CHAPTER 3

GRAVITY AND THE QUANTUM POTENTIAL

Just as we plunged into QM in Chapters 1 and 2 we plunge again into general relativity (GR), Weyl geometry, Dirac-Weyl (DW) theory, and deBroglie-Bohm-Weyl (dBBW) theory. There are many good books available for background in general relativity, especially [**69**] (marvelous for conceptual purposes and for a modern perspective) and [**12**] (a classic masterpiece with all the indices in their place). In addition we mention some excellent books and papers which will arise in references later, namely [**52, 121, 351, 458, 498, 551, 657, 715, 723, 819, 910, 972**]. To develop all the background differential geometry requires a book in itself and the presentation adopted here will in fact include all this implicitly since the topics range over a fairly wide field (see also Chapter 5 where cosmology plays a more central role).

1. INTRODUCTION

A complete description of necessary geometric ideas appears in [**657**] for example and we only make some definitions and express some relations here, usng the venerable tensor notation of indices, etc., since even today much of the physics literature appears in this form. For differential geometry one can refer to [**134, 276, 998**]. First we give some background on Weyl geometry and Brans-Dicke theory following [**12**]; for differential geometry we use the tensor notation of [**12**] and refer to e.g. [**121, 358, 458, 498, 723, 731, 972, 998**] for other notation (see also [**990**] for an interesting variation). One thinks of a differential manifold $M = \{U_i, \phi_i\}$ with $\phi : U_i \to \mathbf{R}^4$ and metric $g \sim g_{ij}dx^i dx^j$ satisfying $g(\partial_k, \partial_\ell) = g_{k\ell} =< \partial_k, \partial_\ell >= g_{\ell k}$. This is for the bare essentials; one can also imagine tangent vectors $X_i \sim \partial_i$ and dual cotangent vectors $\theta^i \sim dx^i$, etc. Given a coordinate change $\tilde{x}^i = \tilde{x}^i(x^j)$ a vector ξ^i transforming via $\tilde{\xi}^i = \sum \partial_i \tilde{x}^j \xi^j$ is called contravariant (e.g. $d\tilde{x}^i = \sum \partial_j \tilde{x}^i dx^j$). On the other hand $\partial\phi/\partial\tilde{x}^i = \sum(\partial\phi/\partial x^j)(\partial x^j/\partial\tilde{x}^i$ leads to the idea of covariant vectors $A_j \sim \partial\phi/\partial x^j$ transforming via $\tilde{A}_i = \sum(\partial x^j/\partial\tilde{x}^i)A_j$ (i.e. $\partial/\partial\tilde{x}^i \sim (\partial x^j/\partial\tilde{x}^j)\partial/\partial x^j$). Now define connection coefficients or Christoffel symbols via (strictly one writes $T^{\gamma}{}_{\alpha} = g_{\alpha\beta}T^{\gamma\beta}$ and $T_{\alpha}{}^{\gamma} = g_{\alpha\beta}T^{\beta\gamma}$ which are generally different - we use that notation here but it is sometimes not used later when it is unnecessary due to symmetries, etc.)

$$\Gamma^r{}_{ki} = -\left\{ \begin{array}{c} r \\ k \;\; i \end{array} \right\} = -\frac{1}{2}\sum(\partial_i g_{k\ell} + \partial_k g_{\ell i} - \partial_\ell g_{ik})g^{\ell r} = \Gamma^r{}_{ik} \tag{1.1}$$

(note this differs by a minus sign from some other authors). Note also that (1.1) follows from equations

$$\partial_\ell g_{ik} + g_{rk}\Gamma^r{}_{i\ell} + g_{ir}\Gamma^r{}_{\ell k} = 0 \tag{1.2}$$

and cyclic permutation; the basic definition of $\Gamma^i{}_{mj}$ is found in the transplantation law $d\xi^i = \Gamma^i{}_{mj}dx^m\xi^j$. Next for tensors $T^\alpha_{\beta\gamma}$ define derivatives $T^\alpha_{\beta\gamma|k} = \partial_k T^\alpha_{\beta\gamma}$ and

$$T^\alpha_{\beta\gamma||\ell} = \partial_\ell T^\alpha_{\beta\gamma} - \Gamma^\alpha_{\ell s}T^s_{\beta\gamma} + \Gamma^s{}_{\ell\beta}T^\alpha_{s\gamma} + \Gamma^s{}_{\ell\gamma}T^\alpha_{\beta s} \tag{1.3}$$

In particular covariant derivatives for contravariant and covariant vectors respectively are defined via

$$\xi^i_{||k} = \partial_k\xi^i - \Gamma^i{}_{k\ell}\xi^\ell = \nabla_k\xi^i;\ \eta_{m||\ell} = \partial_\ell\eta_m + \Gamma^r{}_{m\ell}\eta_r = \nabla_\ell\eta_m \tag{1.4}$$

respectively. Now to describe Weyl geometry one notes first that for Riemannian geometry transplantation holds along with

$$\ell^2 = \|\xi\|^2 = g_{\alpha\beta}\xi^\alpha\xi^\beta;\ \xi^\alpha\eta_\alpha = g_{\alpha\beta}\xi^\alpha\eta^\beta \tag{1.5}$$

For Weyl geometry however one does not demand conservation of lengths and scalar products under affine transplantation as above. Thus assume $d\ell = (\phi_\beta dx^\beta)\ell$ where the covariant vector ϕ_β plays a role analogous to $\Gamma^\alpha_{\beta\gamma}$ and one obtains

$$d\ell^2 = 2\ell^2(\phi_\beta dx^\beta) = d(g_{\alpha\beta}\xi^\alpha\xi^\beta) = \tag{1.6}$$
$$= g_{\alpha\beta|\gamma}\xi^\alpha\xi^\beta dx^\gamma + g_{\alpha\beta}\Gamma^\alpha_{\rho\gamma}\xi^\rho\xi^\beta dx^\gamma + g_{\alpha\beta}\Gamma^\beta_{\rho\gamma}\xi^\alpha\xi^\rho dx^\gamma$$

Rearranging etc. and using (1.5) again gives

$$(g_{\alpha\beta|\gamma} - 2g_{\alpha\beta}\phi_\gamma) + g_{\sigma\beta}\Gamma^\sigma{}_{\alpha\gamma} + g_{\sigma\alpha}\Gamma^\sigma{}_{\beta\gamma} = 0; \tag{1.7}$$
$$\Gamma^\alpha_{\beta\gamma} = -\left\{\begin{matrix}\alpha\\ \beta\ \gamma\end{matrix}\right\} + g^{\sigma\alpha}[g_{\sigma\beta}\phi_\gamma + g_{\sigma\gamma}\phi_\beta - g_{\beta\gamma}\phi_\sigma]$$

Thus we can prescribe the metric $g_{\alpha\beta}$ and the covariant vector field ϕ_γ and determine by (1.7) the field of connection coefficients $\Gamma^\alpha_{\beta\gamma}$ which admits the affine transplantation law as above. If one takes $\phi_\gamma = 0$ the Weyl geometry reduces to Riemannian geometry. This leads one to consider new metric tensors via a metric change $\hat{g}_{\alpha\beta} = f(x^\lambda)g_{\alpha\beta}$ and it turns out that $(1/2)\partial log(f)/\partial x^\lambda$ plays the role of ϕ_λ. Here the metric change is called a gauge transformation and the ordinary connections $\left\{\begin{matrix}\alpha\\ \beta\ \gamma\end{matrix}\right\}$ constructed from $g_{\alpha\beta}$ are equal to the more general connections $\hat{\Gamma}^\alpha_{\beta\gamma}$ constructed according to (1.7) from $\hat{g}_{\alpha\beta}$ and $\hat{\phi}_\lambda = (1/2)\partial log(f)/\partial x^\lambda$. The generalized differential geometry is conformal in that the ratio

$$\frac{\xi^\alpha\eta_\alpha}{\|\xi\|\|\eta\|} = \frac{g_{\alpha\beta}\xi^\alpha\eta^\beta}{[(g_{\alpha\beta}\xi^\alpha\xi^\beta)(g_{\alpha\beta}\eta^\alpha\eta^\beta)]^{1/2}} \tag{1.8}$$

does not change under the gauge transformation $\hat{g}_{\alpha\beta} \to f(x^\lambda)g_{\alpha\beta}$. Again if one has a Weyl geometry characterized by $g_{\alpha\beta}$ and ϕ_α with connections determined by (1.7) one may replace the geometric quantities by use of a scalar field f with

$$\hat{g}_{\alpha\beta} = f(x^\lambda)g_{\alpha\beta},\ \hat{\phi}_\alpha = \phi_\alpha + (1/2)(log(f)_{|\alpha};\ \hat{\Gamma}^\alpha_{\beta\gamma} = \Gamma^\alpha_{\beta\gamma} \tag{1.9}$$

without changing the intrinsic geometric properties of vector fields; the only change is that of local lengths of a vector via $\hat{\ell}^2 = f(x^\lambda)\ell^2$. Note that one can reduce

$\hat{\phi}_\alpha$ to the zero vector field if and only if ϕ_α is a gradient field, namely $F_{\alpha\beta} = \phi_{\alpha|\beta} - \phi_{\beta|\alpha} = 0$ (i.e. $\phi_\alpha = (1/2)\partial_a log(f) \equiv \partial_\beta\phi_\alpha = \partial_\alpha\phi_\beta$). In this case one has length preservation after transplantation around an arbitrary closed curve and the vanishing of $F_{\alpha\beta}$ guarantees a choice of metric in which the Weyl geometry becomes Riemannian; thus $F_{\alpha\beta}$ is an intrinsic geometric quantity for Weyl geometry; note $F_{\alpha\beta} = -F_{\beta\alpha}$ and

$$\{F_{\alpha\beta|\gamma}\} = 0;\ \{F_{\mu\nu|\lambda}\} = F_{\mu\nu|\lambda} + F_{\lambda\mu|\nu} + F_{\nu\lambda|\mu} \tag{1.10}$$

Similarly the concept of covariant differentiation depends only on the idea of vector transplantation. Indeed one can define covariant derivatives via

$$\xi^\alpha_{||\beta} = \xi^\alpha_{|\beta} - \Gamma^\alpha_{\ \beta\gamma}\xi^\gamma \tag{1.11}$$

In Riemannian geometry the curvature tensor is

$$\xi^\alpha_{||\beta|\gamma} - \xi^\alpha_{||\gamma|\beta} = R^\alpha_{\eta\beta\gamma}\xi^\eta;\ R^\alpha_{\ \beta\gamma\delta} = -\Gamma^\alpha_{\ \beta\gamma|\delta} + \Gamma^\alpha_{\ \beta\delta|\gamma} + \Gamma^\alpha_{\ \tau\delta}\Gamma^\tau_{\ \beta\gamma} - \Gamma^\alpha_{\ \tau\gamma}\Gamma^\tau_{\ \beta\delta} \tag{1.12}$$

Using (1.8) one then can express this in terms of $g_{\alpha\beta}$ and ϕ_α but this is complicated. Equations for $R_{\beta\delta} = R^\alpha_{\ \beta\alpha\delta}$ and $R = g^{\beta\delta}R_{\beta\delta}$ are however given in [**12**]. One notes that in Weyl geometry if a vector ξ^α is given, independent of the metric, then $\xi_\alpha = g_{\alpha\beta}\xi^\beta$ will depend on the metric and under a gauge transformation one has $\hat{\xi}_\alpha = f(x^\lambda)\xi_\alpha$. Hence the covariant form of a gauge invariant contravariant vector becomes gauge dependent and one says that a tensor is of weight n if, under a gauge transformation, $\hat{T}^{\alpha\cdots}_{\beta\cdots} = f(x^\lambda)^n T^{\alpha\cdots}_{\beta\cdots}$. Note ϕ_α plays a singular role in (1.9) and has no weight. Similarly $\sqrt{-\hat{g}} = f^2\sqrt{-g}$ (weight 2) and $F^{\alpha\beta} = g^{\alpha\mu}g^{\beta\nu}F_{\mu\nu}$ has weight -2 while $\mathfrak{F}^{\alpha\beta} = F^{\alpha\beta}\sqrt{-g}$ has weight 0 and is gauge invariant. Similarly $F_{\alpha\beta}F^{\alpha\beta}\sqrt{-g}$ is gauge invariant. Now for Weyl's theory of electromagnetism one wants to interpret ϕ_α as an EM potential and one has automatically the Maxwell equations

$$\{F_{\alpha\beta|\gamma}\} = 0;\ \mathfrak{F}^{\alpha\beta}_{|\beta} = \mathfrak{s}^\alpha \tag{1.13}$$

(the latter equation being gauge invariant source equations). These equations are gauge invariant as a natural consequence of the geometric interpretation of the EM field. For the interaction between the EM and gravitational fields one sets up some field equations as indicated in [**12**] and the interaction between the metric quantities and the EM fields is exhibited there (there is much more on EM theory later and see also Section 2.1.1).

REMARK 3.1.1. As indicated earlier in [**12**] $R^i_{\ jk}$ is defined with a minus sign compared with e.g. [**723, 998**] for example. There is also a difference in definition of the Ricci tensor which is taken to be $G^{\beta\delta} = R^{\beta\delta} - (1/2)g^{\beta\delta}R$ in [**12**] with $R = R^\delta_\delta$ so that $G_{\mu\gamma} = g_{\mu\beta}g_{\gamma\delta}G^{\beta\delta} = R_{\mu\gamma} - (1/2)g_{\mu\gamma}R$ with $G^\gamma_\eta = R^\eta_\eta - 2R \Rightarrow G^\eta_\eta = -R$ (recall $n = 4$). In [**723**] the Ricci tensor is simply $R_{\beta\mu} = R^\alpha_{\beta\mu\alpha}$ where $R^\alpha_{\beta\mu\nu}$ is the Riemann curvature tensor and $R = R^\eta_\eta$ again. This is similar to [**998**] where the Ricci tensor is defined as $\rho_{j\ell} = R^i_{ji\ell}$. To clarify all this we note that

$$R_{\eta\gamma} = R^\alpha_{\ \eta\alpha\gamma} = g^{\alpha\beta}R_{\beta\eta\alpha\gamma} = -g^{\alpha\beta}R_{\beta\eta\gamma\alpha} = -R^\alpha_{\ \eta\gamma\alpha} \tag{1.14}$$

which reveals the minus sign difference.

2. SKETCH OF DEBROGLIE-BOHM-WEYL THEORY

From Chapters 1 and 2 we know something about Bohmian mechanics and the quantum potential and we go now to the papers [**869, 870, 871, 872, 873, 874, 875, 876**] by A. and F. Shojai to begin the present discussion (cf. also [**8, 117, 118, 284, 668, 669, 831, 832, 834, 835, 836, 837, 838, 864, 865, 866, 867, 868, 881**] for related work from the Tehran school and [**189, 219, 611, 731, 840, 841, 872**] for linking of dBB theory with Weyl geometry). In nonrelativistic deBroglie-Bohm theory the quantum potential is $Q = -(\hbar^2/2m)(\nabla^2|\Psi|/|\Psi|)$. The particles trajectory can be derived from Newton's law of motion in which the quantum force $-\nabla Q$ is present in addition to the classical force $-\nabla V$. The enigmatic quantum behavior is attributed here to the quantum force or quantum potential (with Ψ determining a "pilot wave" which guides the particle motion). Setting $\Psi = \sqrt{\rho}exp[iS/\hbar]$ one has

$$\frac{\partial S}{\partial t} + \frac{|\nabla S|^2}{2m} + V + Q = 0;\ \frac{\partial \rho}{\partial t} + \nabla\cdot\left(\rho\frac{\nabla S}{m}\right) = 0 \tag{2.1}$$

The first equation in (2.1) is a Hamilton-Jacobi (HJ) equation which is identical to Newton's law and represents an energy condition $E = (|p|^2/2m) + V + Q$ (recall from HJ theory $-(\partial S/\partial t) = E(= H)$ and $\nabla S = p$. The second equation represents a continuity equation for a hypothetical ensemble related to the particle in question. For the relativistic extension one could simply try to generalize the relativistic energy equation $\eta_{\mu\nu}P^\mu P^\nu = m^2c^2$ to the form

$$\eta_{\mu\nu}P^\mu P^\nu = m^2c^2(1+\mathcal{Q}) = \mathcal{M}^2c^2;\ \mathcal{Q} = (\hbar^2/m^2c^2)(\Box|\Psi|/|\Psi|) \tag{2.2}$$

$$\mathcal{M}^2 = m^2\left(1+\alpha\frac{\Box|\Psi|}{|\Psi|}\right);\ \ \alpha = \frac{\hbar^2}{m^2c^2} \tag{2.3}$$

This could be derived e.g. by setting $\Psi = \sqrt{\rho}exp(iS/\hbar)$ in the Klein-Gordon (KG) equation and separating the real and imaginary parts, leading to the relativistic HJ equation $\eta_{\mu\nu}\partial^\mu S\partial^\nu S = \mathfrak{M}^2c^2$ (as in (2.1) - note $P^\mu = -\partial^\mu S$) and the continuity equation is $\partial_\mu(\rho\partial^\mu S) = 0$. The problem of $\mathcal{M}^2$ not being positive definite here (i.e. tachyons) is serious however and in fact (2.2) is not the correct equation (see e.g. [**871, 873, 876**]). One must use the covariant derivatives ∇_μ in place of ∂_μ and for spin zero in a curved background there results ($\mathcal{Q}$ as above)

$$\nabla_\mu(\rho\nabla^\mu S) = 0;\ g^{\mu\nu}\nabla_\mu S\nabla_\nu S = \mathfrak{M}^2c^2;\ \mathfrak{M}^2 = m^2e^{\mathcal{Q}} \tag{2.4}$$

To see this one must require that a correct relativistic equation of motion should not only be Poincaré invariant but also it should have the correct nonrelativistic limit. Thus for a relativistic particle of mass $\mathfrak{M}$ (which is a Lorentz invariant quantity) $\mathfrak{A} = \int d\lambda(1/2)\mathfrak{M}(r)(dr_\mu/d\lambda)(dr^\nu/d\lambda)$ is the action functional where λ is any scalar parameter parametrizing the path $r_\mu(\lambda)$ (it could e.g. be the proper time τ). Varying the path via $r_\mu \to r'_\mu = r_\mu + \epsilon_\mu$ one gets (cf. [**871**])

$$\mathfrak{A} \to \mathfrak{A}' = \mathfrak{A} + \delta\mathfrak{A} = \mathfrak{A} + \int d\lambda\left[\mathfrak{M}\frac{dr_\mu}{d\lambda}\frac{d\epsilon^\mu}{d\lambda} + \frac{1}{2}\frac{dr_\mu}{d\lambda}\frac{dr^\mu}{d\lambda}\epsilon_\nu\partial^\nu\mathfrak{M}\right] \tag{2.5}$$

By least action the correct path satisfies $\delta\mathfrak{A} = 0$ with fixed boundaries so the equation of motion is

$$(d/d\lambda)(\mathfrak{M}u_\mu) = (1/2)u_\nu u^\nu \partial_\mu \mathfrak{M}; \tag{2.6}$$

$$\mathfrak{M}(du_\mu/d\lambda) = ((1/2)\eta_{\mu\nu}u_\alpha u^\alpha - u_\mu u_\nu)\partial^\nu \mathfrak{M}$$

where $u_\mu = dr_\mu/d\lambda$. Now look at the symmetries of the action functional via $\lambda \to \lambda+\delta$. The conserved current is then the Hamiltonian $\mathfrak{H} = -\mathfrak{L}+u_\mu(\partial\mathfrak{L}/\partial u_\mu) = (1/2)\mathfrak{M}u_\mu u^\mu = E$. This can be seen by setting $\delta\mathfrak{A} = 0$ where

$$0 = \delta\mathfrak{A} = \mathfrak{A}' - \mathfrak{A} = \int d\lambda \left[\frac{1}{2}u_\mu u^\mu u^\nu \partial_\nu \mathfrak{M} + \mathfrak{M}u_\mu \frac{du^\mu}{d\lambda}\right]\delta \tag{2.7}$$

which means that the integrand is zero, i.e. $(d/d\lambda)[(1/2)\mathfrak{M}u_\mu u^\mu] = 0$. Since the proper time is defined as $c^2d\tau^2 = dr_\mu dr^\mu$ this leads to $(d\tau/d\lambda) = \sqrt{(2E/\mathfrak{M}c^2)}$ and the equation of motion becomes

$$\mathfrak{M}(dv_\mu/d\tau) = (1/2)(c^2\eta_{\mu\nu} - v_\mu v_\nu)\partial^\nu \mathfrak{M} \tag{2.8}$$

where $v_\mu = dr_\mu/d\tau$. The nonrelativistic limit can be derived by letting the particles velocity be ignorable with respect to light velocity. In this limit the proper time is identical to the time coordinate $\tau = t$ and the result is that the $\mu = 0$ component is satisfied identically via ($r \sim \vec{r}$)

$$\mathfrak{M}\frac{d^2r}{dt^2} = -\frac{1}{2}c^2\nabla\mathfrak{M} \Rightarrow m\left(\frac{d^2r}{dt^2}\right) = -\nabla\left[\frac{mc^2}{2}log\left(\frac{\mathfrak{M}}{\mu}\right)\right] \tag{2.9}$$

where μ is an arbitrary mass scale. In order to have the correct limit the term in parenthesis on the right side should be equal to the quantum potential so $(mc^2/2)log(\mathfrak{M}/\mu) = (\hbar^2/2m)(\nabla^2|\psi|/|\psi|)$ and hence

$$\mathfrak{M} = \mu exp[-(\hbar^2/m^2c^2)(\nabla^2|\Psi|/|\Psi|)] \tag{2.10}$$

One infers that the relativistic quantum mass field is $\mathfrak{M} = \mu exp[(\hbar^2/2m)(\Box|\Psi|/|\Psi|)]$ (manifestly invariant) and setting $\mu = m$ we get (cf. also (2.12) below)

$$\mathfrak{M} = mexp[(\hbar^2/m^2c^2)(\Box|\Psi|/|\Psi|)] \tag{2.11}$$

If one starts with the standard relativistic theory and goes to the nonrelativistic limit one does not get the correct nonrelativistic equations; this is a result of an improper decomposition of the wave function into its phase and norm in the KG equation (cf. also [**110**] for related procedures). One notes here also that (2.11) leads to a positive definite mass squared. Also from [**871**] this can be extended to a many particle version and to a curved spacetime. However, for a particle in a curved background we will take (cf. [**873**] which we follow for the rest of this section)

$$\nabla_\mu(\rho\nabla^\mu S) = 0;\ g^{\mu\nu}\nabla_\mu S\nabla_\nu S = \mathfrak{M}^2c^2;\ \mathfrak{M}^2 = m^2e^{\mathfrak{Q}};\ \mathfrak{Q} = \frac{\hbar^2}{m^2c^2}\frac{\Box_g|\Psi|}{|\Psi|} \tag{2.12}$$

((2.11) suggests that $\mathfrak{M}^2 = m^2exp(2\mathfrak{Q})$ but (2.12) is used for compatibility with the KG approach, etc., where $exp(\mathfrak{Q}) \sim 1 + \mathfrak{Q}$ - cf. remarks after (2.28) below - in any case the qualitative features are close here for either formula). Since, following deBroglie, the quantum HJ equation (QHJE) in (2.12) can be written in the form $(m^2/\mathfrak{M}^2)g^{\mu\nu}\nabla_\mu S\nabla_\nu S = m^2c^2$, **the quantum effects are identical to a change of spacetime metric**

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} = (\mathfrak{M}^2/m^2)g_{\mu\nu} \tag{2.13}$$

which is a conformal transformation. The QHJE becomes then $\tilde{g}^{\mu\nu}\tilde{\nabla}_\mu S\tilde{\nabla}_\nu S = m^2c^2$ where $\tilde{\nabla}_\mu$ represents covariant differentiation with respect to the metric $\tilde{g}_{\mu\nu}$ and the continuity equation is then $\tilde{g}_{\mu\nu}\tilde{\nabla}_\mu(\rho\tilde{\nabla}_\nu S) = 0$. The important conclusion here is that the presence of the quantum potential is equivalent to a curved spacetime with its metric given by (2.13). This is a geometrization of the quantum aspects of matter and it seems that there is a dual aspect to the role of geometry in physics. The spacetime geometry sometimes looks like "gravity" and sometimes reveals quantum behavior. The curvature due to the quantum potential may have a large influence on the classical contribution to the curvature of spacetime. The particle trajectory can now be derived from the guidance relation via differentiation of (2.12) leading to the Newton equations of motion

$$\mathfrak{M}\frac{d^2x^\mu}{d\tau^2} + \mathfrak{M}\Gamma^\mu_{\nu\kappa}u^\nu u^\kappa = (c^2g^{\mu\nu} - u^\mu u^\nu)\nabla_\nu\mathfrak{M} \tag{2.14}$$

Using the conformal transformation above (2.14) reduces to the standard geodesic equation.

Now a general "canonical" relativistic system consisting of gravity and classical matter (no quantum effects) is determined by the action

$$\mathcal{A} = \frac{1}{2\kappa}\int d^4x\sqrt{-g}\mathcal{R} + \int d^4x\sqrt{-g}\frac{\hbar^2}{2m}\left(\frac{\rho}{\hbar^2}\mathcal{D}_\mu S\mathcal{D}^\mu S - \frac{m^2}{\hbar^2}\rho\right) \tag{2.15}$$

where $\kappa = 8\pi G$ and $c = 1$ for convenience. It was seen above that via deBroglie the introduction of a quantum potential is equivalent to introducing a conformal factor $\Omega^2 = \mathfrak{M}^2/m^2$ in the metric. Hence in order to introduce quantum effects of matter into the action (2.15) one uses this conformal transformation to get $(1 + Q \sim exp(Q))$

$$\mathfrak{A} = \frac{1}{2\kappa}\int d^4x\sqrt{-\bar{g}}(\bar{\mathcal{R}}\Omega^2 - 6\bar{\nabla}_\mu\Omega\bar{\nabla}^\mu\Omega) + \tag{2.16}$$

$$+\int d^4x\sqrt{-\bar{g}}\left(\frac{\rho}{m}\Omega^2\bar{\nabla}_\mu S\bar{\nabla}^\mu S - m\rho\Omega^4\right) + \int d^4x\sqrt{-\bar{g}}\lambda\left[\Omega^2 - \left(1 + \frac{\hbar^2}{m^2}\frac{\Box\sqrt{\rho}}{\sqrt{\rho}}\right)\right]$$

where a bar over any quantity means that it corresponds to the nonquantum regime. Here only the first two terms of the expansion of $\mathfrak{M}^2 = m^2 exp(\mathfrak{Q})$ in (2.12) have been used, namely $\mathfrak{M}^2 \sim m^2(1 + \mathfrak{Q})$. No physical change is involved in considering all the terms. λ is a Lagrange multiplier introduced to identify the conformal factor with its Bohmian value. One uses here $\bar{g}_{\mu\nu}$ to raise of lower indices and to evaluate the covariant derivatives; the physical metric (containing the quantum effects of matter) is $g_{\mu\nu} = \Omega^2\bar{g}_{\mu\nu}$. By variation of the action with respect to $\bar{g}_{\mu\nu}$, Ω, ρ, S, and λ one arrives at the following quantum equations of motion:

(1) The equation of motion for Ω

$$\bar{\mathcal{R}}\Omega + 6\Box\Omega + \frac{2\kappa}{m}\rho\Omega(\bar{\nabla}_\mu S\bar{\nabla}^\mu S - 2m^2\Omega^2) + 2\kappa\lambda\Omega = 0 \tag{2.17}$$

(2) The continuity equation for particles $\bar{\nabla}_\mu(\rho\Omega^2\bar{\nabla}^\mu S) = 0$

(3) The equations of motion for particles (here $a' \equiv \bar{a}$)

$$(\bar{\nabla}_\mu S\bar{\nabla}^\mu S - m^2\Omega^2)\Omega^2\sqrt{\rho} + \frac{\hbar^2}{2m}\left[\Box'\left(\frac{\lambda}{\sqrt{\rho}}\right) - \lambda\frac{\Box'\sqrt{\rho}}{\rho}\right] = 0 \tag{2.18}$$

(4) The modified Einstein equations for $\bar{g}_{\mu\nu}$

(2.19)

$$\Omega^2\left[\bar{\mathcal{R}}_{\mu\nu} - \tfrac{1}{2}\bar{g}_{\mu\nu}\bar{\mathcal{R}}\right] - [\bar{g}_{\mu\nu}\Box' - \bar{\nabla}_\mu\bar{\nabla}_\nu]\Omega^2 - 6\bar{\nabla}_\mu\Omega\bar{\nabla}_\nu\Omega + 3\bar{g}_{\mu\nu}\bar{\nabla}_\alpha\Omega\bar{\nabla}^\alpha\Omega +$$
$$+\frac{2\kappa}{m}\rho\Omega^2\bar{\nabla}_\mu S\bar{\nabla}_\nu S - \frac{\kappa}{m}\rho\Omega^2\bar{g}_{\mu\nu}\bar{\nabla}_\alpha S\bar{\nabla}^\alpha S + \kappa m\rho\Omega^4\bar{g}_{\mu\nu} +$$
$$+\frac{\kappa\hbar^2}{m^2}\left[\bar{\nabla}_\mu\sqrt{\rho}\bar{\nabla}_\nu\left(\frac{\lambda}{\sqrt{\rho}}\right) + \bar{\nabla}_\nu\sqrt{\rho}\bar{\nabla}_\mu\left(\frac{\lambda}{\sqrt{\rho}}\right)\right] - \frac{\kappa\hbar^2}{m^2}\bar{g}_{\mu\nu}\bar{\nabla}_\alpha\left[\lambda\frac{\bar{\nabla}^\alpha\sqrt{\rho}}{\sqrt{\rho}}\right] = 0$$

(5) The constraint equation $\Omega^2 = 1 + (\hbar^2/m^2)[(\Box\sqrt{\rho})/\sqrt{\rho}]$

Thus the back reaction effects of the quantum factor on the background metric are contained in these highly coupled equations (cf. also [**27**]). A simpler form of (2.17) can be obtained by taking the trace of (2.19) and using (2.17) which produces $\lambda = (\hbar^2/m^2)\bar{\nabla}_\mu[\lambda(\bar{\nabla}^\mu\sqrt{\rho})/\sqrt{\rho}]$. A solution of this via perturbation methods using the small parameter $\alpha = \hbar^2/m^2$ yields the trivial solution $\lambda = 0$ so the above equations reduce to

$$\bar{\nabla}_\mu(\rho\Omega^2\bar{\nabla}^\mu S) = 0;\ \bar{\nabla}_\mu S\bar{\nabla}^\mu S = m^2\Omega^2;\ \mathfrak{G}_{\mu\nu} = -\kappa\mathfrak{T}^{(m)}_{\mu\nu} - \kappa\mathfrak{T}^{(\Omega)}_{\mu\nu} \tag{2.20}$$

where $\mathfrak{T}^{(m)}_{\mu\nu}$ is the matter energy-momentum (EM) tensor and

$$\kappa\mathfrak{T}^{(\Omega)}_{\mu\nu} = \frac{[g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu]\Omega^2}{\Omega^2} + 6\frac{\nabla_\mu\Omega\nabla_\nu\Omega}{\omega^2} - 2g_{\mu\nu}\frac{\nabla_\alpha\Omega\nabla^\alpha\Omega}{\Omega^2} \tag{2.21}$$

with $\Omega^2 = 1+\alpha(\Box\sqrt{\rho}/\sqrt{\rho})$. Note that the second relation in (2.20) is the Bohmian equation of motion and written in terms of $g_{\mu\nu}$ it becomes $\nabla_\mu S\nabla^\mu S = m^2c^2$.

In the preceeding one has tacitly assumed that there is an ensemble of quantum particles so what about a single particle? One translates now the quantum potential into purely geometrical terms without reference to matter parameters so that the original form of the quantum potential can only be deduced after using the field equations. Thus the theory will work for a single particle or an ensemble and in this connection we make

REMARK 3.2.1. One notes that the use of $\psi\psi^*$ automatically suggests or involves an ensemble if it is to be interpreted as a probability density. Thus the idea that a particle has only a probability of being at or near x seems to mean that some paths take it there but others don't and this is consistent with Feynman's use of path integrals for example. This seems also to say that there is no such thing as a particle, only a collection of versions or cloud connected to the particle idea. Bohmian theory on the other hand for a fixed energy gives a one parameter family of trajectories associated to ψ (see here Section 2.2 and [**197**] for details). This is because the trajectory arises from a third order differential while fixing the solution ψ of the second order stationary Schrödinger equation involves only two "boundary" conditions. As was shown in [**197**] this automatically generates a Heisenberg inequality $\Delta x \Delta p \geq c\hbar$; i.e. the uncertainty is built in when using the wave function ψ and amazingly can be expressed by the operator theoretical framework of quantum mechanics. Thus a one parameter family of paths can be associated with the use of $\psi\psi^*$ and this generates the cloud or ensemble automatically associated with the use of ψ. In fact, based on Remark 2.2.2, one might conjecture that upon using a wave function discription of quantum particle motion, one opens the door to a cloud of particles, all of whose motions are incompletely governed by the SE, since one determining condition for particle motion is ignored. Thus automatically the quantum potential will give rise to a force acting on any such particular trajectory and the "ensemble" idea naturally applies to a cloud of identical particles (cf. also Theorem 1.2.1 and Corollary 1.2.1).

Now first ignore gravity and look at the geometrical properties of the conformal factor given via

$$g_{\mu\nu} = e^{4\Sigma}\eta_{\mu\nu};\ e^{4\Sigma} = \frac{\mathfrak{M}^2}{m^2} = exp\left(\alpha\frac{\Box_\eta\sqrt{\rho}}{\sqrt{\rho}}\right) = exp\left(\alpha\frac{\Box_\eta\sqrt{|\mathfrak{T}|}}{\sqrt{|\mathfrak{T}|}}\right) \tag{2.22}$$

where $\mathfrak{T}$ is the trace of the EM tensor and is substituted for ρ (true for dust). The Einstein tensor for this metric is

$$\mathfrak{G}_{\mu\nu} = 4g_{\mu\nu}\Box_\eta exp(-\Sigma) + 2exp(-2\Sigma)\partial_\mu\partial_\nu exp(2\Sigma) \tag{2.23}$$

Hence as an Ansatz one can suppose that in the presence of gravitational effects the field equation would have a form

$$\mathcal{R}_{\mu\nu} - \frac{1}{2}\mathcal{R}g_{\mu\nu} = \kappa\mathfrak{T}_{\mu\nu} + 4g_{\mu\nu}e^{\Sigma}\Box e^{-\Sigma} + 2e^{-2\Sigma}\nabla_\mu\nabla_\nu e^{2\Sigma} \tag{2.24}$$

This is written in a manner such that in the limit $\mathfrak{T}_{\mu\nu} \to 0$ one will obtain (2.22). Taking the trace of the last equation one gets $-\mathcal{R} = \kappa\mathfrak{T} - 12\Box\Sigma + 24(\nabla\Sigma)^2$ which has the iterative solution $\kappa\mathfrak{T} = -\mathcal{R} + 12\alpha\Box[(\Box\sqrt{\mathcal{R}})/\sqrt{\mathcal{R}}]$ leading to

$$\Sigma = \alpha[(\Box\sqrt{|\mathfrak{T}|}/\sqrt{|\mathfrak{T}|})] \simeq \alpha[(\Box\sqrt{|\mathcal{R}|})/\sqrt{|\mathcal{R}|})] \tag{2.25}$$

to first order in α.

One goes now to the field equations for a toy model. First from the above one sees that $\mathfrak{T}$ can be replaced by $\mathcal{R}$ in the expression for the quantum potential or for the conformal factor of the metric. This is important since the explicit reference to ensemble density is removed and the theory works for a single particle or an ensemble. So from (2.24) for a toy quantum gravity theory one assumes the following field equations

$$\mathfrak{G}_{\mu\nu} - \kappa\mathfrak{T}_{\mu\nu} - \mathfrak{Z}_{\mu\nu\alpha\beta}exp\left(\frac{\alpha}{2}\Phi\right)\nabla^\alpha\nabla^\beta exp\left(-\frac{\alpha}{2}\Phi\right) = 0 \tag{2.26}$$

where $\mathfrak{Z}_{\mu\nu\alpha\beta} = 2[g_{\mu\nu}g_{\alpha\beta} - g_{\mu\alpha}g_{\nu\beta}]$ and $\Phi = (\Box\sqrt{|\mathcal{R}|}/\sqrt{|\mathcal{R}|})$. The number 2 and the minus sign of the second term are chosen so that the energy equation derived later will be correct. Note that the trace of (2.26) is

$$\mathcal{R} + \kappa\mathfrak{T} + 6exp(\alpha\Phi/2)\Box exp(-\alpha\Phi/2) = 0 \tag{2.27}$$

and this represents the connection of the Ricci scalar curvature of space time and the trace of the matter EM tensor. If a perturbative solution is admitted one can expand in powers of α to find $\mathcal{R}^{(0)} = -\kappa\mathfrak{T}$ and $\mathcal{R}^{(1)} = -\kappa\mathfrak{T} - 6exp(\alpha\Phi^0/2)\Box exp(-\alpha\Phi^0/2)$ where $\Phi^{(0)} = \Box\sqrt{|\mathfrak{T}|}/\sqrt{|\mathfrak{T}|}$. The energy relation can be obtained by taking the four divergence of the field equations and since the divergence of the Einstein tensor is zero one obtains

$$\kappa\nabla^\nu\mathfrak{T}_{\mu\nu} = \alpha\mathcal{R}_{\mu\nu}\nabla^\nu\Phi - \frac{\alpha^2}{4}\nabla_\mu(\nabla\Phi)^2 + \frac{\alpha^2}{2}\nabla_\mu\Phi\Box\Phi \tag{2.28}$$

For a dust with $\mathfrak{T}_{\mu\nu} = \rho u_\mu u_\nu$ and u_μ the velocity field, the conservation of mass law is $\nabla^\nu(\rho\mathfrak{M}u_\nu) = 0$ so one gets to first order in α $\nabla_\mu\mathfrak{M}/\mathfrak{M} = -(\alpha/2)\nabla_\mu\Phi$ or $\mathfrak{M}^2 = m^2 exp(-\alpha\Phi)$ where m is an integration constant. This is the correct relation of mass and quantum potential.

In [**873**] there is then some discussion about making the conformal factor dynamical via a general scalar tensor action (cf. also [**867**]) and subsequently one makes both the conformal factor and the quantum potential into dynamical fields and creates a scalar tensor theory with two scalar fields. Thus start with a general action

$$\mathfrak{A} = \int d^4x\sqrt{-g}\left[\phi\mathcal{R} - \omega\frac{\nabla_\mu\phi\nabla^\mu\phi}{\phi} - \frac{\nabla_\mu Q\nabla^\mu Q}{\phi} + 2\Lambda\phi + \mathfrak{L}_m\right] \tag{2.29}$$

The cosmological constant generally has an interaction term with the scalar field and here one uses an ad hoc matter Lagrangian

$$\mathfrak{L}_m = \frac{\rho}{m}\phi^a\nabla_\mu S\nabla^\mu S - m\rho\phi^b - \Lambda(1+Q)^c + \alpha\rho(e^{\ell Q} - 1) \tag{2.30}$$

(only the first two terms $1+Q$ from $exp(Q)$ are used for simplicity in the third term). Here a, b, c are constants to be fixed later and the last term is chosen (heuristically) in such a manner as to have an interaction between the quantum potential field and the ensemble density (via the equations of motion); further the interaction is chosen so that it vanishes in the classical limit but this is ad hoc. Variation of the above action yields

(1) The scalar fields equation of motion

$$\mathcal{R} + \frac{2\omega}{\phi}\Box\phi - \frac{\omega}{\phi^2}\nabla^\mu\phi\nabla_\mu\phi + 2\Lambda + \tag{2.31}$$

$$+\frac{1}{\phi^2}\nabla^\mu Q\nabla_\mu Q + \frac{a}{m}\rho\phi^{a-1}\nabla^\mu S\nabla_\mu S - mb\rho\phi^{b-1} = 0$$

(2) The quantum potential equations of motion

$$(\Box Q/\phi) - (\nabla_\mu Q\nabla^\mu\phi/\phi^2) - \Lambda c(1+Q)^{c-1} + \alpha\ell\rho exp(\ell Q) = 0 \tag{2.32}$$

(3) The generalized Einstein equations

$$\mathfrak{G}^{\mu\nu} - \Lambda g^{\mu\nu} = -\frac{1}{\phi}\mathfrak{T}^{\mu\nu} - \frac{1}{\phi}[\nabla^\mu\nabla^\nu - g^{\mu\nu}\Box]\phi + \frac{\omega}{\phi^2}\nabla^\mu\phi\nabla^\nu\phi - \tag{2.33}$$
$$-\frac{\omega}{2\phi^2}g^{\mu\nu}\nabla^\alpha\phi\nabla_\alpha\phi + \frac{1}{\phi^2}\nabla^\mu Q\nabla^\nu Q - \frac{1}{2\phi^2}g^{\mu\nu}\nabla^\alpha Q\nabla_\alpha Q$$

(4) The continuity equation $\nabla_\mu(\rho\phi^a\nabla^\mu S) = 0$
(5) The quantum Hamilton Jacobi equation

$$\nabla^\mu S\nabla_\mu S = m^2\phi^{b-a} - \alpha m\phi^{-a}(e^{\ell Q} - 1) \tag{2.34}$$

In (2.31) the scalar curvature and the term $\nabla^\mu S\nabla_\mu S$ can be eliminated using (2.33) and (2.34); further on using the matter Lagrangian and the definition of the EM tensor one has

$$(2\omega - 3)\Box\phi = (a+1)\rho\alpha(e^{\ell Q} - 1) - 2\Lambda(1+Q)^c + 2\Lambda\phi - \frac{2}{\phi}\nabla_\mu Q\nabla^\mu Q \tag{2.35}$$

(where $b = a+1$). Solving (2.32) and (2.35) with a perturbation expansion in α one finds

$$Q = Q_0 + \alpha Q_1 + \cdots ;\ \phi = 1 + \alpha Q_1 + \cdots ;\ \sqrt{\rho} = \sqrt{\rho_0} + \alpha\sqrt{\rho_1} + \cdots \tag{2.36}$$

where the conformal factor is chosen to be unity at zeroth order so that as $\alpha \to 0$ (2.34) goes to the classical HJ equation. Further since by (2.34) the quantum mass is $m^2\phi + \cdots$ the first order term in ϕ is chosen to be Q_1 (cf. (2.12)). Also we will see that $Q_1 \sim \Box\sqrt{\rho}/\sqrt{\rho}$ plus corrections which is in accord with Q as a quantum potential field. In any case after some computation one obtains $a = 2\omega k$, $b = a+1$, and $\ell = (1/4)(2\omega k + 1) = (1/4)(a+1) = b/4$ with $Q_0 = [1/c(2c-3)]\{[-(2\omega k+1)/2\Lambda]k\sqrt{\rho_0} - (2c^2 - c + 1)\}$ while ρ_0 can be determined (cf. [**873**] for details). Thus heuristically the quantum potential can be regarded as a dynamical field and perturbatively one gets the correct dependence of quantum potential upon density, modulo some corrective terms.

One goes next to a number of examples and we only consider here the conformally flat solution (cf. also [**869**]). Thus take $g_{\mu\nu} = exp(2\Sigma)\eta_{\mu\nu}$ where $\Sigma << 1$. One obtains from (2.24)

$$\mathcal{R}_{\mu\nu} = \eta_{\mu\nu}\Box\Sigma + 2\partial_\mu\partial_\nu\Sigma \Rightarrow \mathfrak{G}_{\mu\nu} = 2\partial_\mu\partial_\nu\Sigma - 2\eta_{\mu\nu}\Box\Sigma \tag{2.37}$$

One can solve this iteratively to get

$$\mathcal{R}^{(0)} = -\kappa\mathfrak{T} \Rightarrow \Sigma^{(0)} = -\frac{\kappa}{6}\Box^{-1}\mathfrak{T}; \tag{2.38}$$
$$\mathcal{R}^{(1)} = -\kappa\mathfrak{T} + 3\alpha\Box\frac{\Box\sqrt{|\mathfrak{T}|}}{\sqrt{|\mathfrak{T}|}} \Rightarrow \Sigma^{(1)} = -\frac{\kappa}{6}\Box^{-1}\mathfrak{T} + \frac{\alpha}{2}\frac{\Box\sqrt{|\mathfrak{T}|}}{\sqrt{|\mathfrak{T}|}}$$

Consequently

$$\Sigma = -\frac{\kappa}{6}\Box^{-1}\mathfrak{T} + \frac{\alpha}{2}\frac{\Box\sqrt{|\mathfrak{T}|}}{\sqrt{|\mathfrak{T}|}} + \cdots \tag{2.39}$$

The first term is pure gravity, the second pure quantum, and the remaining terms involve gravity-quantum interactions. Other impressive examples are given (cf. also [**869**]).

One goes now to a generalized equivalence principle. The gravitational effects determine the causal structure of spacetime as long as quantum effects give its conformal structure. This does not mean that quantum effects have nothing to do with the causal structure; they can act on the causal structure through back reaction terms appearing in the metric field equations. The conformal factor of the metric is a function of the quantum potential and the mass of a relativistic particle is a field produced by quantum corrections to the classical mass. One has shown that the presence of the quantum potential is equivalent to a conformal mapping of the metric. Thus in different conformally related frames one feels different quantum masses and different curvatures. In particular there are two frames with one containing the quantum mass field and the classical metric while the other contains the classical mass and the quantum metric. In general frames both the spacetime metric and the mass field have quantum properties so one can state that different conformal frames are identical pictures of the gravitational and quantum phenomena. We feel different quantum forces in different conformal frames. The question then arises of whether the geometrization of quantum effects implies conformal invariance just as gravitational effects imply general coordinate invariance. One sees here that Weyl geometry provides additional degrees of freedom which can be identified with quantum effects and seems to create a unified geometric framework for understanding both gravitational and quantum forces. Some features here are: (i) Quantum effects appear independent of any preferred length scale. (ii) The quantum mass of a particle is a field. (iii) The gravitational constant is also a field depending on the matter distribution via the quantum potential (cf. [**867, 874**]). (iv) A local variation of matter field distribution changes the quantum potential acting on the geometry and alters it globally; the nonlocal character is forced by the quantum potential (cf. [**868**]).

2.1. DIRAC-WEYL ACTION. Next (still following [**873**]) one goes to Weyl geometry based on the Weyl-Dirac action

$$\mathfrak{A} = \int d^4x\sqrt{-g}(F_{\mu\nu}F^{\mu\nu} - \beta^2\ {}^W\mathcal{R} + (\sigma+6)\beta_{;\mu}\beta^{;\mu} + \mathfrak{L}_{matter}) \tag{2.40}$$

Here $F_{\mu\nu}$ is the curl of the Weyl 4-vector ϕ_μ, σ is an arbitrary constant and β is a scalar field of weight -1. The symbol ";" represents a covariant derivative under general coordinate and conformal transformations (Weyl covariant derivative) defined as $X_{;\mu} = {}^W\nabla_\mu X - \mathcal{N}\phi_\mu X$ where $\mathcal{N}$ is the Weyl weight of X. The equations of motion are then

$$\mathfrak{G}^{\mu\nu} = -\frac{8\pi}{\beta^2}(\mathfrak{T}^{\mu\nu} + M^{\mu\nu}) + \frac{2}{\beta}(g^{\mu\nu}\,{}^W\nabla^{\alpha}\,{}^W\nabla_\alpha\beta - {}^W\nabla^{\mu}\,{}^W\nabla^\nu\beta) + \tag{2.41}$$

$$+\frac{1}{\beta^2}(4\nabla^\mu\beta\nabla^\nu\beta - g^{\mu\nu}\nabla^\alpha\beta\nabla_\alpha\beta) + \frac{\sigma}{\beta^2}(\beta^{;\mu}\beta^{;\nu} - \frac{1}{2}g^{\mu\nu}\beta^{;\alpha}\beta_{;\alpha});$$

$${}^W\nabla_\mu F^{\mu\nu} = \frac{1}{2}\sigma(\beta^2\phi^\mu + \beta\nabla^\mu\beta) + 4\pi J^\mu;$$

$$\mathcal{R} = -(\sigma+6)\frac{{}^W\Box\beta}{\beta} + \sigma\phi_\alpha\phi^\alpha - \sigma\,{}^W\nabla^\alpha\phi_\alpha + \frac{\psi}{2\beta}$$

where

$$M^{\mu\nu} = (1/4\pi)[(1/4)g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta} - F^{\mu}_{\alpha}F^{\nu\alpha}] \tag{2.42}$$

and

$$8\pi\mathfrak{T}^{\mu\nu} = \frac{1}{\sqrt{-g}}\frac{\delta\sqrt{-g}\mathfrak{L}_{matter}}{\delta g_{\mu\nu}};\ 16\pi J^{\mu} = \frac{\delta\mathfrak{L}_{matter}}{\delta\phi_{\mu}};\ \psi = \frac{\delta\mathfrak{L}_{matter}}{\delta\beta} \tag{2.43}$$

For the equations of motion of matter and the trace of the EM tensor one uses invariance of the action under coordinate and gauge transformations, leading to

$$^{W}\nabla_{\nu}\mathfrak{T}^{\mu\nu} - \mathfrak{T}\frac{\nabla^{\mu}\beta}{\beta} = J_{\alpha}\phi^{\alpha\mu} - \left(\phi^{\mu} + \frac{\nabla^{\mu}\beta}{\beta}\right){}^{W}\nabla_{\alpha}J^{\alpha}; \tag{2.44}$$

$$16\pi\mathfrak{T} - 16\pi{}^{W}\nabla_{\mu}J^{\mu} - \beta\psi = 0$$

The first relation is a geometrical identity (Bianchi identity) and the second shows the mutual dependence of the field equations. Note that in the Weyl-Dirac theory the Weyl vector does not couple to spinors so ϕ_{μ} cannot be interpreted as the EM potential; the Weyl vector is used as part of the spacetime geometry and the auxillary field (gauge field) β represents the quantum mass field. The gravity fields $g_{\mu\nu}$ and ϕ_{μ} and the quantum mass field determine the spacetime geometry. Now one constructs a Bohmian quantum gravity which is conformally invariant in the framework of Weyl geometry. If the model has mass this must be a field (since mass has non-zero Weyl weight). The Weyl-Dirac action is a general Weyl invariant action as above and for simplicity now assume the matter Lagrangian does not depend on the Weyl vector so that $J_{\mu} = 0$. The equations of motion are then

$$\mathfrak{G}^{\mu\nu} = -\frac{8\pi}{\beta^2}(\mathfrak{T}^{\mu\nu} + M^{\mu\nu}) + \frac{2}{\beta}(g^{\mu\nu}{}^{W}\nabla^{\alpha}{}^{W}\nabla_{\alpha}\beta - {}^{W}\nabla^{\mu}{}^{W}\nabla^{\nu}\beta) + \tag{2.45}$$

$$+\frac{1}{\beta^2}(4\nabla^{\mu}\beta\nabla^{\nu}\beta - g^{\mu\nu}\nabla^{\alpha}\beta\nabla_{\alpha}\beta) + \frac{\sigma}{\beta^2}\left(\beta^{;\mu}\beta^{;\nu} - \frac{1}{2}g^{\mu\nu}\beta^{;\alpha}\beta_{;\alpha}\right);$$

$$^{W}\nabla_{\nu}F^{\mu\nu} = \frac{1}{2}\sigma(\beta^2\phi^{\mu} + \beta\nabla^{\mu}\beta);\ \mathcal{R} = -(\sigma+6)\frac{^{W}\Box\beta}{\beta} + \sigma\phi_{\alpha}\phi^{\alpha} - \sigma{}^{W}\nabla^{\alpha}\phi_{\alpha} + \frac{\psi}{2\beta}$$

The symmetry conditions are

$$^{W}\nabla_{\nu}\mathfrak{T}^{\mu\nu} - \mathfrak{T}(\nabla^{\mu}\beta/\beta) = 0;\ 16\pi\mathfrak{T} - \beta\psi = 0 \tag{2.46}$$

(recall $\mathfrak{T} = \mathfrak{T}^{\mu\nu}_{\mu\nu}$). One notes that from (2.45) results ${}^{W}\nabla_{\mu}(\beta^2\phi^{\mu} + \beta\nabla^{\mu}\beta) = 0$ so ϕ_{μ} is not independent of β. To see how this is related to the Bohmian quantum theory one introduces a quantum mass field and shows it is proportional to the Dirac field. Thus using (2.45) and (2.46) one has

$$\Box\beta + \frac{1}{6}\beta\mathcal{R} = \frac{4\pi}{3}\frac{\mathfrak{T}}{\beta} + \sigma\beta\phi_{\alpha}\phi^{\alpha} + 2(\sigma-6)\phi^{\gamma}\nabla_{\gamma}\beta + \frac{\sigma}{\beta}\nabla^{\mu}\beta\nabla_{\mu}\beta \tag{2.47}$$

This can be solved iteratively via

$$\beta^2 = (8\pi\mathfrak{T}/\mathcal{R}) - \{1/[(\mathcal{R}/6) - \sigma\phi_{\alpha}\phi^{\alpha}]\}\beta\Box\beta + \cdots \tag{2.48}$$

Now assuming $\mathfrak{T}^{\mu\nu} = \rho u^\mu u^\nu$ (dust with $\mathfrak{T} = \rho$) we multiply (2.46) by u_μ and sum to get

$$ {}^W\nabla_\nu(\rho u^\nu) - \rho(u_\mu \nabla^\mu \beta/\beta) = 0 \tag{2.49}$$

Then put (2.46) into (2.49) which yields

$$ u^{\nu W}\nabla_\nu u^\mu = (1/\beta)(g^{\mu\nu} - u^\mu u^\nu)\nabla_\nu \beta \tag{2.50}$$

To see this write (assuming $g^{\mu\nu}\nabla_\nu\beta = \nabla^\mu\beta$)

$$ {}^W\nabla_\nu(\rho u^\mu u^\nu) = u^{\mu W}\nabla_\nu \rho u^\mu + \rho u^{\nu W}\nabla_\nu u^\mu \Rightarrow \tag{2.51}$$

$$\Rightarrow u^\mu\left(\frac{u_\mu \nabla^\mu \beta}{\beta}\right) + u^{\nu W}\nabla_\nu u^\mu - \frac{\nabla^\mu\beta}{\beta} = 0 \Rightarrow u^{\nu W}\nabla_\nu u^\mu = (1 - u^\mu u_\mu)\frac{\nabla^\mu \beta}{\beta} =$$

$$(g^{\mu\nu} - u^\mu u_\mu g^{\mu\nu})\frac{\nabla_\nu\beta}{\beta} = (g^{\mu\nu} - u^\mu u^\nu)\frac{\nabla_\nu \beta}{\beta}$$

which is (2.49). Then from (2.48)

$$ \beta^{2(1)} = \frac{8\pi\mathfrak{T}}{\mathcal{R}};\ \beta^{2(2)} = \frac{8\pi\mathfrak{T}}{\mathcal{R}}\left(1 - \frac{1}{(\mathcal{R}/6) - \sigma\phi_\alpha\phi^\alpha}\frac{\Box\sqrt{\mathfrak{T}}}{\sqrt{\mathfrak{T}}}\right);\cdots \tag{2.52}$$

Comparing with (2.14) and (2.3) shows that we have the correct equations for the Bohmian theory provided one identifies

$$ \beta \sim \mathfrak{M};\ \frac{8\pi\mathfrak{T}}{\mathcal{R}} \sim m^2;\ \frac{1}{\sigma\phi_\alpha\phi^\alpha - (\mathcal{R}/6)} \sim \alpha = \frac{\hbar^2}{m^2c^2} \tag{2.53}$$

Thus β is the Bohmian quantum mass field and the coupling constant α (which depends on $\hbar$) is also a field, related to geometrical properties of spacetime. One notes that the quantum effects and the length scale of the spacetime are related. To see this suppose one is in a gauge in which the Dirac field is constant; apply a gauge transformation to change this to a general spacetime dependent function, i.e.

$$ \beta = \beta_0 \to \beta(x) = \beta_0 exp(-\Xi(x));\ \phi_\mu \to \phi_\mu + \partial_\mu \Xi \tag{2.54}$$

Thus the gauge in which the quantum mass is constant (and the quantum force is zero) and the gauge in which the quantum mass is spacetime dependent are related to one another via a scale change. In particular ϕ_μ in the two gauges differ by $-\nabla_\mu(\beta/\beta_0)$ and since ϕ_μ is a part of Weyl geometry and the Dirac field represents the quantum mass one concludes that the quantum effects are geometrized (cf. also (2.45) which shows that ϕ_μ is not independent of β so the Weyl vector is determined by the quantum mass and thus the geometrical aspects of the manifold are related to quantum effects).

2.2. REMARKS ON CONFORMAL GRAVITY. We go here to a series of papers by Arias, Bonal, Cardenas, Gonzalez, Leyva, Martin, and Quiros (cf. [**46, 47, 48, 133, 181, 182, 796, 797, 798, 799**]) and sketch at some length some results concerning Brans-Dicke theory, conformal gravity, and deBroglie-Bohm-Weyl (dBBW) theory (many other topics are also covered in these papers which we omit here - cf. also [**24, 37, 801, 974, 975**]). The presentation in [**188**] of this material is difficult to read and we try here for a smoother development. In [**188**] we started with [**797, 799**] and then gave a later reformulation from [**798**]; we expand upon this now (still in a more or less chronological order [**799, 797, 798, 796**]) and try to make matters clearer. Questions about the physical significance of Riemannian geometry in relativity have been raised in the past (cf. [**150, 301**]) due to the arbitrariness in the metric tensor resulting from the indefiniteness in the choice of units of measure. In fact Brans-Dicke (BD) theory with a changing dimensionless gravitational coupling constant $Gm^2 \sim \phi^{-1}$ (with m the intertial mass of some elementary particle and ϕ the BD field - $\hbar = c = 1$ here) can be formulated in two different ways since either m or G could vary with position in spacetime. The choice $G \sim \phi^{-1}$ with $m = const.$ leads to the Jordan frame (JF) formalism based on the Lagrangian

$$L^{BD}[g,\phi] = \frac{\sqrt{-g}}{16\pi}\left(\phi R - \frac{\omega}{\phi}g^{nm}\nabla_n\phi\nabla_m\phi\right) + L_M[g] \tag{2.55}$$

where R is the curvature scalar, ω is the BD coupling constant, and $L_M[g]$ is the Lagrangian density for ordinary matter minimally coupled to the scalar field. On the other hand the choice $m \sim \phi^{-1/2}$ with G constant leads to the Einstein frame (EF) BD theory based on the Lagrangian

$$\hat{L}^{BD} = \frac{\sqrt{-g}}{16\pi}\left(\hat{R} - \left(\omega + \frac{3}{2}\right)\hat{g}^{nm}\hat{\nabla}_n\hat{\phi}\hat{\nabla}_m\hat{\phi}\right) + \hat{L}_M[\hat{g},\hat{\phi}] \tag{2.56}$$

where now in the EF metric $\hat{g}$ the ordinary matter is nonminimally coupled to the scalar field $\hat{\phi} \equiv log(\phi)$ through the Lagrangian density $\hat{L}_M[\hat{g},\hat{\phi}]$. Both JF and EF formulations of BD gravity are equivalent representations of the same physical situation since they both belong to the same conformal class (cf. [**150**]); in particular $L^{BD}_{EF} \equiv L^{BD}_{JF}$ via a rescaling of spacetime metric $g \to \hat{g} = \phi g$ or $\hat{g}_{ab} = \phi g_{ab}$ where ϕ is smooth and nonvanishing. This rescaling can be interpreted as a particular transformation of the physical units and any dimensionless number (e.g. Gm^2) is invariant; experimental observations are unchanged since spacetime coincidences are not affected. Hence both based formulations (one based on varying G and the other on varying m are indistinguishable) and one has physically equivalent representations of a same physical situation. The same line of reasoning can be applied if minimal and nonminimal coupling to matter are interchanged via

$$(\mathbf{A})\;\; L^{GR}[g,\phi] = \frac{\sqrt{-g}}{16\pi}\left(\phi R - \frac{\omega}{\phi}g^{nm}\nabla_n\phi\nabla_m\phi\right) + L_M[g,\phi]; \tag{2.57}$$

$$(\mathbf{B})\;\; \hat{L}^{GR} = \frac{\sqrt{-g}}{16\pi}\left(\hat{R} - \left(\omega + \frac{3}{2}\right)\hat{g}^{nm}\hat{\nabla}_n\hat{\phi}\hat{\nabla}_m\hat{\phi}\right) + \hat{L}_M[\hat{g}]$$

Both Lagrangians represent equivalent pictures of GR and **(B)** is simply GR with a scalar field as an additional source of gravity (EFGR) and its conformally equivalent Lagrangian **(A)** refers to Jordan frame GR (JFGR). The field equations derivable from Lagrangian **(B)** are

$$\hat{G}_{ab} = 8\pi\hat{T}_{ab} + \left(\omega + \frac{3}{2}\right)\left(\hat{\nabla}_a\hat{\phi}\hat{\nabla}_b\hat{\phi} - \frac{1}{2}\hat{g}_{ab}\hat{g}^{nm}\hat{\nabla}_n\hat{\phi}\hat{\nabla}_m\hat{\phi}\right); \tag{2.58}$$

$$\Box\hat{\phi} = 0;\ \hat{\nabla}_n\hat{T}^{na} = 0;\ \Box = \hat{g}^{nm}\hat{\nabla}_n\hat{\nabla}_m$$

where $\hat{G}_{ab} = \hat{R}_{ab} - (1/2)\hat{g}_{ab}\hat{R}$ and $\hat{T}_{ab} = (2/\sqrt{\hat{g}})(\partial/\partial\hat{g}^{ab})(\sqrt{-\hat{g}}\hat{L}_M)$. Some disadvantages for JFGR historically involve first that the BD scalar field is nonminimally coupled both to scalar curvature and to ordinary matter so the gravitational constant G varies as $G \sim \phi^{-1}$. At the same time the material test particles don't follow the geodesics of the geometry since they are acted on by both the metric field and the scalar field. In particular masses vary from point to point in spacetime so as to preserve a constant Gm^2 (so $m \sim \phi^{1/2}$). The most serious (but illusory) objection is linked with the formulation of the theory in unphysical variables so that the kinetic energy of the scalar field is not positive deffinite (cf. [**351**]). However one shows in [**799**] that the indefiniteness in the sign of the energy density in the Jordan frame is only apparent; in fact once the scalar field energy density is positive definite in the Einstein frame it is also in the Jordan frame.

Usually the JF formulation of BD gravity is linked with Riemannian geometry (cf. [**150**]). This is directly related to the fact that in the JFBD formalism ordinary matter is minimally coupled to the scalar BD field through $L_M[g]$ in (2.55). This means that point particles follow the geodesics of the Riemannian geometry. This geometry is based on the parallel transport law and length preservation law

$$d\xi^a = -\gamma^a_{nm}\xi^m dx^n;\ dg(\xi,\xi) = 0 \tag{2.59}$$

where $g(\xi,\xi) = g_{nm}\xi^n\xi^m$ and γ^a_{nm} are the affine connections of the manifold. These postulates mean that $\gamma^a_{bc} = \Gamma^a_{bc} = (1/2)g^{an}(g_{nb,c} + g_{nc,b} - g_{bc,n})$ (Christoffel symbols). After the rescaling $\hat{g}_{ab} = \phi g_{ab}$ the above parallel transport and length rules become (recall $\hat{\phi} \sim log(\phi)$)

$$d\xi^a = -\hat{\gamma}^a_{nm}\xi^m dx^n;\ d\hat{g}(\xi,\xi) = dx^n\hat{\nabla}_n\hat{\phi}\hat{g}(\xi,\xi); \tag{2.60}$$

$$\hat{\gamma}^a_{bc} = \hat{\Gamma}^a_{bc} - \frac{1}{2}(\hat{\nabla}_b\hat{\phi}\delta^a_c + \hat{\nabla}_c\hat{\phi}\delta^a_b - \hat{\nabla}^a\hat{\phi}\hat{g}_{bc})$$

Thus the affine connections of the manifold don't coincide with the Christoffel symbols of the metric and one has a Weyl type manifold. Thus JF and EF Lagrangians of both BD and GR theories are connected by conformal rescaling of the metric together with scalar field redefinition. This means JF and EF formulations on the one hand and Riemannian and Weyl type geometries on the other form conformal equivalence classes (uniquely defined only after the coupling of matter fields to the metric). In BD theory for example matter minimally couples to the JF so the test particles follow the geodesics of the Riemannian geometry (i.e. JFBD is linked to Riemannian geometry) while EFBD theory (conformal to JFBD) is linked to a Weyl type geometry. Similarly EFGR is linked with Riemannian geometry and JFGR (conformal to EFGR) is linked to a Weyl type geometry. When the matter part of the Lagrangian is absent both BD and GR theories can be interprted on the grounds of either Riemann or Weyl type geometry and one can conclude that BD and GR theory with an extra scalar field coincide.

The field equations of JFGR can be derived, either directly from (2.57) (equation **(A)**) or by conformally mapping (2.58) back to the JF metric according to $\hat{g} = \phi g$, to obtain

$$\Box\phi = 0;\ \nabla_n T^{na} = \frac{1}{2}\phi^{-1}\nabla^a\phi T; \tag{2.61}$$

$$G_{ab} = \frac{8\pi}{\phi}T_{ab} + \frac{\omega}{\phi^2}(\nabla_a\phi\nabla_b\phi - \frac{1}{2}g_{ab}g^{nm}\nabla - n\phi\nabla_m\phi) + \frac{1}{\phi}(\nabla_a\nabla_b\phi - g_{ab}\Box\phi)$$

where $T_{ab} = (2/\sqrt{-g})(\partial/\partial g^{ab})(\sqrt{-g}L_M)$ is the stress energy tensor for ordinary matter in the Jordan frame. The energy is not conserved because the scalar field ϕ exchanges energy with the metric and with the matter fields. The equation of motion of an uncharged spinless mass point that is acted both by the JF metric field g and the scalar field ϕ is

$$\frac{d^2x^a}{ds^2} = -\Gamma^a_{nm}\frac{dx^m}{ds}\frac{dx^n}{ds} - \frac{1}{2}\phi^{-1}\nabla_n\phi\left(\frac{dx^n}{ds}\frac{dx^a}{ds} - g^{an}\right) \tag{2.62}$$

This does not coincide with the geodesic equation of the JF metric and this, together with the more complex structure of (2.61) in comparison to (2.58), introduces additional complications in the dynamics of matter fields. The fact that the Jordan frame does not lead to a well defined energy momentum tensor for the scalar field is perhaps the most serious objection to this representation (cf. [**351**]). Thus the kinetic energy of the JF scalar field is negative definite or indefinite unlike the Einstein frame where for $\omega > -(3/2)$ it is positive definite; this implies no stable ground state and hence unphysical variables (cf. [**351**]). However although the right side of (2.61) does not have a definite sign the scalar field stress energy tensor can be given the canonical form (cf. [**842**] for example). In [**799**] one obtains the same result as in [**842**] by rewriting equation (2.61) in terms of affine magnitudes in the Weyl type manifold. Thus the affine connections of the JF (Weyl type) manifold γ^a_{bc} are related with the Christoffel symbols of the JF metric through $\gamma^a_{bc} = \Gamma^a_{bc} + (1/2)\phi^{-1}(\nabla_b\phi\delta^a_c + \nabla_c\phi\delta^a_b - \nabla^a\phi g_{bc})$ and one can define the affine Einstein tensor ${}^\gamma G_{ab}$ via the γ^a_{bc} instead of the Christoffel symbols of Γ^a_{bc} so that (2.61) becomes

$${}^\gamma G_{ab} = \frac{8\pi}{\phi}T_{ab} + \frac{[\omega + (3/2)]}{\phi^2}(\nabla_a\phi\nabla_b\phi - (1/2)g_{ab}g^{nm}\nabla - n\phi\nabla_m\phi) \tag{2.63}$$

Now $(\phi/8\pi)$ times the second term in the right side has the canonical form for the stress energy tensor (true stress energy tensor) while $(\phi/8\pi)$ times the sum of the second and third terms in the right side will be called the effective stress energy tensor for the BD scalar field ϕ (cf. (2.58)). Thus once the scalar field energy density is positive definite in the Einstein frame it is also in the Jordan frame. This removes the main physical objection to the Jordan frame formulation of GR.

Another remarkable feature of JFGR is the invariance under the following conformal transformations

$$\text{(2.64)}\qquad (\mathbf{A})\ \tilde{g}_{ab} = \phi^2 g_{ab};\ \tilde{\phi} = \phi^{-1}\ \ (\mathbf{B})\ \tilde{g}_{ab} = f g_{ab};\ \tilde{\phi} = f^{-1}\phi$$

where f is smooth. Also JFBD based on **(A)** in (2.57) is invariant under the more general rescaling (cf. [**350, 796**])

$$\text{(2.65)}\qquad \tilde{g}_{ab} = \phi^{2\alpha} g_{ab};\ \tilde{\phi} = \phi^{1-2\alpha};\ \tilde{\omega} = \frac{\omega - 6\alpha(\alpha-1)}{(1-2\alpha)^2}$$

for $\alpha \neq 1/2$. The conformal invariance of a given theory of gravitation under a transformation of physical units is very desirable and in particular **(A)** in (2.57) is thus a better candidate for BD theory than classical theories given by (2.55), (2.56), or **(B)** in (2.57) which are not invariant.

We go now to [**798**] where a number of argments from [**797, 799**] are repeated and amplified for greater clarity. It has been demonstrated already that GR with an extra scalar field and its conformal formulation (JFGR) are different but physically equivalent representations of the same theory. The claim is based on the argument that spacetime coincidences (coordinates) are not affected by a conformal rescaling of the spacetime metric ($\bigstar$) $\hat{g}_{ab} = \Omega^2 g_{ab}$ where Ω^2 is a smooth nonvanishing function on the manifold. Thus the experimental observations (measuements) being nothing but verifications of these coincidences are unchanged too by ($\bigstar$). This means that canonical GR and its conformal image are experimentally indistinguishable. Now a possible objection to this claim could be based on the following argument (which will be refuted). **ARGUMENT**: In canonical GR the matter fields couple minimally to the metric $\hat{g}$ that determines metrical relations on a Riemannian spacetime. Hence matter particles follow the geodesics of the metric $\hat{g}$ (in Riemannian geometry) and their masses are constant over the spacetime manifold, i.e. it is the metric which matter "feels" or perhaps the "physical metric". Under the conformal rescaling the matter fields become non-minimally coupled to the conformal metric g and hence matter particles do not follow the geodesics of this last metric. Furthermore, it is not the metric that determines metrical relations on the Riemannian manifold. This line of reasoning leads to the following conclusion. Although canonical GR and its conformal image may be physically equivalent theories, nevertheless, the physical metric is that which determines metrical relations on a Riemannian spacetime and the conformal metric g is not the physical metric. **REFUTATION - to be developed**: Under the conformal rescaling ($\bigstar$) not only the Lagrangian of the theory is mapped into its conformal Lagrangian but the spacetime geometry itself is mapped too into a conformal geometry. In this last geometry metrical relations involve both the conformal metric g and the conformal factor Ω^2 generating ($\bigstar$). Hence in the conformal Lagrangian the matter fields should "feel" both the metric and the scalar function Ω, i.e. the matter particles would not follow the geodesics of the conformal metric alone. The result is that under ($\bigstar$) the "physical" metric of the untransformed geometry is effectively mapped into the "physical" metric of the conformal geometry. This "missing detail" has apparently been a source of long standing confusion and, although details have been sketched already, more will be provided.

Another question regarding metric theories of spacetime is also clarified, namely the physical content of a given theory of spacetime should be contained in the invariants of the group of position dependent transformations of units and coordinate transformations (cf. [**150**]). All known metric theories of spacetime, including GR, BD, and scalar-tensor theories in general fufill the requirement of invariance under the group of coordinate transformations. It is also evident that any consistent formulation of a given effective theory of spacetime must be invariant also under the group of transformations of units of length, time, and mass. This aspect is treated below and one shows that the only consistent formulation of gravity (among those studied here) is the conformal representation of GR.

Now with some repetition one considers various Lagrangians again. First is that for GR with an extra scalar field, namely ($\alpha \geq 0$ is a free parameter)

$$\hat{\mathfrak{L}}_{GR} = \sqrt{-\hat{g}}(\hat{R} - \alpha(\hat{\nabla}\hat{\phi})^2 + 16\pi\sqrt{-\hat{g}}L_M \tag{2.66}$$

(note $(\hat{\nabla}\hat{\phi})^2 = \hat{g}^{mn}\hat{\phi}_{,m}\hat{\phi}_{,n}$). When $\hat{\phi}$ is constant or $\alpha = 0$ one recovers the usual Einstein theory. Under the conformal rescaling ($\bigstar$) with $\Omega^2 = exp(\hat{\phi})$ the Lagrangian in (2.66) is mapped into its conformal Lagrangian (cf. (2.55))

$$\mathfrak{L}_{GR} = \sqrt{-g}e^{\hat{\phi}}(R - (\alpha - (3/2))(\nabla\hat{\phi})^2 + 16\pi\sqrt{-g}e^{2\hat{\phi}}L_M \equiv \tag{2.67}$$

$$\equiv \sqrt{-g}\left(\phi R - \left(\alpha - \frac{3}{2}\right)\frac{(\nabla\phi)^2}{\phi}\right) + 16\pi\sqrt{-g}L_M$$

(the latter expression having a more usual BD form). Due to the minimal coupling of the scalar field $\hat{\phi}$ to the curvature in canonical GR ((2.66)) the effective gravitational constant $\hat{G}$ (set equal to 1 in (2.66)) is a real constant. The minimal coupling of the matter fields to the metric in (2.66) entails that matter particles follow the geodesics of the metric $\hat{g}$. Hence the inertial mass $\hat{m}$ is constant too over spacetime. This implies that the dimensionless gravitational coupling constant $\hat{G}\hat{m}^2$ ($c = \hbar = 1$) is constant in spacetime - unlike BD theory where this evolves as ϕ^{-1}. This is a conformal invariant feature of GR since dimensionless constants do not change under ($\bigstar$); in other words in conformal GR Gm^2 is constant as well. However in this case ((2.67)) the effective gravitational constant varies like $G \sim \phi^{-1}$ and hence the particle masses vary like $m = exp(\hat{\phi}/2)\hat{m} = \phi^{1/2}\hat{m}$. According to [**301**] the conformal transformation ($\bigstar$) (with $\Omega^2 = \phi$) can be interpreted as a transformation of the units of length time and reciprocal mass; in particular there results $ds = \phi^{-1/2}d\hat{s}$ while $m^{-1} = \phi^{-1/2}\hat{m}^{-1}$. A careful look at (2.66) - (2.67) shows that Einstein's laws of gravity derivable from (2.66) change under the units transformation ($\bigstar$) and this seems to be a serious drawback of canonical GR (and BD theory and scalar-tensor theories in general) since in any consistent theory of spacetime the laws of physics must be invariant under a change of the units of length, time, and mass. This will be clarified below where it is shown that ($\bigstar$) with $\Omega^2 = \phi = exp(\hat{\phi})$ cannot be taken properly as a units transformation. It is just a transformation allowing jumping from one formulation to its conformal equivalent.

In [**797, 799**] one claimed that canonical GR ((2.66)) and its conformal Lagrangian (2.67) are physically equivalent theories since they are indistinguishable from the observational point of view. However it is common to believe that only one of the conformally related metrics is the "physical" metric, i.e. that which detrmines metrical relations on the spacetime manifold. The reasoning leading to this conclusion is based on the following analysis. Take for instance GR with an extra scalar field. Due to the minimal coupling of the matter fields to the metric in (2.66) the matter particles follow the geodesics of the metric $\hat{g}$, namely

$$\frac{d^2x^a}{d\hat{s}^2}+\hat{\Gamma}^a_{mn}\frac{dx^m}{d\hat{s}}\frac{dx^n}{d\hat{s}}=0 \tag{2.68}$$

where $\hat{\Gamma}^a_{bc}=(1/2)\hat{g}^{an}(\hat{g}_{bn,c}+\hat{g}_{cn,b}-\hat{g}_{bc,n})$ are the Christoffel symbols of the metric $\hat{g}$. These coincide with the geodesics of the Riemannian geometry where metrical relations are given by $\hat{g}$ via $\hat{g}(\hat{X},\hat{Y})=\hat{g}_{mn}\hat{X}^m\hat{Y}^n$ and the line element $d\hat{s}^2=\hat{g}_{mn}dx^m dx^n$, etc. It is the reason why canonical GR based on (2.66) is naturally linked with Riemannian geometry (it is the same for JFBD since the matter fields couple minimally to the spacetime metric). The units of this geometry ar constant over the manifold. On the other hand since the matter fields are non-minimally coupled to the metric in the conformal GR the matter particles would not follow the geodesics of the conformal metric g but rather curves which are solutions of the equation conformal to (2.68), namely (2.62) where now Γ^a_{bc} are the Christoffel symbols of the metric g conformal to $\hat{g}$. Hence if one assumes that the spacetime geometry is fixed to be Riemannian and that it is unchanged under the conformal rescaling (★) with $\Omega^2=\phi$ one efffectively arrives at the conclusion that $\hat{g}$ is the "physical" metric. **However this assumption is wrong** and is the source of much long standing confusion (to be further clarified below).

REMARK 3.2.2. One notes that conformal Riemannian geometry (corresponding to Weyl geometry here) develops as follows. Let $\lambda(t)$ be a curve with local coordinates $x^a(t)$ and let X with local coordinates $X^a=dx^a/dt$ be a tangent vector to $\lambda(t)$. The covariant derivative of a given vector field $\hat{Y}$ along λ is given by

$$\frac{\hat{D}\hat{Y}^a}{\partial t}=\frac{\partial\hat{Y}^a}{\partial t}+\hat{\gamma}^a_{mn}\hat{Y}^m\frac{dx^n}{dt} \tag{2.69}$$

where $\hat{\gamma}^a_b$ is a symmetric connection. Given a metric $\hat{g}$ on a manifold $\hat{M}$ the Riemannian geometry is fixed by the condition that there is a unique torsion free (symmetric) connction on $\hat{M}$ such that the covariant derivative of $\hat{g}$ is zero; then parallel transport of vectors $\hat{Y}$ $(\hat{D}\hat{Y}^a/\partial t)=0$ and this preserves scalar products, i.e. $d\hat{g}(\hat{Y},\hat{Y})=0$. The laws of parallel transport and length preservation entail that the symmetric connection $\hat{\gamma}^a_{bc}$ coincides with the Christoffel symbols of the metric $\hat{g}$, so $\hat{\gamma}^a_{bc}=\hat{\Gamma}^a_{bc}$. Suppose now that $\hat{Y}$ transform under (★) (with $\Omega^2=\phi$) as $\hat{Y}=h(\phi)Y^a$; then $dg(Y,Y)=-d[log(\phi h^2)]g(Y,Y)$ which resembles the law of length transport in Weyl geometry. Hence given a Riemannian geometry on $\hat{M}$, under (★) with $\Omega^2=\phi$ one arrives at a Weyl geometry on M. The parallel transport law conformal to (2.69) is

$$\frac{DY^a}{\partial t} + \frac{\partial}{\partial t}(log(h)Y^a;\ \frac{DY^a}{\partial t} = \frac{\partial Y^a}{\partial t} + \gamma^a_{mn}Y^m\frac{dx^n}{dt} \tag{2.70}$$

Here γ^a_{bc} is the symmetric connection on the Weyl manifold M related to the Christoffel symbols of the conformal metric g via

$$\gamma^a_{bc} = \Gamma^a_{bc} + \frac{1}{2}\phi^{-1}(\phi_{,b}\delta^a_c + \phi_{,c}\delta^a_b - g_{bc}g^{an}\phi_{,n}) \tag{2.71}$$

There will be particle motions as in (2.62) and in particular Weyl geometry includes units of measure with point dependent length.

REMARK 3.2.3. We go now to transformations of units following [**796, 798**]. Consider two Lagrangians

$$L_1 = \sqrt{-g}(R - \alpha(\nabla\phi)^2);\ L_2 = \sqrt{-g}\left(\phi R - \left(\alpha - \frac{3}{2}\right)\frac{(\nabla\phi)^2}{\phi}\right) \tag{2.72}$$

with respect to transformations (★) (note L_2 can be obtained from L_1 by rescaling $g \to \phi g$ and $\phi \to log(\phi)$). Consider first conformal transformations $\tilde{g}_{ab} = \phi^\sigma g_{ab}$ (σ arbitrary) leading to

$$\tilde{L}_1 = \sqrt{-\tilde{g}}(\phi^\sigma\tilde{R} + [(3\sigma_(3/2)\sigma^2]\phi^{-2-\sigma} - \alpha\phi^\sigma)(\tilde{\nabla}\phi)^2) \tag{2.73}$$

Hence the laws of gravity it describes change under this transformation; in particular in the conformal (tilde) frame the effective gravitational constant depends on ϕ due to the nonminimal coupling between the scalar field and the curvature. On the other hand L_2 is mapped to

$$\tilde{L}_2 = \sqrt{-\tilde{g}}\left(\phi^{1-\sigma}\tilde{R} - \frac{(\alpha - (3/2) - 3\sigma + (3/2)\sigma^2)}{(1-\sigma)^2}\phi^{\sigma-1}(\tilde{\nabla}\phi^{1-\sigma}\right) \tag{2.74}$$

Consequently if we introduce a new scalar field variable $\tilde{\phi} = \phi^{1-\sigma}$ and redefine the free parameter of the theory via $\tilde{\alpha} = [\alpha + 3\sigma(\sigma - 2)]/(1-\sigma)^2$ the Lagrangian $\tilde{L}_2$ takes the form

$$\tilde{L}_2 = \sqrt{-\tilde{g}}\left(\tilde{\phi}\tilde{R} - \left(\tilde{\alpha} - \frac{3}{2}\right)\frac{(\tilde{\nabla}\tilde{\phi})^2}{\tilde{\phi}}\right) \tag{2.75}$$

Hence the Lagrangian L_2 is invariant in form under the conformal transformation and field transformation indicated. The composition of two such transformations with parameters σ_1 and σ_2 yields a transformation of the same form with parameter $\sigma_3 = \sigma_1 + \sigma_2 - \sigma_1\sigma_2$. The identity transformation involves $\sigma = 0$ and the inverse for σ is a transformation with parameter $\bar{\sigma} = -[\sigma/(1-\sigma)]$. Hence excluding the value $\sigma = 1$ we have a one parameter Abelian group of transformations ($\sigma_3(\sigma_1, \sigma_2) = \sigma_3(\sigma_2, \sigma_1)$). This all leads to the conclusion that, since any consistent theory of spacetime must be invariant under the one parameter group of transformations of units (length, time, mass), spacetime theores based on the Lagrangian for pure GR of the form L_1 ar not consistent theories while those based on Lagrangians of the form L_2 may in principle be consistent formulations of a spacetime theory. In particular canonical GR and the Einstein frame formulation of BD theory are not consistent formulations.

Consider now, separately, matter Lagrangians

$$\text{(2.76)} \qquad (\mathbf{A})\ \sqrt{-g}\phi^2 L_M; \quad (\mathbf{B})\ \sqrt{-g}L_M$$

(**(B)** involves minimal coupling of matter to the metric and **(A)** is non-minimal. Under $\tilde{g}_{ab} = \phi^\sigma g_{ab}$ we have $(\mathbf{A}) \to \sqrt{-\tilde{g}}\phi^{2-2\sigma}L_M$ and hence via scalar field redefinition **(A)** is invariant in form under the group of units transformations. However **(B)** with minimal coupling is not invariant and hence JFBD based on $L_{BD} = L_2 + 16\pi\sqrt{-g}L_M$ coupling (L_2 as in (2.72)) is not a consistent theory of spacetime. The only surviving candidate is conformal GR based on (2.67), namely $L_2 + 16\pi\sqrt{-g}\phi^2 L_M$, and this theory provides a consistent formulation of the laws of gravity. Thus the conformal version of GR involving Weyl type geometry is the object to study and this is picked up again in [**133, 796**] along with some connections to Bohmian theory. Various other topics involving cosmology and singularities are also studied in [**796, 797, 798, 799**] but we omit this here.

REMARK 3.2.4. We make a few comments now following [**133, 796**] about Weyl geometry and the quantum potential. First we have seen that Einstein's GR is incomplete and a Weylian form seems preferable. Secondly there seems to be evidence that a Weylian form can solve (or smooth) various problems involving singularities (cf. [**796, 797, 798, 799**] for some information in this direction). One recalls also the arguments emanating from string theory that a dilaton should coupled to gravity in the low energy limit (cf. [**429**]). It is to be noted that Weyl spacetimes conformally linked to Riemannian structure (such as conformal GR) are called Weyl integrable spacetimes (WISP). The terminology arises from the condition $g_{ab;c} = 0$ for a Riemannian space (where the symbol ";" denotes covariant differentiation. Then if $\hat{g} = \Omega^2 g$ with $\Omega^2 = \phi = exp(\psi)$ (note we are switching the roles of g and $\hat{g}$ used earlier) there results $\hat{g}_{ab;c} = \psi_{,c}\hat{g}_{ab}$ (affine covariant derivative involving $\hat{\Gamma}^a_{bc}$) which are the Weyl affine connection coefficients of the conformal manifold. Comparing to the requirement $\hat{g}_{ab;c} = w_c\hat{g}_{ab}$ for Weyl geometries (w_c is the Weyl gauge vector) we see that Weyl structures conformally linked to Riemannian geometry have the property that $w_c = \psi_{,c}$ (ψ here corresponds to the dilaton) and this is the origin of the term integrable since via $d\ell = \ell dx^n \psi_{,n}$ one arrives at $\oint d\ell = 0$ for WISP (which eliminates the second clock effect often used to critisize Weyl spacetime). Note that the equations of motion of a free particle (or geodesic curves) in the WIST are given by (2.62) (with $\phi = log(\psi)$) and setting e.g. $exp(\psi) = 1 + Q$ where Q is the quantum potential one can regard the last term in (2.62) as the quantum force (see here Section 3.2 for a more refined approach). In any event the moral here is that Weyl geometry implicitly contains the quantum effects of matter - it is already a quantum geometry! In particular a free falling test particle would not "feel" any special quantum force since the effect is built into the free fall.

3. THE SCHRÖDINGER EQUATION IN WEYL SPACE

We go now to Santamato [**840**] and derive the SE from classical mechanics in Weyl space (i.e. from Weyl geometry - cf. also [**63, 188, 189, 219, 224, 490, 841, 989**]). The idea is to relate the quantum force (arising from the quantum potential) to geometrical properties of spacetime; the Klein-Gordon (KG) equation is also treated in this spirit in [**219, 841**] and we discuss this later. One wants to show how geometry acts as a guidance field for matter (as in general relativity). Initial positions are assumed random (as in the Madelung approach) and thus the theory is statistical and is really describing the motion of an ensemble. Thus assume that the particle motion is given by some random process $q^i(t,\omega)$ in a manifold M (where ω is the sample space tag) whose probability density $\rho(q,t)$ exists and is properly normalizable. Assume that the process $q^i(t,\omega)$ is the solution of differential equations

$$\dot{q}^i(t,\omega)=(dq^i/dt)(t,\omega)=v^i(q(t,\omega),t) \tag{3.1}$$

with random initial conditions $q^i(t_0,\omega)=q_0^i(\omega)$. Once the joint distribution of the random variables $q_0^i(\omega)$ is given the process $q^i(t,\omega)$ is uniquely determined by (3.1). One knows that in this situation $\partial_t\rho+\partial_i(\rho v^i)=0$ (continuity equation) with initial Cauchy data $\rho(q,t)=\rho_0(q)$. The natural origin of v^i arises via a least action principle based on a Lagrangian $L(q,\dot{q},t)$ with

$$L^*(q,\dot{q},t)=L(q,\dot{q},t)-\Phi(q,\dot{q},t);\ \Phi=\frac{dS}{dt}=\partial_tS+\dot{q}^i\partial_iS \tag{3.2}$$

Then $v^i(q,t)$ arises by minimizing

$$I(t_0,t_1)=E[\int_{t_0}^{t_1}L^*(q(t,\omega),\dot{q}(t,\omega),t)dt] \tag{3.3}$$

where t_0, t_1 are arbitrary and E denotes the expectation (cf. [**186, 187, 672, 674, 698**] for stochastic ideas). The minimum is to be achieved over the class of all random motions $q^i(t,\omega)$ obeying (3.2) with arbitrarily varied velocity field $v^i(q,t)$ but having common initial values. One proves first

$$\partial_tS+H(q,\nabla S,t)=0;\ v^i(q,t)=\frac{\partial H}{\partial p_i}(q,\nabla S(q,t),t) \tag{3.4}$$

Thus the value of I in (3.3) along the random curve $q^i(t,q_0(\omega))$ is

$$I(t_1,t_0,\omega)=\int_{t_0}^{t_1}L^*(q(,q_0(\omega)),\dot{q}(t,q_0(\omega)),t)dt \tag{3.5}$$

Let $\mu(q_0)$ denote the joint probability density of the random variables $q_0^i(\omega)$ and then the expectation value of the random integral is

$$I(t_1,t_0)=E[I(t_1,t_0,\omega)]=\int_{\mathbf{R}^n}\int_{t_0}^{t_1}\mu(q_0)L^*(q(t,q_0),\dot{q}(t,q_0),t)d^nq_0dt \tag{3.6}$$

Standard variational methods give then

$$\delta I=\int_{\mathbf{R}^n}d^nq_0\mu(_0)\left[\frac{\partial L^*}{\partial\dot{q}^i}(q(t_1,q_0),\partial_tq(t_1,q_0),t)\delta q^i(t_1,q_0)-\right. \tag{3.7}$$

$$\left.-\int_{t_0}^{t_1}dt\left(\frac{\partial}{\partial t}\frac{\partial L^*}{\partial\dot{q}^i}(q(t,q_0),\partial_tq)t,q_0),t)-\frac{\partial L^*}{\partial q^i}(q(t,q_0),\partial_tq(t,q_0),t)\right)\delta q^i(t,q_0)\right]$$

where one uses the fact that $\mu(q_0)$ is independent of time and $\delta q^i(t_0, q_0) = 0$ (recall common initial data is assumed). Therefore

$$(3.8) \qquad (\mathbf{A})\ (\partial L^*/\partial \dot{q}^i)(q(t,q_0), \partial_t q(t,q_0), t) = 0;$$

$$(\mathbf{B})\ \frac{\partial}{\partial t}\frac{\partial L^*}{\partial \dot{q}^i}(q(t,q_0), \partial_t q(t,q_0,t) - \frac{\partial L^*}{\partial q^i}(q(t,q_0), \partial_t q(t,q_0), t) = 0$$

are the necessary conditions for obtaining a minimum of I. Conditions **(B)** are the usual Euler-Lagrange (EL) equations whereas **(A)** is a consequence of the fact that in the most general case one must retain varied motions with $\delta q^i(t_1, q_0)$ different from zero at the final time t_1. Note that since L^* differs from L by a total time derivative one can safely replace L^* by L in **(B)** and putting (3.2) into **(A)** one obtains the classical equations

$$(3.9) \qquad p_i = (\partial L/\partial \dot{q}^i)(q(t,q_0), \dot{q}(t,q_0), t) = \partial_i S(q(t,q_0), t)$$

It is known now that if $det[(\partial^2 L/\partial \dot{q}^i \partial \dot{q}^j] \neq 0$ then the second equation in (3.4) is a consequence of the gradient condition (3.9) and of the definition of the Hamiltonian function $H(q,p,t) = p_i \dot{q}^i - L$. Moreover **(B)** in (3.8) and (3.9) entrain the HJ equation in (3.4). In order to show that the average action integral (3.6) actually gives a minimum one needs $\delta^2 I > 0$ but this is not necessary for Lagrangians whose Hamiltonian H has the form

$$(3.10) \qquad H_C(q,p,t) = \frac{1}{2m} g^{ik}(p_i - A_i)(p_k - A_k) + V$$

with arbitrary fields A_i and V (particle of mass m in an EM field A) which is the form for nonrelativistic applications; given positive definite g_{ik} such Hamiltonians involve sufficiency conditions $det[\partial^2 L/\partial \dot{q}^i \partial \dot{q}^k] = mg > 0$. Finally **(B)** in (3.8) with L^* replaced by L) shows that along particle trajectories the EL equations are satisfied, i.e. the particle undergoes a classical motion with probability one. Notice here that in (3.4) no explicit mention of generalized momenta is made; one is dealing with a random motion entirely based on position. Moreover the minimum principle (3.3) defines a 1-1 correspondence between solutions $S(q,t)$ in (3.4) and minimizing random motions $q^i(t,\omega)$. Provided v^i is given via (3.4) the particle undergoes a classical motion with probability one. Thus once the Lagrangian L or equivalently the Hamiltonian H is given, $\partial_t \rho + \partial_i(\rho v^i) = 0$ and (3.4) uniquely determine the stochastic process $q^i(t,\omega)$. Now suppose that some geometric structure is given on M so that the notion of scalar curvature $R(q,t)$ of M is meaningful. Then we assume (ad hoc) that the actual Lagrangian is

$$(3.11) \qquad L(q,\dot{q},t) = L_C(q,\dot{q},t) + \gamma(\hbar^2/m)R(q,t)$$

where $\gamma = (1/6)(n-2)/(n-1)$ with $n = dim(M)$. Since both L_C and R are independent of $\hbar$ we have $L \to L_C$ as $\hbar \to 0$.

Now for a differential manifold with $ds^2 = g_{ik}(q)dq^i dq^k$ it is standard that in a transplantation $q^i \to q^i + \delta q^i$ one has $\delta A^i = \Gamma^i_{k\ell} A^\ell dq^k$ with $\Gamma^i_{k\ell}$ general affine connection coefficients on M (Riemannian structure is not assumed). In [**840**] it is assumed that for $\ell = (g_{ik} A^i A^k)^{1/2}$ one has $\delta \ell = \ell \phi_k dq^k$ where the ϕ_k are covariant components of an arbitrary vector (Weyl geometry). Then the actual affine connections $\Gamma^i_{k\ell}$ can be found by comparing this with $\delta\ell^2 = \delta(g_{ik}A^iA^k)$ and using $\delta A^i = \Gamma^i_{k\ell}A^\ell dq^k$. A little linear algebra gives then

$$\Gamma^i_{k\ell} = -\left\{ \begin{array}{c} i \\ k\ \ell \end{array} \right\} + g^{im}(g_{mk}\phi_\ell + g_{m\ell}\phi_k - g_{k\ell}\phi_m) \tag{3.12}$$

Thus we may prescribe the metric tensor g_{ik} and ϕ_i and determine via (3.12) the connection coefficients. Note that $\Gamma^i_{k\ell} = \Gamma^i_{\ell k}$ and for $\phi_i = 0$ one has Riemannian geometry. Covariant derivatives are defined via

$$A^k_{,1} = \partial_i A^k - \Gamma^{k\ell}A^\ell;\ A_{k,i} = \partial_i A_k + \Gamma^\ell_{ki}A_\ell \tag{3.13}$$

for covariant and contravariant vectors respectively (where $S_{,i} = \partial_i S$). Note Ricci's lemma no longer holds (i.e. $g_{ik,\ell} \neq 0$) so covariant differentiation and operations of raising or lowering indices do not commute. The curvature tensor $R^i_{k\ell m}$ in Weyl geometry is introduced via $A^i_{,k,\ell} - A^i_{,\ell,k} = F^i_{mk\ell}A^m$ from which arises the standard formula of Riemannian geometry

$$R^i_{mk\ell} = -\partial_\ell\Gamma^i_{mk} + \partial_k\Gamma^i_{m\ell} + \Gamma^i_{n\ell}\Gamma^n_{mk} - \Gamma^i_{nk}\Gamma^n_{m\ell} \tag{3.14}$$

where (3.12) is used in place of the Christoffel symbols. The tensor $R^i_{mk\ell}$ obeys the same symmetry relations as the curvature tensor of Riemann geometry as well as the Bianchi identity. The Ricci symmetric tensor R_{ik} and the scalar curvature R are defined by the same formulas also, viz. $R_{ik} = R^\ell_{i\ell k}$ and $R = g^{ik}R_{ik}$. For completeness one derives here

$$R = \dot{R} + (n-1)[(n-2)\phi_i\phi^i - 2(1/\sqrt{g})\partial_i(\sqrt{g}\phi^i)] \tag{3.15}$$

where $\dot{R}$ is the Riemannian curvature built by the Christoffel symbols. Thus from (3.12) one obtains

$$g^{k\ell}\Gamma^i_{k\ell} = -g^{k\ell}\left\{ \begin{array}{c} i \\ k\ \ell \end{array} \right\} - (n-2)\phi^i;\ \Gamma^i_{k\ell} = -\left\{ \begin{array}{c} i \\ k\ \ell \end{array} \right\} + n\phi_k \tag{3.16}$$

Since the form of a scalar is independent of the coordinate system used one may compute R in a geodesic system where the Christoffel symbols and all $\partial_\ell g_{ik}$ vanish; then (3.12) reduces to $\Gamma^i_{k\ell} = \phi_k\kappa^i_\ell + \phi_\ell\delta^i_k - g_{k\ell}\phi^i$ and hence

$$R = -g^{km}\partial_m\Gamma^i_{k\ell} + \partial_i(g^{k\ell}\Gamma^i_{k\ell}) + g^{\ell m}\Gamma^i_{n\ell}\Gamma^n_{mi} - g^{m\ell}\Gamma^i_{n\ell}\Gamma^n_{m\ell} \tag{3.17}$$

Further one has $g^{\ell m}\Gamma^i_{n\ell}\Gamma^n_{mi} = -(n-2)(\phi_k\phi^k)$ at the point in consideration. Putting all this in (3.17) one arrives at

$$R = \dot{R} + (n-1)(n-2)(\phi_k\phi^k) - 2(n-1)\partial_k\phi^k \tag{3.18}$$

which becomes (3.15) in covariant form. Now the geometry is to be derived from physical principles so the ϕ_i cannot be arbitrary but must be obtained by the same averaged least action principle (3.3) giving the motion of the particle. The minimum in (3.3) is to be evaluated now with respect to the class of all Weyl geometries having arbitrarily varied gauge vectors but fixed metric tensor. Note that once (3.11) is inserted in (3.2) the only term in (3.3) containing the gauge vector is the curvature term. Then observing that $\gamma > 0$ when $n \geq 3$ the minimum principle (3.3) may be reduced to the simpler form $E[R(q(t,\omega),t)] = min$ where only the gauge vectors ϕ_i are varied. Using (3.15) this is easily done. First a little argument shows that $\hat{\rho}(q,t)=\rho(q,t)/\sqrt{g}$ transforms as a scalar in a coordinate change and this will be called the scalar probability density of the random motion of the particle (statistical determination of geometry). Starting from $\partial_t\rho+\partial_i(\rho v^i)=0$ a manifestly covariant equation for $\hat{\rho}$ is found to be $\partial_t\hat{\rho}+(1/\sqrt{g})\partial_i(\sqrt{g}v^i\hat{\rho})=0$. Now return to the minimum problem $E[R(q(t,\omega),t)]=min$; from (3.15) and $\hat{\rho}=\rho/\sqrt{g}$ one obtains

$$E[R(q(t,\omega),t)]=E[\dot{R}(q(t,\omega),t)]+ \tag{3.19}$$

$$+(n-1)\int_M[(n-2)\phi_i\phi^i-2(1/\sqrt{g})\partial_i(\sqrt{g}\phi^i)]\hat{\rho}(q,t)\sqrt{g}d^nq$$

Assuming fields go to 0 rapidly enough on ∂M and integrating by parts one gets then

$$E[R]=E[\dot{R}]-\frac{n-1}{n-2}E[g^{ik}\partial_i(log(\hat{\rho})\partial_k(log(\hat{\rho})]+ \tag{3.20}$$

$$+\frac{n-1}{n-2}E\{g^{ik}[(n-2)\phi_i+\partial_i(log(\hat{\rho})][(n-2)\phi_k+\partial_k(log(\hat{\rho})]\}$$

Since the first two terms on the right are independent of the gauge vector and g^{ik} is positive definite $E[R]$ will be a minimum when

$$\phi_i(q,t)=-[1/(n-2)]\partial_i[log(\hat{\rho})(q,t)] \tag{3.21}$$

This shows that the geometric properties of space are indeed affected by the presence of the particle and in turn the alteration of geometry acts on the particle through the quantum force $f_i=\gamma(\hbar^2/m)\partial_iR$ which according to (3.15) depends on the gauge vector and its derivatives. It is this peculiar feedback between the geometry of space and the motion of the particle which produces quantum effects.

In this spirit one goes now to a geometrical derivation of the SE. Thus inserting (3.21) into (3.16) one gets

$$R=\dot{R}+(1/2\gamma\sqrt{\hat{\rho}})[1/\sqrt{g})\partial_i(\sqrt{g}g^{ik}\partial_k\sqrt{\hat{\rho}})] \tag{3.22}$$

where the value $(n-2)/6(n-1)$ for γ is used. On the other hand the HJ equation (3.4) can be written as

$$\partial_tS+H_C(q,\nabla S,t)-\gamma(\hbar^2/m)R=0 \tag{3.23}$$

where (3.11) has been used. When (3.22) is introduced into (3.23) the HJ equation and the continuity equation $\partial_t\hat{\rho}+(1/\sqrt{g})(\sqrt{g}v^i\hat{\rho})=0$, with velocity field given by (3.4), form a set of two nonlinear PDE which are coupled by the curvature of space. Therefore self consistent random motions of the particle (i.e. random motions compatible with (3.17)) are obtained by solving (3.23) and the continuity equation simultaneously. For every pair of solutions $S(q,t,\hat{\rho}(q,t))$ one gets a possible random motion for the particle whose invariant probability density is $\hat{\rho}$. The present approach is so different from traditional QM that a proof of equivalence is needed and this is only done for Hamiltonians of the form (3.10) (which is not very restrictive). The HJ equation corresponding to (3.10) is

$$\partial_t S + \frac{1}{2m} g^{ik}(\partial_i S - A_i)(\partial_k S - A_k) + V - \gamma \frac{\hbar^2}{m} R = 0 \tag{3.24}$$

with R given by (3.22). Moreover using (3.4) as well as (3.10) the continuity equation becomes

$$\partial_t \hat{\rho} + (1/m\sqrt{g})\partial_i[\hat{\rho}\sqrt{g} g^{ik}(\partial_k S - A_k)] = 0 \tag{3.25}$$

Owing to (3.22),(3.24) and (3.25) form a set of two nonlinear PDE which must be solved for the unknown functions S and $\hat{\rho}$. Now a straightforward calculations shows that, setting

$$\psi(q,t) = \sqrt{\hat{\rho}(q,t)} exp](i/\hbar)S(q,t)], \tag{3.26}$$

the quantity ψ obeys a linear PDE (corrected from [**840**])

$$i\hbar\partial_t \psi = \frac{1}{2m}\left\{\left[\frac{i\hbar\partial_i\sqrt{g}}{\sqrt{g}} + A_i\right] g^{ik}(i\hbar\partial_k + A_k)\right\}\psi + \left[V - \gamma\frac{\hbar^2}{m}\dot{R}\right]\psi \tag{3.27}$$

where only the Riemannian curvature $\dot{R}$ is present (any explicit reference to the gauge vector ϕ_i having disappeared). (3.27) is of course the SE in curvilinear coordinates whose invariance under point transformations is well known. Moreover (3.26) shows that $|\psi|^2 = \hat{\rho}(q,t)$ is the invariant probability density of finding the particle in the volume element $d^n q$ at time t. Then following Nelson's arguments that the SE together with the density formula contains QM the present theory is physically equivalent to traditional nonrelativistic QM. One sees also from (3.26) and (3.27) that the time independent SE is obtained via $S = S_0(q) - Et$ with constant E and $\hat{\rho}(q)$. In this case the scalar curvature of space becomes time independent; since starting data at t_0 is meaningless one replaces the continuity equation with a condition $\int_M \hat{\rho}(q)\sqrt{g} d^n q = 1$.

REMARK 3.3.1. We recall (cf. [**188**]) that in the nonrelativistic context the quantum potential has the form $Q = -(\hbar^2/2m)(\partial^2\sqrt{\rho}/\sqrt{\rho})$ ($\rho \sim \hat{\rho}$ here) and in more dimensions this corresponds to $Q = -(\hbar^2/2m)(\Delta\sqrt{\rho}/\sqrt{\rho})$. Here we have a SE involving $\psi = \sqrt{\rho} exp[(i/\hbar)S]$ with corresponding HJ equation (3.24) which corresponds to the flat space 1-D $S_t + (s')^2/2m + V + Q = 0$ with continuity equation $\partial_t\rho + \partial(\rho S'/m) = 0$ (take $A_k = 0$ here). The continuity equation in (3.25) corresponds to $\partial_t\rho + (1/m\sqrt{g})\partial_i[\rho\sqrt{g} g^{ik}(\partial_k S)] = 0$. For $A_k = 0$ (3.24) becomes

$$\partial_t S + (1/2m) g^{ik}\partial_i S \partial_k S + V - \gamma(\hbar^2/m)R = 0 \tag{3.28}$$

This leads to an identification $Q \sim -\gamma(\hbar^2/m)R$ where R is the Ricci scalar in the Weyl geometry (related to the Riemannian curvature built on standard Christoffel symbols via (3.15)). Here $\gamma = (1/6)[(n-2)/(n-2)]$ as above which for $n = 3$ becomes $\gamma = 1/12$; further the Weyl field $\phi_i = -\partial_i log(\rho)$. Consequently (see below).

PROPOSITION 3.1. For the SE (3.27) in Weyl space the quantum potential is $Q = -(\hbar^2/12m)R$ where R is the Weyl-Ricci scalar curvature. For Riemannian flat space $\dot{R} = 0$ this becomes via (3.22)

$$R = \frac{1}{2\gamma\sqrt{\rho}}\partial_i g^{ik}\partial_k\sqrt{\rho} \sim \frac{1}{2\gamma}\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}} \Rightarrow Q = -\frac{\hbar^2}{2m}\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}} \quad (3.29)$$

as it should and the SE (3.27) reduces to the standard SE in the form $i\hbar\partial_t\psi = -(\hbar^2/2m)\Delta\psi + V\psi$ $(A_k = 0)$.

3.1. FISHER INFORMATION REVISITED. Via Remarks 1.1.4, 1.1.5, and 1.1.6 from Chapter 1 (based on [**395, 446, 447, 448, 449, 805, 806**]) we recall the derivation of the SE in Theorem 1.1.1. Thus with some repetition recall first that the classical Fisher information associated with translations of a 1-D observable X with probability density $P(x)$ is

$$F_X = \int dx\, P(x)([log(P(x)]')^2 > 0 \quad (3.30)$$

One has a well known Cramer-Rao inequality $Var(X) \geq F_X^{-1}$ where $Var(X) \sim$ variance of X. A Fisher length for X is defined via $\delta X = F_X^{-1/2}$ and this quantifies the length scale over which $p(x)$ (or better $log(p(x))$) varies appreciably. Then the root mean square deviation ΔX satisfies $\Delta X \geq \delta X$. Let now P be the momentum observable conjugate to X, and P_{cl} a classical momentum observable corresponding to the state ψ given via $p_{cl}(x) = (\hbar/2i)[(\psi'/\psi) - (\bar{\psi}'/\bar{\psi})]$. One has then the identity $< p >_\psi = < p_{cl} >_\psi$ following via integration by parts. Now define the nonclassical momentum by $p_{nc} = p - p_{cl}$ and one shows then

$$\Delta X \Delta p \geq \delta X \Delta p \geq \delta X \Delta p_{nc} = \hbar/2 \quad (3.31)$$

Then consider a classical ensemble of n-dimensional particles of mass m moving under a potential V. The motion can be described via the HJ and continuity equations

$$\frac{\partial s}{\partial t} + \frac{1}{2m}|\nabla s|^2 + V = 0; \ \frac{\partial P}{\partial t} + \nabla \cdot \left[P\frac{\nabla s}{m}\right] = 0 \quad (3.32)$$

for the momentum potential s and the position probability density P (note that there is no quantum potential and this will be supplied by the information term). These equations follow from the variational principle $\delta L = 0$ with Lagrangian $L = \int dt\, d^n x\, P\left[(\partial s/\partial t) + (1/2m)|\nabla s|^2 + V\right]$. It is now assumed that the classical Lagrangian must be modified due to the existence of random momentum fluctuations. The nature of such fluctuations is immaterial and one can assume that the momentum associated with position x is given by $p = \nabla s + N$ where the fluctuation term N vanishes on average at each point x. Thus s changes to being an average momentum potential. It follows that the average kinetic energy $< |\nabla s|^2 > /2m$ appearing in the Lagrangian above should be replaced by $< |\nabla s + N|^2 > /2m$ giving rise to

$$L' = L + (2m)^{-1}\int dt < N \cdot N >= L + (2m)^{-1}\int dt(\Delta N)^2 \quad (3.33)$$

where $\Delta N = < N\cdot N >^{1/2}$ is a measure of the strength of the quantum fluctuations. The additional term is specified uniquely, up to a multiplicative constant, by the three assumptions given in Remark 1.1.4 This leads to the result that

$$(\Delta N)^2 = c\int d^n x\, P|\nabla log(P)|^2 \tag{3.34}$$

where c is a positive universal constant (cf. [**446**]). Further for $\hbar = 2\sqrt{c}$ and $\psi = \sqrt{P}exp(is/\hbar)$ the equations of motion for p and s arising from $\delta L' = 0$ are $i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi$.

A second derivation is given in Remark 1.1.5. Thus let $P(y^i)$ be a probability density and $P(y^i + \Delta y^i)$ be the density resulting from a small change in the y^i. Calculate the cross entropy via

$$J(P(y^i+\Delta y^i):P(y^i)) = \int P(y^i+\Delta y^i)log\frac{P(y^i+\Delta y^i)}{P(y^i)}d^n y \simeq \tag{3.35}$$

$$\simeq \left[\frac{1}{2}\int\frac{1}{P(y^i)}\frac{\partial P(y^i)}{\partial y^i}\frac{\partial P(y^i)}{\partial y^k)}d^n y\right]\Delta y^i\Delta y^k = I_{jk}\Delta y^i\Delta y^k$$

The I_{jk} are the elements of the Fisher information matrix. The most general expression has the form

$$I_{jk}(\theta^i) = \frac{1}{2}\int\frac{1}{P(x^i|\theta^i)}\frac{\partial P(x^i|\theta^i)}{\partial\theta^j}\frac{\partial P(x^i|\theta^i)}{\partial\theta^k}d^n x \tag{3.36}$$

where $P(x^i|\theta^i)$ is a probability distribution depending on parameters θ^i in addition to the x^i. For $P(x^i|\theta^i) = P(x^i+\theta^i)$ one recovers (3.35). If P is defined over an n-dimensional manifold with positive inverse metric g^{ik} one obtains a natural definition of the information associated with P via

$$I = g^{ik}I_{ik} = \frac{g^{ik}}{2}\int\frac{1}{P}\frac{\partial P}{\partial y^i}\frac{\partial P}{\partial y^k}d^n y \tag{3.37}$$

Now in the HJ formulation of classical mechanics the equation of motion takes the form

$$\frac{\partial S}{\partial t} + \frac{1}{2}g^{\mu\nu}\frac{\partial S}{\partial x^\mu}\frac{\partial S}{\partial x^\nu} + V = 0 \tag{3.38}$$

where $g^{\mu\nu} = diag(1/m,\cdots,1/m)$. The velocity field u^μ is then $u^\mu = g^{\mu\nu}(\partial S/\partial x^\nu)$. When the exact coordinates are unknown one can describe the system by means of a probability density $P(t,x^\mu)$ with $\int Pd^n x = 1$ and

$$(\partial P/\partial t) + (\partial/\partial x^\mu)(Pg^{\mu\nu}(\partial S/\partial x^\nu) = 0 \tag{3.39}$$

These equations completely describe the motion and can be derived from the Lagrangian

$$L_{CL} = \int P\left\{(\partial S/\partial t) + (1/2)g^{\mu\nu}(\partial S/\partial x^\mu)(\partial S/\partial x^\nu) + V\right\}dtd^n x \tag{3.40}$$

using fixed endpoint variation in S and P. Quantization is obtained by adding a term proportional to the information I defined in (3.37). This leads to

(3.41)
$$L_{QM} = L_{CL} + \lambda I = \int P\left\{\frac{\partial S}{\partial t} + \frac{1}{2}g^{\mu\nu}\left[\frac{\partial S}{\partial x^\mu}\frac{\partial S}{\partial x^\nu} + \frac{\lambda}{P^2}\frac{\partial P}{\partial x^\mu}\frac{\partial P}{\partial x^\nu}\right] + V\right\} dtd^nx$$

Fixed endpoint variation in S leads again to (3.39) while variation in P leads to

$$\frac{\partial S}{\partial t} + \frac{1}{2}g^{\mu\nu}\left[\frac{\partial S}{\partial x^\mu}\frac{\partial S}{\partial x^\nu} + \lambda\left(\frac{1}{P^2}\frac{\partial P}{\partial x^\mu}\frac{\partial P}{\partial x^\nu} - \frac{2}{P}\frac{\partial^2 P}{\partial x^\mu \partial x^\nu}\right)\right] + V = 0 \quad (3.42)$$

These equations are equivalent to the SE if $\psi = \sqrt{P}exp(iS/\hbar)$ with $\lambda = (2\hbar)^2$ (recall also Remark 1.1.6 for connections to entropy). Now following ideas in [**219, 223, 715**] we note in (3.41) for $\phi_\mu \sim A_\mu = \partial_\mu log(P)$ (which arises in (3.21)) and $p_\mu = \partial_u S$, a complex velocity can be envisioned leading to (cf. also [**224**])

$$|p_\mu + i\sqrt{\lambda}A_\mu|^2 = p_\mu^2 + \lambda A_\mu^2 \sim g^{\mu\nu}\left(\frac{\partial S}{\partial x^\mu}\frac{\partial S}{\partial x^\nu} + \frac{\lambda}{P^2}\frac{\partial P}{\partial x^\mu}\frac{\partial P}{\partial x^\nu}\right) \quad (3.43)$$

Further I in (3.37) is exactly known from ϕ_μ so one has a direct connection between Fisher information and the Weyl field ϕ_μ, along with motivation for a complex velocity (cf. Sections 1.2 and 1.3).

REMARK 3.3.2. Comparing now with [**189**] and quantum geometry in the form $ds^2 = \sum(dp_j^2/p_j)$ on a space of probability distributions (to be discussed in Chapter 5) we can define (3.37) as a Fisher information metric in the present context. This should be positive definite in view of its relation to $(\Delta N)^2$ in (3.34) for example. Now for $\psi = Rexp(iS/\hbar)$ one has ($\rho \sim \hat{\rho}$ here)

$$-\frac{\hbar^2}{2m}\frac{R''}{R} \equiv -\frac{\hbar^2}{2m}\frac{\partial^2\sqrt{\rho}}{\sqrt{\rho}} = -\frac{\hbar^2}{8m}\left[\frac{2\rho''}{\rho} - \left(\frac{\rho'}{\rho}\right)^2\right] \quad (3.44)$$

in 1-D while in more dimensions we have a form ($\rho \sim P$)

$$Q \sim -2\hbar^2 g^{\mu\nu}\left[\frac{1}{P^2}\frac{\partial P}{\partial x^\mu}\frac{\partial P}{\partial x^\nu} - \frac{2}{P}\frac{\partial^2 P}{\partial x^\mu \partial x^\nu}\right] \quad (3.45)$$

as in (3.44) (arising from the Fisher metric I of (3.37) upon variation in P in the Lagrangian). It can also be related to an osmotic velocity field $u = D\nabla log(\rho)$ via $Q = (1/2)u^2 + D\nabla \cdot u$ connected to Brownian motion where D is a diffusion coefficient (cf. [**223, 395, 715**]). For $\phi_\mu = -\partial_\mu log(P)$ we have then $\mathbf{u} = -D\phi$ with $Q = D^2((1/2)(|\phi|^2 - \nabla\cdot\phi)$, expressing Q directly in terms of the Weyl vector. This enforces the idea that QM is built into Weyl geometry!

3.2. THE KG EQUATION. The formulation above from [**840**] was modified in [**841**] to a derivation of the Klein-Gordon (KG) equation via an average action principle. The spacetime geometry was then obtained from the average action principle to obtain Weyl connections with a gauge field ϕ_μ (thus the geometry has a statistical origin). The Riemann scalar curvature $\dot{R}$ is then related to the Weyl scalar curvature R via an equation

$$R = \dot{R} - 3[(1/2)g^{\mu\nu}\phi_\mu\phi_\nu + (1/\sqrt{-g})\partial_\mu(\sqrt{-g}g^{\mu\nu}\phi_\nu)] \quad (3.46)$$

Explicit reference to the underlying Weyl structure disappears in the resulting SE (as in (3.27)). The HJ equation in [**841**] has this form (for $A_\mu = 0$ and $V = 0$) $g^{\mu\nu}\partial_\mu S\partial_\nu S = m^2 - (R/6)$ so in some sense (recall here $\hbar = c = 1$) $m^2 - (R/6) \sim \mathfrak{M}^2$ where $\mathfrak{M}^2 = m^2 exp(Q)$ and $Q = (\hbar^2/m^2c^2)(\Box\sqrt{\rho}/\sqrt{\rho}) \sim (\Box\sqrt{\rho}/m^2\sqrt{\rho})$ via Section 3.2 (for signature $(-,+,+,+)$ - recall here $g^{\mu\nu}\nabla_\mu S\nabla_\nu S = \mathfrak{M}^2c^2$). Thus for $exp(Q) \sim 1 + Q$ one has $m^2 - (R/6) \sim m^2(1+Q) \Rightarrow (R/6) \sim -Qm^2 \sim -(\Box\sqrt{\rho}/\sqrt{\rho})$. This agrees also with [**219**] where the whole matter is analyzed incisively (cf. also Remark 3.3.5). We recall also here from [**798**] (cf. Section 3.2.2) that in the conformal geometry the particles do not follow geodesics of the conformal metric alone. We will sketch an elaboration of this now from [**841**] (paper one). Thus summarizing [**840**] and the second paper in [**841**] one shows that traditional QM is equivalent (in some sense) to classical statistical mechanics in Weyl spaces. The following two points of view are taken to be equivalent

(1) **(A)** The spacetime is a Riemannian manifold and the statistical behavior of a spinless particle is described by the KG equation while probabilities combine according to Feynman quantum rules.
(2) **(B)** The spacetime is a generic affinely connected manifold whose actual geometric structure is determined by the matter content. The statistical behavior of a spinless particle is described by classical statistical mechanics and probabilities combine according to Laplace rules.
(3) In nonrelativistic applications the words spacetime, Riemannian, and KG are to be replacedby space, Euclidean, and SE.

We are skipping over the second paper in [**841**] here and going to the first paper which treates matters in a gauge invariant manner. The moral seems to be (loosely) that quantum mechanics in Riemannian spacetime is the same as classical statistical mechanics in a Weyl space. In particular one wants to establish that traditional QM, based on wave equations and ad hoc probability calculus (as in (1) above) is merely a convenient mathematical construction to overcome the complications arising from a nontrivial spacetime geometric structure. Here one works from first principles and includes gauge invariance (i.e. invariance with respect to an arbitrary choice of the spacetime calibration). The spacetime is supposed to be a generic 4-dimensional differential manifold with torsion free connections $\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}$ and a metric tensor $g_{\mu\nu}$ with signature $(+,-,-,-)$ (one takes $\hbar = c = 1$ - which I deem unfortunate since the role and effect of such quantities is not revealed). Here the (restrictive) hypothesis of assuming a Weyl geometry from the beginning is released, both the particle motion and the spacetime geometric structure are derived from a single average action principle. A result of this approach is that the spacetime connections are forced to be integrable Weyl connections by the extremization principle.

The particle is supposed to undergo a motion in spacetime with deterministic trajectories and random initial conditions taken on an arbitrary spacelike 3-dimensional hypersurface; thus the theory describes a relativistic Gibbs ensemble of particles (cf. Remark 3.3.3). Both the particle motion and the spacetime connections can be obtained from the average stationary action principle

$$\delta\left[E\left(\int_{\tau_1}^{\tau_2} L(x(\tau,\dot{x}(\tau))d\tau\right)\right]=0 \tag{3.47}$$

This action integral must be parameter invariant, coordinate invariant, and gauge invariant. All of these requirements are met if L is positively homogeneous of the first degree in $\dot{x}^\mu = dx^\mu/d\tau$ and transforms as a scalar of Weyl type $w(L) = 0$. The underlying probability measure must also be gauge invariant. A suitable Lagrangian is then

$$L(x,dx) = (m^2 - (R/6))^{1/2}ds + A_\mu dx^\mu \tag{3.48}$$

where $ds = (g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu)^{1/2}d\tau$ is the arc length and R is the space time scalar curvature; m is a parameterlike scalar field of Weyl type (or weight) $w(m) = -(1/2)$. The factor 6 is essentially arbitrary and has been chosen for future convenience. The vector field A_μ can be interpreted as a 4-potential due to an externally applied EM field and the curvature dependent factor in front of ds is an effective particle mass. This seems a bit ad hoc but some feeling for the nature of the Lagrangian can be obtained from Section 3.2 (cf. also [**63**]). The Lagrangian will be gauge invariant provided the A_μ have Weyl type $w(A_\mu) = 0$. Now one can split A_μ into its gradient and divergence free parts $A_\mu = \bar{A}_\mu - \partial_\mu S$, with both S and $\bar{A}_\mu$ having Weyl type zero, and with $\bar{A}_\mu$ interpreted as and EM 4-potential in the Lorentz gauge. Due to the nature of the action principle regarding fixed endpoints in variation one notes that the average action principle is not invariant under EM gauge transformations $A_\mu \to A_\mu + \partial_\mu S$; but one knows that QM is also not invariant under EM gauge transformations (cf. [**17**]) so there is no incompatability with QM here.

Now the set of all spacetime trajectories accessible to the particle (the particle path space) may be obtained from (3.47) by performing the variation with respect to the particle trajectory with fixed metric tensor, connections, and an underlying probability measure. Thus (cf. Remark 3.3.3) the solution is given by the so-called Carathéodory complete figure (cf. [**826**]) associated with the Lagrangian

$$\bar{L}(x,dx) = (m^2 - (R/6))^{1/2}ds + \bar{A}_\mu dx^\mu \tag{3.49}$$

(note this leads to the same equations as (3.48) since the Lagrangians differ by a total differential dS). The resulting complete figure is a geometric entity formed by a one parameter family of hypersurfaces $S(x) = const.$ where S satisfies the HJ equation

$$g^{\mu\nu}(\partial_\mu S - \bar{A}_\mu)(\partial_\nu S - \bar{A}_\nu) = m^2 - \frac{R}{6} \tag{3.50}$$

and by a congruence of curves intersecting this family given by

$$\frac{dx^\mu}{ds} = \frac{g^{\mu\nu}(\partial_\nu S - \bar{A}_\nu)}{[g^{\rho\sigma}(\partial_\rho S - \bar{A}_\rho)(\partial_\sigma S - \bar{A}_\sigma)]^{1/2}} \tag{3.51}$$

The congruence yields the actual particle path space and the underlying probability measure on the path space may be defined on an arbitrary 3-dimensional hypersurface intersecting all of the members of the congruence without tangencies (cf. [**443**]). The measure will be completely identified by its probability current density j^μ (see [**841**] and Remark 3.3.3). Moreover, since the measure is independent of the arbitrary choice of the hypersurface, j^μ must be conserved, i.e. $\partial_\mu j^\mu = 0$ (see Remark 3.3.3). Since the trajectories are deterministically defined by (3.51), j^μ must be parallel to the particle 4-velocity (3.51), and hence

$$j^\mu = \rho\sqrt{-g}g^{\mu\nu}(\partial_\nu S - \bar{A}_\nu) \tag{3.52}$$

with some $\rho > 0$. Now gauge invariance of the underlying measure as well as of the complete figure requires that j^μ transforms as a vector density of Weyl type $w(j^\mu) = 0$ and S as a scalar of Weyl type $w(S) = 0$. From (3.52) one sees then that ρ transforms as a scalar of Weyl type $w(\rho) = -1$ and ρ is called the scalar probability density of the particle random motion.

The actual spacetime affine connections are obtained from (3.47) by performing the variation with respect to the fields $\Gamma^\lambda_{\mu\nu}$ for a fixed metric tensor, particle trajectory, and probability measure. It is expedient to tranform the average action principle to the form of a 4-volume integral

$$\delta\left[\int_\Omega d^4x[(m^2 - (R/6))(g_{\mu\nu}j^\mu j^\nu]^{1/2} + A_\mu j^\mu\right] = 0 \tag{3.53}$$

where Ω is the spacetime region occupied by the congruence (3.51) and j^μ is given by (3.52) (cf. [**841**] and Remark 3.3.3 for proofs). Since the connection fields $\Gamma^\lambda_{\mu\nu}$ are contained only in the curvature term R the variational problem (3.53) can be further reduced to

$$\delta\left[\int_\Omega \rho R\sqrt{-g}d^4x\right] = 0 \tag{3.54}$$

(here the HJ equation (3.50) has been used). This states that the average spacetime curvature must be stationary under a variation of the fields $\Gamma^\lambda_{\mu\nu}$ (principle of stationary average curvature). The extremal connections $\Gamma^\lambda_{\mu\nu}$ arising from (3.54) are derived in [**841**] using standard field theory techniques and the result is

$$\Gamma^\lambda_{\mu\nu} = \left\{ \begin{array}{c} \lambda \\ \mu\ \nu \end{array} \right\} + \frac{1}{2}(\phi_\mu\delta^\lambda_\nu + \phi_\nu\delta^\lambda_\mu - g_{\mu\nu}g^{\lambda\rho}\phi_\rho);\ \ \phi_\mu = \partial_\mu log(\rho) \tag{3.55}$$

This shows that the resulting connections are integrable Weyl connections with a gauge field ϕ_μ (cf. [**840**], Section 3, and Section 3.1). The HJ equation (3.50) and the continuity equation $\partial_\mu j^\mu = 0$ can be consolidated in a single complex equation for S, namely

$$e^{iS}g^{\mu\nu}(iD_\mu - \bar{A}_\mu)(iD_\nu - \bar{A}_\nu)e^{-iS} - (m^2 - (R/6)) = 0;\ \ D_\mu\rho = 0 \tag{3.56}$$

Here D_μ is (doubly covariant - i.e. gauge and coordinate invariant) Weyl derivative given by (cf. [**63**])

$$D_\mu T^\alpha_\beta = \partial_\mu T^\alpha_\beta + \Gamma^\alpha_{\mu\epsilon}T^\epsilon_\beta - \Gamma^\epsilon_{\mu\beta}T^\alpha_\epsilon + w(T)\phi_\mu T^\alpha_\beta \tag{3.57}$$

It is to be noted that the probability density (but not the rest mass) remains constant relative to D_μ. When written out (3.56) for a set of two coupled partial differential equations for ρ and S. To any solution corresponds a particular random motion of the particle.

Next one notes that (3.56) can be cast in the familiar KG form, i.e.

$$[(i/\sqrt{-g})\partial_\mu\sqrt{-g} - \bar{A}_\mu]g^{\mu\nu}(i\partial_\nu - \bar{A}_\nu)\psi - (m^2 - (\dot{R}/6))\psi = 0 \tag{3.58}$$

where $\psi = \sqrt{\rho}exp(-iS)$ and $\dot{R}$ is the Riemannian scalar curvature built out of $g_{\mu\nu}$ only. We have the (by now) familiar formula

$$R = \dot{R} - 3[(1/2)g^{\mu\nu}\phi_\mu\phi_\nu + (1/\sqrt{-g})\partial_\mu(\sqrt{-g}g^{\mu\nu}\phi_\nu)] \tag{3.59}$$

According to point of view **(A)** above in the KG equation (3.58) any explicit reference to the underlying spacetime Weyl structure has disappeared; thus the Weyl structure is hidden in the KG theory. However we note that no physical meaning is attributed to ψ or to the KG equation. Rather the dynamical and statistical behavior of the particle, regarded as a classical particle, is determined by (3.56), which, although completely equivalent to the KG equation, is expressed in terms of quantities having a more direct physical interpretation.

REMARK 3.3.3. We extract here from the Appendices to paper 1 of [**841**]. In Appendix A one shows that the Carathéodory complete figure formed by the congruence (3.51) solves the variational problem (3.47). One needs the notion of the Gibbs ensemble in relativistic mechanics (cf. [**443**]). Roughly a relativistic Gibbs ensemble of particles may be assimilated to an incoherent globule of matter moving in spacetime. More precisely a relativistic Gibbs ensemble is given by (i) A congruence of timelike curves in spacetime (the path space of the particles) and (ii) A probability measure defined on this congruence (note a congruence of spacelike curves could also be envisioned but causality is affected - a physical intepretation is unclear although it could be related to a statistical formulation of virtual phenomena). The construction here goes as follows. Let K be a 3-parameter congruence of time like curves in spacetime be given via (♦) $x^\mu = x^\mu(\tau, u^k)$ where $k = 1, 2, 3$ and τ is an arbitrary parameter along each curve of the congruence. For simplicity assume that the congruence covers a region Ω of spacetime simply (i.e. one and only one curve of K passes through each point of Ω). Then one can regard (♦) as a change of coordinates from x^μ to y^μ where $y^0 = t,\ y^k = u^k$ (assume the Jacobian is nonzero in Ω). Consider then the action integral $L = \int_{\tau_1}^{\tau_2} L(x(\tau, u^k), \dot{x}(\tau, u^k)d\tau$ with L homogeneous of the first degree in the derivatives $\dot{x}^\mu = \partial x^\mu/\partial\tau$. Given a 1-1 correspondence between the u^k and members of the congruence K one may introduce a formula for the probability that the particle follows a sample path having parameters u^k in some 3-dimensional region B as $prob(B) = \int_{B\subset\mathbf{R}} \mu(u^k)du^1du^2du^3$ where $\mu(u^k)$ is some probability density defined on $\mathbf{R}^3$. Hence the average action integral in (3.47) may be written as

$$I = E\left[\int_{\tau_1}^{\tau_2} Ld\tau\right] = \int_{\mathbf{R}^3}\int_{\tau_1}^{\tau_2} \mu(u^k)L(x^\mu(\tau, u^k), \dot{x}^\mu(\tau, u^k)d\tau \prod du^i \tag{3.60}$$

The last term is a 4-dimensional volume integral over the zone between the hyperplanes $y^0 = \tau_1$ and $y^0 = \tau_2$ in the y coordinate. In the x coordinates these hyperplanes are mapped on two 3-dimensional hypersurfaces $\tau(x^\mu) = \tau_1$ and $\tau(x^\mu) = \tau_2$ where $\tau(x^\mu)$ is obtained by solving (♦) with respect to τ; since they are merely a result of the parametrization of K they can be regarded as essentially arbitrary.

The integrand in (3.60) depends on the 4 unknown functions $x^\mu(y^\nu)$ and on their first derivatives $\partial x^\mu/\partial y^0$, and on the coordinates y^ν themselves. Therefore the variational problem $\delta I = 0$ is reduced to a standard variational problem whose solution will yield the functions $x^\mu(\tau, u^k)$, i.e. the actual congruence that renders the average action stationary.

Now the Lagrangian density in (3.60) is $\Lambda = \mu(u^k)L(x^\mu(\tau, u^k), x^\mu_{,\tau}(\tau, u^k)$ in which $x^\mu_{,\tau} = \dot{x}^\mu$ with τ and u^k are the independent variables. By standard methods the EL expressions are ($x^\mu_{,k} = \partial x^\mu/\partial u^k$)

$$E(\Lambda) = \frac{\partial}{\partial u^k}\left[\frac{\partial \Lambda}{\partial x^\mu_{,k}}\right] + \frac{\partial}{\partial \tau}\left[\frac{\partial \Lambda}{\partial x^\mu_{,\tau}}\right] - \frac{\partial \Lambda}{\partial x^\mu} \tag{3.61}$$

In this case howver $\partial\Lambda/\partial x^\mu_{,k} = 0$ and hence the fixed equations $E(\Lambda) = 0$ reduce to (note μ does not depend explicitly on τ)

$$\frac{\partial}{\partial \tau}\left[\mu\frac{\partial L}{\partial x^\mu_{,\tau}}\right] - \mu\frac{\partial L}{\partial x^\mu} = 0 \Rightarrow \frac{\partial}{\partial \tau}\left[\frac{\partial L}{\partial \dot{x}^\mu}\right] - \frac{\partial L}{\partial x^\mu} = 0 \tag{3.62}$$

and this coincides with the EL equations associated with the action integral above. This means that the actual congruence must be a congruence of extremals or equivalently that the particle obeys equations of motion (3.62) with probability one. Even if the congruence is extremal however we are left with nonvanishing surface terms in the variation of I, namely

(3.63)
$$\delta I = \int_{\mathbf{R}^3} \mu(u^k)\prod du^i\left[\frac{\partial L}{\partial \dot{x}^\mu}(\tau_2, u^k)\delta x^\mu(\tau_2, u^k) - \frac{\partial L}{\partial \dot{x}^\mu}(\tau_1, u^k)\delta x^\mu(\tau_1, u^k)\right] = 0$$

In (3.63) the quantities δx^μ at $\tau = \tau_2$ and $\tau = \tau_1$ are displacements between points P and $P + \delta P$ where the curves x^μ and $x^\mu + \delta x^\mu$ intersect the hypesurfaces $\tau = \tau_2$ and $\tau = \tau_1$ so $\delta x^\mu(\tau_1, u^k)$ and $\delta x^\mu(\tau_1, u^k)$ are tangential to the hypersurfaces. Since the hypersurfaces $\tau(x^\mu) = const.$ are essentially arbitrary so must be the displacements δx^μ and $\delta I = 0$ implies then ($\bullet$) $\partial L/\partial \dot{x}^\mu(\tau, u^k) = 0$. Finally relating L with the Lagrangian (3.48) and comparing with $\bar{L}$ as defined in (3.49) one has $\partial L/\partial \dot{x}^\mu = \partial \bar{L}/\partial \dot{x}^\mu - \partial_\mu S$ so ($\bullet$) yields $\partial \bar{L}/\partial \dot{x}^\mu = \partial_\mu S$. Moreover L and $\bar{L}$, differing only by a total differential dS, lead to the same EL equations and hence one can replace L by $\bar{L}$ in (3.62). In conclusion the congruence that renders the average action stationary must be (i) A congruence of curves that are extremal with respect to Lagrangian $\bar{L}$ and (ii) A congruence satisfying the integrability conditions $\partial \bar{L}/\partial \dot{X}^\mu = \partial_\mu S$. However by standard HJ theory such a congruence is given by (3.51) provided $S(x^\mu)$ obeys the HJ equation associated with $\bar{L}$, namely (3.50).

In appendix B the current density j^μ is introduced and the equivalence between the average action (3.47) and the 4-volume integral (3.53) is proved. This provides a useful connection between ensemble averages and 4-volume integrals appearing in field theories. Here (3.60) is expressed in terms of the y coordinates (τ, u^k) and it can also be expressed in terms of the x coordinates. For this one introduces the current density j^μ associated with the relativistic Gibbs ensemble. The surface element normal to the hypersurface $\tau(u^k) = const.$ is given by $d\sigma_\mu = \pi_\mu du^1 du^2 du^3$ where π_μ are Jacobians

$$\pi_0 = \frac{\partial(x^1, x^2, x^3)}{\partial(u^1, u^2, u^3)};\ \pi_1 = \frac{\partial(x^0, x^2, x^3)}{\partial(u^1, u^2, u^3)}, \cdots \tag{3.64}$$

Then define the current density via $\mu = j^\mu \pi_\mu$ so that $prob(B)$ becomes

$$prob(B) = \int_{B\subset\mathbf{R}^3} \mu du^1 du^2 du^3 = \int_{B\subset\mathbf{R}^3} j^\mu d\sigma_\mu \tag{3.65}$$

The direction of j^μ is still not defined so one is free to choose the current direction parallel to the congruence K, i.e. $j^\mu = \lambda\dot{x}^\mu$. The independence of the underlying measure on the chosen hypersurface $\tau = const.$ is exprssed analytically by the fact that $\mu = \mu(u^1, u^2, u^3)$ does not depend on τ explicitly. Consequently $\partial_\mu j^\mu = 0$ since by the Gauss theorem

$$\int_{\tau(x^\mu)=\tau_2} j^\mu d\sigma_\mu - \int_{\tau(x^\mu)=\tau_1} j^\mu d\sigma_\mu = \int_\Omega \partial_\mu j^\mu d^4x = 0 \tag{3.66}$$

where Ω is the strip between the essentially arbitrary hypersurfaces $\tau = \tau_1$ and $\tau = \tau_2$. The same result could be obtained by differentiating $\mu = j^\mu \pi_\mu$ and using properties of Jacobians. Passing to x coordinates (3.60) becomes

$$I = \int_\Omega \mu L J^{-1} d^4x;\ J = \frac{\partial(x^0, x^1, x^2, x^3)}{\partial(\tau, u^1, u^2, u^3)} \tag{3.67}$$

Note that by definition $J = (\partial x^\mu/\partial\tau)\pi_\mu$ so

$$I = \int_\Omega \mu[L(x^\mu, \dot{x}^\mu)/(\dot{x}^\mu \pi_\mu)] d^4x \tag{3.68}$$

Since L is homogeneous of the first degree in the $\dot{x}^\mu$ the term in square brackets in (3.68) is homogeneous of degree zero in the $\dot{x}^\mu$. Hence we can replace $\dot{x}^\mu$ with the current $j^\mu = \lambda\dot{x}^\mu$ without affecting the integral to obtain $I = \int_\Omega L(x^\mu j^\mu) d^4x$ where $\mu = j^\mu \pi_\mu$ has been used. Thus the average action I may be converted to a four volume integral of $L(x^\mu, j^\mu)$. When this formal substitution is made in (3.48), (3.53) is obtained. This substitution does not alter the functional dependence of the average action integral I on the connection fields $\Gamma^\lambda_{\mu\nu}$ so the variational problems (3.47) and (3.53) are equivalent as long as the variation is performed with respect to these fields.

In Appendix C one derives (3.55); since similar calculations have already been used earlier (and will recur again) we omit this here.

REMARK 3.3.4. The formula (3.59) goes back to Weyl [**986**] and the connection of matter to geometry arises from (3.55). The time variable is treated in a special manner here related to a Gibbs ensemble and $\rho > 0$ is built into the theory. Thus problems of statistical transparancy as in Remark 2.3.3 will apparently not arise.

REMARK 3.3.5. As mentioned at the beginning of Section 3.2, in [**219**] the Santamato theory is analyzed in depth from several points of view and a number of directions for further study are indicated (in [**224**] the importance of a complex velocity is emphasized - see also Section 7.1.2). There is also a related development for the Dirac equation using an approach related to [**397, 463, 464**], where both relativistic and nonrelativistic spin 1/2 particles can be classically treated using anticommuting Grassmanian variables. However we prefer to treat the Dirac equation in a different manner later (cf. also [**89**] and Section 2.1.1).

4. SCALE RELATIVITY AND KG

In [**186**] and Section 1.2 we sketched a few developments in the theory of scale relativity. This is by no means the whole story and we want to give a taste of some further main ideas while deriving the KG equation in this context (cf. [**11, 232, 272, 273, 274, 715, 716, 717, 718, 719, 720, 721**]). A main idea here is that the Schrödinger, Klein-Gordon, and Dirac equations are all geodesic equations in the fractal framework. They have the form $D^2/ds^2 = 0$ where D/ds represents the appropriate covariant derivative. The complex nature of the SE and KG equation arises from a discrete time symmetry breaking based on nondifferentiability. For the Dirac equation further discrete symmetry breakings are needed on the spacetime variables in a biquaternionic context (cf. here [**232**]). First we go back to [**715, 716, 720**] and sketch some of the fundamentals of scale relativity. This is a very rich and beautiful theory extending in both spirit and generality the relativity theory of Einstein (cf. also [**225**] for variations involving Clifford theory). The basic idea here is that (following Einstein) the laws of nature apply whatever the state of the system and hence the relevant variables can only be defined relative to other states. Standard scale laws of power-law type correspond to Galilean scale laws and from them one actually recovers quantum mechanics (QM) in a nondifferentiable space. The quantum behavior is a manifestation of the fractal geometry of spacetime. In particular (as indicated in Section 1.2) the quantum potential is a manifestation of fractality in the same way as the Newton potential is a manifestation of spacetime curvature. In this spirit one can also conjecture (cf. [**720**]) that this quantum potential may explain various dynamical effects presently attributed to dark matter (cf. also [**16**] and Chapter 4). Now for basics one deals with a continuous but nondifferentiable physics. It is known for example that the length of a continuous nondifferentiable curve is dependent on the resolution ϵ. One approach now involves smoothing a nondifferentiable function f via $f(x,\epsilon) = \int_{-\infty}^{\infty} \phi(x,y,\epsilon) f(y)dy$ where ϕ is smooth and say "centered" at x (we refer also to Remark 1.2.8 and [**11, 272, 273, 274**] for a more refined treatment of such matters). There will now arise differential equations involving $\partial f/\partial log(\epsilon)$ and $\partial^2 f/\partial x \partial log(\epsilon)$ for example and the $log(\epsilon)$ term arises as follows. Consider an infinitesimal dilatation $\epsilon \to \epsilon' = \epsilon(1 + d\rho)$ with a curve length

$$\ell(\epsilon) \to \ell(\epsilon') = \ell(\epsilon + \epsilon d\rho) = \ell(\epsilon) + \epsilon \ell_\epsilon d\rho = (1 + \tilde{D} d\rho)\ell(\epsilon) \tag{4.1}$$

Then $\tilde{D} = \epsilon \partial_\epsilon = \partial/\partial log(\epsilon)$ is a dilatation operator and in the spirit of renormalization (multiscale approach) one can assume $\partial \ell(x,\epsilon)/\partial log(\epsilon) = \beta(\ell)$ (where $\ell(x,\epsilon)$ refers to the curve defined by $f(x,\epsilon)$). Now for Galilean scale relativity consider

$\partial\ell(x,\epsilon)/\partial log(\epsilon) = a + b\ell$ which has a solution

$$\ell(x,\epsilon) = \ell_0(x)\left[1 + \zeta(x)\left(\frac{\lambda}{\epsilon}\right)^{-b}\right] \tag{4.2}$$

where $\lambda^{-b}\zeta(x)$ is an integration constant and $\ell_0 = -a/b$. One can choose $\zeta(x)$ so that $<\zeta^2(x)>= 1$ and for $a \neq 0$ there are two regimes (for $b < 0$)

(1) $\epsilon << \lambda \Rightarrow \zeta(x)(\lambda/\epsilon)^{-b} >> 1$ and ℓ is given by a scale invariant fractal like power with dimension $D = 1 - b$, namely $\ell(x,\epsilon) = \ell_0(\lambda/\epsilon)^{-b}$.
(2) $\epsilon >> \lambda \Rightarrow \zeta(x)(\lambda/\epsilon)^{-b} << 1$ and ℓ is independent of scale.

Here $\epsilon = \lambda$ constitutes a transition point between fractal and nonfractal behavior. Only the special case $a = 0$ yields unbroken scale invariance of $\ell = \ell_0(\lambda/\epsilon)^{\delta}$ ($\delta = -b$) and one has then $\tilde{D}\ell = b\ell$ so the scale dimension is an eigenvalue of $\tilde{D}$. Finally the case $b > 0$ corresponds to the cosmological domain.

Now one looks for scale covariant laws and checks this for power laws $\phi = \phi_0(\lambda/\epsilon)^{\delta}$. Thus a scale transformation for $\delta(\epsilon') = \delta(\epsilon)$ will have the form

$$log\frac{\phi(\epsilon')}{\phi_0} = log\frac{\phi(\epsilon)}{\phi_0} + V\delta(\epsilon);\ V = log\frac{\epsilon}{\epsilon'} \tag{4.3}$$

In the same way that only velocity differences have a physical meaning in Galilean relativity here only V differences or scale differences have a physical meaning. Thus V is a "state of scale" just as velocity is a state of motion. In this spirit laws of linear transformation of fields in a scale transformation $\epsilon \to \epsilon'$ amount to finding $A, B, C, D(V)$ such that

$$log\frac{\phi(\epsilon')}{\phi_0} = A(V)log\frac{\phi(\epsilon)}{\phi_0} + B(V)\delta(\epsilon);\ \delta(\epsilon') = C(V)log\frac{\phi(\epsilon)}{\phi_0} + D(V)\delta(\epsilon) \tag{4.4}$$

Here $A = 1$, $B = V$, $C = 0$, $D = 1$ corresponds to the Galileo group. Note also $\epsilon \to \epsilon' \to \epsilon'' \Rightarrow V'' = V + V'$. Now for the analogue of Lorentz transformations there is a need to preserve the Galilean dilatation law for scales larger than the quantum classical transition. Note $V = log(\epsilon/\epsilon') \sim \epsilon/\epsilon' = exp(-V)$ and set $\rho = \epsilon'/\epsilon$ with $\rho' = \epsilon''/\epsilon'$ and $\rho'' = \epsilon''/\epsilon$; then $log\rho'' = \log\rho + \log\rho'$ and one is thinking here of $\rho : \epsilon \to \epsilon'$, $\rho' : \epsilon' \to \epsilon''$ and $\rho'' : \epsilon \to \epsilon''$ with compositions (the notation is meant to somehow correspond to (4.1)). Now recall the Einstein-Lorentz law $w = (u+v)/[1+(uv/c^2)]$ but one now has several regimes to consider. Following [**716, 720**] small scale symmetry is broken by mass via the emergence of $\lambda_c = \hbar/mc$ (Compton length) and $\lambda_{dB} = \hbar/m\nu$ (deBroglie length), while for extended objects $\lambda_{th} = \hbar/m < \nu^2 >^{1/2}$ (thermal deBroglie length) affects transitions. The transition scale in (4.2) is the Einstein-deBroglioe scale (in rest frame $\lambda \sim \tau = \hbar/mc^2$) and in the cosmological realm the scale symmetry is broken by the emergence of static structure of typical size $\lambda_g = (1/3)(GM/ < \nu^2 >)$. The scale space consists of three domains (quantum, classical - scale independent, and cosmological). Another small scale transition factor appears in the Planck length scale $\lambda_P = (\hbar G/c^3)^{1/2}$ and at large scales the cosmological constant Λ comes into play. With this background the composition of dilatations is taken to be

$$(4.5)\qquad log\frac{\epsilon'}{\lambda}=\frac{log\rho+\log\frac{\epsilon}{\lambda}}{1+\frac{log\rho log\frac{\epsilon}{\lambda}}{log^2(L/\lambda)}}=\frac{log\rho+\log\frac{\epsilon}{\lambda}}{1+\frac{log\rho log(\epsilon/\lambda)}{C^2}}$$

where $L\sim\lambda_P$ near small scales and $L\sim\Lambda$ near large scales (note $\epsilon=L\Rightarrow\epsilon'=L$ in (A.4)). Comparing with $w=(u+v)/(1+(uv/c^2))$ one thinks of $log(L/\lambda)=C\sim c$ (note here $log^2(a/b)=log^2(b/a)$ in comparing formulas in [**716, 720**]). Lengths now change via

$$(4.6)\qquad log\frac{\ell'}{\ell_0}=\frac{log(\ell/\ell_0)+\delta log\rho}{\sqrt{1-\frac{\log^2\rho}{C^2}}}$$

and the scale variable δ (or djinn) is no longer constant but changes via

$$(4.7)\qquad \delta(\epsilon')=\frac{\delta(\epsilon)+\frac{log\rho log(\ell/\ell_0)}{C^2}}{\sqrt{1-\frac{log^2\rho}{C^2}}}$$

where $\lambda\sim$ fractal-nonfractal transition scale.

We have derived the SE in Section 1.2 (cf. also [**186**]) and go now to the KG equation via scale relativity. The derivation in the first paper of [**232**] seems the most concise and we follow that at first (cf. also [**716**]). All of the elements of the approach for the SE remain valid in the motion relativistic case with the time replaced by the proper time s, as the curvilinear parameter along the geodesics. Consider a small increment dX^μ of a nondifferentiable four coordinate along one of the geodesics of the fractal spacetime. One can decompose this in terms of a large scale part $\overline{LS}<dX^\mu>=dx^\mu=v_\mu ds$ and a fluctuation $d\xi^\mu$ such that $\overline{LS}<d\xi^\mu>=0$. One is led to write the displacement along a geodesic of fractal dimension $D=2$ via

$$(4.8)\qquad dX^\mu_\pm=d_\pm x^\mu+d\xi^\mu_\pm=v^\mu_\pm ds+u^\mu_\pm\sqrt{2\mathcal{D}}ds^{1/2}$$

Here $u^\mu_\pm$ is a dimensionless fluctuation andd the length scale $2\mathcal{D}$ is introduced for dimensional purposes. The large scale forward and backward derivatives d/ds_+ and d/ds_- are defined via

$$(4.9)\qquad \frac{d}{ds_\pm}f(s)=lim_{s\to 0_\pm}\overline{LS}\left\langle\frac{f(s+\delta s)-f(s)}{\delta s}\right\rangle$$

Applied to x^μ one obtains the forward and backward large scale four velocities of the form

$$(4.10)\qquad (d/dx_+)x^\mu(s)=v^\mu_+;\ (d/ds_-)x^\mu=v^\mu_-$$

Combining yields

$$(4.11)\qquad \frac{d'}{ds}=\frac{1}{2}\left(\frac{d}{ds_+}+\frac{d}{ds_-}\right)-\frac{i}{2}\left(\frac{d}{ds_+}-\frac{d}{ds_-}\right);$$

$$\mathcal{V}^\mu=\frac{d'}{ds}x^\mu=V^\mu-iU^\mu=\frac{v^\mu_++v^\mu_-}{2}-i\frac{v^\mu_+-v^\mu_-}{2}$$

For the fluctuations one has

$$\overline{LS} < d\xi_{\pm}^{\mu} d\xi_{\pm}^{\nu} >= \mp 2\mathcal{D}\eta^{\mu\nu} ds \tag{4.12}$$

One chooses here $(+,-,-,-)$ for the Minkowski signature for $\eta^{\mu\nu}$ and there is a mild problem because the diffusion (Wiener) process makes sense only for positive definite metrics. Various solutions were given in [**314, 859, 1013**] and they are all basically equivalent, amounting to the transformatin a Laplacian into a D'Alembertian. Thus the two forward and backward differentials of $f(x,s)$ should be written as

$$(df/ds_{\pm}) = (\partial_s + v_{\pm}^{\mu}\partial_{\mu} \mp \mathcal{D}\partial^{\mu}\partial_{\mu})f \tag{4.13}$$

One considers now only stationary functions f, not depending explicitly on the proper time s, so that the complex covariant derivative operator reduces to

$$(d'/ds) = (\mathcal{V}^{\mu} + i\mathcal{D}\partial^{\mu})\partial_{\mu} \tag{4.14}$$

Now assume that the large scale part of any mechanical system can be characterized by a complex action $\mathfrak{S}$ leading one to write

$$\delta\mathfrak{S} = -mc\delta\int_a^b ds = 0;\ ds = \overline{LS} < \sqrt{dX^{\nu}dX_{\nu}} > \tag{4.15}$$

This leads to $\delta\mathfrak{S} = -mc\int_a^b \mathcal{V}_{\nu} d(\delta x^{\nu})$ with $\delta x^{\nu} = \overline{LS} < dX^{\nu} >$. Integrating by parts yields

$$\delta\mathfrak{S} = -[mc\delta x^{\nu}]_a^b + mc\int_a^b \delta x^{\nu}(d\mathcal{V}_{\mu}/ds)ds \tag{4.16}$$

To get the equations of motion one has to determine $\delta\mathfrak{S} = 0$ between the same two points, i.e. at the limits $(\delta x^{\nu})_a = (\delta x^{\nu})_b = 0$. From (4.16) one obtains then a differential geodesic equation $d\mathcal{V}/ds = 0$. One can also write the elementary variation of the action as a functional of the coordinates. So consider the point a as fixed so $(\delta x^{\nu})_a = 0$ and consider b as variable. The only admissable solutions are those satisfying the equations of motion so the integral in (4.16) vanishes and writing $(\delta x^{\nu})_b$ as δx^{ν} gives $\delta\mathfrak{S} = -mc\mathcal{V}_{\nu}\delta x^{\nu}$ (the minus sign comes from the choice of signature). The complex momentum is now

$$\mathcal{P}_{\nu} = mc\mathcal{V}_{\nu} = -\partial_{\nu}\mathfrak{S} \tag{4.17}$$

and the complex action completely characterizes the dynamical state of the particle. Hence introduce a wave function $\psi = exp(i\mathfrak{S}/\mathfrak{S}_0)$ and via (4.17) one gets

$$\mathcal{V}_{\nu} = (i\mathfrak{S}_0/mc)\partial_{\nu} log(\psi) \tag{4.18}$$

Now for the scale relativistic prescription replace the derivative in d/ds by its covariant expression d'/ds. Using (4.18) one transforms $d\mathcal{V}/ds = 0$ into

$$-\frac{\mathfrak{S}_0^2}{m^2c^2}\partial^{\mu} log(\psi)\partial_{\mu}\partial_{\nu} log(\psi) - \frac{\mathfrak{S}_0\mathcal{D}}{mc}\partial^{\mu}\partial_{\mu}\partial_{\nu} log(\psi) = 0 \tag{4.19}$$

The choice $\mathfrak{S}_0 = \hbar = 2mc\mathcal{D}$ allows a simplification of (4.19) when one uses the identity

$$\frac{1}{2}\left(\frac{\partial_\mu\partial^\mu\psi}{\psi}\right) = \left(\partial_\mu log(\psi) + \frac{1}{2}\partial_\mu\right)\partial^\mu\partial^\nu log(\psi) \quad (4.20)$$

Dividing by $\mathcal{D}^2$ one obtains the equation of motion for the free particle $\partial^\nu[\partial^\mu\partial_\mu\psi/\psi] = 0$. Therefore the KG equation (no electromagnetic field) is

$$\partial^\mu\partial_\mu\psi + (m^2c^2/\hbar^2)\psi = 0 \quad (4.21)$$

and this becomes an integral of motion of the free particle provided the integration constant is chosen in terms of a squared mass term $m^2c^2/\hbar^2$. Thus the quantum behavior described by this equation and the probabilistic interpretation given to ψ is reduced here to the description of a free fall in a fractal spacetime, in analogy with Einstein's general relativity. Moreover these equations are covariant since the relativistic quantum equation written in terms of d'/ds has the same form as the equation of a relativistic macroscopic and free particle using d/ds. One notes that the metric form of relativity, namely $V^\mu V_\mu = 1$ is not conserved in QM and it is shown in [**775**] that the free particle KG equation expressed in terms of $\mathcal{V}$ leads to a new equality

$$\mathcal{V}^\mu\mathcal{V}_\mu + 2i\mathcal{D}\partial^\mu\mathcal{V}_\mu = 1 \quad (4.22)$$

In the scale relativistic framework this expression defines the metric that is induced by the internal scale structures of the fractal spacetime. In the absence of an electromagnetic field $\mathcal{V}^\mu$ and $\mathfrak{S}$ are related by (4.17) which can be writen as $\mathcal{V}_\mu = -(1/mc)\partial_\mu\mathfrak{S}$ so (4.22) becomes

$$\partial^\mu\mathfrak{S}\partial_\mu\mathfrak{S} - 2imc\mathcal{D}\partial^\mu\partial_\mu\mathfrak{S} = m^2c^2 \quad (4.23)$$

which is the new form taken by the Hamilton-Jacobi equation.

REMARK 3.4.1. We go back to [**716, 775**] now and repeat some of their steps in a perhaps more primitive but revealing form. Thus one omits the $\overline{LS}$ notation and uses $\lambda \sim 2\mathcal{D}$; equations (4.8) - (4.14) and (4.11) are the same and one writes now $\mathfrak{d}/ds$ for d'/ds. Then $\mathfrak{d}/ds = \mathcal{V}^\mu\partial_\mu + (i\lambda/2)\partial^\mu\partial_\mu$ plays the role of a scale covariant derivative and one simply takes the equation of motion of a free relativistic quantum particle to be given as $(\mathfrak{d}/ds)\mathcal{V}^\nu = 0$, which can be interpreted as the equations of free motion in a fractal spacetime or as geodesic equations. In fact now $(\mathfrak{d}/ds)\mathcal{V}^\nu = 0$ leads directly to the KG equation upon writing $\psi = exp(i\mathfrak{S}/mc\lambda)$ and $\mathfrak{P}^\mu = -\partial^\mu\mathfrak{S} = mc\mathcal{V}^\mu$ so that $i\mathfrak{S} = mc\lambda log(\psi)$ and $\mathcal{V}^\mu = i\lambda\partial^\mu log(\psi)$. Then

$$\left(\mathcal{V}^\mu\partial_\mu + \frac{i\lambda}{2}\partial^\mu\partial_\mu\right)\partial^\nu log(\psi) = 0 = i\lambda\left(\frac{\partial^\mu\psi}{\psi}\partial_\mu + \frac{1}{2}\partial^\mu\partial_\mu\right)\partial^\nu \log(\psi) \quad (4.24)$$

Now some identities are given in [**775**] for aid in calculation here, namely

$$\frac{\partial^\mu\psi}{\psi}\partial_\mu\frac{\partial^\nu\psi}{\psi} = \frac{\partial^\mu\psi}{\psi}\partial^\nu\left(\frac{\partial_\mu\psi}{\psi}\right) = \quad (4.25)$$
$$= \frac{1}{2}\partial^\nu\left(\frac{\partial^\mu\psi}{\psi}\frac{\partial_\mu\psi}{\psi}\right);\ \partial_\mu\left(\frac{\partial^\mu\psi}{\psi}\right) + \frac{\partial^\mu\psi}{\psi}\frac{\partial_\mu\psi}{\psi} = \frac{\partial^\mu\partial_\mu\psi}{\psi}$$

The first term in the last equation of (4.24) is then $(1/2)[(\partial^\mu\psi/\psi)(\partial_\mu\psi/\psi)]$ and the second is

$$(4.26)\qquad (1/2)\partial^\mu\partial_\mu\partial^\nu log(\psi) = (1/2)\partial^\mu\partial^\nu\partial_\mu log(\psi) =$$

$$= (1/2)\partial^\nu\partial^\mu\partial_\mu log(\psi) = (1/2)\partial^\nu\left(\frac{\partial^\mu\partial_\mu\psi}{\psi} - \frac{\partial^\mu\psi\partial_\mu\psi}{\psi^2}\right)$$

Combining we get $(1/2)\partial^\nu(\partial^\mu\partial_\mu\psi/\psi) = 0$ which integrates then to a KG equation

$$(4.27)\qquad -(\hbar^2/m^2c^2)\partial^\mu\partial_\mu\psi = \psi$$

for suitable choice of integration constant (note $\hbar/mc$ is the Compton wave length).

Now in this context or above we refer back to Section 2.2 for example and write $Q = -(1/2m)(\Box R/R)$ (cf. Section 2.2 before Remark 2.2.1 and take $\hbar = c = 1$ for convenience here). Then recall $\psi = exp(i\mathfrak{S}/m\lambda)$ and $\mathfrak{P}_\mu = m\mathcal{V}_\mu = -\partial_\mu\mathfrak{S}$ with $i\mathfrak{S} = m\lambda log(\psi)$. Also $\mathcal{V}_\mu = -(1/m)\partial_\mu\mathfrak{S} = i\lambda\partial_\mu log(\psi)$ with $\psi = Rexp(iS/m\lambda)$ so $log(\psi) = i\mathfrak{S}/m\lambda = log(R) + iS/m\lambda$, leading to

$$(4.28)\qquad \mathcal{V}_\mu = i\lambda[\partial_\mu log(R) + (i/m\lambda)\partial_\mu S] = -\frac{1}{m}\partial_\mu S + i\lambda\partial_\mu log(R) = V_\mu + iU_\mu$$

Then $\Box = \partial^\mu\partial_\mu$ and $U_\mu = \lambda\partial_\mu log(R)$ leads to

$$(4.29)\qquad \partial^\mu U_\mu = \lambda\partial^\mu\partial_\mu log(R) = \lambda\Box log(R)$$

Further $\partial^\mu\partial_\nu log(R) = (\partial^\mu\partial_\nu R/R) - (R_\nu R_\mu/R^2)$ so

$$(4.30)\qquad \Box log(R) = \partial^\mu\partial_\mu log(R) = (\Box R/R) - (\sum R_\mu^2/R^2) =$$

$$= (\Box R/R) - \sum(\partial_\mu R/R)^2 = (\Box R/R) - |U|^2$$

for $|U|^2 = \sum U_\mu^2$. Hence via $\lambda = 1/2m$ for example one has

$$(4.31)\qquad Q = -(1/2m)(\Box R/R) = -\frac{1}{2m}\left[|U|^2 + \frac{1}{\lambda}\Box log(R)\right] =$$

$$= -\frac{1}{2m}\left[|U|^2 + \frac{1}{\lambda}\partial^\mu U_\mu\right] = -\frac{1}{2m}|U|^2 - \frac{1}{2}div(\vec{U})$$

(cf. Section 2.2).

REMARK 3.4.2. The words fractal spacetime as used in the scale relativity methods of Nottalle et al for producing geodesic equations (SE or KG equation) are somewhat misleading in that essentially one is only looking at continuous nondifferentiable paths for example. Scaling as such is of course considered extensively at other times. It would be nice to create a fractal derivative based on scaling properties and H-dimension alone for example which would permit the powerful techniques of calculus to be used in a fractal context. There has been of course some work in this direction already in e.g. [**187, 257, 411, 437, 466, 471, 562, 721, 748, 816**].

5. QUANTUM MEASUREMENT AND GEOMETRY

We consider here a paper [**989**], which is based in part on a famous paper of London [**611**] (reprinted in [**731**]). In [**611**] it was shown that the ratio of the Weyl scale factor to the Schrödinger wave function is constant if the proportionality constant between the Weyl potential and the EM potential is taken to be imaginary; this observation gave birth to modern gauge theories and the original Weyl theory was absorbed into QM with the original scale freedom becoming invariance under unitary gauge transformations (cf. also Section 3.5.1). Both the Weyl theory and the Schrödinger theory describe the evolution of a field in time and given the factor of i and the Kaluza-Klein framework used by London, those evolutions are the same. In the Weyl picture the field characterizes the length scales of fundamental matter, while in the Schrödinger picture it is the wave function corresponding to a fundamental particle. This analogy is pursued further in [**989**] with a main theme being the equivalence between Weyl measurement and quantum measurement; a complete theory of measurement in a Weyl geometry is said to contain the crucial elements of quantization and analogies of the following sort are indicated.

(5.1)

$Weyl - quantum\ correspondence$	$Quantum\ mechanics$
$Zero - Weyl - weight\ number$	$Real\ eigenvalue$
$Diffusion\ equation$	SE
$Weiner\ path\ integral$	$Feynman\ path\ integral$
$Weightful\ length\ field\ \psi_w$	$Complex\ state\ function\ \psi$
$Weyl\ conjugate\ \psi_{-w}$	ψ^*
$Probability\ \psi_w\psi_{-w}$	$Probability\ \vert\psi\vert^2$
$\psi_w \to e^{w\phi}\psi_w\ (conformal)$	$\psi \to e^{i\phi}\psi\ (unitary)$

We will try to make sense out of this following [**989**] (cf. also [**63, 64**]). Begin with a real 4-dimensional manifold $(M,[g])$ where $[g]$ is a conformal equivalence class of Lorentz metrics. In addition to local coordinate transformations one has Weyl (conformal) transformations given via $T(x)' = exp[w(T)\Lambda(x)]T(x)$ where T is a tensor field and $w(T)$ is the Weyl weight (a real number). One takes a coordinate basis $E_\alpha = \partial/\partial x^\alpha$ and $E^\alpha = dx^\alpha$ in the tangent and cotangent space satisfying $w(E_\alpha) = w(E^\alpha) = 0$.

DEFINITION 5.1. One defines a torsion free derivative D via

- Linearity: $D(aT_1 + bT_2) = aDT_1 + bDT_2$ for real a, b
- Leibniz: $D(T_1T_2) = (DT_1)T_2 + T_1(DT_2)$
- Weyl covariant: $D(fT) = [df + w(f)Wf]T + fDT$ where W is a real 1-form (Weyl potential)
- Zero weight: $w(DT) = w(T)$

Under a Weyl transformation $W \to W' = W - d\Lambda$ and one has

$$DT = D_\mu T^\alpha_{\ \beta} E^\mu \otimes E_\alpha \otimes E^\beta;\ D_\mu T^\alpha_{\ \beta} = \tag{5.2}$$

$$= \partial_\mu T^\alpha_{\ \beta} + T^\rho_{\ \beta}\Gamma^\alpha_{\ \rho\mu} - T^\alpha_{\ \rho}\Gamma^\rho_{\ \beta\mu} + w(T)W_\mu T^\alpha_{\ \beta}$$

There is no unique metric on the space; instead the metric is to be taken of the Weyl type $w(g) = 2$ so that under a Weyl transformation $g' = exp[2\Lambda(x)]g$.

The principle fields of the theory are related by the requirement $Dg = 0$, or in components

$$D_\mu g_{\alpha\beta} = 0 = \partial_\mu g_{\alpha\beta} - g_{\rho\beta}\Gamma^\rho{}_{\alpha\mu} - g_{\alpha\rho}\Gamma^\rho{}_{\beta\mu} + 2W_\mu g_{\alpha\beta} \tag{5.3}$$

This can be solved to give

$$\Gamma^\alpha{}_{\beta\mu} = \left\{ \begin{array}{c} \alpha \\ \beta\ \ \mu \end{array} \right\} + (\delta^\alpha{}_\beta W_\mu + \delta^\alpha{}_\mu W_\beta - g_{\beta\mu} W^\alpha) \tag{5.4}$$

Vanishing torsion has been assumed in (5.4) so that the bracket expression is the usual Christoffel connection. The curvature tensor is then

$$R^\alpha{}_{\beta\mu\nu} = \Gamma^\alpha{}_{\beta\nu,\mu} - \Gamma^\alpha{}_{\beta\mu,\nu} + \Gamma^\alpha{}_{\rho\mu}\Gamma^\rho{}_{\beta\nu} - \Gamma^\alpha{}_{\rho\nu}\Gamma^\rho{}_{\beta\mu} \tag{5.5}$$

Unlike the Riemannian curvature tensor the Weyl curvature has nonvanishing trace on the first pair of indices so that $(1/2)R^\alpha{}_{\alpha\mu\nu} = W_{\nu,\mu} - W_{\mu,\nu} = W_{\mu\nu}$ where $W_{\mu\mu}$ is the gauge invariant field strength of the Weyl potential. One says that two fields are Weyl conjugate if they have the same Lorenz transformation properties but opposite Weyl weights.

Now for a theory of measurement one first looks at zero weight fields. In this direction note that fields with nonvanishing Weyl weight will experience changes under parallel transport. For example the mass squared transported along a path with unit tangent vector $u^\mu = dx^\mu/d\tau$ satisfies

$$0 = u^\mu D_\mu(m^2) = u^\mu \partial_\mu(m^2) + w(m^2)u^\mu W_\mu m^2 \tag{5.6}$$

Integrating along the path of motion one finds a path dependence of the form $m^2 = m_0^2 exp[w(m^2)\int W_\mu u^\mu d\tau]$ where the line integral has been written in terms of the path parameter τ. Note this is analogous to $m^2 = m_0^2 exp(Q)$ in the Shojai theory of Section 3.2 suggesting some relation to a quantum potential $Q \sim w(m^2)\int W_\mu u^\mu d\tau$. However at this point there is no quantum matter posited and no density ρ so a Weyl vector $W_\mu \sim \partial_\mu log(\rho)$ as in Remark 3.3.1 is untenable and no comparison to (3.28) can be undertaken. However this does show a geometrical dependence of mass in general and in the flat space of Remark 3.3.1 it is replaced by a quantum potential. Indeed this (Schouten-Haantjes) conformal mass thus depends on the Weyl vector and if two particles of identical mass are allowed to propagate freely (by parallel transport) along different paths and brought together there will be a mass difference

$$\Delta m^2 = m_0^2 e^{w(m^2)\oint W_\mu u^\mu d\tau} \equiv m_0^2 e^{w(m^2)\int_S W_{\mu\nu} dS^{\mu\nu}} \tag{5.7}$$

where $dS^{\mu\nu}$ is an element of any 2-surface S bounded by the closed curve defined by the two particles. Hence unless the surface integral of the Weyl field strength vanishes there will be a path dependence for masses and of any other field of nonzero weight. One postulates now **(I)** that all quantities of vanishing Weyl weight should be physically meaningful (observables) and **(II)** that all fields occur in conjugate pairs satisfying conjugate equations of motion. Assume that $M_\pm$ evolves by parallel transport along a path as above via

$$0 = u^\mu D_\mu D_\pm = u^\mu \bar{D}_\mu M_\pm \pm w(M)M_\pm W_\mu u^\mu \tag{5.8}$$

where $\bar{D}$ is a derivation using the full connection (5.4) and one sets $w(M) = w(M_+) > 0$ for convenience. One has also

$$M_{\pm} = \mathfrak{M}exp[\mp w(M)\int W_\mu u^\mu d\tau] \tag{5.9}$$

where $\mathfrak{M}$ is weightless with $u^\mu \bar{D}_\mu \mathfrak{M} = 0$. Now suppose one wants to measure some characteristic of M (i.e. of M_+ or M_-). M can be scaled by an arbitrary gauge function and one transports M along a path so that its covariant derivative in the dirction of motion vanishes. Then the change in size is specified by (5.9) but it is not clear that we can tell what path a particle has taken. In a Riemannian space there are geodesics determining the paths of classical matter but that is not true in a Weyl space (in this regard we refer to [**188**], Section 3.2, and to [**133, 796, 797, 798, 799**]).

In order to study the motion of M one begins with the observation that a Weyl geometry provides a probability $P_{AB}(M)$ of finding a value M at a point B for a system which is known to have had a value M_0 at point A. Finding $P_{AB}(M)$ is tantamount to finding the fraction of paths which the system may follow leading to any given value of M. Since there may be no special paths in a Weyl geometry one has to settle for moments of the distribution. To find the average value of magnitude of M denoted by $<M>$ one integrates (5.9) over all paths via

$$<M>=\int \mathcal{D}[x]M_0 exp[w(M)\int_A^B W_\mu u^\mu d\tau] \tag{5.10}$$

where the usual path integral normalization is included implicitly in $\mathcal{D}[x]$ (see e.g. [**362, 457, 855**]) for path integrals). However this gives no information as to whether one should expect M to actualy reach B. In [**989**] there is then a long discussion (and a detailed Appendix) involving path averages, probability, Wiener integrals, etc. plus a postulate **(III)** that the probability a system will undergo a given infinitesimal displacement x^μ is inversely proportional to the change in length such a displacement produces in the system. Now $d\ell = w(M)W_\mu dx^\mu = w(M)W_\mu u^\mu d\tau$ and a plausible (rigorous) argument is given then to represent the probability of the system reaching any spacetime point x from x_0 as

$$G(x_0;x) = \int \mathcal{D}[x]exp[w(M)\int_{x_0}^x W_\mu u^\mu d\tau] \tag{5.11}$$

(which bears an obvious resemblence to (5.10)). Comparison of (5.10) and (5.11) involves noting first that (5.11) is gauge dependent but the gauge dependence comes out of the path integral since it depends only on the end points. Thus

$$G'(x_0;x) = \int \mathcal{D}[x]exp[w(M)\int_{x_0}^x (W_\mu - \partial_\mu\phi)u^\mu d\tau] = \tag{5.12}$$

$$= e^{-w(M)[\phi(x)-\phi(x_0)]}\int \mathcal{D}[x]exp[w(M)\int_{x_0}^x W_\mu u^\mu d\tau]$$

This means that one can eleiminate the gauge factor by multiplying by the Weyl conjugate expression

$$\bar{G}'(x_0;x) = e^{w(M)[\phi(x)-\phi(x_0)]}\int \mathcal{D}[x]exp[-w(M)\int_{x_0}^{x} W_\mu u^\mu d\tau] \tag{5.13}$$

to give a meaningful gauge invariant probability $P(x_0,x) = \bar{G}(x_0;x)G(x_0;x)$ which is the probability of detecting the dilating system at x given its presence at x_0. It may be thought of as the joint probability of finding both M and $\bar{M}$ at x. Here one is dealing with a real path integral, unlike QM, and the phase invariance of a wave function $\psi' = exp(i\phi)\psi$ is replaced by conformal invariance $M' = exp(\phi)M$ (this is the same factor of i introduced by London in 1927). Since that time gauge transformations have appeared as phases and the wave interpretation has been maintained; now one maintains a real gauge transformation and changes the interpretation of physical phenomena (see [**989**] for more discussion in this direction).

Now one shows the equivalence to QM of the nonrelativistic limit of (5.11) when the exponent in the path integral is identified with a multiple of the classical action, i.e. $\int W_\mu u^\mu d\tau = \lambda S = \lambda\int L d\tau$. The integrands here may also be equated except for the possible addition of the total derivative of a function of τ. But such a derivative is already known to be both a gauge freedom of W_μ and a transformation of L that leaves the equations of motion unaltered. So the possible equivalent versions of L may be understood as gauge changes of the underlying geometry. This identification fixes the physical interpretation of W_μ up to the gauge choice and since $u^\mu = \dot{x}^\mu$ equating the integrands gives

$$\lambda P_\mu = \lambda(\partial L/\partial u^\mu) = W_\mu \tag{5.14}$$

so that W_μ is proportional to the generalized momentum P_μ conjugate to x^μ. Now Weyl had originally identified W_μ with the derivative of an EM potential $\partial_\mu U \sim A_\mu$ and the present approach suggests $W_\mu = \lambda(p_\mu + A_\mu)$ so that all energy provides a surce of expansion rather than just EM energy. This still allows gauge transformations of W_μ to be identified with gauge transformations of A_μ. Next one goes to the nonrelativistic limit of the path integral to find a differential equation for the amplitudes $G(x_0;x)$. It is convenient to explicitly separate the kinetic term $p_\mu u^\mu$ from $W_\mu u^\mu$ which will enable one to identify the path integral in (5.11) with a Wiener integral. Thus with full generality one writes $W_\mu = \lambda(p_\mu + \tilde{W}_\mu)$ where any gauge transformation is understood to apply to $\tilde{W}_\mu$. Now consider the nonrelativistic limit where the integral $\int p_\mu u^\mu d\tau \sim mc^2\int d\tau$ so that $mc^2\int d\tau \sim \int[mc^2 - (m/2)\mathbf{v}^2]dt$. To this order the path integral becomes (suppressing limits of integration)

$$G(x_0;x) = \int \mathcal{D}[x]e^{\lambda w(M)\int[(1/2)m\mathbf{v}^2 + \tilde{W}\cdot\mathbf{v} - \tilde{W}^0 - mc^2)]dt} \tag{5.15}$$

This is of the form

$$P(x_0;x) = \int \mathcal{D}[x]exp[-(1/2)\int((\dot{\mathbf{q}}+\mathbf{w})^2 - \nabla\cdot\mathbf{w})dt] \tag{5.16}$$

where $P(x_0;x)$ is the propagator for the Fokker-Planck equation $\partial_t P = (1/2)\nabla^2 P + \nabla\cdot(\mathbf{w}P)$ provided one makes the identifications

$$\dot{\mathbf{q}} = \sqrt{-w(M)\lambda m}\mathbf{v};\ \nabla_x = \sqrt{-w(M)\lambda m}\nabla_q;\ \psi = Pe^{-2mc^2}; \tag{5.17}$$
$$\mathbf{w} = \sqrt{-w(M)\lambda/m}\tilde{\mathbf{W}};\ 2w(M)\lambda(mc^2+\tilde{\mathbf{W}}^0) = \mathbf{w}^2 - \nabla\cdot\mathbf{w}$$

(cf. [**457, 672, 674, 698, 856**]). Carrying out the substitutions and setting $\lambda\tilde{\mathbf{W}}_\mu = -U(\lambda\phi,\mathbf{A})$ one obtains $\psi(x) = \int\psi(x')G(x,x')dx'$ as a solution to

$$\frac{1}{w(M)\lambda}\partial_t\psi = -\frac{1}{2m[w(M)\lambda]^2}[\nabla + w(M)\lambda\mathbf{A}]^2 + (mc^2+U\phi)\psi \tag{5.18}$$

with initial condition $\psi = \psi(x')$ (this should be checked to clarify the roles of U and ϕ). If one sets $\lambda = \hbar^{-1}$ and the time is allowed to become imaginary the SE minimally coupled to EM arises. Thus choose $\lambda = \hbar^{-1}$ but leave time alone since it is not needed; then (5.18) can be interpreted as a stochastic form of QM. Evidently the Weyl weight serves the function of i, changing sign appropriately for the conjugate field. The emergence of the Fokker-Planck equation indicates diffusion and this is discussed at length in [**186, 672, 674, 698, 856**]. In addition the matter is discussed in [**989**] from various points of view. In particular one takes $(1/\hbar)S = \int W_\mu u^\mu d\tau$ and observes that a classical limit of the Weyl geometry will exist whenever there is an extremum to the action (as in the Feynman path integral). Thus a classical limit of (5.11) occurs whenever $\Psi = exp[w(M)\int_{x_0}^x W_\mu u^\mu d\tau]$ is extremal. However there is a difference here involving Ψ as a length factor. One shows that $\delta\Psi = 0$ corresponds to a special case of the Weyl field since $\int_A^B d\tau(W_{\mu,\nu} - W_{\nu,\mu})u^\mu\delta x^\nu = 0$ arises via variation which means $W_{\mu\nu}u^\nu = 0$. Some calculation then shows that $W_\alpha = \xi\partial_\alpha\chi$ (up to a gauge transformation) for any appropriately normalized functions ξ, χ satisfying

$$(D_\mu\chi)u^\nu = (D_\mu\chi)v^\mu = (D_\mu\xi)u^\nu = (D_\mu\xi)v^\mu = 0; \tag{5.19}$$
$$(1/2)\epsilon^{\mu\nu\alpha\beta}W_{\alpha\beta} = u^\mu v^\nu - u^\nu v^\mu$$

with ϵ the Levi-Civita tensor (cf. [**302, 989**]). Now $W_\alpha = \xi\partial_\alpha\chi$ is a rather remarkable relation; it represents a restricted form of W^α since it is easy to find a Weyl vector such that $W_{\mu\nu}u^\nu \sim W_{\mu 0} \neq 0$ for all nonspacelike u^ν. Since this formula arises for an arbitrary set of paths u^α it is clear that not all Weyl fields will have a classical limit. Thus as argued at the beginning the generic Weyl geometry lacks preferred paths and requires a path average. On the other hand if one chooses a gauge where $W_\alpha u^\alpha = 0$ (which is possible) then weightful bodies followed the preferred classical trajectories and experience no dilation. There is considerable discussion along these lines in [**989**] which is omitted here; there is also interesting material on relations to general relativity. In particular it is pointed out that size changes associated with nonvanishing Weyl field strength are not necessarily classically observable. However the Weyl field itself must be present and consequently must be detectable. Finding the physical field that it corresponds to simply requires substituting the appropriate conjugate momentum for W_μ in the classical equation of motion $W_{\mu\nu}u^\nu = 0$. Since the only long range forces are gravity and EM and gravity is still accounted for by the Riemannian curvature, W_μ must be electromagnetic. The most general classical conjugate momentum is therefore that of a point particle with charge q moving in an EM field. Then in an arbitrary gauge

$$W_\mu = (1/\hbar)(p_\mu + qA_\mu + \partial_\mu \Lambda) \tag{5.20}$$

where $p_\mu = mu_\mu$ and $u_\mu u^\mu = -1$. Then

$$0 = W_{\mu\nu}u^\nu = (1/\hbar)(p_{\mu,\nu} - p_{\nu,\mu} + qA_{\mu,\nu} - qA_{\nu\mu})u^\nu \tag{5.21}$$

or (using $(u_\mu u^\mu)_{,\nu} = 0$) $dp^\mu/d\tau = qu_\nu F^{\mu\nu}$ which is the Lorenz force law (note that Planck's constant drops out). For the interpretation of W_μ itself one can combine the curl of (5.20) with

$$W_{\alpha\beta} = D_\alpha \chi D_\beta \xi - D_\beta \chi D_\alpha \xi = \partial_\alpha \chi \partial_\beta \xi - \partial_\beta \chi \partial_\alpha \xi \tag{5.22}$$

(cf. (5.19) and the surrounding discussion); this leads to

$$\partial_\alpha \chi \partial_\beta \xi - \partial_\beta \chi \partial_\alpha \xi = (1/\hbar)(p_{\alpha,\beta} - p_{\beta,\alpha} + qA_{\alpha,\beta} - qA_{\beta,\alpha}) \tag{5.23}$$

the time component of which gives again the Lorenz law. The spatial components can be solved for the magnetic field to give

$$\mathbf{B} = (\hbar/q)(\nabla\chi \times \nabla\xi) - (m/q)(\nabla \times \mathbf{v}) \tag{5.24}$$

The two fields χ and ξ on the right side of $\mathbf{B}$ are sufficient to guarantee the existence of any type of physical magnetic field. Conversely one can use (5.24) to solve for the Weyl field in terms of $\mathbf{B}$ and $\mathbf{v}$ (which of course depend on $\hbar$). One notes that for vanishing Weyl field (5.24) reduces to the London equation for superconductivity. This means that matter fields which conspire to produce a Riemannian geometry become superconducting.

5.1. MEASUREMENT ON A BICONFORMAL SPACE. We continue the theme of Section 3.5 with a more general perspective from [**35**] based on biconformal geometry (cf. Appendix E for some background material and see also [**35, 36, 113, 497, 558, 987, 980, 981, 989, 990, 991, 992, 993, 994, 1010**]). We regard this approach via biconformal geometry as very interesting and will try to present it faithfully. The background material in Appendix E should be read first; results in [**994**] for example create a unified geometrical theory of gravity and electromagnetism based on biconformal geometry. One develops in [**35**] an interpretation for quantum behavior within the context of biconformal gauge theory based on the following postulates:

1. A σ_C biconformal space provides the physical arena for quantum and classical physics.
2. Quantities of vanishing conformal weight comprise the class of physically meaningful observables.
3. The probability that a system will follow any given infinitesimal displacement is inversely proportional to the dilatation the displacement produces in the system.

From these assumptions follow the basic properties of classical and quantum mechanics. The symplectic structure of biconformal space is similar to classical phase space and also gives rise to Hamilton's equations, Hamilton's principal function, conjugate variables, fundamental Poisson brackets, and Liouville theory when postulate 3 is replaced by a postulate of extremal motion. We sketch this here (somewhat brutally) and refer to [**35**] for details, philosophy, and further references; the details for the biconformal geometry are spelled out in [**992, 994**]. Thus one wants a physical arena which contains 4-D spacetime in a straightforward manner but which is large enough and structured so as to contain both general relativity (GR) and quantum theory (QT) at the same time. One demands therefore invariance under global Lorentz transformations, translations, and scalings (see below) and the Lie group characterizing this is the conformal group $O(4,2)$ or its covering group $SU(2,2)$. In Appendix E the basic facts about Lorentz transformations $M^a_{\ b} = -M_{ba} = \eta_{ac}M^c_{\ b}$, translations P_a, special conformal transformations K^a, and dilatations D are exhibited in the context of conformal gauge theory $(a,b = 0,1,2,3)$. One has two involutive automorphisms of the conformal algebra, first

$$\sigma_1 : \ (M^a_{\ b}, P_a, K^a, D) \to (M^a_{\ b}, -P_a, -K^a, D) \tag{5.25}$$

which identifies the residual local Lorentz and dilatation symmetry characteristic of biconformal gauging and this corresponds (resp. for the Poincaré Lie algebra or the Weyl algebra) to

$$\sigma_1 : \ (M^a_{\ b}, P_a) \to (M^a_{\ b}, -P_a) \ \textit{or} \ \sigma_1 : \ (M^a_{\ b}, P_a, D) \to (M^a_{\ b}, -P_a, D) \tag{5.26}$$

There is also a second involution for the conformal group, namely

$$\sigma_2 : \ (M^a_{\ b}, P_a, K^a, D) \to (M^a_{\ b}, K_a, P^a, -D) \tag{5.27}$$

Some representations of the conformal algebra, namely $su(2,2)$, are necessarily complex and σ_2 can be realized as complex conjugation. Specifically one thinks of a representation in which P_a and K^a are complex conjugates while $M^a_{\ b}$ is real and D is purely imaginary and such representations will be called σ_C representations. Biconformal spaces for which the connection 1-forms (and hence curvatures) have this property are then called σ_C spaces (see Appendix E for examples). This leads to postulate 1 above, namely the physical arena for QT and classical physics is a σ_C biconformal space. Now biconformal gauging of the conformal group provides in particular a symplectic structure as follows. Gauging D introduces a single gauge 1-form ω (the Weyl vector) and the corresponding dilatational curvature 2-form is

$$\Omega = d\omega - 2\omega^a\omega_a \tag{5.28}$$

where ω^a, ω_a are 1-form gauge fields for the translation and special conformal transformations respectively, which span an 8-dimensional space as an orthonormal basis (note $\omega_a = \eta_{ab}\bar{\omega}^b$ for σ_C representations and products are wedge products). Now for all torsion free solutions to the biconformal field equations (i.e. $*d*d\omega^0_0 = J$, $\omega^0_a = T_a + \cdot$, etc. - cf. Appendix E) the dilatational curvature takes the form ($\bullet$) $\Omega = \kappa\omega^a\omega_a$ with κ constant, so the structure equation becomes ($\bullet\bullet$) $d\omega = (\kappa + 2)\omega^a\omega_a$. As a result $d\omega$ is closed and nondegenerate and hence symplectic (since ω^a, ω_a span the space). There is also a biconformal metric arising from the group invariant Killing metric $K_{\Sigma\Pi} = c^{\Lambda}_{\Delta\Sigma}c^{\Delta}_{\Lambda\Pi}$ where $c^{\Lambda}_{\Delta\Sigma}$ $(\Sigma, \Pi, \cdots = 1, 2, \cdots, 15)$ are the real structure constants from the Lie algebra. This metric has a nondegenerate projection to the 8-D subspace spanned by P_a, K^a and provides a natural pseudo-Riemannian metric on biconformal manifolds. The projection takes the form

$$K_{AB} = \begin{pmatrix} & \eta_{ab} \\ \eta_{ab} & \end{pmatrix} \ (A, B = 0, 1, \cdots, 7) \tag{5.29}$$

One defines now conformal weights w of a definite weight field F via (♦) $D_\phi : F \to [exp(w\phi)]F$ where D_ϕ is dilatation by $exp(\phi)$ (cf. [**989**] and Section 3.5). One assumes now postulate 2 and concludes that for a field with nontrivial Weyl weight to have physical meaning it must be possible to construct weightless scalars by combining it with other fields (easily done with conjugate fields); one notes that zero weight fields are self conjugate. The symplectic form $\Theta = \omega^a\omega_a$ defines a symplectic bracket via

$$\{f, g\} = \Theta^{MN}\frac{\partial f}{\partial u^M}\frac{\partial g}{\partial u^N} \tag{5.30}$$

where $u^M = (x^a, y^b)$. For real solutions f, g to the field equations f and g are conjugate if they satisfy $\{f, f\} = 1$, $\{f, f\} = \{g, g\} = 0$. However for σ_C representations ω is a pure imaginary 1-form since it is defined as the dual to the dilatation generator D which is pure imaginary. One sees then that

$$\overline{\omega^a\omega_a} = \bar{\omega}^a\bar{\omega}_a = \eta^{ab}\omega_b\eta_{ac}\omega^c = -\omega^a\omega_a \tag{5.31}$$

so the dilatational curvature and the symplectic form are imaginary (cf. also [**35, 529**]). Consequently, for use of a complex gauge vector with real gauge transformations, the fundamental brackets should take here the form

$$\{f, g\} = i;\ \{f, f\} = \{g, g\} = 0;\ w_f = -w_g \tag{5.32}$$

In an arbitrary biconformal space one sets either

$$\frac{1}{\hbar}S = \frac{1}{\hbar}\int Ld\lambda = \int\omega = \int(W_a dx^a + \bar{W}_a dy^a)\ or \tag{5.33}$$
$$\frac{i}{\hbar}S = \frac{i}{\hbar}\int Ld\lambda = \int\omega = \int(W_a dx^a + \bar{W}_a dy^a)$$

The second form holds in a σ_C representation for the conformal group. An arbitrary parameter λ is OK since the integral of the Weyl 1-form is independent of parametrization. This integral also governs measurable size change since under parallel transport the Minkowski length of a vector V^a changes by

$$\ell = \ell_0 exp\int\omega;\ \ell^2 = \eta_{ab}V^aV^b \tag{5.34}$$

(cf. Appendix E). This change occurs because $\eta_{ab} = (-1, 1, 1, 1)$ is not a natural structure for biconformal space. This is in contrast to the Killing metric K_{AB} where lengths are of zero conformal weight. In a σ_C representation the Weyl vector is imaginary so the measurable part of the change in ℓ is not a real dilatation - rather, it is a change of phase. Now for classical mechanics one uses a variation of postulate 3, namely: **The motion of a (classical) physical system is given by extrema of the integral of the Weyl vector**. Biconformal spaces are real symplectic manifolds so the Weyl vector can be chosen so that the symplectic form satisfies the Darboux theorem $\omega = W_a dx^a = -y_z dx^a$; for σ_C representations the Darboux equations still holds but now with

$$\omega = W_a dx^a = -iy_a dx^a \tag{5.35}$$

and the classical motion is independent of which form is chosen. Thus the symplectic form for the σ_C case is $\Theta = d\omega = -idy_a dx^a$ and one has (♦♦) $\{x^a, y_b\} = i\delta^a_b$. Thus from (♦♦) it follows that y_b is the conjugate variable to the position coordinate x^b and in mechanical units one may set $y_a = \alpha p_a$ with

$$i\alpha S = \int \omega = -i\alpha \int (p_0 dt + p_i dx^i) \tag{5.36}$$

(α can be any constant with appropriate dimensions). Now if one requires t as an invariant parameter (so $\delta t = 0$) one can vary the corresponding canonical bracket to find

$$0 = \delta\{t, p_0\} = \{\delta t, p_0\} + \{t, \delta p_0\} = \frac{\partial(\delta p_0)}{\partial p_0} \tag{5.37}$$

Thus δp_0 can depend only on the remaining coordinates so $\delta p_0 = -\delta H(y_i, x^j, t)$ and the existence of a Hamiltonian is a consequence of choosing time as a nonvaried parameter of the motion. Applying the postulate $\delta S = 0$ variation leads to

$$0 = i\alpha\delta S = -i\alpha \int (\delta p_0 dt + \delta p_i dx^i - dp_i \delta x^i) = \tag{5.38}$$
$$= -i\alpha \int \left(-\frac{\partial H}{\partial x^i}\delta x^i dt - \frac{\partial H}{\partial p_i}\delta p_i dt + \delta p_i dx^i - dp_i \delta x^i \right)$$

and this gives the standard Hamilton's equations

$$0 = -\frac{\partial H}{\partial p_i} dt + dx^i;\; 0 = -\frac{\partial H}{\partial x^i} dt - dp_i \tag{5.39}$$

(note i and α drop out of the equations).

In the presence of nonvanishing dilatational curvature one then considers a classical experiment to measure size (or phase) change along C_1, while a ruler measured by λ moves along C_2 (C_i are classical paths between two fixed points). Some argument (see [**35**]) leads to an unchanged ratio of lengths via

$$\frac{\ell}{\lambda} = \frac{\ell_0}{\lambda_0} exp \int_{C-1-C-2} \omega = \frac{\ell_0}{\lambda_0} exp \oint \omega = \frac{\ell_0}{\lambda_0} exp \int\int_S d\omega = \frac{\ell_0}{\lambda_0} \tag{5.40}$$

where S is any surface bounded by the closed curve $C_1 - C_2$ (cf. also Section 3.5). Thus no dilatations are observable along classical paths. This calculation also shows that the restriction of ω to classical paths is exact and proves the existence of Hamilton's principal function S with

$$\alpha S(x) = \int^x W_a dx^a = \int^x W_a \frac{dx^a}{dt} dt \tag{5.41}$$

There is further argument in [**35**] via gauge freedom to show that classical objects do not exhibit measurable length change (in the complex case the phase changes cannot be removed by gauge choice but they are unobservable). Relations between phase space and biconformal space are discussed and one arrives at QM.

From the above one knows that there is no measurable size change along classical pathes in a biconformal geometry but for systems evolving along other than extremal paths (where the Hamilton equations do not apply for example) there may be measurable dilatation. To deal with this one needs nonclassical motion and one goes to the basic postulate 3, namely that the probability a system will follow any given infinitesimal displacement is inversely proportional to the dilatation the displacement produces in the system. The properties of biconformal space determine the evolution of Minkowski lengths along arbitrary curves and the imaginary Weyl vector produces measurable phase changes in the same way as the wave function. Combining this with the classically probabilistic motion of postulate 3, together with the necessary use of a standard of length to comply with postulate 2, one concludes that the probability of a system at x_0^a arriving at the point x_1^a is given by

$$P(x_1^i) = \int \mathcal{D}[x_{C'}] exp\left(\int_{C'} \omega\right) \int \mathcal{D}[x_C] exp\left(-\int_C \omega\right) = \tag{5.42}$$

$$= \mathcal{P}(x_1^i)\mathcal{P}(-x_1^i) = \mathcal{P}(x_1^i)\bar{\mathcal{P}}(x_1^i)$$

where a path average over all paths connecting the two points is involved and $\bar{\mathcal{P}}(x)$ is simulaneously the probability amplitude of the conformally conjugate system reaching x_1^i. Here one considers ratios ℓ/λ as above and includes all possible ruler paths. These are standard Feynman path integrals which are known to lead to the Schrödinger equation (not Wiener integrals as in [**989**]) and it is the requirement of a length standard that forces the product structure in (5.42). Note that the phase invariance of a wave function $\psi' = exp(i\phi)\psi$ is created by the σ_C conformal invariance $M' = exp(\lambda w)M$. The i in the Weyl vector is the crucial i noted by London in [**611**] (cf. [**989**] and Section 3.5). Note also that the path integral in (5.42) and the biconformal paths depend generically on the spacetime and momentum variables so one can immediately generalize to the usual integrals of QM, namely

$$\mathcal{P}(x_1^i) = \int \mathcal{D}[x_C]\mathcal{D}[y_C] exp(\int C\omega) \tag{5.43}$$

Note also that the failure of the base space to break into space like and momentum like submanifolds indicates a fundamental coupling between position and momentum and suggests a connection to the Heisenberg uncertainty principle. The arguments in [**35**] have a somewhat heuristic flavor at times but are certainly plausible and do refine the techniques of [**989**] (sketched in Section 3.5) in many ways. Given the success of biconformal geometry in unifying GR and EM it would seem only natural and just that QM could be encompassed as well in the same framework and further developments are eagerly awaited.

REMARK 3.5.1 We note from [**993**] that when identifying biconformal coordinates (x^μ, y_ν) with phase space coordinates (x^μ, p_ν) one sets naturally $y_\nu = \beta p_\nu$. This β must account for a sign difference in $\eta^{\mu\nu}\beta p_\mu \beta p_\nu = -\eta^{\mu\nu} y_\mu y_\nu$ (cf. [**993**]) so β is pure imaginary. Further to account for the different units of y_ν ($length^{-1}$) and p_ν (momentum) one chooses $y_\nu = (i/\hbar)p_\nu$ and this relation between the geometric variables of conformal gauge theory and the physical momentum variables is the source of complex quantities in QM.