\partial x_i}\frac{\partial S}{\partial x_i} + V + V_Q\right) = 0 \tag{2.42}$$

The quantity in brackets is thus a function of time and since this does not affect the velocity field one may absorb it in S (i.e. set it equal to zero) leading to

$$\frac{\partial S}{\partial t} + \frac{1}{2m}\frac{\partial S}{\partial x_i}\frac{\partial S}{\partial x_i} + V + V_Q = 0 \tag{2.43}$$

Combining all this the function $\psi = \sqrt{\rho} exp(iS/\hbar)$ satisfies the SE

$$i\hbar\frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x_i \partial x_i} + V\psi \tag{2.44}$$

This has all been deduced from (2.22) subject to the quasi potential requirement. The quantization condition (•) becomes (••) $\int_{C_0}(\partial S(x,t)/\partial x_i)dx_i = n\hbar$ ($n \in \mathbf{Z}$ where C_0 is a closed curve fixed in space that does not pass through nodes. Given the initial wave function $\psi_0(a)$ one can now compute ψ for all (x,t) as follows. first solve (2.22) subject to initial conditions $q_0(a) = a,\ \partial q_{i0}(a)/\partial t = m^{-1}\partial S_0(a)/\partial a$ to get the set of trajectories for all (x,t). Then substitute $q(a,t)$ and q_t in (2.36) to find ρ and $\partial S/\partial x$ which gives S up to an additive function of time $f(t)$. To fix this function up t a constant use (2.43) and one gets finally

$$\psi(x,t) = \sqrt{(J^{-1}\rho_0|_{a(x,t)}} exp\left[\frac{i}{\hbar}\left(\int m(\partial q_i(a,t)/\partial t)|_{a(x,t)} dx_i + f(t)\right)\right] \tag{2.45}$$

The Eulerian equations (2.40) and (2.41) form a closed system of four coupled PDE to determine the four independent fields ($\rho(x)$, $v_i(x)$) and do not refer to the material paths. One notes that the Lagrangian theory from which the Eulerian system was derived comprises seven independent fields (ρ, $q(a)$, $p(a)$). In the case of quasi potential flow there are respectively 2 or 5 independent fields. This may be regarded as an incompleteness in the Eulerian description or a redundancy in the Lagrangian description; it could also be viewed in terms of refinement. One notes also that the law of motion (2.29) for the fluid elements coincides with that of the dBB interpretation of QM and one must be careful to discriminate between the two points of view.

2.2. ELECTROMAGNETISM AND THE SE. We go next to the second paper in [**474**] which connects the electromagnetic (EM) fields to hydrodynamics and relates this to the quantum potential. Thus the source free Maxwell equations in free space are

$$\epsilon_{ijk}\partial_j E_k = -\frac{\partial B_i}{\partial t};\ \epsilon_{ijk}\partial_j B_k = \frac{1}{c^2}\frac{\partial E_i}{\partial t};\ \partial_i E_i = \partial_i B_i = 0 \tag{2.46}$$

One regards the last two equations as constraints rather than dynamical equations. First one goes to a representation of these equations in Schrödinger form and begins with the Riemann-Silberstein 3-vector $F_i = \sqrt{\epsilon_0/2}(E_i + icB_i)$ and 3×3 angular momentum matrices s_i so that

$$(s_i)_{jk} = -i\hbar\epsilon_{ijk};\ [s_i, s_j] = i\hbar\epsilon_{ijk}s_k \tag{2.47}$$

so that (2.46) is equivalent to

$$i\hbar\frac{\partial F_i}{\partial t} = -ic(s_j)_{ik}\partial_j F_k;\ \partial_i F_i = 0 \tag{2.48}$$

To formulate groundwork for continuous representation of the spin freedoms one transforms to a representation of the s_i where the z-component is diagonal via the unitary matrix

$$U_{ai} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & i & 0 \\ 0 & 0 & \sqrt{2} \\ 1 & i & 0 \end{pmatrix} \tag{2.49}$$

and Maxwell's equations become

$$i\hbar\frac{\partial G_a}{\partial t} = -ic(J_j)_{ab}\partial_j G_b;\ G_a = U_{ai}F_i;\ J_i = Us_iU^{-1};\ (a, b = 1, 0, -1) \tag{2.50}$$

Here one has

$$\begin{pmatrix} G_1 \\ G_0 \\ G_{-1} \end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} -F_1 + iF_2 \\ \sqrt{2}F_3 \\ F_1 + iF_2 \end{pmatrix};\ J_1 = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}; \tag{2.51}$$

$$J_2 = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix};\ J_3 = \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Next one passes to an angular coordinate representation using the Euler angles $(\alpha_r) = (\alpha, \beta, \gamma)$ and conventions of [**471**] so that

$$\hat{M}_1 = i\hbar(Cos(\beta)\partial_\alpha - Sin(\beta)Ctn(\alpha)\partial_\beta + Sin(\beta)Csc(\alpha)\partial_\gamma); \tag{2.52}$$

$$\hat{M}_2 = i\hbar(-Sin(\beta)\partial_\alpha - Cos(\beta)Ctn(\alpha)\partial_\beta + Cos(\beta)Csc(\alpha)\partial_\gamma);\ \hat{M}_3 = i\hbar\partial_\beta$$

The SE (2.50) becomes then

$$i\hbar\frac{\partial\psi(x,\alpha)}{\partial t} = -ic\hat{M}_i\partial_i\psi(x,\alpha) \equiv i\hbar\frac{\partial\psi}{\partial t} = -c\hbar\hat{\lambda}_i\partial_i\psi \quad \left(\hat{\lambda}_i = \frac{\hat{M}_i}{-i\hbar}\right) \tag{2.53}$$

where ψ is a function on the 6-dimensional manifold $M = \mathbf{R}^3 \otimes SO(3)$ whose points are labeled by (x,α). The wave function may be expanded in terms of an orthornormal set of spin 1 basis functions $u_a(\alpha)$ (cf. [**471**]) in the form

$$\psi(x,\alpha,t) = G_a(x,t)u_a(\alpha)\ (a=1,0,-1);\ u_1(\alpha) = \frac{\sqrt{3}}{4\pi}Sin(\alpha)e^{-i\beta}; \tag{2.54}$$

$$u_0(\alpha) = \frac{ii\sqrt{3}}{2\sqrt{2}\pi}Cos(\alpha);\ u_{-1}(\alpha) = \frac{\sqrt{3}}{4\pi}Sin(\alpha)e^{i\beta};$$

where $\int u_a^*(\alpha)u_b(\alpha)d\Omega = \delta_{ab}$ with $d\Omega = Sin(\alpha)d\alpha d\beta d\gamma$ and $\alpha \in [0,\pi]$, $\beta \in [0,2\pi]$, $\gamma \in [0,2\pi]$. One can show that $\int u_a^*\hat{M}_i u_b(\alpha)d\Omega = (J_i)_{ab}$ and multiplying (2.53) (with use of (2.54)) one recovers the Maxwell equations in the form (2.50). In this formulation the field equations (2.53) come out as second order PDE and summation over i or a is replaced by integration in α_r. For example the energy density and Poynting vector have the alternate expressions

$$\frac{\epsilon_0}{2}(\mathbf{E}^2 + c^2\mathbf{B}^2) = F_i^*F_i = G_a^*G_a = \int|\psi(x,\alpha)|^2 d\Omega; \tag{2.55}$$

$$\epsilon_0 c^2(\mathbf{E}\times\mathbf{B})_i = \frac{c}{\hbar}F_j^*(s_i)_{jk}F_k = \frac{c}{\hbar}G_a^*(J_i)_{ab}G_b = \frac{c}{\hbar}\int\psi^*(x,\alpha)\hat{M}_i\psi(x,\alpha)d\Omega$$

For the hydrodynamic model one writes $\psi = \sqrt{\rho}exp(iS/\hbar)$ and splitting (2.53) into real and imaginary parts we get

$$\frac{\partial S}{\partial t} + \frac{c}{\hbar}\hat{\lambda}_i S\partial_i S + Q = 0;\ \frac{\partial\rho}{\partial t} + \frac{c}{\hbar}\partial_i(\rho\hat{\lambda}_i S) + \frac{c}{\hbar}\hat{\lambda}(\rho\partial_i S) = 0;\ Q = -c\hbar\frac{\hat{\lambda}_i\partial_i\sqrt{\rho}}{\sqrt{\rho}} \tag{2.56}$$

These equations are equivalent to the Maxwell equations provided ρ and S obey certain conditions; in particular single valuedness of the wave function requires

$$\oint_{C_0}\partial_i S dx_i + \partial_r S d\alpha_r = n\hbar\ (n \in \mathbf{Z}) \tag{2.57}$$

where C_0 is a closed curve in M. In the hydrodynamic model n is interpreted as the net strength of the vortices contained in C_0 (these occur in nodal regions ($\psi = 0$) where S is singular). Comparing (2.56) with the Eulerian continuity equation corresponding to a fluid of density ρ with translational and rotational freedom one expects

$$\frac{\partial\rho}{\partial t} + \partial_i(\rho v_i) + \hat{\lambda}_i(\rho\omega_i) = 0;\ v_i \sim (c/\hbar)\hat{\lambda}S;\ \omega_i \sim (c/\hbar)\partial_i S \tag{2.58}$$

Thus one obtains a kind of potential flow (strictly quasi-potential in view of (2.57)) with potential $(c/\hbar)S$. The quantity Q in (2.56) is of course the analogue for the Maxwell equations of the quantum potential and will have the classical form $-\nabla^2\sqrt{\rho}/\sqrt{\rho}$ when the appropriate metric on M is identified. From the Bernoulli-like (or HJ-like) equation in (2.56) we obtain the analogue of Euler's force law for the EM field. Thus applying first ∂_i and using (2.58) one gets

$$\left(\frac{\partial}{\partial t} + v_j\partial_j + \omega_j\hat{\lambda}_j\right)\omega_i = -\frac{c}{\hbar}\partial_i Q \tag{2.59}$$

Acting on this with $\hat{\lambda}_i$ and using $[\hat{\lambda}_i, \hat{\lambda}_j] = -\epsilon_{ijk}\hat{\lambda}_k$ yields

$$\left(\frac{\partial}{\partial t} + v_j\partial_j + \omega_i\hat{\lambda}_j\right) v_i = \epsilon_{ijk}\omega_j v_k - \frac{c}{\hbar}\hat{\lambda}_i Q \tag{2.60}$$

which contains a Coriolis type force in addition to the quantum contribution. The paths $x = x(x_0, \alpha_0, t)$ and $\alpha = \alpha(x_0, \alpha_0, t)$ of the fluid particles in M are obtained by solving the differential equations

$$v_i(x, \alpha, t) = \frac{\partial x_i}{\partial t};\ v_r(x, \alpha, t) = \frac{\partial \alpha_r}{\partial t} \tag{2.61}$$

These paths are an analogue in the full wave theory of a ray.

Now one generalizes this to coordinates x^μ in an N-dimensional Riemannian manifold M with (static) metric $g_{\mu\nu}(x)$. The history of the fluid is encoded in the positions $\xi(\xi_0, t)$ of distinct fluid elements and one assumes a single valued and differentiable map between coordinates (cf. Section 7.2.1). Let $P_0(\xi_0)$ be the initial density of some continuously distributed quantity in M (mass in ordinary hydrodynamics, energy here) and set $g = det(g_{\mu\nu})$. Then the quantity in an elementary volume $d^N\xi_0$ attached to the point ξ_0 is $P_0(\xi_0)\sqrt{-g(\xi_0)}d^N\xi_0$ and conservation of this quantity is expressed via

$$P(\xi(\xi_0, t))\sqrt{-g(\xi(\xi_0, t))}d^N\xi(\xi_0, t) = P_0(\xi_0)\sqrt{-g(\xi_0}d^N\xi_0 \equiv \tag{2.62}$$

$$\equiv P(\xi_0, t) = D^{-1}(\xi_0, t)P_0(\xi_0);\ D(\xi_0, t) = \sqrt{g(\xi)/g(\xi_0)}J(\xi_0, t)$$

where J is the Jacobian

$$J = \frac{1}{N!}\epsilon_{\mu_1\cdots\mu_n}\epsilon^{\nu_1\cdots\nu_n}\frac{\partial \xi^{\mu_1}}{\partial \xi_0^{\nu_1}}\cdots\frac{\partial \xi^{\mu_N}}{\partial \xi_0^{\nu_N}} \tag{2.63}$$

One assumes the Lagrangian for the set of fluid particles has a kinetic term and an internal potential representing a certain kind of particle interaciton

$$L = \int P_0(\xi_0)\left(\frac{1}{2}g_{\mu\nu}(\xi)\frac{\partial \xi^\mu}{\partial t}\frac{\partial \xi^\nu}{\partial t} - g^{\mu\nu}\frac{c^2\ell^2}{8}\frac{1}{P^2}\frac{\partial P}{\partial \xi^\mu}\frac{\partial P}{\partial \xi^\nu}\right)\sqrt{-g(\xi_0)}d^N\xi_0 \tag{2.64}$$

Here ℓ is a constant with the dimension of length (introduced for dimensional reasons) and $\xi = \xi(\xi_0, t)$; one substitutes now for P from (2.62) and writes

$$\frac{\partial}{\partial \xi^\mu} = J^{-1}J^\nu_\mu\frac{\partial}{\partial \xi_0^\nu};\ J^\nu_\mu = \frac{\partial J}{\partial(\partial \xi^\mu/\partial \xi_0^\nu)};\ \frac{\partial \xi^\mu}{\partial \xi_0^\nu}J^\sigma_\mu = J\delta^\sigma_\nu \tag{2.65}$$

One assumes suitable behavior at infinity so that surface terms in the variational calculations vanish and the Euler-Lagrange equations are then

(2.66)

$$\frac{\partial^2 \xi^\mu}{\partial t^2} + \left\{\begin{array}{c}\mu\\ \nu\ \sigma\end{array}\right\}\frac{\partial \xi^\nu}{\partial t}\frac{\partial \xi^\sigma}{\partial t} = -\frac{c\ell}{\hbar}g^{\mu\nu}\frac{\partial Q}{\partial \xi^\nu};\ Q = \frac{-\hbar c\ell}{2\sqrt{-g P}}\frac{\partial}{\partial \xi^\mu}\left(\sqrt{-g}g^{\mu\nu}\frac{\partial\sqrt{P}}{\partial \xi^\nu}\right)$$

$$\left\{\begin{array}{c}\mu\\ \nu\ \sigma\end{array}\right\} = \frac{1}{2}g^{\mu\rho}\left(\frac{\partial g_{\sigma\rho}}{\partial xi^\nu} + \frac{\partial g_{\nu\rho}}{\partial \xi^\sigma} - \frac{\partial g_{\nu\sigma}}{\partial \xi^\rho}\right)$$

Now one restricts to quasi-potential flows with conditions

$$g_{\mu\nu}(\xi_0)\frac{\partial\xi_0^\mu}{\partial t}=\frac{c\ell}{\hbar}\frac{\partial S_0(\xi_0)}{\partial\xi_0^\nu};\ \oint_C\frac{\partial S_0(\xi_0)}{\partial\xi_0^\mu}d\xi_0^\mu=h\hbar\ (n\in\mathbf{Z})\tag{2.67}$$

One follows the same procedures as in Section 7.2.1 so multiplying (2.66) by $g_{\sigma\mu}(\partial\xi^\sigma/\partial\xi_0^\rho)$ and integrating gives
(2.68)

$$g_{\sigma\mu}(\xi(\xi_0,t))\frac{\partial\xi^\sigma}{\partial\xi_0^\rho}\frac{\partial\xi^\mu}{\partial t}=g_{\rho\mu}(\xi_0)\frac{\partial\xi_0^\mu}{\partial t}+\frac{\partial}{\partial\xi_0^\rho}\int_0^t\left(\frac{1}{2}g_{\mu\nu}(\xi(\xi_0,t))\frac{\partial\xi^\mu}{\partial t}\frac{\partial\xi^\nu}{\partial t}-\frac{c\ell}{\hbar}Q\right)dt$$

Then substituting (2.67) one has

$$g_{\sigma\mu}\frac{\partial\xi^\sigma}{\partial\xi_0^\rho}\frac{\partial\xi^\mu}{\partial t}=\frac{c\ell}{\hbar}\frac{\partial S}{\partial\xi_0^\rho};\ S=S_0+\int_0^t\left(\frac{\hbar}{2c\ell}g_{\mu\nu}\frac{\partial\xi^\mu}{\partial t}\frac{\partial\xi^\nu}{\partial t}-Q\right)dt\tag{2.69}$$

The left side gives the velocity at time t relative to ξ_0 and this is a gradient. To obtain the ξ components multiply by $J^{-1}J_\nu^\rho$ and use (2.65) to get $g_{\mu\nu}(\partial\xi^\nu/\partial t)=(c\ell/\hbar)(\partial S/\partial\xi^\mu)$ where $S=S(\xi_0(\xi,t),t)$. Thus for all time the covariant velocity of each particle is the gradient of a potential with respect to the current position. Finally to see that the motion is quasi-potential since (2.66) holds and the value in (2.67) of the circulation is preserved following the flow, i.e.

$$\frac{\partial}{\partial t}\oint_C(t)g_{\mu\nu}\frac{\partial\xi^\nu}{\partial t}d\xi^\mu=0\tag{2.70}$$

Finally for the SE one defines a fundamental link between the particle (Lagrangian) and wave-mechanical (Eulerian) pictures via

$$P(x,t)\sqrt{-g(x)}=\int\delta(x-\xi(\xi_0,g))P_0(\xi_0)\sqrt{-g(\xi_0)}d^N\xi_0\tag{2.71}$$

The corresponding formla for the Eulerian velocity is contained in the current expression

$$P(x,t)\sqrt{-g(x)}v^\mu(x,t)=\int\frac{\partial\xi^\mu}{\partial t}\delta(x-\xi(\xi_0,t))P_0(\xi_0)\sqrt{-g(\xi_0)}d^N\xi_0\tag{2.72}$$

These are equivalent to the following local expressions ($\xi_0\sim\xi_0(x,t)$)
(2.73)

$$P(x,t)\sqrt{-g(x)}=J^{-1}\big|_{\xi_0}P_0(\xi_0(x,t))\sqrt{-g(\xi_0(x,t))};\ v^\mu(x,t)=\left.\frac{\partial\xi^\mu(\xi_0,t)}{\partial t}\right|_{\xi_0}$$

We can now translate the Lagrangian flow equations into Eulerian language. First differentiate (2.71) in t and use (2.72) to get

$$\frac{\partial P}{\partial t}+\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\mu}(P\sqrt{-g}v^\mu)=0\tag{2.74}$$

Then, differentiating (2.72) and using (2.66) and (2.74) one obtains the analogue of the classical Euler equation

$$\frac{\partial v^\mu}{\partial t}+v^\nu\frac{\partial v^\mu}{\partial x^\nu}+\left\{\begin{array}{c}\mu\\ \nu\ \sigma\end{array}\right\}v^\nu v^\sigma=-\frac{c\ell}{\hbar}g^{\mu\nu}\frac{\partial Q}{\partial x^\nu}\tag{2.75}$$

where Q is given by (2.66) with ξ replaced by x. Finally the quasi-potential condition becomes

$$v^{\mu} = \frac{c\ell}{\hbar} g^{\mu\nu} \frac{\partial S(x,t)}{\partial x^{\nu}} \tag{2.76}$$

Formula (2.73) give the general solution of the coupled continuity and Euler equations (2.74) and (2.75) in terms of the paths and initial density. To establish the connection between the Eulerian equations and the SE note that (2.75) and (2.76) can be written

$$\frac{\partial}{\partial x^{\mu}} \left(\frac{\partial S}{\partial t} + \frac{c\ell}{2\hbar} g^{\nu\sigma} \frac{\partial S}{\partial x^{\nu}} \frac{\partial S}{\partial x^{\sigma}} + Q \right) = 0 \tag{2.77}$$

Again the quantity in brackets is a function of time which is incorporated into S if necessary and one arrives at

$$\frac{\partial S}{\partial t} + \frac{c\ell}{2\hbar} g^{\nu\sigma} \frac{\partial S}{\partial x^{\nu}} \frac{\partial S}{\partial x^{\sigma}} + Q = 0 \tag{2.78}$$

Combining (2.78) with (2.74) and using (2.76) one finds for $\psi = \sqrt{P} exp(i/\hbar)$ the equation

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{-\hbar c\ell}{2\sqrt{-g}} \frac{\partial}{\partial x^{\mu}} \left(\sqrt{-g} g^{\mu\nu} \frac{\partial \psi}{\partial x^{\nu}} \right) \tag{2.79}$$

(for a system of mass $\hbar/c\ell$). The quantization condition (2.70) becomes

$$\oint_{C_0} \frac{\partial S(x,t)}{\partial x^{\mu}} dx^{\mu} = n\hbar \;\; (n \in \mathbf{Z}) \tag{2.80}$$

Now an alternate representation of the internal angular motion can be given in terms of the velocity fields $v_r(x,\alpha,t)$ conjugate to the Euler angles. One has

$$\omega_i = (A^{-1})_{ir} v_r; \;\; v_r = A_{ri}\omega_i; \tag{2.81}$$

$$(A^{-1})_{ir} = \begin{pmatrix} -Cos(\beta) & 0 & -Sin(\alpha)Sin(\beta) \\ Sin(\beta) & 0 & -Sin(\alpha)Cos(\beta) \\ 0 & -1 & -Cos(\alpha) \end{pmatrix};$$

The relations (2.52) may be written as $\hat{\lambda}_i = A_{ir}\partial_r$ and hence $\omega_j \hat{\lambda}_j = v_r \partial_r$. In terms of the conjugate velocities Euler's equations (2.59) and (2.60) become

$$\left(\frac{\partial}{\partial t} + v_j\partial_j + v_r\partial_r \right) v_s + A_{si}\partial_r (A^{-1})_{iq} v_q v_r = -\frac{c}{\hbar} A_{si}\partial_i Q; \tag{2.82}$$

$$\left(\frac{\partial}{\partial t} + v_j\partial_j + v_r\partial_r \right) v_i + \epsilon_{ijk} (A^{-1})_{kr} v_j v_r = -\frac{c}{\hbar} \hat{\lambda}_i Q$$

Now one specializes the general treatment to the manifold $M = \mathbf{R}^3 \times SO(3)$ with coordinates $x^{\mu} = (x_i, \alpha_r)$ and metric

$$g^{\mu\nu} = \begin{pmatrix} 0 & \ell^{-1}A_{ir} \\ \ell^{-1}A_{ri} & 0 \end{pmatrix}; \; g_{\mu\nu} = \begin{pmatrix} 0 & \ell(A^{-1})_{ir} \\ \ell(A^{-1})_{ri} & 0 \end{pmatrix} \tag{2.83}$$

and density $P = \rho/\ell^3$. From $\partial_r(\sqrt{-g} g^{ir}) = 0$ and $\ell g^{ir}\partial_r = \hat{\lambda}_i$ one gets via $[\hat{\lambda}_i, \hat{\lambda}_j] = -\epsilon_{ijk}\hat{\lambda}_k$ the relation $g^{ir}(\partial_s g_{rj} - \partial_r g_{sj}) = \ell^{-1}\epsilon_{ijk} g_{sk}$. Then the relation (2.76) becomes (2.58), (2.74) becomes (2.56), (2.75) becomes (2.82), (2.66) (with ξ

replaced by x) becomes (2.56), (2.78) becomes (2.56), (2.79) becomes (2.53), and (2.80) becomes (2.57). Writing $\xi^\mu = (q_i, \theta_r)$ for the Lagrangian coordinates (2.64) becomes

$$(2.84)\quad L = \int \ell\rho_0(q_0,\theta_0)\left((A^{-1})_{ir}\frac{\partial q_i}{\partial t}\frac{\partial \theta_r}{\partial t} - A_{ir}\frac{c^2}{4\rho^2}\frac{\partial \rho}{\partial q_i}\frac{\partial \rho}{\partial \theta_r}\right)Sin(\theta_0)d^3\theta_0 d^3q_0$$

Newton's law (2.66) reduces to the coupled relations

$$(2.85)\qquad \frac{\partial^2 q_i}{\partial t^2} + \epsilon_{ijk}(A^{-1})_{ir}\frac{\partial q_j}{\partial t}\frac{\partial \theta_r}{\partial t} = -\frac{c}{\hbar}A_{ir}\frac{\partial Q}{\partial \theta_r};$$

$$\frac{\partial^2 \theta_s}{\partial g^2} + A_{si}\frac{\partial}{\partial \theta_r}(A^{-1})_{iq}\frac{\partial \theta_q}{\partial t}\frac{\partial \theta_r}{\partial t} = -\frac{c}{\hbar}A_{si}\frac{\partial Q}{\partial q_i}$$

where A_{ir} is given via (2.81) with α_r replaced by $\theta_r(q_0,\theta_0,t)$ and one substitutes $\rho(q_0,\theta_0,t) = D^{-1}(q_0,\theta_0,t)\rho_0(q_0,\theta_0)$ along with

$$(2.86)\qquad Q = -c\hbar A_{ir}\frac{1}{\sqrt{\rho}}\frac{\partial^2\sqrt{\rho}}{\partial\theta_r\partial q_i};\ \ \frac{\partial}{\partial q_i} = J^{-1}\left(J_{ij}\frac{\partial}{\partial q_{0j}} + J_{is}\frac{\partial}{\partial \theta_{0s}}\right);$$

$$\frac{\partial}{\partial \theta_r} = J^{-1}\left(J_{rj}\frac{\partial}{\partial q_{0j}} + J_{rs}\frac{\partial}{\partial \theta_{0s}}\right)$$

Given the initial wavefunction $\psi_0(x,\alpha) = G_{0a}(x)u_a(\alpha) = \sqrt{\rho_0}exp(iS_0/\hbar)$ one can solve (2.85) subject to initial conditions $\partial q_{0i}/\partial t = (c/\hbar)A_{ir}(\partial S_0/\partial\theta_{0r})$ and $\partial\theta_{0r}/\partial t = (c/\hbar)A_{ri}(\theta_0)(\partial S_0/\partial q_{0i})$ to get the set of trajectories for all (q_0,θ_0,t). Then invert these functions and substitute $q_0(x,\alpha,t)$ and $\theta_0(x,\alpha,t)$ in the right side of (2.85) to find $\rho(x,\alpha,t)$ and in the right sides of the equations

$$(2.87)\qquad \partial_r S = \frac{\hbar}{c}(A^{-1})_{ir}\frac{\partial q_i}{\partial t};\ \partial_i S = \frac{\hbar}{c}(A^{-1})_{ri}\frac{\partial \theta_r}{\partial t}$$

to get S up to an additive function of time $\hbar f(t)$, which is adjusted as before (cf. (2.56)). There results

(2.88)

$$\psi = \sqrt{D^{-1}\rho_0)|_{q_0,\theta_0}}\,e^{[(i/c)\int(A^{-1})_{ir}(\partial q_i/\partial t)|_{q_0,\theta_0}d\alpha_r + (A^{-1})_{ri}(\partial\theta_r/\partial t)|_{q_0,\theta_0}dx_i + if(t)]}$$

Finally the components of the time dependent EM field may be read off from (2.50) where

$$(2.89)\qquad G_a = \int \psi(x,\alpha)u_a^*(\alpha)d\Omega$$

EXAMPLE 2.1. One computes the time dependence of the EM field whose initial form is $E_{0i} = (ECos(kz),0,0)$ with $B_{0i} = (0,(1/c)ECos(kz),0)$. The initial wavefunction is $\psi_0 = G_{01}u_1$ or

$$(2.90)\qquad \psi_0(q_0,\theta_0) = -\frac{\sqrt{3}}{2\sqrt{2\pi}}ECos(kq_{03})Sin(\theta_{01})e^{-i\theta_{02}}$$

One looks for solutions to (2.85) that generate a time dependent wavefunction whose spatial dependence is on z alone. The Hamiltonian in the SE (2.53) then reduces to $-ic\hat{M}_3\partial_3\psi(x,\alpha)$ alone which preserves the spin dependence of ψ_0. Hence, since ρ is independent of θ_3, the quantum potential Q in (2.86) vanishes. Some calculation leads to

$$\psi(x,\alpha,t) = -\frac{\sqrt{3}}{2\sqrt{2}\pi} ECos(k(z-ct))Sin(\alpha)e^{-i\beta}; \tag{2.91}$$

$$E_i = (Ecos(k(ct-z)),0,0);\ B_i = (0,(1/c)ECos(k(ct-z)),0)$$

Note that one obtains oscillatory behavior of the Eulerian variables from a model in which the individual fluid elements do not oscillate! This circumvents one of the classical problems where it was considered necessary for the elements of a continuum to vibrate in order to support a wave motion. Another interesting feature is that the speed of each element $|\mathbf{v}| = |cCosc(\theta_{01})|$ obeys $c \leq |\mathbf{v}| < \infty$. One might regard the occurence of superluminal speeds as evidence that the Lagrangian model is only a mathematical tool. Indeed performing a weighed sum of the velocity over the angles to get the Poynting vector $\epsilon_0 c^2(\mathbf{E}\times\mathbf{B})_i = \int \rho v_i d\Omega$ the collective x and y motions cancel to give the conventional geometrical optics rays propagating at speed c in the z-direction.

3. SOME SPECULATIONS ON THE AETHER

We give first some themes and subsequently some details and speculations.

(1) In a hauntingly appealing paper [**494**] P. Isaev makes conjectures, with supporting arguments, which arrive at a definition of the aether as a Bose-Einstein condensate of neutrino-antineutrino pairs of Cooper type (Bose-Einstein condensates of various types have been considered by others in this context - cf. [**262, 263, 338, 398, 482, 510, 606, 893, 960**] and Remark 5.3.1). The equation for the ψ-aether is then a solution of the massless Klein-Gordon (KG) equation (photon equation)

$$\left(\hbar^2\Delta - \frac{\hbar^2}{c^2}\partial_t^2\right)\psi = 0 \tag{3.1}$$

(cf. also [**911**]). This ψ field heuristically acts as a carrier of waves (playground for waves) and one might say that special relativity (SR) is a way of including the influence of the aether on physical processes and consequently SR does not see the aether (cf. here also the idea of a Dirac aether in [**215, 216, 302, 537, 727**] and Einstein-aether theories as in [**338, 510**] - some of this is developed below). In the electromagnetic (EM) theory one looks at $\vec{\psi} = (\phi, \vec{A})$ with $\Box\psi_i = 0$ as the defining equation for a real ψ-aether, in terms of the potentials ϕ and $\vec{A}$ which therefore define the ψ-aether. EM waves are then considered as oscillations of the ψ aether and wave processes in the aether accompanying a moving particle determine wave properties of the particle. The ensemble point of view can be considered artificial, in accord wth a conclusion we made in [**194, 197, 203**], based on Theme 3 below, that uncertainty and the ensemble "cloud" are based on the lack of deterimation of particle trajectories when using the SE instead of a third order equation. Thus it is the SE context which automatically (and gratuitously) introduces probability; nevertheless, given the limitations of measurement, it produces an amazingly accurate theory.

(2) Next there is the classical deBroglie-Bohm (dBB) theory (cf. [**191, 186, 187, 188, 189, 205, 203, 471**] - and references in these papers) where, working from a Schrödinger equation (SE)

$$-\frac{\hbar^2}{2m}\Delta\psi + V\psi = i\hbar\psi_t;\ \psi = Re^{iS/\hbar} \tag{3.2}$$

one arrives at a quantum potential $Q = -(\hbar^2/2m)(\Delta\sqrt{\rho}/\sqrt{\rho})$ $(R = \sqrt{\rho})$ associated to a quantum Hamilton-Jacobi equation (QHJE)

$$S_t + \frac{(\nabla S)^2}{2m} + V + Q = 0 \tag{3.3}$$

The ensuing particle theory exhibits trajectory motion choreographed by ψ via $Q = -(\hbar^2/2m)(\Delta|\psi|/|\psi|)$ or directly via the guidance equation

$$\dot{\vec{x}} = \vec{v} = \hbar\Im\frac{\psi^*\nabla\psi}{\psi^*\psi} \tag{3.4}$$

(cf. [**186, 187, 188, 203**] for extensive references). Relativistic and geometrical aspects are also provided below.

(3) In [**346**] Faraggi and Matone develop a theory of $x-\psi$ duality, related to Seiberg-Witten theory in the string arena, which was expanded in various ways in [**2, 41, 110, 198, 194, 191, 249, 640, 751, 958**]. Here one works from a stationary SE $[-(\hbar^2/2m)\Delta + V(x)]\psi = E\psi$, and, assuming for convenience one space dimension, the space variable x is determined by the wave function ψ from a prepotential $\mathfrak{F}$ via Legendre transformations. The theory suggests that x plays the role of a macroscopic variable for a statistical system with a scaling term $\hbar$. Thus define a prepotential $\mathfrak{F}_E(\psi) = \mathfrak{F}(\psi)$ such that the dual variable $\psi^D = \partial\mathfrak{F}/\partial\psi$ is a (linearly independent) solution of the same SE. Take V and E real so that $\bar{\psi} = \psi^D$ qualifies and write $\partial_x\mathfrak{F} = \psi^D\partial_x\psi = (1/2)[\partial_x(\psi\psi^D) + W)]$ where W is the Wronskian. This leads to $(\psi^D = \bar{\psi})$ the relation $\mathfrak{F} = (1/2)\psi\bar{\psi} + (W/2)x$ (setting the integration constant to zero). Consequently, scaling W to $-2i\sqrt{2m}/\hbar$ one obtains

$$\frac{i\sqrt{2m}}{\hbar}x = \frac{1}{2}\psi\frac{\partial\mathfrak{F}}{\partial\psi} - \mathfrak{F} \equiv \frac{i\sqrt{2m}}{\hbar}x = \psi^2\frac{\partial\mathfrak{F}}{\partial\psi^2} - \mathfrak{F} \tag{3.5}$$

which exhibits x as a Legendre transform of $\mathfrak{F}$ with respect to ψ^2. Duality of the Legendre transform then gives also

$$\mathfrak{F} = \phi\partial_\phi\left(\frac{i\sqrt{2m}\,x}{\hbar}\right) - \left(\frac{i\sqrt{2m}\,x}{\hbar}\right);\ \phi = \partial_{\psi^2}\mathfrak{F} = \frac{\bar{\psi}}{2\psi} \tag{3.6}$$

so that $\mathfrak{F}$ and $(i\sqrt{2m}\,x/\hbar)$ form a Legendre pair. In particular one has $\rho = |\psi|^2 = \frac{2i\sqrt{2m}}{\hbar}x + 2\mathfrak{F}$ which also relates x and the probability density (but indirectly since the x term really only cancels the imaginary part of $2\mathfrak{F}$). In any event one sees that the wave function ψ specifically determines the exact location of the "particle" whose quantum evolution is described by

ψ. We mention here also that the (stationary) SE can be replaced by a third order equation

$$4\mathfrak{F}''' + (V(x) - E)(\mathfrak{F}' - \psi\mathfrak{F}'')^3 = 0;\ \mathfrak{F}' \sim \frac{\partial \mathfrak{F}}{\partial \psi} \tag{3.7}$$

and a dual stationary SE has the form

$$\frac{\hbar^2}{2m}\frac{\partial^2 x}{\partial \psi^2} = \psi[E - V]\left(\frac{\partial x}{\partial \psi}\right)^3 \tag{3.8}$$

A noncommutative version of this is developed in the second paper of [**958**].

(4) We also note for comparison and analogy some relations between Legendre duals in mechanics, thermodynamics, and (x,ψ) duality. Thus (cf. [**202, 596**]) one has in mechanics $p\dot{x} - L = H$ via $L = (/2)m\dot{x}^2 - V$ and $H = (p^2/2m) + V$ with $p = \partial L/\partial \dot{x}$ and $\dot{x} = \partial H/\partial p$. In thermodynamics one has a Helmholtz free energy F with $F = U - TS$ for energy U, entropy S, and temperature T. Set $\mathcal{F} = -F$ to obtain $\mathcal{F} = T(\partial\mathcal{F}/\partial T) - U$ and $U = S\partial_S U - \mathcal{F}$ (where $\partial_T \mathcal{F} = S$ and $\partial_S U = T$). Now put this in a table where we write the (x,ψ) duality in the form $\chi = \psi^2(\partial\mathfrak{F}/\partial\psi^2) - \mathfrak{F}$ with $\mathfrak{F} = \phi(\partial\chi/\partial\phi) - \chi$ (for $\chi = (i\sqrt{2m}/\hbar)x$ and $\phi = (\partial\mathfrak{F}/\partial\psi^2)$). This leads to a table

$$\begin{array}{ccc} \textit{Mechanics} & \textit{Thermodynamics} & (x,\psi)\ \textit{duality} \\ \dot{x},\ p,\ L,\ H & T,\ S,\ \mathcal{F},\ U & \psi^2,\ \phi,\ \mathfrak{F},\ \chi \\ p\dot{x} - H = L & TS - U = \mathcal{F} & \psi^2\phi - \mathfrak{F} = \chi \\ L = \dot{x}\frac{\partial H}{\partial p} - H & \mathcal{F} = S\frac{\partial U}{\partial S} - U & \mathfrak{F} = \phi\frac{\partial \chi}{\partial \phi} - \chi \\ H = p\frac{\partial L}{\partial \dot{x}} - L & U = T\frac{\partial \mathcal{F}}{\partial T} - \mathcal{F} & \chi = \psi^2\frac{\partial \mathfrak{F}}{\partial \psi^2} - \mathfrak{F} \end{array} \tag{3.9}$$

One says that e.g. $(\mathfrak{F},\chi)$ or $(\mathcal{F},U)$ or (L,H) form a Legendre dual pair and in the first situation one refers to (x,ψ) duality. One sees in particular that $\mathfrak{F} = \psi^2\phi - \chi$ where $\psi^2\phi \sim \dot{x}p$ in mechanics. Note that $\phi = \partial\mathfrak{F}/\partial\psi^2 = (1/2\psi)(\partial\mathfrak{F}/\partial\psi) = \bar{\psi}/2\psi$ with $\psi^2\phi = (1/2)\psi\bar{\psi} = (1/2)|\psi|^2$. In any event $\chi = -i(\sqrt{2m}/\hbar)x$ and we will see below how the physics can be expressed via ψ, $\partial/\partial\psi$, $d\psi$ etc. without mentioning x. This allows one to think of the coordinate x as an emergent entity and we like to think of $x - \psi$ duality in this spirit.

3.1. DISCUSSION OF A PUTATIVE PSI AETHER. We mention [**650, 753, 935**] for some material on the aether and the vacuum and refer to the bibliography for other references. We sketch first some material from [**2, 41, 110, 249, 751, 958**] which extends theme 3 to the Klein-Gordon (KG) equation. Following [**958**] take a spacetime manifold M with a metric field g and a scalar field ϕ satisfying the KG equation. Locally one has cartesian coordinates x^α ($\alpha = 0, 1. \cdots, n-1$) in which the metric is diagonal with $g_{\alpha\beta}(x) = \eta_{\alpha\beta}(x)$ and the KG equation has the form $(\Box_x + m^2)\phi(x) = 0$ ($\Box_x \stackrel{?}{\sim} (\hbar^2/c^2)[(\partial_t^2/c^2) - \nabla^2]$ - cf. (2.26)). Defining prepotentials such that $\tilde{\phi}^{(\alpha)} = \partial\mathfrak{F}^{(\alpha)}[\phi^{(\alpha)}]/\partial\phi^{(\alpha)}$ where $\phi^{(\alpha)}$ and $\tilde{\phi}^{(\alpha)}$ are two linearly independent solutions of the KG equation depending on a variable x^α (where the x^β for $\beta \neq \alpha$ enter ϕ^α and $\tilde{\phi}^\alpha$ as parameters) one has as above (with a different scaling factor)

$$\frac{\sqrt{2m}}{\hbar}x^{\alpha} = \frac{1}{2}\phi^{(\alpha)}\frac{\partial\mathfrak{F}^{(\alpha)}[\phi^{\alpha)}]}{\partial\phi^{(\alpha)}} - \mathfrak{F}^{(\alpha)};\ (\partial^{\alpha}\partial_{\alpha}-V^{\alpha})\,\phi^{\alpha} = 0 \tag{3.10}$$

This is suggested in [**346**] and used in [**958**]; the factor $\sqrt{2m}/\hbar$ is simply a scaling factor and it may be more appropriate to scale $x^0 \sim ct$ differently or in fact to scale all variables as indicated below with factors $\beta^i(x^j, t)$. Locally now $\mathfrak{F}^{(\alpha)}$ satisfies the third order equation

$$4\mathfrak{F}^{(\alpha)'''} + [V^{(\alpha)}(x^{\alpha}) + m^2](\phi^{(\alpha)}\mathfrak{F}^{(\alpha)''} - \mathfrak{F}^{(\alpha)'})^3 = 0 \tag{3.11}$$

where $' \sim \partial/\partial\phi^{(\alpha)}$ and the "effective" potential V has the form

$$V^{(\alpha)}(x^{\alpha}) = \left[\frac{1}{\phi(x)}\sum_{\beta=0,\,\beta\neq\alpha}^{n-1}\partial^{\beta}\partial_{\beta}\phi(x)\right]\Bigg|_{x^{\beta\neq\alpha}\ fixed} \tag{3.12}$$

REMARK 7.3.1. Strictly speaking V^{α} does not have the form $\Box R/R$ of a quantum potential; however since it is created by the wave function ϕ we could well think of it as a form of quantum potential. We will refer to it as the effective potential as in [**346**] and note from Section 3.2 that with $\eta^{\mu\nu} = (-1, 1, 1, 1)$ and $\Box = -(1/c^2)\partial_t^2 + \Delta$, one has for $\phi = Rexp(iS/\hbar)$

$$\frac{1}{2m}(\partial S)^2 = \frac{\hbar^2}{2m}\frac{\Box\phi}{\phi} + \frac{\hbar^2}{2m}\frac{\Box R}{R}; \tag{3.13}$$

$$\partial(R^2\partial S) = 0;\ Q_{rel} = -\frac{\hbar^2}{2m}\frac{\Box R}{R}$$

The discussion below indicates that much further development of these themes should be possible.

As indicated in [**346**], once x^{α} is replaced with its functional dependence on $\mathfrak{F}^{\alpha}$ given in (3.10), (3.11) becomes an ordinary differential equation for $\mathfrak{F}^{\alpha}(\phi^{\alpha})$; further the functional structure of $\mathfrak{F}^{\alpha}$ does not depend on the parameters x^{β} for $\beta \neq \alpha$ (which enter ϕ^{α} as parameters). Now as a consequence of (3.10) one has

$$\frac{\partial}{\partial x^{\alpha}} = \frac{(8m)^{1/2}}{\hbar}\frac{1}{E^{(\alpha)}}\frac{\partial}{\partial\phi^{(\alpha)}};\ dx^{\alpha} = \frac{\hbar}{(8m)^{1/2}}E^{(\alpha)}d\phi^{(\alpha)} \tag{3.14}$$

where $E^{(\alpha)} = \phi^{(\alpha)}\mathfrak{F}^{(\alpha)''} - \mathfrak{F}^{(\alpha)'}$. Here (3.14) represents an induced parametrization on the spaces $T_P(U)$ and $T_P^*(U)$ ($P \in U$ - local tangent and cotangent spaces). Note there is no summation over α in (3.14). Now using the linearity of the metric tensor field (cf. [**322**]) one sees that the components of the metric in the $\{(\phi^{(\alpha)}, \mathfrak{F}^{(\alpha)}\}$ are

$$G_{\alpha\beta}(\phi) = \frac{\hbar^2}{8m}E^{(\alpha)}E^{(\beta)}\eta_{\alpha\beta}(x) \tag{3.15}$$

Now let z^{μ} ($\mu = 0, 1, \cdots, n-1$) be a general coordinate system in U and write the coordinate transformation matrices via

$$A^{\alpha}_{\mu} = \frac{\partial x^{\alpha}}{\partial z^{\mu}};\ (A^{-1})^{\mu}_{\alpha} = \frac{\partial z^{\mu}}{\partial x^{\alpha}} \tag{3.16}$$

The metric then takes the form

$$g_{\mu\nu}(z) = \frac{8m}{\hbar^2}\frac{1}{E^{(\alpha)}E^{(\beta)}}A^{\alpha}_{\mu}A^{\beta}_{\nu}G_{\alpha\beta}(\phi) \tag{3.17}$$

The components of the metric connection can be computed via

$$\Gamma^{\rho}_{\mu\nu} = \frac{1}{2}g^{\rho\sigma}(z)\sum_{\mathcal{P}}\epsilon_{\mathcal{P}}\mathcal{P}\left[\partial g_{\sigma\nu}(z)/\partial z^{\mu}\right] \tag{3.18}$$

where $\mathcal{P}$ is a cyclic permutation of the ordered set of indices $\{\sigma\nu\mu\}$ and $\epsilon_{\mathcal{P}}$ is the signature of $\mathcal{P}$. Via the coordinate transformation (3.16) the function $\phi^{(\alpha)}$ depends on all the z^{μ}. The metric connection (3.18) can be expressed in the $\{\phi^{(\alpha)}, \mathfrak{F}^{(\alpha)}\}$ parametrization via

$$\Gamma^{\rho}_{\mu\nu} = \left(\frac{2m}{\hbar}\right)^{1/2}\frac{E^{(\rho)}E^{(\sigma)}}{E^{(\gamma)}}(A^{-1})^{\rho}_{\tau}(A^{-1})^{\sigma}_{\chi}G^{\tau\chi}\times \tag{3.19}$$
$$\times\sum_{\mathcal{P}}\epsilon_{\mathcal{P}}\mathcal{P}\left[A^{\gamma}_{\mu}\frac{\partial}{\partial\phi^{(\gamma)}}\left(\frac{1}{E^{(\alpha)}E^{(\beta)}}A^{\alpha}_{\sigma}A^{\beta}_{\nu}G_{\alpha\beta}\right)\right]$$

In [**958**] one computes, in the $(\phi^{(\alpha)}, \mathfrak{F}^{(\alpha)})$ parametrization, the components of the curvature tensor, the Ricci tensor, and the scalar curvature and gives an expression for the Einstein equations (we omit the details here). These matters are taken up again in [**41**] for a general curved spacetime and some sufficient constraints are isolated which make the theory work. Also in both papers a quantized version of the KG equation is also treated and the relevant $x - \psi$ duality is spelled out in operator form. We omit this also in remarking that the main feature here for our purposes is the fact that one can describe spacetime geometry (at least locally) in terms of (field) solutions of a KG equation and prepotentials (which are themselves functions of the fields). In other words the coordinates are programmed by fields and if the motion of some particle of mass m is involved then its coordinates are choreographed by the fields with a quantum potential entering the picture via (3.12). In [**2**] a similar duality is worked out for the Dirac field and cartesian coordinates and to connect this with the aether idea one should examine the above formulas for $m \to 0$.

Let us do some rescaling now and recall the origin of equations such as (3.10). Thus (cf. [**191, 198, 206, 346**]) one writes (in 1-D)

$$\partial_{\psi}\mathfrak{F} = \psi^D \sim \bar{\psi};\ \partial_x\mathfrak{F} = \partial_{\psi}\mathfrak{F}\partial_x\psi = \psi^D\psi_x = \frac{1}{2}[\partial_x(\psi^D\psi) + W];\ W = \psi^D\psi_x - \psi^D_x\psi \tag{3.20}$$

and $W = constant$ (this is the scaling factor). For example with $x \sim ct$ we write

$$\mathfrak{F} = \frac{1}{2}\psi^E\psi + Wct = \frac{1}{2}\psi^D\psi + \gamma ct \tag{3.21}$$

to find $(\chi^0 \sim \gamma ct)$

$$\gamma ct = \frac{1}{2}\phi^0\frac{\partial\mathfrak{F}^0}{\partial\phi^0} - \mathfrak{F}^0;\ E^0 = \phi^0\frac{\partial^2\mathfrak{F}^0}{\partial(\phi^0)^2} - \frac{\partial\mathfrak{F}^0}{\partial\phi^0} \tag{3.22}$$

$$dt = \frac{E^0 d\phi^0}{2\gamma c};\ \partial_t = \frac{2\gamma c}{E^0}\frac{\partial}{\partial\phi^0} \tag{3.23}$$

For the other variables x^1, x^2, x^3 we write $\gamma c = \beta$ and

$$dx^i = \frac{E^i d\phi^i}{2\beta};\ \frac{\partial}{\partial x^i} = \frac{2\beta}{E^i}\frac{\partial}{\partial\phi^i} \tag{3.24}$$

Again (3.11) holds along with (3.12). Now simply replace $\sqrt{2m}/\hbar$ by γc when $ct \sim t = x^0$ and by β^i for the x^i ($1 \leq i \leq 3$) to obtain for example (here $\beta^i = \beta^i(x^j, t)$ and $\beta^0 = \gamma c$)

$$G_{\alpha\sigma}(\phi) = \frac{4E^\alpha E^\sigma}{\beta^\alpha\beta^\sigma}\eta_{\alpha\sigma} \tag{3.25}$$

in place of (2.15). Consequently we obtain an heuristic result

THEOREM 3.1. One can then continue this process to find analogues of (3.16) - (3.19) and we think now of the KG equation as $[(\partial_t^2/c^2) - \nabla_x^2 + (c^2m^2/\hbar^2)]\phi = 0$ so letting $m \to 0$ we obtain the photon (or aether) equation (2.10) of Isaev (assuming the scaling factors β^i can be taken independently of m). Now however we have in addition a geometry for this putative aether via (3.22) - (3.25) and their continuations (see [**1021**]).

Let us sketch next some arguments from [**494**] where we omit the historical and philosophical introduction describing some opinions and ideas of famous people, e.g. Dirac, Einstein, Faraday, Lorentz, Maxwell, Planck, Poincaré, Schwinger, et al. One begins with a KG equation

$$\left(\hbar^2\nabla^2 - \frac{\hbar^2}{c^2}\frac{\partial^2}{\partial t^2} - m^2c^2\right)\psi(s,t) = 0 \equiv (\hbar^2\Box - m^2c^2)\psi = 0 \tag{3.26}$$

One asserts that any relativistic equation for a free particle with mass m whould be understood not as an equation in vacuum but as an equation for a particle with mass m in the aether; thence setting the mass equal to zero one arrives at (3.1) for the equation of the aether itself. This is called the ψ-aether in contrast to the (impossible!) Lorentz-Maxwell aether. Now consider the case of an EM field with $\mathbf{H} = curl(\mathbf{A})$ and $\mathbf{E} = -(1/c)\partial_t\mathbf{A} - \nabla\phi$ and use the Lorentz condition $div\mathbf{A} + (1/c)\partial_t\phi = 0$. Then the potentials $\mathbf{A}$ and ϕ satisfy

$$\Box\mathbf{A} = \nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = 0;\ \Box\phi = \nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t} = 0 \tag{3.27}$$

Using the Lorentz gauge one can take $\phi = 0$ so the charge independent part of the potentials is determined via

$$\Box\mathbf{A} = 0;\ div\mathbf{A} = 0;\ \phi = 0;\ \mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t};\ \mathbf{H} = curl\mathbf{A} \tag{3.28}$$

This system (3.28) is completely equivalent to the Maxwell-Lorentz equations and the general solution is given by a superposition of transverse waves. For a more symmetric representation one can write $\vec{\psi} = (\phi, \mathbf{A})$ and (3.27) becomes $\Box\psi_i(x,t) = 0$; these are called the equations for the real ψ-aether.

The classical unphysicality of ϕ, $\mathbf{A}$ now is removed in attaching them to the physically observable reality of the ψ-aether. Indeed the KG equation can be written as a product of two commuting matrix operators

$$I_{\alpha\beta}(\Box - m^2) = \sum_\delta \left(i\gamma^n \frac{\partial}{\partial x_n} + m\right)_{\alpha\delta} \left(i\gamma^k \frac{\partial}{\partial x_k} - m\right)_{\delta\beta} \tag{3.29}$$

and in order that the field function satisfy the KG equation one could require that it satisfy also one of the first order equations

$$\left(i\gamma^n \frac{\partial}{\partial x_n} + m\right)\psi = 0 \;\; or \;\; \left(i\gamma^n \frac{\partial}{\partial x_n} - m\right)\psi = 0 \tag{3.30}$$

Putting $m = 0$ in (3.30) one has possible equations for the neutrino-anti-neutrino field (there may be some question about $m = 0$ here). Recall that particle solutions of the KG equation corresponding to single valued representations of the Lorentz group have integer spins while particles with half-integer spin are described by a spinor representation. One also knows that the neutrino has spin $\hbar/2$. In any event (cf. (3.27)-(3.28)) the potentials ϕ and $\mathbf{A}$ are not merely auxilliary functions but are connected to physical reality in the form of the ψ-aether by neutrino-anti-neutrino pairs (cf. [**494**] for further arguments along these lines).

For an interesting connection of the ψ-aether with QM consider a hydrogen atom with spherically symetric and time independent potential $V(\mathbf{r}) = V(r)$ where $r = |\mathbf{r}|$. The solution to the SE $-i\hbar\partial_t\psi = -(\hbar^2/2m)\nabla^2\psi + V(r)\psi$ is obtained by separation of variables $\psi = u(\mathbf{r})f(t)$ with $u(\mathbf{r}) = R(r)Y(\theta,\phi)$. This is a problem of two body interaction (a proton and an electron) and for stationary states with energy E one looks at $\psi(x,t) = Cexp(-iEt/\hbar)$ satisfying

$$\frac{1}{sin(\theta)}\frac{\partial}{\partial\theta}\left(sin(\theta)\frac{\partial Y}{\partial\theta} + \frac{1}{sin^2(\theta)}\frac{\partial^2 Y}{\partial\phi^2}\right) + \lambda Y = 0; \tag{3.31}$$
$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \left\{\frac{2\mu}{\hbar^2}[E - V(r)] - \frac{\lambda}{r^2}\right\}R = 0$$

Here μ is the reduced mass of the system (proton + electron), E is the energy level for the bound state $p + e$ $(E < 0)$, and $V(r) = e^2/r$ is the potential energy. (3.31) is solved by further separation of variables $Y = \Theta(\theta)\Phi(\phi)$ leading to

$$\frac{\partial^2\Phi}{\partial\phi^2} + \nu\Phi = 0; \;\; \frac{1}{sin(\theta)}\frac{d}{d\theta}\left(sin(\theta)\frac{\partial\Theta}{\partial\theta}\right) + \left(\lambda - \frac{\nu}{sin^2\theta}\right)\Theta = 0 \tag{3.32}$$

The solution for Φ is $\Phi_m(\phi) = (1/2\pi)exp(im\phi)$ with $\nu = m^2$ and physically admissible solutions for Θ (associated Legendre polynomials) require $\lambda = \ell(\ell+1)$ with $|m| \leq \ell$. For R one has

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{2\mu}{\hbar^2}\frac{e^2}{r}R(r) + \frac{2\mu}{\hbar^2}ER(r) - \frac{\ell(\ell+1)}{r^2}R = 0 \tag{3.33}$$

Now here $V(r) = (2\mu/\hbar^2)(e^2/r) = [\ell(\ell+1)/r^2]$ and the term involving e^2/r is responsible for the Coulomb interaction of a proton with an electron; however the second term $\ell(\ell+1)/r^2$ does not depend on any physical interaction (even though in [**847**] it is said to be connected with angular momentum). Now putting the Coulomb interaction to zero the $\ell(\ell+1)/r^2$ term does not disappear and it makes no sense to attribute it to angular momentum. It is now claimed that in fact this term arises because of the ψ-aether and an argument based on standing waves in a spherical resonator is given. Thus following [**155**] one considers an associated Borgnis function $U(r,\theta,\phi)$, having definite connections to **E** and **H**, and when it satisfies

$$\frac{\partial^2 U}{\partial r^2}+\frac{1}{r^2 sin(\theta)}\left[\frac{\partial}{\partial\theta}sin(\theta)\frac{\partial U}{\partial\theta}+\frac{\partial}{\partial\phi}\frac{1}{sin(\theta)}\frac{\partial U}{\partial\phi}\right]+k^2U=0 \tag{3.34}$$

the Maxwell equations are also valid. Further U is connected by definite relations with $\mathbf{A}$ and ϕ, i.e. with the ψ-aether (presumably all this is spelled out in [**155**]). To solve (3.34) one writes $U=F_1(r)F_2(\theta,\phi)$ (following the notation of [**155**]) and there results

$$(\mathbf{A})\ \ \frac{1}{sin(\theta)}\frac{\partial}{\partial\theta}sin(\theta)\frac{\partial F_2}{\partial\theta}+\frac{1}{sin^2\theta}\frac{\partial^2 F_2}{\partial\phi^2}+\gamma F_2=0 \tag{3.35}$$

$$(\mathbf{B})\ \ r^2\frac{\partial^2 F_1}{\partial r^2}+k^2r^2F_1-\gamma F_1=0$$

One considers here EM waves harmonic in time and characterized either by the frequency $\nu=kc/2\pi$ or by the wave vector $k=2\pi\nu/c$ with $[k]=1/cm$. Now **(A)** in (3.35) is the same as (3.31) with spherical function solutions and regular solutions of **(B)** in (3.35) exist when $\gamma=n(n+1)$. Setting $F_1(r)=rf(r)$ one obtains then

$$\frac{d^2f}{dr^2}+\frac{2}{r}\frac{df}{dr}+\left[k^2-\frac{n(n+1)}{r^2}\right]f(r)=0 \tag{3.36}$$

A little calculation puts (3.33) into the form

$$\frac{d^2R}{dr^2}+\frac{2}{r}\frac{dR}{dr}+\left(\frac{2\mu E}{\hbar^2}+\frac{2\mu e^2}{\hbar^2 r}-\frac{\ell(\ell+1)}{r^2}\right)R=0 \tag{3.37}$$

Setting $2\mu e^2/\hbar^2 r=0$ and replacing E by $E=p^2/2\mu$ in (3.37) one obtains

$$\frac{d^2R}{dr^2}+\frac{2}{r}\frac{dR}{dr}+\left(k^2-\frac{\ell(\ell+1)}{r^2}\right)R=0 \tag{3.38}$$

where $2\mu^2p^2/2\mu\hbar^2=k^2\hbar^2/\hbar^2=k^2$ with k the wave vector). Now (3.36) and (3.38) are identical and are solved under the same boundary conditions (i.e. $f(r)$ should be finite as $r\to 0$ and when $r\to\infty$ one wants $f(r)\to 0$ on the boundary of a sphere). The corresponding solutions to (3.36) represent standing waves inside the sphere at values $n=0,1,\cdots$ with $m\leq n$. Since EM waves are nothing but oscillations of the ψ-aether the term $n(n+1)/r^2$ in (3.36) is responsible for standing waves of the ψ-aether in a sphere resonator. Thus (mathematically at least) one can say that the problem of finding the energy levels in a hydrogen atom via the SE is equivalent to the problem of finding natural EM oscillations in a spherical resonator. One recalls that one of the basic postulates of QM (quantization of orbits in a hydrogen atom à la Bohr with $mvr=n\hbar/2\pi$) is equivalent to determination of conditions for existence of standing waves of the ψ-aether in a spherical resonator. This suggests that QM may be equivalent to "mechanics" of the ψ-aether. One remarks that until now only a small part of the alleged ψ-aether properties have been observed, namely in superfluidity and superconductivity (see e.g. [**968**]). It is suggested that one might well rethink a lot of physics in terms of the aether, rather than, for example, the standard model. In any event there is much further discussion in [**494**], related to real physical situations, and well worth reading.

4. REMARKS ON TRAJECTORIES

There have been a number of papers written involving microstates and Bohmian mechanics (cf. [**138, 140, 139, 191, 194, 197, 203, 305, 306, 307, 308, 309, 347, 373, 374, 375, 376, 348, 349, 520**]) and we sketch here some features of the Bouda-Djama method following [**309**]. There are some disagreements regarding quantum trajectories, discussed in [**138, 375**], which we will not deal with here. Generally we have followed [**347**] in our previous discussion and microstates were not explicitly considered (beyond mentioning the third order equation and the comments in Remark 2.2.2). Thus, referring to [**309**] for philosophy, one begins with the SE $-(\hbar^2/2m)\Delta\psi + V\psi = i\hbar\psi_t$ where $\psi = Rexp(iS/\hbar)$ in 3-D and arrives at the standard

$$\frac{1}{2m}(\nabla S)^2 - \frac{\hbar^2}{2m}\frac{\Delta R}{R} + V = -S_t;\ \nabla\cdot\left(R^2\frac{\nabla S}{m}\right) + V = -\partial(R^2) \tag{4.1}$$

witeh $Q = -(\hbar^2/2m)(\Delta R/R)$. Then one sets

$$\mathbf{j} = \frac{\hbar}{2mi}(\psi^*\nabla\psi - \psi\nabla\psi^*) = R^2\frac{\nabla S}{m} \Rightarrow \nabla\cdot\mathbf{j} + \partial_t R^2 = 0 \tag{4.2}$$

and $\rho = |\psi|^2 = R^2$ as usual. The velocity $\mathbf{v}$ is taken as $\mathbf{v} = \mathbf{j}/\rho = \nabla S/m$ here in the spirit of Bohm (and Dürr, Goldstein, Zanghi, et al). Working in 1-D with $S = S_0(x,E) - Et$ one recovers the stationary HJ equation of Section 2.2 for example and there is some discussion about the situation $S_0 = constant$ referring to Floyd and Farragi-Matone. Explicit calculations for microstates are considered and comparisons are indicated. The EP of Faraggi-Matone is then discussed as in Section 2.2 and the quantum mass field $m_Q = m(1-\partial_E Q)$ is introduced. This leads to the third order differential equation for $\dot{x}$ (where $P = \partial_x S_0 = m_Q\dot{x}$)

$$\frac{m_Q^2}{2m} + V(x) - E + \frac{|hbar^2}{4m}\left(\frac{m_Q''}{m_Q} - \frac{3}{2}\frac{(m_Q')^2}{m_Q^2} - \frac{m_Q'}{m_Q}\frac{\ddot{x}}{\dot{x}^2} + \frac{\dddot{x}}{\dot{x}^3} - \frac{5}{2}\frac{\ddot{x}^2}{\dot{x}^4}\right) = 0 \tag{4.3}$$

It is observed correctly that (4.3) is a difficult equation to manipulate, requiring a priori a solution of the QSHJE.

Now one proposes a Lagrangian which depends on x, $\dot{x}$ and the set of hidden variables Γ which is connected to constants of integration from an equation like (4.3). This approach was developed in order to avoid dealing with the Jacobi type formula $t - t_0 = \partial S_0/\partial E$ which the authors felt should be restricted to HJ equations of first order. Then one looks for a quantum Lagrangian L_q such that $(d/dt)(\partial L_q/\partial\dot{x}) - \partial_x L_q = 0$ and writes

$$L_q(x,\dot{x},\Gamma) = \frac{m}{2}\dot{x}^2 f(x,\Gamma) - V(x);\ \frac{\partial L_q}{\partial\dot{x}} = m\dot{x}f(x,\Gamma);\ \frac{\partial L_q}{\partial x} = \frac{m}{2}\dot{x}^2 f_x - V_x \tag{4.4}$$

This leads to

$$mf(x,\Gamma)\ddot{x} + \frac{m\dot{x}^2}{2}f_x + V_x = 0 \tag{4.5}$$

Then set $H_q = (\partial_x L_q)\dot{x} - L_q$ and $P = \partial L_q/\partial\dot{x} = m\dot{x}f$ so

$$H_q = \frac{m\dot{x}^2}{2}f(x,\Gamma) + V(x) = \frac{P^2}{2mf} + V(x) \tag{4.6}$$

Working with the stationary situation $S = S_0(x,\Gamma) - Et$ some calculation gives then

$$\frac{1}{2mf}S_x^2 + V = -S_t \Rightarrow \frac{1}{2mf}(\partial_x S_0)^2 + V(x) - E = 0 \tag{4.7}$$

Now referring to the general equation (2.18) in Chapter 2 (extracted from [**347**]) one writes here $w = \tilde{\theta}/\tilde{\phi} \sim \psi^D/\psi \in \mathbf{R}$ with $(\alpha \sim \omega)$ so that (cf. [**?, ?**])

$$e^{2iS_0/\hbar} = e^{i\omega}\frac{(\tilde{\theta}/\tilde{\phi}) + i\bar{\ell}}{(\tilde{g}t/\tilde{\phi}) - i\ell} \rightsquigarrow S_0 = \hbar Tan^{-1}\frac{\theta + \mu\phi}{\nu\theta + \phi} \tag{4.8}$$

(cf. [**139**] for details). For the QSHJE the basic equation is (2.2.17) which we repeat as

(4.9)

$$\frac{1}{2m}(S_0')^2 + \mathfrak{W} + Q = 0;\ \mathfrak{W} = -\frac{\hbar^2}{4m}\left\{e^{2iS_0/\hbar}, x\right\} \sim V - E;\ Q = \frac{\hbar^2}{4m}\{S_0, x\}$$

There is a “quantum” transformation $x \to \hat{x}$ described in [**347, 348**] with the QSHJE arising then from a conformal modification of the CSHJE. Thus note (•) $\{x, S_0\} = -(S_0')^{-2}\{S_0, x\}$ and define $\mathfrak{U}(S_0) = \{x, S_0\}/2 = -(1/2)(S_0')^{-2}\{S_0, x\}$. This gives a conformal rescaling $\frac{1}{2m}(S_0')^2[1 - \hbar^2\mathfrak{U}] + V - E = 0$ since

(4.10)

$$\frac{1}{2m}(S_0')^2[1 - \hbar^2\mathfrak{U}] = \frac{1}{2m}(S_0')^2[1 - \frac{\hbar^2}{2}\{x, S_0\}] = \frac{1}{2m}(S_0')^2[1 + \frac{\hbar^2}{2}(S_0')^{-2}\{S_0, x\}] =$$

$$= \frac{1}{2m}(S_0')^2 + \frac{\hbar^2}{4m}\{S_0, x\} = \frac{1}{2m}(S_0')^2 + Q \Rightarrow Q = -\frac{\hbar^2}{2m}(S_0')^2\mathfrak{U}$$

which agrees with $Q = (\hbar^2/4m)\{S_0, x\}$ using (•). Then from (4.10)

$$\left(\frac{\partial x}{\partial\hat{x}}\right)^2 = 1 = \hbar^2\mathfrak{U}(S_0) = 1 + 2m(S_0')^2 Q \Rightarrow \hat{x} = \int^x \frac{dx}{\sqrt{1 + 2m(\partial S_0)^{-2}Q}} \tag{4.11}$$

Similarly using the QSHJE (2.2.17) we have

$$\left(\frac{\partial x}{\partial\hat{x}}\right)^2 = (S_0')^{-2}[(S_0')^2 + 2mQ] = [(S_0')^2 - 2m\mathfrak{W}](S_0')^{-2} \Rightarrow \tag{4.12}$$

$$\Rightarrow \hat{x} = \int^x \frac{S_0' dx}{\sqrt{2m(E - V)}}$$

This all follows from [**347, 348**] and is used in [**309**]. Now from (4.7), (4.9), (4.10), and the QSHJE one can write (correcting a sign in [**309**])

$$f(x,\Gamma)=\left[1+\frac{\hbar^2}{2}(S_0')^{-2}\{S_0,x\}\right]^{-1}\Rightarrow f=\frac{(S_0')^2}{2m(E-V)} \tag{4.13}$$

and via (4.8) $\Gamma=\Gamma(E,\mu,\nu)$ with $f=f(x,E,\mu,\nu)$. Putting this in (4.7) gives then

$$E=\frac{m\dot{x}^2}{2}\frac{(S_0')^2}{2m(E-V)}+V\Rightarrow \dot{x}S_0'=2(E-V) \tag{4.14}$$

Note that this equation also follows from (4.5), namely $m\dot{x}f=\partial L_q/\partial\dot{x}$, and integration (cf. [**309**]). Now for the appropriate third order trajectory equation in this framework, one finds from (4.14) and the QSHJE

$$(E-V)^4-\frac{m\dot{x}^2}{2}(E-V)^3+\frac{\hbar^2}{8}\left[\frac{3}{2}\left(\frac{\ddot{x}}{\dot{x}}\right)^2-\frac{\dddot{x}}{\dot{x}}\right](E-V)^2- \tag{4.15}$$

$$-\frac{\hbar^2}{8}\left[\dot{x}^2\frac{d^2V}{dx^2}+\ddot{x}\frac{dV}{dx}\right](E-V)-\frac{3\hbar^2}{16}\left[\dot{x}\frac{dV}{dx}\right]^2=0$$

(cf. [**138, 309**]). This is somewhat simpler to solve that (4.3) since it is independent of the SE and the QSHJE. We refer now to [**138, 140, 139, 305, 306, 307, 308, 309**] for more in this direction.