CHAPTER 1

THE SCHRÖDINGER EQUATION

Perhaps no subject has been the focus of as much mystery as "classical" quantum mechanics (QM) even though the standard Hilbert space framework provides an eminently satisfactory vehicle for determining accurate conclusions in many situations. This and other classical viewpoints provide also seven decimal place accuracy in quantum electrodynamics (QED) for example. So why all the fuss? The erection of the Hilbert space edifice and the subsequent development of operator algebras (extending now into noncommutative (NC) geometry) has an air of magic. It works but exactly why it works and what it really represents remain shrouded in ambiguity. Also geometrical connections of QM and classical mechanics (CM) are still a source of new work and a modern paradigm focuses on the emergence of CM from QM (or below). Below could mean here a microstructure of space time, or quantum foam, or whatever. Hence we focus on other approaches to QM and will recall any needed Hilbert space ideas as they arise.

1. DIFFUSION AND STOCHASTIC PROCESSES

There are some beautiful stochastic theories for diffusion and QM mainly concerned with origins of the Schrödinger equation (SE). For background information we mention for example [**33, 98, 131, 191, 192, 241, 242, 258, 471, 555, 589, 591, 615, 628, 647, 672, 674, 698, 715, 719, 726, 783, 810, 860, 1025, 1026, 1027**]. The present development focuses on certain aspects of the SE involving the wave function form $\psi = Rexp(iS/\hbar)$, hydro dynamical versions, diffusion processes, quantum potentials, and fractal methods. The aim is to envision "structure", both mathematical and physical, and we sometimes avoid detailed technical discussion of mathematical fine points (cf. [**241, 242, 243, 271, 315, 345, 531, 591, 592, 607, 615, 672, 674, 810, 918**] for various delicate matters). For example, rather than looking at such topics as Markov processes with jumps we prefer to seek "meaning" for the Schrödinger equation via microstructure and fractals in connection with diffusion processes and kinetic theory.

First consider the SE in the form $-(\hbar^2/2m)\psi'' + V\psi = i\hbar\psi_t$ so that for $\psi = Rexp(iS/\hbar)$ one obtains

$$S_t + \frac{S_X^2}{2m} + V - \frac{\hbar^2 R''}{2mR} = 0;\ \partial_t(R^2) + \frac{1}{m}(R^2 S')' = 0 \tag{1.1}$$

where $S' \sim \partial S/\partial X$. Writing $P = R^2$ (probability density $\sim |\psi|^2$) and $Q = -(\hbar^2/2m)(R''/R)$ (quantum potential) this becomes

$$S_t + \frac{(S')^2}{2m} + Q + V = 0;\ P_t + \frac{1}{m}(PS')' = 0 \tag{1.2}$$

and this has some hydrodynamical interpretations in the spirit of Madelung. Indeed going to [**294**] for example we take $p = S'$ with $p = m\dot{q}$ for $\dot{q}$ a velocity (or "collective" velocity - unspecified). Then (1.2) can be written as ($\rho = mP$ is an unspecified mass density)

$$S_t + \frac{p^2}{2m} + Q + V = 0;\ P_t + \frac{1}{m}(Pp)' = 0;\ p = S';\ P = R^2; \tag{1.3}$$

$$Q = -\frac{\hbar^2}{2m}\frac{R''}{R} = -\frac{\hbar^2}{2m}\frac{\partial^2\sqrt{\rho}}{\sqrt{\rho}}$$

Note here

$$\frac{\partial^2\sqrt{\rho}}{\sqrt{\rho}} = \frac{1}{4}\left[\frac{2\rho''}{\rho} - \left(\frac{\rho'}{\rho}\right)^2\right] \tag{1.4}$$

Now from $S' = p = m\dot{q} = mv$ one has

$$P_t + (P\dot{q})' = 0 \equiv \rho_t + (\rho\dot{q})' = 0;\ S_t + \frac{p^2}{2m} + V - \frac{\hbar^2}{2m}\frac{\partial^2\sqrt{\rho}}{\sqrt{\rho}} = 0 \tag{1.5}$$

Differentiating the second equation in X yields ($\partial \sim \partial/\partial X,\ v = \dot{q}$)

$$mv_t + mvv' + \partial V - \frac{\hbar^2}{2m}\partial\left(\frac{\partial\sqrt{\rho}}{\sqrt{\rho}}\right) = 0 \tag{1.6}$$

Consequently, multiplying by $p = mv$ and ρ respectively in (1.5) and (1.6), we obtain

$$m\rho v_t + m\rho vv' + \rho\partial V - \frac{\hbar^2}{2m}\rho\partial\left(\frac{\partial^2\sqrt{\rho}}{\sqrt{\rho}}\right) = 0;\ mv\rho_t + mv(\rho' v + \rho v') = 0 \tag{1.7}$$

Then adding in (1.7) we get

$$\partial_t(\rho v) + \partial(\rho v^2) + \frac{\rho}{m}\partial V - \frac{\hbar^2}{2m^2}\rho\partial\left(\frac{\partial^2\sqrt{\rho}}{\sqrt{\rho}}\right) = 0 \tag{1.8}$$

This is similar to an equation in [**294**] (called an "Euler" equation) and it definitely has a hydrodynamic flavor (cf. also [**434**] and see Section 6.2 for more details and some expansion).

Now go to [**743**] and write (1.6) in the form ($mv = p = S'$)

$$\frac{\partial v}{\partial t} + (v\cdot\nabla)v = -\frac{1}{m}\nabla(V+Q);\ v_t + vv' = -(1/m)\partial(V+Q) \tag{1.9}$$

The higher dimensional form is not considered here but matters are similar there. This equation (and (1.8)) is incomplete as a hydrodynamical equation as a consequence of a missing term $-\rho^{-1}\nabla\mathfrak{p}$ where $\mathfrak{p}$ is the pressure (cf. [**607**]). Hence one "completes" the equation in the form

$$m\left(\frac{\partial v}{\partial t}+(v\cdot\nabla)v\right)=-\nabla(V+Q)-\nabla F;\ mv_t+mvv'=-\partial(V+Q)-F' \tag{1.10}$$

where $\nabla F=(1/R^2)\nabla\mathfrak{p}$ (or $F'=(1/R^2)\mathfrak{p}'$). By the derivations above this would then correspond to an extended SE of the form

$$i\hbar\frac{\partial\psi}{\partial t}=-\frac{\hbar^2}{2m}\Delta\psi+V\psi+F\psi \tag{1.11}$$

provided one can determine F in terms of the wave function ψ. One notes that it a necessary condition here involves $curlgrad(F)=0$ or $curl(R^{-2}\nabla\mathfrak{p})=0$ which enables one to take e.g. $\mathfrak{p}=-bR^2=-b|\psi|^2$. For one dimension one writes $F'=-b(1/R^2)\partial|\psi|^2=-(2bR'/R)\Rightarrow F=-2blog(R)=-blog(|\psi|^2)$. Consequently one has a corresponding SE

$$i\hbar\frac{\partial\psi}{\partial t}=-\frac{\hbar^2}{2m}\psi''+V\psi-b(log|\psi|^2)\psi \tag{1.12}$$

This equation has a number of nice features discussed in [**743**] (cf. also [**223, 280, 292, 311, 312, 413, 691, 692, 693, 1028, 1029, 1030, 1031, 1032, 1033, 1034**]). For example $\psi=\beta G(x-vt)exp(ikx-i\omega t)$ is a solution of (2.28) with $V=0$ and for $v=\hbar k/m$ one gets $\psi=cexp[-(B/4)(x-vt+d)^2]\ exp(ikx-i\omega t)$ where $B=4mb/\hbar^2$. Normalization $\int_{-\infty}^{\infty}|\psi|^2=1$ is possible with $|\psi|^2=\delta_m(\xi)=\sqrt{m\alpha/\pi}exp(-\alpha m\xi^2)$ where $\alpha=2b/\hbar^2$, $d=0$, and $\xi=x-vt$ F or $m\to\infty$ we see that δ_m becomes a Dirac delta and this means that motion of a particle with big mass is strongly localized. This is impossible for ordinary QM since $exp(ikx-i\omega t)$ cannot be localized as $m\to\infty$. Such behavior helps to explain the so-called collapse of the wave function and since superposition does not hold Schrödinger's cat is either dead or alive. Further $v=k\hbar/m$ is equivalent to the deBroglie relation $\lambda=h/p$ since $\lambda=(2\pi/k)=2\pi(\hbar/mv)=2\pi(h/2\pi)(1/p)$.

REMARK 1.1.1. We go now to [**530**] and the linear SE in the form $i(\partial\psi/\partial t)=-(1/2m)\Delta\psi+U(\vec{r})\psi$; such a situation leads to the Ehrenfest equations which have the form

$$<\vec{v}>=(d/dt)<\vec{r}>;\ <\vec{r}>=\int d^3x|\psi(\vec{r},t)|^2\vec{r};\ m(d/dt)<\vec{v}>= \tag{1.13}$$

$$=\vec{F}(t)\vec{F}(t)=-\int d^3x|\psi(\vec{r},t)|^2\vec{\nabla}U(\vec{r})$$

Thus the quantum expectation values of position and velocity of a suitable quantum system obey the classical equations of motion and the amplitude squared is a natural probability weight. The result tells us that besides the statistical fluctuations quantum systems posess an extra source of indeterminacy, regulated in a very definite manner by the complex wave function. The Ehrenfest theorem can be extended to many point particle systems and in [**530**] one singles out the kind of nonlinearities that violate the Ehrenfest theorem. A theorem is proved that connects Galilean invariance, and the existence of a Lagrangian whose Euler-Lagrange equation is the SE, to the fulfillment of the Ehrenfest theorem.

REMARK 1.1.2. There are many problems with the quantum mechanical theory of derived nonlinear SE (NLSE) but many examples of realistic NLSE arise in the study of superconductivity, Bose-Einstein condensates, stochastic models of quantum fluids, etc. and the subject demands further study. We make no attempt to survey this here but will give an interesting example later from [**223**] related to fractal structures where a number of the difficulties are resolved. For further information on NLSE, in addition to the references above, we refer to [**100, 281, 392, 413, 530, 534, 535, 536, 788, 789, 790, 956, 957**] for some typical situations (the list is not at all complete and we apologize for omissions). Let us mention a few cases.

- The program of [**530**] introduces a Schrödinger Lagrangian for a free particle including self-interactions of any nonlinear nature but no explicit dependence on the space of time coordinates. The corresponding action is then invariant under spatial coordinate transformations and by Noether's theorem there arises a conserved current and the physical law of conservation of linear momentum. The Lagrangian is also required to be a real scalar depending on the phase of the wave function only through its derivatives. Phase transformations will then induce the law of conservation of probability identified as the modulus squared of the wave function. Galilean invariance of the Lagrangian then determines a connection between the probability current and the linear momentum which insures the validity of the Ehrenfest theorem.
- We turn next to [**535**] for a statistical origin for QM (cf. also [**191, 281, 534, 536, 698, 723, 809, 849**]). The idea is to build a program in which the microscopic motion, underlying QM, is described by a rigorous dynamics different from Brownian motion (thus avoiding unnecessary assumptions about the Brownian nature of the underlying dynamics). The Madelung approach gives rise to fluid dynamical type equations with a quantum potential, the latter being capable of interpretation in terms of a stress tensor of a quantum fluid. Thus one shows in [**535**] that the quantum state corresponds to a subquantum statistical ensemble whose time evolution is governed by classical kinetics in the phase space. The equations take the form

$$\rho_t + \partial_x(\rho u) = 0;\ \partial_t(\mu\rho u_i) + \partial_j(\rho\phi_{ij}) + \rho\partial_{x_i}V = 0; \quad (1.14)$$

$$\partial_t(\rho E) + \partial_x(\rho S) - \rho\partial_t V = 0$$

$$\frac{\partial S}{\partial t} + \frac{1}{2\mu}\left(\frac{\partial S}{\partial x}\right)^2 + \mathcal{W} + V = 0 \quad (1.15)$$

for two scalar fields ρ, S determining a quantum fluid. These can be rewritten as

$$\frac{\partial \xi}{\partial t} + \frac{1}{\mu}\frac{\partial^2 S}{\partial x^2} + \frac{1}{\mu}\frac{\partial \xi}{\partial x}\frac{\partial S}{\partial x} = 0; \quad (1.16)$$

$$\frac{\partial S}{\partial t} - \frac{\eta^2}{4\mu}\frac{\partial^2 \xi}{\partial x^2} - \frac{\eta^2}{8\mu}\left(\frac{\partial \xi}{\partial x}\right)^2 + \frac{1}{2\mu}\left(\frac{\partial S}{\partial x}\right)^2 + V = 0$$

where $\xi = log(\rho)$ and for $\Omega = (\xi/2) + (i/\eta)S = log\Psi$ with $m = N\mu$, $\mathcal{V} = NV$, and $\hbar = N\eta$ one arrives at a SE

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + \mathcal{V}\Psi \tag{1.17}$$

Further one can write $\Psi = \rho^{1/2}exp(i\mathfrak{S}/\hbar)$ with $\mathfrak{S} = NS$ and here $N = \int |\Psi|^2 d^n x$. The analysis is very interesting.

We will return to this later.

REMARK 1.1.3. Now in [**324**] one is obliged to use the form $\psi = Rexp(iS/\hbar)$ to make sense out of the constructions (this is no problem with suitable provisos, e.g. that S is not constant - cf. [**110, 191, 197, 198, 346, 347**]). Thus note $\psi'/\psi = (R'/R) + i(S'/\hbar)$ with $\Im(\psi'/\psi) = (1/m)S' \sim p/m$ (see also (1.22) below). Note also $J = (\hbar/m)\Im\psi^*\psi'$ and $\rho = R^2 = |\psi|^2$ represent a current and a density respectively. Then using $p = mv = m\dot{q}$ one can write

$$p = (\hbar/m)\Im(\psi'/\psi);\ J = (\hbar/m)\Im|\psi|^2(\psi^*\psi'/|\psi|^2) = (\hbar/m)\Im(\rho p) \tag{1.18}$$

Then look at the SE in the form $i\hbar\psi_t = -(\hbar^2/2m)\psi'' + V\psi$ with $\psi_t = (R_t + iS_t R/\hbar)exp(iS/\hbar)$ and

$$\psi_{xx} = [(R' + (iS'R/\hbar)exp(iS/\hbar)]' = \tag{1.19}$$

$$[R'' + (2iS'R'/\hbar) + (iS''R/\hbar) + (iS'/\hbar)^2 R]exp(iS/\hbar)$$

which means

$$-\frac{\hbar^2}{2m}\left[R'' - \left(\frac{S'}{\hbar}\right)^2 + \frac{2iS'r'}{\hbar} + \frac{iS''R}{\hbar}\right] + VR = i\hbar\left[R_t + \frac{iS_t R}{\hbar}\right] \Rightarrow \tag{1.20}$$

$$\Rightarrow \partial_t R^2 + \frac{1}{m}(R^2 S')' = 0;\ S_t + \frac{(S')^2}{2mR} - \frac{\hbar^2 R''}{2mR} + V = 0$$

This can also be written as (cf. (1.3))

$$\partial_t\rho + \frac{1}{m}\partial(p\rho) = 0;\ S_t + \frac{p^2}{2m} + Q + V = 0 \tag{1.21}$$

where $Q = -\hbar^2 R''/2mR$. Now we sketch the philosophy of [**324, 325**] in part. Most of such aspects are omitted here and we try to isolate the essential mathematical features (see Section 1.2 for more). First one emphasizes configurations based on coordinates whose motion is choreographed by the SE according to the rule (1-D only here)

$$\dot{q} = v = \frac{\hbar}{m}\Im\frac{\psi^*\psi'}{|\psi|^2} \tag{1.22}$$

where $i\hbar\psi_t = -(\hbar^2/2m)\psi'' + V\psi$. The argument for (1.22) is based on obtaining the simplest Galilean and time reversal invariant form for velocity, transforming correctly under velocity boosts. This leads directly to (1.22) (cf. (1.18))) so that Bohmian mechanics (BM) is governed by (1.22) and the SE. It's a fairly convincing argument and no recourse to Floydian time seems possible (cf. [**191, 347, 373, 374**]). Note however that if $S = c$ then $\dot{q} = v = (\hbar/m)\Im(R'/R) = 0$ while $p = S' = 0$ so perhaps this formulation avoids the $S = 0$ problems indicated in

[**191, 347, 373, 374**]. One notes also that BM depends only on the Riemannian structure $g = (g_{ij}) = (m_i\delta_{ij})$ in the form

$$\dot{q} = \hbar\Im(grad\psi/\psi);\ i\hbar\psi_t = -(\hbar^2/2)\Delta\psi + V\psi \tag{1.23}$$

What makes the constant $\hbar/m$ in (1.22) important here is that with this value the probability density $|\psi|^2$ on configuration space is equivariant. This means that via the evolution of probability densities $\rho_t + div(v\rho) = 0$ (as in (1.21) with $v \sim p/m$) the density $\rho = |\psi|^2$ is stationary relative to ψ, i.e. $\rho(t)$ retains the form $|\psi(q,t)|^2$. One calls $\rho = |\psi|^2$ the quantum equilibrium density (QED) and says that a system is in quantum equilibrium when its coordinates are randomly distributed according to the QED. The quantum equilibrium hypothesis (QHP) is the assertion that when a system has wave function ψ the distribution ρ of its coordinates satisfies $\rho = |\psi|^2$.

REMARK 1.1.4. We extract here from [**446, 447, 448**] (cf. also the references there for background and [**381, 382, 523**] for some information geometry). There are a number of interesting results connecting uncertainty, Fisher information, and QM and we make no attempt to survey the matter. Thus first recall that the classical Fisher information associated with translations of a 1-D observable X with probability density $P(x)$ is

$$F_X = \int dx\, P(x)([log(P(x)]')^2 > 0 \tag{1.24}$$

Recall now the Cramer-Rao inequality $Var(X) \geq F_X^{-1}$ where $Var(X) \sim$ variance of X. A Fisher length for X is defined via $\delta X = F_X^{-1/2}$ and this quantifies the length scale over which $p(x)$ (or better $log(p(x))$) varies appreciably. Then the root mean square deviation ΔX satisfies $\Delta X \geq \delta X$. Let now P be the momentum observable conjugate to X, and P_{cl} a classical momentum observable corresponding to the state ψ given via $p_{cl}(x) = (\hbar/2i)[(\psi'/\psi) - (\bar{\psi}'/\bar{\psi})]$ (cf. (1.22)). One has then the identity $< p >_\psi = < p_{cl} >_\psi$ via integration by parts. Now define the nonclassical momentum by $p_{nc} = p - p_{cl}$ and one shows that $\Delta X \Delta p \geq \delta X \Delta p \geq \delta X \Delta p_{nc} = \hbar/2$. Then go to [**447**] now where two proofs are given for the derivation of the SE from the exact uncertainty principle ($\delta X \Delta p_{nc} = \hbar/2$). Thus consider a classical ensemble of n-dimensional particles of mass m moving under a potential V. The motion can be described via the HJ and continuity equations

$$\frac{\partial s}{\partial t} + \frac{1}{2m}|\nabla s|^2 + V = 0;\ \frac{\partial P}{\partial t} + \nabla \cdot \left[P\frac{\nabla s}{m}\right] = 0 \tag{1.25}$$

for the momentum potential s and the position probability density P (note that we have interchanged p and P from [**447**] - note also there is no quantum potential and this will be supplied by the information term). These equations follow from the variational principle $\delta L = 0$ with Lagrangian

$$L = \int dt\, d^n x\, P\left[\frac{\partial s}{\partial t} + \frac{1}{2m}|\nabla s|^2 + V\right] \tag{1.26}$$

It is now assumed that the classical Lagrangian must be modified due to the existence of random momentum fluctuations. The nature of such fluctuations is immaterial for (cf. [**447**] for discussion) and one can assume that the momentum associated with position x is given by $p = \nabla s + N$ where the fluctuation term N vanishes on average at each point x. Thus s changes to being an average momentum potential. It follows that the average kinetic energy $< |\nabla s|^2 > /2m$ appearing in (1.26) should be replaced by $< |\nabla s + N|^2 > /2m$ giving rise to

$$L' = L + (2m)^{-1}\int dt < N\cdot N >= L + (2m)^{-1}\int dt(\Delta N)^2 \tag{1.27}$$

where $\Delta N =< N\cdot N >^{1/2}$ is a measure of the strength of the fluctuations. The additional term is specified uniquely, up to a multiplicative constant, by the following three assumptions

(1) Action principle: L' is a scalar Lagrangian with respect to the fields P and s where the principle $\delta L' = 0$ yields causal equations of motion. Thus $(\Delta N)^2 = \int d^n x\, pf(P, \nabla P, \partial P/\partial t, s, \nabla s, \partial s/\partial t, x, t)$ for some scalar function f.
(2) Additivity: If the system comprises two independent noninteracting subsystems with $P = P_1P_2$ then the Lagrangian decomposes into additive subsystem contributions; thus $f = f_1 + f_2$ for $P = P_1P_2$.
(3) Exact uncertainty: The strength of the momentum fluctuation at any given time is determined by and scales inversely with the uncertainty in position at that time. Thus $\Delta N \to k\Delta N$ for $x \to x/k$. Moreover since position uncertainty is entirely characterized by the probability density P at any given time the function f cannot depend on s, nor explicitly on t, nor on $\partial P/\partial t$.

The following theorem is then asserted (see [**447**] for the proofs).

THEOREM 1.1. The above 3 assumptions imply the relation $(\Delta N)^2 = c\int d^n x\, P|\nabla log(P)|^2$ where c is a positive universal constant.

COROLLARY 1.1. It follows from (1.27) that the equations of motion for p and s corresponding to the principle $\delta L' = 0$ are

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + V\psi \tag{1.28}$$

where $\hbar = 2\sqrt{c}$ and $\psi = \sqrt{P}exp(is/\hbar)$.

REMARK 1.1.5. We sketch here for simplicity and clarity another derivation of the SE along similar ideas following [**805**]. Let $P(y^i)$ be a probability density and $P(y^i + \Delta y^i)$ be the density resulting from a small change in the y^i. Calculate the cross entropy via

$$J(P(y^i + \Delta y^i) : P(y^i)) = \int P(y^i + \Delta y^i)log\frac{P(y^i + \Delta y^i)}{P(y^i)}d^n y \simeq \tag{1.29}$$

$$\simeq \left[\frac{1}{2}\int \frac{1}{P(y^i)}\frac{\partial P(y^i)}{\partial y^i}\frac{\partial P(y^i)}{\partial y^k)}d^n y\right]\Delta y^i\Delta y^k = I_{jk}\Delta y^i\Delta y^k$$

The I_{jk} are the elements of the Fisher information matrix. The most general expression has the form

$$I_{jk}(\theta^i) = \frac{1}{2}\int \frac{1}{P(x^i|\theta^i)}\frac{\partial P(x^i|\theta^i)}{\partial \theta^j}\frac{\partial P(x^i|\theta^i)}{\partial \theta^k}d^n x \tag{1.30}$$

where $P(x^i|\theta^i)$ is a probability distribution depending on parameters θ^i in addition to the x^i. For $P(x^i|\theta^i) = P(x^i + \theta^i)$ one recovers (1.29) (straightforward - cf. [**805**]). If P is defined over an n-dimensional manifold with positive inverse metric g^{ik} one obtains a natural definition of the information associated with P via

$$I = g^{ik}I_{ik} = \frac{g^{ik}}{2}\int \frac{1}{P}\frac{\partial P}{\partial y^i}\frac{\partial P}{\partial y^k}d^n y \tag{1.31}$$

Now in the HJ formulation of classical mechanics the equation of motion takes the form

$$\frac{\partial S}{\partial t} + \frac{1}{2}g^{\mu\nu}\frac{\partial S}{\partial x^\mu}\frac{\partial S}{\partial x^\nu} + V = 0 \tag{1.32}$$

where $g^{\mu\nu} = diag(1/m, \cdots, 1/m)$. The velocity field u^μ is given by $u^\mu = g^{\mu\nu}(\partial S/\partial x^\nu)$. When the exact coordinates are unknown one can describe the system by means of a probability density $P(t, x^\mu)$ with $\int P d^n x = 1$ and

$$(\partial P/\partial t) + (\partial/\partial x^\mu)(P g^{\mu\nu}(\partial S/\partial x^\nu)) = 0 \tag{1.33}$$

These equations completely describe the motion and can be derived from the Lagrangian

$$L_{CL} = \int P\left\{\frac{\partial S}{\partial t} + \frac{1}{2}g^{\mu\nu}\frac{\partial S}{\partial x^\mu}\frac{\partial S}{\partial x^\nu} + V\right\}dtd^n x \tag{1.34}$$

using fixed endpoint variation in S and P. Quantization is obtained by adding a term proportional to the information I defined in (1.31). This leads to

$$L_{QM} = L_{CL} + \lambda I = \int P\left\{\frac{\partial S}{\partial t} + \frac{1}{2}g^{\mu\nu}\left[\frac{\partial S}{\partial x^\mu}\frac{\partial S}{\partial x^\nu} + \frac{\lambda}{P^2}\frac{\partial P}{\partial x^\mu}\frac{\partial P}{\partial x^\nu}\right] + V\right\}dtd^n x \tag{1.35}$$

Fixed endpoint variation in S leads again to (1.33) while variation in P leads to

$$\frac{\partial S}{\partial t} + \frac{1}{2}g^{\mu\nu}\left[\frac{\partial S}{\partial x^\mu}\frac{\partial S}{\partial x^\nu} + \lambda\left(\frac{1}{P^2}\frac{\partial P}{\partial x^\mu}\frac{\partial P}{\partial x^\nu} - \frac{2}{P}\frac{\partial^2 P}{\partial x^\mu \partial x^\nu}\right)\right] + V = 0 \tag{1.36}$$

These equations are equivalent to the SE if $\psi = \sqrt{P}exp(iS/\hbar)$ with $\lambda = (2\hbar)^2$.

REMARK 1.1.6. In Remarks 1.1.6 - 1.1.8 one uses $Q = \pm(1/m)$ times the standard $Q = -(\hbar^2/2m)(\Delta\sqrt{\rho}/\sqrt{\rho}$. The SE gives to a probability distribution $\rho = |\psi|^2$ (with suitable normalization) and to this one can associate an information entropy $S(t)$ (actually configuration information entropy) $S = -\int \rho log(\rho)d^3x$ which is typically not a conserved quantity (S is an unfortunate notation here but we retain it momentarily since no confusion should arise). The rate of change in time of S can be readily found by using the continuity equation $\partial_t \rho = -\nabla \cdot (v\rho)$ where v is a current velocity field. Note here (cf. also [**752**])

$$\frac{\partial S}{\partial t} = -\int \rho_t (1 + log(\rho))dx = \int (1 + log(\rho))\partial(v\rho) \tag{1.37}$$

Note that a formal substitution of $v = -u$ in the contintuity equation implies the standard free Browian motion outcome $dS/dt = D \cdot \int [(\nabla\rho)^2/\rho)d^3x = D \cdot Tr\mathfrak{F} \geq 0$ - use here $u = D\nabla log(\rho)$ with $D = \hbar/2m$) and (1.37) with $\int (1 + log(\rho))\partial(v\rho) = -\int v\rho\partial log(\rho) = -\int v\rho' \sim \int ((\rho')^2/\rho)$ modulo constants involving D etc. Recall here $\mathfrak{F} \sim -(2/D^2)\int \rho Q dx = \int dx[(\nabla\rho)^2/\rho]$ is a functional form of Fisher information. A high rate of information entropy production corresponds to a rapid spreading (flattening down) of the probablity density. This delocalization feature is concomitant with the decay in time property quantifying the time rate at which the far from equilibrium system approaches its stationary state of equilibrium $(d/dt)Tr\mathfrak{F} \leq 0$.

REMARK 1.1.7. Now going back to the quantum context one admits general forms of the current velocity v. For example consider a gradient field $v = b - u$ where the so-called forward drift $b(x,t)$ of the stochastic process depends on a particular diffusion model. Then one can rewrite the continuity equation as a standard Fokker-Planck equation $\partial \rho = D\Delta\rho - \nabla \cdot (b\rho)$. Boundary restrictions requiring ρ, $v\rho$, and $b\rho$ to vanish at spatial infinities or at boundaries yield the general entropy balance equation

$$\frac{dS}{dt} = \int \left[\rho(\nabla \cdot b) + D \cdot \frac{(\nabla\rho)^2}{\rho}\right] d^3x \equiv -D\frac{dS}{dt} = \int \rho(v \cdot u)d^3x =< v \cdot u > \tag{1.38}$$

The first term in the first equation is not positive definite and can be interpreted as an entropy flux while the second term refers to the entropy production proper. The flux term represents the mean value of the drift field divergence $\nabla \cdot b$ which by itself is a local measure of the flux incoming to or outgoing from an infinitesimal surrounding of x at time t. If locally $(\nabla \cdot b)(x,t) > 0$ on an infinitesimal time scale we would encounter a local entropy increase in the system (increasing disorder) while in case $(\nabla \cdot b)(x,t) < 0$ one thinks of local entropy loss or restoration or order. Only in the situation $< \nabla \cdot b >= 0$ is there no entropy production. Quantum dynamics permits more complicated behavior. One looks first for a general criterion under which the information entropy S is a conserved quantity. Consider (1.8) and invoke the diffusion current to write (recall $u = D(\nabla\rho)/\rho$)

$$D\frac{dS}{dt} = -\int [\rho^{-1/2}(\rho v)] \cdot [\rho^{-1/2}(D\nabla\rho)]d^3x \tag{1.39}$$

Then by means of the Schwarz inequality one has $D|dS/dt| \leq < v^2 >^{1/2} < u^2 >^{1/2}$ so a necessary (but insufficient) condition for $dS/dt \neq 0$ is that both $< v^2 >$ and $< u^2 >$ are nonvanishing. On the other hand a sufficient condition for $dS/dt = 0$ is that either one of these terms vanishes. Indeed in view of $< u^2 >= D^2 \int [(\nabla\rho)^2/\rho]d^3x$ the vanishing information entropy production implies $dS/dt = 0$; the vanishing diffusion current does the same job.

REMARK 1.1.8. We develop a little more perspective now (following [**395**] - first paper). Recall Q written out as

$$-Q = 2D^2\frac{\Delta\rho^{1/2}}{\rho^{1/2}} = D^2\left[\frac{\Delta\rho}{\rho} - \frac{1}{2\rho^2}(\nabla\rho)^2\right] = \frac{1}{2}u^2 + D\nabla\cdot u \tag{1.40}$$

where $u = D\nabla log(\rho)$ is called an osmotic velocity field. The standard Brownian motion involves $v = -u$, known as the diffusion current velocity and (up to a dimensional factor) is identified with the thermodynamic force of diffusion which drives the irreversible process of matter exchange at the macroscopic level. On the other hand, even while the thermodynamic force is a concept of purely statistical origin associated with a collection of particles, in contrast to microscopic forces which have a direct impact on individual particles themselves, it is well known that this force manifests itself as a Newtonian type entry in local conservation laws describing the momentum balance; in fact it pertains to the average (local average) momentum taken over by the particle cloud, a statistical ensemble property quantified in terms of the probability distribution at hand. It is precisely the (negative) gradient of the above potential Q in (1.40) which plays the Newtonian force role in the momentum balance equations. The second analytical expression of interest here involves

$$-\int Q\rho dx = (1/2)\int u^2\rho dx = (1/2)D^2\cdot F_X;\ F_X = \int\frac{(\nabla\rho)^2}{\rho}dx \tag{1.41}$$

where F_X is the Fisher information, encoded in the probability density ρ which quantifies its gradient content (sharpness plus localization/disorder) Note that

$$-\int Q\rho = -\int[(1/2)u^2\rho + D\rho u'] = -\int(1/2)u^2\rho + \int Du\rho' = \tag{1.42}$$

$$= -(1/2)\int D^2(\rho'/\rho)^2\rho + D^2\int\rho'(\rho'/\rho) = (D^2/2)\int(\rho')^2/\rho = (1/2)\int u^2\rho$$

On the other hand the local entropy production inside the system sustaining an irreversible process of diffusion is given via

$$\frac{dS}{dt} = D\cdot\int\frac{(\nabla\rho)^2}{\rho}dx = D\cdot F_X \geq 0 \tag{1.43}$$

This stands for an entropy production rate when the Fick law induced diffusion current (standard Brownian motion case) $j = -D\nabla\rho$, obeying $\partial_t\rho + \nabla j = 0$, enters the scene. Here $S = -\int\rho log(\rho)dx$ plays the role of (time dependent) information entropy in the nonequilibrium statistical mechanics framework for the thermodynamics of irreversible processes. It is clear that a high rate of entropy increase coresponds to a rapid spreading (flattening) of the probability density. This explicitly depends on the sharpness of density gradients. The potential Q(x,t), the Fisher information F_X, the nonequilibrium measure of entropy production dS/dt, and the information entropy $S(t)$ are thus mutually entangled quantities, each being exclusively determined in terms of ρ and its derivatives.

In the standard statistical mechanics setting the Euler equation gives a prototypical momentum balance equation in the (local) mean

$$(\partial_t + v\cdot\nabla)v = \frac{F}{m} - \frac{\nabla P}{\rho} \tag{1.44}$$

where $F = -\nabla F$ represents normal Newtonian force and P is a pressure term. Q appears in the hydrodynamical formalism of QM via

$$(\partial_t + v\cdot\nabla)v = \frac{1}{m}F - \nabla Q = \frac{1}{m}F + \frac{\hbar^2}{2m^2}\nabla\frac{\Delta\rho^{1/2}}{\rho^{1/2}} \tag{1.45}$$

Another spectacular example pertains to the standard free Brownian motion in the strong friction regime (Smoluchowski diffusion), namely

$$(\partial_t + v\cdot\nabla)v = -2D^2\nabla\frac{\Delta\rho^{1/2}}{\rho^{1/2}} = -\nabla Q \tag{1.46}$$

where $v = -D(\nabla\rho/\rho)$ (formally $D = \hbar/2m$).

REMARK 1.1.9. The papers in [**291, 292**] contain very interesting derivations of Schrödinger equations via diffusion ideas à la Nelson, Markov wave equations, and suitable "applied" forces (e.g. radiative reactive forces).

We go now to Nagasawa [**670, 671, 672, 673, 674**] to see how diffusion and the SE are really connected (cf. also [**15, 141, 223, 421, 676, 681, 698, 726, 732, 733, 734, 735, 736**] for related material, some of which is discussed later in detail); for now we simply sketch some formulas for a simple Euclidean metric where $\Delta = \sum(\partial/\partial x^i)^2$. Then $\psi(t,x) = exp[R(t,x) + iS(t,x)]$ satisfies a SE $i\partial_t\psi + (1/2)\Delta\psi + ia(t,x)\cdot\nabla\psi - V(t,x)\psi = 0$ ($\hbar$ and m omitted with $a(t,x)$ a drift coefficient) if and only if

$$V = -\frac{\partial S}{\partial t} + \frac{1}{2}\Delta R + \frac{1}{2}(\nabla R)^2 - \frac{1}{2}(\nabla S)^2 - a\cdot\nabla S; \tag{1.47}$$

$$0 = \frac{\partial R}{\partial t} + \frac{1}{2}\Delta S + (\nabla S)\cdot(\nabla R) + a\cdot\nabla R$$

in the region $D = \{(s,x) : \psi(s,x) \neq 0\}$ (a harmless gauge factor in the divergence is also being omitted). Solutions are often referred to as weak or distributional but we do not belabor this point. From [**671, 672, 673**] there results

THEOREM 1.2. Let $\psi(t,x) = exp[R(t,x) + iS(t,x)]$ be a solution of the SE above; then $\phi(t,x) = exp[R(t,x) + S(t,x)]$ and $\hat{\phi} = exp[R(t,x) - S(t,x)]$ are solutions of

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}\Delta\phi + a(t,x)\cdot\nabla\phi + c(t,x,\phi)\phi = 0; \tag{1.48}$$

$$-\frac{\partial\hat{\phi}}{\partial t} + \frac{1}{2}\Delta\hat{\phi} - a(t,x)\cdot\nabla\hat{\phi} + c(t,x,\phi)\hat{\phi} = 0$$

where the creation and annihilation term $c(t,x,\phi)$ is given via

$$c(t,x,\phi) = -V(t,x) - 2\frac{\partial S}{\partial t}(t,x) - (\nabla S)^2(t,x) - 2a\cdot\nabla S(t,x) \tag{1.49}$$

Conversely given $(\phi, \hat{\phi})$ as in Theorem 1.2 satisfying (1.48) it follows that ψ satisfies the SE with V as in (1.49) (note $R = (1/2)log(\hat{\phi}\phi)$ and $S = (1/2)log(\phi/\hat{\phi})$ with $exp(R) = (\hat{\phi}\phi)^{1/2}$).

We note that the equations (1.48) are not imaginary time SE and from all this one can conclude that nonrelativistic QM is diffusion theory in terms of Schrödinger processes (described by $(\phi, \hat{\phi})$ - more details later). Further it is shown that certain key postulates in Nelson's stochastic mechanics or Zambrini's Euclidean QM (cf. [**1011**]) can both be avoided in connecting the SE to diffusion processes (since they are automatically valid). Look now at Theorem 1.2 for one dimension and write $T = \hbar t$ with $X = (\hbar/\sqrt{m})x$ and $A = a\hbar/\sqrt{m}$; then the SE becomes

$$i\hbar\psi_T = -(\hbar^2/2m)\psi_{XX} - iA\psi_X + V\psi; \tag{1.50}$$

$$i\hbar R_T + (\hbar^2/m^2)R_X S_X + (\hbar^2/2m^2)S_{XX} + AR_X = 0;$$

$$V = -i\hbar S_T + (\hbar^2/2m)R_{XX} + (\hbar^2/2m^2)R_X^2 - (\hbar^2/2m^2)S_X^2 - AS_X$$

Hence

PROPOSITION 1.1. The SE of Theorem 1.2, written in the variables $X = (\hbar/\sqrt{m})x$, $T = \hbar t$, with $A = (\sqrt{m}/\hbar)a$ and $V = V(X,T) \sim V(x,t)$ is equivalent to (2.2).

Making a change of variables in (1.48) now, as in Proposition 1.1, yields

COROLLARY 1.2. Equation (1.48), written in the variables of Proposition 1.2, becomes

$$\hbar\phi_T + \frac{\hbar^2}{2m}\phi_{XX} + A\phi_X + \tilde{c}\phi = 0; \ -\hbar\hat{\phi}_T + \frac{\hbar^2}{2m}\hat{\phi}_{XX} - A\hat{\phi}_X + \tilde{c}\hat{\phi} = 0; \tag{1.51}$$

$$\tilde{c} = -\tilde{V}(X,T) - 2\hbar S_T - \frac{\hbar^2}{m}S_X^2 - 2AS_X$$

Thus the diffusion processes pick up factors of $\hbar$ and $\hbar/\sqrt{m}$.

REMARK 1.1.10. We extract here from the Appendix to [**672**] for some remarks on competing points of view regarding diffusion and the the SE. First some work of Fenyes [**360**] is cited where a Lagrangian is taken as

$$L(t) = \int \left[\frac{\partial S}{\partial t} + \frac{1}{2}(\nabla S)^2 + V + \frac{1}{2}\left(\frac{1}{2}\frac{\nabla\mu}{\mu}\right)^2\right]\mu dx \tag{1.52}$$

where $\mu_t(x) = exp(2R(t,x))$ denotes the distribution density of a diffusion process and V is a potential function. The term $\Pi(\mu) = (1/2)[(1/2)(\nabla\mu/\mu)]^2$ is called a diffusion pressure and since $(1/2)(\nabla\mu/\mu) \sim \nabla R$ the Lagrangian can be written as

$$L = \int \left[\frac{\partial S}{\partial t} + \frac{1}{2}(\nabla S)^2 + \frac{1}{2}(\nabla R)^2 + V\right]\mu dx \tag{1.53}$$

Applying the variational principle $\delta \int_a^b L(t)dt = 0$ one arrives at

$$(1.54)\quad \frac{\partial S}{\partial t} + \frac{1}{2}\left[(\nabla(R+S))\right]^2 - (\nabla(R+S))\cdot\left(\frac{1}{2}\frac{\nabla\mu}{\mu}\right) + \left(\frac{1}{2}\frac{\nabla\mu}{\mu}\right)^2 - \frac{1}{4}\frac{\Delta\mu}{\mu} + V = 0$$

which is called a motion equation of probability densities. From this he shows that the function $\psi = exp(R+iS)$ satisfies the SE $i\partial_t + (1/2)\Delta\psi - V(t,x)\psi = 0$. Indeed putting $\Pi(\mu)$ and the formula $(1/2)(\Delta\mu/\mu) + (1/2)\Delta R + (\nabla R)^2$ into (1.53) one obtains

$$(1.55)\quad \frac{\partial S}{\partial t} + \frac{1}{2}(\nabla S)^2 - \frac{1}{2}(\nabla R)^2 - \frac{1}{2}\Delta R + V = 0$$

which goes along with the duality relation $R_t + (1/2)\Delta S + \nabla S\cdot\nabla R + b\cdot\nabla R = 0$ where $u = (1/2)(a+\hat{a}) = \nabla R$ and $v = (1/2)(a-\hat{a}) = \nabla S$ as derived in the Nagasawa theory. Hence $\psi = exp(R+iS)$ satisfies the SE by previous calculations. One can see however that the equation (1.53) is not needed since the SE and diffusion equations are equivalent and in fact the equations of motion are the diffusion equations. Moreover it is shown in [**672**] that (1.53) is an automatic consequence in diffusion theory with $V = -c - 2S_t - (\nabla S)^2$ and therefore it need not be postulated or derived by other means. This is a simple calculation from the theory developed above.

REMARK 1.1.11. Nelson's important work in stochastic mechanics [**698**] produced the SE from diffusion theory but involved a stochastic Newtonian equation which is shown in [**672**] to be automatically true. Thus Nelson worked in a general context which for our purposes here can be considered in the context of Brownian motions

$$(1.56)\quad B(t) = \partial_t + (1/2)\Delta + b\cdot\nabla + a\cdot\nabla;\ \hat{B}(t) = -\partial_t + (1/2)\Delta - b\cdot\nabla + \hat{a}\cdot\nabla$$

and used a mean acceleration $\alpha(t,x) = -(1/2)[B(t)\hat{B}(t)x + \hat{B}(t)B(t)x]$. Assuming the duality relations after (1.55) he obtains a formula

$$(1.57)\quad \alpha(t,x) = -\frac{1}{2}[B(t)(-b+\hat{a}) + \hat{B}(b+a)] = b_t + (1/2)\nabla(b)^2 - (b+v)\times curl(b) -$$

$$-[-v_t + (1/2)\Delta u + (1/2)(\hat{a}\cdot\nabla)a + (1/2)(a\cdot\nabla)\hat{a} - (b\cdot\nabla)v - (v\cdot\nabla)b - v\times curl(b)]$$

Then it is shown that the SE can be deduced from the stochastic Newton's equation

$$(1.58)\quad \alpha(t,x) = -\nabla V + \frac{\partial b}{\partial t} + \frac{1}{2}\nabla(b^2) - (b+v)\times curl(b)$$

Nagasawa shows that this serves only to reproduce a known formula for V yielding the SE; he also shows that (1.57) also is an automatic consequence of the duality formulation of diffusion equations above. This equation (1.57) is often called stochastic quantization since it leads to the SE and it is in fact correct with the V specified there. However the SE is more properly considered as following directly from the diffusion equations in duality and is not correctly an equation of motion. There is another discussion of Euclidean QM developed by Zambrini [**1011**]. This involves $\tilde{\alpha}(t,x) = (1/2)[B(t)B(t)x + \hat{B}(t)\hat{B}(t)x]$ (with $(\sigma\sigma^T)^{ij} = \delta^{ij}$). It is postulated that this equals $-\nabla c + b_t + (1/2)\nabla(b)^2 - b + v) \times curl(b)$ which in fact leads to the same equation for V as above with $V = -c - 2S_t - (\nabla S)^2 - 2b \cdot \nabla S$ so there is nothing new. Indeed it is shown in [**672**] that the postulated equivalence holds automatically as a simple consequence of time reversal of diffusion processes.

2. SCALE RELATIVITY

Scale relativity (SR) is due to L. Nottale (cf. [**715, 716, 717, 718, 719, 720, 721**]) and somehow has not been accorded any real recognition by the "establishment". We only touch here on derivations of the SE and will develop further aspects later; the arguments are evidently heuristic but have a compelling interest. More general relativistic and cosmological features are discussed in Chapter 2 where further discussion is given. The ideas involve spacetime having a fractal microstructure containing in particular continuous (self-similar) nondifferentiable paths which serve as geodesic quantum paths of Hausdorff dimension $D = 2$. This is in fact a good notion of quantum path (following Feynman for example - cf. [**1**]) and we will see how it leads to a lovely (heuristic) derivation of the SE which automatically creates a complex wave function.

REMARK 1.2.1. One considers quantum paths à la Feynman so that (**E1**) $lim_{t \to t'}[X(t) - X(t')]^2/(t - t')$ exists. This implies $X(t) \in H^{1/2}$ where H^α means $c\epsilon^\alpha \leq |X(t) - X(t')| \leq C\epsilon^\alpha$ and from [**345**] for example this means $dim_H X[a,b] = 1/2$. Now one "knows" (see e.g. [**1**]) that quantum and Brownian motion paths (in the plane) have H-dimension 2 and some clarification is needed here. We refer to [**625**] where there is a paper on Wiener Brownian motion (WBM), random walks, etc. discussing Hausdorff and other dimensions of various sets. Thus given $0 < \lambda < 1/2$ with probability 1 a Browian sample function X satisfies $|X(t+h) - X(t)| \leq b|h|^\lambda$ for $|h| \leq h_0$ where $b = b(\lambda)$. This leads to the result that with probability 1 the graph of a Brownian sample function has Hausdorff and box dimension 3/2. On the other hand a Browian trail (or path) in 2 dimensions has Hausdorff and box dimension 2 (note a quantum path can have self intersections, etc.).

There are now several excellent approaches. The method of Nottale [**700, 715, 718**] is preeminent (cf. also [**732, 733, 734, 735**]) and there is also a nice derivation of a nonlinear SE via fractal considerations in [**223**] (indicated below). The most elaborate and rigorous approach is due to Cresson [**272**], with elaboration and updating in [**3, 273, 274**]. There are various derivations of the SE and we follow [**715**] here (cf. also [**718, 828**]). The philosophy of scale relativity will be discussed later and we just write down equations here pertaining to the SE. First a bivelocity structure is defined (recall that one is dealing with fractal paths). One defines first

$$\frac{d_+}{dt}y(t) = lim_{\Delta t \to 0_+}\left\langle \frac{y(t+\Delta t) - y(t)}{\Delta t} \right\rangle; \tag{2.1}$$

$$\frac{d_-}{dt}y(t) = lim_{\Delta t \to 0_+}\left\langle \frac{y(t) - y(t-\Delta t)}{\Delta t} \right\rangle$$

Applied to the position vector x this yields forward and backward mean velocities, namely $(d_+/dt)x(t) = b_+$ and $(d_-/dt)x(t) = b_-$. Here these velocities are defined as the average at a point q and time t of the respective velocities of the outgoing and incoming fractal trajectories; in stochastic QM this corresponds to an average on the quantum state. The position vector $x(t)$ is thus "assimilated" to a stochastic process which satisfies respectively after $(dt > 0)$ and before $(dt < 0)$ the instant t a relation $dx(t) = b_+[x(t)]dt + d\xi_+(t) = b_-[x(t)]dt + d\xi_-(t)$ where $\xi(t)$ is a Wiener process (cf. [**698**]). It is in the description of ξ that the $D = 2$ fractal character of trajectories is inserted; indeed that ξ is a Wiener process means that the $d\xi$'s are assumed to be Gaussian with mean 0, mutually independent, and such that

$$< d\xi_{+i}(t)d\xi_{+j}(t) >= 2\mathcal{D}\delta_{ij}dt;\ < d\xi_{-i}(t)d\xi_{-j}(t) >= -2\mathcal{D}\delta_{ij}dt \tag{2.2}$$

where $< >$ denotes averaging ($\mathcal{D}$ is now the diffusion coefficient). Nelson's postulate (cf. [**698**]) is that $\mathcal{D} = \hbar/2m$ and this has considerable justification (cf. [**715**]). Note also that (2.2) is indeed a consequence of fractal (Hausdorff) dimension 2 of trajectories follows from $< d\xi^2 > /dt^2 = dt^{-1}$, i.e. precisely Feynman's result $< v^2 >^{1/2} \sim \delta t^{-1/2}$ (the discussion here in [**715**] is unclear however - cf. [**29**]). Note also that Brownian motion (used in Nelson's postulate) is known to be of fractal (Hausdorff) dimension 2. Note also that any value of $\mathcal{D}$ may lead to QM and for $\mathcal{D} \to 0$ the theory becomes equivalent to the Bohm theory. Now expand any function $f(x,t)$ in a Taylor series up to order 2, take averages, and use properties of the Wiener process ξ to get

$$\frac{d_+f}{dt} = (\partial_t + b_+ \cdot \nabla + \mathcal{D}\Delta)f;\ \frac{d_-f}{dt} = (\partial_t + b_- \cdot \nabla - \mathcal{D}\Delta)f \tag{2.3}$$

Let $\rho(x,t)$ be the probability density of $x(t)$; it is known that for any Markov (hence Wiener) process one has $\partial_t\rho + div(\rho b_+) = \mathcal{D}\Delta\rho$ (forward equation) and $\partial_t\rho + div(\rho b_-) = -\mathcal{D}\Delta\rho$ (backward equation). These are called Fokker-Planck equations and one defines two new average velocities $V = (1/2)[b_+ + b_-]$ and $U = (1/2)[b_+ - b_-]$. Consequently adding and subtracting one obtains $\rho_t + div(\rho V) = 0$ (continuity equation) and $div(\rho U) - \mathcal{D}\Delta\rho = 0$ which is equivalent to $div[\rho(U - \mathcal{D}\nabla log(\rho))] = 0$. One can show, using (2.3) that the term in square brackets in the last equation is zero leading to $U = \mathcal{D}\nabla log(\rho)$. Now place oneself in the (U,V) plane and write $\mathcal{V} = V - iU$. Then write $(d_{\mathcal{V}}/dt) = (1/2)(d_+ + d_-)/dt$ and $(d_{\mathcal{U}}/dt) = (1/2)(d_+ - d_-)/dt$. Combining the equations in (2.3) one defines $(d_{\mathcal{V}}/dt) = \partial_t + V \cdot \nabla$ and $(d_{\mathcal{U}}/dt) = \mathcal{D}\Delta + U \cdot \nabla$; then define a complex operator $(d'/dt) = (d_{\mathcal{V}}/dt) - i(d_{\mathcal{U}}/dt)$ which becomes

$$\frac{d'}{dt} = \left(\frac{\partial}{\partial t} - i\mathcal{D}\Delta\right) + \mathcal{V} \cdot \nabla \tag{2.4}$$

One now postulates that the passage from classical mechanics to a new nondifferentiable process considered here can be implemented by the unique prescription of replacing the standard d/dt by d'/dt. Thus consider $\mathfrak{S} = \left\langle \int_{t_1}^{t_2} \mathcal{L}(x,\mathcal{V},t)dt \right\rangle$ yielding by least action $(d'/dt)(\partial\mathcal{L}/\partial\mathcal{V}_i) = \partial\mathcal{L}/\partial x_i$. Define then $\mathcal{P}_i = \partial\mathcal{L}/\partial\mathcal{V}_i$ leading to $\mathcal{P} = \nabla\mathfrak{S}$ (recall the classical action principle with $dS = pdq - Hdt$). Now for Newtonian mechanics write $L(x,v,t) = (1/2)mv^2 - \mathbf{U}$ which becomes

$\mathcal{L}(x,\mathcal{V},t) = (1/2)m\mathcal{V}^2 - \mathfrak{U}$ leading to $-\nabla\mathfrak{U} = m(d'/dt)\mathcal{V}$. One separates real and imaginary parts of the complex acceleration $\gamma = (d'\mathcal{V}/dt$ to get

$$d'\mathcal{V} = (d_{\mathcal{V}} - id_{\mathcal{U}})(V - iU) = (d_{\mathcal{V}}V - d_{\mathcal{U}}U) - i(d_{\mathcal{U}}V + d_{\mathcal{V}}U) \tag{2.5}$$

The force $F = -\nabla\mathfrak{U}$ is real so the imaginary part of the complex acceleration vanishes; hence

$$\frac{d_{\mathcal{U}}}{dt}V + \frac{d_{\mathcal{V}}}{dt}U = \frac{\partial U}{\partial t} + U\cdot\nabla V + V\cdot\nabla U + \mathcal{D}\Delta V = 0 \tag{2.6}$$

from which $\partial U/\partial t$ may be obtained. This is a weak point in the derivation since one has to assume e.g. that $U(x,t)$ has certain smoothness properties (see below for refinements). Differentiating the expression $U = \mathcal{D}\nabla log(\rho)$ and using the continuity equation yields another expression $(\partial U/\partial t) = -\mathcal{D}\nabla(divV) - \nabla(V\cdot U)$. Comparison of these relations yields $\nabla(divV) = \Delta V - U\wedge curlV$ where the $curlU$ term vanishes since U is a gradient. However in the Newtonian case $\mathcal{P} = m\mathcal{V}$ so $\mathcal{P}\nabla\mathfrak{S}$ implies that $\mathcal{V}$ is a gradient and hence a generalization of the classical action S can be defined. Recall $V = 2\mathcal{D}\nabla S$ and $\nabla(divV) = \Delta V$ with $curlV = 0$; combining this with the expression for U one obtains $\mathfrak{S} = log(\rho^{1/2}) + iS$. One notes that this is compatible with [**698**] for example. Finally set $\psi = \sqrt{\rho}exp(iS) = exp(i\mathfrak{S})$ with $\mathcal{V} = -2i\mathcal{D}\nabla(log\psi)$ and note

$$U = \mathcal{D}\nabla log(\rho);\ V = 2\mathcal{D}\nabla S; \tag{2.7}$$

$$\mathcal{V} = -2i\mathcal{D}\nabla log\psi = -i\mathcal{D}\nabla log(\rho) + 2\mathcal{D}\nabla S = V - iU$$

Thus for $\mathcal{P} = m\mathcal{V}$ the relation $\mathcal{P} \sim -i\hbar\nabla$ or $\mathcal{P}\psi = -i\hbar\nabla\psi$ has a natural interpretation. Putting ψ in the equation $-\nabla\mathfrak{U} = m(d'/dt)\mathcal{V}$, which generalizes Newton's law to fractal space the equation of motion takes the form $\nabla\mathfrak{U} = 2i\mathcal{D}m(d'/dt)(\nabla log(\psi))$. Then noting that d' and ∇ do not commute one replaces d'/dt by (2.4) to obtain

$$\nabla\mathfrak{U} = 2i\mathcal{D}m\left[\partial_t\nabla log(\psi) - i\mathcal{D}\Delta(\nabla log(\psi)) - 2i\mathcal{D}(\nabla log(\psi)\cdot\nabla)(\nabla log(\psi)\right] \tag{2.8}$$

This expression can be simplified via

$$\nabla\Delta = \Delta\nabla;\ (\nabla f\cdot\nabla)(\nabla f) = (1/2)\nabla(\nabla f)^2;\ \frac{\Delta f}{f} = \Delta log(f) + (\nabla log(f))^2 \tag{2.9}$$

which implies

$$\frac{1}{2}\Delta(\nabla log(\psi)) + (\nabla log(\psi)\cdot\nabla)(\nabla log(\psi)) = \frac{1}{2}\nabla\frac{\Delta\psi}{\psi} \tag{2.10}$$

Integrating this equation yields $\mathcal{D}^2\Delta\psi + i\mathcal{D}\partial_t\psi - (\mathfrak{U}/2m)\psi = 0$ up to an arbitrary phase factor $\alpha(t)$ which can be set equal to 0 by a suitable choice of phase S. Replacing $\mathcal{D}$ by $\hbar/2m$ one arrives at the SE $i\hbar\psi_t = -(\hbar^2/2m)\Delta\psi + \mathfrak{U}\psi$ and this suggests an interpretation of QM as mechanics in a nondifferentiable (fractal) space.

In fact (using one space dimension for convenience) we see that if $\mathfrak{U} = 0$ then the free motion $m(d'/dt)\mathcal{V} = 0$ yields the SE $i\hbar\psi_t = -(\hbar^2/2m)\psi_{xx}$ as a geodesic equation in "fractal" space. Further from $U = (\hbar/m)(\partial\sqrt{\rho}/\sqrt{\rho})$ and $Q = -(\hbar^2/2m)(\Delta\sqrt{\rho}/\sqrt{\rho})$ one arrives at a lovely relation, namely

PROPOSITION 2.1. The quantum potential Q can be written in the form $Q = -(m/2)U^2 - (\hbar/2)\partial U$ (cf. (1.40) multiplied by $-m$). Hence the quantum potential arises directly from the fractal nonsmooth nature of the quantum paths. Since Q can be thought of as a quantization of a classical motion we see that the quantization corresponds exactly to the existence of nonsmooth paths. Consequently smooth paths imply no quantum mechanics.

REMARK 1.2.2. In [**15**] (to be discussed later) one writes again $\psi = Rexp(iS/\hbar)$ with field equations in the hydrodynamical picture (1-D for convenience)

$$d_t(m_0\rho v) = \partial_t(m_0\rho v) + \nabla(m_0\rho v) = -\rho\nabla(u + Q);\ \partial_t\rho + \nabla\cdot(\rho v) = 0 \tag{2.11}$$

where $Q = -(\hbar^2/2m_0)(\Delta\sqrt{\rho}/\sqrt{\rho})$. The Nottale approach is used as above with $d_v \sim d_{\mathcal{V}}$ and $d_u \sim d_{\mathcal{U}}$. One assumes that the velocity field from the hydrodynamical model agrees with the real part v of the complex velocity $V = v - iu$ so $v = (1/m_0)\nabla s \sim 2\mathcal{D}\partial s$ and $u = -(1/m_0)\nabla\sigma \sim \mathcal{D}\partial log(\rho)$ where $\mathcal{D} = \hbar/2m_0$. In this context the quantum potential $Q = -(\hbar^2/2m_0)\Delta\mathcal{D}\sqrt{\rho}/\sqrt{\rho}$ becomes

$$Q = -m_0\mathcal{D}\nabla\cdot u - (1/2)m_0u^2 \sim -(\hbar/2)\partial u - (1/2)m_0u^2 \tag{2.12}$$

Consequently Q arises from the fractal derivative and the nondifferentiability of spacetime again, as in Proposition 2.1. Further one can relate u (and hence Q) to an internal stress tensor whereas the v equations correspond to systems of Navier-Stokes type.

REMARK 1.2.3. Some of the relevant equations for dimension one are collected together later. We note that it is the presence of $\pm$ derivatives that makes possible the introduction of a complex plane to describe velocities and hence QM; one can think of this as the motivation for a complex valued wave function and the nature of the SE.

We go now to [**223**] and will sketch some of the material. Here one extends ideas of Nottale and Ord in order to derive a nonlinear Schrödinger equation (NLSE). Using the hydrodynamic model in [**743**] one added a hydrostatic pressure term to the Euler-Lagrange equations and another possibility is to add instead a kinematic pressure term. The hydrostatic pressure is based on an Euler equation $-\nabla p = \rho g$ where ρ is density and g the gravitational acceleration (note this gives $-p = \rho gx$ in 1-D). In [**743**] one took $\rho = \psi^*\psi$, b a mass-energy parameter, and $-p = \rho$; then the hydrostatic potential is (for $\rho_0 = 1$)

$$b\int g(x)\cdot dr = -b\int\frac{\nabla p}{\rho}\cdot dr = -blog(\rho/\rho_0) = -blog(\psi^*\psi) \tag{2.13}$$

Here $-blog(\psi^*\psi)$ has energy units and explains the nonlinear term of [**111**] which involved

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + U\psi - b[log(\psi^*\psi)]\psi \tag{2.14}$$

A derivation of this equation from the Nelson stochastic QM was given by Lemos (cf. [**588**]). There are moreover some problems since this equation does not obey the homogeneity condition saying that the state $\lambda|\psi>$ is equivalent to $|\psi>$; moreover

(2.14) is not invariant under $\psi \to \lambda\psi$. Further, plane wave solutions to (2.14) do not seem to have a physical interpretion due to extraneous dispersion relations. Finally one would like to have a SE in terms of ψ alone. Note that another NLSE could be obtained by adding kinetic pressure terms $(1/2)\rho v^2$ and taking $\rho = a\psi^*\psi$ where $v = p/m$. Now using the relations from HJ theory $(\psi/\psi^*) = exp[2i\mathfrak{S}(x)/\hbar]$ and $p = \nabla\mathfrak{S}(x) = mv$ one can write $v = -i(\hbar/2m)\nabla log(\psi/\psi^*)$ so that the energy density becomes

$$(1/2)\rho|v|^2 = (a\hbar^2/8m^2)\psi\psi^*\nabla log(\psi/\psi^*)\cdot\nabla log(\psi^*/\psi) \tag{2.15}$$

This leads to a corresponding nonlinear potential associated with the kinematical pressure via $(a\hbar^2/8m^2)\nabla log(\psi/\psi^*)\cdot\nabla log(\psi^*/\psi)$. Hence a candidate NLSE is

$$i\hbar\partial_t = -\frac{\hbar^2}{2m}\nabla^2\psi + U\psi - b[log(\psi^*\psi)]\psi + \frac{a\hbar^2}{8m^2}\left(\nabla log\frac{\psi}{\psi^*}\cdot\nabla log\frac{\psi^*}{\psi}\right) \tag{2.16}$$

Here the Hamiltonian is Hermitian and $a \neq b$ are both mass-energy parameters to be determined experimentally. The new term can also be written in the form $\nabla log(\psi/\psi^*)\cdot\nabla log(\psi^*/\psi) = -[\nabla log(\psi/\psi^*)]^2$. The goal now is to derive a NLSE directly from fractal space time dynamics for a particle undergoing Brownian motion. This does not require a quantum potential, a hydrodynamic model, or any pressure terms as above.

REMARK 1.2.4. One should make some comments about the kinematic pressure terms $(1/2)\rho v^2 \sim (\hbar^2/2m)(a/m)|\nabla log(\psi)|^2$ versus hydrostatic pressure terms of the form $\int(\nabla p/\rho) \sim -blog(\psi^*\psi)$. The hydrostatic term breaks homogeneity whereas the kinematic pressure term preserves homogeneity (scaling with a λ factor). The hydrostatic pressure term is also not compatible with the motion kinematics of a particle executing a fractal Brownian motion. The fractal formulation will enable one to relate the parameters a, b to $\hbar$.

Following Nottale, nondifferentiability implies a loss of causality and one is thinking of Feynmann paths with $< v^2 > \propto (dx/dt)^2 \propto dt^{2[(1/D)-1)}$ with $D = 2$. Now a fractal function $f(x,\epsilon)$ could have a derivative $\partial f/\partial\epsilon$ and renormalization group arguments lead to $(\partial f(x,\epsilon)/\partial log\epsilon) = a(x) + bf(x,\epsilon)$ (cf. [**715**]). This can be integrated to give $f(x,\epsilon) = f_0(x)[1-\zeta(x)(\lambda/\epsilon)^{-b}]$. Here $\lambda^{-b}\zeta(x)$ is an integration constant and $f_0(x) = -a(x)/b$. This says that any fractal function can be approximated by the sum of two terms, one independent of the resolution and the other resolution dependent; $\zeta(x)$ is expected to be a flucuating function with zero mean. Provided $a \neq 0$ and $b < 0$ one has two interesting cases (i) $\epsilon << \lambda$ with $f(x,\epsilon) \sim f_0(x)(\lambda/\epsilon)^{-b}$ and (ii) $\epsilon >> \lambda$ with f independent of scale. Here λ is the deBroglie wavelength. Now one writes

$$r(t+dt,dt) - r(t,dt) = b_+(r,t)dt + \xi_+(t,dt)\left(\frac{dt}{\tau_0}\right)^\beta; \tag{2.17}$$

$$r(t,dt) - r(t-dt,dt) - b_-(r,t)dt + \xi_-(t,dt)\left(\frac{dt}{\tau_0}\right)^\beta$$

where $\beta = 1/D$ and $b_\pm$ are average forward and backward velocities. This leads to $v_\pm(r,t,dt) = b_\pm(r,t) + \xi_\pm(t,dt)(dt/\tau_0)^{\beta-1}$. In the quantum case $D = 2$ one has $\beta = 1/2$ so $dt^{\beta-1}$ is a divergent quantity (i.e. nondifferentiability ensues). Following [**588, 715, 698**] one defines

$$\frac{d_{\pm}r(t)}{dt} = lim_{\Delta t \to \pm 0}\left\langle \frac{r(t+\Delta t)-r(t)}{\Delta t}\right\rangle \tag{2.18}$$

from which $d_{\pm}r(t)/dt = b_{\pm}$. Now following Nottale one writes

$$\frac{\delta}{dt} = \frac{1}{2}\left(\frac{d_+}{dt}+\frac{d_-}{dt}\right) - \frac{i}{2}\left(\frac{d_+}{dt}-\frac{d_-}{dt}\right) \tag{2.19}$$

which leads to $(\delta/dt) = (\partial/\partial t) + v\cdot\nabla - iD\nabla^2$. Here in principle $\mathcal{D}$ is a real valued diffusion constant to be related to $\hbar$. and $< d\xi_{\pm i}d\xi_{\pm j} >= \pm 2\mathcal{D}\delta_{ij}dt$. Now for the complex time dependent wave function we take $\psi = exp[i\mathfrak{S}/2m\mathcal{D}]$ with $p = \nabla\mathfrak{S}$ so that $v = -2iD\nabla log(\psi)$. The SE is obtained from the Newton equation ($F = ma$) via $-\nabla U = m(\delta/dt)v = -2im\mathcal{D}(\delta/dt)\nabla log(\psi)$ which yields

$$-\nabla U = -2im[\mathcal{D}\partial_t\nabla log(\psi)] - 2\mathcal{D}\nabla\left(\mathcal{D}\frac{\nabla^2\psi}{\psi}\right) \tag{2.20}$$

(see [**715**] for identities involving ∇). Integrating yields $\mathcal{D}^2\nabla^2\psi + i\mathcal{D}\partial_t\psi - (U/2m)\psi = 0$ up to an arbitrary phase factor which may be set equal to zero. Now replacing D by $\hbar/2m$ one gets the SE $i\hbar\partial_t\psi + (\hbar^2/2m)\nabla^2\psi = U\psi$. Here the Hamiltonian is Hermitian, the equation is linear, and the equation is homogeneous of degree 1 under the substitution $\psi \to \lambda\psi$.

Next one generalizes this by relaxing the assumption that the diffusion coefficient is real. Some comments on complex energies are needed - in particular constraints are often needed (cf. [**788**]). However complex energies are not alien in ordinary QM (cf. [**223**] for references). Now the imaginary part of the linear SE yields the continuity equation $\partial_t\rho + \nabla\cdot(\rho v) = 0$ and with a complex potential the imaginary part of the potential will act as a source term in the continuity equation. Instead of $< d\zeta_{\pm}d\zeta_{\pm} >= \pm 2\mathcal{D}dt$ with $\mathcal{D}$ and $2mD = \hbar$ real one sets

$$< d\zeta_{\pm}d\zeta_{\pm} >= \pm(\mathcal{D}+\mathcal{D}^*)dt;\ 2m\mathcal{D} = \hbar = \alpha + i\beta \tag{2.21}$$

The complex time derivative operator becomes $(\delta/dt) = \partial_t + v\cdot\nabla - (i/2)(\mathcal{D}+\mathcal{D}^*)\nabla^2$. Writing again $\psi = exp[i\mathfrak{S}/2mD] = exp(i\mathfrak{S}/\hbar)$ one obtains $v = -2i\mathcal{D}\nabla log(\psi)$. The NLSE is then obtained (via the Newton law) via the relation $-\nabla U = m(\delta/dt)v = -2im\mathcal{D}(\delta/dt)\nabla log(\psi)$. Combining equations yields then

$$\nabla U = 2im[\mathcal{D}\partial_t\nabla log(\psi) - 2i\mathcal{D}^2(\nabla log(\psi)\cdot\nabla)(\nabla log(\psi) - \tag{2.22}$$
$$-\frac{i}{2}(\mathcal{D}+\mathcal{D}^*)\mathcal{D}\nabla^2(\nabla log(\psi)]$$

Now using the identities (i) $\nabla\nabla^2 = \nabla^2\nabla$, (ii) $2(\nabla log(\psi)\cdot\nabla)(\nabla log(\psi) = \nabla(\nabla log(\psi))^2$ and (iii) $\nabla^2 log(\psi) = \nabla^2\psi/\psi - (\nabla log(\psi))^2$ leads to a NLSE with nonlinear (kinematic pressure) potential, namely

$$i\hbar\partial_t\psi = -\frac{\hbar^2}{2m}\frac{\alpha}{\hbar}\nabla^2\psi + U\psi - i\frac{\hbar^2}{2m}\frac{\beta}{\hbar}(\nabla log(\psi))^2\psi \tag{2.23}$$

Note the crucial minus sign in front of the kinematic pressure term and also that $\hbar = \alpha + i\beta = 2m\mathcal{D}$ is complex. When $\beta = 0$ one recovers the linear SE. The nonlinear potential is complex and one defines $W = -(\hbar^2/2m)(\beta/\hbar)(\nabla log(\psi))^2$ with U the ordinary potential; then the NLSE is

$$i\hbar\partial_t\psi = [-(\hbar^2/2m)(\alpha/\hbar)\nabla^2 + U + iW]\psi \tag{2.24}$$

This is the fundamental result of [**223**]; it has the form of an ordinary SE with complex potential $U+iW$ and complex $\hbar$. The Hamiltonian is no longer Hermitian and the potential itself depends on ψ. Nevertheless one can have meaningful physical solutions with real valued energies and momenta; the homogeneity breaking hydrostatic pressure term $-b(log(\psi^*\psi)\psi$ is not present (it would be meaningless) and the NLSE is invariant under $\psi \to \lambda\psi$.

REMARK 1.2.5. One could ask why not simply propose as a valid NLSE an equation

$$i\hbar\partial_t\psi = -\frac{\hbar^2}{2m}\nabla^2\psi + \frac{\hbar^2}{2m}\frac{a}{m}|\nabla log(\psi)|^2\psi$$

Here one has a real Hamiltonian satisfying the homogeneity condition and the equation admits soliton solutions of the form $\psi = CA(x-vt)exp[i(kx-\omega t)]$ where $A(x-vt)$ is to be determined by solving the NLSE. The problem here is that the equation suffers from an extraneous dispersion relation. Thus putting in the plane wave solution $\psi \sim exp[-i(Et-px)]$ one gets an extraneous energy momentum (EM) relation (after setting $U=0$), namely $E = (p^2/2m)[1+(a/m)]$ instead of the usual $E = p^2/2m$ and hence $E_{QM} \neq E_{FT}$ where FT means field theory.

REMARK 1.2.6. It has been known since e.g. [**788**] that the expression for the energy functional in nonlinear QM does not coincide with the QM energy functional, nor is it unique. To see this write down the NLSE of [**111**] in the form $i\hbar\partial_t\psi = \partial H(\psi,\psi^*)/\partial\psi^*$ where the real Hamiltonian density is

$$H(\psi,\psi^*) = -\frac{\hbar^2}{2m}\psi^*\nabla^2\psi + U\psi^*\psi - b\psi^*log(\psi^*\psi)\psi + b\psi^*\psi$$

Then using $E_{FT} = \int Hd^3r$ we see it is different from $<\hat{H}>_{QM}$ and in fact $E_{FT} - E_{QM} = \int b\psi^*\psi d^3r = b$. This problem does not occur in the fractal based NLSE since it is written entirely in terms of ψ.

REMARK 1.2.7. In the fractal based NLSE there is no discrepancy between the QM energy functional and the FT energy functional. Both are given by

$$N^{NLSE}_{fractal} = -\frac{\hbar^2}{2m}\frac{\alpha}{\hbar}\psi^*\nabla^2\psi + U\psi^*\psi - i\frac{\hbar^2}{2m}\frac{\beta}{\hbar}\psi^*(\nabla log(\psi)^2\psi$$

The NLSE is unambiguously given by in Remark 1.2.5 and $H(\psi,\psi^*)$ is homogeneous of degree 1 in λ. Such equations admit plane wave solutions with dispersion relation $E = p^2/2m$; indeed, inserting the plane wave solution into the fractal based NLSE one gets (after setting $U=0$)

$$E = \frac{\hbar^2}{2m}\frac{\alpha}{\hbar}\frac{p^2}{2m} + i\frac{\beta}{\hbar}\frac{p^2}{2m} = \frac{p^2}{2m}\frac{\alpha+i\beta}{\hbar} = \frac{p^2}{2m} \tag{2.25}$$

since $\hbar = \alpha + i\beta$. The remarkable feature of the fractal approach versus all other NLSE considered sofar is that the QM energy functional is precisely the FT one. The complex diffusion constant represents a truly new physical phenomenon insofar as a small imaginary correction to the Planck constant is the hallmark of nonlinearity in QM (see [**223**] for more on this).

REMARK 1.2.8. Some refinements of the Nottale derivation are given in [**272**] and we consider $x \to f(x(t),t) \in C^n$ with $X(t) \in H^{1/n}$ (i.e. $c\epsilon^{1/n} \leq |X(t') - X(t)| \leq C\epsilon^{1/n}$). Define ($f$ real valued)

$$\nabla^\epsilon_\pm f(t) = \frac{f(t \pm \epsilon) - f(t)}{\pm\epsilon};\ \frac{\Box_\epsilon f}{\Box t}(f) = \frac{1}{2}(\nabla^\epsilon_+ + \nabla^\epsilon_-)f - \frac{i}{2}(\nabla^\epsilon_+ - \nabla^\epsilon_- f); \tag{2.26}$$

$$a_{\epsilon,j}(t) = \frac{1}{2}[(\Delta^\epsilon_+ x)^j - (-1)^j(\Delta^\epsilon_- x)^j] - \frac{i}{2}[(\Delta^\epsilon_+ x)^j + (-1)^j(\Delta^\epsilon_- x)^j]$$

Here one assumes $h > 0$ and $\epsilon(f,h) \geq \epsilon > 0$ where $\epsilon(f,h)$ is the minimal resolution defined via $inf_\epsilon\{a_\epsilon(f) < h\}$ for $a_\epsilon f(t) = |[f(t+\epsilon)+f(t-\epsilon)-2f(t)]/\epsilon|$. If $\epsilon(f,h)$ is not 0 then f is not differentiable (but not conversely). Now assume some minimal control over the lack of differentiability (cf. [**272**]) and then for f now complex valued with $\Box_\epsilon f/\Box t = (\Box_\epsilon f_\Re/\Box t) + i(\Box_\epsilon f_\Im/\Box t)$ (note the mixing of i terms is not trivial) one has

$$\frac{\Box_\epsilon f}{\Box t} = \frac{\partial f}{\partial t} + \frac{\Box_\epsilon x}{\Box t}\frac{\partial f}{\partial x} + \sum_2^n \frac{1}{j!}a_{\epsilon,j}(t)\frac{\partial^j f}{\partial x^j}\epsilon^{j-1} + o(\epsilon^{1/n}) \tag{2.27}$$

We sketch now the derivation of a SE in the spirit of Nottale but with more mathematical polish. Going to [**272**] one defines (for a nondifferentiable function f)

$$f_\epsilon(t) = \frac{1}{2\epsilon}\int_{t-\epsilon}^{t+\epsilon} f(s)ds; \tag{2.28}$$

$$f^+_\epsilon(t) = \frac{1}{2\epsilon}\int_t^{t+\epsilon} f(s)ds;\ f^-_\epsilon(t) = \frac{1}{2\epsilon}\int_{t-\epsilon}^t f(s)ds$$

One considers quantum paths à la Feynman so that $lim_{t\to t'}[X(t)-X(t')]^2/(t-t')$ exists. This implies $X(t) \in H^{1/2}$ where H^α means $c\epsilon^\alpha \leq |X(t)-X(t')| \leq C\epsilon^\alpha$ and from Remark 1.2.1 for example this means $dim_H X[a,b] = 1/2$. Next, thinking of classical Lagrangians $L(x,v,t) = (1/2)mv^2 + \mathbf{U}(x,t)$, one defines an operator Q via ($(x,t,v) \sim$ classical variables)

$$Q(t) = t;\ Q(x(t)) = X(t);\ Q(v(t)) = \mathcal{V}(t);\ Q\left(\frac{df}{dt}\right) = Q\left(\frac{d}{dt}\right)\cdot Q(f) \tag{2.29}$$

where $Q(d/dt) = d/dt$ if $Q(f)(t)$ is differentiable and $Q(d/dt) = \Box_\epsilon/\Box t$ where $\epsilon(x,h) > \epsilon > 0$ if $Q(f)(t)$ is nondifferentiable. Note $\mathcal{V}(t) = Q(d/dt)[X(t)]$ so regularity of X determines the form of Q here and for $Q(x) = X \in H^{1/2}$ one has $\mathcal{V} = \Box_\epsilon X/\Box t$. The scalar Euler-Lagrange (EL) equation associated to $\mathcal{L}(X(t),\mathcal{V}(t),t) = Q(L(x(t),v(t),t)$ is

$$\frac{\Box_\epsilon}{\Box t}\left(\frac{\partial\mathcal{L}}{\partial\mathcal{V}}(X(t),\mathcal{V}(t),t)\right) = \frac{\partial\mathcal{L}}{\partial X}(X(t),\mathcal{V}(t),t) \tag{2.30}$$

Now given a classical $v \sim (1/m)(\partial S/\partial x)$ one gets $\mathcal{V} = (1/m)(\partial \mathfrak{S}/\partial X)$ and $\mathcal{L} = (\partial \mathfrak{S}/\partial t)$ with $\psi(X,t) = exp[i\mathfrak{S}(X,t)/2m\gamma]$. For $\mathcal{L} \sim (1/2)m\mathcal{V}^2 + \mathfrak{U}$ then the quantum (EL) equation is $m(\Box_\epsilon \mathcal{V}/\Box t) = (\partial \mathfrak{U}/\partial X)$ leading to

(2.31)
$$2i\gamma m\left[-\frac{\psi_X^2}{\psi}\left(i\gamma + \frac{a_\epsilon(t)}{2}\right) + \partial_t\psi + \frac{a_\epsilon(t)}{2}\frac{\partial^2\psi}{\partial X^2}\right] = (\mathfrak{U}(X) + \alpha(X))\psi + o(\epsilon^{1/2})$$

where

$$a_\epsilon(t) = \frac{1}{2}\{[(\nabla_+^\epsilon X(t))^2 - (\nabla_-^\epsilon X(t))^2] - i[(\nabla_+^\epsilon X(t))^2 + (\nabla_-^\epsilon X(t))^2]\} \quad (2.32)$$

Then (2.32) is called the generalized SE and the nonlinear character of such equations is discussed in [**192, 223**] for example. In [**272**] one then arrives at a conventional looking SE under the assumption $a_\epsilon = -2i\gamma$, leading to

$$\gamma^2\frac{\partial^2\psi}{\partial X^2} + i\gamma\frac{\partial\psi}{\partial t} = [\mathfrak{U}(X,t) + \alpha(X)]\frac{\psi}{2m} + o(\epsilon^{1/2}) \quad (2.33)$$

One can then always take $\alpha(X) = 0$ and choosing $\gamma = \hbar/2m$ one arrives at $i\hbar\psi_t + (\hbar^2/2m)(\partial^2\psi/\partial t^2) = \mathfrak{U}\psi$. However the requirement $a_\epsilon(t) = -2i\gamma$ seems quite restrictive.

- Note here that the argument using $a_\pm$ is rigorous via [**272**]. $a_\epsilon = -i\hbar/m$ is permissible and in fact can have solutions of $\nabla_\sigma^\epsilon X(t) = constant$ via $X_c(t) = \pm\sqrt{\hbar/2m)}(t - c - (\epsilon/2)) + P_\epsilon(t)$ where $P_\epsilon \in H^{1/2}$ is an arbitrary periodic function.

Referring back to Example 1.2.3 we have $b_\pm(t)(t) \sim \Box_\pm x(t)$ and $V = (/2)(\Box_+ x + \Box_- x)(t)$ with $U = (1/2)(\Box_+ x - \Box_- x)(t)$. The relation between U and the quantum potential Q will formally still hold (cf. also [**273**] on nondifferentiable variational principles) and one can rewrite this as $\sqrt{\rho}U = (\hbar/m)\partial\sqrt{\rho}$; $\sqrt{\rho}Q = -(\hbar^2/2m)\partial^2\sqrt{\rho}$ along with $\partial(\sqrt{\rho}U) = -(2/\hbar)\sqrt{\rho}Q$. If U is not differentiable one could also look at $\sqrt{\rho}U = -(2/\hbar)\int_0^X \sqrt{\rho}QdX' + f(t)$ with $f(t)$ possibly determinable via the term $(\sqrt{\rho}U)(0,t)$.

3. REMARKS ON FRACTAL SPACETIME

There have been a number of articles and books involving fractal methods in spacetime or fractal spacetime itself with impetus coming from quantum physics and relativity. We refer here especially to [**1, 186, 187, 225, 422, 675, 676, 677, 678, 679, 680, 681, 682, 683, 684, 685, 686, 687, 688, 689, 690**] for background to this paper. Many related papers are omitted here and we refer in particular to the journal Chaos, Solitons, and Fractals CSF) for further information. For information on fractals and stochastic processes we refer for example to [**33, 83, 241, 242, 243, 345, 423, 555, 562, 592, 643, 625, 697, 725, 748, 763, 810, 918, 942, 985**]. We discuss here a few background ideas and constructions in order to indicate the ingredients for El Naschie's Cantorian spacetime $\mathfrak{E}^\infty$, whose exact nature is elusive. Suitable references are given but there are many more papers in the journal CSF by El Naschie (and others) based on these fundamental ideas and these are either important in a revolutionary sense or a fascinating refined form of science fiction. In what appears at times to be pure numerology one manages to (rather hastily) produce amazingly close numerical approximations to virtually all the fundamental constants of physics (including string theory). The key concepts revolve around the famous golden ratio $(\sqrt{5}-1)/2$ and a strange Cantorian space $\mathfrak{E}^{\infty}$ which we try to describe below. It is very tempting to want all of these (heuristic) results to be true and the approach seems close enough and universal enough to compel one to think something very important must be involved. Moreover such scope and accuracy cannot be ignored so we try to examine some of the constructions in a didactic manner in order to possibly generate some understanding.

3.1. COMMENTS ON CANTOR SETS.

EXAMPLE 3.1. In the paper [**643**] one discusses random recursive constructions leading to Cantor sets, etc. Associated with each such construction is a universal number α such that almost surely the random object has Hausdorff dimension α (we assume that ideas of Hausdorff and Minkowski-Bouligand (MB) or upper box dimension are known - cf. [**83, 186, 345, 592**]). One construction of a Cantor set goes as follows. Choose x from $[0,1]$ according to the uniform distribution and then choose y from $[x,1]$ according to the uniform distribution on $[x,1]$. Set $J_0=[0,x]$ and $J_1=[y,1]$ and recall the standard 1/3 construction for Cantor sets. Continue this procedure by rescaling to each of the intervals already obtained. With probability one one then obtains a Cantor set S_c^0 with Hausdorff dimension $\alpha=\phi=(\sqrt{5}-1)/2\sim .618$. Note that this is just a particular random Cantor set; there are others with different Hausdorff dimensions (there seems to be some - possibly harmless - confusion on this point in the El Naschie papers). However the golden ratio ϕ is a very interesting number whose importance rivals that of π or e. In particular (cf. [**1**]) ϕ is the hardest number to approximate by rational numbers and could be called the most irrational number. This is because its continued fraction represention involves all $1's$.

EXAMPLE 3.2. From [**676**] the Hausdorff (H) dimension of a traditional triadic Cantor set is $d_c^{(0)}=log(2)/log(3)$. To determine the equivalent to a triadic Cantor set in 2 dimensions one looks for a set which is triadic Cantorian in all directions. The analogue of an area $A=1\times 1$ is a quasi-area $A_c=d_c^{(0)}\times d_c^{(0)}$ and to normalize A_c one uses $\rho_2=(A/A_c)_2=1/(d_c^{(0)})^2$ (for n-dimensions $\rho_n=1/(d_c^{(0)})^{n-1}$). Then the n^{th} Cantor like H dimension $d_c^{(n)}$ will have the form $d_c^{(n)}=\rho_n d_c^{(0)}=1/(d_c^{(0)})^{n-1}$. Note also that the H dimension of a Sierpinski gasket is $d_c^{(n+1)}/d_c^{(n)}=1/d_c^{(0)}=log(3)/log(2)$ and in any event the straight-forward interpretation of $d_c^{(2)}=log(3)/log(2)$ is a scaling of $d_c^{(0)}=log(2)/log(3)$ proportional to the ratio of areas $(A/A_c)_2$. One notes that $d_c^{(4)}=1/(d_c^{(0)})^3=(log(3)/log(2))^3\simeq 3.997\sim 4$ so the 4-dimensional Cantor set is essentially “space filling”.

Another derivation goes as follows. Define probability quotients via $\Omega=dim(subset)/dim(set)$. For a triadic Cantor set in 1-D $\Omega^{(1)}=d_c^{(0)}/d_c^{(1)}=d_c^{(0)}$ $(d_c^{(1)}=1)$. To lift the Cantor set to n-dimensions look at the multiplicative probability law $\Omega^{(n)} = (\Omega^{(1)})^n = (d_c^{(0)})^n$. However since $\Omega^{(1)} = d_c^{(0)}/d_c^{(n)}$ we get

$$d_c^{(0)}/d_c^{(n)} = (d_c^{(0)})^n \Rightarrow d_c^{(n)} = 1/(d_c^{(0)})^{n-1} \tag{3.1}$$

Since $\Omega^{(n-1)}$ is the probability of finding a Cantor point (Cantorian) one can think of the H dimension $d_c^{(n)} = 1/\Omega^{(n-1)}$ as a measure of ignorance. One notes here also that for $d_c^{(0)} = \phi$ (the Cantor set $S_c^{(0)}$ of Example 3.1) one has $d_c^{(4)} = 1/\phi^3 = 4 + \phi^3 \simeq 4.236$ which is surely space filling.

Based on these ideas one proves in [**676, 680, 682**] a number of theorems and we sketch some of this here. One picks a "backbone" Cantor set with H dimension $d_c^{(0)}$ (the choice of $\phi = d_c^{(0)}$ will turn out to be optimal for many arguments). Then one imagines a Cantorian spacetime $\mathfrak{E}^\infty$ built up of an infinite number of spaces of dimension $d_c^{(n)}$ $(-\infty \leq n < \infty)$. The exact form of embedding etc. here is not specified so one imagines e.g. $\mathfrak{E}^\infty = \cup \mathfrak{E}^{(n)}$ (with unions and intersections) in some amorphous sense. There are some connections of this to vonNeumann's continuous geometries indicated in [**684**]. In this connection we remark that only $\mathfrak{E}^{(-\infty)}$ is the completely empty set ($\mathfrak{E}^{(-1)}$ is not empty). First we note that $\phi^2 + \phi - 1 = 0$ leading to

$$1 + \phi = 1/\phi,\ \phi^3 = (2+\phi)/\phi,\ (1+\phi)/(1-\phi) = 1/\phi(1-\phi) = 4 + \phi^3 = 1/\phi^3 \tag{3.2}$$

(a very interesting number indeed). Then one asserts that

THEOREM 3.1. Let $(\Omega^{(1)})^n$ be a geometrical measure in n-dimensional space of a multiplicative point set process and $\Omega^{(1)}$ be the Hausdorff dimension of the backbone (generating) set $d_c^{(0)}$. Then $< d >= 1/d_c^{(0)}(1 - d_c^{(0)})$ (called curiously an average Hausdorff dimension) will be exactly equal to the average space dimension $\tilde{}< n >= (1 + d_c^{(0)})(1 - d_c^{(0)})$ and equivalent to a 4-dimensional Cantor set with H-dimension $d_c^{(4)} = 1/(d_c^{(0)})^3$ if and only if $d_c^{(0)} = \phi$.

To see this take $\Omega^{(n)} = (\Omega^{(1)})^n$ again and consider the total probability of the additive set described by the $\Omega^{(n)}$, namely $Z_0 = \sum_0^\infty (\Omega^{(1)})^n = 1/(1 - \Omega^{(1)})$. It is conceptually easier here to regard this as a sum of weighted dimensions (since $d_c^{(n)} = 1/(d_c^{(0)})^{n-1}$) and consider $w_n = n(d_c^{(0)})^n$. Then the expectation of n becomes (note $d_c^{(n)} \sim 1/(d_c^{(0)})^{n-1} \sim 1/\Omega^{(n-1)}$ so $n(d_c^{(0)})^{n-1} \sim n/d_c^{(n)}$)

$$E(n) = \frac{\sum_1^\infty n^2 (d_c^{(0)})^{n-1}}{\sum_1^\infty n (d_c^{(0)})^{n-1}} = \tilde{}< n >= \frac{1 + d_c^{(0)}}{1 - d_c^{(0)}} \tag{3.3}$$

Another average here is defined via (blackbody gamma distribution)

$$< n >= \frac{\int_0^\infty n^2 (\Omega^{(1)})^n dn}{\int_0^\infty n (\Omega^{(1)})^n dn} = \frac{-2}{log(\Omega^{(1)})} \tag{3.4}$$

which corresponds to $\tilde{}<n>$ after expanding the logarithm and omitting higher order terms. However $\tilde{}<n>$ seems to be the more valid calculation here. Similarly one defines (somewhat ambiguously) an expected value for $d_c^{(n)}$ via

$$<d>=\frac{\sum_1^\infty n(d_c^{(0)})^{n-1}}{\sum_1^\infty (d_c^{(0)})^n}=\frac{1}{d_c^{(0)}(1-d_c^{(0)})} \tag{3.5}$$

This is contrived of course (and cannot represent $E(d_c^{(n)})$ since one is computing reciprocals $\sum(n/d_c^{(n)})$ but we could think of computing an expected ignorance and identifying this with the reciprocal of dimension. Thus the label $<d>$ does not seem to represent an expected dimension but if we accept it as a symbol then for $d_c^{(0)}=\phi$ one has

$$\tilde{}<n>=\frac{1+\phi}{1-\phi}=<d>=\frac{1}{\phi(1-\phi)}=d_c^{(4)}=4+\phi^3=\frac{1}{\phi^3}\sim 4.236 \tag{3.6}$$

REMARK 1.3.1. We note that the normalized probability $N=\Omega^{(1)}/Z_0=\Omega^{(1)}(1-\Omega^{(1)})=1/<d>$ for any $d_c^{(0)}$. Further if $<d>=4=1/d_c^{(0)}(1-d_c^{(0)})$ one has $d_c^{(0)}=1/2$ while $\tilde{}<n>=3<4=<d>$. One sees also that $d_c^{(0)}=1/2$ is the minimum (where $d<d>/d(d_c^{(0)})=0$).

REMARK 1.3.2. The results of Theorem 3.1 should really be phrased in terms of $\mathfrak{E}^\infty$ (cf. [**685**]). thus ($H\sim$ Hausdorff dimension and $T\sim$ topological dimension)

$$dim_H\mathfrak{E}^{(n)}=d_c^{(n)}=\frac{1}{(d_c^{(0)})^{n-1}}; \tag{3.7}$$

$$<d>=\frac{1}{d_c^{(0)}(1-d_c^{(0)})};\ \tilde{}<dim_T\mathfrak{E}^\infty>=\frac{1+d_c^{(0)}}{1-d_c^{(0)}}=\tilde{}<n>$$

In any event $\mathfrak{E}^\infty$ is formally infinite dimensional but effectively it is $4\pm$ dimensional with an infinite number of internal dimensions. We emphasize that $\mathfrak{E}^\infty$ appears to be constructed from a fixed backbone Cantor set with H dimension $1/2\le d_c^{(0)}<1$; thus each such $d_c^{(0)}$ generates an $\mathfrak{E}^\infty$ space. Note that in [**685**] $\mathfrak{E}^\infty$ is looked upon as a transfinite discretum underpinning the continuum (whatever that means).

REMARK 1.3.3. An interesting argument from [**684**] goes as follows. Thinking of $d_c^{(0)}$ as a geometrical probability one could say that the spatial (3-dimensional) probability of finding a Cantorian "point" in $\mathfrak{E}^\infty$ must be given by the intersection probability $P=(d_c^{(0)})^3$ where $3\sim 3$ topological spatial dimension. P could then be regarded as a Hurst exponent (cf. [**1, 715, 985**]) and the Hausdorff dimension of the fractal path of a Cantorian would be $d_{path}=1/H=1/P=1/(d_c^{(0)})^3$. Given $d_c^{(0)}=\phi$ this means $d_{path}=4+\phi^3\sim 4^+$ so a Cantorian in 3-D would sweep out a 4-D world sheet; i.e. the time dimension is created by the Cantorian space $\mathfrak{E}^\infty$ (! - ?). Conjecturing further (wildly) one could say that perhaps space (and gravity) is created by the fractality of time. This is a typical form of conjecture to be found in the El Naschie papers - extremely thought provoking but ultimately heuristic. Regarding the Hurst exponent one recalls that for Feynmann trajectories in $1+1$ dimensions $d_{path} = 1/H = 1/d_c^{(0)} = d_c^{(2)}$. Thus we are concerned with relating the two determinations of d_{path} (among other matters). Note that path dimension is often thought of as a fractal dimension (M-B or box dimension), which is not necessarily the same as the Hausdorff dimension. However in [**29**] one shows that quantum mechanical free motion produces fractal paths of Hausdorff dimension 2 (cf. also [**583**]).

REMARK 1.3.4. Following [**226**] let $S_c^{(0)}$ correspond to the set with dimension $d_c^{(0)} = \phi$. Then the complementary dimension is $\tilde{d}_c^{(0)} = 1 - \phi = \phi^2$. The path dimension is given as in Remark 1.3.3 by $d_{path} = d_c^{(2)} = 1/\phi = 1+\phi$ and $\tilde{d}_{path} = \tilde{d}_c^{(2)} = 1/(1-\phi) = 1/\phi^2 = (1+\phi)^2$. Following El Naschie for an equivalence between unions and intersections in a given space one requires (in the present situation) that

(3.8)
$$d_{crit} = d_c^{(2)} + \tilde{d}_c^{(2)} = \frac{1}{\phi} + \frac{1}{\phi^2} = \frac{\phi(1+\phi)}{\phi^3} = \frac{1}{\phi^3} = \frac{1}{\phi} \cdot \frac{1}{\phi^2} = d_c^{(2)} \cdot \tilde{d}_c^{(2)} = 4 + \phi^3$$

where $d_{crit} = 4 + \phi^3 = d_c^{(4)} \sim 4.236$. Thus the critical dimension coincides with the Hausdorff dimension of $S_c^{(4)}$ which is embedded densely into a smooth space of topological dimension 4. On the other hand the backbone set of dimension $d_c^{(0)} = \phi$ is embedded densely into a set of topological dimension zero (a point). Thus one thinks in general of $d_c^{(n)}$ as the H dimension of a Cantor set of dimension ϕ embedded into a smooth space of integer topological dimension n.

REMARK 1.3.5. In [**226**] it is also shown that realization of the spaces $\mathfrak{E}^{(n)}$ comprising $\mathfrak{E}^{\infty}$ can be expressed via the fractal sprays of Lapidus-van Frankenhuysen (cf. [**592**]). Thus we refer to [**592**] for graphics and details and simply sketch some ideas here (with apologies to M. Lapidus). A fractal string is a bounded open subset of $\mathbf{R}$ which is a disjoint union of an infinite number of open intervals $\mathfrak{L} = \ell_1, \ell_2, \cdots$. The geometric zeta function of $\mathfrak{L}$ is $\zeta_{\mathfrak{L}}(s) = \sum_1^{\infty} \ell_j^{-s}$. One assumes a suitable meromorphic extension of $\zeta_{\mathfrak{L}}$ and the complex dimensions of $\mathfrak{L}$ are defined as the poles of this meromorphic extension. The spectrum of $\mathfrak{L}$ is the sequence of frequencies $f = k \cdot \ell_j^{-1}$ $(k = 1, 2, \cdots)$ and the spectral zeta function of $\mathfrak{L}$ is defined as $\zeta_{\nu}(s) = \sum_f f^{-s}$ where in fact $\zeta_{\nu}(s) = \zeta_{\mathfrak{L}}(s)\zeta(s)$ (with $\zeta(s)$ the classical Riemann zeta function). Fractal sprays are higher dimensional generalizations of fractal strings. As an example consider the spray Ω obtained by scaling an open square B of size 1 by the lengths of the standard triadic Cantor string CS. Thus Ω consists of one open square of size 1/3, 2 open squares of size 1/9, 4 open squares of size 1/27, etc. (see [**592**] for pictures and explanations). Then the spectral zeta function for the Dirichlet Laplacian on the square is $\zeta_B(s) = \sum_{n_1,n_2=1}^{\infty} (n_1^2 + n_2^2)^{s/2}$ and the spectral zeta function of the spray is $\zeta_{\nu}(s) = \zeta_{CS}(s) \cdot \zeta_B(s)$. Now $\mathfrak{E}^{\infty}$ is composed of an infinite hierarchy of sets $\mathfrak{E}^{(j)}$ with dimension $(1+\phi)^{j-1} = 1/\phi^{j-1}$ $(j = 0, \pm 1, \pm 2, \cdots)$ and these sets correspond to a special case of boundaries $\partial\Omega$ for fractal sprays Ω whose scaling ratios are suitable binary powers of $2^{-\phi^{j-1}}$. Indeed for $n = 2$ the spectral zeta function of the fractal golden spray indicated above is $\zeta_\nu(s) = (1/(1 - 2 \cdot 2^{s\phi})\zeta_B(s)$. The poles of $\zeta_B(s)$ do not coincide with the zeros of the denominator $1 - 2 \cdot 2^{-s\phi}$ so the (complex) dimensions of the spray correspond to those of the boundary $\partial\Omega$ of Ω. One finds that the real part $\Re s$ of the complex dimensions coincides with $dim\, \mathfrak{E}^{(2)} = 1 + \phi = 1/\phi^2$ and one identifies then $\partial\Omega$ with $\mathfrak{E}^{(2)}$. The procedure generalizes to higher dimensions (with some stipulations) and for dimension n there results $\Re s = 1/\phi^{n-1} = dim\, \mathfrak{E}^{(n)}$. This produces a physical model of the Cantorian fractal space from the boundaries of fractal sprays (see [**226**] for further details and [**592**] for precision). Other (putative) geometric realizations of $\mathfrak{E}^\infty$ are indicated in [**688**] in terms of wild topologies, etc.

3.2. COMMENTS ON HYDRODYNAMICS. We sketch first some material from [**15**] (see also [**294, 715, 718, 720**] and Sections 1-2 for background). Thus let ψ be the wave function of a test particle of mass m_0 in a force field $U(r,t)$ determined via $i\hbar\partial_t\psi = U\psi - (\hbar^2/2m)\nabla^2\psi$ where $\nabla^2 = \Delta$. One writes $\psi(r,t) = R(r,t)exp(iS(r,t))$ with $v = (\hbar/2m)\nabla S$ and $\rho = R \cdot R$ (one assumes $\rho \neq 0$ for physical meaning). Thus the field equations of QM in the hydrodynamic picture are

$$d_t(m_0\rho v) = \partial_t(m_0\rho v) + \nabla(m_0\rho v) = -\rho\nabla(U + Q);\ \partial_t\rho + \nabla \cdot (\rho v) = 0 \tag{3.9}$$

where $Q = -(\hbar^2/2m_0)(\Delta\sqrt{\rho}/\sqrt{\rho})$ is the quantum potential (or interior potential). Now because of the nondifferentiability of spacetime an infinity of geodesics will exist between any couple of points A and B. The ensemble will define the probability amplitude (this is a nice assumption but geodesics should be defined here). At each intermediate point C one can consider the family of incoming (backward) and outgoing (forward) geodesics and define average velocities $b_+(C)$ and $b_-(C)$ on these families. These will be different in general and following Nottale this doubling of the velocity vector is at the origin of the complex nature of QM. Even though Nottale reformulates Nelson's stochastic QM the former's interpretation is profoundly different. While Nelson (cf. [**698**]) assumes an underlying Brownian motion of unknown origin which acts on particles in Minkowskian spacetime, and then introduces nondifferentiability as a byproduct of this hypothesis, Nottale assumes as a fundamental and universal principle that spacetime itself is no longer Minkowskian nor differentiable. An interesting comment here from [**15**] is that with Nelson's Browian motion hypothesis, nondifferentiability is but an approximation which expected to break down at the scale of the underlying collisions, where a new physics should be introduced, while Nottale's hypothesis of nondifferentiability is essential and should hold down to the smallest possible length scales. Following Nelson one defines now the mean forward and backward derivatives

$$\frac{d_\pm}{dt}y(t) = lim_{\Delta t \to 0_\pm}\left\langle \frac{y(t + \Delta t) - y(t)}{\Delta t} \right\rangle \tag{3.10}$$

This gives forward and backward mean velocities $(d_+/dt)x(t) = b_+$ and $(d_-/dt)x(t) = b_-$ for a position vector x. Now in Nelson's stochastic mechanics one writes two systems of equations for the forward and backward processes and combines them in the end in a complex equation, Nottale works from the beginning with a complex derivative operator

$$\frac{\delta}{dt} = \frac{(d_+ + d_-) - i(d_+ - d_-)}{2dt} \tag{3.11}$$

leading to $V = (\delta/dt)x(t) = v - iu = (1/2)(b_+ + b_-) - (i/2)(b_+ - b_-)$. One defines also $(d_v/dt) = (1/2)(d_+ + d_-)/dt$ and $(d_u/dt) = (1/2)(d_+ - d_-)/dt$ so that $d_v x/dt = v$ and $d_u x/dt = u$. Here v generalizes the classical velocity while u is a new quantity arising from nondifferentiability. This leads to a stochastic process satisfying (respectively for the forward ($dt > 0$) and backward ($dt < 0$) processes) $dx(t) = b_+[x(t)] + d\xi_+(t) = b_-[x(t)] + d\xi_-(t)$. The $d\xi(t)$ terms can be seen as fractal functions and they amount to a Wiener process when the fractal dimension $D = 2$. Then the $d\xi(t)$ are Gaussian with mean zero, mutually independent, and satisfy $< d\xi_{\pm i} d\xi_{\pm j} >= \pm 2D\delta_{ij}dt$ where $\mathcal{D}$ is a diffusion coefficient determined as $\mathcal{D} = \hbar/2m_0$ when $\tau_0 = \hbar/(m_0c^2)$ (deBroglie time scale in the rest frame (cf. [**15**]). This allows one to give a general expression for the complex time derivative, namely

$$df = \frac{\partial f}{\partial t} + \nabla f \cdot dx + \frac{1}{2}\frac{\partial^2 f}{\partial x_i \partial x_j} dx_i dx_j \tag{3.12}$$

Next compute the forward and backward derivatives of f; then one will arrive at $< dx_i dx_j > \to < d\xi_{\pm i} d\xi_{\pm j} >$ so the last term in (3.12) amounts to a Laplacian and one obtains $(d_\pm f/dt) = [\partial_t + b_\pm \cdot \nabla \pm \mathcal{D}\Delta]f$ which is an important result. Thus assume the fractal dimension is not 2 in which case there is no longer a cancellation of the scale dependent terms in (3.12) and instead of $\mathcal{D}\Delta f$ one would obtain an explicitly scale dependent behavior proportional to $\delta t^{(2/D)-1}\Delta f$. In other words the value $D = 2$ for the fractal dimension implies that the scale symmetry becomes hidden in the operator formalism. One obtains the complex time derivative operator in the form $(\delta/dt) = \partial_t + V \cdot \nabla - iD\Delta$ (V as above). Nottale's prescription is then to replace d/dt by δ/dt. In this spirit one can write now $\psi = exp(i(\mathfrak{S}/2m_0\mathcal{D}))$ so that $V = -2i\mathcal{D}\nabla(log(\psi))$ and then the generalized Newton equation $-\nabla U = m_0(\delta/dt)V$ reduces to the SE ($L = (1/2)mv^2 - U$).

Now assume the velocity field from the hydrodynamic model agrees with the real part v of the complex velocity V and equate the wave functions from the two models $\psi = exp(i\mathfrak{S}/2m_0\mathcal{D})$ and $\psi = Rexp(iS)$ with $m = m_0$; one obtains for $\mathfrak{S} = s + i\sigma$ the formulas $s = 2m_0\mathcal{D}S$, $\mathcal{D} = (\hbar/2m_0)$, and $\sigma = -m_0\mathcal{D}log(\rho)$. Using the definition $V = (1/m_0)\nabla\mathfrak{S} = (1/m_0)\nabla s + (i/m_0)\nabla\sigma = v - iu$ (which results from the above equations) we get

$$v = (1/m_0)\nabla s = 2D\nabla S;\ u = -(1/m_0)\nabla\sigma = \mathcal{D}\nabla log(\rho) \tag{3.13}$$

Note that the imaginary part of the complex velocity coincides with Nottale. Dividing the time dependent SE $i\hbar\psi_t = U\psi - (\hbar^2/2m_0)\Delta\psi$ by $2m_0$ and taking the gradient gives $\nabla U/m_0 = 2\mathcal{D}\nabla[i\partial_t log(\psi) + \mathcal{D}(\Delta\psi/\psi)]$ where $\hbar/2m_0$ has been replaced by $\mathcal{D}$. Then consider the identities

$$\Delta\nabla = \nabla\Delta;\ (\nabla f \cdot \nabla)(\nabla f) = (1/2)\nabla(\nabla f)^2;\ \frac{\Delta f}{f} = \Delta log(f) + (\nabla log(f))^2 \tag{3.14}$$

Then the second term in the right of the equation for $\nabla U/m_0$ becomes $\nabla(\Delta\psi/\psi) = \Delta(\nabla log(\psi)) + 2(\nabla log(\psi)\cdot\nabla)(\nabla log(\psi))$ so we obtain

$$\nabla U = 2iDm_0[\partial_t \nabla log(\psi) - i\mathcal{D}\Delta(\nabla log(\psi) - 2i\mathcal{D}(\nabla log(\psi)\cdot\nabla)(\nabla log(\psi))] \tag{3.15}$$

One can show that this is nothing but the generalized Newton equation $-\nabla U = m_0(\delta/dt)V$. Now replacing the complex velocity $V = -2i\mathcal{D}\nabla log(\psi)$ and taking into account the form of V, we get

$$-\nabla U = m_0\{\partial_t(v - i\mathcal{D}\nabla log(\rho) + [i(v - i\mathcal{D}\nabla log(\rho)\cdot\nabla](v - i\mathcal{D}\nabla log(\rho)) - \\ -i\mathcal{D}\Delta(v - i\mathcal{D}\nabla log(\rho))\} \tag{3.16}$$

Equation (3.16) is a complex differential equation and reduces to

$$m_0[\partial_t v + (v\cdot\nabla)v] = -\nabla\left(U - 2m_0\mathcal{D}^2\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right);\ \nabla\left\{\frac{1}{\rho}\left[\partial_t\rho + \nabla\cdot(\rho v)\right]\right\} \tag{3.17}$$

The last equation in (3.17) reduces to the continuity equation up to a phase factor $\alpha(t)$ which can be set equal to zero (note again that $\rho \neq 0$ is posited). Thus (3.17) is nothing but the fundamental equations (3.9) of the hydrodynamic model. Further combining the imaginary part of the complex velocity with the quantum potential, and using (3.14), one gets $Q = -m_0\mathcal{D}\nabla\cdot u - (1/2)m_0u^2$ (as indicated in Remark 1.2.2). Since u arises from nondifferentiability according to our nondifferentiable space model of QM it follows that the quantum potential comes from the nondifferentiability of the quantum spacetime (note that the x derivatives should be clarified and $\mathfrak{E}^\infty$ has not been utilized).

Putting $U = 0$ in the first equation of (3.17), multiplying by ρ, and taking the second equation into account yields

$$\partial_t(m_0\rho\nu_k) + \frac{\partial}{\partial x_i}(m_0\rho\nu_i\nu_k) = -\rho\frac{\partial}{x_k}\left[2m_0\mathcal{D}^2\frac{1}{\sqrt{\rho}}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_i}(\sqrt{\rho})\right] \tag{3.18}$$

(here $\nu_k \sim v_k$ seems indicated). Now set $\Pi_{ik} = m_0\rho\nu_i\nu_k - \sigma_{ik}$ along with $\sigma_{ik} = m_0\rho D^2(\partial/\partial x_i)(\partial/\partial x_k)(log(\rho))$. Then (3.18) takes the simple form

$$\partial_t(m_0\rho\nu_k) = -\partial\Pi_{ik}/\partial x_i \tag{3.19}$$

The analogy with classical fluid mechanics works well if one introduces the kinematic $\mu = D/2$ and dynamic $\eta = (1/2)m_0\mathcal{D}\rho$ viscosities. Then Π_{ik} defines the momentum flux density tensor and σ_{ik} the internal stress tensor $\sigma_{ik} = \eta[(\partial u_i/\partial x_k) + (\partial u_k/\partial x_i)]$. One can see that the internal stress tensor is build up using the quantum potential while the equations (3.18) or (3.19) are nothing but systems of Navier-Stokes type for the motion where the quantum potential plays the role of an internal stress tensor. In other words the nondifferentiability of the quantum spacetime manifests itself like an internal stress tensor. For clarity in understanding (3.19) we put this in one dimensional form so (3.18) becomes

$$\partial_t(m_0\rho v) + \partial_x(m_0\rho v^2) = -\rho\partial\left(2m_0\mathcal{D}^2\frac{1}{\sqrt{\rho}}\partial^2\sqrt{\rho}\right) = \rho\partial Q \tag{3.20}$$

and $\Pi = m_0\rho v^2 - \sigma$ with $\sigma = m_0\rho D^2\partial^2 log(\rho)$ which agrees with standard formulas. Now note $\partial\sqrt{\rho} = (1/2)\rho^{-1/2}\rho'$ and $\partial^2\sqrt{\rho} = (1/2)[-(1/2)\rho^{-3/2}(\rho')^2 + \rho^{-1/2}\rho'']$ with $\partial^2 log(\rho) = \partial(\rho'/\rho) = (\rho''/\rho) - (\rho'/\rho)^2$ while

(3.21)

$$-\rho\partial\left[2m_0\mathcal{D}^2\frac{1}{\sqrt{\rho}}\left(\partial^2\sqrt{\rho}\right)\right] = -2m_0\mathcal{D}^2\rho\partial\left[\frac{1}{2\sqrt{\rho}}\left(-\frac{1}{2}\rho^{-3/2}(\rho')^2 + \rho^{-1/2}\rho''\right)\right] =$$

$$= -2m_0\mathcal{D}^2\rho\partial\left[\frac{\rho''}{2\rho} - \frac{1}{4}\left(\frac{(\rho'}{\rho}\right)^2\right] = -m_0\mathcal{D}^2\rho\partial\left[\frac{\rho''}{\rho} - \frac{1}{2}\left(\frac{\rho'}{\rho}\right)^2\right]$$

One wants to show then that (3.19) holds or equivalently $-\partial\sigma =$ (3.21). However

(3.22)

$$-\partial\sigma = -\partial[m_0\rho D^2\partial^2 log(\rho)] = -m_0\mathcal{D}^2\left[\rho'\left(\frac{\rho''}{\rho} - \left(\frac{\rho'}{\rho}\right)^2\right) + \rho\partial\left(\frac{\rho''}{\rho} - \frac{(\rho')^2}{\rho}\right)\right]$$

so we want (3.22) = (3.21) which is easily verified.

4. REMARKS ON FRACTAL CALCULUS

We sketch first (in summary form) from [**748**] where a calculus based on fractal subsets of the real line is formulated. A local calculus based on renormalizing fractional derivatives à la [**562**] is subsumed and embellished. Consider first the concept of content or α-mass for a (generally fractal) subset $F \subset [a,b]$ (in what follows $0 < \alpha \leq 1$). Then define the flag function for a set F and a closed interval I as $\theta(F,I) = 1$ ($F \cap I \neq \emptyset$ and otherwise $\theta = 0$. Then a subdivision $P_{[a,b]} \sim P$ of $[a,b]$ $(a < b)$ is a finite set of points $\{a = x_0, x_1, \cdots, x_n = b\}$ with $x_i < x_{i+1}$. If Q is any subdivision with $P \subset Q$ it is called a refinement and if $a = b$ the set $\{a\}$ is the only subdivision. Define then

$$\sigma^\alpha[F,p] = \sum_0^{n-1}\frac{(x_{i+1}-x_i)^\alpha}{\Gamma(\alpha+1)}\theta(F,[x_i,x_{i+1}]) \tag{4.1}$$

For $a = b$ one defines $\sigma^\alpha[F,P] = 0$. Next given $\delta > 0$ and $a \leq b$ the coarse grained mass $\gamma^\alpha_\delta(F,a,b)$ of $F \cap [a,b]$ is given via

$$\gamma^\alpha_\delta(F,a,b) = inf_{|P|\leq\delta}\sigma^\alpha[F,P]\ \ (|P| = max_{0\leq i\leq n-1}(x_{i+1}-x_i)) \tag{4.2}$$

where the infimum is over P such that $|P| \leq \delta$. Various more or less straightforward properties are:

- For $a \leq b$ and $\delta_1 < \delta_2$ one has $\gamma^\alpha_{\delta_1}(F,a,b) \geq \gamma^\alpha_{\delta_2}(F,a,b)$.
- For $\delta > 0$ and $a < b < c$ one has $\gamma^\alpha_\delta(F,a,b) \leq \gamma^\alpha_\delta(F,a,c)$ and $\gamma^\alpha_\delta(F,b,c) \leq \gamma^\alpha_\delta(F,a,c)$.
- γ^α_δ is continuous in b and a.

Now define the mass function $\gamma^\alpha(F,a,b)$ via $\gamma^\alpha(F,a,b) = lim_{\delta\to 0}\gamma^\alpha_\delta(F,a,b)$. The following results are proved

(1) If $F \cap (a,b) = \emptyset$ then $\gamma^\alpha(F,a,b) = 0$.
(2) Let $a < b < c$ and $\gamma^\alpha(F,a,c) < \infty$. Then $\gamma^\alpha(F,a,c) = \gamma^\alpha(F,a,b) + \gamma^\alpha(F,b,c)$. Hence $\gamma^\alpha(F,a,b)$ is increasing in b and decreasing in a.

(3) Let $a < b$ and $\gamma^\alpha(F, a, b) \neq 0$ be finite. If $0 < y < \gamma^\alpha(F, a, b)$ then there exists c, $a < c < b$ such that $\gamma^\alpha(F, a, c) = y$. Further if $\gamma^\alpha(F, a, b)$ is finite then $\gamma^\alpha(F, a, x)$ is continuous for $x \in (a, b)$.
(4) For $F \subset \mathbf{R}$ and $\lambda \in \mathbf{R}$ let $F + \lambda = \{x + \lambda;\ x \in F\}$. Then $\gamma^\alpha(F + \lambda, a + \lambda, b + \lambda) = \gamma^\alpha(F, a, b)$ and $\gamma^\alpha(\lambda F, \lambda a, \lambda b) = \lambda^\alpha \gamma^\alpha(F, a, b)$.

Now for a_0 an arbitrary fixed real number one defines the integral staircase function of order α for F is

$$S_F^\alpha(x) = \begin{cases} \gamma^\alpha(F, a_0, x) & x \geq a_0 \\ -\gamma^\alpha(F, x, a_0) & otherwise \end{cases} \tag{4.3}$$

The following properties of S_F are restatements of properties for γ^α. thus

- $S_F^\alpha(x)$ is increasing in x.
- If $F \cap (x, y) = 0$ then S_F^α is constant in $[x, y]$.
- $S_F^\alpha(y) - S_F^\alpha(x) = \gamma^\alpha(F, x, y)$.
- S_F^α is continuous on (a, b).

Now one considers the sets F for which the mass function $\gamma^\alpha(F, a, b)$ gives the most useful information. Indeed one can use the mass function to define a fractal dimension. If $0 < \alpha < \beta \leq 1$ one writes
(4.4)

$$\sigma^\beta[F, P] \leq |P|^{\beta-\alpha}\sigma^\alpha[F, P]\frac{\Gamma(\alpha+1)}{\Gamma(\beta+1)};\ \gamma_\delta^\beta(F, a, b) \leq \delta^{\beta-\alpha}\gamma_\delta^\alpha(F, a, b)\frac{\Gamma(\alpha+1)}{\Gamma(\beta+1)}$$

Thus in the limit $\delta \to 0$ one gets $\gamma^\beta(F, a, b) = 0$ provided $\gamma^\alpha(F, a, b) < \infty$ and $\alpha < \beta$. It follows that $\gamma^\alpha(F, a, b)$ is infinite up to a certain value α_0 and then jumps down to zero for $\alpha > \alpha_0$ (if $\alpha_0 < 1$). This number is called the γ-dimension of F; $\gamma^{\alpha_0}(F, a, b)$ may itself be zero, finite, or infinite. To make the definition precise one says that the γ-dimension of $F \cap [a, b]$, denoted by $dim_\gamma(F \cap [a, b])$, is

$$dim_\gamma(F \cap [a, b]) = \begin{cases} inf\{\alpha;\ \gamma^\alpha(F, a, b) = 0\} \\ sup\{\alpha;\ \gamma^\alpha(F, a, b) = \infty\} \end{cases} \tag{4.5}$$

One shows that $dim_H(F \cap [a, b]) \leq dim_\gamma(F \cap [a, b])$ where dim_H denotes Hausdorff dimension. Further $dim_\gamma(F \cap [a, b]) \leq dim_B(F \cap [a, b])$ where dim_B is the box dimension. Some further analysis shows that for $F \subset \mathbf{R}$ compact $dim_\gamma F = dim_H F$.

Next one notes that the correspondence $F \to S_F^\alpha$ is many to one (examples from Cantor sets) and one calls the sets giving rise to the same staircase function "staircasewise congruent". The equivalence class of congruent sets containing F is denoted by $\mathcal{E}_F$; thus if $G \in \mathcal{E}_F$ it follows that $S_G^\alpha = S_F^\alpha$ and $\mathcal{E}_G^\alpha = \mathcal{E}_F^\alpha$. One says that a point x is a point of change of f if f is not constant over any open interval (c, d) containing x. The set of all points of change of f is denoted by $Sch(f)$. In particular if $G \in \mathcal{E}_F^\alpha$ then $S_G^\alpha(x) = S_F^\alpha(x)$ so $Sch(S_G^\alpha) = Sch(S_F^\alpha)$. Thus if $F \subset \mathbf{R}$ is such that $S_F^\alpha(x)$ is finite for all x ($\alpha = dim_\gamma F$) then $H = Sch(S_F^\alpha) \in \mathcal{E}_F^\alpha$. This takes some proving which we omit (cf. [**748**]). As a consequence let $F \subset \mathbf{R}$ be such that $S_F^\alpha(x)$ is finite for all $x \in \mathbf{R}$ ($\alpha = dim_\gamma F$). Then the set $H = Sch(S_F^\alpha)$ is perfect (i.e. H is closed and every point is a limit point). Hence given $S_F^\alpha(x)$ finite for all x ($\alpha = dim_\gamma F$) one calls $Sch(S_F^\alpha)$ the α-perfect representative of $\mathcal{E}_F^\alpha$ and one proves that it is the minimal closed set in $\mathcal{E}_F^\alpha$. Indeed an α-perfect set in $\mathcal{E}_F^\alpha$ is the intersection of all closed sets G in $\mathcal{E}_F^\alpha$. One can also say that if $F \subset \mathbf{R}$ is α-perfect and $x \in F$ then for $y < x < z$ either $S_F^\alpha(y) < S_F^\alpha(x)$ or $S_F^\alpha(x) < S_F^\alpha(z)$ (or both). Thus for an α-perfect set it is assured that the values of $S_F^\alpha(y)$ must be different from $S_F^\alpha(x)$ at all points y on at least one side of x. As an example one shows that the middle third Cantor set $C = E_{1/3}$ is α-perfect for $\alpha = log(2)/log(3) = d_H(C)$ so $C = Sch(S_C^\alpha)$.

Now look at F with the induced topology from $\mathbf{R}$ and consider the idea of F-continuity.

DEFINITION 4.1. Let $F \subset \mathbf{R}$ and $f : \mathbf{R} \to \mathbf{R}$ with $x \in F$. A number ℓ is said to be the limit of f through the points of F, or simply F-limit, as $y \to x$ if given $\epsilon > 0$ there exists $\delta > 0$ such that $y \in F$ and $|y - x| < \delta \Rightarrow |f(y) - \ell| < \epsilon$. In such a case one writes $\ell = F - limit_{y \to x} f(y)$. A function f is F-continuous at $x \in F$ if $f(x) = F - limit_{y \to x} f(y)$ and uniformly F-continuuous on $E \subset F$ if for $\epsilon > 0$ there exists $\delta > 0$ such that $x \in F$, $y \in E$ and $|y - x| < \delta \Rightarrow |f(y) - f(x)| < \epsilon$. One sees that if f is F-continuous on a compact set $E \subset F$ then it is uniformly F-continuous on E.

DEFINITION 4.2. The class of functions $f : \mathbf{R} \to \mathbf{R}$ which are bounded on F is denoted by $B(F)$. Define for $f \in B(F)$ and I a closed interval

$$M[f, F, I] = \begin{cases} sup_{x \in F \cap I} f(x) & F \cap I \neq \emptyset \\ 0 & otherwise \end{cases} \tag{4.6}$$

$$m[f, F, I] = \begin{cases} inf_{x \in F \cap I} f(x) & F \cap I \neq \emptyset \\ 0 & otherwise \end{cases}$$

DEFINITION 4.3. Let $S_F^\alpha(x)$ be finite for $x \in [a, b]$ and P be a subdivision with points $x_0, \cdots, x_n$. The upper F^α and lower F^α sums over P are given respectively by

$$U^\alpha[f, F, P] = \sum_0^{n-1} M[f, F, [x_i, x_{i+1}]](S_F^\alpha(x_{i+1}) - S_F^\alpha(x_i)); \tag{4.7}$$

$$L^\alpha[f, F, P] = \sum_0^{n-1} m[f, F, [x_i, x_{i+1}]](S_F^\alpha(x_{i+1}) - S_F^\alpha(x_i))$$

This is sort of like Riemann-Stieltjes integration and in fact one shows that if Q is a refinement of P then $U^\alpha[f, F, Q] \leq U^\alpha[f, F, P]$ and $L^\alpha[f, F, Q] \geq L^\alpha[f, F, P]$. Further $U^\alpha[f, F, P] \geq L^\alpha[f, F, Q]$ for any subdivisions of $[a, b]$ and this leads to the idea of F-integrability. Thus assume S_F^α is finite on $[a, b]$ and for $f \in B(F)$ one defines lower and upper F^α-integrals via

$$\underline{\int_a^b} f(x) d_F^\alpha x = sup_P L^\alpha[f, F, P]; \ \overline{\int_a^b} f(x) d_F^\alpha x = inf_P U^\alpha[f, F, P] \tag{4.8}$$

One then says that f is F^α-integrable if (**D15**) $\underline{\int_a^b} f(x) d_F^\alpha x = \overline{\int_a^b} f(x) d_F^\alpha x = \int_a^b f(x) d_F^\alpha x$.

One shows then

(1) $f \in B(F)$ is F^α-integrable on $[a,b]$ if and only if for any $\epsilon > 0$ there is a subdivision P of $[a,b]$ such that $U^\alpha[f,F,P] < L^\alpha[f,F,P] + \epsilon$.
(2) Let $F \cap [a,b]$ be compact with S_F^α finite on $[a,b]$. Let $f \in B(F)$ and $a < b$; then if f is F-continuous on $F \cap [a,b]$ it follows that f is F^α-integrable on $[a,b]$.
(3) Let $a < b$ and f be F^α-integrable on $[a,b]$ with $c \in (a,b)$. Then f is F^α-integrable on $[a,c]$ and $[c,b]$ with $\int_a^b f(x)d_F^\alpha x = \int_a^c f(x)d_F^\alpha x + \int_c^b f(x)d_F^\alpha x$.
(4) If f is F^α-integrable then $\int_a^b \lambda f(x)d_F^\alpha x = \lambda \int_a^b f(x)d_F^\alpha x$ and, for g also F^α-integrable, $\int_a^b (f(x)+g(x))d_F^\alpha x = \int_a^b f(x)d_F^\alpha x + \int_a^b g(x)d_F^\alpha x$.
(5) If f,g are F^α-integrable and $f(x) \geq g(x)$ for $x \in F \cap [a,b]$ then $\int_a^b f(x)d_F^\alpha x \geq \int_a^b g(x)d_F^\alpha x$.

One specifies also $\int_b^a f(x)d_F^\alpha x = -\int_a^b f(x)d_F^\alpha x$ and it is easily shown that if $\chi_F(x)$ is the characteristic function of F then $\int_a^b \chi_F(x)d_F^\alpha x = S_F^\alpha(b) - S_F^\alpha(a)$. Now for differentiation one writes

$$\mathcal{D}_F^\alpha f(x) = \begin{cases} F - lim_{y\to x}\frac{f(y)-f(x)}{S_F^\alpha(y)-S_F^\alpha(x)} & x \in F \\ 0 & otherwise \end{cases} \tag{4.9}$$

if the limit exists. One shows then

(1) If $\mathcal{D}_F^\alpha f(x)$ exists for all $x \in (a,b)$ then $f(x)$ is F-continuous in (a,b).
(2) With obvious hypotheses $\mathcal{D}_F^\alpha(\lambda f(x)) = \lambda \mathcal{D}_F^\alpha f(x)$ and $\mathfrak{D}_F^\alpha(f+g)(x) = \mathcal{D}_F^\alpha f(x) + \mathcal{D}_F^\alpha g(x)$. Further if f is constant then $\mathcal{D}_F^\alpha f = 0$.
(3) $\mathfrak{D}_F^\alpha(S_F^\alpha(x)) = \chi_F(x)$.
(4) (Rolle's theorem) Let $f : \mathbf{R} \to \mathbf{R}$ be continuous with $Sch(f) \subset F$ where F is α-perfect and assume $\mathcal{D}_F^\alpha f(x)$ is defined for all $x \in [a,b]$ with $f(a) = f(b) = 0$. Then there is a point $c \in F \cap [a,b]$ such that $\mathcal{D}_F^\alpha f(c) \geq 0$ and a point $d \in F \cap [a,b]$ where $\mathcal{D}_F^\alpha f(d) \leq 0$.

EXAMPLE 4.1. This is the best that can be done with Rolle's theorem since for C the Cantor set $E_{1/3}$ take $f(x) = S_C^\alpha(x)$ for $0 \leq x \leq 1/2$ and $f(x) = 1 - S_C^\alpha(x)$ for $1/2 < x \leq 1$. This function is continuous with $f(0) = f(1) = 0$ and the set of change $(Sch(f))$ is C. The C^α-derivative is given by $\mathcal{D}_C^\alpha f(x) = \chi_C(x)$ for $0 \leq x \leq 1/2$ and by $-\chi_C(x)$ for $1/2 < x \leq 1$. Thus $x \in C$ which implies $\mathcal{D}_C^\alpha f(x) = \pm 1 \neq 0$.

As a corollary one has the following result: Let f be continuous with $Sch(f) \subset F$ where F is α-perfect; assume $\mathcal{D}_F^\alpha f(s)$ exists at all points of $[a,b]$ and that $S_F^\alpha(b) \neq S_F^\alpha(a)$. Then there are points $c,d \in F$ such that

$$\mathcal{D}_F^\alpha f(c) \geq \frac{f(b)-f(a)}{S_F^\alpha(b)-S_F^\alpha(b)};\ \mathcal{D}_F^\alpha f(d) \leq \frac{f(b)-f(a)}{S_F^\alpha(b)-S_F^\alpha(a)} \tag{4.10}$$

Similarly if f is continuous with $Sch(f) \subset F$ and $\mathcal{D}_F^\alpha f(x) = 0\ \forall x \in [a,b]$ then $f(x)$ is constant on $[a,b]$. There are also other fundamental theorems as follows

(1) (Leibniz rule) If $u,v : \mathbf{R} \to \mathbf{R}$ are F^α-differentiable then $\mathcal{D}_F^\alpha(uv)(x) = (\mathcal{D}_F^\alpha u(x))v(x) + u(x)\mathcal{D}_F^\alpha v(x)$.

(2) Let $F \subset \mathbf{R}$ be α-perfect. If $f \in B(F)$ is F-continuous on $F \cap [a,b]$ with $g(x) = \int_a^x f(y) d_F^\alpha y$ for all $x \in [a,b]$ then $\mathcal{D}_F^\alpha g(x) = f(x)\chi_F(x)$.
(3) Let $f : \mathbf{R} \to \mathbf{R}$ be continuous and F^α-differentiable with $Sch(f)$ contained in an α-perfect set F; let also $h : \mathbf{R} \to \mathbf{R}$ be F-continuous such that $h(x)\chi_F(x) = \mathcal{D}_F^\alpha f(x)$. Then $\int_a^b h(x) d_F^\alpha x = f(b) - f(a)$.
(4) (Integration by parts) Assume: (i) u is continuous on $[a,b]$ and $Sch(u) \subset F$. (ii) $\mathcal{D}_F^\alpha u(x)$ exists and is F-continuous on $[a,b]$. (iii) v is F-continuous on $[a,b]$. Then

$$\int_a^b uv d_F^\alpha x = \left[u(x) \int_a^x v(x') d_F^\alpha x'\right]\Big|_a^b - \int_a^b \mathcal{D}_F^\alpha u(x) \int_a^x v(x') d_F^\alpha x' d_F^\alpha x \tag{4.11}$$

Some examples are given relative to applications and we mention e.g.

EXAMPLE 4.2. Following [**562**] one has a local fractal diffusion equation

$$\mathcal{D}_{F,t}^\alpha(W(x,t)) = \frac{\chi_F(t)}{2} \frac{\partial^2}{\partial x^2} W(x,t) \tag{4.12}$$

with solution

$$W(x,t) = \frac{1}{(2\pi S_F^\alpha(t))^{1/2}} exp\left(\frac{-x^2}{2S_F^\alpha(t)}\right) \tag{4.13}$$

The appendix to [**748**] also gives some formulas for repeated integration and differentiation. For example it is shown that

$$(\mathcal{D}_F^\alpha)^2 (S_F^\alpha(x))^2 = 2\chi_F(x); \quad \int_a^{x'} (S_F^\alpha(x))^n d_F^\alpha x = \frac{1}{n+1}(S_F^\alpha(x'))^{n+1} \tag{4.14}$$

We refer to [**562, 748**] for other interesting material.

5. A BOHMIAN APPROACH TO QUANTUM FRACTALS

The powerful exact uncertainty method of Hall and Reginatto for passing from classical to quantum mechanics has been further embellished and deepened in recent years (see e.g. [**186, 187, 189, 203, 396, 444, 445, 446, 447, 448, 449, 450, 749, 805, 806, 807, 844, 845**] and Sections 1.1, 3.1, and 4.7. In [**445**] one finds an apparent incompleteness in the traditional trajectory based Bohmian mechanics when dealing with a quantum particle in a box. It turns out that there is no suitable HJ equation for describing the motion which in fact has a fractal character. After reviewing the material on scale relativity in Section 1.2 for example it is not surprising to encounter such situations and in [**844**] the Bohmian point of view is reinstated for fractal trajectories. One should also remark in passing that there is much material available on weak or distribution solutions of HJ type equations and some of this should come into play here (cf. [**211**]). The main issue here however is that in order to treat wave functions displaying fractal features (quantum fractals) one needs to enlarge the picture via limiting processes. One derives the quantum trajectories by means of limiting procedures that involve the expansion of the wave function in a series of eigenvectors of the Hamiltonian.

Consider first the quantum analogue of the Weierstrass function

$$W(x) = \sum_0^\infty b^r Sin(a^r x);\ \ a > 1 > b > 0;\ \ ab \geq 1 \tag{5.1}$$

Then in the problem of a particle in a 1-dimensional box of length L (with $0 < x < L$) one can construct wave functions of the form

$$\Phi_t(x;R) = A\sum_{r=0}^R n^{r(s-2)} Sin(p_{n,r}x/\hbar)e^{-iE_{n,r}t/\hbar} \tag{5.2}$$

with $2 > s > 0$ and $n \geq 2$. Here $p_{n,r} = n^r\pi\hbar/L$ is the quantized momentum (with integer quantum number given by $n' = n^r$). $E_{n,r} = p_{n,r}^2/2m$ is the eigenenergy and a is a normalization constant. This wave function, which is a solution of the time dependent SE, is continuous and differentiable everywhere. However the wave function resulting in the limit, namely $\Phi_t(x) = lim_{R\to\infty}\Phi_t(x;R)$ is a fractal object in both space and time (cf. [**626**]). This method for generating quantum fractals basically involves (given s) choosing a quantum number, say n, and then considering the series that contains its powers $n' = n^r$. There is also another related method (cf. [**142**]) of generating quantum fractals based on the presence of discontinuities in the wave function. The emergence of fractal features arises from the perturbations that such discontinuities cause in the wave function during propagation. This generating process can be easily understood by considering a wave function initially uniform along a certain interval $\ell = x_2 - x_1 \leq L$ inside the box

$$\Psi_0(x) = \begin{cases} \frac{1}{\sqrt{\ell}} & x_1 < x < x_2 \\ 0 & otherwise \end{cases} \tag{5.3}$$

The Fourier decomposition of this wave function is

$$\Psi_0(x) = \frac{2}{\pi\sqrt{\ell}}\sum_1^\infty \frac{1}{n}[Cos(p_n x_1/\hbar) - Cos(p_n x_2/\hbar)]Sin(p_n x/\hbar) \tag{5.4}$$

whose time evolved form is

$$\Psi_t(x) = \frac{2}{\pi\sqrt{\ell}}\sum_1^\infty \frac{1}{n}[Cos(p_n x_1/\hbar) - Cos(p_n x_2/\hbar)]Sin(p_n x/\hbar)e^{-iE_n t/\hbar} \tag{5.5}$$

It is equivalent to consider $r = R = 1$ in (5.2) and sum over n from 1 to N; the quantum fractal is then obtained in the limit $N \to \infty$. This equivalence is based on the fact that the Fourier decomposition of Ψ_0 gives precisely its expansion in terms of the eigenvectors of the Hamiltonian in the problem of a particle in a box (this is not a general situation).

EXAMPLE 5.1. The fractality of wave functions like $\Phi_t(x)$ or $\Psi_t(x)$ can be analytically estimated (cf. [**142**]) by taking advantage of a result for Fourier series. Thus given an arbitrary function $f(x) = \sum_1^N a_n exp(-inx)$ its real and imaginary pars are fractals (and also $|f(x)|^2$) with dimension $D_f = (5-\beta)/2$ if its power spectrum has the asymptotic form $|a_n|^2 \sim n^{-\beta}$ for $N \to \infty$ with $1 < \beta \leq 3$. Alternatively the fractality of $f(x)$ can also be calculated by measuring the length

$\mathfrak{L}$ of its real and imaginary parts (or $|f(x)|^2$) as a function of the number of terms N considered in the generating sries. Asymptotically the relation bewteen $\mathfrak{L}$ and N can be expressed as $\mathfrak{L}(N) \propto N^{D_f-1}$ which diverges if $f(x)$ is a fractal object. One notes that to increase the number of terms contributing to $f(x)$ is analogous to measuring its length with more precision.

It is known that for quantum fractals the corresponding expected value of the energy $<\hat{H}>$ becomes infinite. This is related to the fact that the familiar form of the SE

$$i\hbar\partial_t\Psi_t(x) = \hat{H}\Psi_t(x) \tag{5.6}$$

does not hold in general (cf. [**445, 1003**]). In this case neither the left side of (5.6) nor the right side belong to the Hilbert space; however the identity

$$[\hat{H} - i\hbar\partial_t]\Psi_t(x) = 0 \tag{5.7}$$

still remains valid. In this situation one says that $\Psi_t(x)$ is a weak solution of the SE (note weak solutions have many meanings and have been extensively studied in PDE - cf. [**211**]).

The formal basis of Bohmian mechanics (BM) is usually established via

$$\Psi_t(x) = \rho_t^{1/2}(x)e^{iS_t(x)/\hbar};\ \frac{\partial\rho_t}{\partial t} + \nabla\cdot\left(\rho_t\frac{\nabla S_t}{m}\right) = 0; \tag{5.8}$$

$$\frac{\partial S_t}{\partial t} + \frac{(\nabla S_t)^2}{2m} + V + Q_t = 0;\ Q_t = -\frac{\hbar^2}{2m}\frac{\nabla^2\rho_t^{1/2}}{\rho_t^{1/2}}$$

One postulates also the trajectory velocity as

$$\dot{x} = \frac{\nabla S_t}{m} = \frac{\hbar}{m}\Im[\Psi_t^{-1}\nabla\Psi_t] \tag{5.9}$$

Now Q_t in (5.8) is well defined provided that the quantum state is also well defined (i.e. continuous and differentiable). However this is not the case for quantum fractals and the theory seems incomplete; the solution is to take into account the decomposition of the quantum fractal in terms of differentiable eigenvectors and redefining Q_t in (5.8). Thus any wave function Ψ_t is expressible as

$$\Psi_t(x:N) = \sum_1^N c_n\xi_n(x)e^{-iE_nt/\hbar} \tag{5.10}$$

in the limit $N \to \infty$ (cf. Φ_t above and (5.5)) where the $\xi_n(x)$ are eigenvectors with eigenvalues E_n of the corresponding Hamiltonian. One can then define the quantum trajectories evolving under the guidance of this wave as

$$x_t = lim_{N\to\infty}x_N(t);\ \dot{x}_N = \frac{\hbar}{m}\Im\left[\Psi_t^{-1}(x;N)\frac{\partial\Psi_t(x;N)}{\partial x}\right] \tag{5.11}$$

Note the calculation of trajectories is not based on S_t, which has no trivial decomposition in a series of nice functions, but this kind of velocity formulation is common in e.g. [**324, 325, 326, 327, 328, 329, 415, 416, 418, 419, 420**] where one modern version of BM is being developed.

EXAMPLE 5.2. A numerical example is given in [**844**] and we only mention a few features here. Thus one considers a highly delocalized particle in a box with wave function (5.4) and $x_1 = 0$ with $x_2 = L$. Then (5.5) becomes

$$\Psi_t(x) = \frac{4}{\pi\sqrt{L}} e^{-iE_1 t/\hbar} \sum_{n\ odd} \frac{1}{n} Sin(p_n x/\hbar) e^{i\omega_{n,1} t} \tag{5.12}$$

where $\lambda_{n,1} = (E_n - E_1)/\hbar$ (in the numerical calculations one uses $L = m = \hbar = 1$). Here the probability density ρ_t is periodic in time but the wave function is not periodic (this does not affect (5.11)). Various features are observed (e.g. Cantor set structures, Gibbs phenomena, etc.) and graphs are displayed - we omit any further discussion here.

In summary, although the SE is not satisfied by quantum fractals as a whole, it is when one considers its decomposition in terms of the eigenvectors of the Hamiltonian. The contributing eigenvectors are continuous and differentiable and any wave function (regular or not) admits a decomposition in terms of eigenvectors. Correspondingly the Bohmian equation of motion must be reformulated in terms of such decompositions via (5.11) and this can be regarded also as a generalization of (5.8). We mention in passing that from time to time there are papers claiming contradictions between BM and QM and we refer here to [**436, 629, 712**] for some refutations.

REMARK 5.3. Let us mention here a suggestion of 't Hooft [**475**] about establishing the physical link between classical and quantum mechanics by employing the underlying equations of classical mechanics and including into them a specially chosen dissipative function. The wave like QM turns out to follow from the particle like classical mechanics due to embedding in the latter a dissipation "device" responsible for loss of information. Thus the initial precise information about the classical trajectory is lost in QM due to the "dissipative spread" of the trajectory and its transformation into a fuzzy object such as the fractal Hausdorff path of dimension 2 in a simple case of a spinless particle. Some rough calculations in this direction appear in [**426**]. and we refer also to [**122, 427, 749**].