CHAPTER 2

DEBROGLIE-BOHM IN VARIOUS CONTEXTS

The quantum potential arises in various forms, some of which were discussed in Sections 1.1 and 1.2. We return to this now in a somewhat more systematic manner. The original theory goes back to deBroglie and D. Bohm (see e.g. [**94, 95, 128, 129, 154, 471, 472, 532**]) and in its modern version the dominant themes seem to be contained in [**88, 102, 288, 295, 324, 325, 326, 327, 328, 329, 387, 402, 414, 415, 927, 948**] with variations as in [**110, 186, 187, 188, 189, 191, 194, 195, 196, 197, 198, 346, 347, 373, 374, 375**] based on work of Bertoldi, Faraggi, and Matone (cf. also [**68, 138, 148, 165, 236, 305, 520, 574, 575, 576, 873, 881**]) and cosmology following [**123, 188, 189, 219, 498, 499, 500, 501, 571, 709, 710, 711, 840, 841, 871, 872, 873, 875, 876, 895, 989, 990**]. In any event the quantum potential does enter into any trajectory theory of deBroglie-Bohm (dBB) type. The history is discussed for example in [**471**] (cf. also [**68, 126, 127, 129, 154**]) and we have seen how this quantum potential idea can be formulated in various ways in terms of statistical mechanics, hydrodynamics, information and entropy, etc. when dealing with different versions and origins of the SE. Given the existence of particles we finds the pilot wave of thinking very attractive, with the wave function serving to choreograph the particle motion (or perhaps to "create" particles and/or spacetime paths). However the existence of particles itself is not such an assured matter and in field theory approaches for example one will deal with particle currents (cf. [**701**] and see also e.g. [**94, 95, 326, 402, 948**]). The whole idea of quantum particle path seems in any case to be either fractal (cf. [**1, 3, 14, 15, 186, 223, 232, 273, 676, 715, 717, 720, 733, 734, 735, 736**], stochastic (see e.g. [**68, 148, 186, 381, 382, 446, 447, 448, 449, 534, 536, 671, 674, 698, 805, 806**], or field theoretic (cf. [**94, 95, 326, 402, 701, 702, 703, 704, 705, 706, 707, 948**]. The fractal approach sometimes imagines an underlying micro-spacetime where paths are perhaps fractals with jumps, etc. and one possible advantage of a field theoretic approach would be to let the fields sense the ripples, which as e.g. operator valued Schwartz type distributions, they could well accomplish. In fact what comes into question here is the structure of the vacuum and/or of spacetime itself. One can envision microstructures as in [**186, 422, 676, 690**] for example, textures (topological defects) as in [**71, 74, 75, 170, 978**], Planck scale structure and QFT, along with space-time uncertainty relations as in [**71, 316, 317, 604, 1008**], vacuum structures and conformal invariance as in [**668, 669, 835, 831, 837, 838**], pilot wave cosmology as in [**834, 881**], ether theories as in [**851, 919**], etc. Generally there seems to be a sense in which particles cannot be measured as such

and hence the idea of particle currents (perhaps corresponding to fuzzy particles or ergodic clumps) should prevail perhaps along with the idea of probability packets. A number of arguments work with a (representative) trajectory as if it were a single particle but there is no reason to take this too seriously; it could be thought of perhaps as a "typical" particle in a cloud but conclusions should perhaps always be constructed from an ensemble point of view. We will try to develop some of this below. The sticky point as we see it now goes as follows. Even though one can write stochastic equations for (typical) particle motion as in the Nelson theory for example one runs into the problem of ever actually being able to localize a particle. Indeed as indicated in [**316, 317**] (working in a relativistic context but this should hold in general) one expects space time uncertainty relations even at a semiclassical level since any localization experiment will generate a gravitational field and deform spacetime. Thus there are relations $[q_\mu, q_\nu] = i\lambda_P^2 Q_{\mu\nu}$ where λ_P is the Planck length and the picture of spacetime as a local Minkowski manifold should break down at distances of order λ_P. One wants the localization experiment to avoid creating a black hole (putting the object out of "reach") for example and this suggests $\Delta x_0(\sum_1^3 \Delta x_i) \gtrsim \lambda_P^2$ with $\Delta x_1 \Delta x_2 + \Delta x_2 \Delta x_3 + \Delta x_3 \Delta x_1 \gtrsim \lambda_P^2$ (cf. [**316, 317**]). On the other hand in [**701**] it is shown that in a relativistic bosonic field theory for example one can speak of currents and n-particle wave functions can have particles attributed to them with well defined trajectories, even though the probability of their experimental detection is zero. Thus one enters an arena of perfectly respectible but undetectible particle trajectories. The discussion in [**256, 326, 920, 953, 961**] is also relevant here; some recourse to the idea of beables, reality, and observables as beables, etc. is also involved (cf. [**94, 95, 256, 961**]). We will have something to say about all these matters.

The dominant approach as in [**324, 325, 326, 327, 402, 948**] will be discussed as needed (a thorough discussion would take a book in itself) and we only note here that one is obliged to use the form $\psi = Rexp(iS/\hbar)$ to make sense out of the constructions (this is no problem with suitable provisos, e.g. that S is not constant - cf. [**110, 191, 346, 347, 373, 374**] and comments later). This leads to

$$(\bigstar)\ S_t + \frac{(S')^2}{2m} - \left(\frac{\hbar^2}{2m}\right)\left(\frac{R''}{R}\right) + V = 0;\ \partial_t R^2 + \partial\left(\frac{R^2 S'}{m}\right) = 0$$

(cf. (1.1.1)) where $Q = -\hbar^2 R''/2mR$ arising from a SE $i\hbar\partial_t\psi = -(\hbar^2/2m)\psi_{xx} + V\psi$ (we use 1-D for simplicity here). In [**324**] one emphasizes configurations based on coordinates whose motion is choreographed by the SE according to the rule

$$(\bigstar\bigstar)\ \dot{q} = v = \frac{\hbar}{m}\Im\frac{\psi^*\psi'}{|\psi|^2} = \frac{\hbar}{m}\Im\left(\frac{\psi'}{\psi}\right)$$

The argument for ($\bigstar\bigstar$) is based on obtaining the simplest Galilean and time reversal invariant form for velocity, transforming correctly under velocity boosts. This leads directly to ($\bigstar\bigstar$) so that Bohmian mechanics (BM) is governed by ($\bigstar\bigstar$) and the SE. It's a fairly convincing argument and no recourse to Floydian need be involved (cf. [**110, 191, 347, 373, 374**]). Note however that if $S = c$ then $\dot{q} = v = (\hbar/m)\Im(R'/R) = 0$ while $p = S' = 0$ so this formulation seems to avoid the $S = constant$ problems indicated in [**110, 191, 347, 373, 374**].

What makes the constant $\hbar/m$ in (★★) important here is that with this value the probability density $|\psi|^2$ on configuration space is equivariant. This means that via the evolution of probability densities $\rho_t + div(v\rho) = 0$ (as in (1.1.5)) the density $\rho = |\psi|^2$ is stationary relative to ψ, i.e. $\rho(t)$ retains the form $|\psi(q,t)|^2$. One calls $\rho = |\psi|^2$ the quantum equilibrium density (QEDY) and says that a system is in quantum equilibrium when its coordinates are randomly distributed according to the QEDY. The quantum equilibrium hypothesis (QEHP) is the assertion that when a system has wave function ψ the distribution ρ of its coordinates satisfies $\rho = |\psi|^2$.

1. THE KLEIN-GORDON AND DIRAC EQUATIONS

Before embarking on further discussion of QM it is necessary to describe some aspects of quantum field theory (QFT) and in particular to give some foundation for the Klein-Gordon (KG) and Dirac equations. For QFT we rely on [**120, 457, 528, 764, 827, 1015**] and concentrate on aspects of general quantum theory that are expressed through such equations. We alternate between signature $(-,+,+,+)$ and $(+,-,-,-)$ in Minkowski space, depending on the source. It is hard to avoid using units $\hbar = c = 1$ when sketching theoretical matters (which is personally repugnant) but we will set $\hbar = c = 1$ and shift to the general notation whenever any real meaning is desired. Thus $|length| \sim |time| \sim |energy|^{-1} \sim |mass|^{-1}$ and m = the inverse Compton wavelength ($mc/\hbar = \ell_C^{-1}$). The approaches in [**457, 764**] seem best adapted to our needs and in particular [**457**] gives a nice discussion motivating second quantization of a nonrelativistic SE (cf. [**701**] for first quantization). The resulting second quantization would be Galiean invariant but not Lorentz invariant so we go directly to the KG equation as follows. Note that there are often notational differences in various treatments of QFT and we use that of [**457**] in general. Start now from $E^2 = p^2 + m^2$ (which is the relativistic form of $E = p^2/2m$) to arrive, via $E \to i\partial_t$ and $p_j \to -i\partial_j = -i\partial/\partial x^j$, at the KG equation

$$(\partial_t^2 - \nabla^2)\phi + m^2\phi = 0 \tag{1.1}$$

where $\phi = \phi(\mathbf{x}, t)$ is a scalar wave function. This can also be derived from an action

$$S(\phi) = \int d^4x \mathfrak{L}(\phi, \partial_\mu\phi) = \frac{1}{2}\int d^4x(\partial^\mu\partial_\mu\phi - m^2\phi) \tag{1.2}$$

($x^0 = t,\ x = (\mathbf{x}, t)$), provided ϕ transforms as a Lorentz scalar (required also in (1.1)). The first problems arise from negative energy solutions (e.g. $exp[i(\mathbf{k}\cdot\mathbf{x} + \omega t)]$ is a solution of (1.1) with $E = -\omega = -(\mathbf{k}^2 + m^2)^{1/2}$). Secondly the energy spectrum is not bounded below (i.e. one could extract an arbitrary amount of energy from a single particle system). Further, using a positve square root of $E^2 = p^2 + m^2$ would involve a square root of a differential operator and nonlocal terms. Next observe that conserved currents j_μ (with $\partial^\mu j_\mu = 0$) arise à la E. Noether in the form

$$j_0 = \rho = \frac{i}{2m}(\phi^*\phi_t - \phi_t^*\phi);\ j_i = \frac{1}{2im}(\phi^*\partial_i\phi - (\partial_i\phi^*)\phi) \tag{1.3}$$

(where normal ordering is implicit here in order to avoid dealing with a vacuum energy term - to be discussed later). We note that for the plane wave solution above $\rho = -\omega/m = -(1/m)(\mathbf{k}^2 + m^2)^{1/2}$ and this is not a good probability density. The difficulties are resolved by giving up the idea of a one particle theory; it is not compatible with Lorentz invariance and the solution is to quantize the field ϕ.

Thus take $S(\phi)$ as in (1.2) with $\pi = \partial\mathfrak{L}/\partial(\partial_\mu\phi) = \partial_t\phi = \dot{\phi}$ and construct a Hamiltonian

$$H = \frac{1}{2}\int d^3x[\pi^2(x) + |\nabla\phi(x)|^2 + m^2\phi^2(x)] \tag{1.4}$$

In analogy with QM where $[x, p] = i$ one stipulates

$$[\phi(\mathbf{x}, t), \pi(\mathbf{y}, t)] = i\delta(\mathbf{x} - \mathbf{y});\ [\phi(\mathbf{x}, t), \phi(\mathbf{y}, t)] = [\pi(\mathbf{x}, t), \pi(\mathbf{y}, t)] = 0 \tag{1.5}$$

The operator equation $\dot{\phi} = i[H, \phi]$ then yields $\pi = \dot{\phi}$ and $\dot{\pi} = i[H, \pi]$ reproduces the KG equation. This is now a quantum field theory and for a particle interpretation one expands $\phi(\mathbf{x}, t)$ in terms of classical solutions of the KG equation via

$$\phi(\mathbf{x}, t) = \sum a(\mathbf{k})\phi_k^+(x) + b(\mathbf{k})\phi_k^-(x) = \tag{1.6}$$

$$= \int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{1}{2\omega_k}(a(\mathbf{k})e^{-i[\omega_k t - \mathbf{k}\cdot\mathbf{x}]} + b(\mathbf{k})e^{i[\omega_k t - \mathbf{k}\cdot\mathbf{x}]})$$

where $\omega_k = (\mathbf{k}^2 + m^2)^{1/2}$ and $\phi_k^\pm$ denotes a classical positive (resp. negative) energy plane wave solution of (1.1) ($k\cdot x = k_0x_0 - \mathbf{k}\cdot\mathbf{x} = \omega_k t - \mathbf{k}\cdot\mathbf{x}$). With $\phi(\mathbf{x}, t)$ an operator one has operators $a(\mathbf{k})$ and $b(\mathbf{k})$; further since $\phi(\mathbf{x}, t)$ is classically a real field we must have a Hermitian operator here and hence $b(\mathbf{k}) = a^\dagger(\mathbf{k})$. The normalization factor $1/2\omega_k$ is chosen for Lorentz invariance (cf. [**457**] for details). It follows immediately from $\pi = \partial_t\phi$ that

$$\pi(\mathbf{x}, t) = \int \frac{k^3k}{(2\pi)^3}\frac{1}{2\omega_k}(-i\omega_k a(\mathbf{k})e^{-ik\cdot x} + i\omega_k a^\dagger(\mathbf{k})e^{ik\cdot x}) \tag{1.7}$$

Some calculation (via Fourier formulas) leads then to

$$a(\mathbf{k}) = \int d^3x e^{ik\cdot x}[\omega_k\phi(\mathbf{x}, t) + i\pi(\mathbf{x}, t)] \tag{1.8}$$

and the algebra of a, $a^\dagger$ is then determined by $[a(\mathbf{k}), a^\dagger(\mathbf{k}')] = (2\pi)^3 2\omega_k\delta^3(\mathbf{k} - \mathbf{k}')$. The Hamiltonian (1.4) yields

$$H = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}\omega_k[a^\dagger(\mathbf{k})a(\mathbf{k}) - a(\mathbf{k})a^\dagger(\mathbf{k})] \tag{1.9}$$

There is a bit of hocus-pocus here since the calculation gives $a^\dagger a + (1/2)[a, a^\dagger]$ ($= (1/2)(a^\dagger a + aa^\dagger)$ formally) but $[a, a^\dagger] \sim \delta(0)$ corresponds to the sum oveer all modes of zero point energies $\omega_k/2$. This infinite energy cannot be detected experimaentally since experiments only measure differences from the ground state of H. In any event the zero point field (ZPF) will be discussed in some detail later.

Now the ground state is defined via $a(\mathbf{k})|0 >= 0$ with $< 0|0 >= 1$ ($a^\dagger(\mathbf{k})|0 >$ is a one particle state with energy ω_k and momentum $\mathbf{k}$ while $(a^\dagger(\mathbf{k}))^2|0 >$ contains two such particles, etc.). One notes however that the state $a^\dagger(\mathbf{k})|0 >$ is not

normalizable since $< 0|a(\mathbf{k})a^\dagger(\mathbf{k})|0 >= \delta(0)$ is not normalizable. This is not surprising since $a^\dagger(\mathbf{k})$ creates a particle of definite energy and momentum and by the uncertainty principle its location is unknown. Thus its wave function is a plane wave and such states are not normalizable. In fact $a^\dagger(\mathbf{k})$ is an operator valued distribution and one can do calculations by "smearing" and considering states $\int d^3k f(\mathbf{k})a^\dagger|0 >$ for functions f such that $\int d^3k|f(\mathbf{k})|^2 < \infty$ for example. One sees also that the bare vacuum $|0 >$ is an eigenstate of the Hamiltonian but its energy is divergent via $< 0|H|0 >= (1/2)\int d^3k\omega_k\delta^3(0)$ (where $(2\pi)^3\delta^3(0) \sim \int d^3x$). To deal with such infinities one subtracts them away, i.e. $H \to H - < 0|H|0 >$ and this corresponds to normal ordering the Hamiltonian via $: aa^\dagger :=: a^\dagger a := a^\dagger a$ leading to

$$: H := \int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}\omega_k a^\dagger(\mathbf{k})a(\mathbf{k}) \tag{1.10}$$

with vanishing vacuum expectation.

REMARK 2.1.1. Regarding Lorentz invariance one recalls that the Lorentz group $O(3,1)$ is the set of 4×4 matrices leaving the form $s^2 = (x^0)^2 - \sum(x^i)^2 = x^\mu g_{\mu\nu}x^\nu$ invariant. One writes $(x')^\mu = \Lambda^\mu_\nu x^\nu$ and notes that $g_\mu = \Lambda^\rho_\mu g_{\rho\sigma}\Lambda^\sigma_\nu \sim g = \Lambda^T g\Lambda$. Since s^2 can be plus or minus there is a splitting into regions $(x-y)^2 > 0$ (time-like), $(x-y)^2 < 0$ (space-like), and $(x-y)^2 = 0$ (light-like). A standard parametrization for Lorentz boosts involves $(x^0 = ct)$

$$x' = \frac{x+vt}{\sqrt{1-(v/c)^2}};\ y' = y;\ z' = z;\ t' = \frac{t+(vx/c^2)}{\sqrt{1-(v/c)^2}} \tag{1.11}$$

One writes e.g. $\gamma = 1/\sqrt{1-(v/c)^2} = cosh(\phi)$ with $sinh(\phi) = \beta\gamma = v\gamma/c$.

REMARK 2.1.2. The total 4-momentum operator is

$$P^\mu = \int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}k^\mu a^\dagger(\mathbf{k})a(\mathbf{k}) \tag{1.12}$$

and the total angular momentum operator is

$$M^{\mu\nu} = \int d^3x(x^\mu p^\nu - x^\nu p^\mu) \tag{1.13}$$

The Lorentz algebra (for infinitesimal Lorentz transformations) is

$$[M^{\mu\nu}, M^{\lambda\sigma}] = i(\eta^{\mu\lambda}M^{\nu\sigma} - \eta^{\nu\lambda}M^{\mu\sigma} - \eta^{\mu\sigma}M^{\nu\lambda} + \eta^{\nu\sigma}M^{\mu\lambda}) \tag{1.14}$$

where $\eta^{\mu\nu} = diag(1,-1,-1,-1)$.

REMARK 2.1.3. The commutator rules (1.5) are not manifestly Lorentz covariant. However one can verify that the same quantum theory is obtained regardless of what Lorentz frame is chosen; to do this one shows that the QM operator forms of the Lorentz generators satisfy the Lorentz algebra after quantization (this is given as an exercise in [**457**]).

EXAMPLE 1.1. Quantum fields are also discussed briefly in [**471**] and we extract here from this source. The approach follows [**128**] and one takes $\mathcal{L} = (1/2)\partial_\mu\psi\partial^\mu\psi = (1/2)[\dot\psi^2 - (\nabla\psi)^2]$ as Lagrangian where $\dot\psi = \partial_t\psi$ and variational

technique yields the wave equation $\Box\psi = 0$ ($\hbar = c = 1$). Define conjugate momentum as $\pi = \partial\mathcal{L}/\partial\dot{\psi}$, the Hamiltonian via $\mathcal{H} = \pi\dot{\psi} - \mathcal{L} = (1/2)[\pi^2 + (\nabla\psi))^2]$, and the field Hamiltonian by $\mathfrak{H} = \int \mathcal{H} d^3x$. Replacing π by $\delta S/\delta\psi$ where $S[\psi]$ is a functional the classical HJ equation of the field $\partial_t S + H = 0$ becomes

$$\frac{\partial S}{\partial t} + \frac{1}{2}\int d^3x \left[\left(\frac{\delta S}{\delta\psi}\right)^2 + (\nabla S)^2\right] = 0 \tag{1.15}$$

The term $(1/2)\int d^3x(\nabla\psi)^2$ plays the role of an external potential. To quantize the system one treats $\psi(\mathbf{x})$ and $\pi(\mathbf{x})$ as Schrödinger operators with $[\psi(\mathbf{x}), \psi(\mathbf{x}')] = [\pi(\mathbf{x}), \pi(\mathbf{x}')] = 0$ and $[\psi(\mathbf{x}), \pi(\mathbf{x}')] = i\delta(\mathbf{x}-\mathbf{x}')$. Then one works in a representation $|\psi(\mathbf{x}) >$ in which the Hermitian operator $\psi(\mathbf{x})$ is diagonal. The Hamiltonian becomes an operator $\hat{H}$ acting on a wavefunction $\Psi[\psi(\mathbf{x}), t) =< \psi(\mathbf{x})|\Psi(t) >$ which is a functional of the real field ψ and a function of t. This is not a point function of $\mathbf{x}$ since Ψ depends on the variable ψ for all $\mathbf{x}$. Now the SE for the field is $i\partial_t\Psi = \hat{H}\Psi$ or explicitly

$$i\frac{\partial\Psi}{\partial t} = \int d^3x \frac{1}{2}\left[-\frac{\delta^2}{\delta\psi^2} + (\nabla\psi)^2\right]\Psi \tag{1.16}$$

Thus ψ is playing the role of the space variable $\mathbf{x}$ in the particle SE and the continuous index $\mathbf{x}$ here is analogous to a discrete index n in the many particle theory. To arrive at a causal interpretation now one writes $\Psi = Rexp(iS)$ for $R, S[\psi, t]$ real functionals and decomposes (1.16) as

$$\frac{\partial S}{\partial t} + \frac{1}{2}\int d^3x \left[\left(\frac{\delta S}{\delta\psi}\right)^2 + (\nabla\psi)^2\right] + Q = 0;\ \ \frac{\partial R^2}{\partial t} + \int d^3x \frac{\delta}{\delta\psi}\left(R^2\frac{\delta S}{\delta\psi}\right) = 0 \tag{1.17}$$

where the quantum potential is now $Q[\psi, t] = -(1/2R)\int d^3x(\delta^2 R/\delta\psi^2)$. (1.17) now gives a conservation law wherein, at time t, $R^2 D\psi$ is the probability for the field to lie in an element of volume $D\psi$ around ψ, where $D\psi$ means roughly $\prod_{\mathbf{x}} d\psi$ and there is a normalization $\int |\Psi|^2 D\psi = 1$. Now introduce the assumption that at each instant t the field ψ has a well defined value for all $\mathbf{x}$ as in classical field theory, whatever the state Ψ. Then the time evolution is obtained from the solution of the "guidance" formula

$$\frac{\partial\psi(\mathbf{x}, t)}{\partial t} = \left.\frac{\delta S[\psi(\mathbf{x}), t]}{\delta\psi(\mathbf{x})}\right|_{\psi(\mathbf{x})=\psi(\mathbf{x},t)} \tag{1.18}$$

(analogous to $m\ddot{\mathbf{x}} = \nabla S$) once one has specified the initial function $\psi_0(\mathbf{x})$ in the HJ formalism. To find the equation of motion for the field coordinates apply $\delta/\delta\psi$ to the HJ equation (1.17) to get formally ($\dot{\psi} \sim \delta S/\delta\psi$)

$$\frac{d}{dt}\dot{\psi} = -\frac{\delta}{\delta\psi}\left[Q + \frac{1}{2}\int d^3x(\nabla\psi)^2\right];\ \ \frac{d}{dt} = \frac{\partial}{\partial t} + \int d^3x \frac{\partial\psi}{\partial t}\frac{\delta}{\delta\psi} \tag{1.19}$$

This is analogous to $m\ddot{\mathbf{x}} = -\nabla(V+Q)$ and, noting that $d\dot{\psi}/dt = \partial\dot{\psi}/dt$ and taking the classical external force term to the right one arrives, via standard variational

methods, at

$$\Box\psi(\mathbf{x}),t) = -\left.\frac{\delta Q[\psi(\mathbf{x},t]}{\delta\psi(\mathbf{x})}\right|_{\psi(\mathbf{x})=\psi(\mathbf{x},t)} \tag{1.20}$$

(note $(\delta/\delta\psi)\int d^3x(\nabla\psi)^2 \sim -2\Delta\psi$ and $(\delta/\delta\psi 0\partial_t\psi = \partial_t(\delta/\delta\psi)\psi = 0)$. The quantum force term on the right side is responsible for all the characteristic effects of QFT. In particular comparing to a classical massive KG equation $\Box\psi + m^2\psi = 0$ with suitable initial conditions one can argue that the quantum force generates mass in the sense that the massless quantum field acts as if it were a classical field with mass given via the quantum potential (cf. Remark 2.2.1 below).

1.1. ELECTROMAGNETISM AND THE DIRAC EQUATION. It will be useful to have a differential form discription of EM fields and we supply this via [**723**]. Thus one thinks of tensors $T = T^{\sigma}_{\mu\nu}\partial_\sigma \otimes dx^\mu \otimes dx^\nu$ with contractions of the form $T(dx^\sigma, \partial_\sigma) \sim T_\nu dx^\nu$. For $\eta = \eta_{\mu\nu}dx^\mu \otimes dx^\nu$ one has $\eta^{-1} = \eta^{\mu\nu}\partial_\mu \otimes \partial_\nu$ and $\eta\eta^{-1} = 1 \sim diag(\delta^\mu_\mu)$. Note also e.g.

$$\eta_{\mu\nu}dx^\mu \otimes dx^\nu(\mathbf{u},\mathbf{w}) = \eta_{\mu\nu}dx^\mu(\mathbf{u})dx^\nu(\mathbf{w}) = \tag{1.21}$$

$$= \eta_{\mu\nu}dx^\mu(u^\alpha\partial_\alpha)dx^\nu(w^\tau\partial_\tau) = \eta_{\mu\nu}u^\mu w^\nu$$

$$\eta(\mathbf{u}) = \eta_{\mu\nu}dx^\mu \otimes dx^\nu(\mathbf{u}) = \eta_{\mu\nu}dx^\mu(\mathbf{u})dx^\nu = \tag{1.22}$$

$$= \eta_{\mu\nu}dx^\mu(u^\alpha\partial_\alpha)dx^\nu = \eta_{\mu\nu}u^\mu dx^\nu = u_\nu dx^\nu$$

for a metric η. Recall $\alpha\wedge\beta = \alpha\otimes\beta - \beta\otimes\alpha$ and

$$\alpha\wedge\beta = \alpha_\mu dx^\mu \wedge \beta_\nu dx^\nu = (1/2)(\alpha_\mu\beta_\nu - \alpha_\nu\beta_\mu)dx^\mu\wedge dx^\nu \tag{1.23}$$

The EM field tensor is $F = (1/2)F_{\mu\nu}dx^\mu\wedge dx^\nu$ where

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}; \tag{1.24}$$

$$F = E_x dx^0\wedge dx^1 + E_y dx^0\wedge dx^2 + E_z dx^0\wedge dx^3 - B_z dx^1\wedge dx^2 +$$

$$+B_y dx^1\wedge dx^3 - B_x dx^2\wedge dx^3$$

The equations of motion of an electric charge is then $d\mathbf{p}/d\tau = (e/m)\mathbf{F}(\mathbf{p})$ where $\mathbf{p} = p^\mu\partial_\mu$. There is only one 4-form, namely $\epsilon = dx^0\wedge dx^1\wedge dx^2\wedge dx^3 = (1/4!)\epsilon_{\mu\nu\sigma\tau}dx^\mu\wedge dx^\nu\wedge dx^\sigma\wedge dx^\tau$ where $\epsilon_{\mu\nu\sigma\tau}$ is totally antisymmetric. Recall also for $\alpha = \alpha_{\mu\nu\ldots}dx^\mu\wedge dx^\nu\cdots$ one has $d\alpha = d\alpha_{\mu\nu\ldots}\wedge dx^\mu\wedge dx^\nu\cdots = \partial_\alpha\alpha_{\mu\nu\ldots}dx^\sigma\wedge dx^\mu\wedge dx^\nu\cdots$ and $dd\alpha = 0$. Define also the Hodge star operator on F and j via ${}^*F = (1/4)\epsilon_{\mu\nu\sigma\tau}F^{\sigma\tau}dx^\mu\wedge dx^\nu$ and ${}^*j = (1/3!)\epsilon_{\mu\nu\sigma\tau}j^\tau dx^\mu\wedge dx^\nu\wedge dx^\sigma$; these are called dual tensors. Now the Maxwell equations are

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c}j^\nu;\ \partial^\alpha F^{\mu\nu} + \partial^\mu F^{\nu\alpha} + \partial^\nu F^{\alpha\mu} = 0 \tag{1.25}$$

and this can now be written in the form

$$dF = 0;\ d^*F = \frac{4\pi}{c} * j \tag{1.26}$$

and $0 = d^*j = 0$ is automatic. In terms of $A = A_\mu dx^\mu$ where $F = dA$ the relation $dF = 0$ is an identity $ddA = 0$.

A few remarks about the tensor nature of j^μ and $F^{\mu\nu}$ are in order and we write $n = n(x)$ and $\mathbf{v} = \mathbf{v}(x)$ for number density and velocity with charge density $\rho(x) = qn(x)$ and current density $\mathbf{j} = qn(x)\mathbf{v}(x)$. The conservation of particle number leads to $\nabla \cdot \mathbf{j} + \rho_t = 0$ and one writes

$$j^\nu = (c\rho, j_x, j_y, j_z) = (c\rho n, qnv_x, qnv_y, qnv_z) \equiv j^\nu = n_0 q u^\nu \equiv j^\nu = \rho_0 u^\nu \tag{1.27}$$

where $n_0 = n\sqrt{1 - (v^2/c^2)}$ and $\rho_0 = qn_0$ (ρ_0 here is charge density). Since j^ν consists of u^ν multiplied by a scalar it must have the transformation law of a 4-vector $j^{'\beta} = a^\beta_{\ \nu} j^\nu$ under Lorentz transformations ($a^\beta_\nu \sim \Lambda^\beta_\nu$). Then the conservation law can be written as $\partial_\nu j^\nu = 0$ with obvious Lorentz invariance. After some argument one shows also that $F^{\mu\nu} = a^{\ \nu}_\beta a^{\ \mu}_\alpha F^{'\alpha\beta}$ under Lorentz transformations so $F^{\mu\nu}$ is indeed a tensor. The equation of motion for a charged particle can be written now as

$$(d\mathbf{p}/dt) = q\mathbf{E} + (q/c)\mathbf{v} \times \mathbf{B};\ \mathbf{p} = m\mathbf{v}/\sqrt{1 - (v^2/c^2)} \tag{1.28}$$

This is equivalent to $dp^\mu/dt = (q/m)p_\nu F^{\mu\nu}$ with obvious Lorentz invariance. The energy momentum tensor of the EM field is

$$T^{\mu\nu} = -(1/4\pi)[F^{\mu\alpha}F^{\ \nu}_\alpha - (1/4)\eta^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}] \tag{1.29}$$

(cf. [**723**] for details) and in particular $T^{00} = (1/8\pi)(\mathbf{E}^2 + \mathbf{B}^2)$ while the Poynting vector is $T^{0k} = (1/4\pi)(\mathbf{E} \times \mathbf{B})^k$.

One can equally well work in a curved space where e.g. covariant derivatives are defined via $\nabla_n T = lim_{d\lambda \to 0}[(T(\lambda + d\lambda) - T(\lambda) - \delta T]/d\lambda$ where δT is the change in T produced by parallel transport. One has then the usual rules $\nabla_u(T \otimes R) = \nabla_u T \otimes R + T \otimes \nabla_u R$ and for $\mathbf{v} = v^\nu \partial_\nu$ one finds $\nabla_\mu \mathbf{v} = \partial_\mu v^\nu \partial_\nu + v^\nu \nabla_\mu \partial_\nu$. Now if $\mathbf{v}$ was constructed by parallel transport its covariant derivative is zero so, acting with the dual vector dx^α gives

$$\frac{\partial x^\nu}{\partial x^\mu} dx^\alpha(\partial_\nu) + v^\nu dx^\alpha(\nabla_\mu \partial_\nu) = 0 \equiv \partial_\mu v^\alpha + v^\nu dx^\alpha(\nabla_\mu \partial_\nu) = 0 \tag{1.30}$$

Comparing this with the standard $\partial_\mu v^\alpha + \Gamma^\alpha_{\ \mu\nu} v^\nu = 0$ gives $dx^\alpha(\nabla_\mu \partial_\nu) = \Gamma^\alpha_{\ \mu\nu}$. One can show also for vectors u, v, w (boldface omitted) and a 1-form α

$$(\nabla_u \nabla_v - \nabla_v \nabla_u - uv + vu)\alpha(w) = R(\alpha, u, v, w); \tag{1.31}$$

$$R = F^\sigma_{\ \beta\mu\nu} \partial_\sigma \otimes dx^\beta \otimes dx^\mu \otimes dx^\nu$$

so R represents the Riemann tensor.

For the nonrelativistic theory we recall from [**649**] that one can define a transverse and longitudinal component of a field F via

$$F^{||}(r) = -\frac{1}{4\pi} \int d^3r' \frac{\nabla' \cdot F(r')}{|r - r'|};\ F^\perp(r) = \frac{1}{4\pi} \nabla \times \nabla \times \int d^3r' \frac{F(r')}{|r - r'|} \tag{1.32}$$

For a point particle of mass m and charge e in a field with potentials A and ϕ one has nonrelativistic equations $m\ddot{x} = eE + (e/c)v \times B$ (boldface is suppressed

here) where one recalls $B = \nabla \times A$, $v = \dot{x}$, and $E = -\nabla\phi - (1/c)A_t$ with $H = (1/2m)(p - (e/c)A)^2 + e\phi$ leading to

$$\dot{x} = \frac{1}{2m}\left(p - \frac{e}{c}A\right);\ \dot{p} = \frac{e}{c}[v \times B + (v \cdot \nabla)A] - e\nabla\phi \tag{1.33}$$

Recall here also

$$B = \nabla \times A;\ \nabla \cdot E = 0;\ \nabla \cdot B = 0;\ \nabla \times E = -(1/c)B_t; \tag{1.34}$$
$$\nabla \times B = (1/c)E_t;\ E = -(1/c)A_t - \nabla\phi$$

(the Coulomb gauge $\nabla \cdot A = 0$ is used here). One has now $E = E^{\perp} + E^{||} \sim E^T + E^L$ with $\nabla \cdot E^{\perp} = 0$ and $\nabla \times E^{||} = 0$ and in Coulomb gauge $E^{\perp} = -(1/c)A_t$ and $E^{||} = -\nabla\phi$. Further

$$H \sim \frac{1}{2m}\left(p - \frac{e}{c}A\right)^2 + e\phi + \frac{1}{8\pi}\int d^3r((E^{\perp})^2 + B^2) \tag{1.35}$$

(covering time evolution of both particle and fields).

For the relativistic theory one goes to the Dirac equation

$$i(\partial_t + \alpha \cdot \nabla)\psi = \beta m\psi \tag{1.36}$$

which, to satisfy $E^2 = \mathbf{p}^2 + m^2$ with $E \sim i\partial_t$ and $\mathbf{p} \sim -i\nabla$, implies $-\partial_t^2\psi = (-i\alpha \cdot \nabla + \beta m)^2\psi$ and ψ will satisfy the KG equation if $\beta^2 = 1$, $\alpha_i\beta + \beta\alpha_i \equiv \{\alpha_i, \beta\} = 0$, and $\{\alpha_i, \alpha_j\} = 2\delta_{ij}$ (note $c = \hbar = 1$ here with $\alpha \cdot \nabla \sim \sum \alpha_\mu\partial_\mu$ and cf. [**647, 650**] for notations and background). This leads to matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix};\ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \tag{1.37}$$
$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix};\ \alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix};\ \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

where α_i and β are 4×4 matrices. Then for convenience take $\gamma^0 = \beta$ and $\gamma^i = \beta\alpha_i$ which satisfy $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ (Lorentz metric) with $(\gamma^i)^\dagger = -\gamma^i$, $(\gamma^i)^2 = -1$, $(\gamma^0)^\dagger = \gamma^0$, and $(\gamma^0)^2 = 1$. The Dirac equation for a free particle can now be written

$$\left(i\gamma^\mu\frac{\partial}{\partial x^\mu} - m\right)\psi = 0 \equiv (i\not{\partial} - m)\psi = 0 \tag{1.38}$$

where $\not{A} = g_{\mu\nu}\gamma^\mu A^\nu = \gamma^\mu A_\mu$ and $\not{\partial} = \gamma^\mu\partial_\mu$. Taking Hermitian conjugates in (1.36), noting that α and β are Hermitian, one gets $\bar{\psi}(i\overleftarrow{\not{\partial}} + m) = 0$ where $\bar{\psi} = \psi^\dagger\beta$. To define a conserved current one has an equation $\bar{\psi}\gamma^\mu\partial_\mu\psi + \gamma^\mu\bar{\psi}_\mu\psi = \partial_\mu(\bar{\psi}\gamma^\mu\psi) = 0$ leading to the conserved current $j^\mu = \bar{\psi}\gamma^\mu\psi = (\psi^\dagger\psi, \psi^\dagger\alpha\psi)$ (this means $\rho = \psi^\dagger\psi$ and $\mathbf{j} = \psi^\dagger\alpha\psi$ with $\partial_t\rho + \nabla \cdot \mathbf{j} = 0$). The Dirac equation has the Hamiltonian form

$$i\partial_t\psi = -i\alpha \cdot \nabla\psi + \beta m\psi = (\alpha \cdot \mathbf{p} + \beta m)\psi \equiv H\psi \tag{1.39}$$

($\alpha \cdot \mathbf{p} \sim \sum \alpha_\mu p_\mu$). To obtain a Dirac equation for an electron coupled to a prescribed external EM field with vector and scalar potentials A and ϕ one substitutes $p^\mu \to p^\mu - eA^\mu$, i.e. $\mathbf{p} \to \mathbf{p} - e\mathbf{A}$ and $p^0 = i\partial_t \to i\partial_t - e\Phi$, to obtain

$$i\partial_t\psi = [\alpha \cdot (\mathbf{p} - e\mathbf{A}) + e\Phi + \beta m]\psi \tag{1.40}$$

This identifies the Hamiltonian as $H = \alpha\cdot(\mathbf{p}-e\mathbf{A})+e\Phi+\beta m = \alpha\cdot\mathbf{p}+\beta m+H_{int}$ where $H_{int} = -e\alpha\cdot\mathbf{A}+e\Phi$, suggesting α as the operator corresponding to the velocity v/c; this is strengthened by the Heisenberg equations of motion

$$\dot{\mathbf{r}} = \left(\frac{1}{i\hbar}\right)[\mathbf{r},H] = \alpha;\ \dot{\pi} = \left(\frac{1}{i\hbar}\right)[\pi,H] = e(\mathbf{E}+\alpha\times\mathbf{B}) \tag{1.41}$$

Another bit of notation now from [**650**] is useful. Thus (again with $c=\hbar=1$) one can define e.g.

$$\sigma_z = -i\alpha_x\alpha_y;\ \sigma_x = -i\alpha_y\alpha_z;\ \sigma_y = -i\alpha_z\alpha_x;\ \rho_3 = \beta; \tag{1.42}$$
$$\rho_1 = \sigma_z\alpha_z = -i\alpha_x\alpha_y\alpha_z;\ \rho_2 = i\rho_1\rho_3 = \beta\alpha_x\alpha_y\alpha_z$$

so that $\beta=\rho_3$ and $\alpha^k = \rho_1\sigma^k$. Recall also that the angular momentum $\vec{\ell}$ of a particle is $\vec{\ell} = \mathbf{r}\times\mathbf{p}$ ($\sim(-i)\mathbf{r}\times\nabla$) with components ℓ_k satisfying $[\ell_x,\ell_y] = i\ell_z,\ [\ell_y,\ell_z] = i\ell_x$, and $[\ell_z,\ell_x] = i\ell_y$. Any vector operator L satisfying such relations is called an angular momentum. Next one defines $\sigma_{\mu\nu} = (1/2)i[\gamma_\mu,\gamma_\nu] = i\gamma_\mu\gamma_\nu$ ($\mu\neq\nu$) and $S_{\alpha\beta} = (1/2)\sigma_{\alpha\beta}$. Then the 6 components $S_{\alpha\beta}$ satisfy

$$S_{10} = (i/2)\alpha_x;\ S_{20} = (i/2)\alpha_y;\ S_{30} = (i/1)\alpha_z; \tag{1.43}$$
$$S_{23} = (1/2)\sigma_x;\ ,\ S_{31} = (1/2)\sigma_y;\ S_{12} = (1/2)\sigma_z$$

The $S_{\alpha\beta}$ arise in representing infinitesimal rotations for the orthochronous Lorentz group via matrices $I+i\epsilon S_{\alpha\beta}$. Further one can represent total angular momentum J in the form $J = L+S$ where $L = \mathbf{r}\times\mathbf{p}$ and $S = (1/2)\sigma$ (L is orbital angular momentum and S represents spin). We recall that the gamma matrices are given via $\gamma = \beta\alpha$. Finally $[(i\partial_t - e\phi)-\alpha\cdot(-i\nabla - e\mathbf{A})-\beta m]\psi = 0$ (cf. (1.40)) and one gets

$$[i\gamma^\mu D_\mu - m)\psi = [\gamma^\mu(i\partial_\mu - eA_\mu)-m]\psi = 0 \tag{1.44}$$
$$D_\mu = \partial_\mu + ieA_\mu \equiv (\partial_0 + ie\phi, \nabla - ie\mathbf{A})$$

Working on the left with $(-i\gamma^\lambda D_\lambda - m)$ gives then $[\gamma^\lambda\gamma^\mu D_\lambda D_\mu + m^2]\psi = 0$ where $\gamma^\lambda\gamma^\mu = g^{\lambda\mu} + (1/2)[\gamma^\lambda,\gamma^\mu]$. By renaming the dummy indices one obtains $[\gamma^\lambda,\gamma^\mu]D_\lambda D_\mu = -[\gamma^\lambda,\gamma^\mu]D_\mu D_\lambda = (1/2)[\gamma^\lambda,\gamma^\mu][D_\lambda,D_\mu]$ leading to

$$[D_\lambda,D_\mu] = ie[\partial_\lambda,A_\mu] + ie[A_\lambda,\partial_\mu] = ie(\partial_\lambda A_\mu - \partial_\mu A_\lambda) = ieF_{\lambda\mu} \tag{1.45}$$

This yields then $\gamma^\lambda\gamma^\mu D_\lambda D_\mu = D_\mu D_\lambda + eS^{\lambda\mu}F_{\lambda\mu}$ where $S^{\lambda\mu}$ represents the spin of the particle. Therefore one can write $[D_\mu D^\mu + eS^{\lambda\mu}F_{\lambda\mu} + m^2]\psi = 0$. Comparing with the standard form of the KG equation we see that this differs by the term $eS^{\lambda\mu}F_{\lambda\mu}$ which is the spin coupling of the particle to the EM field and has no classical analogue.

2. BERTOLDI-FARAGGI-MATONE THEORY

The equivalence principle (EP) of Faraggi-Matone (cf. [**110, 191, 193, 198, 347, 641**]) is based on the idea that all physical systems can be connected by a coordinate transformation to the free situation with vanishing energy (i.e. all potentials are equivalent under coordinate transformations). This automatically leads to the quantum stationary Hamilton-Jacobi equation (QSHJE) which is a third order nonlinear differential equation providing a trajectory representation of

quantum mechanics (QM). The theory transcends in several respects the Bohm theory and in particular utilizes a Floydian time (cf. [**373, 374**]) leading to $\dot{q} = p/m_Q \neq p/m$ where $m_Q = m(1 - \partial_E Q)$ is the "quantum mass" and Q the "quantum potential" (cf. also Section 7.4). Thus the EP is reminscient of the Einstein equivalence of relativity theory. This latter served as a midwife to the birth of relativity but was somewhat inaccurate in its original form. It is better put as saying that all laws of physics should be invariant under general coordinate transformations (cf. [**723**]). This demands that not only the form but also the content of the equations be unchanged. More precisely the equations should be covariant and all absolute constants in the equations are to be left unchanged (e.g. c, $\hbar$, e, m and $\eta_{\mu\nu}$ = Minkowski tensor). Now for the EP, the classical picture with $S^{cl}(q, Q^0, t)$ the Hamilton principal function ($p = \partial S^{cl}/\partial q$) and P^0, Q^0 playing the role of initial conditions involves the classical HJ equation (CHJE) $H(q, p = (\partial S^{cl}/\partial q), t) + (\partial S^{cl}/\partial t) = 0$. For time independent V one writes $S^{cl} = S_0^{cl}(q, Q^0) - Et$ and arrives at the classical stationary HJ equation (CSHJE) $(1/2m)(\partial S_0^{cl}/\partial q)^2 + \mathfrak{W} = 0$ where $\mathfrak{W} = V(q) - E$. In the Bohm theory one looked at Schrödinger equations $i\hbar\psi_t = -(\hbar^2/2m)\psi'' + V\psi$ with $\psi = \psi(q)exp(-iEt/\hbar)$ and $\psi(q) = R(qexp(i\hat{W}/\hbar)$ leading to

$$\left(\frac{1}{2m}\right)(\hat{W}')^2 + V - E - \frac{\hbar^2 R''}{2mR} = 0;\ (R^2\hat{W}')' = 0 \tag{2.1}$$

where $\hat{Q} = -\hbar^2 R''/2mR$ was called the quantum potential; this can be written in the Schwartzian form $\hat{Q} = (\hbar^2/4m)\{\hat{W}; q\}$ (via $R^2\hat{W}' = c$). Here $\{f; q\} = (f'''/f') - (3/2)(f''/f')^2$. Writing $\mathfrak{W} = V(q) - E$ as in above we have the quantum stationary HJ equation (QSHJE)

$$(1/2m)(\partial\hat{W}'/\partial q)^2 + \mathfrak{W}(q) + \hat{Q}(q) = 0 \equiv \mathfrak{W} = -(\hbar^2/4m)\{exp(2iS_0/\hbar); q\} \tag{2.2}$$

This was worked out in the Bohm school (without the Schwarzian connections) but $\psi = Rexp(i\hat{W}/\hbar)$ is not appropriate for all situations and care must be taken ($\hat{W} =$ *constant* must be excluded for example - cf. [**347, 373, 374**]). The technique of Faraggi-Matone (FM) is completely general and with only the EP as guide one exploits the relations between Schwarzians, Legendre duality, and the geometry of a second order differential operator $D_x^2 + V(x)$ (Möbius transformations play an important role here) to arrive at the QSHJE in the form

$$\frac{1}{2m}\left(\frac{\partial S_0^v(q^v)}{\partial q^v}\right)^2 + \mathfrak{W}(q^v) + \mathfrak{V}^v(q^v) = 0 \tag{2.3}$$

where $v : q \to q^v$ represents an arbitrary locally invertible coordinate transformation. Note in this direction for example that the Schwarzian derivative of the the ratio of two linearly independent elements in $ker(D_x^2 + V(x))$ is twice $V(x)$. In particular given an arbitrary system with coordinate q and reduced action $S_0(q)$ the system with coordinate q^0 corresponding to $V - E = 0$ involves $\mathfrak{W}(q) = (q^0; q)$ where (q^0, q) is a cocycle term which has the form $(q^a; q^b) = -(\hbar^2/4m)\{q^a; q^b\}$. In fact it can be said that the essence of the EP is the cocycle condition

$$(q^a; q^c) = (\partial_{q^c} q^b)^2[(q^a; q^b) - (q^c; q^b)] \tag{2.4}$$

In addition FM developed a theory of (x,ψ) duality (cf. [**346**])) which related the space coordinate and the wave function via a prepotential (free energy) in the form $\mathfrak{F}=(1/2)\psi\bar{\psi}+iX/\epsilon$ for example. A number of interesting philosophical points arise (e.g. the emergence of space from the wave function) and we connected this to various features of dispersionless KdV in [**191, 198**] in a sort of extended WKB spirit (cf. also Section 7.3). One should note here that although a form $\psi = Rexp(i\hat{W}/\hbar)$ is not generally appropriate it is correct when one is dealing with two independent solutions of the Schrödinger equation ψ and $\bar{\psi}$ which are not proportional. In this context we utilized some interplay between various geometric properties of KdV which involve the Lax operator $L^2=D_x^2+V(x)$ and of course this is all related to Schwartzians, Virasoro algebras, and vector fields on S^1 (see e.g. [**191, 192, 198, 200, 201**]). Thus the simple presence of the Schrödinger equation (SE) in QM automatically incorporates a host of geometrical properties of $D_x=d/dx$ and the circle S^1. In fact since the FM theory exhibits the fundamental nature of the SE via its geometrical properties connected to the QSHJE one could speculate about trivializing QM (for 1-D) to a study of S^1 and ∂_x.

We import here some comments based on [**110**] concerning the Klein-Gordon (KG) equation and the equivalence principle (EP) (details are in [**110**] and cf. also [**164, 165, 166, 298, 472, 474, 473, 478, 479, 480, 666, 667**] for the KG equation which is treated in some detail later at several places in this book). One starts with the relativistic classical Hamilton-Jacobi equation (RCHJE) with a potential $V(q,t)$ given as

$$\frac{1}{2m}\sum_1^D(\partial_k S^{cl}(q,t))^2+\mathfrak{W}_{rel}(q,t)=0; \tag{2.5}$$

$$\mathfrak{W}_{rel}(q,t)=\frac{1}{2mc^2}[m^2c^4-(V(q,t)+\partial_t S^{cl}(q,t))^2]$$

In the time-independent case one has $S^{cl}(q,t)=S_0^{cl}(q)-Et$ and (2.3) becomes

$$\frac{1}{2m}\sum_1^D(\partial_k S_0^{cl})^2+\mathfrak{W}_{rel}=0;\ \mathfrak{W}_{rel}(q)=\frac{1}{2mc^2}[m^2c^4-(V(q)-E)^2] \tag{2.6}$$

In the latter case one can go through the same steps as in the nonrelativistic case and the relativistic quantum HJ equation (RQHJE) becomes

$$(1/2m)(\nabla S_0)^2+\mathfrak{W}_{rel}-(\hbar^2/2m)(\Delta R/R)=0;\ \nabla\cdot(R^2\nabla S_0)=0 \tag{2.7}$$

these equations imply the stationary KG equation

$$-\hbar^2c^2\Delta\psi+(m^2c^4-V^2+2EV-E^2)\psi=0 \tag{2.8}$$

where $\psi=Rexp(iS_0/\hbar)$. Now in the time dependent case the (D+1)-dimensional RCHJE is ($\eta^{\mu\nu}=diag(-1,1,\cdots,1)$

$$(1/2m)\eta^{\mu\nu}\partial_\mu S^{cl}\partial_\nu S^{cl}+\mathfrak{W}'_{rel}=0; \tag{2.9}$$

$$\mathfrak{W}'_{rel}=(1/2mc^2)[m^2c^4-V^2(q)-2cV(q)\partial_0 S^{cl}(q)]$$

with $q=(ct,q_1,\cdots,q_D)$. Thus (2.9) has the same structure as (2.6) with Euclidean metric replaced by the Minkowskian one. We know how to implement the EP by

adding Q via $(1/2m)(\partial S)^2 + \mathfrak{W}_{rel} + Q = 0$ (cf. [**347**] and remarks above). Note now that $\mathfrak{W}'_{rel}$ depends on S^{cl} requires an identification

$$\mathfrak{W}_{rel} = (1/2mc^2)[m^2c^4 - V^2(q) - 2cV(q)\partial_0 S(q)] \tag{2.10}$$

(S replacing S^{cl}) and implementation of the EP requires that for an arbitrary $\mathfrak{W}^a$ state ($q \sim q^a$) one must have

$$\mathfrak{W}^b_{rel}(q^b) = (p^b|p^a)\mathfrak{W}^a_{rel}(q^a) + (q^q; q^b);\ Q^b(q^b) = (p^b|p^a)Q(q^a) - (q^a; q^b) \tag{2.11}$$

where

$$(p^b|p) = [\eta^{\mu\nu}p^b_\mu p^b_\nu/\eta^{\mu\nu}p_\mu p_\nu] = p^T J\eta J^T p/p^T\eta p;\ J^\mu_\nu = \partial q^\mu/\partial q^{b^\nu} \tag{2.12}$$

(J is a Jacobian and these formulas are the natural multidimensional generalization - see [**110**] for details). Furthermore there is a cocycle condition $(q^a; q^c) = (p^c|p^b)[(q^a; q^b) - (q^c; q^b)]$.

Next one shows that $\mathfrak{W}_{rel} = (\hbar^2/2m)[\Box(Rexp(iS/\hbar))/Rexp(iS/\hbar)]$ and hence the corresponding quantum potential is $Q_{rel} = -(\hbar^2/2m)[\Box R/R]$. Then the RQHJE becomes $(1/2m)(\partial S)^2 + \mathfrak{W}_{rel} + Q = 0$ with $\partial \cdot (R^2\partial S) = 0$ (here $\Box R = \partial_\mu\partial^\mu R$) and this reduces to the standard SE in the classical limit $c \to \infty$ (note $\partial \sim (\partial_0, \partial_1, \cdots, \partial_D)$ with $q_0 = ct$, etc. - cf. (2.9)). To see how the EP is simply implemented one considers the so called minimal coupling prescription for an interaction with an electromagnetic four vector A_μ. Thus set $P^{cl}_\mu = p^{cl}_\mu + eA_\mu$ where p^{cl}_μ is a particle momentum and $P^{cl}_\mu = \partial_\mu S^{cl}$ is the generalized momentum. Then the RCHJE reads as $(1/2m)(\partial S^{cl} - eA)^2 + (1/2)mc^2 = 0$ where $A_0 = -V/ec$. Then $\mathfrak{W} = (1/2)mc^2$ and the critical case $\mathfrak{W} = 0$ corresponds to the limit situation where $m = 0$. One adds the standard Q correction for implementation of the EP to get $(1/2m)(\partial S - eA)^2 + (1/2)mc^2 + Q = 0$ and there are transformation properties (here $(\partial S - eA)^2 \sim \sum(\partial_\mu S - eA_\mu)^2$)

$$\mathfrak{W}(q^b) = (p^b|p^a)\mathfrak{W}^a(q^a) + (q^a; q^b);\ Q^b(q^b) = (p^q|p^a)Q^a(q^a) - (q^a; q^b) \tag{2.13}$$

$$(p^b|p) = \frac{(p^b - eA^b)^2}{(p - eA)^2} = \frac{(p - eA)^T J\eta J^T(p - eA)}{(p - eA)^T\eta(p - eA)}$$

Here J is a Jacobian $J^\mu_\nu = \partial q^\mu/\partial q^{b^\nu}$ and this all implies the cocycle condition again. One finds now that (recall $\partial \cdot (R^2(\partial S - eA)) = 0$ - continuity equation)

$$(\partial S - eA)^2 = \hbar^2\left(\frac{\Box R}{R} - \frac{D^2(Re^{iS/\hbar})}{Re^{iS/\hbar}}\right);\ D_\mu = \partial_\mu - \frac{i}{\hbar}eA_\mu \tag{2.14}$$

and it follows that

$$\mathfrak{W} = \frac{\hbar^2}{2m}\frac{D^2(Re^{iS/\hbar})}{Re^{iS/\hbar}};\ Q = -\frac{\hbar^2}{2m}\frac{\Box R}{R};\ D^2 = \Box - \frac{2ieA\partial}{\hbar} - \frac{e^2A^2}{\hbar^2} - \frac{ie\partial A}{\hbar} \tag{2.15}$$

$$(\partial S - eA)^2 + m^2c^2 - \hbar^2\frac{\Box R}{R} = 0;\ \partial \cdot (R^2(\partial S - eA)) = 0 \tag{2.16}$$

Note also that (2.9) agrees with $(1/2m)(\partial S^{cl} - eA)^2 + (1/2)mc^2 = 0$ after setting $\mathfrak{W}_{rel} = mc^2/2$ and replacing $\partial_\mu S^{cl}$ by $\partial_\mu S^{cl} - eA_\mu$. One can check that (2.16) implies the KG equation $(i\hbar\partial + eA)^2\psi + m^2c^2\psi = 0$ with $\psi = Rexp(iS/\hbar)$.

REMARK 2.2.1. We extract now a remark about mass generation and the EP from [**110**]. Thus a special property of the EP is that it cannot be implemented in classical mechanics (CM) because of the fixed point corresponding to $\mathfrak{W} = 0$. One is forced to introduce a uniquely determined piece to the classical HJ equation (namely a quantum potential Q). In the case of the RCHJE the fixed point $\mathfrak{W}(q^0) = 0$ corresponds to $m = 0$ and the EP then implies that all the other masses can be generated by a coordinate transformation. Consequently one concludes that masses correspond to the inhomogeneous term in the transformation properties of the $\mathfrak{W}^0$ state, i.e. $(1/2)mc^2 = (q^0; q)$. Furthermore by (2.13) masses are expressed in terms of the quantum potential $(1/2)mc^2 = (p|p^0)Q^0(q^0) - Q(q)$. In particular in [**347**] the role of the quantum potential was seen as a sort of intrinsic self energy which is reminiscent of the relativistic self energy and this provides a more explicit evidence of such an interpretation.

REMARK 2.2.2. In a previous paper [**194**] (working with stationary states and ψ satisfying the Schrödinger equation (SE) $-(\hbar^2/2m)\psi'' + V\psi = E\psi$) we suggested that the notion of uncertainty in quantum mechanics (QM) could be phrased as incomplete information. The background theory here is taken to be the trajectory theory of Bertoldi-Faraggi-Matone (and Floyd) as above and the idea in [**194**] goes as follows. First recall that microstates satisfy a third order quantum stationary Hamilton-Jacobi equation (QSHJE)

$$\frac{1}{2m}(S_0')^2 + \mathfrak{W}(q) + Q(q) = 0;\ Q(q) = \frac{\hbar^2}{4m}\{S_0; q\}; \tag{2.17}$$

$$\mathfrak{W}(q) = -\frac{\hbar^2}{4m}\{exp(2iS_0/\hbar); q\} \sim V(q) - E$$

where $\{f; q\} = (f'''/f') - (3/2)(f''/f')^2$ is the Schwarzian and S_0 is the Hamilton principle function. Also one recalls that the EP of Faraggi-Matone can only be implemented when $S_0 \neq const$; thus consider $\psi = Rexp(iS_0/\hbar)$ with $Q = -\hbar^2 R''/2mR$ and $(R^2 S_0')' = 0$ where $S_0' = p$ and $m_Q\dot{q} = p$ with $m_Q = m(1 - \partial_E Q)$ and $t \sim \partial_E S_0$ (Q in (2.17) is the definitive form - cf. [**349**]). Thus microstates require three initial or boundary conditions in general to determine S_0 whereas the SE involves only two such conditions (cf. also Section 7.4 and [**138, 140, 139, 305, 306, 307, 308, 309, 347, 348, 349, 373, 374, 375, 520**]). Hence in dealing with the SE in the standard QM Hilbert space formulation one is not using complete information about the "particles" described by microstate trajectories. The price of underdetermination is then uncertainty in q, p, t for example. In the present note we will make this more precise and add further discussion. Following [**197**] we now make this more precise and add further discussion. For the stationary SE $-(\hbar^2/2m)\psi'' + V\psi = E\psi$ it is shown in [**347**] that one has a general formula

$$e^{2iS_0(\delta)/\hbar} = e^{i\alpha}\frac{w + i\bar{\ell}}{w - i\ell} \tag{2.18}$$

($\delta \sim (\alpha, \ell)$) with three integration constants, α, ℓ_1, ℓ_2 where $\ell = \ell_1 + i\ell_2$ and $w \sim \psi^D/\psi \in \mathbf{R}$. Note ψ and ψ^D are linearly independent solutions of the SE and

one can arrange that $\psi^D/\psi \in \mathbf{R}$ in describing any situation. Here p is determined by the two constants in ℓ and has a form

$$p = \frac{\pm\hbar\Omega\ell_1}{|\psi^D - i\ell\psi|^2} \tag{2.19}$$

(where $w \sim \psi^D/\psi$ above and $\Omega = \psi'\psi^D - \psi(\psi^D)'$). Now let p be determined exactly with $p = p(q, E)$ via the Schrödinger equation and S_0'. Then $\dot{q} = (\partial_E p)^{-1}$ is also exact so $\Delta q = (\partial_E p)^{-1}(\tau)\Delta t$ for some τ with $0 \leq \tau \leq t$ is exact (up to knowledge of τ). Thus given the wave function ψ satisfying the stationary SE with two boundary conditions at $q = 0$ say to fix uniqueness, one can create a probability density $|\psi|^2(q, E)$ and the function S_0'. This determines p uniquely and hence $\dot{q}$. The additional constant needed for S_0 appears in (2.18) and we can write $S_0 = S_0(\alpha, q, E)$ since from (2.18) one has

$$S_0 - (\hbar/2)\alpha = -(i\hbar/2)log(\beta) \tag{2.20}$$

and $\beta = (w + i\bar{\ell})/(w - i\ell)$ with $w = \psi^D/\psi$ is to be considered as known via a determination of suitable ψ, ψ^D. Hence $\partial_\alpha S_0 = -\hbar/2$ and consequently $\Delta S_0 \sim \partial_\alpha S_0 \delta\alpha = -(\hbar/2)\Delta\alpha$ measures the indeterminacy or uncertainty in S_0.

Let us expand upon this as follows. Note first that the determination of constants necessary to fix S_0 from the QSHJE is not usually the same as that involved in fixing ℓ, $\bar{\ell}$ in (2.18). In paricular differentiating in q one gets

$$S_0' = -\frac{i\hbar\beta'}{\beta};\ \beta' = -\frac{2i\Re\ell w'}{(w - i\ell)^2} \tag{2.21}$$

Since $w' = -\Omega/\psi^2$ where $\Omega = \psi'\psi^D - \psi(\psi^D)'$ we get $\beta' = -2i\ell_1\Omega/(\psi^D - i\ell\psi)^2$ and consequently

$$S_0' = -\frac{\hbar\ell_1\Omega}{|\psi^D - i\ell\psi|^2} \tag{2.22}$$

which agrees with p in (2.19) ($\pm\hbar$ simply indicates direction). We see that e.g. $S_0(x_0) = i\hbar\ell_1\Omega/|\psi^D(x_0) - i\ell\psi(x_0)|^2 = f(\ell_1, \ell_2, x_0)$ and $S_0'' = g(\ell_1, \ell_2, x_0)$ determine the relation between $(p(x_0), p'(x_0))$ and (ℓ_1, ℓ_2) but they are generally different numbers. In any case, taking α to be the arbitrary unknown constant in the determination of S_0, we have $S_0 = S_0(q, E, \alpha)$ with $q = q(S_0, E, \alpha)$ and $t = t(S_0, E, \alpha) = \partial_E S_0$ (emergence of time from the wave function). One can then write e.g.

$$\Delta q = (\partial q/\partial S_0)(\hat{S}_0, E, \alpha)\Delta S_0 = (1/p)(\hat{q}, E)\Delta S_0 = -(1/p)(\hat{q}, E)(\hbar/2)\Delta\alpha \tag{2.23}$$

(for intermediate values $(\hat{S}_0, \hat{q})$) leading to

THEOREM 2.1. With p determined uniquely by two "initial" conditions so that Δp is determined and q given via (2.18) we have from (2.23) the inequality $\Delta p\Delta q = O(\hbar)$ which resembles the Heisenberg uncertainty relation.

COROLLARY 2.1. Similarly $\Delta t = (\partial t/\partial S_0)(\hat{S}_0, E, \alpha)\Delta S_0$ for some intermediate value $\hat{S}_0$ and hence as before $\Delta E\Delta t = O(\hbar)$ (ΔE being precise).

Note that there is no physical argument here; one is simply looking at the number of conditions necessary to fix solutions of a differential equation. In fact (based on some corresondence with E. Floyd) it seems somewhat difficult to produce a viable physical argument. We refer also to Remark 3.1.2 for additional discussion.

REMARK 2.2.3. In order to get at the time dependent SE from the BFM (Bertoldi-Faraggi-Matone) theory we proceed as follows. From the previous discussion on the KG equation one sees that (dropping the EM terms) in the time independent case one has $S^{cl}(q,t) = S_0^{cl}(q) - Et$

(2.24)
$$(1/2m)\sum_1^D(\partial_k S_0^{cl})^2 + \mathfrak{W}_{rel} = 0;\ \mathfrak{W}_{rel}(q) = (1/2mc^2)[m^2c^4 - (V(q)-E)^2]$$

leading to a stationary RQHJE

$$(1/2m)(\nabla S_0)^2 + \mathfrak{W}_{rel} - (\hbar^2/2m)(\Delta R/R) = 0;\ \nabla\cdot(R^2\nabla S_0) = 0 \tag{2.25}$$

This implies also the stationary KG equation

$$-\hbar^2c^2\Delta\psi + (m^2c^4 - V^2 + 2VE - E^2)\psi = 0 \tag{2.26}$$

Now in the time dependent case one can write $(1/2m)\eta^{\mu\nu}\partial_\mu S^{cl}\partial_\nu S^{cl} + \mathfrak{W}'_{rel} = 0$ where $\eta \sim diag(-1,1,\cdots,1)$ and

$$\mathfrak{W}'_{rel}(q) = (1/2mc^2)[m^2c^4 - V^2(q) - 2cV(q)\partial_0 S^{cl}(q)] \tag{2.27}$$

with $q \equiv (ct, q_1, \cdots, q_D)$. Thus we have the same structure as (2.24) with Euclidean metric replaced by a Minkowskian one. To implement the EP we have to modify the classical equation by adding a function to be determined, namely $(1/2m)(\partial S)^2 + \mathfrak{W}_{rel} + Q = 0$ ($(\partial S)^2 \sim \sum(\partial_\mu S)^2$ etc.). Observe that since $\mathfrak{W}'_{rel}$ depends on S^{cl} we have to make the identification $\mathfrak{W}_{rel} = (1/2mc^2)[m^2c^4 - V^2(q) - 2cV(q)\partial_0 S(q)]$ which differs from $\mathfrak{W}'_{rel}$ since S now appears instead of S^{cl}. Implementation of the EP requires that for an arbitrary $\mathfrak{W}^a$ state

$$\mathfrak{W}^b_{rel}(q^b) = (p^b|p^a)\mathfrak{W}^a_{rel}(q^a) + (q^q;q^b);\ Q^b(q^b) = (p^b|p^a)Q^a(q^a) - (q^a;q^b) \tag{2.28}$$

where now $(p^b|p) = \eta^{\mu\nu}p^b_\mu p^b_\nu/\eta^{\mu\nu}p_\mu p_\nu = p^TJ\eta J^Tp/p^T\eta p$ and $J^\mu_\nu = \partial q^\mu/\partial(q^b)^\nu$. This leads to the cocycle condition $(q^a;q^c) = (p^c|p^b)[(q^q;q^b) - (q^c;q^b)]$ as before. Now consider the identity

$$\alpha^2(\partial S)^2 = \Box(Rexp(\alpha S))/Rexp(\alpha S) - (\Box R/R) - (\alpha\partial\cdot(R^2\partial S)/R^2) \tag{2.29}$$

and if R satisfies the continuity equation $\partial\cdot(R^2\partial S) = 0$ one sets $\alpha = i/\hbar$ to obtain

$$\frac{1}{2m}(\partial S)^2 = -\frac{\hbar^2}{2m}\frac{\Box(Re^{iS/\hbar})}{Re^{iS/\hbar}} + \frac{\hbar^2}{2m}\frac{\Box R}{R} \tag{2.30}$$

Then it is shown that $\mathfrak{W}_{rel} = (\hbar^2/2m)(\Box(Rexp(iS/\hbar))/Rexp(iS/\hbar)$ so $Q_{rel} = -(\hbar^2/2m)(\Box R/R)$. Thus the RQHJE has the form (cf. (2.14) - (2.16))

$$\frac{1}{2m}(\partial S)^2 + \mathfrak{W}_{rel} - \frac{\hbar^2}{2m}\frac{\Box R}{R} = 0;\ \partial\cdot(R^2\partial S) = 0 \tag{2.31}$$

Now for the time dependent SE one takes the nonrelativistic limit of the RQHJE. For the classical limit one makes the usual substitution $S = S' - mc^2t$ so as $c \to \infty$ $\mathfrak{W}_{rel} \to (1/2)mc^2 + V$ and $-(1/2m)(\partial_0 S)^2 \to \partial_t S' - (1/2)mc^2$ with $\partial(R^2\partial S) = 0 \to m\partial_t(R')^2 + \nabla \cdot ((R')^2\nabla S') = 0$. Therefore (removing the primes) (2.31) becomes $(1/2m)(\nabla S)^2 + V + \partial_t S - (\hbar^2/2m)(\Delta R/R) = 0$ with the time dependent nonrelativistic continuity equation being $m\partial_t R^2 + \nabla \cdot (R^2\nabla S) = 0$. This leads then (for $\psi \sim Rexp(iS/\hbar)$) to the SE

$$i\hbar\partial_t\psi = \left(-\frac{\hbar^2}{2m}\Delta + V\right)\psi \tag{2.32}$$

One sees from all this that the BFM theory is profoundly governed by the equivalence principle and produces a usable framework for computation. It is surprising that it has not attracted more adherents.

3. FIELD THEORY MODELS

In trying to imagine particle trajectories of a fractal nature or in a fractal medium we are tempted to abandon (or rather relax) the particle idea and switch to quantum fields (QF). Let the fields sense the bumps and fractality; if one can think of fields as operator valued distributions for example then fractal supports for example are quite reasonable. There are other reasons of course since the notion of particle in quantum field theory (QFT) has a rather fuzzy nature anyway. Then of course there are problems with QFT itself (cf. [**973**]) as well as arguments that there is no first quantization (except perhaps in the Bohm theory - cf. [**701, 1016**]). We review here some aspects of particles arising from QF and QFT methods, especially in a Bohmian spirit (cf. [**77, 110, 256, 324, 325, 326, 454, 472, 478, 479, 480, 494, 532, 634, 701, 702, 703, 704, 705, 706, 707, 708, 709, 710, 711, 984**]). We refer to [**454, 973**] for interesting philosophical discussion about particles and localized objects in a QFT and will extract here from [**77, 256, 326, 703, 704**]; for QFT we refer to [**457, 912, 935, 1015**]. Many details are omitted and standard QFT techniques are assumed to be known and we will concentrate here on derivations of KG type equations and the nature of the quantum potential (the Dirac equation will be treated later).

3.1. EMERGENCE OF PARTICLES. The papers [**704**] are impressive in producing a local operator describing the particle density current for scalar and spinor fields in an arbitrary gravitational and electromagnetic background. This enables one to describe particles in a local, general covariant, and gauge invariant manner. The current depends on the choice of a 2-point Wightman function and a most natural choice based on the Green's function à la Schwinger- deWitt leads to local conservation of the current provided that interaction with quantum fields is absent. Interactions lead to local nonconservation of current which describes local particle production consistent with the usual global description based on the interaction picture. The material is quite technical but we feel it is important and will sketch some of the main points; the discussion should provide a good exercise in field theoretic technique. The notation is indicated as we proceed and we make no attempt to be consistent with other notation. Thus let $g_{\mu\nu}$ be a

classical background metric, g the determinant, and R the curvature. The action of a Hermitian scalar field ϕ can be written as

$$S = \frac{1}{2}\int d^4x|g|^{1/2}[g^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) - m^2\phi^2 - \xi R\phi^2] \tag{3.1}$$

where ξ is a coupling constant. Writing this as $S = \int d^4x|g|^{1/2}\mathcal{L}$ the canonical mommentum vector is $\pi_\mu = [\partial\mathcal{L}/\partial(\partial^\mu\phi)] = \partial_\mu\phi$ (standard $g_{\mu\nu}$). The corresponding equation of motion is $(\nabla^\mu\partial_\mu + m^2 + \xi R)\phi = 0$ where ∇^μ is the covariant derivative. Let Σ be a spacelike Cauchy hypersurface with unit normal vector n^μ; the canonical momentum scalar is defined as $\pi = n^\mu\pi_\mu$ and the volume element on Σ is $d\Sigma^\mu = d^3x|g^{(3)}|^{1/2}n^\mu$ with scalar product $(\phi_1,\phi_2) = i\int_\Sigma d\Sigma^\mu\phi_1^*\overleftrightarrow{\partial_\mu}\phi_2$ where $a\overleftrightarrow{\partial_\mu}b = a\partial_\mu g - (\partial_\mu a)b$. If ϕ_i are solutions of the equation of motion then the scalar product does not depend on Σ. One chooses coordinates (t,x) such that $t = c$ on Σ so that $n^\mu = g_0^\mu/\sqrt{g_{00}}$ and the canonical commutation relations become

$$[\phi(x),\phi(x')]_\Sigma = [\pi(x),\pi(x')]_\Sigma = 0;\ [\phi(x),\pi(x')]_\Sigma = |g^{(3)}|^{-1/2}i\delta^3(x-x') \tag{3.2}$$

(here x, x' lie on Σ). This can be written in a manifestly covariant form via

$$\int_\Sigma d\Sigma^{'\mu}[\phi(x),\partial'_\mu\phi(x')]\chi(x') = \int_\Sigma d\Sigma^{'\mu}[\phi(x'),\partial_\mu\phi(x)]\chi(x') = i\chi(x) \tag{3.3}$$

for an arbitrary test function χ. For practical reasons one writes $\tilde{n}^\mu = |g^{(3)}|^{1/2}n^\mu$ where the tilde indicates that it is not a vector. Then $\nabla_\mu\tilde{n}_\nu = 0$ and in fact $\tilde{n}^\mu = (|g^{(3)}|^{1/2}/\sqrt{g_{00}}, 0, 0, 0)$. It follows that $d\Sigma^\mu = d^3x\tilde{n}^\mu$ while (2.11) can be written as $\tilde{n}^0(x')[\phi(x),\partial'_0\phi(x')]_\Sigma = i\delta^3(x-x')$. Consequently

$$[\phi(x),\tilde{\pi}(x')]_\Sigma = i\delta^3(x-x');\ \tilde{\pi} = |g^{(3)}|^{1/2}\pi \tag{3.4}$$

Now choose a particular complete orthonormal set of solutions $\{f_k(x)\}$ of the equation of motion satisfying therefore

$$(f_k, f_{k'}) = -(f_k^*, f_{k'}^*) = \delta_{kk'};\ (f_k^*, f_{k'}) = (f_k, f_{k'}^*) = 0 \tag{3.5}$$

One can then write $\phi(x) = \sum_k a_k f_k(x) + a_k^\dagger f_k^*(x)$ from which we deduce that $a_k = (f_k,\phi)$ and $a_k^\dagger = -(f_k^*,\phi)$ while $[a_k, a_{k'}^\dagger] = \delta_{kk'}$ and $[a_k, a_{k'}] = [a_k^\dagger, a_{k'}^\dagger] = 0$. The lowering and raising operators a_k and $a_k^\dagger$ induce the representation of the field algebra in the usual manner and $a_k|0>= 0$. The number operator is $N = \sum a_k^\dagger a_k$ and one defines a two point function $W(x,x') = \sum f_k(x)f_k^*(x')$ (different definitions appear later). Using the equation of motion one finds that W is a Wightman function $W(x,x') =< 0|\phi(x)\phi(x')|0 >$ and one has $W^*(x,x') = W(x',x)$. Further, via the equation of motion, for f_k, f_k^* one has

$$(\nabla^\mu\partial_\mu + m^2 + \xi R(x))W(x,x') = 0 = (\nabla^{'\mu}\partial'_\mu + m^2 + \xi R(x'))W(x,x') \tag{3.6}$$

From the form of W and the commutation relations there results also

$$W(x,x')|_\Sigma = W(x',x)|_\Sigma;\ \partial_0\partial'_0W(x,x')|_\Sigma = \partial_0\partial'_0W(x',x)|_\Sigma; \tag{3.7}$$

$$\tilde{n}^0\partial'_0[W(x,x') - W(x',x)]_\Sigma = i\delta^3(x-x')$$

The number operator given by $N = \sum a_k^\dagger a_k$ is a global quantity. However a new way of looking into the concept of particles emerges when $a_k = (f_k, \phi)$, etc. is put into N; using the scalar product along with the expression for W leads to

$$N = \int_\Sigma d\Sigma^\mu \int_\Sigma d\Sigma^{'\nu} W(x,s') \overleftrightarrow{\partial_\mu} \overleftrightarrow{\partial'_\nu} \phi(x)\phi(x') \tag{3.8}$$

By interchanging the names of the coordinates x, x' and the names of the indices $\mu\,\nu$ this can be written as a sum of two equal terms

$$N = \frac{1}{2}\int_\Sigma d\Sigma^\mu \int_\Sigma d\Sigma^{'\nu} W(x,x') \overleftrightarrow{\partial_\mu} \overleftrightarrow{\partial'_\nu} \phi(x)\phi(x') + \tag{3.9}$$

$$\frac{1}{2}\int_\Sigma d\Sigma^\mu \int_\Sigma d\Sigma^{'\nu} W(x',x) \overleftrightarrow{\partial_\mu} \overleftrightarrow{\partial'_\nu} \phi(x')\phi(x)$$

Using also $W^*(x,x') = W(x',x)$ one sees that (3.9) can be written as $N = \int_\Sigma d\Sigma^\mu j_\mu(x)$ where

$$j_\mu(x) = (1/2)\int_\Sigma d\Sigma^{'\nu} \{W(x,x') \overleftrightarrow{\partial_\mu} \overleftrightarrow{\partial'_\nu} \phi(x)\phi(x') + h.c.\} \tag{3.10}$$

(where *h.c.* denotes hermitian conjugate). Evidently the vector $j_\mu(x)$ should be interpreted as the local current of particle density. This representation has three advantages over $N = \sum a_k^\dagger a_k$: (i) It avoids the use of $a_k,\ a_k^\dagger$ related to a particular choice of modes $f_k(x)$. (ii) It is manifestly covariant. (iii) The local current $j_\mu(x)$ allows one to view the concept of particles in a local manner. If now one puts all this together with the antisymmetry of $\overleftrightarrow{\partial_\mu}$ we find

$$j_\mu = i \sum_{k,k'} f_k^* \overleftrightarrow{\partial_\mu} f_{k'} a_k^\dagger a_{k'} \tag{3.11}$$

From this we see that j_μ is automatically normally ordered and has the property $j_\mu|0>=0$ (not surprising since $N = \sum a_k^\dagger a_k$ is normally ordered). Further one finds $\nabla^\mu j_\mu = 0$ (covariant conservation law) so the background gravitational field does not produce particles provided that a unique vacuum defined by $a_k|0>=0$ exists. This also implies global conservation since it provides that $N = \int_\Sigma d\Sigma^\mu j_\mu(x)$ does not depend on time. The extra terms in $\nabla^\mu j_\mu = 0$ originating from the fact that $\nabla_\mu \neq \partial_\mu$ are compensated by the extra terms in $N = \int_\Sigma d\Sigma^\mu j_\mu$ that originate from the fact that $d\Sigma^\mu$ is not written in "flat" coordinates. The choice of vacuum is related to the choice of $W(x,x')$. Note that although $j_\mu(x)$ is a local operator some nonlocal features of the particle concept still remain because (3.10) involves an integration over Σ on which x lies. Since $\phi(x')$ satisfies the equation of motion and $W(x,x')$ satisfies (3.6) this integral does not depend on Σ. However it does depend on the choice of $W(x,x')$. Note also the separation between x and x' in (3.10) is spacelike which softens the nonlocal features because $W(x,x')$ decreases rapidly with spacelike separation - in fact it is negligible when the space like separation is much larger than the Compton wavelength.

We pick this up now in [**702**]. Thus consider a scalar Hermitian field $\phi(x)$ in a curved background satisfying the equation of motion and choose a particular complete orthonormal set $\{f_k(x)\}$ having relations and scalar product as before. The

field ϕ can be expanded as $\phi(x) = \phi^+(x) + \phi^-(x)$ where $\phi^+(x) = \sum a_k f_k(x)$ and $\phi^-(x) = \sum a_k^\dagger f_k^*(x)$. Introducing the two point function $W^+(x,x') = \sum f_k(x) f_k^*(x')$ with $W^-(x,x') = \sum f_k^*(x) f_k(x')$ one finds the remarkable result that

(3.12)
$$\phi^+(x) = i\int_\Sigma d\Sigma'^\nu W^+(x,x') \overleftrightarrow{\partial}'_\nu \phi(x');\ \phi^-(x) = -i\int_\Sigma d\Sigma'^\nu W^-(x,x') \overleftrightarrow{\partial}'_\nu \phi(x')$$

We see that the extraction of $\phi^\pm(x)$ from $\phi(x)$ is a nonlocal procedure. Note however that the integrals in (3.12) do not depend on the choice of the timelike Cauchy hypersurface Σ because $W^\pm(x,x')$ satisfies the equation of motion with respect to x' just as $\phi(x')$ does. However these integrals do depend on the choice of $W^\pm(x,x')$, i.e. on the choice of the set $\{f_k(x)\}$. Now define normal ordering in the usual way, putting ϕ^- on the left, explicitly : $\phi^+\phi^- := \phi^-\phi^+$ while the ordering of the combinations $\phi^-\phi^+$, $\phi^+\phi^+$, and $\phi^-\phi^-$ leaves these combinations unchanged. Generalize this now by introducing 4 different orderings $N_{(\pm)}$ and $A_{(\pm)}$ defined via

(3.13)
$$N_+\phi^+\phi^- = \phi^-\phi^+;\ N_-\phi^+\phi^- = -\phi^-\phi^+;$$
$$A_+\phi^-\phi^+ = \phi^+\phi^-;\ A_-\phi^-\phi^+ = -\phi^+\phi^-$$

Thus N_+ is normal ordering, N_- will be useful, and the antinormal orderings $A_\pm$ can be used via symmetric orderings $S_+ = (1/2)[N_+ + A_+]$ and $S_- = (1/2)[N_- + A_-]$. When S_+ acts on a bilinear combination of fields it acts as the default ordering, i.e. $S_+\phi\phi = \phi\phi$.

Now the particle current for scalar Hermitian fields can be written as (cf. (3.10))

(3.14)
$$j_\mu(x) = \frac{1}{2}\int_\Sigma d\Sigma'^\nu \left[W^+(x,x') \overleftrightarrow{\partial_\mu} \overleftrightarrow{\partial}'_\nu \phi(x)\phi(x;) + W^-(x,x') \overleftrightarrow{\partial_\mu} \overleftrightarrow{\partial}'_\nu \phi(x')\phi(x)\right]$$

(3.14) can be written in a local form as $j_\mu(x) = (i/2)[\phi(x) \overleftrightarrow{\partial_\mu} \phi^+(x) + \phi^-(x) \overleftrightarrow{\partial_\mu} \phi(x)]$ (via (3.12)). Using the identities $\phi^+ \overleftrightarrow{\partial_\mu} \phi^+ = \phi^- \overleftrightarrow{\partial_\mu} \phi^-$ this can be written in the elegant form $j_\mu = i\phi^- \overleftrightarrow{\partial_\mu} \phi^+$. Similarly using (3.13) this can be written in another elegant form without explicit use of $\phi^\pm$, namely $j_\mu = (i/2)N_-\phi \overleftrightarrow{\partial_\mu} \phi$. Note that the expression on the right here without the ordering N_- vanishes identically - this peculiar feature may explain why the particle current was not previously discovered. The normal ordering N_- provides that $j_\mu|0>=0$ which is related to the fact that the total number of particles is $N = \int_\Sigma d\Sigma^\mu j_\mu = \sum a_k^\dagger a_k$. Alternatively one can choose the symmetric ordering S_- and define the particle current as $j_\mu = (i/2)S_-\phi \overleftrightarrow{\partial_\mu} \phi$. This leads to the total number of particles $N = (1/2)\sum(a_k^\dagger a_k + a_k a_k^\dagger) = \sum[a_k^\dagger a_k + (1/2)]$.

When the gravitational background is time dependent one can introduce a new set of solutions $u_k(x)$ for each time t, such that the $u_k(x)$ are positive frequency modes at that time. This leads to functions with an extra time dependence $u_k(x,t)$

that do not satisfy the equation of motion (cf. [**704**]). Define $\phi^{\pm}$ as in (3.12) but with the two point functions

$$W^{+}(x,x') = \sum u_k(x,t)u_k^*(x',t');\ W^{-}(x,x') = \sum u_k^*(x,t)u_k(x',t') \tag{3.15}$$

As shown in [**704**] such a choice leads to a local description of particle creation consistent with the conventional global description based on the Bogoliubov transformation. Putting $\phi(x) = \sum a_k f_k(x) + a_k^{\dagger} f_k^*(x)$ in (3.12) with (3.15) yields $\phi^{+}(x) = \sum A_k(t)u_k(x,t)$ and $\phi^{-}(x) = \sum A_k^{\dagger}(t)u_k^*(x)$ where

$$A_k(t) = \sum \alpha_{kj}^*(t)a_j - \beta_{kj}^*(t)a_k^{\dagger};\ \alpha_{jk} = (f_j,u_k);\ \beta_{jk}(t) = -(f_j^*,u_k) \tag{3.16}$$

Putting these $\phi^{\pm}$ in $j_\mu = i\phi^{-}\overleftarrow{\partial}\phi^{+}$ one finds

$$j_\mu(x) = i\sum_{k,k'} A_k^{\dagger}(t)u_k^*(x,t)\overleftrightarrow{\partial_\mu}A_{k'}(t)u_{k'}(x,t) \tag{3.17}$$

Note that because of the extra time dependence the fields $\phi^{\pm}$ do not satisy the equation of motion $(\nabla^{\mu}\partial_\mu + m^2 + \xi R)\phi = 0$ and hence the current (3.17) is not conserved, i.e. $\nabla^{\mu}j_\mu$ is a nonvanishing local scalar function describing the creation of particles in a local and invariant manner as in [**704**]. In [**702**] there follows a discussion about where and when particles are created with conclusion that this happens at the spacetime points where the metric is time dependent. Hawking radiation is then cited as an example. Generally the choice of the 2-point function (3.15) depends on the choice of time coordinate. Therefore in general a natural choice of the 2-point function (3.15) does not exist. In [**704**] an alternative choice is introduced via $W^{\pm}(x,x') = G^{\pm}(x,x')$ where $G^{\pm}(x,x')$ is determined by the Schwinger- deWitt function. As argued in [**704**] this choice seems to be the most natural since the $G^{\pm}$ satisfy the equation of motion and hence the particle current in which $\phi^{\pm}$ are calculated by putting : $\phi^{+}\phi^{-} := \phi^{-}\phi^{+}$ in (3.12) is conserved; this suggests that classical gravitational backgrounds do not create particles (see below).

A complex scalar field $\phi(x)$ and its Hermitian conjugate $\phi^{\dagger}$ in an arbitrary gravitational background can be expanded as

$$\phi = \phi^{P+} + \phi^{A-};\ \phi^{\dagger} = \phi^{P-} + \phi^{A+};\ \phi^{P+} = \sum a_k f_k(x); \tag{3.18}$$

$$\phi^{P-} = \sum a_k^{\dagger} f_k^*;\ \phi^{A+} = \sum b_k f_k(x);\ \phi^{A-} = \sum b_k^{\dagger} f_k^*(x)$$

In a similar manner to the preceeding one finds also

(3.19)

$$\phi^{P+} = i\int d\Sigma'^{\nu}W^{+}(x,x')\overleftrightarrow{\partial}'_{\nu}\phi(x');\ \phi^{A+} = i\int_{\Sigma} d\Sigma'^{\nu}W^{+}(x,x')\overleftrightarrow{\partial}'_{\nu}\phi^{\dagger}(x');$$

$$\phi^{P-} = -i\int_{\Sigma} d\Sigma'^{\nu}W^{-}(x,x')\overleftrightarrow{\partial}'_{\nu}\phi^{\dagger}(x');\ \phi^{A-} = -i\int_{\Sigma} d\Sigma'^{\nu}W^{-}(x,x')\overleftrightarrow{\partial}'_{\nu}\phi(x')$$

The particle current $j^P_\mu(x)$ and the antiparticle current $j^A_\mu(x)$ are then (cf. [**704**]) (3.20)

$$j^P_\mu(x) = \frac{1}{2}\int_\Sigma d\Sigma'^\nu \left[W^+(x,x')\overleftrightarrow{\partial_\mu}\overleftrightarrow{\partial'_\nu}\phi^\dagger(x)\phi(x') + W^-(x,x')\overleftrightarrow{\partial_\mu}\overleftrightarrow{\partial'_\nu}\phi^\dagger(x')\phi(x)\right];$$

$$j^A_\mu(x) = \frac{1}{2}\int_\Sigma d\Sigma'^\nu \left[W^+(x,x')\overleftrightarrow{\partial_\mu}\overleftrightarrow{\partial'_\nu}\phi(x)\phi^\dagger(x') + W^-(x,x')\overleftrightarrow{\partial_\mu}\overleftrightarrow{\partial'_\nu}\phi(x')\phi^\dagger(x)\right]$$

Consequently they can be written in a purely local form as

$$j^P_\mu = i\phi^{P-}\overleftrightarrow{\partial_\mu}\phi^{P+} + j^{mix}_\mu;\ j^A_\mu = i\phi^{A-}\overleftrightarrow{\partial_\mu}\phi^{A+} - j^{mix}_\mu; \tag{3.21}$$

$$j^{mix}_u = \frac{i}{2}\left[\phi^{P-}\overleftrightarrow{\partial_\mu}\phi^{A-} - \phi^{P+}\overleftrightarrow{\partial_\mu}\phi^{A+}\right]$$

The current of charge j^-_μ has the form $j^-_\mu = j^P_\mu - j^A_\mu$ which can be written as (cf. [**704**]) $j^-_\mu =: i\phi^\dagger\overleftrightarrow{\partial_\mu}\phi := \frac{i}{2}\left[\phi^\dagger\overleftrightarrow{\partial_\mu}\phi - \phi\overleftrightarrow{\partial_\mu}\phi^\dagger\right]$. Using (3.13) this can also be written as

$$j^-_\mu = N_+ i\phi^\dagger\overleftrightarrow{\partial_\mu}\phi = (i/2)N_+[\phi^\dagger\overleftrightarrow{\partial_\mu}\phi - \phi\overleftrightarrow{\partial_\mu}\phi^\dagger] \tag{3.22}$$

The current of total number of particles is now defined as $j^+_\mu = j^P_\mu + j^A_\mu$ and it is shown in [**704**] that j^+_μ can be written as $j_\mu = j^1_\mu + j^2_\mu$ where $\phi = (1/\sqrt{2})(\phi_1 + i\phi_2)$ (j^i_μ are two currents of the form (3.14). Therefore using $j_\mu = (i/2)N_-\phi\overleftrightarrow{\partial_\mu}\phi$ one can write j_μ as $j^+_\mu = (i/2)N_-[\phi_1\overleftrightarrow{\partial_\mu}\phi_1 + \phi_2\overleftrightarrow{\partial_\mu}\phi_2]$. Finally one shows that this can be written in a form analogous to (3.22) as $j^+_\mu = (i/2)N_-[\phi^\dagger\overleftrightarrow{\partial_\mu}\phi + \phi\overleftrightarrow{\partial_\mu}\phi^\dagger]$. This can be summarized by defining currents $q^\pm_\mu = (1/2)[\phi^\dagger\overleftrightarrow{\partial_\mu}\phi \pm \phi\overleftrightarrow{\partial_\mu}\phi^\dagger]$ leading to $j^\pm_\mu = N_\mu q^\pm_\mu$. The current q^+_μ vanishes but $N_- q^+_\mu$ does not vanish. These results can be easily generalized to the case where the field interacts with a backgound EM field (as in [**704**]). The equations are essentially the same but the derivatives ∂_μ are replaced by the corresponding gauge covariant derivatives and the particle 2-point functions $W^{P\pm}$ are not equal to the antiparticle 2-point functions $W^{A\pm}$. As in the gravitational case in the case of interaction with an EM background three different choices for the 2-point functions exist and we refer to [**704**] for details.

REMARK 2.3.1 In a classical field theory the energy-momentum tensor (EMT) of a real scalar field is

$$T_{\mu\nu} = (\partial_\mu\phi)(\partial_\nu\phi) - (1/2)g_{\mu\nu}[g^{\alpha\beta}(\partial_\alpha\phi)(\partial_\beta\phi) - m^2\phi^2] \tag{3.23}$$

Contrary to the conventional idea of particles in QFT the EMT is a local quantity. Therefore the relation between the definition of particles and that of EMT is not clear in the conventional approach to QFT in curved spacetime. Here one can exploit the local and covariant description of particles to find a clear relation between particles and EMT. One has to choose some ordering of the operators in (3.23) just as a choice of ordering is needed in order to define the particle current. Although the choice is not obvious it seems natural that the choice for one quantity should determine the choice for the other. Thus if the quantum EMT is defined via : $T_{\mu\nu}$:= $N_+ T_{\mu\nu}$ then the particle current should be defined as $N_- i\phi\overleftrightarrow{\partial_\mu}\phi$. The nonlocalities related to the extraction of ϕ^+ and ϕ^- from ϕ needed for the

definitions of the normal orderings N_+ and N_- appear both in the EMT and in the particle current. Similarly if $W^{\pm}$ is chosen as in $W^+(x,x') = \sum f_k(x) f_k^*(x')$ for one quantity then it should be chosen in the same way for the other. The choices as above lead to a consistent picture in which both the energy and the number of particles vanish in the vacuum $|0>$ defined by $a_k|0>=0$. Alternatively if $W^{\pm}$ is chosen as in (3.15) for the definition of particles it should be chosen in the same way for the definition of the EMT. Assume for simplicity that spacetime is flat at some late time t. Then the normally ordered operator of the total number of particles at t is $N(t) = \sum_q A_q^{\dagger}(t) A_q(t)$ (cf. (3.16)) while the normally ordered operator of energy is $H(t) = \sum_q \omega_q A^{\dagger}(t) A_q(t)$ (note here $q \sim \mathbf{q}$). Owing to the extra time dependence it is clear that both the particle current and the EMT are not conserved in this case. Thus it is clear that the produced energy exactly corresponds to the produced particles. A similar analysis can be caried out for the particle-antiparticle pair creation caused by a classical EM background. Since the energy should be conserved this suggests that $W^{\pm}$ should not be chosen as in (3.15), i.e. that classical backgrounds do not cause particle creation (see [**702**] for more discussion). The main point in all this is that particle currents as developed above can be written in a purely local form. The nonlocalities are hidden in the extraction of $\phi^{\pm}$ from ϕ. The formalism also reveals a relation between EM and particles suggesting that it might not be consistent to use semiclassical methods to describe particle creation; it also suggests that the vacuum energy might contribute to dark matter that does not form structures, instead of contributing to the cosmological constant.

3.2. BOSONIC BOHMIAN THEORY. We follow here [**703**] concerning Bohmian particle trajectories in relativistic bosonic and fermionic QFT. First we recall that there is no objection to a Bohmian type theory for QFT and no contradiction to Bell's theorems etc. (see e.g. [**77, 126, 256, 326**]). Without discussing all the objections to such a theory we simply construct one following Nikolic (cf. also [**180, 453, 588**] for related information). Thus consider first particle trajectories in relativistic QM and posit a real scalar field $\phi(x)$ satisfying the Klein-Gordon equation in a Minkowski metric $\eta_{\mu\nu} = diag(1,-1,-1,-1)$ written as $(\partial_0^2 - \nabla^2 + m^2)\phi = 0$. Let $\psi = \phi^+$ with $\psi^* = \phi^-$ correspond to positive and negative frequency parts of $\phi = \phi^+ + \phi^-$. The particle current is $j_\mu = i\psi^* \overleftrightarrow{\partial_\mu} \psi$ and $N = \int d^3x j_0$ is the positive definite number of particles (not the charge). This is most easily seen from the plane wave expansion $\phi^+(x) = \int d^3k a(\kappa) exp(-ikx)/\sqrt{(2\pi)^3 2k_0}$ since then $N = \int d^3k a^{\dagger}(\kappa) a(\kappa)$ (see above and [**702, 704**] where it is shown that the particle current and the decomposition $\phi = \phi^+ + \phi^-$ make sense even when a background gravitational field or some other potential is present). One can write also $j_0 = i(\phi^- \pi^+ - \phi^+ \pi^-)$ where $\pi = \pi^+ + \pi^-$ is the canonical momentum (cf. [**471**]). Alternatively ϕ may be interpreted not as a field containg an arbitrary number of particles but rather as a one particle wave function. Here we note that contrary to a field a wave function is not an observable and so doing we normalize ϕ here so that $N = 1$. The current j_μ is conserved via $\partial_\mu j^\mu = 0$ which implies that $N = \int d^3x j_0$ is also conserved, i.e. $dN/dt = 0$. In the causal interpretation one postulates that the particle has

the trajectory determined by $dx^\mu/d\tau = j^\mu/2m\psi^*\psi$. The affine parameter τ can be eliminated by writing the trajectory equation as $d\mathbf{x}/dt = \mathbf{j}(t,\mathbf{x})/j_0(t,\mathbf{x})$ where $t = x^0,\ \mathbf{x} = (x^1,x^2,x^3)$ and $\mathbf{j} = (j^1,j^2,j^3)$. By writing $\psi = Rexp(iS)$ where $R,\ S)$ are real one arrives at a Hamilton-Jacobi (HJ) form $dx^\mu/d\tau = -(1/m)\partial^{mu}S$ and the KG equation is equivalent to

$$\partial^\mu(R^2\partial_\mu S) = 0;\ \frac{(\partial^\mu S)(\partial_\mu S)}{2m} - \frac{m}{2} + Q = 0 \tag{3.24}$$

Here $Q = -(1/2m)(\partial^\mu\partial_\mu R/R$ is the quantum potential. One has put here $c = \hbar = 1$ and reinserted we would have

$$\frac{(\partial^\mu S)(\partial_\mu S)}{2m} - \frac{c^2 m}{2} - \frac{\hbar^2}{2m}\frac{\partial^\mu\partial_\mu R}{R} = 0 \tag{3.25}$$

From the HJ form and (3.24) plus the identity $d/d\tau = (dx^\mu/dt)\partial_\mu$ one arrives at the equations of motion $m(d^2x^\mu/d\tau^2) = \partial^\mu Q$. A typical trajectory arising from $d\mathbf{x}/dt = \mathbf{j}/j_0$ could be imagined as an S shaped curve in the $t-x$ plane (with t horizontal) and cut with a vertical line through the middle of the S. The velocity may be superluminal and may move backwards in time (at points where $j_0 < 0$). There is no paradox with backwards in time motion since it is physically indistinguishable from a motion forwards with negative energy. One introduces a physical number of particles via $N_{phys} = \int d^3x|j_0|$. Contrary to $N = \int d^3x j_0$ the physical number of particles is not conserved. A pair of particles one with positive and the other with negative energy may be created or annihilated; this resembles the behavior of virtual particles in convential QFT.

Now go to relativistic QFT where in the Heisenberg picture the Hermitian field operator $\hat\phi(x)$ satisfies

$$(\partial_0^2 - \nabla^2 + m^2)\hat\phi = J(\hat\phi) \tag{3.26}$$

where J is a nonlinear function describing the interaction. In the Schrödinger picture the time evolution is determined via the Schrödinger equation (SE) in the form $H[\phi, -i\delta/\delta\phi]\Psi[\phi,t] = i\partial_t\Psi[\phi,t]$ where Ψ is a functional with respect to $\phi(\mathbf{x})$ and a function of t. A normalized solution of this can be expanded as $\Psi[\phi,t] = \sum_{-\infty}^{\infty}\tilde\Psi_n[\phi,t]$ where the $\tilde\Psi_n$ are unnormalized n-particle wave functionals. Since any (reasonable) $\phi(\mathbf{x})$ can be Fourier expanded one can write

$$\tilde\Psi_n[\phi,t] = \int d^3k_1\cdots d^3k_n c_n(\mathbf{k}^{(n)},t)\Psi_{n,\mathbf{k}^{(n)}}[\phi] \tag{3.27}$$

where $\mathbf{k}^{(n)} = \{\mathbf{k}_1,\cdots,\mathbf{k}_n\}$. These functionals in (3.27) constitute a complete orthonormal basis which generalizes the basis of Hermite functions and they satisfy

$$\int \mathcal{D}\phi\Psi_0^*[\phi]\phi(\mathbf{x}_1)\cdots\phi(\mathbf{x}_{n'})\Psi_{n,\mathbf{k}^{(n)}}[\phi] = 0\ \ (n \neq n) \tag{3.28}$$

For free fields (i.e. when $J = 0$ in (3.26) one has

$$c_n(\mathbf{k}^{(n)},t) = c_n(\mathbf{k}^{(n)})e^{-i\omega_n(\mathbf{k}^{(n)})t};\ \omega_n = E_0 + \sum_1^n\sqrt{\mathbf{k}_j^2 + m^2} \tag{3.29}$$

where E_0 is the vacuum energy. In this case the quantities $|c_n(\mathbf{k}^{(n)},t)|^2$ do not depend on time so the number of particles (corresponding to the quantized version of $N=\int d^3x j_0$) is conserved. In a more general situation with interactions the SE leads to a more complicated time depenedence of the coefficients c_n and the number of particles is not conserved. Now the n-particle wave function is

$$\psi_n(\mathbf{x}^{(n)},t) =< 0|\hat{\phi}(t,\mathbf{x}_1)\cdots\hat{\phi}(t,\mathbf{x}_n)|\Psi > \tag{3.30}$$

(the multiplication of the right side by $(n!)^{-1/2}$ would lead to a normalized wave function only if $\Psi=\tilde{\Psi}_n$). The generalization of (3.30) to the interacting case is not trivial because with an unstable vacuum it is not clear what is the analogue of $<0|$. Here the Schrödinger picture is more convenient where (3.30) becomes

$$\psi_n(\mathbf{x}^{(n)},t)=\int \mathcal{D}\phi\Psi_0^*[\phi]exp(-i\phi_0(t))\phi(\mathbf{x}_1)\cdots\phi(\mathbf{x}_n)\Psi[\phi,t] \tag{3.31}$$

where $\phi_0(t)=-E_0t$. For the interacting case one uses a different phase $\phi_0(t)$ defined via an expansion, namely

$$\hat{U}(t)\Psi_0[\phi]=r_0(t)exp(i\phi_0(t))\Psi_0[\phi]+\sum_1^\infty\cdots \tag{3.32}$$

where $r_0(t)\geq 0$ and $\hat{U}(t)=U(\phi,-i\delta/\delta\phi,t]$ is the unitary time evolution operator. One sees that even in the interacting case only the $\tilde{\Psi}_n$ part of Ψ contributes to (3.31) so $\tilde{\Psi}_n$ can be called the n-particle wave functional. The wave function (3.30) can also be generalized to a nonequaltime wave function $\psi_n(x^{(n)})=S_{\{x_j\}}<0|\hat{\phi}(x_1)\cdots\hat{\phi}(x_n)|\Psi>$ (here $S_{\{x_j\}}$ denotes symmetrization over all x_j which is needed because the field operators do not commute for nonequal times. For the interacting case the nonequaltime wave function is defined as a generalization of (3.30) with the replacements

$$\phi(\mathbf{x}_j)\rightarrow\hat{U}^\dagger(t_j)\phi(\mathbf{x}_j)\hat{U}(t_j);\ \Psi[\phi,t]\rightarrow\hat{U}^\dagger(t)\Psi[\phi,t]=\Psi[\phi]; \tag{3.33}$$

$$e^{-i\phi_0(t)}\rightarrow e^{-i\phi_0(t_1)}\hat{U}(t_1)$$

followed by symmetrization.

In the deBroglie-Bohm (dBB) interpretation the field $\phi(x)$ has a causal evolution determined by

$$(\partial_0^2-\nabla^2+m^2)\phi(x)=J(\phi(x))-\left(\frac{\delta Q[\phi,t]}{\delta\phi(\mathbf{x})}\right)_{\phi(\mathbf{x})=\phi(x)}; \tag{3.34}$$

$$Q=-\frac{1}{2|\Psi|}\int d^3x\frac{\delta^2|\Psi|}{\delta\phi^2(\mathbf{x})}$$

where Q is the quantum potential again. However the n particles attributed to the wave function ψ_n also have causal trajectories determined by a generalization of $d\mathbf{x}/dt=\mathbf{j}/j_0$ as

$$\frac{d\mathbf{x}_{n,j}}{dt}=\left(\frac{\psi_n^*(x^{(n)})\overleftrightarrow{\nabla_j}\psi_n(x^{(n)})}{\psi_n^*(x^{(n)})\overleftrightarrow{\partial_{t_j}}\psi_n(x^{(n)})}\right)_{t_1=\cdots=t_n=t} \tag{3.35}$$

These n-particles have well defined trajectories even when the probability (in the conventional interpretation of QFT) of the experimental detection is equal to zero. In the dBB interpretation of QFT we can introduce a new causally evolving "effectivity" parameter $e_n[\phi, t]$ defined as

$$e_n[\phi,t] = |\tilde{\Psi}_n[\phi,t]|^2 / \sum_{n'}^{\infty} |\tilde{\Psi}_{n'}[\phi,t]|^2 \tag{3.36}$$

The evolution of this parameter is determined by the evolution of ϕ given via (3.34) and by the solution $\Psi = \sum \tilde{\Psi}$ of the SE. This parameter might be interpreted as a probability that there are n particles in the system at time t if the field is equal (but not measured!) to be $\phi(\mathbf{x})$ at that time. However in the dBB theory one does not want a stochastic interpretation. **Hence assume that e_n is an actual property of the particles guided by the wave function ψ_n and call it the effectivity of these n particles.** This is a nonlocal hidden variable attributed to the particles and it is introduced to provide a deterministic description of the creation and destruction of particles. One postulates that the effective mass of a particles guided by ψ_n is $m_{eff} = e_n m$ and similarly for the energy, momentum, charge, etc. This is achieved by postulating that the mass density is $\rho_{mass}(\mathbf{x}, t) = m \sum_1^{\infty} e_n \sum_1^n \delta^3(\mathbf{x} - \mathbf{x}_{n,j}(t))$ and similarly for other quantities. Thus if $e_n = 0$ such particles are ineffective, i.e. their effect is the same as if they didn't exist while if $e_n = 1$ they exist in the usual sense. However the trajectories are defined even for the particles for which $e_n = 0$ and QFT is a theory of an infinite number of particles although some of them may be ineffective (conventionally one would say they are virtual). We will say more about this later.

3.3. FERMIONIC THEORY. This extraction from [**701**] (cf. also [**325**]) becomes even more technical but a sketch should be rewarding; there is more detail and discussion in [**701**]. The Dirac equation in Minkowski space $\eta_{\mu\nu} = diag(1, -1, -1, -1)$ is $i\gamma^\mu \partial_\mu - m)\psi(x) = 0$ where $x = (x^i) = (t, \mathbf{x})$ with $\mathbf{x} \in \mathbf{R}^3$ (cf. Section 2.1.1). A general solution can be written as $\psi(x) = \psi^P(x) + \psi^A(x)$ where the particle and antiparticle parts can be expanded as $\psi^P = \sum b_k u_k(x)$ and $\psi^A = \sum d_k^* v_k(x)$. Here u_k (resp. v_k) are positive (resp. negative) frequency 4-spinors that, together, form a complete orthonormal set of solutions to the Dirac equation. The label k means $(\mathbf{k}, s)$ where $s = \pm 1/2$ is the spin label. Writing $\Omega^P(x, x') = \sum u_k(x) u_k^\dagger(x')$ and $\Omega^A(x, x') = \sum v_k(x) v_k^\dagger(x')$ one can write

$$\psi^P = \int d^3x' \Omega^P(x, x')\psi(x');\ \psi^A(x) = \int d^3x' \Omega^A(x, x')\psi(x') \tag{3.37}$$

where $t = t'$. The particle and antiparticle currents are $j_\mu^P = \bar{\psi}^P \gamma_\mu \psi^P$ and $j_\mu^A = \bar{\psi}^A \gamma_\mu \psi^A$ where $\bar{\psi} = \psi^\dagger \gamma_0$. Since ψ^P and ψ^A satisfy the Dirac equation the currents j_μ^P, j_μ^A are separately conserved, i.e. $\partial^\mu j_\mu^P = \partial^\mu j_\mu^A = 0$. One postulates then trajectories of the form

$$\frac{d\mathbf{x}^P}{dt} = \frac{\mathbf{j}^P(t, \mathbf{x}^P)}{j_0^P(t, \mathbf{x})};\ \frac{d\mathbf{x}^A}{dt} = \frac{\mathbf{j}^A(t, \mathbf{x}^A)}{j_0^A(t, \mathbf{x}^A)} \tag{3.38}$$

where $\mathbf{j} = (j^1, j^2, j^3)$ for a causal interpretation of the Dirac equation. Now in QFT the coefficients b_k and d_k^* become anticommuting operators with $\hat{b}_k^\dagger$ and $\hat{d}_k^\dagger$ creating particles and antiparticles while $\hat{b}_k$ and $\hat{d}_k$ annihilate them. In the Schrödinger picture the field opperators $\hat{\psi}(\mathbf{x})$ and $\hat{\psi}^\dagger(\mathbf{x})$ satisfy the commutation relations $\{\hat{\psi}_a(\mathbf{x}), \hat{\psi}_{a'}^\dagger(\mathbf{x}')\} = \delta_{aa'}\delta^3(\mathbf{x}-\mathbf{x}')$ while other commutators vanish (a is the spinor index). These relations can be represented via

$$\hat{\psi}_a(\mathbf{x}) = \frac{1}{\sqrt{2}}\left[\eta_a(\mathbf{x}) + \frac{\delta}{\delta\eta_a^*(\mathbf{x})}\right];\ \hat{\psi}_a^\dagger(\mathbf{x}) = \frac{1}{\sqrt{2}}\left[\eta_a^*(\mathbf{x}) + \frac{\delta}{\delta\eta_z(\mathbf{x})}\right] \tag{3.39}$$

where $\eta_a,\ \eta_a^*$ are anticommuting Grassmann numbers satisfying $\{\eta_a(\mathbf{x}), \eta_{a'}(\mathbf{x}')\} = \{\eta_a^*(\mathbf{x}), \eta_{a'}^*(\mathbf{x}')\} = \{\eta_a(\mathbf{x}), \eta_{a'}^*(\mathbf{x}')\} = 0$. Next introduce a complete orthonormal set of spinors $u_k(\mathbf{x})$ and $v_k(\mathbf{x})$ which are equal to the spinors $u_k(x)$ and $v_k(x)$ at $t = 0$. An arbitrary quantum state may then be obtained by acting with creation operators

$$b_k^\dagger = \int d^3x \hat{\psi}^\dagger(\mathbf{x})u_k(\mathbf{x});\ \hat{d}_k^\dagger = \int d^3x v_k^\dagger(\mathbf{x})\hat{\psi}(\mathbf{x}) \tag{3.40}$$

on the vacuum $|0>=|\Psi_0>$ represented by

$$\Psi_0[\eta, \eta^\dagger] = Nexp\{\int d^3x \int d^3x' \eta^\dagger(\mathbf{x})\Omega(\mathbf{x}, \mathbf{x}')\eta(\mathbf{x}')\} \tag{3.41}$$

Here $\Omega(\mathbf{x}, \mathbf{x}') = (\Omega^A - \Omega^P)(\mathbf{x}, \mathbf{x}')$, N is a constant such that $<\Psi_0|\Psi_0>= 1$ and the scalar product is $<\Psi|\Psi>=\int \mathcal{D}^2\eta\Psi^*[\eta, \eta^\dagger\Psi'[\eta, \eta^\dagger]$; also $\mathcal{D}^2 = \mathcal{D}\eta\mathcal{D}\eta^\dagger$ and Ψ^* is dual (not simply the complex conjugate) to Ψ. The vacuum is chosen such that $\hat{b}_k\Psi_0 = \hat{d}_k\Psi_0 = 0$. A functional $\Psi[\eta, \eta^\dagger]$ can be expanded as $\Psi[\eta, \eta^\dagger] = \sum c_K\Psi_K[\eta, \eta^\dagger]$ where the set $\{\psi_K\}$ is a complete orthonormal set of Grassmann valued functionals. This is chosen so that each Ψ_K is proportional to a functional of the form $\hat{b}_{k_1}^\dagger \cdots \hat{b}_{k_{n_P}}^\dagger \hat{d}_{k'_1}^\dagger \cdots \hat{d}_{k'_{n_A}}^\dagger\ \Psi_0$ which means that each Ψ_K has a definite number n_P of particles and n_A of antiparticles. Therefore one can write $\Psi[\eta, \eta^\dagger] = \sum_{n_P, n_A=0}^\infty \tilde{\Psi}_{n_P, n_A}[\eta, \eta^\dagger]$ where the tilde denotes that these functionals, in contrast to Ψ and Ψ_K, do not have unit norm. Time dependent states $\Psi[\eta, \eta^\dagger, t]$ can be expanded as

$$\Psi[\eta, \eta^\dagger, t] = \sum_K c_K(t)\Psi_K[\eta, \eta^\dagger] = \sum_{n_p, n_A=0}^\infty \tilde{\Psi}_{n_P, n_A}[\eta, \eta^\dagger, t] \tag{3.42}$$

The time dependence of the c-number coefficients $c_K(t)$ is governed by the functional SE

$$H[\hat{\psi}, \hat{\psi}^\dagger]\Psi[\eta, \eta^\dagger, t] = i\partial_t\Psi[\eta, \eta^\dagger, t] \tag{3.43}$$

Since the Hamiltonian H is a Hermitian operator the norms $<\Psi(t)|\Psi(t)>= \sum |c_K(t)|^2$ do not depend on time. In particular if H is the free Hamiltonian (i.e. the Hamiltonian that generates the second quantized free Dirac equation) then the quantities $|c_K(t)|$ do not depend on time, which means that the average number of particles and antiparticles does not change with time when there are no interactions.

Next introduce the wave function of n_P particles and n_A antiparticles via

$$\psi_{n_P,n_A} \equiv \psi_{b_1\cdots b_{n_P} d_1\cdots d_{n_A}}(\mathbf{x}_1,\cdots,\mathbf{x}_{n_P},\mathbf{y}_1\cdots,\mathbf{y}_{n_A},t) \tag{3.44}$$

It has n_P+n_A spinor indices and for free fields the (unnormalized) wave function can be calculated using the Heisenberg picture as

$$\psi_{n_P,n_A} =< 0|\hat{\psi}^P_{b_1}(t,\mathbf{x}_1)\cdots\hat{\psi}^{A\dagger}_{d_{n_A}}(t,\mathbf{y}_{n_A})|\Psi > \tag{3.45}$$

where $\hat{\psi}^P$ and $\hat{\psi}^A$ are extracted from $\hat{\psi}$ using (3.37). In the general interacting case the wave function can be calculated using the Schrödinger picture as

$$\psi_{n_P,n_A} = \int \mathcal{D}^2\eta\Psi^*_0[\eta,\eta^\dagger]e^{-i\phi_0(t)}\hat{\psi}^P_{b_1}(\mathbf{x}_1)\cdots\hat{\psi}^{A\dagger}_{d_{n_A}}(y_{n_A})\Psi[\eta,\eta^\dagger,t] \tag{3.46}$$

Here the phase $\phi_0(t)$ is defined by an expansion as in (3.42), namely

$$\hat{U}(t)\Psi_0[\eta,\eta^\dagger] = r_0(t)exp(i\phi_0(t))\Psi_0[\eta,\eta^\dagger] + \sum_{(n_P,n_A)\neq(0,0)}\cdots \tag{3.47}$$

where $r_0(t)\geq 0$ and $\hat{U}(t)=U[\hat{\psi},\hat{\psi}^\dagger,t]$ is the unitary time evolution operator that satisfies the SE (3.43). The current attributed to the i^{th} corpuscle (particle or antiparticle) in the wave function ψ_{n_P,n_A} is $j_{\mu(i)}=\bar{\psi}_{n_P,n_A}\gamma_{\mu(i)}\psi_{n_P,n_A}$ where one writes

$$\bar{\psi}\Gamma_i\psi = \bar{\psi}_{a_1\cdots a_i\cdots a_n}(\Gamma)_{a_i a'_i}\psi_{a_1\cdots a'_i\cdots a_n}; \tag{3.48}$$

$$\bar{\psi}_{a_1\cdots a_n} = \psi^*_{a'_1\cdots a'_n}(\gamma_0)_{a'_1 a_1}\cdots(\gamma_0)_{a'_n a_n}$$

Hence the trajectory of the i^{th} corpuscle guided by the wave function ψ_{n_P,n_A} is given by the generalization of (3.38), namely $d\mathbf{x}_i/dt=\mathbf{j}_i/j_{0(i)}$.

We now need a causal interpretation of the processes of creation and destruction of particles and antiparticles. For bosonic fields this was achieved by introducing the effectivity parameter in Section 1.3.2 but this cannot be done for the Grassmann fields $\eta,\ \eta^\dagger$ because $\Psi^*[\eta,\eta^\dagger,t]\Psi[\eta,\eta^\dagger,t]$ is Grassmann valued and cannot be interpreted as a probability density. Hence another formulation of fermionic states is developed here, more similar to the bosonic states. First the notion of the scalar product can be generalized in such a way that it may be Grassmann valued which allows one to write $\Psi[\eta,\eta^\dagger,t]=<\eta,\eta^\dagger|\Psi(t)>$ and $1=\int\mathcal{D}^2\eta|\eta,\eta^\dagger><\eta,\eta^\dagger|$ (cf. [**457**]). We can also introduce

$$<\phi,\phi^\dagger|\eta,\eta^\dagger>=\sum_K <\phi,\phi^\dagger|\Psi_K><\Psi_K|\eta,\eta^\dagger>=\sum_K \Psi_K[\phi,\phi^\dagger]\Psi^*_K[\eta,\eta^\dagger] \tag{3.49}$$

so one sees that the sets $\{\Psi_K[\eta,\eta^\dagger]\}$ and $\{\Psi_K[\phi,\phi^\dagger]\}$ are two representations of the same orthonormal basis $\{|\Psi_K>\}$ for the same Hilbert space of fermionic states. In other words the state $|\Psi(t)>$ can be represented as $\Psi[\phi,\phi^\dagger,t]=<\phi,\phi^\dagger|\Psi(t)>$ which can be expanded as

$$\Psi[\phi,\phi^\dagger,t]=\sum_K c_K(t)\Psi_K[\phi,\phi^\dagger]=\sum_{n_P,n_A=0}^{\infty}\tilde{\psi}_{n_P,n_A}[\phi,\phi^\dagger,t] \tag{3.50}$$

Putting the unit operator $1 = \int \mathcal{D}^2\phi|\phi,\phi^\dagger >< \phi,\phi^\dagger|$ in the expression for $< \Psi(t),\Psi(t) >$ we see that the time independent norm can be written as

$$< \Psi(t)|\Psi(t) >= \int \mathcal{D}^2\phi\Psi^*[\phi,\phi^\dagger,t)]\Psi[\phi,\phi^\dagger,t] \tag{3.51}$$

Therefore the quantity $\rho[\phi,\phi^\dagger,t] = \Psi^*[\phi,\phi^\dagger,t]\Psi[\phi,\phi^\dagger,t]$ can be interpreted as a positive definite probability density for spinors ϕ, $\phi^\dagger$ to have space dependence $\phi(\mathbf{x})$ and $\phi^\dagger(\mathbf{x})$ respectively at time t. The SE (3.43) can also be written in the ϕ-representation as $\hat{H}_\phi\Psi[\phi,\phi^\dagger,t] = i\partial_t\Psi[\phi,\phi^\dagger,t]$ where the Hamiltonian $\hat{H}_\phi$ is defined by its action on wave functionals $\Psi[\phi,\phi^\dagger,t]$ determined via

(3.52)
$$\hat{H}_\phi[\phi,\phi^\dagger,t] = \int \mathcal{D}^2\eta \int \mathcal{D}^2\phi' < \phi,\phi^\dagger|\eta,\eta^\dagger > \hat{H} < \eta,\eta^\dagger|\phi',\phi'^\dagger > \Psi[\phi',\phi'^\dagger,t]$$

where $\hat{H} = H[\hat{\psi},\hat{\psi}^\dagger]$ is the Hamiltonian of (3.43).

One can now obtain a causal interpretation of a quantum system described by a c-number valued wave function satisfying a SE. The material is written for an n-dimensional vector $\vec{\phi}$ but in a form that generalizes to infinite dimensions. The wave function $\psi(\vec{\phi},t)$ satisfies the SE $\hat{H}\psi = i\partial_t\psi$ where $\hat{H}$ is an arbitrary Hermitian Hamiltonian written in the $\vec{\phi}$ representation. The quantity $\rho = \psi^*\psi$ is the probability density for the variables $\vec{\phi}$ and the average velocity is

$$d < \vec{\phi} > (t)/dt = \int d^n\phi\rho(\vec{\phi},t)\vec{u}(\vec{\phi},t);\ \vec{u} = i\psi^*[\hat{H},\vec{\phi}]\psi/\psi^*\psi \tag{3.53}$$

Introduce a source J via $J = (\partial\rho/\partial t)+\vec{\nabla}(\rho\vec{u})$ (note e.g. for the example of (3.52) J does not vanish even though it frequently will vanish). One wants to find a quantity $\vec{v}(\vec{\phi},t)$ that has the property (3.53) in the form $d < \vec{\phi} > (t)/dt = \int d^n\phi\rho(\vec{\phi},t)\vec{v}(\vec{\phi},t)$ but at the same time satisfies the equivariance property $\partial_t\rho+\vec{\nabla}(\rho\vec{v}) = 0$. These two properties allow one to postulate a consistent causal interpretation of QM in which $\vec{\phi}$ has definite values at each time t determined via $d\vec{\phi}/dt = \vec{v}(\vec{\phi},t)$. In particular the equivariance provides that the statistical distribution of the variables $\vec{\phi}$ is given by ρ for any time t provided that it is given by ρ for some initial time t_0. When $J = 0$ then $\vec{v} = \vec{u}$ which corresponds to the dBB interpretation. The aim now is to generalize this to the general case of $\vec{v}$ in the form $\vec{v} = \vec{u}+\rho^{-1}\vec{\mathcal{E}}$ where $\vec{\mathcal{E}}(\vec{\phi},t)$ is the quantity to be determined. From $\partial_t\rho+\vec{\nabla}(\rho\vec{v}) = 0$ we see that $\vec{\mathcal{E}}$ must be a solution of the equation $\vec{\nabla}\vec{\mathcal{E}} = -J$. Now let $\vec{\mathcal{E}}$ be some particular solution of this equation; then $\vec{\mathcal{E}}(\vec{\phi},t) = \vec{e}(t) + \vec{E}(\vec{\phi},t)$ is also a solution for an arbitrary $\vec{\phi}$ independent function $\vec{e}(t)$. Comparing with (3.53) one sees that $\int d^n\phi\vec{\mathcal{E}} = 0$ is required. This fixes the function $\vec{e}$ to be $\vec{e}(t) = -V^{-1}\int d^n\phi\vec{E}(\vec{\phi},t)$ where $V = \int d^n\phi$. Thus it remains to choose $\vec{E}$ and in [**703**] one takes $\vec{E}$ such that $\vec{E} = 0$ when $J = 0$ so that $\vec{\mathcal{E}} = 0$ when $J = 0$ as well; thus $\vec{v} = \vec{u}$ when $J = 0$. There is still some arbitrariness in $\vec{E}$ so take $\vec{E} = \vec{\nabla}\Phi$ where $\vec{\nabla}^2\Phi = -J$, which is solved via $\Phi(\vec{\phi},t)\int d^n\phi' G(\vec{\phi},\vec{\phi}')J(\vec{\phi}',t)$, so that $\vec{\nabla}^2G(\vec{\phi},\vec{\phi}') = -\delta^n(\vec{\phi}-\vec{\phi}')$. The solution can be expressed as a Fourier transform $G(\vec{\phi},\vec{\phi}') = \int(d^nk/(2\pi)^n)exp[i\vec{k}(\vec{\phi}-\vec{\phi}')]/\vec{k}^2$.

To eleminate the factor $1/(2\pi)^n$ one uses a new integration variable $\vec{\chi} = \vec{k}/2\pi$ and we obtain

$$\Phi(\vec{\phi}, t) = \int d^n\chi \int d^n\phi' [exp(2i\pi\vec{\chi}(\vec{\phi} - \vec{\phi}')]J(\vec{\phi}', t)/(2\pi)^2\vec{\chi}^2] \tag{3.54}$$

Now for a causal interpretation of fermionic QFT one writes first for simplicity $A[\mathbf{x}]$ for functionals of the form $A[\phi, \phi^\dagger, t, \mathbf{x}]$ and introduces

$$u_a[\mathbf{x}] = i\frac{\Psi^*[\hat{H}_\phi, \phi_a(\mathbf{x})]\Psi}{\Psi^*\Psi};\ u_a^*[\mathbf{x}] = i\frac{\Psi^*[\hat{H}_\phi, \phi_a^*(\mathbf{x})]\Psi}{\Psi^*\Psi} \tag{3.55}$$

where $\Psi = \Psi[\phi, \phi^\dagger, t]$. Next introduce the source

$$J = \frac{\partial\rho}{\partial t} + \sum_a \int d^3x \left[\frac{\delta(\rho u_a[\mathbf{x}])}{\delta\phi_z(\mathbf{x})} + \frac{\delta(\rho u_a^*[\mathbf{x}])}{\delta\phi_a^*(\mathbf{x})}\right] \tag{3.56}$$

where $\rho = \Psi^*\Psi$. Introduce now the notation $\alpha \cdot \beta = \sum_a \int d^3x[\alpha_a(\mathbf{x})\beta_a(\mathbf{x}) + \alpha_a^*(\mathbf{x})\beta_a^*(\mathbf{x})]$ and (3.51) generalizes to

$$\Phi[\phi, \phi^\dagger, t] = \int \mathcal{D}^2\chi \int \mathcal{D}^2\phi' \frac{e^{2\pi i\chi\cdot(\phi-\phi')}}{(2\pi)^2\chi\cdot\chi} J[\phi', \phi'^\dagger, t] \tag{3.57}$$

Then write for $V = \int \mathcal{D}^2\phi$

$$E_a[\mathbf{x}] = \frac{\delta\Phi}{\delta\phi_a(\mathbf{x})};\ E_a^*[\mathbf{x}] = \frac{\delta\Phi}{\delta\phi_a^*(\mathbf{x})}; \tag{3.58}$$

$$e_a(t, \mathbf{x}) = -V^{-1}\int \mathcal{D}^2\phi E_a[\phi, \phi^\dagger, t, \mathbf{x}];\ e_a^*(t, \mathbf{x}) = -V^{-1}\int \mathcal{D}^2\phi E_a^*[\phi, \phi^\dagger, t, \mathbf{x}]$$

The corresponding velocities are then

$$v_a[\mathbf{x}] = u_a[\mathbf{x}] + \rho^{-1}(e_a(t, \mathbf{x}) + E_a[\mathbf{x}]);\ v_a^*[\mathbf{x}] = u_a^*[\mathbf{x}] + \rho^{-1}(e_a^*(t, \mathbf{x}) + E_a^*[\mathbf{x}]) \tag{3.59}$$

Next introduce hidden variables $\phi(t, \mathbf{x})$ and $\phi^\dagger(t, \mathbf{x})$ with causal evolution given then by

$$\frac{\partial\phi_a(t, \mathbf{x})}{\partial t} = v_a[\phi, \phi^\dagger, t, \mathbf{x}];\ \frac{\partial\phi_a^*(t, \mathbf{x})}{\partial t} = v_a^*[\phi, \phi^\dagger, t, \mathbf{x}] \tag{3.60}$$

where it is understood that the right sides are calculated at $\phi(\mathbf{x}) = \phi(t, \mathbf{x})$ etc. In analogy with the bosonic fields treated earlier one introduces effectivity parameters guided by the wave function ψ_{n_P,n_A} given by

$$e_{n_P,n_A}[\phi, \phi^\dagger, t] = \frac{|\tilde{\Psi}_{n_P,n_A}[\phi, \phi^\dagger, t]|^2}{\sum_{n'_P,n'_A} |\tilde{\Psi}_{n'_P,n'_A}[\phi, \phi^\dagger, t]|^2} \tag{3.61}$$

REMARK 2.3.2. Concerning the nature of the effectivity parameter we extract from [**701**] as follows. In the bosonic theory the analogue of (3.61) is

$$e_n[\{\phi\}, t] = \frac{|\tilde{\Psi}_n[\{\phi\}], t|^2}{\sum_{n'} |\tilde{\Psi}_{n'}[\{\phi\}], t|^2} \tag{3.62}$$

$\{\phi\} = \{\phi_1, \cdots, \phi_{N_s}\}$ where N_s is the number of different particle species. Now the measured effectivity can be any number between 0 and 1 and this is no contradiction since if different $\tilde{\Psi}_n$ in the expansion do not overlap in the ϕ space

then they represent a set of nonoverlapping "channels" for the causally evolving field ϕ. The field necessarily enters one and only one of the channels and one sees that $e_n = 1$ for the nonempty channel with $e_{n'} = 0$ for all empty channels. The effect is the same as if the wave functional Ψ "collapsed" into one of the states $\tilde{\psi}_n$ with a definite number of particles. In a more general situation different $\tilde{\Psi}_n$ of the measured particles may overlap. However the general theory of ideal quantum measurements (cf. [**126**]) provides that the total wave functional can be written again as a sum of nonoverlappiing wave functionals in the $\{\phi\}$ space, where one of the fields represents the measured field, while the others represent fields of the measuring apparatus. Thus only one of the $\tilde{\Psi}$ in (3.62) becomes nonempty with the corresponding $e_n = 1$ while all the other $e_{n'} = 0$. The essential point is that from the point of view of an observer who does not know the actual field configuration the probability for such an effective collapse of the wave functional is exactly equal to the usual quantum mechanical probability for such a collapse. Hence the theory has the same statistical properties as the usual theory. In the case when all the effectivities are less than 1 (i.e. the wave functional has not collapsed) the theory does not agree nor disagree with standard theory; effectivity is a hidden variable. This agrees with the Bohmian particle positions which agree with the standard quantum theory only when the wave function effectively collapses into a state with a definite particle position. Similar comments apply to the fermionic picture. In an ideal experiment in which the number of particles is measured, different Ψ_{n_P,n_A} do not overlap in the $(\phi, \phi^\dagger)$ space and the fields ϕ, $\phi^\dagger$ necessarily enter into a unique "channel" $\tilde{\Psi}_{n_P,n_A}$, etc.

REMARK 2.3.3. In [**711**] one addresses the question of statistical transparency. Thus the probabilitistic interpretation of the nonrelativistic SE does not work for the relativistic KG equation $(\partial^\mu \partial_\mu + m^2)\psi = 0$ (where $x = (\mathbf{x}, t)$ and $\hbar = c = 1$) since $|\psi|^2$ does not correspond to a probability density. There is a conserved current $j^\mu = i\psi^* \overleftrightarrow{\partial^\mu} \psi$ (where $a \overleftrightarrow{\partial^\mu} b = a\partial^\mu b - b\partial^\mu a$) but the time component j^0 is not positive definite. In [**701, 703**] the equations that determine the Bohmain trajectories of relativistic quantum particles described by many particle wave functions were written in a form requiring a preferred time coordinate. However a preferred Lorentz frame is not necessare (cf. [**105**]) and this is developed in [**711**] following [**105, 703**]. First note that as in [**105, 703**] it appears that particles may be superluminal and the principle of Lorentz covariance does not forbid superluminal velocities and conversly superluminal velocities do not lead to causal paradoxes (cf. [**105, 711**]). As noted in [**105**] the Lorentz-covariant Bohmian interprtation of the many particle KG equation is not statistically transparent. This means that the statistical distribution of particle positions cannot be calculated in a simple way from the wave function alone without the knowledge of particle trajetories. One knows that classcal QM is statistically transparent of course and this perhaps helps to explain why Bohmian mechanics has not attracted more attention. However statistical transparency (ST) may not be a fundamental property of nature as the following facts suggest:

- Classical mechanics, relativistic or nonrelativistic, is not ST.

- Relativistic QM based on the KG equation (or some of its generalizations) is not ST.
- The relativistic Dirac equation is ST but its many particle relativistic generalization is not (unless a preferred time coordinate is determined in an as yet unknown dynamical manner).
- Nonrelativistic QM is ST but not completely so since it distinguishes the time variable (e.g. $\rho(x^1, x^2, t)$ is not a probability density).
- The background independent quantum gravity based on the Wheeler-DeWitt (WDW) equation lacks the notion of time and is not ST.

The upshot is that since statistical probabilities can be calculated via Bohmian trajectories that theory is more powerful than other interpretations of general QM (see [**711**] for discussion on this). Now let $\hat{\phi}(x)$ be a scalar field operator satisfying the KG equation (an Hermitian uncharged field for simplicity so that negative values of the time component of the current cannot be interpreted as negatively charged particles). The corresponding n-particle wave function is (cf. [**703**])

$$\psi(x_1, \cdots, x_n) = (n!)^{-1/2} S_{\{x_a\}} < 0|\hat{\phi}(x_1) \cdots \hat{\phi}(x_n)|n > \tag{3.63}$$

Here $S_{\{x_a\}}$ $(a = 1, \cdots, n)$ denotes the symmetrization over all x_a which is needed because the field operators do not commute for nonequal times. The wave function ψ satisfies n KG equations

$$(\partial_a^\mu \partial_{a\mu} + m^2)\psi(x_1, \cdots, x_n) = 0 \tag{3.64}$$

Although the operator $\hat{\phi}$ is Hermitian the nondiagonal matrix element ψ defined by (3.63) is complex and one can introduce n real 4-currrents $j_a^\mu = i\psi^* \overleftrightarrow{\partial_a^\mu} \psi$ each of which is separately conserved via $\partial_a^\mu j_{a\mu} = 0$. Equation (3.64) also implies

$$\left(\sum_a \partial_a^\mu \partial_{a\mu} + nm^2 \right) \psi(x_1, \cdots, x_n) = 0 \tag{3.65}$$

and the separate conservation equations imply that $\sum_a \partial_a^\mu j_{a\mu} = 0$. Now write $\psi = Rexp(iS)$ with R and S real. Then (3.65) is equivalent to a set of two equations

$$\sum_a \partial_a^\mu ({}^2\partial_{a\mu} S) = 0; \; -\frac{\sum_a (\partial_a^\mu S)(\partial_{a\mu} S)}{2m} + \frac{nm}{2} + Q = 0; \; Q = \frac{1}{2m} \frac{\sum_a \partial_a^\mu \partial_{a\mu} R}{2mR} \tag{3.66}$$

where Q is the quantum potential. The first equation is equivalent to a current conservation equation while the second is the quantum analogue of the relativistic HJ equation for n particles. The Bohmian interpretation consistists in postulating the existence of particle trajectories $x_a^\mu(s)$ satisfying $dx_a^\mu/ds = -(1/m)\partial_a^\mu s$ where s is an affine parameter along the n curves in the 4-dimensional Minkowski space. This equation has a form identical to the corresponding classical relativistic equation and can also be written as $dx_a^\mu/ds = j_a^\mu/2m\psi^*\psi$. Hence using $d/ds = \sum_a (dx_z^\mu/ds)\partial_{a\mu}$ one finds the equations of motion

$$m \frac{d^2 x_a^\mu}{ds^2} = \partial_a^\mu Q \tag{3.67}$$

Note that the equations above for the particle trajectories are nonlocal but still Lorentz covariant. The Lorentz covariance is a consequence of the fact that the trajectories in spacetime do not depend on the choice of affine parameter s (cf. [**105**]). Instead, by choosing n "initial" spacetime positions x_a, the n trajectories are uniquely determined by the vector fields j_a^μ or $-\partial_a^\mu S$ (i.e. the trajectories are integral curves of these vector fields). The nonlocality is encoded in the fact that the right hand side of (3.67) depends not only on x_a but also on all the other $x_{a'}$. This is a consequence of the fact that $Q(x_1, \cdots, x_n)$ in (3.66) is not of the form $\sum_a Q_a(x_1, \cdots, x_n)$, which in turn is related to the fact that $S(x_1, \cdots, x_n)$ is not of the form $\sum_a S(x_a)$. Note also that the fact that we parametrize all trajectories with the same parameter s is not directly related to the nonlocality, because such a parametrization can be used even in local classical physics. When the interactions are local then one can even use another parameter s_a for each curve but when the interactions are not local one must use a single parameter s; new separate parameters could only be used after the equations are solved. In the nonrelativistic limit all wave function frequencies are (approximately) equal to m so from $j_a^\mu \psi^* \overleftrightarrow{\partial_a^\mu} \psi$ all time components are equal and given by $j_a^0 = 2m\psi^*\psi = \tilde{\rho}$ which does not depend on a. Writing then $\rho(\mathbf{x}_1, \cdots, \mathbf{x}_n) = \tilde{\rho}(x_1, \cdots, x_n)|_{t_1=\cdots=t_n=t}$ one obtains $\partial_t \rho + \sum_a \partial_a^i j_{ai} = 0$ and this implies that ρ can be interpreted as a probability density. In the full relativistic there is generally no analogue of such a function ρ. We refer to [**711**] for more discussion.

4. DeDONDER, WEYL, AND BOHM

We go here to a fascinating paper [**708**] which gives a manifestly covariant canonical method of field quantization based on the classical DeDonder-Weyl (DW) formulation of field theory (cf. also Appendix A for some background on DW theory following [**586**]). The Bohmian formulation is not postulated for intepretational purposes here but derived from purely technical requirements, namely covariance and consistency with standard QM. It arises automatically as a part of the formalism without which the theory cannot be formulated consistently. This together with the results of [**701, 711**] suggest that it is Bohmian mechanics that might be the missing bridge between QM and relativity; further (as will be seen later) it should play an important role in cosmology. The classical covariant canonical DeDonder-Weyl formalism is given first following [**586**] and for simplicity one real scalar field in Minkowski spacetime is used. Thus let $\phi(x)$ be a real scalar field described by

$$\mathfrak{A} = \int d^4x \mathfrak{L};\ \mathfrak{L} = \frac{1}{2}(\partial^\mu \phi)(\partial_\mu \phi) - V(\phi) \tag{4.1}$$

As usual one has

$$\pi^\mu = \frac{\partial \mathfrak{L}}{\partial(\partial_\mu \phi)} = \partial^\mu \phi;\ \partial_\mu \phi = \frac{\partial \mathfrak{H}}{\partial \pi^\mu};\ \partial_\mu \pi^\mu = -\frac{\partial \mathfrak{H}}{\partial \phi} \tag{4.2}$$

where the scalar DeDonder-Weyl (DDW) Hamilonian (not related to the energy density) is given by the Legendre transform $\mathfrak{H}(\pi^\mu, \phi) = \pi^\mu \partial_\mu \phi - \mathfrak{L} = (1/2)\pi^\mu \pi_\mu + V$. The equations (4.2) are equivalent to the standard Euler-Lagrange (EL) equations and by introducing the local vector $S^\mu(\phi(x), x)$ the dynamics can also be

described by the covariant DDW HJ equation and equations of motion

$$\mathfrak{H}\left(\frac{\partial S^\alpha}{\partial\phi},\phi\right)+\partial_\mu S^\mu=0;\ \partial^\mu\phi=\pi^\mu=\frac{\partial S^\mu}{\partial\phi} \tag{4.3}$$

Note here ∂_μ is the partial derivative acting only on the second argument of $S^\mu(\phi(x),x)$; the corresonding total derivative is $d_\mu=\partial_\mu+(\partial_\mu\phi)(\partial/\partial\phi)$. Note that the first equation in (4.3) is a single equation for four quantities S^μ so there is a lot of freedom in finding solutions. Nevertheless the theory is equivalent to other formulations of classical field theory. Now following [**533**] one considers the relation between the covariant HJ equation and the conventional HJ equation; the latter can be derived from the former as follows. Using (4.2), (4.3) takes the form $(1/2)\partial_\phi S_\mu\partial_\phi S^\mu+V+\partial_\mu S^\mu=0$. Then using the equation of motion in (4.3) write the first term as

$$\frac{1}{2}\frac{\partial S_\mu}{\partial\phi}\frac{\partial S^\mu}{\partial\phi}=\frac{1}{2}\frac{\partial S^0}{\partial\phi}\frac{\partial S^0}{\partial\phi}+\frac{1}{2}(\partial_i\phi)(\partial^i\phi) \tag{4.4}$$

Similarly using (4.3) the last term is $\partial_\mu S^\mu=\partial_0 S^0+d_iS^i-(\partial_i\phi)(\partial^i\phi)$. Now introduce the quantity $\mathfrak{S}=\int d^3xS^0$ so $[\partial S^0(\phi(x),x)/\partial\phi(x)]=[\delta\mathfrak{S}([\phi(\mathbf{x},t)],t)/\delta\phi(\mathbf{x},t)]$ where $\delta/\delta\phi(\mathbf{x},t)\equiv[\delta/\delta\phi(x)]_{\phi(x)=\phi(\mathbf{x},t)}$ is the space functional derivative. Putting this together gives then

$$\int d^3x\left[\frac{1}{2}\left(\frac{\delta\mathfrak{S}}{\delta\phi(\mathbf{x},t)}\right)^2+\frac{1}{2}(\nabla\phi)^2+V(\phi)\right]+\partial_t\mathfrak{S}=0 \tag{4.5}$$

which is the standard noncovariant HJ equation. The time evolution of $\phi(\mathbf{x},t)$ is given by $\partial_t\phi(\mathbf{x},t)=\delta\mathfrak{S}/\delta\phi(\mathbf{x},t)$ which arises from the time component of (4.3). Note that in deriving (4.5) it was necessary to use the space part of the equations of motion (4.3) (this does not play an important role in classical physics but is important here). Now for the Bohmian formulation look at the SE $\hat{H}\Psi=i\hbar\partial_t\Psi$ where we write

$$\hat{H}=\int d^3x\left[-\frac{\hbar^2}{2}\left(\frac{\delta}{\delta\phi(\mathbf{x})}\right)^2+\frac{1}{2}(\nabla\phi)^2+V(\phi)\right]; \tag{4.6}$$

$$\Psi([\phi(\mathbf{x})],t)=\Re([\phi(\mathbf{x})],t)e^{i\mathfrak{S}(([\phi(\mathbf{x})],t)/\hbar}$$

Then the complex SE equation is equivalent to two real equations

$$\int d^3x\left[\frac{1}{2}\left(\frac{\delta\mathfrak{S}}{\delta\phi(\mathbf{x})}\right)^2+\frac{1}{2}(\nabla\phi)^2+V(\phi)+Q\right]+\partial_tS=0; \tag{4.7}$$

$$\int d^3x\left[\frac{\delta\Re}{\delta\phi(\mathbf{x})}\frac{\delta\mathfrak{S}}{\delta\phi(\mathbf{x})}+J\right]+\partial_t\Re=0;\ Q=-\frac{\hbar^2}{2\Re}\frac{\delta^2\Re}{\delta\phi^2(\mathbf{x})};\ J=\frac{\Re}{2}\frac{\delta^2\mathfrak{S}}{\delta\phi^2(\mathbf{x})}$$

The second equation is also equivalent to

$$\partial_t\Re^2+\int d^3x\frac{\delta}{\delta\phi(\mathbf{x})}\left(\Re^2\frac{\delta\mathfrak{S}}{\delta\phi(\mathbf{x})}\right)=0 \tag{4.8}$$

and this exhibits the unitarity of the theory because it provides that the norm $\int[d\phi(\mathbf{x})]^2\Psi^*\Psi=\int[d\phi(\mathbf{x})]\Re^2$ does not depend on time. The quantity $\Re^2([\phi(\mathbf{x})],t)$ represents the probability density for fields to have the configuration $\phi(\mathbf{x})$ at time t.

One can take (4.7) as the starting point for quantization of fields (note $exp(i\mathfrak{S}/\hbar)$ should be single valued). Equations (4.7) and (4.8) suggest a Bohmian interpretation with deterministic time evolution given via $\partial_t\phi$. Remarkably the statistical predictions of this deterministic interpretation are equivalent to those of the conventional interpretation. All quantum uncertainties are a consequence of the ignorance of the actual initial field configuration $\phi(\mathbf{x}, t_0)$. The main reason for the consistency of this interpretation is the fact that (4.8) with $\partial_t\phi$ as above represents the continuity equation which provides that the statistical distribution $\rho([\phi(\mathbf{x})], t)$ of field configurations $\phi(\mathbf{x})$ is given by the quantum distribution $\rho = \mathfrak{R}^2$ at any time t, provided that ρ is given by $\mathfrak{R}^2$ at some initial time. The initial distribution is arbitrary in principle but a quantum H theorem explains why the quantum distribution is the most probable (cf. [**954**]). Comparing (4.7) with (4.5) we see that the quantum field satisfies an equation similar to the classical one, with the addition of a term resulting from the nonlocal quantum potential Q. The quantum equation of motion then turns out to be

$$\partial^\mu\partial_\mu\phi + \frac{\partial V(\phi)}{\partial\phi} + \frac{\delta\mathfrak{Q}}{\delta\phi(\mathbf{x};t)} = 0 \tag{4.9}$$

where $\mathfrak{Q} = \int d^3xQ$. A priori perhaps the main unattractive feature of the Bohmian formulation appears to be the lack of covariance, i.e. a preferred Lorentz frame is needed and this can be remedied with the DDW presentation to follow.

Thus one wants a quantum substitute for the classical covariant DDW HJ equation $(1/2)\partial_\phi S_\mu\partial_\phi S^\mu + V + \partial_\mu S^\mu = 0$. Define then the derivative

$$\frac{dA([\phi],x)}{d\phi(x)} = \int d^4x' \frac{\delta A([\phi],x')}{\delta\phi(x)} \tag{4.10}$$

where $\delta/\delta\phi(x)$ is the spacetime functional derivative (not the space functional derivative used before in (4.5)). In particular if $A([\phi],x)$ is a local functional, i.e. if $A([\phi],x) = A(\phi(x),x)$ then

$$\frac{dA(\phi(x),x)}{d\phi(x)} = \int d^4x' \frac{\delta A(\phi(x'),x')}{\delta\phi(x)} = \frac{\partial A(\phi(x),x)}{\partial\phi(x)} \tag{4.11}$$

Thus $d/d\phi$ is a generalization of $\partial/\partial\phi$ such that its action on nonlocal functionals is also well defined. An example of interest is a functional nonlocal in space but local in time so that

$$\frac{\delta A([\phi],x')}{\delta\phi(x)} = \frac{\delta A([\phi],x')}{\delta\phi(\mathbf{x}),x^0)}\delta((x')^0 - x^0) \Rightarrow$$
$$\Rightarrow \frac{dA([\phi],x)}{d\phi(x)} = \frac{\delta}{\delta\phi(\mathbf{x},x^0)}\int d^3x' A([\phi],\mathbf{x}',x^0) \tag{4.12}$$

Now the first equation in (4.3) and the equations of motion become

$$\frac{1}{2}\frac{dS_\mu}{d\phi}\frac{dS^\mu}{d\phi} + V + \partial_\mu S^\mu = 0;\ \partial^\mu\phi = \frac{dS^\mu}{d\phi} \tag{4.13}$$

which is appropriate for the quantum modification. Next one proposes a method of quantization that combines the classical covariant canonical DDW formalism with the standard specetime asymmetric canonical quantization of fields. The starting

point is the relation between the noncovariant classical HJ equation (4.5) and its quantum analogue (4.7). Suppressing the time dependence of the field in (4.5) we see that they differ only in the existence of the Q term in the quantum case. This suggests the following quantum analogue of the classical covariant equation (4.13)

$$\frac{1}{2}\frac{dS_\mu}{d\phi}\frac{dS^\mu}{d\phi} + V + Q + \partial_\mu S^\mu = 0 \tag{4.14}$$

Here $S^\mu = S^\mu([\phi], x)$ is a functional of $\phi(x)$ so S^μ at x may depend on the field $\phi(x')$ at all points x'. One can also allow for time nonlocalities (cf. [**711**]). Thus (4.15) is manifestly covariant provided that Q given by (4.7) can be written in a covariant form. The quantum equation (4.14) must be consistent with the conventional quantum equation (4.7); indeed by using a similar procedure to that used in showing that (4.3) implies (4.5) one can show that (4.14) implies (4.7) provided that some additional conditions are fulfilled. First S^0 must be local in time so that (4.12) can be used. Second S^i must be completely local so that $dS^i/d\phi = \partial S^i/\partial\phi$, which implies

$$d_i S^i = \partial_i S^i + (\partial_i \phi)\frac{dS^i}{d\phi} \tag{4.15}$$

However just as in the classical case in this procedure it is necessary to use the space part of the equations of motion (4.3). Therefore these classical equations of motion must be valid even in the quantum case. Since we want a covariant theory in which space and time play equal roles the validity of the space part of the (4.3) implies that its time part should also be valid. Consequently in the covariant quantum theory based on the DDW formalism one must require the validity of the second equation in (4.13). This requirement is nothing but a covariant version of the Bohmian equation of motion written for an arbitrarily nonlocal S^μ (this clarifies and generalizes results in [**533**]). The next step is to find a covariant substitute for the second equation in (4.7). One introduces a vector $R^\mu([\phi], x)$ which will generate a preferred foliation of spacetime such that the vector R^μ is normal to the leaves of the foliation. Then define

$$\mathfrak{R}([\phi], \Sigma) = \int_\Sigma d\Sigma_\mu R^\mu;\ \mathfrak{S}([\phi], x) = \int_\Sigma d\Sigma_\mu S^\mu \tag{4.16}$$

where Σ is a leaf (a 3-dimensional hypersurface) generated by R^μ. Hence the covariant version of $\Psi = \mathfrak{R}exp(i\mathfrak{S})$ is $\Psi([\phi], \Sigma) = \mathfrak{R}([\phi], \Sigma)exp(i\mathfrak{S}([\phi], \Sigma)/\hbar)$. For R^μ one postulates the equation

$$\frac{dR^\mu}{d\phi}\frac{dS^\mu}{d\phi} + J + \partial_\mu R^\mu = 0 \tag{4.17}$$

In this way a preferred foliation emerges dynamically as a foliation generated by the solution R^μ of the equations (4.17) and (4.14). Note that R^μ does not play any role in classical physics so the existence of a preferred foliation is a purely quantum effect. Now the relation between (4.17) and (4.7) is obtained by assuming that nature has chosen a solution of the form $R^\mu = (R^0, 0, 0, 0)$ where R^0 is local in time. Then integrating (4.17) over d^3x and assuming again that S^0 is local in time

one obtains (4.7). Thus (4.17) is a covariant substitute for the second equation in (4.7). It remains to write covariant versions for Q and J and these are

$$Q = -\frac{\hbar^2}{2\mathfrak{R}}\frac{\delta^2\mathfrak{R}}{\delta_\Sigma\phi^2(x)};\ J = \frac{\mathfrak{R}}{2}\frac{\delta^2\mathfrak{S}}{\delta_\Sigma\phi^2(x)} \tag{4.18}$$

where $\delta/\delta_\Sigma\phi(x)$ is a version of the space functional derivative in which Σ is generated by R^μ. Thus (4.17) and (4.14) with (4.18) represent a covariant substitute for the functional SE equivalent to (4.8). The covariant Bohmain equations (4.13) imply a covariant version of (4.9), namely

$$\partial^\mu\partial_\mu\phi + \frac{\partial V}{\partial\phi} + \frac{dQ}{d\phi} = 0 \tag{4.19}$$

Since the last term can also be written as $\delta(\int d^4xQ)/\delta\phi(x)$ the equation of motion (4.19) can be obtained by varying the quantum action

$$\mathfrak{A}_Q = \int d^4x\mathfrak{L}_Q = \int d^4x(\mathfrak{L} - Q) \tag{4.20}$$

Thus in summary the covariant canonical quantization of fields is given by equations (4.13), (4.14), (4.17), and (4.18). The conventional functional SE corresponds to a special class of solutions for which $R^i = 0,\ S^i$ are local, while R^0 and S^0 are local in time. In [**708**] a multifield generalization is also spelled out, a toy model is considered, and applications to quantum gravity are treated. The main result is that a manifestly covariant method of field quantization based on the DDW formalism is developed which treats space and time on an equal footing. Unlike the conventional canonical quantization it is not formulated in terms of a single complex SE but in terms of two coupled real equations. The need for a Bohmian formulation emerges from the requirement that the covariant method should be consistent with the conventional noncovariant method. This suggests that Bohmian mechanics (BM) might be a part of the formalism without which the covariant quantum theory cannot be formulated consistently.

5. QFT AND STOCHASTIC JUMPS

The most extensive treatment of Bohmian theory is due to a group based in Germany, Italy, and the USA consisting of V. Allori, A. Barut, K. Berndl, M. Daumer, D. Dürr, H. Georgi, S. Goldstein, J. Lebowitz, S. Teufel, R. Tumulka, and N. Zanghi (cf. [**1, 26, 88, 102, 103, 104, 105, 288, 324, 325, 326, 327, 328, 329, 330, 414, 415, 437, 417, 418, 415, 416, 417, 418, 419, 927, 928, 948**]). There is also of course the pioneering work of deBroglie and Bohm (see e.g. [**154, 126, 127, 128, 129, 154**]) as well as important work of many other people (cf. [**68, 94, 95, 110, 138, 148, 164, 165, 166, 186, 187, 188, 189, 191, 197, 198, 236, 277, 295, 298, 305, 306, 346, 347, 373, 374, 375, 438, 472, 471, 474, 478, 479, 480, 873, 905, 953, 961**]). We make no attempt to survey the philosophy of Bohmian mechanics (BM), or better deBroglie-Bohm theory (dBB theory), here. This involves many issues, some of them delicate, which are discussed at length in the references cited. The book [**111**] by Holland provides a good beginning and in view of recent work perhaps another book on this subject alone would be welcome. There is a lot of associated "philosophy",

involving hidden variables, nonlocality, EPR ideas, wave function collapse, pilot waves, implicate order, measurement problems, decoherence, etc., much of which has been resolved or might well be forgotten. Many matters are indeed clarified already in the literature above (cf. in particular [**94, 95, 330, 415, 471**]) and we will not belabor philosophical matters. It may well be that a completely unified mathematical theory is beyond reach at the moment but thre are already quite accurate and workable models available and the philosophy of dBB theory as developed by the American-German-Italian school mentioned is quite sophisticated and convincing.

Basically, following [**415**], for the nonrelativistic theory, GM for N particles is determined by the two equations

$$i\hbar\psi_t = H\psi;\ \frac{dq_k}{dt} = \frac{\hbar}{2m_k}\Im\left[\frac{\psi^*\partial_k\psi}{\psi^*\psi}\right] \tag{5.1}$$

The latter equation is called the guidance or pilot equation which choreographs the motion of the particles. If ψ is spinor valued the products in the numerator and denominator are scalar products and if external magnetic fields are present the gradient $\nabla \sim (\partial_k)$ should be understood as the covariant derivative involving the vector potential (thus accomodating some versions of field theory - more on this later). Since the denominator vanishes at nodes of ψ existence and uniqueness of solutions for Bohmian dynamics is nontrivial but this is proved in [**104, 106**]. This formula extends to spin and the right side corresponds to J/ρ which is the ratio of the quantum probability current to the quantum probability density. Further from the quantum continuity condition $\partial_t\rho + div(J) = 0$ (derivable from the SE) it follows that if the configuration of particles is random at the initial time t_0 with probability distribution $\psi^*\psi$ then this remains true for all times (assuming no interaction with the environment). Upon setting $\psi = Rexp(iS/\hbar)$ one identifies $p_k = m_k v_k$ with $\partial_k S$ (which is equivalent to the guiding equation for particles without spin) and this corresponds to particles being acted upon by the force $\partial_k Q$ generated by the quantum potential (in addition to any "classical" forces).

REMARK 2.5.1. Recall from Section 2.3.2 that in the BFM theory of Bohmian type $\dot{q} = p/m_Q \neq p/m$ where $m_Q = m(1-\partial_E Q)$ in stationary situations with energy E. Here one is using a Floydian time and there has been a great deal of discussion, involving e.g. tunneling times (see e.g. [**138, 139, 140, 191, 194, 197, 198, 305, 306, 307, 296, 309, 347, 373, 374, 375, 376, 520**]). We do not attempt to resolve any issues here and refer to the references for up to date information.

In any event we proceed with BM or dBB theory in full confidence not only that it works but that it is probably the best way to look at QM. We regard the quantum potential Q as being a quantization vehicle which expresses the influence of quantum fluctuations (cf. Chapters 1,4,5); it also arises in describing Weyl curvature (cf. Chapters 4,5) and thus we regard it as perhaps the fundamental object of QM. Returning now to [**415**] one notes that the predictions of

BM for measurements must agree with those of standard QM provided configurations are random with distributions given by the quantum equilibrium distribution $|\psi|^2$. Then a probability distribution ρ^ψ depending on ψ is called equivariant if $(\rho^\psi)_t = \rho^{\psi_t}$ where the right side comes from the SE and the left from the guiding equation (since $\rho_t + div(J) = 0$ with $v = J/\rho$ arises in (5.1)). This has been studied in detail and we summarize some results below. Further BM can handle spin via (5.1) (as mentioned above) and nonlocality is no problem; however Lorentz invariance, even for standard QM, is tricky and one views it as an emergent symmetry. Further QFT with particle creation and annihilation is a current topic of research (cf. Sections 2.3 and 2.4) and some additional remarks in this direction will follow. The papers [**324, 325**] are mainly about quantum equilibrium, absolute uncertainty, and the nature of operators. There are two long papers here (75 and 77 pages) and an earlier paper of 35 pages so we make no attempt to cover this here. We mention briefly some results of the two more recent papers however. Thus from the abstract to the second paper of [**324**] the quantum formalism is treated as a measurement formalism, i.e. a phenomenological formalism describing certain macroscopic regularities. One argues that it can be regarded and best be understood as arising from Bohmian mechanics, which is what emerges from the SE for a system of particles when one merely insists that "particles' means particles. BM is a fully deterministic theory of particles in motion, a motion choreographed by the wave function. One finds that a Bohmian universe, although deterministic, evolves in such a manner that an appearance of randomness emerges, precisely as described by the quantum formalism and given by $\rho = |\psi|^2$. A crucial ingredient in the analysis of the origin of this randomness is the notion of the effective wave function of a subsystem. When the quantum formalism is regarded as arising in this way the paradoxes and perplexities so often associated with (nonrelativistic) quantum theory evaporate. A fundamental fact here is that given a SE $i\hbar\psi_t = -(\hbar^2/2)\sum(\Delta_k\psi/m_k)+V\psi$ one can derive a velocity formula $v_k^\psi = (\hbar/m_k)\Im(\nabla_k\psi/\psi)$ by general arguments based on symmetry considerations and this yields (5.1) without any recourse to a formula $\psi = Rexp(iS/\hbar)$. Further the continuity equation $\rho_t + div(\rho v^\psi) = 0$ holds and this implies the equivariance $\rho(q,t) = |\psi(q,t)|^2$ provided this is true at (t_0, q_0). The distribution $\rho = |\psi|^2$ is called the quantum equilibrium distribution (QELD) and a system is in quantum equilibrium when the QELD is appropriate for its description. The quantum equilibrium hypothesis (QEH) is that if a system has wave function ψ then $\rho = |\psi|^2$. It is necessary to discuss wave functions of systems and subsystems at some length and it is argued that in a universe governed by BM it is impossible to know more about the configuration of any subsystem than what is expressed via $\rho = |\psi|^2$ (despite the fact that for BM the actual configuration is an objective property, beyond the wave function). Moreover, this uncertainty, of an absolute and precise character, emerges with complete ease, the structure of BM being such that it allows for the formulation and clean demonstration of statistical statements of a purely objective character which nontheless imply the claims concerning the irreducible limitations on possible knowledge, whatever this knowledge may precisely mean and however one might attempt to obtain this knowledge, provided it is consistent with BM. This limitation on what can be known is called absolute

uncertainty. One proceeds by analysis of systems and subsystems and we refer to [**324**] for details. In [**325**] one shows how the entire quantum formalism, operators as observables, etc. naturally emerges in BM from the analysis of measurments. It is however quite technical, with considerable important and delicate reasoning, and we cannot possibly deal with it in a reasonable number of pages.

We go to [**326**] now where a comprehensive theory is developed for Bohmian mechanics and QFT (cf. also [**330**]). Bohm and subsequently Bell had proposed such models and the latter is model is modified and expanded in [**326, 330**] in the context of what are called Bell models. One will treat the configuration space variables in terms of Markov processes with jumps (which is reminiscent of the diffusion picture in [**673, 674**] (cf. also [**186**]). Roughly one thinks of world lines involving particle creation and annihilation, hence jumps, and writes $\mathfrak{Q} = \cup_0^\infty \mathfrak{Q}^n$ where, taking identical particles, the sector $\mathfrak{Q}^n$ is best defined as $\mathbf{R}^{3n}/S_n$ where $S \sim$ permutations. For several particle species one forms several copies of $\mathfrak{Q}$, one for each species, and obtains a union of sectors $\mathfrak{Q}^{(n)}$ where now $n \sim (n_1, \ldots, n_k)$ for the k species of particles. Note that a path $Q(t)$ will typically have discontinuities, even if there is nothing discontinuous in the world line pattern, because it jumps to a different sector at every creation or annihilation event. One can think of the bosonic Fock space as a space of L^2 functions on $\cup_n \mathbf{R}^{3n}/S_n$ with the fermionic Fock space being L^2 functions on $\cup_n \mathbf{R}^{3n}$, antisymmetric under permutation. A Bell type QFT specifies such world line patterns or histories in configuration space by specifying three sorts of "laws of motion": when to jump, where to jump, and how to move between jumps. One consequence of these laws (to be enumerated) is the property of preservation of $|\Psi_0|^2$ at time t_0 to be equal to $|\Psi_t|^2$ at time t; this is called equivariance (see above and cf. [**325, 327**] for more detail on equivariance for Bohmian mechanics - the same sort of reasoning will apply here). One will use the quantum state vector Ψ to determine the laws of motion and here a state described by the pair (Ψ_t, Q_t) where Ψ evolves according to the SE $i\hbar\partial_t\Psi_t = H\Psi$. Typically $H = H_0 + H_I$ and it is important to note that although there is an aactual particle number $N(t) = \#Q(t)$ or $Q(t) \in \mathfrak{Q}^{N(t)}$, Ψ need not be a number eigenstate (i.e. concentrated in one sector). This is similar to the usual double-slit experiment in which the particle passes through only one slit although the wavefunction passes through both. As with this experiment, the part of the wave function that passes through another sector of $\mathfrak{Q}$ (or another slit) may well influence the behavior of $Q(t)$ at a later time. The laws of motion of Q_t depend on Ψ_t (and on H) and the continuous part of the motion is governed by

$$\frac{dQ_t}{dt} = v^{\Psi_t}(Q_t) = \Re\frac{\Psi_t^*(Q_t)(\dot{\hat{q}}\Psi_t)(Q_t)}{\Psi_t^*(Q_t)\Psi_t(Q_t)};\ \dot{\hat{q}} = \frac{d}{d\tau}e^{iH_0\tau/\hbar}\Big|_{\tau=0} = \frac{i}{\hbar}[H_0, \hat{q}] \tag{5.2}$$

Here $\dot{\hat{q}}$ is the time derivative of the $\mathfrak{Q}$ valued Heisenberg position operator $\hat{q}$ evolved with H_0 alone. One should understand this as saying that for any smooth function $f : \mathfrak{Q} \to \mathbf{R}$

$$\frac{df(Q_t)}{dt} = \Re\frac{\Psi_t^*(Q_t)(i/\hbar)[H_0, \hat{f}]\Psi_t)(Q_t)}{\Psi_t^*(Q_t)\Psi_t(Q_t)} \tag{5.3}$$

This expression is of the form $v^\Psi \cdot \nabla f(Q_t)$ (as it must be for defining a dynamics for Q_t) if the free Hamiltonian is a differential operator of up to second order (more on this later). **Note that the KG equation is not covered by** (5.2) **or** (5.3). The numerator and denominators above involve, when appropriate, scalar products in spin space. One may view v as a vector field on $\mathfrak{Q}$ and thus as consisting of one vector field v^n on every manifold $\mathfrak{Q}^n$; it is then $v^{N(t)}$ that governs the motion of $Q(t)$ in (5.2). If H_0 were the Schrödinger operator $-\sum_1^n(\hbar^2/2m)\Delta_i + V$ (5.2) yields the Bohm velocities $v_i^\Psi = (\hbar/m_i)\Im[\Psi^*\nabla_i\Psi/\Psi^*\Psi]$. When H_0 is the "second quantization" of a 1-particle Schrödinger operator (5.2) involves equal masses in every sector $\mathfrak{Q}^n$. Similarly in case H_0 is the second quantization of the Dirac opertor $-ic\hbar\vec{\alpha}\cdot\nabla + \beta mc^2$ (5.2) says that a configuration $Q(t)$ (with N particles) moves according to (the N-particle version of) the known variant of Bohm's velocity formula for Dirac wavefunctions $v^\Psi = (\Psi^*\alpha\Psi/\Psi^*\Psi)c$ (cf. [**127**]). The jumps now are stochastic in nature, i.e. they occur at random times and lead to random destinations. In Bell type QFT God does play dice. There are no hidden variables which would fully determine the time and destination of a jump (cf. here Section 2.3 and the effectivity parameters). The probability of jumping, with the next dt seconds to the volume dq in $\mathfrak{Q}$ is $\sigma^\Psi(dq|Q_t)dt$ with

$$\sigma^\Psi(dq|q') = \frac{2}{\hbar}\frac{[\Im\Psi^*(q) < q|H_I|q' > \Psi(q')]^+}{\Psi^*(q')\Psi(q')}dq \tag{5.4}$$

where $x^+ = max(x, 0)$. Thus the jump rate σ^Ψ depends on the present configuration Q_t, on the state vector Ψ_t which has a guiding role similar to that in the Bohm theory, and of course on the overall setup of the QFT as encoded in the interaction Hamiltonian H_I (cf. [**326**] for a simple example). There is a striking similarity between (5.4) and (5.2) in that they are both cases of "minimal" Markov processes associated with a given Hamiltonian (more on this below). When H_0 is replaced by H_I in the right side of (5.3) one obtains an operator on functions $f(q)$ that is naturally associated with the process defined by the jump rates (5.4).

The field operators (operator valued fields on spacetime) provide a connection, the only connection in fact, between spacetime and the abstract Hilbert space containing the quantum states $|\Psi>$, which are usually regarded not as functions but as abstract vectors. What is crucial now is that (i) The field operators naturally correspond to the spatial structure provided by a projection valued (PV) measure on configuration space $\mathfrak{Q}$, and (ii) The process defined here can be efficiently expressed in terms of a PV measure. Thus consider a PV measure P on $\mathfrak{Q}$ acting on $\mathcal{H}$ where for $B \subset \mathfrak{Q}$, $P(B)$ means the projection to the space of states localized in B. Then one can rewrite the formulas above in terms of P and $|\Psi>$ and we get

$$\frac{df(Q_t)}{dt} = \Re\frac{<\Psi|P(dq)\frac{i}{\hbar}[H_0, \hat{f}]|\Psi>}{<\Psi|P(dq)|\Psi>}\Bigg|_{q=Q_t}\ ;\ \hat{f} = \int_{q\in\mathfrak{Q}} f(q)P(dq) \tag{5.5}$$

(for smooth functions $f : \mathfrak{Q} \to \mathbf{R}$) and

$$\sigma^\Psi(dq|q') = \frac{2}{\hbar}\frac{\Im <\Psi|P(dq)H_IP(dq')|\Psi>]^+}{<\Psi|P(dq')|\Psi>} \tag{5.6}$$

Note that $< \Psi|P(dq)|\Psi >$ is the probability distribution analogous to the standard $|\Psi(q)|^2 dq$. The next question is how to obtain the PV measure P from the field operators. Such a measure is equivalent to a system of number operators (more on this below); thus an additive operator valued set function $N(R),\ R \in \mathbf{R}^3$ such that the $N(R)$ commute pairwise and have spectra in the nonnegative integers. By virtue of the canonical commutation and anticommutation relations for the field operators $\phi(\mathbf{x})$ the easiest way to obtain such a system of number operators is via $N(R) = \int_R \phi^*(\mathbf{x})\phi(\mathbf{x})d^3\mathbf{x}$. Thus what one needs from a QFT in order to construct trajectories are: (i) a Hilbert space $\mathcal{H}$ (ii) a Hamiltonian $H = H_0 + H_I$ (iii) a configuration space $\mathfrak{Q}$ (or measurable space), and (iv) a PV measure on $\mathfrak{Q}$ acting on $\mathcal{H}$. This will be done below following [**326**].

We go now to the last paper in [**326**] which is titled quantum Hamiltonians and stochastic jumps. The idea is that for the Hamiltonian of a QFT there is associated a $|\Psi|^2$ distributed Markov process, typically a jump process, on the configuration space of a variable number of particles. A theory is developed generalizing work of J. Bell and the authors of [**326**]. The central formula of the paper is

$$\sigma(dq|q') = \frac{[(2/\hbar)\Im < \Psi|P(dq)HP(dq')|\Psi >]^+}{< \Psi|P(dq')|\Psi >} \tag{5.7}$$

It plays a role similar to that of Bohm's equation of motion

$$\frac{dQ}{dt} = v(Q);\ v = \hbar\Im\frac{\Psi^*\nabla\Psi}{\Psi^*\Psi} \tag{5.8}$$

Together these two equations make possible a formulation of QFT that makes no reference to observers or measurements, while implying that observers, when making measurements, will arrive at precisely the results that QFT is known to predict. This formulation takes up ideas from the seminal papers of J. Bell [**94, 95**] and such theories will be referred to as Bell-type QFT's. The aim is to present methods for constructing a canonical Bell type model for more of less any regularized QFT. One assumes a well defined Hamiltonian as given (with cutoffs included if needed). The primary variables of such theories are particle positions and Bell suggested a dynamical law governing the motion of the particles in which the Hamiltonian H and the state vector Ψ determine the jump rates σ. These rates are in a sense the smallest choice possible (explained below) and are called minimal jump rates; they preserve the $|\Psi|^2$ distribution. Bell type QFT's can also be regarded as extensions of Bohmian mechanics which cover particle creation and annihilation; the quantum equilibrium distribution more or less dictates that creation of a particle occurs in a stochastic manner as in the Bell model. We recall that for Bohmian mechanics in addition to (5.8) one has an evolution equation $i\hbar\partial_t\Psi = H\Psi$ for the wave function with $H = -(\hbar^2/2\Delta + V$ for spinless particles ($\Delta = div\nabla$). For particles with spin Ψ takes values in the appropriate spin space $\mathbf{C}^k$, V may be matrix valued, and inner products in (1.66) are understood as involving inner products in spin spaces. The success of the Bohmian method is based on the preservation of $|\Psi|^2$, called equivariance and this follows immediately from comparing the continuity equation for a probability distribution ρ associated with (5.8), namely $\partial_t\rho = -div(\rho v)$, with the equation for $|\Psi|^2$ following from the

SE, namely

$$\partial_t|\Psi|^2(q,t) = (2/\hbar)\Im[\Psi^*(q,t)(H\Psi)(q,t)] \tag{5.9}$$

In fact it follows from the continuity equation that

$$(2/\hbar)\Im[\Psi^*(q,t)(H\Psi)(q,t)] = -div[\hbar\Im\Psi^*(q,t)\nabla\Psi(q,t)] \tag{5.10}$$

so recalling (5.8), one has $\partial_t|\Psi|^2 = -div(|\Psi|^2 v)$, and hence if $\rho + |\Psi_t|^2$ as some time t there results $\rho = |\psi_t|^2$ for all times. One is led naturaly to the consideration of Markov processes as candidates for the equivariant motion of the configuration Q for more general Hamiltonians (see e.g. [**506, 674, 810, 815**] for Markov processes - [**674**] is especially good for Markov processes with jumps and dynamics but we follow [**506, 815**] for background since the ideas are more or less clearly stated without a deathly deluge of definitions and notation - of course for a good theory much of the verbiage is actually important).

DEFINITION 5.1. Let ($\mathfrak{E}$ be a Borel σ-algebra of subsets of E. For Ω generally a path space (e.g. $\Omega \sim C(\mathbf{R}^+, E)$ with $X_t(\omega) = X(t,\omega) = \omega(t)$ and $\mathfrak{F}_t = \sigma(X_s(\omega),\ s \le t)$ a filtration by Borel sub σ-algebras) a Markov process (**C11**) $X = (\Omega, \{\mathfrak{F}_t\}, \{X_t\}, \{P_t\}, \{P^x,\ x\}E\})$ with $t \ge 0$ and state space $(E, \mathfrak{E})$, is an E valued stochastic process adapted to $\{\mathfrak{F}_t\}$ such that for $0 \le s \le t$, $f \in b\mathfrak{E}$ ($b\mathfrak{E}$ means bounded $\mathfrak{E}$ measurable functions), and $x \in E$, $E^x[f(X_{s+t})|\mathfrak{F}_t] = (P_t f)(X_s)$, $P^x ae$ (ae means almost everywhere). Here $\{P_t\}$ is a transition function on $(E, \mathfrak{E})$, i.e. a family of kernels $P_t : E \times \mathfrak{E} \to [0,1]$ such that

(1) For $t \ge 0$ and $x \in E$, $P_t(x,\cdot)$ is a measure on $\mathfrak{E}$ with $P_t(x,E) \le 1$
(2) For $t \ge 0$ and $\Gamma \in \mathfrak{E}$ $P_t(\cdot,\Gamma)$ is $\mathfrak{E}$ measurable
(3) For $x, t \ge 0$, $x \in E$, and $\Gamma \in \mathfrak{E}$ one has $P_{t+s}(x,\Gamma) = \int_E P_s(x,dy)P_t(y,\Gamma)$

The equation in #3 is called the Chapman-Kolmogorov (CK) equation and, thinking of the transition functions as inducing a family $\{P_t\}$ of positive bounded operators or norm less than or equal to 1 on $b\mathfrak{E}$ one has $P_t f(x) = (P_t f)(x) = \int_E P_t(x,dy)f(y)$ in which case the CK equation has the semigroup property $P_s P_t = P_{s+t}$ for $s, t \ge 0$.

Under mild regularity conditions if a transition semigroup $\{P_t\}$ is given there will exist on some probability space a Markov process X with suitable paths such that the strong Markov property holds, i.e. $E^x[f(X_{S+t})]|\mathfrak{F}_x] = (P_t f)(X_S)\ P^x\ ae$ whenever S is a finite stopping time (here $S : \Omega \to [0,\infty]$ is a $\mathfrak{E}$ stopping time if $\{S \le t\} = \{\omega;\ S(\omega) \le t\} \in \mathfrak{E}_t$ for every $t < \infty$).

EXAMPLE 5.1. A Markov process with countable state space is called a Markov chain. One writes $p_{ij}(t) = P_t(i,\{j\})$ with $P(t) = \{p_{ij}(t);\ i,j \in E\}$. Assume $P_t(i,E) = 1$ and $p_{ij}(t) \to \delta_{ij}$ as $t \downarrow 0$. This will imply that in fact $p'_{ij}(0) = q_{ij}$ exists and the matrix $Q = (q_{ij})$ is called an infinitesimal generator of $\{P_t\}$ with $q_{ij} \ge 0$ $(i \ne j)$ and $\sum_j q_{ik} = 0$ $(i \in E)$. This illustrates some important structure for Markov processes. Thus when $P'(0) = Q$ exists one can write $P'(t) = lim_{\epsilon\to 0}\epsilon^{-1}[P(t+\epsilon) - P(t)] = lim_{\epsilon\to 0}P(t)[P(\epsilon) - I] = P(t)Q$. Then solving this equation one has $P(t) = exp(tQ$ as a semigroup generated by Q. The

resolvent is defined via $R_\lambda = \int_0^\infty exp(-\lambda t)P_t dt$ and one can regard it as

$$(\lambda R_\lambda)_{ij} = \int_0^\infty \lambda exp(-\lambda t)p_{ij}(t)dt = \mathbf{P}(X_{\mathbf{T}} = j | X_0 = i) \tag{5.11}$$

where $\mathbf{T}$ is a random variable independent of X with the exponential distribution of rate λ. It follows then that $R_\lambda = (\lambda - Q)^{-1}$ and $R_\lambda - R_\mu = (\mu - \lambda)R_\lambda R_\mu$. The whole subject is full of pathological situations however and we make no attempt to describe this.

REMARK 2.5.2. [**674**] is oriented toward diffusion processes and departs from the concept that the kinematics of quantum particles is stochastic calculus (in particular Markov processes) while the kinematics of classical particles is classical differential calculus. The relation between these two calculi must be established. Thus classically $x(t) = x(a) + \int_a^t v(s, x(s))ds$ while for a particle with say Brownian noise B_t and a drift field $a(t, x(t))$ one has

$$X_t = X_a + \int_a^t a(s, X_s)ds + \int_a^t \sigma(s, X_s)dB_s \tag{5.12}$$

We recall $dB_t^2 \sim dt$ so X_t has no velocity and the drift field $a(t, x)$ is not an average speed. However $P[X] = \int_\Omega X dP$ is the expectation (since $P[\sigma(t, X_t)dB_t] = 0$). Now the notation for Markov processes involves nonnegative transition functions $P(s, x; t, B)$ with $a \leq s \leq t \leq b,\ x \in \mathbf{R}^d$, and $B \in \mathfrak{B}(\mathbf{R}^d)$ which are measures in B, measurable in x, and satisfy the CK equation

$$P(s, x; t, B) = \int_{\mathbf{R}^d} P(s, x; r, dy)P(r, y; t, B);\ P(s, x; t, \mathbf{R}^d) = 1 \tag{5.13}$$

If there is a measurable function p such that $P(s, x; t, B) = \int_B p(s, x; t, y)dy$ $(t - s > 0)$ then p is a transition density. One defines a probability measure P on a path space $\Omega = (\mathbf{R}^d)^{[a,b]}$ via finite dimensional distributions

(5.14)

$$P[f(X_a, X_{t_1}, \cdots, X_{t_n}, X_b)] = \int \mu_a(dx_0)P(a, x_0; t_1, dx_1)P(t_1, x_1; t_2, dx_2) \cdots \times$$

$$\times \cdots P(t_{n-1}, x_{n-1}; b, dx_n)f(x_0, \cdots, x_n)$$

Moreover one defines a family $\{X_t;\ t \in [a, b]\}$ on Ω via $X_t(\omega) = \omega(t),\ \omega \in \Omega$. Note one assumes the right continuity of $X_t(\omega)$ ae. This representation can be written as $P = [\mu_a P >>$ and is called the Kolmogorov representation of P. Let now $\{\mathfrak{F}_s^t\}$ be a filtration as before, i.e. a family of σ-fields generated by $\{X_r(\omega);\ s \leq r \leq t\}$. Then we have a Markov process $\{X_t, t \in [a, b], \mathfrak{F}_s^t, P\}$. Replacing μ_a by δ_x and a by s with $s < t_1 < \cdots < t_{n-1} < t_n \leq b$ one defines probability measures $P_{(s,x)},\ (s, x) \in [a, b] \times \mathbf{R}^d$ from (1.67) via

$$P_{(s,x)}[f(X_{t_1}, \cdots, X_{t_{n-1}}, X_{t_n})] = \tag{5.15}$$

$$= \int P(s, x; t_1, dx_1) \cdots P(t_{n-1}, x_{n-1}; t_n, dx_n)f(x_1, \cdots, x_n)$$

As a special case one has $P_{(s,x)}[f(t, X_t)] = \int P(s, x; t, dy)f(t, y)$ and one can also prove that $P[GF] = P[GP_{(s,X_s)}[F]]$ for any bounded $\mathfrak{F}_a^s$ measurable G and any bounded $\mathfrak{F}_s^b$ measurable F. This is the time inhomogeneous Markov property which

can be written in terms of conditional expectations as $P[F|\mathfrak{F}_a^s] = P_{(s,X_s)}[F],\ P\ ae.$ There is a great deal of material in [**674**] about Markov processes with jumps but we prefer to stay here with [**326**] for notational convenience.

Going back to [**326**] we consider a Markov process Q_t on configuration space with transition probabilities characterized by the backward generator L_t, a time dependent linear operator acting on functions f via $L_t f(q) = (d/ds)E(f(Q_{t+s}|Q_t = q)$ where d/ds means the right derivative at $s = 0$ and $E(\cdot|\cdot)$ is conditional expectation. Equivalently the transition probabilities are characterized by the forward generator $\mathcal{L}_t$ (or simply generator) which is also a linear operator but acts on (signed) measures on the configuration space. Its defining property is that for every process Q_t with the given transition properties $\partial_t \rho_t = \mathcal{L}_t \rho_t$. Thus $\mathcal{L}$ is dual to L_t in the sense

$$\int f(q)\mathcal{L}_t \rho(dq) = \int L_t f(q)\rho(dq) \tag{5.16}$$

Given equivariance for $|\Psi|^2$, one says that the corresponding transition probabilities are equivariant and this is equivalent to $\mathcal{L}_t|\Psi|^2 = \partial_y|\Psi|^2$ for all t; when this holds one says that $\mathcal{L}_t$ is an equivariant generator (with respect to Ψ_t and H). One says that a Markov process is Q equivariant if and only if for every t the distribution ρ_t of Q_t equals $|\Psi_t|^2$. For this equivariant transition probabilities are necessary but not sufficient; however for equivariant transition probabilities there is a unique equivariant Markov process. The crucial idea here for construction of an equivariant Markov process is to note that (5.9) is completely general and to find a generator $\mathcal{L}_t$ such that the right side of (5.9) can be read as the action of $\mathcal{L}$ on $\rho = |\Psi|^2$ means $(2/\hbar)\Im\Psi^* H\Psi = \mathcal{L}|\Psi|^2$. This will be implemented later. For H of the form $-(\hbar^2/2)\Delta + V$ one has (5.10) and hence

$$\frac{2}{\hbar}\Im\Psi^* H\Psi = -div(\hbar\Im\Psi^*\nabla\Psi) = -div\left(|\Psi|^2\hbar\Im\frac{\Psi^*\nabla\Psi}{|\Psi|^2}\right) \tag{5.17}$$

Since the generator of the (deterministic) Markov process corresponding to the dynamical system $dQ/dt = v(Q)$ is given by a velocity vector field is $\mathcal{L}\rho = -div(\rho v)$ we may recognize the last term of (1.67) as $\mathcal{L}|\Psi|^2$ with $\mathcal{L}$ the generator of the deterministic process defined by (5.8). Thus Bohmian mechanics arises as the natural equivariant process on configuration space associated with H and Ψ. One notes that Bohmian mechanics is not the only solution of $(2/\hbar)\Im\Psi^* H\Psi - \mathfrak{L}|\Psi|^2$; there are alternatives such as Nelson's stochastic mechanics (and hence Nagasawa's theory of [**672, 674**]) and other velocity formulas (cf. [**295**]).

For equivariant jump processes one says that a (pure) jump process is a Markov process on $\mathfrak{Q}$ for which the only motion that occurs is via jumps. Given that $Q_t = q$ the probability for a jump to q' (i.e. into the infinitesimal volume dq' around q') by time $t+dt$ is $\sigma_t(d'q|q)dt$ where σ is called the jump rate. Here σ is a finite measure in the first variable; $\sigma(B|q)$ is the rate (i.e. the probability per unit time) of jumping to somewhere in the set $B \subset \mathfrak{Q}$ given that the present location is q. The overall jump rate is $\sigma(\mathfrak{Q}|q)$ (sometimes one writes $\rho(dq) = \rho(q)dq$). A jump first occurs when a random waiting time T has elapsed, after the time t_0 at which the process

was started or at which the most recent previous jump has occured. For purposes of simulating or constructing the process, the destination q' can be chosen at the time of jumping, $t_0 + T$, with probability distribution $\sigma_{t_0+T}(\mathfrak{Q}|q)^{-1}\sigma_{t_0+T}(\cdot|q)$. In case the overall jump rate is time independent T is exponentially distributed with mean of $\sigma(\mathfrak{Q}|q)^{-1}$. When the rates are time dependent (as they will typically in what follows) the waiting time remains such that $\int_{t_0}^{t_0+T} \sigma_t(\mathfrak{Q}|q)dt$ is exponentially distributed with mean 1, i.e. T becomes exponential after a suitable (time dependent) rescaling of time. The generator of a pure jump process can be expressed in terms of the rates

$$\mathcal{L}\rho(dq) = \int_{q'\in\mathfrak{Q}} (\sigma(dq|q')\rho(dq') - \sigma(dq')|q)\rho(dq)) \tag{5.18}$$

which is a balance or master equation expressing $\partial_t\rho$ as the gain due to jumps to dq minus the loss due to jumps away from dq. One says the jump rates are equivariant if $\mathcal{L}_\sigma$ is an equivariant generator.

Given a Hamiltonian $H = H_0 + H_I$ one obtains

$$(2/\hbar)\Im\Psi^* H_0\Psi + (2/\hbar)\Im\Psi^* H_I\Psi - \mathcal{L}|\Psi|^2 \tag{5.19}$$

This opens the possibility of finding a generator $\mathcal{L} = \mathcal{L}_0+\mathcal{L}_I$ given $(2/\hbar)\Im\Psi^* H_0\Psi = \mathcal{L}_0|\Psi|^2$ and $(2/\hbar)\Im\Psi^* H_I\Psi = \mathcal{L}_I|\Psi|^2$; this will be called process additivity and correspondingly $L = L_0 + L_I$. If one has two deterministic processes of the form $\mathcal{L}\rho = -div(\rho v)$ then adding generators corresponds to $v = v_+v_2$. For a pure jump process adding generators corresponds to adding rates σ_i which is equivalent to saying there are two kinds of jumps. Now add generators for a deterministic and a jump process via

$$\mathcal{L}\rho(q) = -div(\rho v)(q) + \int_{q'\in\mathfrak{Q}} (\sigma(q|q')\rho(q') - \sigma(q'|q)\rho(q))\, dq' \tag{5.20}$$

This process moves with velocity $v(q)$ until it jumps to q' where it continues moving with velocity $v(q')$. One can understand (5.20) in terms of gain or loss of probability density due to motion and jumps; the process is piecewise deterministic with random intervals between jumps and random destinations. Note that for a Wiener process the generator is the Laplacian and adding to it the generator of a deterministic process means introducing a drift.

Now consider H_I and note that in QFT's with cutoffs it is usually the case that H_I is an integral operator. Hence one writes here $H \sim H_I$ and thinks of it as an integral operator with $\mathfrak{Q} \sim \mathbf{R}^n$. What characterizes jump processes is that some amount of probability that vanishes at $q \in \mathfrak{Q}$ can reappear in an entirely different region say at $q' \in \mathfrak{Q}$. This suggests that the Hamiltonians for which the expression (5.9) for $\partial_t|\Psi|^2$ is naturally an integral over q' correspond to pure jump processes. Thus when is the left side of $(2/\hbar)\Im|psi^* H\Psi = \mathfrak{L}|\Psi|^2$ an integral over q' or $(H\Psi)(q) = \int dq' < q|H|q' > \Psi(q')$. In this case one should choose the jump rates so that when $\rho = |\Psi|^2$ one has

$$\sigma(q|q')\rho(q') - \sigma(q'|q)\rho(q) = (2/\hbar)\Im\Psi^*(q) < q|H|q' > \Psi(q') \tag{5.21}$$

This suggests, since jump rates are nonnegative and the right side of (5.21) is antisymmetric) that $\sigma(q|q')\rho(q') = [(2/\hbar)\Im\Psi^*(q) < q|H|q' > \Psi(q')]^+$ or

$$\sigma(q|q') = \frac{(2/\hbar)\Im\Psi^*(q) < q|H|q' > \Psi(q')]^+}{\Psi^*(q')\Psi(q')} \tag{5.22}$$

These rates are an instance of what can be called minimal jump rates associated with H (and Ψ). They are actually the minimal possible values given (5.21) and this is discussed further in [**326**]. Minimality entails that at any time t one of the transitions $q_1 \to q_2$ or $q_2 \to q_1$ is forbidden and this will be called a minimal jump process. One summarizes motions via

H	*motion*
integral operator	*jumps*
differential operator	*deterministic continuous motion*
multiplication operator	*no motion* ($\mathcal{L} = 0$)

The reasoning above applies to the more general setting of arbitrary configuration spaces $\mathfrak{Q}$ and generalized observables - POVM's - defining what the "position" representation is to be. One takes the following ingredients from QFT

(1) A Hilbert space $\mathcal{H}$ with scalar product $< \Psi|\Phi >$.
(2) A unitary one parameter group U_t in $\mathcal{H}$ with Hamiltonian H, i.e. $U_t = exp[-(i/\hbar)tH]$, so that in the Schrödinger picture the state Ψ evolves via $i\hbar\partial_t\Psi = H\Psi$. U_t could be part of a representation of the Poincaré group.
(3) A positive operator valued measure (POVM) $P(dq)$ on $\mathfrak{Q}$ acting on $\mathcal{H}$ so that the probability that the system in the state Ψ is localized in dq at time t is $\mathbf{P}_t(dq) =< \Psi_t|P(dq)|\Psi_t >$.

Mathematically a POVM on $\mathfrak{Q}$ is a countably additive set function (measure) defined on measurable subsets of $\mathfrak{Q}$ with values in the positive (bounded self adjoint) operators on a Hilbert space $\mathcal{H}$ such that $P(\mathfrak{Q}) = Id$. Physically for purposes here $P(\cdot)$ represents the (generalized) position observable, with values in $\mathfrak{Q}$. The notion of POVM generalizes the more familiar situation of observables given by a set of commuting self adjoint operators, corresponding by means of the spectral theorm to a projection valued measure (PVM) - the case where the positive operators are projection operators (see [**326**] for discussion). The goal now is to specify equivariant jump rates $\sigma = \sigma^{\Psi,H,P}$ so that $\mathcal{L}_\sigma\mathbf{P} = d\mathbf{P}/dt$. To this end one could take the following steps.

(1) Note that $(d\mathbf{P}_t(dq)/dt) = (2/\hbar)\Im < \Psi_t|P(dq)H|\Psi_t >$.
(2) Insert the resolution of the identity $I = \int_{q'\in\mathfrak{Q}} P(dq')$ and obtain

$$(d\mathbf{P}_t(dq)/dt) = \int_{q'\in\mathfrak{Q}} \mathbf{J}_t(dq, dq'); \tag{5.23}$$

$$\mathbf{J}_t(dq, dq') = (2/\hbar)\Im < \Psi_t|P(dq)HP(dq')|\Psi_t >$$

(3) Observe that **J**is antisymmetric so since $x = x^+ - (-x)^+$ one has

$$\mathbf{J}(dq, dq') = [(2/\hbar)\Im < \Psi|P(dq)HP(dq')|\Psi]^+ - \tag{5.24}$$

$$-[(2/\hbar)\Im < \Psi|P(dq')HP(dq)|\Psi >]^+$$

(4) Multiply and divide both terms by $\mathbf{P}(\cdot)$ obtaining

$$\int_{q'\in\mathfrak{Q}} \mathbf{J}(dq,dq') = \int_{q'\in\mathfrak{Q}} \Big(\frac{(2/\hbar)\Im < \Psi|P(ddq)HP(dq')|\Psi >]^+}{< \Psi|P(dq')|\Psi >} \mathbf{P}(dq') - \\ - \frac{[(2/\hbar)\Im < \Psi|P(dq')HP(dq)|\Psi >]^+}{< \Psi|P(dq)|\Psi >} \mathbf{P}(dq) \Big) \tag{5.25}$$

(5) By comparison with (5.18) recognize the right side of the above equation as $\mathcal{L}_\sigma \mathbf{P}$ with $\mathcal{L}_\sigma$ the generator of a Markov jump process with jump rates (5.7) (minimal jump rates).

Note the right side of (5.7) should be understood as a density (Radon-Nikodym derivative).

When H_0 is made of differential operators of up to second order one can characterize the process associated with H_0 in a particularly succinct manner as follows. Define for any H, P, Ψ an operator L acting on functions $f : \mathfrak{Q} \to \mathbf{R}$ which may or may not be the backward generator of a process via

$$Lf(q) = \Re \frac{< \Psi|P(dq)\hat{L}\hat{f}|\Psi >}{< \Psi|P(dq)\Psi >} = \Re \frac{< \Psi|P(dq)(i/\hbar)[H,\hat{f}]|\Psi >}{< \Psi|P(dq)|\Psi >} \tag{5.26}$$

where [,] means the commutator and $\hat{f} = \int_{q\in\mathfrak{Q}} f(q)P(dq)$ with $\hat{L}$ the generator of the (Heisenberg) evolution $\hat{f}$,

$$\hat{L}\hat{f} = (d/d\tau)exp(iH\tau/\hbar)\hat{f}exp(-iH\tau/\hbar)|_{\tau=0} = (i/\hbar)[H,\hat{f}] \tag{5.27}$$

Note if P is a PVM then $\hat{f} = f(\hat{q})$. (5.26) could be guessed in the following manner: Since Lf is in a certain sense the time derivative of f it might be expected to be related to $\hat{L}\hat{f}$ which is in a certain sense (cf. (5.27)) the time derivative of $\hat{f}$. As a way of turning the operator $\hat{L}\hat{f}$ into a function $Lf(q)$ the middle term in (5.26) is an obvious possibility. Note also that this way of arriving at (5.26) does not make use of equivariance. The formula for the forward generator equivalent to (5.26) reads

$$\mathcal{L}\rho(dq) = \Re < \Psi|\widehat{(d\rho/d\mathbf{P})}(i/\hbar)[H,P(dq)]|\Psi > \tag{5.28}$$

Whenever L is indeed a backward generator we call it the minimal free (backward) generator associated with Ψ, H, P. Then the corresponding process is equivariant and this is the case if (and there is reason to expect, only if) P is a PVM and H is a differential operator of up to second order in the position representation, in which P is diagonal. In that case the process is deterministic and the backward generator has the form $L = v \cdot \nabla$ where v is the velocity field; thus (5.26) directly specifies the velocity in the form of a first order differential operator $v \cdot \nabla$. In case H is the N-particle Schrödinger operator with or without spin (5.26) yields the Bohmian velocity (5.8) and if H is the Dirac operator the Bohm-Dirac velocity emerges. Thus in some cases (5.26) leads to just the right backward generator. In [**326**] there are many examples and mathematical sections designed to prove various assertions but we omit this here..

6. BOHMIAN MECHANICS IN QFT

We extract here from a fascinating paper [**713**] by H. Nikolić. Quantum field theory (QFT) can be formulated in the Schrödinger picture by using a functional time dependent SE but this requires a choice of time coordinate and the corresponding choice of a preferred foliation of spacetime producing a relativistically noncovariant theory. The problem of noncovariance can be solved by replacing the usual time dependent SE with the many fingered time (MFT) Tomonaga-Schwinger equation, which does not require a preferred foliation and the quantum state is a functional of an arbitrary timelike hypersurface. In a manifestly covariant formulation introduced in [**316**] the hypersurface does not even have to be timelike. The paper [**713**] develops a Bohmian interpretation for the MFT theory for QFT and refers to [**77, 108, 255, 473, 478, 587, 711, 708, 769, 772, 774, 819, 820, 876, 910**] for background and related information. Thus let $x = \{x^\mu\} = (x^0, \mathbf{x})$ be spacetime coordinates. A timelike Cauchy hypersurface Σ can be defined via $x^0 = T(\mathbf{x})$ with $\mathbf{x}$ denoting coordinates on Σ. Let $\phi(\mathbf{x})$ be a dynamical field on Σ (a real scalar field for convenience) and write T, ϕ without an argument for the functions themselves with $\phi = \phi|_\Sigma$ etc. Let $\hat{\mathfrak{H}}(\mathbf{x})$ be the Hamiltonian density operator and then the dynamics of a field ϕ is described by the MFT Tomonaga-Schwinger equation

$$\hat{\mathfrak{H}}\Psi[\phi, T] = i\frac{\delta\Psi[\phi, T]}{\delta T(\mathbf{x})} \tag{6.1}$$

Note $\delta T(\mathbf{x})$ denotes an infinitesimal change of the hypersurface Σ. The quantity $\rho[\phi, T] = |\Psi[\phi, T]|^2$ represents the probability density for the field to have a value ϕ on Σ or equivalently the probability density for the field to have a value ϕ at time T. One can say that ϕ has a definite value φ at some time T_0 if

$$\Psi[\phi, T_0] = \delta(\phi - \varphi) = \prod_{\mathbf{x}\in\Sigma} \delta(\phi(\mathbf{x}) - \varphi(\mathbf{x})) \tag{6.2}$$

[**713**] then provides an important discussion of measurement and contextuality in QM which we largly omit here in order to go directly to the Bohmian formulation.

For simplicity take a free scalar field with

$$\hat{\mathfrak{H}}(\mathbf{x}) = -\frac{1}{2}\frac{\delta^2}{\delta\phi^2(\mathbf{x})} + \frac{1}{2}[(\nabla\phi(\mathbf{x}))^2 + m^2\phi^2(\mathbf{x})] \tag{6.3}$$

Writing $\Psi = Rexp(iS)$ with R and S real functionals the complex equation (6.1) is equivalent to two real equations with

$$\frac{1}{2}\left(\frac{\delta S}{\delta\phi(\mathbf{x})}\right)^2 + \frac{1}{2}[(\nabla\phi(\mathbf{x}))^2 + m^2\phi^2(\mathbf{x})] + \mathfrak{Q}(\mathbf{x}, \phi, T] + \frac{\delta S}{\delta T(\mathbf{x})} = 0; \tag{6.4}$$

$$\frac{\delta\rho}{\delta T(\mathbf{x})} + \frac{\delta}{\delta\phi(\mathbf{x})}\left(\rho\frac{\delta S}{\delta T(\mathbf{x})}\right) = 0;\ \mathfrak{Q}(\mathbf{x}, \phi, T] = -\frac{1}{2R}\frac{\delta^2 R}{\delta\phi^2(\mathbf{x})}$$

The conservation equation shows that it is consistent to interpret $\rho[\phi, T]$ as the probability density for the field to have the value ϕ at the hypersurface determined

by the time T. Now let σ_x be a small region around $\mathbf{x}$ and define the derivative

$$\frac{\partial}{\partial T(\mathbf{x})} = \lim_{\sigma_x \to 0} \int_{\sigma_x} d^3x \frac{\delta}{\delta T(\mathbf{x})} \tag{6.5}$$

where $\sigma_x \to 0$ means that the 3-volume goes to zero (note $\partial T(\mathbf{y})/\partial T(\mathbf{x}) = \delta_{xy}$). It is convenient to integrate (6.4) inside a small σ_x leading to

$$\frac{\partial \rho}{\partial T(\mathbf{x})} + \frac{\partial}{\partial \phi(\mathbf{x})} \left(\rho \frac{\delta S}{\delta \phi(\mathbf{x})} \right) = 0 \tag{6.6}$$

where $\partial/\partial\phi(\mathbf{x})$ is defined as in (6.5). The Bohmian interpretation consists now in introducing a deterministic time dependent hidden variable such that the time evolution of this variable is consistent with the probabilistic interpretation of ρ. From (6.6) one sees that this is naturally achieved by introducing a MFT field $\Phi(x, T]$ that satisfies the MFT Bohmian equations of motion

$$\frac{\partial \Phi(\mathbf{x}, T]}{\partial T(\mathbf{x})} = \left. \frac{\delta S}{\delta \phi(\mathbf{x})} \right|_{\phi=\Phi} \tag{6.7}$$

From (6.7) and the quantum MFT HJ equation (6.4) results

$$\left[\left(\frac{\partial}{\partial T(\mathbf{x})} \right)^2 - \nabla_x^2 + m^2 \right] \Phi(\mathbf{x}, T] = - \left. \frac{\partial \mathfrak{Q}(\mathbf{x}, \phi, T]}{\partial \psi(\mathbf{x})} \right|_{\phi=\Phi} \tag{6.8}$$

This can be viewed as a MFT KG equation modified with a nonlocal quantum term on the right side. The general solution of (6.7) has the form

$$\Phi_{gen}(\mathbf{x}), T] = F(\mathbf{x}, c(\mathbf{x}, T]; T] \tag{6.9}$$

where F is a function(al) that depends on the right side of (6.7) and $c(\mathbf{x}, T]$ is an arbitrary function(al) with the property

$$\frac{\delta c(\mathbf{x}, T]}{\delta T(\mathbf{x})} = 0 \tag{6.10}$$

This quantity can be viewed as an arbitrary MFT integration constant - it is constant in the sense that it does not depend on $T(\mathbf{x})$, but it may depend on T at other points $\mathbf{x}' \neq \mathbf{x}$. To provide the correct classical limit (indicated below) one restricts $c(\mathbf{x}, T]$ to satisfy

$$c(\mathbf{x}, T] = c(\mathbf{x}) \tag{6.11}$$

where $c(\mathbf{x})$ is an arbitrary function. Here it is essential to realize that $\Phi(\mathbf{x}, T]$ is a function of $\mathbf{x}$ but a functional of T; the field Φ depends not only on $(\mathbf{x}, T(\mathbf{x})) \equiv (\mathbf{x}, x^0) \equiv x$ but also on the choice of the whole hypersurface Σ that contains the point $\mathbf{x}$. Consequently the MFT Bohmian interpretation does not in general assign a value of the field at the point x unless the whole hypersurface containing x is specified. On the other hand if e.g. $\delta S/\delta\phi(\mathbf{x})$ on the right side in (6.7) is a local functional, i.e. of the form $V(\mathbf{x}, \phi(\underline{x}), T(\mathbf{x})$, then the solution of (6.7) is a local functional of the form

$$\Phi(\mathbf{x}, T(\mathbf{x}) = \Phi(\mathbf{x}, x^0) = \Phi(x) \tag{6.12}$$

This occurs for example when the wave functional is a local product $\Psi[\phi, T] = \prod_{\mathbf{x}} \psi_{\mathbf{x}}(\phi(\mathbf{x}, T(\mathbf{x})$. Interactions with the measuring apparatus can also produce

locality. As for the classical limit one can formulate the classical HJ equation as a MFT theory (cf. [**819, 820**]) without of course the $\mathfrak{Q}$ term. Hence by imposing a restriction similar to (6.11) the solution $S[\phi, T]$ can be chosen so that $\delta S/\delta\phi(\mathbf{x})$ is a local functional; the restriction (6.11) again implies that the classical solution Φ is also a local functional.

The MFT formalism was introduced by Tomonaga and Schwinger to provide the manifest covariance of QFT in the interaction picture. The picture here is so far not manifestly covariant since time is not treated on an equal footing with space. However the MFT formalism can be here also in a manifestly covariant manner via [**316, 819, 820**]. One starts by introducing a set of 3 real parameters $\{s^1, s^2, s^3\} \equiv \mathbf{s}$ to serve as coordinates on a 3-dimensional manifold (a priori $\mathbf{s}$ is not related to $\mathbf{x}$). The 3-dimensional manifold Σ can be embedded in the 4-dimensional spacetime by introducing 4 functions $X^\mu(\mathbf{s})$ and a 3-dimensional hypersurface is given via $x^\mu = X^\mu(\mathbf{s})$. The 3 parameters s^i can be eliminated leading to an equation of the form $f(x^0, x^1, x^2, x^3) = 0$ and assuming that the background spacetime metric $g_{\mu\nu}(x)$ is given the induced metric $q_{ij}(\mathbf{s})$ on this hypersurface is

$$q_{ij}(\mathbf{s}) = g_{\mu\nu}(X(\mathbf{s}))\frac{\partial X^\mu(\mathbf{s})}{\partial s^i}\frac{\partial X^\nu(\mathbf{s})}{\partial s^j} \tag{6.13}$$

Similarly a normal to the surface is

$$\tilde{n}(\mathbf{s}) = \epsilon_{\mu\alpha\beta\gamma}\frac{\partial X^\alpha}{\partial s^1}\frac{\partial X^\beta}{\partial s^2}\frac{\partial X^\gamma}{\partial s^3} \tag{6.14}$$

and the unit normal transforming as a spacetime vector is

$$n^\mu(\mathbf{s}) = \frac{g^{\mu\nu}\tilde{n}_\nu}{\sqrt{|g^{\alpha\beta}\tilde{n}_\alpha\tilde{n}_\beta|}} \tag{6.15}$$

Now some of the original equations above can be written in a covariant form by making the replacements

$$\mathbf{x} \to \mathbf{s};\ \frac{\delta}{\delta T(\mathbf{x})} \to n^\mu(\mathbf{s})\frac{\delta}{\delta X^\mu(\mathbf{s})} \tag{6.16}$$

The Tomonaga-Schwinger equation (6.1) becomes

$$\hat{\mathfrak{H}}(\mathbf{s})\Psi[\phi, X] = in^\mu(\mathbf{s})\frac{\delta\Psi[\phi, X]}{\delta X^\mu(\mathbf{s})} \tag{6.17}$$

For free fields the Hamiltonian density operator in curved spacetime is

$$\hat{\mathfrak{H}} = \frac{-1}{2|q|^{1/2}}\frac{\delta^2}{\delta\phi^2(\mathbf{s})} + \frac{|q|^{1/2}}{2}[-q^{ij}(\partial_i\phi)(\partial_j\phi) + m^2\phi^2] \tag{6.18}$$

The Bohmian equations of motion (6.7) becomes

$$\frac{\partial\Phi(\mathbf{s}, T]}{\partial\tau(\mathbf{s})} = \frac{1}{|q(\mathbf{s})|^{1/2}}\frac{\delta S}{\delta\phi(\mathbf{s})}\bigg|_{\phi=\Phi};\ \frac{\partial}{\partial\tau(\mathbf{s})} \equiv lim_{\sigma_x\to 0}\int_{\sigma_x} d^3s n^\mu(\mathbf{s})\frac{\delta}{\delta X^\mu(\mathbf{s})} \tag{6.19}$$

Similarly (6.8) becomes

$$\left[\left(\frac{\partial}{\partial\tau(\mathbf{s})}\right)^2+\nabla^i\nabla_i+m^2\right]\Phi(\mathbf{s},X]=-\frac{1}{|q(\mathbf{s})|^{1/2}}\frac{\partial\mathfrak{Q}(\mathbf{s},\phi,X]}{\partial\phi\mathbf{s})}\Big|_{\phi=\Phi} \tag{6.20}$$

where ∇_i is the covariant derivative with respect to s^i and

$$\mathfrak{Q}(\mathbf{s},\phi,X]=-\frac{1}{|q(\mathbf{s})|^{1/2}}\frac{1}{2R}\frac{\delta^2R}{\delta\phi^2(\mathbf{s})} \tag{6.21}$$

corresponding to a quantum potential. The same hypersurface Σ can be parametrized by different sets of 4 functions $X^\mu(\mathbf{s})$ of course but quantities such as $\Psi[\phi,X]$ and $\Phi(\mathbf{s},X]$ depend on Σ, not on the parametrization. The freedom in choosing functions $X^\mu(\mathbf{s})$ is sort of a gauge freedom related to the covariance. Now to find a solution of the covariant equations above it is convenient to fix a gauge and for a timelike surface the simplest choice is $X^i(\mathbf{s})=s^i$. This implies $\delta X(\mathbf{s})=0$ which leads to equations similar to those obtained previously. For example (6.19) becomes

$$(g^{00}(\mathbf{x}))^{1/2}\frac{\partial\Phi(\mathbf{x},X^0]}{\partial X^0(\mathbf{x})}=\frac{1}{|q(\mathbf{x})|^{1/2}}\frac{\delta S}{\delta\phi(\mathbf{x})}\Big|_{\phi=\Phi} \tag{6.22}$$

which is the curved spacetime version of (6.7).