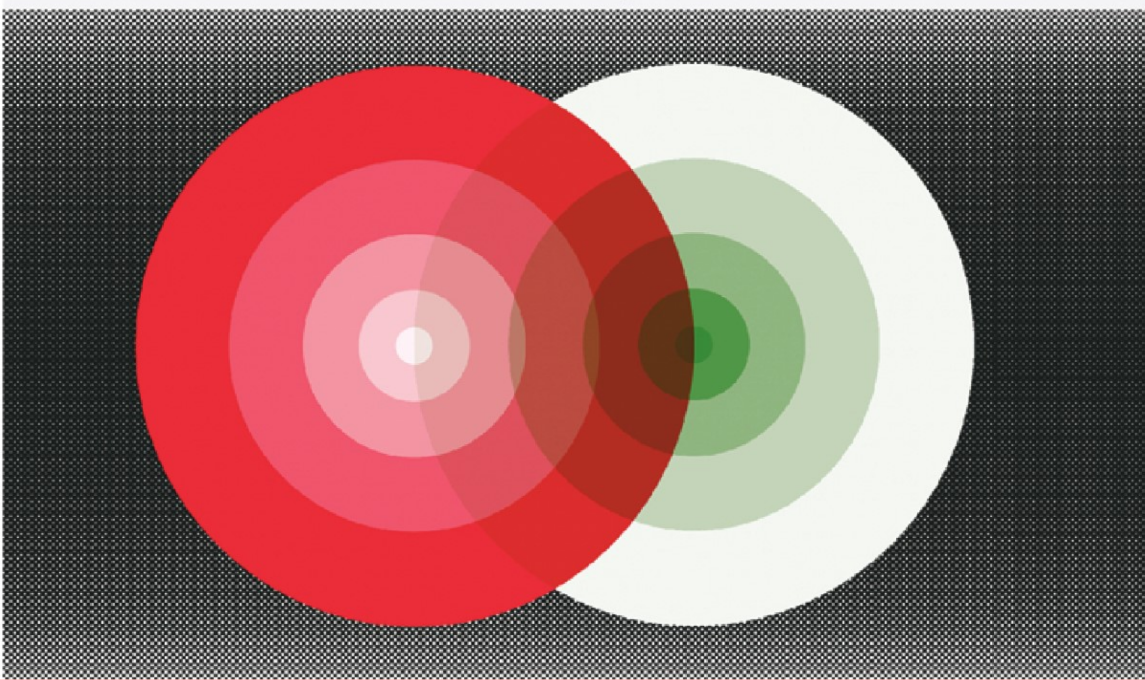
Fluctuations, Information, Gravity and the Quantum Potential

by

R.W. Carroll

Springer



Fundamental Theories of Physics

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PREFACE

The quantum potential is a recurring theme and Leitmotiv for this book (along with variations on the Bohmian trajectory representation for quantum mechanics (QM)); it appears in many guises and creates connections between quantum mechanics (QM), general relativity (GR), information, and Bohmian trajectories (as well as diffusion, thermodynamics, hydrodynamics, fractal structure, entropy, etc.). We make no claim to create quantum gravity for which there are several embryonic theories presently under construction (cf. [**53, 429, 783, 819, 829, 896, 929**]) and we will not deal with much of this beyond some use of the Ashtekar variables for general relativity (cf. also [**192, 228, 275, 258, 430, 460, 461, 462, 578, 645, 744, 802, 803, 829, 902, 1012**] for approaches involving posets, discrete differential manifolds, quantum causal spaces, groupoids, noncommutative geometry, etc.). The main theme of the book is to relate ideas of quantum fluctuations (expressed via Fisher information for example, or diffusion processes, or fractal structures, or particle creation and annihilation, or whatever) in terms of the so called quantum potential (which arises most conspicuously in deBroglie-Bohm theories). This quantum potential can be directly connected to the diffusion version of the Schrödinger equation as in [**672, 674, 698**] and is also related to the Ricci-Weyl curvature of Dirac-Weyl theory ([**186, 187, 188, 189, 219, 872, 873**]). Recent work of A. and F. Shojai also develops a quantum potential and Schrödinger equation relative to the Wheeler-deWitt equation using Ashtekar variables (cf. [**876**]). Further, work of Israelit-Rosen on Weyl-Dirac theory leads to matter production by geometry (cf. [**498, 499, 500, 501**]) which could also be related to the quantum potential via the Dirac field as a matter field. In following this theme we have also been led to developments in scale relativity (cf. [**715, 716, 718**]) where the Schrödinger equation arises from fractal structure and the quantum potential clearly determines a quantization. Quantum potentials also arise in the (x, ψ) duality theme of Faraggi and Matone and we give a variation on this related to the massless KG equation (or putative aether equation) following Vancea et al. (cf. [1021]). The quantum fluctuation theme then leads also to stochastic electrodynamics (SED) and the energy of the vacuum (zero point field - ZPF) and thence to further examination of electrodynamics, massless particles, etc. (cf. [**190, 650, 753**]). There is throughout the book an involvement with quantum field theory (QFT) where in particular we extract from work of Nikolić, and there is considerable material devoted to entropy and information. In a sense the magical structure of quantum mechanics (QM) à la von Neumann and others is too perfect; one cannot see what is "really" going on and this makes the deBroglie-Bohm theory attractive, where one has at least the illusion of having particle trajectories, etc. In fact we develop the theme that the Schrödinger mechanics of Hilbert space etc. is inherently incomplete (or uncertain); in particular the uncertainty can arise from its inability to see a third "initial" condition in the microstate trajectories (see Sections 2.2 and 7.4). We show in Remarks 2.2.2 and 3.2.1 how this produces exactly the Heisenberg uncertainty principle and how it is consequently natural to consider ensembles of particles and probability densities for the Schrödinger picture (cf. also [**194, 197, 203, 346, 347, 373, 374**]). The Hilbert space uncertainty is in fact automatic, due to the operator approach in Hilbert space, and is thus independent of quantum fluctuations, diffusion, fractals, microstates, or whatever, so the Hilbert space formulation is indifferent to the interpretation of uncertainty. However, we are interested in microcausality here and, furthermore, quantum fluctuations generating a quantum potential also have a direct relation to uncertainty via the exact uncertainty principle of Hall-Reginatto; this is discussed at some length.

We want to make a few remarks about writing style. With a mathematical background my writing style has acquired a certain flavor based on equations and almost devoid of physical motivation or insights. I have tried to temper this when writing about physics but it is impossible to achieve the degree of physical insight characteristic of natural born physicists. I am comforted by the words of Dirac who seemed to be guided and motivated by equations (although he of course possessed more physical insight than I could possibly claim). In any case the "meaning" of physics for me lies largely in beautiful equations (Einstein, Maxwell, Schrödinger, Dirac, Hamilton-Jacobi, etc.) and the revelations about the universe presumably therein contained. I hope to convey this spirit in writing and perhaps validate somewhat this approach. Physics is a vast garden of delights and we can only gape at some of the wonders (as expressed here via equations in directions of personal esthetic appeal). We refrain from rashly suggesting that nature is simply a manifestation of underlying mathematical structure (i.e. symmetry, combinatorics, topology, geometry, etc.) played out on a global stage with energy and matter as actors. On the other hand we are aware that much of physics, both experimental and theoretical, apparently has little to do with equations. Many phenomena are now recognized as emergent (cf. [**594, 860**]) and one has to deal with phase transitions, self organized criticality, chaos, etc. I know little about such matters and hope that the mathematical approach is not mistaken for arrogance; it is only a dominant reverence for that beauty which I am able to perceive. The book of course is not finished; it probably cannot ever be finished since there is new material appearing every weekday on the electronic bulletin boards. Hence we have had to declare it finished as of April 10, 2005. It has reached the goal of roughly 450 pages and includes whatever I conceived of as most important about the quantum potential and Bohmian mechanics. I have learned a lot in writing this and there is some original material (along with over 1000 references). Philosophy has not been treated with much respect since I can only sense meaning in physics through the equations. The fact that one can envision and manipulate "composite" and abstract concepts or entities such as EM fields, energy, entropy, force, gravity, mass, pressure, spacetime, spin, temperature, wave functions, etc. and that there should turn out to be equations and relations among these "creatures" has always staggered my imagination. So does the fact that various combinations of large and small numbers (such as c, e, G, $\hbar$, H etc.) can be combined into dimensionless form. Abstract mathematics has its own garden of concepts and relations also of course but physics seems so much more real (even though I have resisted any desire to experiment with these beautiful concepts, mainly because I seem to have difficulty even in setting an alarm clock for example).

We will present a number of ideas which lack establishment deification, either because they are too new or, in heuristic approaches of considerable import, because of obvious conceptual flaws requiring futher consideration. There is also some speculative material, clearly labeled as such, which we hope will be productive. The need for crazy ideas has been immortalized in a famous comment of N. Bohr so there is no shame attached to exploratory ventures, however unconventional they may seem at first appearance. The idea of a quantum potential as a link between classical and quantum phenomena, and the related attachment to Bohmian type mechanics, seems compelling and is pursued throughout the book; that it should also have links to quantum fluctuations (and information) and Weyl curvature makes it irresistible.

There is no attempt to present a final version of anything. For a time in the 1990's the connection of string theory to soliton mathematics (e.g. in Seiberg-Witten theory) seemed destined to solve everything but it was only preliminary. Some of this is summarized nicely in a lovely paper of A. Morozov [**662**] where connections to matrix models and special functions (and much more) are indicated and the need for mathematical development in various directions is emphasized (cf. [**191, 662**] and references there along with a few comments in Chapter 8). The arena of noncommutative geometry and quantum groups is personally very appealing (see e.g. [**192, 258, 589, 615, 619, 620**]) and we refer to [**157, 158, 159, 160, 161, 162, 260, 580, 581, 582, 722**] for some fascinating work on Feynman diagrams, quantum groups, and quantum field theory (some of this is sketched in Chapter 8). We have not gone into here (and know little about) several worlds of phenomenological physics regarding e.g. QCD, quarks, gluons, etc. (whose story is partly described in [**997**] by F. Wilczek in his Nobel lecture). We have also been fascinated by the mathematics of superfluids à la G. Volovik ([**968, 969**]) and the mathematics is often related to other topics in this book; however our understanding of the physics is very limited. There are also a few remarks about cosmology and here again we know little and have only indicated a few places where Dirac-Weyl geometry, and hence perhaps the quantum potential, play a role.

As to the book itself we mainly develop the theme of the quantum potential and with it the Bohmian trajectory representation of QM. The quantum potential arises most innocently in the Bohmian theory and the Schrödinger equation (SE) as an expression $Q = -(\hbar^2/2m)(\Delta|\psi|/|\psi|)$ where ψ is the wave function and Q appears then in the corresponding Hamilton-Jacobi (HJ) equation as a potential term. It is possible to relate this term to Fisher information, entropy, and quantum fluctuations in a natural manner and further to hydrodynamics, stress tensors, diffusion, Weyl-Ricci scalar curvatures, fractal velocities, osmotic pressure, etc. It arises in relativistic form via $Q = (\hbar^2/m^2c^2))\Box|\psi|/|\psi|)$ and in field theoretic models as e.g. $Q = -(\hbar^2/2|\Psi|)(\delta^2|\Psi|/\delta|\Psi|^2)$. In terms of lapse and shift functions one has a WDW version in the form $Q = h^2NqG_{ijk\ell}(\delta^2|\Psi|/\delta q_{ij}\delta q_{k\ell})$ (q is a surface metric). There is also a way in which the quantum potential can be considered as a mass generation term (cf. Remark 2.2.1 and [**110**]) and this is surely related to its role in Weyl-Dirac geometry in determining the matter field. In purely Weyl geometry one can use the quantum matter field (determined by Q) as a metric multiplier to create the conformal geometry. In Chapter 7 we give a resumé of various aspects of the quantum potential contained in Chapters 1-7, followed by further information on the quantum potential. We had hoped to include material on the mathematics of acoustics and superfluids à la Volovik and thermo-field-theory à la Blassone, Celeghini, Das, Iorio, Jizba, Rasetti, Vitiello, Umezawa, et al. (see e.g. [**122, 231, 493, 519, 912, 913, 963, 964, 965**]). This however is too much (and no new insights into the quantum potential were visible); in any case there are already books [**286, 950, 968**] available. Aside from a few short sketches we have also felt that there is not enough room here to properly cover the nonlinear Schrödinger equation (NLSE). (cf [**223, 280, 292**, **311, 312, 313, 413**, **691**, **692**, **693**, **1028, 1029, 1030, 1031, 1032, 1033, 1034**]).We have had recourse to quantum field theory (QFT) at several places in the book, referring to [**120, 457, 935, 1015**] for example, and we provide in Chapter 8 a survey article written in 2003-2004 (updated a bit) on quantum field theory (QFT), tau functions, Hopf algebras, and vertex operators. Although this does not touch on the quantum potential (and delves mainly into taming the combinatorics of QFT via quantum groups for example), some of this material seems to be relevant to the program suggested in [**662**] and should be of current interest (cf. [**191, 192**] for more background). We recently became aware of important work in various directions by Brown, Hiley, deGosson, Padmanabhan, and Smolin (see [**741, 1018 1019, 1022, 1023, 1024**] and cf. also [**1020, 1021**]).

I would like to acknowledge interesting conversations about various topics in physics, with my sons David and Malcolm, and with my wife's son Jim Bredt; they all know much more than I about real physics. I have also benefitted from correspondence with C. Castro, M. Davidson, D. Delphenich, V. Dobrev, E. Floyd, G. Grössing, G. Kaniadakis, M. Kozlowski, M. Lapidus, M. Matone, H. Nikolić, F. Shojai, S. Tiwari, and J.T. Wheeler and from conversations with M. Bergvelt, L. Bogdanov, F. Calogero, H. Doebner, J. Edelstein, P. Grinevich, Y. Kodama, B. Konopelchenko, A. Morozov, R. Parthasarathy, O. Pashaev, P. Sabatier, and M. Stone. A glance at the contents and index will indicate the important role of H. Weyl in this book and it is perhaps appropriate to mention a "familial" influence as well, via my thesis advisor A. Weinstein who was Weyl's only Privatdozent. The book is dedicated to my wife Denise Rzewska-Bredt-Carroll who has guided me in a passionate love affair through the perils of our golden years.

CHAPTER 1

THE SCHRÖDINGER EQUATION

Perhaps no subject has been the focus of as much mystery as "classical" quantum mechanics (QM) even though the standard Hilbert space framework provides an eminently satisfactory vehicle for determining accurate conclusions in many situations. This and other classical viewpoints provide also seven decimal place accuracy in quantum electrodynamics (QED) for example. So why all the fuss? The erection of the Hilbert space edifice and the subsequent development of operator algebras (extending now into noncommutative (NC) geometry) has an air of magic. It works but exactly why it works and what it really represents remain shrouded in ambiguity. Also geometrical connections of QM and classical mechanics (CM) are still a source of new work and a modern paradigm focuses on the emergence of CM from QM (or below). Below could mean here a microstructure of space time, or quantum foam, or whatever. Hence we focus on other approaches to QM and will recall any needed Hilbert space ideas as they arise.

1. DIFFUSION AND STOCHASTIC PROCESSES

There are some beautiful stochastic theories for diffusion and QM mainly concerned with origins of the Schrödinger equation (SE). For background information we mention for example [**33, 98, 131, 191, 192, 241, 242, 258, 471, 555, 589, 591, 615, 628, 647, 672, 674, 698, 715, 719, 726, 783, 810, 860, 1025, 1026, 1027**]. The present development focuses on certain aspects of the SE involving the wave function form $\psi = Rexp(iS/\hbar)$, hydro dynamical versions, diffusion processes, quantum potentials, and fractal methods. The aim is to envision "structure", both mathematical and physical, and we sometimes avoid detailed technical discussion of mathematical fine points (cf. [**241, 242, 243, 271, 315, 345, 531, 591, 592, 607, 615, 672, 674, 810, 918**] for various delicate matters). For example, rather than looking at such topics as Markov processes with jumps we prefer to seek "meaning" for the Schrödinger equation via microstructure and fractals in connection with diffusion processes and kinetic theory.

First consider the SE in the form $-(\hbar^2/2m)\psi'' + V\psi = i\hbar\psi_t$ so that for $\psi = Rexp(iS/\hbar)$ one obtains

$$(1.1)\qquad S_t + \frac{S_X^2}{2m} + V - \frac{\hbar^2 R''}{2mR} = 0;\ \partial_t(R^2) + \frac{1}{m}(R^2 S')' = 0$$

where $S' \sim \partial S/\partial X$. Writing $P = R^2$ (probability density $\sim |\psi|^2$) and $Q = -(\hbar^2/2m)(R''/R)$ (quantum potential) this becomes

$$S_t + \frac{(S')^2}{2m} + Q + V = 0;\ P_t + \frac{1}{m}(PS')' = 0 \tag{1.2}$$

and this has some hydrodynamical interpretations in the spirit of Madelung. Indeed going to [**294**] for example we take $p = S'$ with $p = m\dot{q}$ for $\dot{q}$ a velocity (or "collective" velocity - unspecified). Then (1.2) can be written as ($\rho = mP$ is an unspecified mass density)

$$S_t + \frac{p^2}{2m} + Q + V = 0;\ P_t + \frac{1}{m}(Pp)' = 0;\ p = S';\ P = R^2; \tag{1.3}$$

$$Q = -\frac{\hbar^2}{2m}\frac{R''}{R} = -\frac{\hbar^2}{2m}\frac{\partial^2\sqrt{\rho}}{\sqrt{\rho}}$$

Note here

$$\frac{\partial^2\sqrt{\rho}}{\sqrt{\rho}} = \frac{1}{4}\left[\frac{2\rho''}{\rho} - \left(\frac{\rho'}{\rho}\right)^2\right] \tag{1.4}$$

Now from $S' = p = m\dot{q} = mv$ one has

$$P_t + (P\dot{q})' = 0 \equiv \rho_t + (\rho\dot{q})' = 0;\ S_t + \frac{p^2}{2m} + V - \frac{\hbar^2}{2m}\frac{\partial^2\sqrt{\rho}}{\sqrt{\rho}} = 0 \tag{1.5}$$

Differentiating the second equation in X yields ($\partial \sim \partial/\partial X,\ v = \dot{q}$)

$$mv_t + mvv' + \partial V - \frac{\hbar^2}{2m}\partial\left(\frac{\partial\sqrt{\rho}}{\sqrt{\rho}}\right) = 0 \tag{1.6}$$

Consequently, multiplying by $p = mv$ and ρ respectively in (1.5) and (1.6), we obtain

$$m\rho v_t + m\rho vv' + \rho\partial V - \frac{\hbar^2}{2m}\rho\partial\left(\frac{\partial^2\sqrt{\rho}}{\sqrt{\rho}}\right) = 0;\ mv\rho_t + mv(\rho' v + \rho v') = 0 \tag{1.7}$$

Then adding in (1.7) we get

$$\partial_t(\rho v) + \partial(\rho v^2) + \frac{\rho}{m}\partial V - \frac{\hbar^2}{2m^2}\rho\partial\left(\frac{\partial^2\sqrt{\rho}}{\sqrt{\rho}}\right) = 0 \tag{1.8}$$

This is similar to an equation in [**294**] (called an "Euler" equation) and it definitely has a hydrodynamic flavor (cf. also [**434**] and see Section 6.2 for more details and some expansion).

Now go to [**743**] and write (1.6) in the form ($mv = p = S'$)

$$\frac{\partial v}{\partial t} + (v\cdot\nabla)v = -\frac{1}{m}\nabla(V+Q);\ v_t + vv' = -(1/m)\partial(V+Q) \tag{1.9}$$

The higher dimensional form is not considered here but matters are similar there. This equation (and (1.8)) is incomplete as a hydrodynamical equation as a consequence of a missing term $-\rho^{-1}\nabla\mathfrak{p}$ where $\mathfrak{p}$ is the pressure (cf. [**607**]). Hence one "completes" the equation in the form

$$(1.10)\quad m\left(\frac{\partial v}{\partial t}+(v\cdot\nabla)v\right)=-\nabla(V+Q)-\nabla F;\ mv_t+mvv'=-\partial(V+Q)-F'$$

where $\nabla F=(1/R^2)\nabla\mathfrak{p}$ (or $F'=(1/R^2)\mathfrak{p}'$). By the derivations above this would then correspond to an extended SE of the form

$$i\hbar\frac{\partial\psi}{\partial t}=-\frac{\hbar^2}{2m}\Delta\psi+V\psi+F\psi \tag{1.11}$$

provided one can determine F in terms of the wave function ψ. One notes that it a necessary condition here involves $curlgrad(F)=0$ or $curl(R^{-2}\nabla\mathfrak{p})=0$ which enables one to take e.g. $\mathfrak{p}=-bR^2=-b|\psi|^2$. For one dimension one writes $F'=-b(1/R^2)\partial|\psi|^2=-(2bR'/R)\Rightarrow F=-2blog(R)=-blog(|\psi|^2)$. Consequently one has a corresponding SE

$$i\hbar\frac{\partial\psi}{\partial t}=-\frac{\hbar^2}{2m}\psi''+V\psi-b(log|\psi|^2)\psi \tag{1.12}$$

This equation has a number of nice features discussed in [**743**](cf. also [**223, 280, 292, 311, 312, 413, 691, 692, 693, 1028, 1029, 1030, 1031, 1032, 1033, 1034**]). For example $\psi=\beta G(x-vt)exp(ikx-i\omega t)$ is a solution of (2.28) with $V=0$ and for $v=\hbar k/m$ one gets $\psi=cexp[-(B/4)(x-vt+d)^2]\ exp(ikx-i\omega t)$ where $B=4mb/\hbar^2$. Normalization $\int_{-\infty}^{\infty}|\psi|^2=1$ is possible with $|\psi|^2=\delta_m(\xi)=\sqrt{m\alpha/\pi}exp(-\alpha m\xi^2)$ where $\alpha=2b/\hbar^2$, $d=0$, and $\xi=x-vt$ F or $m\to\infty$ we see that δ_m becomes a Dirac delta and this means that motion of a particle with big mass is strongly localized. This is impossible for ordinary QM since $exp(ikx-i\omega t)$ cannot be localized as $m\to\infty$. Such behavior helps to explain the so-called collapse of the wave function and since superposition does not hold Schrödinger's cat is either dead or alive. Further $v=k\hbar/m$ is equivalent to the deBroglie relation $\lambda=h/p$ since $\lambda=(2\pi/k)=2\pi(\hbar/mv)=2\pi(h/2\pi)(1/p)$.

REMARK 1.1.1. We go now to [**530**] and the linear SE in the form $i(\partial\psi/\partial t)=-(1/2m)\Delta\psi+U(\vec{r})\psi$; such a situation leads to the Ehrenfest equations which have the form

$$(1.13)\quad <\vec{v}>=(d/dt)<\vec{r}>;\ <\vec{r}>=\int d^3x|\psi(\vec{r},t)|^2\vec{r};\ m(d/dt)<\vec{v}>=$$

$$=\vec{F}(t)\vec{F}(t)=-\int d^3x|\psi(\vec{r},t)|^2\vec{\nabla}U(\vec{r})$$

Thus the quantum expectation values of position and velocity of a suitable quantum system obey the classical equations of motion and the amplitude squared is a natural probability weight. The result tells us that besides the statistical fluctuations quantum systems posess an extra source of indeterminacy, regulated in a very definite manner by the complex wave function. The Ehrenfest theorem can be extended to many point particle systems and in [**530**] one singles out the kind of nonlinearities that violate the Ehrenfest theorem. A theorem is proved that connects Galilean invariance, and the existence of a Lagrangian whose Euler-Lagrange equation is the SE, to the fulfillment of the Ehrenfest theorem.

REMARK 1.1.2. There are many problems with the quantum mechanical theory of derived nonlinear SE (NLSE) but many examples of realistic NLSE arise in the study of superconductivity, Bose-Einstein condensates, stochastic models of quantum fluids, etc. and the subject demands further study. We make no attempt to survey this here but will give an interesting example later from [**223**] related to fractal structures where a number of the difficulties are resolved. For further information on NLSE, in addition to the references above, we refer to [**100, 281, 392, 413, 530, 534, 535, 536, 788, 789, 790, 956, 957**] for some typical situations (the list is not at all complete and we apologize for omissions). Let us mention a few cases.

- The program of [**530**] introduces a Schrödinger Lagrangian for a free particle including self-interactions of any nonlinear nature but no explicit dependence on the space of time coordinates. The corresponding action is then invariant under spatial coordinate transformations and by Noether's theorem there arises a conserved current and the physical law of conservation of linear momentum. The Lagrangian is also required to be a real scalar depending on the phase of the wave function only through its derivatives. Phase transformations will then induce the law of conservation of probability identified as the modulus squared of the wave function. Galilean invariance of the Lagrangian then determines a connection between the probability current and the linear momentum which insures the validity of the Ehrenfest theorem.
- We turn next to [**535**] for a statistical origin for QM (cf. also [**191, 281, 534, 536, 698, 723, 809, 849**]). The idea is to build a program in which the microscopic motion, underlying QM, is described by a rigorous dynamics different from Brownian motion (thus avoiding unnecessary assumptions about the Brownian nature of the underlying dynamics). The Madelung approach gives rise to fluid dynamical type equations with a quantum potential, the latter being capable of interpretation in terms of a stress tensor of a quantum fluid. Thus one shows in [**535**] that the quantum state corresponds to a subquantum statistical ensemble whose time evolution is governed by classical kinetics in the phase space. The equations take the form

$$\rho_t + \partial_x(\rho u) = 0;\ \partial_t(\mu\rho u_i) + \partial_j(\rho\phi_{ij}) + \rho\partial_{x_i}V = 0;$$
$$\partial_t(\rho E) + \partial_x(\rho S) - \rho\partial_t V = 0 \tag{1.14}$$

$$\frac{\partial S}{\partial t} + \frac{1}{2\mu}\left(\frac{\partial S}{\partial x}\right)^2 + \mathcal{W} + V = 0 \tag{1.15}$$

for two scalar fields ρ, S determining a quantum fluid. These can be rewritten as

$$\frac{\partial\xi}{\partial t} + \frac{1}{\mu}\frac{\partial^2 S}{\partial x^2} + \frac{1}{\mu}\frac{\partial\xi}{\partial x}\frac{\partial S}{\partial x} = 0; \tag{1.16}$$
$$\frac{\partial S}{\partial t} - \frac{\eta^2}{4\mu}\frac{\partial^2\xi}{\partial x^2} - \frac{\eta^2}{8\mu}\left(\frac{\partial\xi}{\partial x}\right)^2 + \frac{1}{2\mu}\left(\frac{\partial S}{\partial x}\right)^2 + V = 0$$

where $\xi = log(\rho)$ and for $\Omega = (\xi/2) + (i/\eta)S = log\Psi$ with $m = N\mu$, $\mathcal{V} = NV$, and $\hbar = N\eta$ one arrives at a SE

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2\Psi}{\partial x^2} + \mathcal{V}\Psi \tag{1.17}$$

Further one can write $\Psi = \rho^{1/2}exp(i\mathfrak{S}/\hbar)$ with $\mathfrak{S} = NS$ and here $N = \int|\Psi|^2 d^n x$. The analysis is very interesting.

We will return to this later.

REMARK 1.1.3. Now in [**324**] one is obliged to use the form $\psi = Rexp(iS/\hbar)$ to make sense out of the constructions (this is no problem with suitable provisos, e.g. that S is not constant - cf. [**110, 191, 197, 198, 346, 347**]). Thus note $\psi'/\psi = (R'/R) + i(S'/\hbar)$ with $\Im(\psi'/\psi) = (1/m)S' \sim p/m$ (see also (1.22) below). Note also $J = (\hbar/m)\Im\psi^*\psi'$ and $\rho = R^2 = |\psi|^2$ represent a current and a density respectively. Then using $p = mv = m\dot{q}$ one can write

$$p = (\hbar/m)\Im(\psi'/\psi);\; J = (\hbar/m)\Im|\psi|^2(\psi^*\psi'/|\psi|^2) = (\hbar/m)\Im(\rho p) \tag{1.18}$$

Then look at the SE in the form $i\hbar\psi_t = -(\hbar^2/2m)\psi'' + V\psi$ with $\psi_t = (R_t + iS_tR/\hbar)exp(iS/\hbar)$ and

$$\psi_{xx} = [(R' + (iS'R/\hbar)exp(iS/\hbar)]' = \tag{1.19}$$
$$[R'' + (2iS'R'/\hbar) + (iS''R/\hbar) + (iS'/\hbar)^2R]exp(iS/\hbar)$$

which means

$$-\frac{\hbar^2}{2m}\left[R'' - \left(\frac{S'}{\hbar}\right)^2 + \frac{2iS'r'}{\hbar} + \frac{iS''R}{\hbar}\right] + VR = i\hbar\left[R_t + \frac{iS_tR}{\hbar}\right] \Rightarrow \tag{1.20}$$
$$\Rightarrow \partial_t R^2 + \frac{1}{m}(R^2S')' = 0;\; S_t + \frac{(S')^2}{2mR} - \frac{\hbar^2 R''}{2mR} + V = 0$$

This can also be written as (cf. (1.3))

$$\partial_t\rho + \frac{1}{m}\partial(p\rho) = 0;\; S_t + \frac{p^2}{2m} + Q + V = 0 \tag{1.21}$$

where $Q = -\hbar^2 R''/2mR$. Now we sketch the philosophy of [**324, 325**] in part. Most of such aspects are omitted here and we try to isolate the essential mathematical features (see Section 1.2 for more). First one emphasizes configurations based on coordinates whose motion is choreographed by the SE according to the rule (1-D only here)

$$\dot{q} = v = \frac{\hbar}{m}\Im\frac{\psi^*\psi'}{|\psi|^2} \tag{1.22}$$

where $i\hbar\psi_t = -(\hbar^2/2m)\psi'' + V\psi$. The argument for (1.22) is based on obtaining the simplest Galilean and time reversal invariant form for velocity, transforming correctly under velocity boosts. This leads directly to (1.22) (cf. (1.18))) so that Bohmian mechanics (BM) is governed by (1.22) and the SE. It's a fairly convincing argument and no recourse to Floydian time seems possible (cf. [**191, 347, 373, 374**]). Note however that if $S = c$ then $\dot{q} = v = (\hbar/m)\Im(R'/R) = 0$ while $p = S' = 0$ so perhaps this formulation avoids the $S = 0$ problems indicated in

[**191, 347, 373, 374**]. One notes also that BM depends only on the Riemannian structure $g = (g_{ij}) = (m_i \delta_{ij})$ in the form

$$\dot{q} = \hbar \Im(grad\psi/\psi);\ i\hbar\psi_t = -(\hbar^2/2)\Delta\psi + V\psi \tag{1.23}$$

What makes the constant $\hbar/m$ in (1.22) important here is that with this value the probability density $|\psi|^2$ on configuration space is equivariant. This means that via the evolution of probability densities $\rho_t + div(v\rho) = 0$ (as in (1.21) with $v \sim p/m$) the density $\rho = |\psi|^2$ is stationary relative to ψ, i.e. $\rho(t)$ retains the form $|\psi(q,t)|^2$. One calls $\rho = |\psi|^2$ the quantum equilibrium density (QED) and says that a system is in quantum equilibrium when its coordinates are randomly distributed according to the QED. The quantum equilibrium hypothesis (QHP) is the assertion that when a system has wave function ψ the distribution ρ of its coordinates satisfies $\rho = |\psi|^2$.

REMARK 1.1.4. We extract here from [**446, 447, 448**] (cf. also the references there for background and [**381, 382, 523**] for some information geometry). There are a number of interesting results connecting uncertainty, Fisher information, and QM and we make no attempt to survey the matter. Thus first recall that the classical Fisher information associated with translations of a 1-D observable X with probability density $P(x)$ is

$$F_X = \int dx\, P(x)([log(P(x)]')^2 > 0 \tag{1.24}$$

Recall now the Cramer-Rao inequality $Var(X) \geq F_X^{-1}$ where $Var(X) \sim$ variance of X. A Fisher length for X is defined via $\delta X = F_X^{-1/2}$ and this quantifies the length scale over which $p(x)$ (or better $log(p(x))$) varies appreciably. Then the root mean square deviation ΔX satisfies $\Delta X \geq \delta X$. Let now P be the momentum observable conjugate to X, and P_{cl} a classical momentum observable corresponding to the state ψ given via $p_{cl}(x) = (\hbar/2i)[(\psi'/\psi) - (\bar{\psi}'/\bar{\psi})]$ (cf. (1.22)). One has then the identity $< p >_\psi = < p_{cl} >_\psi$ via integration by parts. Now define the nonclassical momentum by $p_{nc} = p - p_{cl}$ and one shows that $\Delta X \Delta p \geq \delta X \Delta p \geq \delta X \Delta p_{nc} = \hbar/2$. Then go to [**447**] now where two proofs are given for the derivation of the SE from the exact uncertainty principle ($\delta X \Delta p_{nc} = \hbar/2$). Thus consider a classical ensemble of n-dimensional particles of mass m moving under a potential V. The motion can be described via the HJ and continuity equations

$$\frac{\partial s}{\partial t} + \frac{1}{2m}|\nabla s|^2 + V = 0;\ \frac{\partial P}{\partial t} + \nabla \cdot \left[P\frac{\nabla s}{m}\right] = 0 \tag{1.25}$$

for the momentum potential s and the position probability density P (note that we have interchanged p and P from [**447**] - note also there is no quantum potential and this will be supplied by the information term). These equations follow from the variational principle $\delta L = 0$ with Lagrangian

$$L = \int dt\, d^n x\, P\left[\frac{\partial s}{\partial t} + \frac{1}{2m}|\nabla s|^2 + V\right] \tag{1.26}$$

It is now assumed that the classical Lagrangian must be modified due to the existence of random momentum fluctuations. The nature of such fluctuations is immaterial for (cf. [**447**] for discussion) and one can assume that the momentum associated with position x is given by $p = \nabla s + N$ where the fluctuation term N vanishes on average at each point x. Thus s changes to being an average momentum potential. It follows that the average kinetic energy $< |\nabla s|^2 > /2m$ appearing in (1.26) should be replaced by $< |\nabla s + N|^2 > /2m$ giving rise to

$$L' = L + (2m)^{-1} \int dt < N \cdot N >= L + (2m)^{-1} \int dt (\Delta N)^2 \tag{1.27}$$

where $\Delta N =< N \cdot N >^{1/2}$ is a measure of the strength of the fluctuations. The additional term is specified uniquely, up to a multiplicative constant, by the following three assumptions

(1) Action principle: L' is a scalar Lagrangian with respect to the fields P and s where the principle $\delta L' = 0$ yields causal equations of motion. Thus $(\Delta N)^2 = \int d^n x\, p f(P, \nabla P, \partial P/\partial t, s, \nabla s, \partial s/\partial t, x, t)$ for some scalar function f.
(2) Additivity: If the system comprises two independent noninteracting subsystems with $P = P_1 P_2$ then the Lagrangian decomposes into additive subsystem contributions; thus $f = f_1 + f_2$ for $P = P_1 P_2$.
(3) Exact uncertainty: The strength of the momentum fluctuation at any given time is determined by and scales inversely with the uncertainty in position at that time. Thus $\Delta N \to k\Delta N$ for $x \to x/k$. Moreover since position uncertainty is entirely characterized by the probability density P at any given time the function f cannot depend on s, nor explicitly on t, nor on $\partial P/\partial t$.

The following theorem is then asserted (see [**447**] for the proofs).

THEOREM 1.1. The above 3 assumptions imply the relation $(\Delta N)^2 = c \int d^n x\, P |\nabla log(P)|^2$ where c is a positive universal constant.

COROLLARY 1.1. It follows from (1.27) that the equations of motion for p and s corresponding to the principle $\delta L' = 0$ are

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + V\psi \tag{1.28}$$

where $\hbar = 2\sqrt{c}$ and $\psi = \sqrt{P} exp(is/\hbar)$.

REMARK 1.1.5. We sketch here for simplicity and clarity another derivation of the SE along similar ideas following [**805**]. Let $P(y^i)$ be a probability density and $P(y^i + \Delta y^i)$ be the density resulting from a small change in the y^i. Calculate the cross entropy via

$$J(P(y^i + \Delta y^i) : P(y^i)) = \int P(y^i + \Delta y^i) log \frac{P(y^i + \Delta y^i)}{P(y^i)} d^n y \simeq \tag{1.29}$$

$$\simeq \left[\frac{1}{2} \int \frac{1}{P(y^i)} \frac{\partial P(y^i)}{\partial y^i} \frac{\partial P(y^i)}{\partial y^k)} d^n y \right] \Delta y^i \Delta y^k = I_{jk} \Delta y^i \Delta y^k$$

The I_{jk} are the elements of the Fisher information matrix. The most general expression has the form

$$I_{jk}(\theta^i) = \frac{1}{2}\int \frac{1}{P(x^i|\theta^i)}\frac{\partial P(x^i|\theta^i)}{\partial \theta^j}\frac{\partial P(x^i|\theta^i)}{\partial \theta^k}d^n x \tag{1.30}$$

where $P(x^i|\theta^i)$ is a probability distribution depending on parameters θ^i in addition to the x^i. For $P(x^i|\theta^i) = P(x^i + \theta^i)$ one recovers (1.29) (straightforward - cf. [**805**]). If P is defined over an n-dimensional manifold with positive inverse metric g^{ik} one obtains a natural definition of the information associated with P via

$$I = g^{ik}I_{ik} = \frac{g^{ik}}{2}\int \frac{1}{P}\frac{\partial P}{\partial y^i}\frac{\partial P}{\partial y^k}d^n y \tag{1.31}$$

Now in the HJ formulation of classical mechanics the equation of motion takes the form

$$\frac{\partial S}{\partial t} + \frac{1}{2}g^{\mu\nu}\frac{\partial S}{\partial x^\mu}\frac{\partial S}{\partial x^\nu} + V = 0 \tag{1.32}$$

where $g^{\mu\nu} = diag(1/m, \cdots, 1/m)$. The velocity field u^μ is given by $u^\mu = g^{\mu\nu}(\partial S/\partial x^\nu)$. When the exact coordinates are unknown one can describe the system by means of a probability density $P(t, x^\mu)$ with $\int P d^n x = 1$ and

$$(\partial P/\partial t) + (\partial/\partial x^\mu)(P g^{\mu\nu}(\partial S/\partial x^\nu)) = 0 \tag{1.33}$$

These equations completely describe the motion and can be derived from the Lagrangian

$$L_{CL} = \int P\left\{\frac{\partial S}{\partial t} + \frac{1}{2}g^{\mu\nu}\frac{\partial S}{\partial x^\mu}\frac{\partial S}{\partial x^\nu} + V\right\}dtd^n x \tag{1.34}$$

using fixed endpoint variation in S and P. Quantization is obtained by adding a term proportional to the information I defined in (1.31). This leads to

$$L_{QM} = L_{CL} + \lambda I = \int P\left\{\frac{\partial S}{\partial t} + \frac{1}{2}g^{\mu\nu}\left[\frac{\partial S}{\partial x^\mu}\frac{\partial S}{\partial x^\nu} + \frac{\lambda}{P^2}\frac{\partial P}{\partial x^\mu}\frac{\partial P}{\partial x^\nu}\right] + V\right\}dtd^n x \tag{1.35}$$

Fixed endpoint variation in S leads again to (1.33) while variation in P leads to

$$\frac{\partial S}{\partial t} + \frac{1}{2}g^{\mu\nu}\left[\frac{\partial S}{\partial x^\mu}\frac{\partial S}{\partial x^\nu} + \lambda\left(\frac{1}{P^2}\frac{\partial P}{\partial x^\mu}\frac{\partial P}{\partial x^\nu} - \frac{2}{P}\frac{\partial^2 P}{\partial x^\mu \partial x^\nu}\right)\right] + V = 0 \tag{1.36}$$

These equations are equivalent to the SE if $\psi = \sqrt{P}exp(iS/\hbar)$ with $\lambda = (2\hbar)^2$.

REMARK 1.1.6. In Remarks 1.1.6 - 1.1.8 one uses $Q = \pm(1/m)$ times the standard $Q = -(\hbar^2/2m)(\Delta\sqrt{\rho}/\sqrt{\rho}$. The SE gives to a probability distribution $\rho = |\psi|^2$ (with suitable normalization) and to this one can associate an information entropy $S(t)$ (actually configuration information entropy) $S = -\int \rho log(\rho)d^3x$ which is typically not a conserved quantity (S is an unfortunate notation here but we retain it momentarily since no confusion should arise). The rate of change in time of S can be readily found by using the continuity equation $\partial_t \rho = -\nabla \cdot (v\rho)$ where v is a current velocity field. Note here (cf. also [**752**])

$$\frac{\partial S}{\partial t} = -\int \rho_t (1 + log(\rho))dx = \int (1 + log(\rho))\partial(v\rho) \tag{1.37}$$

Note that a formal substitution of $v = -u$ in the contintuity equation implies the standard free Browian motion outcome $dS/dt = D \cdot \int [(\nabla\rho)^2/\rho) d^3x = D \cdot Tr\mathfrak{F} \geq 0$ - use here $u = D\nabla log(\rho)$ with $D = \hbar/2m)$ and (1.37) with $\int (1 + log(\rho))\partial(v\rho) = -\int v\rho\partial log(\rho) = -\int v\rho' \sim \int ((\rho')^2/\rho)$ modulo constants involving D etc. Recall here $\mathfrak{F} \sim -(2/D^2)\int \rho Q dx = \int dx[(\nabla\rho)^2/\rho]$ is a functional form of Fisher information. A high rate of information entropy production corresponds to a rapid spreading (flattening down) of the probablity density. This delocalization feature is concomitant with the decay in time property quantifying the time rate at which the far from equilibrium system approaches its stationary state of equilibrium $(d/dt)Tr\mathfrak{F} \leq 0$.

REMARK 1.1.7. Now going back to the quantum context one admits general forms of the current velocity v. For example consider a gradient field $v = b - u$ where the so-called forward drift $b(x, t)$ of the stochastic process depends on a particular diffusion model. Then one can rewrite the continuity equation as a standard Fokker-Planck equation $\partial\!\!\!/ \rho = D\Delta\rho - \nabla \cdot (b\rho)$. Boundary restrictions requiring ρ, $v\rho$, and $b\rho$ to vanish at spatial infinities or at boundaries yield the general entropy balance equation

$$\frac{dS}{dt} = \int \left[\rho(\nabla \cdot b) + D \cdot \frac{(\nabla\rho)^2}{\rho}\right] d^3x \equiv -D\frac{dS}{dt} = \int \rho(v \cdot u)d^3x = < v \cdot u > \tag{1.38}$$

The first term in the first equation is not positive definite and can be interpreted as an entropy flux while the second term refers to the entropy production proper. The flux term represents the mean value of the drift field divergence $\nabla \cdot b$ which by itself is a local measure of the flux incoming to or outgoing from an infinitesimal surrounding of x at time t. If locally $(\nabla \cdot b)(x, t) > 0$ on an infinitesimal time scale we would encounter a local entropy increase in the system (increasing disorder) while in case $(\nabla \cdot b)(x, t) < 0$ one thinks of local entropy loss or restoration or order. Only in the situation $< \nabla \cdot b >= 0$ is there no entropy production. Quantum dynamics permits more complicated behavior. One looks first for a general criterion under which the information entropy S is a conserved quantity. Consider (1.8) and invoke the diffusion current to write (recall $u = D(\nabla\rho)/\rho$)

$$D\frac{dS}{dt} = -\int [\rho^{-1/2}(\rho v)] \cdot [\rho^{-1/2}(D\nabla\rho)]d^3x \tag{1.39}$$

Then by means of the Schwarz inequality one has $D|dS/dt| \leq < v^2 >^{1/2} < u^2 >^{1/2}$ so a necessary (but insufficient) condition for $dS/dt \neq 0$ is that both $< v^2 >$ and $< u^2 >$ are nonvanishing. On the other hand a sufficient condition for $dS/dt = 0$ is that either one of these terms vanishes. Indeed in view of $< u^2 >= D^2 \int [(\nabla\rho)^2/\rho]d^3x$ the vanishing information entropy production implies $dS/dt = 0$; the vanishing diffusion current does the same job.

REMARK 1.1.8. We develop a little more perspective now (following [**395**] - first paper). Recall Q written out as

$$-Q = 2D^2 \frac{\Delta\rho^{1/2}}{\rho^{1/2}} = D^2 \left[\frac{\Delta\rho}{\rho} - \frac{1}{2\rho^2}(\nabla\rho)^2\right] = \frac{1}{2}u^2 + D\nabla\cdot u \tag{1.40}$$

where $u = D\nabla log(\rho)$ is called an osmotic velocity field. The standard Brownian motion involves $v = -u$, known as the diffusion current velocity and (up to a dimensional factor) is identified with the thermodynamic force of diffusion which drives the irreversible process of matter exchange at the macroscopic level. On the other hand, even while the thermodynamic force is a concept of purely statistical origin associated with a collection of particles, in contrast to microscopic forces which have a direct impact on individual particles themselves, it is well known that this force manifests itself as a Newtonian type entry in local conservation laws describing the momentum balance; in fact it pertains to the average (local average) momentum taken over by the particle cloud, a statistical ensemble property quantified in terms of the probability distribution at hand. It is precisely the (negative) gradient of the above potential Q in (1.40) which plays the Newtonian force role in the momentum balance equations. The second analytical expression of interest here involves

$$-\int Q\rho dx = (1/2)\int u^2\rho dx = (1/2)D^2\cdot F_X;\ F_X = \int \frac{(\nabla\rho)^2}{\rho}dx \tag{1.41}$$

where F_X is the Fisher information, encoded in the probability density ρ which quantifies its gradient content (sharpness plus localization/disorder) Note that

$$-\int Q\rho = -\int[(1/2)u^2\rho + D\rho u'] = -\int(1/2)u^2\rho + \int Du\rho' = \tag{1.42}$$

$$= -(1/2)\int D^2(\rho'/\rho)^2\rho + D^2\int \rho'(\rho'/\rho) = (D^2/2)\int(\rho')^2/\rho = (1/2)\int u^2\rho$$

On the other hand the local entropy production inside the system sustaining an irreversible process of diffusion is given via

$$\frac{dS}{dt} = D\cdot\int\frac{(\nabla\rho)^2}{\rho}dx = D\cdot F_X \geq 0 \tag{1.43}$$

This stands for an entropy production rate when the Fick law induced diffusion current (standard Brownian motion case) $j = -D\nabla\rho$, obeying $\partial_t\rho + \nabla j = 0$, enters the scene. Here $S = -\int \rho log(\rho)dx$ plays the role of (time dependent) information entropy in the nonequilibrium statistical mechanics framework for the thermodynamics of irreversible processes. It is clear that a high rate of entropy increase coresponds to a rapid spreading (flattening) of the probability density. This explicitly depends on the sharpness of density gradients. The potential Q(x,t), the Fisher information F_X, the nonequilibrium measure of entropy production dS/dt, and the information entropy $S(t)$ are thus mutually entangled quantities, each being exclusively determined in terms of ρ and its derivatives.

In the standard statistical mechanics setting the Euler equation gives a prototypical momentum balance equation in the (local) mean

$$(\partial_t + v\cdot\nabla)v = \frac{F}{m} - \frac{\nabla P}{\rho} \tag{1.44}$$

where $F = -\nabla F$ represents normal Newtonian force and P is a pressure term. Q appears in the hydrodynamical formalism of QM via

$$(\partial_t + v\cdot\nabla)v = \frac{1}{m}F - \nabla Q = \frac{1}{m}F + \frac{\hbar^2}{2m^2}\nabla\frac{\Delta\rho^{1/2}}{\rho^{1/2}} \tag{1.45}$$

Another spectacular example pertains to the standard free Brownian motion in the strong friction regime (Smoluchowski diffusion), namely

$$(\partial_t + v\cdot\nabla)v = -2D^2\nabla\frac{\Delta\rho^{1/2}}{\rho^{1/2}} = -\nabla Q \tag{1.46}$$

where $v = -D(\nabla\rho/\rho)$ (formally $D = \hbar/2m$).

REMARK 1.1.9. The papers in [**291, 292**] contain very interesting derivations of Schrödinger equations via diffusion ideas à la Nelson, Markov wave equations, and suitable "applied" forces (e.g. radiative reactive forces).

We go now to Nagasawa [**670, 671, 672, 673, 674**] to see how diffusion and the SE are really connected (cf. also [**15, 141, 223, 421, 676, 681, 698, 726, 732, 733, 734, 735, 736**] for related material, some of which is discussed later in detail); for now we simply sketch some formulas for a simple Euclidean metric where $\Delta = \sum(\partial/\partial x^i)^2$. Then $\psi(t,x) = exp[R(t,x) + iS(t,x)]$ satisfies a SE $i\partial_t\psi + (1/2)\Delta\psi + ia(t,x)\cdot\nabla\psi - V(t,x)\psi = 0$ ($\hbar$ and m omitted with $a(t,x)$ a drift coefficient) if and only if

$$V = -\frac{\partial S}{\partial t} + \frac{1}{2}\Delta R + \frac{1}{2}(\nabla R)^2 - \frac{1}{2}(\nabla S)^2 - a\cdot\nabla S; \tag{1.47}$$

$$0 = \frac{\partial R}{\partial t} + \frac{1}{2}\Delta S + (\nabla S)\cdot(\nabla R) + a\cdot\nabla R$$

in the region $D = \{(s,x) : \psi(s,x) \neq 0\}$ (a harmless gauge factor in the divergence is also being omitted). Solutions are often referred to as weak or distributional but we do not belabor this point. From [**671, 672, 673**] there results

THEOREM 1.2. Let $\psi(t,x) = exp[R(t,x) + iS(t,x)]$ be a solution of the SE above; then $\phi(t,x) = exp[R(t,x) + S(t,x)]$ and $\hat{\phi} = exp[R(t,x) - S(t,x)]$ are solutions of

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}\Delta\phi + a(t,x)\cdot\nabla\phi + c(t,x,\phi)\phi = 0; \tag{1.48}$$

$$-\frac{\partial\hat{\phi}}{\partial t} + \frac{1}{2}\Delta\hat{\phi} - a(t,x)\cdot\nabla\hat{\phi} + c(t,x,\phi)\hat{\phi} = 0$$

where the creation and annihilation term $c(t,x,\phi)$ is given via

$$c(t,x,\phi) = -V(t,x) - 2\frac{\partial S}{\partial t}(t,x) - (\nabla S)^2(t,x) - 2a\cdot\nabla S(t,x) \tag{1.49}$$

Conversely given $(\phi, \hat{\phi})$ as in Theorem 1.2 satisfying (1.48) it follows that ψ satisfies the SE with V as in (1.49) (note $R = (1/2)log(\hat{\phi}\phi)$ and $S = (1/2)log(\phi/\hat{\phi})$ with $exp(R) = (\hat{\phi}\phi)^{1/2}$).

We note that the equations (1.48) are not imaginary time SE and from all this one can conclude that nonrelativistic QM is diffusion theory in terms of Schrödinger processes (described by $(\phi, \hat{\phi})$ - more details later). Further it is shown that certain key postulates in Nelson's stochastic mechanics or Zambrini's Euclidean QM (cf. [**1011**]) can both be avoided in connecting the SE to diffusion processes (since they are automatically valid). Look now at Theorem 1.2 for one dimension and write $T = \hbar t$ with $X = (\hbar/\sqrt{m})x$ and $A = a\hbar/\sqrt{m}$; then the SE becomes

$$i\hbar\psi_T = -(\hbar^2/2m)\psi_{XX} - iA\psi_X + V\psi; \tag{1.50}$$

$$i\hbar R_T + (\hbar^2/m^2)R_X S_X + (\hbar^2/2m^2)S_{XX} + AR_X = 0;$$

$$V = -i\hbar S_T + (\hbar^2/2m)R_{XX} + (\hbar^2/2m^2)R_X^2 - (\hbar^2/2m^2)S_X^2 - AS_X$$

Hence

PROPOSITION 1.1. The SE of Theorem 1.2, written in the variables $X = (\hbar/\sqrt{m})x$, $T = \hbar t$, with $A = (\sqrt{m}/\hbar)a$ and $V = V(X,T) \sim V(x,t)$ is equivalent to (2.2).

Making a change of variables in (1.48) now, as in Proposition 1.1, yields

COROLLARY 1.2. Equation (1.48), written in the variables of Proposition 1.2, becomes

$$\hbar\phi_T + \frac{\hbar^2}{2m}\phi_{XX} + A\phi_X + \tilde{c}\phi = 0;\ -\hbar\hat{\phi}_T + \frac{\hbar^2}{2m}\hat{\phi}_{XX} - A\hat{\phi}_X + \tilde{c}\hat{\phi} = 0; \tag{1.51}$$

$$\tilde{c} = -\tilde{V}(X,T) - 2\hbar S_T - \frac{\hbar^2}{m}S_X^2 - 2AS_X$$

Thus the diffusion processes pick up factors of $\hbar$ and $\hbar/\sqrt{m}$.

REMARK 1.1.10. We extract here from the Appendix to [**672**] for some remarks on competing points of view regarding diffusion and the the SE. First some work of Fenyes [**360**] is cited where a Lagrangian is taken as

$$L(t) = \int\left[\frac{\partial S}{\partial t} + \frac{1}{2}(\nabla S)^2 + V + \frac{1}{2}\left(\frac{1}{2}\frac{\nabla\mu}{\mu}\right)^2\right]\mu dx \tag{1.52}$$

where $\mu_t(x) = exp(2R(t,x))$ denotes the distribution density of a diffusion process and V is a potential function. The term $\Pi(\mu) = (1/2)[(1/2)(\nabla\mu/\mu)]^2$ is called a diffusion pressure and since $(1/2)(\nabla\mu/\mu) \sim \nabla R$ the Lagrangian can be written as

$$L = \int\left[\frac{\partial S}{\partial t} + \frac{1}{2}(\nabla S)^2 + \frac{1}{2}(\nabla R)^2 + V\right]\mu dx \tag{1.53}$$

Applying the variational principle $\delta\int_a^b L(t)dt = 0$ one arrives at

$$\frac{\partial S}{\partial t}+\frac{1}{2}\left[(\nabla(R+S))\right]^2-(\nabla(R+S))\cdot\left(\frac{1}{2}\frac{\nabla\mu}{\mu}\right)+\left(\frac{1}{2}\frac{\nabla\mu}{\mu}\right)^2-\frac{1}{4}\frac{\Delta\mu}{\mu}+V=0 \quad (1.54)$$

which is called a motion equation of probability densities. From this he shows that the function $\psi = exp(R+iS)$ satisfies the SE $i\partial_t + (1/2)\Delta\psi - V(t,x)\psi = 0$. Indeed putting $\Pi(\mu)$ and the formula $(1/2)(\Delta\mu/\mu)+(1/2)\Delta R+(\nabla R)^2$ into (1.53) one obtains

$$\frac{\partial S}{\partial t}+\frac{1}{2}(\nabla S)^2-\frac{1}{2}(\nabla R)^2-\frac{1}{2}\Delta R+V=0 \quad (1.55)$$

which goes along with the duality relation $R_t + (1/2)\Delta S + \nabla S\cdot\nabla R + b\cdot\nabla R = 0$ where $u = (1/2)(a+\hat{a}) = \nabla R$ and $v = (1/2)(a-\hat{a}) = \nabla S$ as derived in the Nagasawa theory. Hence $\psi = exp(R+iS)$ satisfies the SE by previous calculations. One can see however that the equation (1.53) is not needed since the SE and diffusion equations are equivalent and in fact the equations of motion are the diffusion equations. Moreover it is shown in [**672**] that (1.53) is an automatic consequence in diffusion theory with $V = -c - 2S_t - (\nabla S)^2$ and therefore it need not be postulated or derived by other means. This is a simple calculation from the theory developed above.

REMARK 1.1.11. Nelson's important work in stochastic mechanics [**698**] produced the SE from diffusion theory but involved a stochastic Newtonian equation which is shown in [**672**] to be automatically true. Thus Nelson worked in a general context which for our purposes here can be considered in the context of Brownian motions

$$B(t) = \partial_t + (1/2)\Delta + b\cdot\nabla + a\cdot\nabla;\;\; \hat{B}(t) = -\partial_t + (1/2)\Delta - b\cdot\nabla + \hat{a}\cdot\nabla \quad (1.56)$$

and used a mean acceleration $\alpha(t,x) = -(1/2)[B(t)\hat{B}(t)x + \hat{B}(t)B(t)x]$. Assuming the duality relations after (1.55) he obtains a formula

$$\alpha(t,x) = -\frac{1}{2}[B(t)(-b+\hat{a})+\hat{B}(b+a)] = b_t+(1/2)\nabla(b)^2-(b+v)\times curl(b)- \quad (1.57)$$

$$-[-v_t+(1/2)\Delta u+(1/2)(\hat{a}\cdot\nabla)a+(1/2)(a\cdot\nabla)\hat{a}-(b\cdot\nabla)v-(v\cdot\nabla)b-v\times curl(b)]$$

Then it is shown that the SE can be deduced from the stochastic Newton's equation

$$\alpha(t,x) = -\nabla V + \frac{\partial b}{\partial t} + \frac{1}{2}\nabla(b^2) - (b+v)\times curl(b) \quad (1.58)$$

Nagasawa shows that this serves only to reproduce a known formula for V yielding the SE; he also shows that (1.57) also is an automatic consequence of the duality formulation of diffusion equations above. This equation (1.57) is often called stochastic quantization since it leads to the SE and it is in fact correct with the V specified there. However the SE is more properly considered as following directly from the diffusion equations in duality and is not correctly an equation of motion. There is another discussion of Euclidean QM developed by Zambrini [**1011**]. This involves $\tilde{\alpha}(t,x) = (1/2)[B(t)B(t)x + \hat{B}(t)\hat{B}(t)x]$ (with $(\sigma\sigma^T)^{ij} = \delta^{ij}$). It is postulated that this equals $-\nabla c + b_t + (1/2)\nabla(b)^2 - b + v) \times curl(b)$ which in fact leads to the same equation for V as above with $V = -c - 2S_t - (\nabla S)^2 - 2b \cdot \nabla S$ so there is nothing new. Indeed it is shown in [**672**] that the postulated equivalence holds automatically as a simple consequence of time reversal of diffusion processes.

2. SCALE RELATIVITY

Scale relativity (SR) is due to L. Nottale (cf. [**715, 716, 717, 718, 719, 720, 721**]) and somehow has not been accorded any real recognition by the "establishment". We only touch here on derivations of the SE and will develop further aspects later; the arguments are evidently heuristic but have a compelling interest. More general relativistic and cosmological features are discussed in Chapter 2 where further discussion is given. The ideas involve spacetime having a fractal microstructure containing in particular continuous (self-similar) nondifferentiable paths which serve as geodesic quantum paths of Hausdorff dimension $D = 2$. This is in fact a good notion of quantum path (following Feynman for example - cf. [**1**]) and we will see how it leads to a lovely (heuristic) derivation of the SE which automatically creates a complex wave function.

REMARK 1.2.1. One considers quantum paths à la Feynman so that (**E1**) $lim_{t \to t'}[X(t) - X(t')]^2/(t - t')$ exists. This implies $X(t) \in H^{1/2}$ where H^α means $c\epsilon^\alpha \leq |X(t) - X(t')| \leq C\epsilon^\alpha$ and from [**345**] for example this means $dim_H X[a,b] = 1/2$. Now one "knows" (see e.g. [**1**]) that quantum and Brownian motion paths (in the plane) have H-dimension 2 and some clarification is needed here. We refer to [**625**] where there is a paper on Wiener Brownian motion (WBM), random walks, etc. discussing Hausdorff and other dimensions of various sets. Thus given $0 < \lambda < 1/2$ with probability 1 a Browian sample function X satisfies $|X(t+h) - X(t)| \leq b|h|^\lambda$ for $|h| \leq h_0$ where $b = b(\lambda)$. This leads to the result that with probability 1 the graph of a Brownian sample function has Hausdorff and box dimension 3/2. On the other hand a Browian trail (or path) in 2 dimensions has Hausdorff and box dimension 2 (note a quantum path can have self intersections, etc.).

There are now several excellent approaches. The method of Nottale [**700, 715, 718**] is preeminent (cf. also [**732, 733, 734, 735**]) and there is also a nice derivation of a nonlinear SE via fractal considerations in [**223**] (indicated below). The most elaborate and rigorous approach is due to Cresson [**272**], with elaboration and updating in [**3, 273, 274**]. There are various derivations of the SE and we follow [**715**] here (cf. also [**718, 828**]). The philosophy of scale relativity will be discussed later and we just write down equations here pertaining to the SE. First a bivelocity structure is defined (recall that one is dealing with fractal paths). One defines first

$$\frac{d_+}{dt}y(t) = lim_{\Delta t \to 0_+}\left\langle \frac{y(t+\Delta t) - y(t)}{\Delta t}\right\rangle; \tag{2.1}$$

$$\frac{d_-}{dt}y(t) = lim_{\Delta t \to 0_+}\left\langle \frac{y(t) - y(t-\Delta t)}{\Delta t}\right\rangle$$

Applied to the position vector x this yields forward and backward mean velocities, namely $(d_+/dt)x(t) = b_+$ and $(d_-/dt)x(t) = b_-$. Here these velocities are defined as the average at a point q and time t of the respective velocities of the outgoing and incoming fractal trajectories; in stochastic QM this corresponds to an average on the quantum state. The position vector $x(t)$ is thus "assimilated" to a stochastic process which satisfies respectively after ($dt > 0$) and before ($dt < 0$) the instant t a relation $dx(t) = b_+[x(t)]dt + d\xi_+(t) = b_-[x(t)]dt + d\xi_-(t)$ where $\xi(t)$ is a Wiener process (cf. [**698**]). It is in the description of ξ that the $D = 2$ fractal character of trajectories is inserted; indeed that ξ is a Wiener process means that the $d\xi$'s are assumed to be Gaussian with mean 0, mutually independent, and such that

$$(2.2)\qquad < d\xi_{+i}(t)d\xi_{+j}(t) >= 2\mathcal{D}\delta_{ij}dt;\ < d\xi_{-i}(t)d\xi_{-j}(t) >= -2\mathcal{D}\delta_{ij}dt$$

where $< >$ denotes averaging ($\mathcal{D}$ is now the diffusion coefficient). Nelson's postulate (cf. [**698**]) is that $\mathcal{D} = \hbar/2m$ and this has considerable justification (cf. [**715**]). Note also that (2.2) is indeed a consequence of fractal (Hausdorff) dimension 2 of trajectories follows from $< d\xi^2 > /dt^2 = dt^{-1}$, i.e. precisely Feynman's result $< v^2 >^{1/2} \sim \delta t^{-1/2}$ (the discussion here in [**715**] is unclear however - cf. [**29**]). Note also that Brownian motion (used in Nelson's postulate) is known to be of fractal (Hausdorff) dimension 2. Note also that any value of $\mathcal{D}$ may lead to QM and for $\mathcal{D} \to 0$ the theory becomes equivalent to the Bohm theory. Now expand any function $f(x, t)$ in a Taylor series up to order 2, take averages, and use properties of the Wiener process ξ to get

$$(2.3)\qquad \frac{d_+ f}{dt} = (\partial_t + b_+ \cdot \nabla + \mathcal{D}\Delta)f;\ \frac{d_- f}{dt} = (\partial_t + b_- \cdot \nabla - \mathcal{D}\Delta)f$$

Let $\rho(x, t)$ be the probability density of $x(t)$; it is known that for any Markov (hence Wiener) process one has $\partial_t\rho + div(\rho b_+) = \mathcal{D}\Delta\rho$ (forward equation) and $\partial_t\rho + div(\rho b_-) = -\mathcal{D}\Delta\rho$ (backward equation). These are called Fokker-Planck equations and one defines two new average velocities $V = (1/2)[b_+ + b_-]$ and $U = (1/2)[b_+ - b_-]$. Consequently adding and subtracting one obtains $\rho_t + div(\rho V) = 0$ (continuity equation) and $div(\rho U) - \mathcal{D}\Delta\rho = 0$ which is equivalent to $div[\rho(U - \mathcal{D}\nabla log(\rho))] = 0$. One can show, using (2.3) that the term in square brackets in the last equation is zero leading to $U = \mathcal{D}\nabla log(\rho)$. Now place oneself in the (U, V) plane and write $\mathcal{V} = V - iU$. Then write $(d_{\mathcal{V}}/dt) = (1/2)(d_+ + d_-)/dt$ and $(d_{\mathcal{U}}/dt) = (1/2)(d_+ - d_-)/dt$. Combining the equations in (2.3) one defines $(d_{\mathcal{V}}/dt) = \partial_t + V \cdot \nabla$ and $(d_{\mathcal{U}}/dt) = \mathcal{D}\Delta + U \cdot \nabla$; then define a complex operator $(d'/dt) = (d_{\mathcal{V}}/dt) - i(d_{\mathcal{U}}/dt)$ which becomes

$$(2.4)\qquad \frac{d'}{dt} = \left(\frac{\partial}{\partial t} - i\mathcal{D}\Delta\right) + \mathcal{V} \cdot \nabla$$

One now postulates that the passage from classical mechanics to a new nondifferentiable process considered here can be implemented by the unique prescription of replacing the standard d/dt by d'/dt. Thus consider $\mathfrak{S} = \left\langle \int_{t_1}^{t_2} \mathcal{L}(x, \mathcal{V}, t)dt \right\rangle$ yielding by least action $(d'/dt)(\partial\mathcal{L}/\partial\mathcal{V}_i) = \partial\mathcal{L}/\partial x_i$. Define then $\mathcal{P}_i = \partial\mathcal{L}/\partial\mathcal{V}_i$ leading to $\mathcal{P} = \nabla\mathfrak{S}$ (recall the classical action principle with $dS = pdq - Hdt$). Now for Newtonian mechanics write $L(x, v, t) = (1/2)mv^2 - \mathbf{U}$ which becomes

$\mathcal{L}(x,\mathcal{V},t) = (1/2)m\mathcal{V}^2 - \mathfrak{U}$ leading to $-\nabla\mathfrak{U} = m(d'/dt)\mathcal{V}$. One separates real and imaginary parts of the complex acceleration $\gamma = (d'\mathcal{V}/dt$ to get

$$d'\mathcal{V} = (d_{\mathcal{V}} - id_{\mathcal{U}})(V - iU) = (d_{\mathcal{V}}V - d_{\mathcal{U}}U) - i(d_{\mathcal{U}}V + d_{\mathcal{V}}U) \tag{2.5}$$

The force $F = -\nabla\mathfrak{U}$ is real so the imaginary part of the complex acceleration vanishes; hence

$$\frac{d_{\mathcal{U}}}{dt}V + \frac{d_{\mathcal{V}}}{dt}U = \frac{\partial U}{\partial t} + U\cdot\nabla V + V\cdot\nabla U + \mathcal{D}\Delta V = 0 \tag{2.6}$$

from which $\partial U/\partial t$ may be obtained. This is a weak point in the derivation since one has to assume e.g. that $U(x,t)$ has certain smoothness properties (see below for refinements). Differentiating the expression $U = \mathcal{D}\nabla log(\rho)$ and using the continuity equation yields another expression $(\partial U/\partial t) = -\mathcal{D}\nabla(divV) - \nabla(V\cdot U)$. Comparison of these relations yields $\nabla(divV) = \Delta V - U \wedge curlV$ where the $curlU$ term vanishes since U is a gradient. However in the Newtonian case $\mathcal{P} = m\mathcal{V}$ so $\mathcal{P}\nabla\mathfrak{S}$ implies that $\mathcal{V}$ is a gradient and hence a generalization of the classical action S can be defined. Recall $V = 2\mathcal{D}\nabla S$ and $\nabla(divV) = \Delta V$ with $curlV = 0$; combining this with the expression for U one obtains $\mathfrak{S} = log(\rho^{1/2}) + iS$. One notes that this is compatible with [**698**] for example. Finally set $\psi = \sqrt{\rho}exp(iS) = exp(i\mathfrak{S})$ with $\mathcal{V} = -2i\mathcal{D}\nabla(log\psi)$ and note

$$U = \mathcal{D}\nabla log(\rho);\ V = 2\mathcal{D}\nabla S; \tag{2.7}$$
$$\mathcal{V} = -2i\mathcal{D}\nabla log\psi = -i\mathcal{D}\nabla log(\rho) + 2\mathcal{D}\nabla S = V - iU$$

Thus for $\mathcal{P} = m\mathcal{V}$ the relation $\mathcal{P} \sim -i\hbar\nabla$ or $\mathcal{P}\psi = -i\hbar\nabla\psi$ has a natural interpretation. Putting ψ in the equation $-\nabla\mathfrak{U} = m(d'/dt)\mathcal{V}$, which generalizes Newton's law to fractal space the equation of motion takes the form $\nabla\mathfrak{U} = 2i\mathcal{D}m(d'/dt)(\nabla log(\psi))$. Then noting that d' and ∇ do not commute one replaces d'/dt by (2.4) to obtain

$$\nabla\mathfrak{U} = 2i\mathcal{D}m\left[\partial_t\nabla log(\psi) - i\mathcal{D}\Delta(\nabla log(\psi)) - 2i\mathcal{D}(\nabla log(\psi)\cdot\nabla)(\nabla log(\psi)\right] \tag{2.8}$$

This expression can be simplified via

$$\nabla\Delta = \Delta\nabla;\ (\nabla f\cdot\nabla)(\nabla f) = (1/2)\nabla(\nabla f)^2;\ \frac{\Delta f}{f} = \Delta log(f) + (\nabla log(f))^2 \tag{2.9}$$

which implies

$$\frac{1}{2}\Delta(\nabla log(\psi)) + (\nabla log(\psi)\cdot\nabla)(\nabla log(\psi)) = \frac{1}{2}\nabla\frac{\Delta\psi}{\psi} \tag{2.10}$$

Integrating this equation yields $\mathcal{D}^2\Delta\psi + i\mathcal{D}\partial_t\psi - (\mathfrak{U}/2m)\psi = 0$ up to an arbitrary phase factor $\alpha(t)$ which can be set equal to 0 by a suitable choice of phase S. Replacing $\mathcal{D}$ by $\hbar/2m$ one arrives at the SE $i\hbar\psi_t = -(\hbar^2/2m)\Delta\psi + \mathfrak{U}\psi$ and this suggests an interpretation of QM as mechanics in a nondifferentiable (fractal) space.

In fact (using one space dimension for convenience) we see that if $\mathfrak{U} = 0$ then the free motion $m(d'/dt)\mathcal{V} = 0$ yields the SE $i\hbar\psi_t = -(\hbar^2/2m)\psi_{xx}$ as a geodesic equation in "fractal" space. Further from $U = (\hbar/m)(\partial\sqrt{\rho}/\sqrt{\rho})$ and $Q = -(\hbar^2/2m)(\Delta\sqrt{\rho}/\sqrt{\rho})$ one arrives at a lovely relation, namely

PROPOSITION 2.1. The quantum potential Q can be written in the form $Q = -(m/2)U^2 - (\hbar/2)\partial U$ (cf. (1.40) multiplied by $-m$). Hence the quantum potential arises directly from the fractal nonsmooth nature of the quantum paths. Since Q can be thought of as a quantization of a classical motion we see that the quantization corresponds exactly to the existence of nonsmooth paths. Consequently smooth paths imply no quantum mechanics.

REMARK 1.2.2. In [**15**] (to be discussed later) one writes again $\psi = Rexp(iS/\hbar)$ with field equations in the hydrodynamical picture (1-D for convenience)

$$d_t(m_0\rho v) = \partial_t(m_0\rho v) + \nabla(m_0\rho v) = -\rho\nabla(u + Q);\ \partial_t\rho + \nabla\cdot(\rho v) = 0 \tag{2.11}$$

where $Q = -(\hbar^2/2m_0)(\Delta\sqrt{\rho}/\sqrt{\rho})$. The Nottale approach is used as above with $d_v \sim d_{\mathcal{V}}$ and $d_u \sim d_{\mathcal{U}}$. One assumes that the velocity field from the hydrodynamical model agrees with the real part v of the complex velocity $V = v - iu$ so $v = (1/m_0)\nabla s \sim 2\mathcal{D}\partial s$ and $u = -(1/m_0)\nabla\sigma \sim \mathcal{D}\partial log(\rho)$ where $\mathcal{D} = \hbar/2m_0$. In this context the quantum potential $Q = -(\hbar^2/2m_0)\Delta\mathcal{D}\sqrt{\rho}/\sqrt{\rho}$ becomes

$$Q = -m_0\mathcal{D}\nabla\cdot u - (1/2)m_0u^2 \sim -(\hbar/2)\partial u - (1/2)m_0u^2 \tag{2.12}$$

Consequently Q arises from the fractal derivative and the nondifferentiability of spacetime again, as in Proposition 2.1. Further one can relate u (and hence Q) to an internal stress tensor whereas the v equations correspond to systems of Navier-Stokes type.

REMARK 1.2.3. Some of the relevant equations for dimension one are collected together later. We note that it is the presence of $\pm$ derivatives that makes possible the introduction of a complex plane to describe velocities and hence QM; one can think of this as the motivation for a complex valued wave function and the nature of the SE.

We go now to [**223**] and will sketch some of the material. Here one extends ideas of Nottale and Ord in order to derive a nonlinear Schrödinger equation (NLSE). Using the hydrodynamic model in [**743**] one added a hydrostatic pressure term to the Euler-Lagrange equations and another possibility is to add instead a kinematic pressure term. The hydrostatic pressure is based on an Euler equation $-\nabla p = \rho g$ where ρ is density and g the gravitational acceleration (note this gives $-p = \rho g x$ in 1-D). In [**743**] one took $\rho = \psi^*\psi$, b a mass-energy parameter, and $-p = \rho$; then the hydrostatic potential is (for $\rho_0 = 1$)

$$b\int g(x)\cdot dr = -b\int\frac{\nabla p}{\rho}\cdot dr = -blog(\rho/\rho_0) = -blog(\psi^*\psi) \tag{2.13}$$

Here $-blog(\psi^*\psi)$ has energy units and explains the nonlinear term of [**111**] which involved

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi + U\psi - b[log(\psi^*\psi)]\psi \tag{2.14}$$

A derivation of this equation from the Nelson stochastic QM was given by Lemos (cf. [**588**]). There are moreover some problems since this equation does not obey the homogeneity condition saying that the state $\lambda|\psi>$ is equivalent to $|\psi>$; moreover

(2.14) is not invariant under $\psi \to \lambda\psi$. Further, plane wave solutions to (2.14) do not seem to have a physical interpretion due to extraneous dispersion relations. Finally one would like to have a SE in terms of ψ alone. Note that another NLSE could be obtained by adding kinetic pressure terms $(1/2)\rho v^2$ and taking $\rho = a\psi^*\psi$ where $v = p/m$. Now using the relations from HJ theory $(\psi/\psi^*) = exp[2i\mathfrak{S}(x)/\hbar]$ and $p = \nabla\mathfrak{S}(x) = mv$ one can write $v = -i(\hbar/2m)\nabla log(\psi/\psi^*)$ so that the energy density becomes

$$(1/2)\rho|v|^2 = (a\hbar^2/8m^2)\psi\psi^*\nabla log(\psi/\psi^*)\cdot\nabla log(\psi^*/\psi) \tag{2.15}$$

This leads to a corresponding nonlinear potential associated with the kinematical pressure via $(a\hbar^2/8m^2)\nabla log(\psi/\psi^*)\cdot\nabla log(\psi^*/\psi)$. Hence a candidate NLSE is

$$i\hbar\partial_t = -\frac{\hbar^2}{2m}\nabla^2\psi + U\psi - b[log(\psi^*\psi)]\psi + \frac{a\hbar^2}{8m^2}\left(\nabla log\frac{\psi}{\psi^*}\cdot\nabla log\frac{\psi^*}{\psi}\right) \tag{2.16}$$

Here the Hamiltonian is Hermitian and $a \neq b$ are both mass-energy parameters to be determined experimentally. The new term can also be written in the form $\nabla log(\psi/\psi^*)\cdot\nabla log(\psi^*/\psi) = -[\nabla log(\psi/\psi^*)]^2$. The goal now is to derive a NLSE directly from fractal space time dynamics for a particle undergoing Brownian motion. This does not require a quantum potential, a hydrodynamic model, or any pressure terms as above.

REMARK 1.2.4. One should make some comments about the kinematic pressure terms $(1/2)\rho v^2 \sim (\hbar^2/2m)(a/m)|\nabla log(\psi)|^2$ versus hydrostatic pressure terms of the form $\int(\nabla p/\rho) \sim -blog(\psi^*\psi)$. The hydrostatic term breaks homogeneity whereas the kinematic pressure term preserves homogeneity (scaling with a λ factor). The hydrostatic pressure term is also not compatible with the motion kinematics of a particle executing a fractal Brownian motion. The fractal formulation will enable one to relate the parameters a, b to $\hbar$.

Following Nottale, nondifferentiability implies a loss of causality and one is thinking of Feynmann paths with $< v^2 >\propto (dx/dt)^2 \propto dt^{2[(1/D)-1)}$ with $D = 2$. Now a fractal function $f(x,\epsilon)$ could have a derivative $\partial f/\partial\epsilon$ and renormalization group arguments lead to $(\partial f(x,\epsilon)/\partial log\epsilon) = a(x) + bf(x,\epsilon)$ (cf. [**715**]). This can be integrated to give $f(x,\epsilon) = f_0(x)[1-\zeta(x)(\lambda/\epsilon)^{-b}]$. Here $\lambda^{-b}\zeta(x)$ is an integration constant and $f_0(x) = -a(x)/b$. This says that any fractal function can be approximated by the sum of two terms, one independent of the resolution and the other resolution dependent; $\zeta(x)$ is expected to be a flucuating function with zero mean. Provided $a \neq 0$ and $b < 0$ one has two interesting cases (i) $\epsilon << \lambda$ with $f(x,\epsilon) \sim f_0(x)(\lambda/\epsilon)^{-b}$ and (ii) $\epsilon >> \lambda$ with f independent of scale. Here λ is the deBroglie wavelength. Now one writes

$$r(t+dt,dt) - r(t,dt) = b_+(r,t)dt + \xi_+(t,dt)\left(\frac{dt}{\tau_0}\right)^\beta; \tag{2.17}$$

$$r(t,dt) - r(t-dt,dt) - b_-(r,t)dt + \xi_-(t,dt)\left(\frac{dt}{\tau_0}\right)^\beta$$

where $\beta = 1/D$ and $b_\pm$ are average forward and backward velocities. This leads to $v_\pm(r,t,dt) = b_\pm(r,t) + \xi_\pm(t,dt)(dt/\tau_0)^{\beta-1}$. In the quantum case $D = 2$ one has $\beta = 1/2$ so $dt^{\beta-1}$ is a divergent quantity (i.e. nondifferentiability ensues). Following [**588, 715, 698**] one defines

$$\frac{d_{\pm}r(t)}{dt} = lim_{\Delta t\to\pm 0}\left\langle\frac{r(t+\Delta t)-r(t)}{\Delta t}\right\rangle \tag{2.18}$$

from which $d_{\pm}r(t)/dt = b_{\pm}$. Now following Nottale one writes

$$\frac{\delta}{dt} = \frac{1}{2}\left(\frac{d_+}{dt}+\frac{d_-}{dt}\right) - \frac{i}{2}\left(\frac{d_+}{dt}-\frac{d_-}{dt}\right) \tag{2.19}$$

which leads to $(\delta/dt) = (\partial/\partial t) + v\cdot\nabla - iD\nabla^2$. Here in principle $\mathcal{D}$ is a real valued diffusion constant to be related to $\hbar$. and $< d\xi_{\pm i}d\xi_{\pm j} >= \pm 2\mathcal{D}\delta_{ij}dt$. Now for the complex time dependent wave function we take $\psi = exp[i\mathfrak{S}/2m\mathcal{D}]$ with $p = \nabla\mathfrak{S}$ so that $v = -2iD\nabla log(\psi)$. The SE is obtained from the Newton equation ($F = ma$) via $-\nabla U = m(\delta/dt)v = -2im\mathcal{D}(\delta/dt)\nabla log(\psi)$ which yields

$$-\nabla U = -2im[\mathcal{D}\partial_t\nabla log(\psi)] - 2\mathcal{D}\nabla\left(\mathcal{D}\frac{\nabla^2\psi}{\psi}\right) \tag{2.20}$$

(see [**715**] for identities involving ∇). Integrating yields $\mathcal{D}^2\nabla^2\psi + i\mathcal{D}\partial_t\psi - (U/2m)\psi = 0$ up to an arbitrary phase factor which may be set equal to zero. Now replacing D by $\hbar/2m$ one gets the SE $i\hbar\partial_t\psi + (\hbar^2/2m)\nabla^2\psi = U\psi$. Here the Hamiltonian is Hermitian, the equation is linear, and the equation is homogeneous of degree 1 under the substitution $\psi \to \lambda\psi$.

Next one generalizes this by relaxing the assumption that the diffusion coefficient is real. Some comments on complex energies are needed - in particular constraints are often needed (cf. [**788**]). However complex energies are not alien in ordinary QM (cf. [**223**] for references). Now the imaginary part of the linear SE yields the continuity equation $\partial_t\rho + \nabla\cdot(\rho v) = 0$ and with a complex potential the imaginary part of the potential will act as a source term in the continuity equation. Instead of $< d\zeta_{\pm}d\zeta_{\pm} >= \pm 2\mathcal{D}dt$ with $\mathcal{D}$ and $2mD = \hbar$ real one sets

$$< d\zeta_{\pm}d\zeta_{\pm} >= \pm(\mathcal{D}+\mathcal{D}^*)dt;\ 2m\mathcal{D} = \hbar = \alpha + i\beta \tag{2.21}$$

The complex time derivative operator becomes $(\delta/dt) = \partial_t + v\cdot\nabla - (i/2)(\mathcal{D}+\mathcal{D}^*)\nabla^2$. Writing again $\psi = exp[i\mathfrak{S}/2mD] = exp(i\mathfrak{S}/\hbar)$ one obtains $v = -2i\mathcal{D}\nabla log(\psi)$. The NLSE is then obtained (via the Newton law) via the relation $-\nabla U = m(\delta/dt)v = -2im\mathcal{D}(\delta/dt)\nabla log(\psi)$. Combining equations yields then

$$\nabla U = 2im[\mathcal{D}\partial_t\nabla log(\psi) - 2i\mathcal{D}^2(\nabla log(\psi)\cdot\nabla)(\nabla log(\psi) - \\ -\frac{i}{2}(\mathcal{D}+\mathcal{D}^*)\mathcal{D}\nabla^2(\nabla log(\psi)] \tag{2.22}$$

Now using the identities (i) $\nabla\nabla^2 = \nabla^2\nabla$, (ii) $2(\nabla log(\psi)\cdot\nabla)(\nabla log(\psi) = \nabla(\nabla log(\psi))^2$ and (iii) $\nabla^2 log(\psi) = \nabla^2\psi/\psi - (\nabla log(\psi))^2$ leads to a NLSE with nonlinear (kinematic pressure) potential, namely

$$i\hbar\partial_t\psi = -\frac{\hbar^2}{2m}\frac{\alpha}{\hbar}\nabla^2\psi + U\psi - i\frac{\hbar^2}{2m}\frac{\beta}{\hbar}(\nabla log(\psi))^2\psi \tag{2.23}$$

Note the crucial minus sign in front of the kinematic pressure term and also that $\hbar = \alpha + i\beta = 2m\mathcal{D}$ is complex. When $\beta = 0$ one recovers the linear SE. The nonlinear potential is complex and one defines $W = -(\hbar^2/2m)(\beta/\hbar)(\nabla log(\psi))^2$ with U the ordinary potential; then the NLSE is

$$i\hbar\partial_t\psi = [-(\hbar^2/2m)(\alpha/\hbar)\nabla^2 + U + iW]\psi \tag{2.24}$$

This is the fundamental result of [**223**]; it has the form of an ordinary SE with complex potential $U+iW$ and complex $\hbar$. The Hamiltonian is no longer Hermitian and the potential itself depends on ψ. Nevertheless one can have meaningful physical solutions with real valued energies and momenta; the homogeneity breaking hydrostatic pressure term $-b(log(\psi^*\psi)\psi$ is not present (it would be meaningless) and the NLSE is invariant under $\psi \to \lambda\psi$.

REMARK 1.2.5. One could ask why not simply propose as a valid NLSE an equation

$$i\hbar\partial_t\psi = -\frac{\hbar^2}{2m}\nabla^2\psi + \frac{\hbar^2}{2m}\frac{a}{m}|\nabla log(\psi)|^2\psi$$

Here one has a real Hamiltonian satisfying the homogeneity condition and the equation admits soliton solutions of the form $\psi = CA(x-vt)exp[i(kx-\omega t)]$ where $A(x-vt)$ is to be determined by solving the NLSE. The problem here is that the equation suffers from an extraneous dispersion relation. Thus putting in the plane wave solution $\psi \sim exp[-i(Et-px)]$ one gets an extraneous energy momentum (EM) relation (after setting $U=0$), namely $E = (p^2/2m)[1+(a/m)]$ instead of the usual $E = p^2/2m$ and hence $E_{QM} \neq E_{FT}$ where FT means field theory.

REMARK 1.2.6. It has been known since e.g. [**788**] that the expression for the energy functional in nonlinear QM does not coincide with the QM energy functional, nor is it unique. To see this write down the NLSE of [**111**] in the form $i\hbar\partial_t\psi = \partial H(\psi,\psi^*)/\partial\psi^*$ where the real Hamiltonian density is

$$H(\psi,\psi^*) = -\frac{\hbar^2}{2m}\psi^*\nabla^2\psi + U\psi^*\psi - b\psi^* log(\psi^*\psi)\psi + b\psi^*\psi$$

Then using $E_{FT} = \int Hd^3r$ we see it is different from $< \hat{H} >_{QM}$ and in fact $E_{FT} - E_{QM} = \int b\psi^*\psi d^3r = b$. This problem does not occur in the fractal based NLSE since it is written entirely in terms of ψ.

REMARK 1.2.7. In the fractal based NLSE there is no discrepancy between the QM energy functional and the FT energy functional. Both are given by

$$N^{NLSE}_{fractal} = -\frac{\hbar^2}{2m}\frac{\alpha}{\hbar}\psi^*\nabla^2\psi + U\psi^*\psi - i\frac{\hbar^2}{2m}\frac{\beta}{\hbar}\psi^*(\nabla log(\psi)^2\psi$$

The NLSE is unambiguously given by in Remark 1.2.5 and $H(\psi,\psi^*)$ is homogeneous of degree 1 in λ. Such equations admit plane wave solutions with dispersion relation $E = p^2/2m$; indeed, inserting the plane wave solution into the fractal based NLSE one gets (after setting $U=0$)

$$E = \frac{\hbar^2}{2m}\frac{\alpha}{\hbar}\frac{p^2}{2m} + i\frac{\beta}{\hbar}\frac{p^2}{2m} = \frac{p^2}{2m}\frac{\alpha+i\beta}{\hbar} = \frac{p^2}{2m} \tag{2.25}$$

since $\hbar = \alpha + i\beta$. The remarkable feature of the fractal approach versus all other NLSE considered sofar is that the QM energy functional is precisely the FT one. The complex diffusion constant represents a truly new physical phenomenon insofar as a small imaginary correction to the Planck constant is the hallmark of nonlinearity in QM (see [**223**] for more on this).

REMARK 1.2.8. Some refinements of the Nottale derivation are given in [**272**] and we consider $x \to f(x(t),t) \in C^n$ with $X(t) \in H^{1/n}$ (i.e. $c\epsilon^{1/n} \leq |X(t') - X(t)| \leq C\epsilon^{1/n}$). Define ($f$ real valued)

$$(2.26)\quad \nabla_{\pm}^{\epsilon} f(t) = \frac{f(t \pm \epsilon) - f(t)}{\pm\epsilon};\ \frac{\Box_\epsilon f}{\Box t}(f) = \frac{1}{2}(\nabla_+^\epsilon + \nabla_-^\epsilon)f - \frac{i}{2}(\nabla_+^\epsilon - \nabla_-^\epsilon f);$$

$$a_{\epsilon,j}(t) = \frac{1}{2}[(\Delta_+^\epsilon x)^j - (-1)^j(\Delta_-^\epsilon x)^j] - \frac{i}{2}[(\Delta_+^\epsilon x)^j + (-1)^j(\Delta_-^\epsilon x)^j]$$

Here one assumes $h > 0$ and $\epsilon(f,h) \geq \epsilon > 0$ where $\epsilon(f,h)$ is the minimal resolution defined via $inf_\epsilon\{a_\epsilon(f) < h\}$ for $a_\epsilon f(t) = |[f(t+\epsilon)+f(t-\epsilon)-2f(t)]/\epsilon|$. If $\epsilon(f,h)$ is not 0 then f is not differentiable (but not conversely). Now assume some minimal control over the lack of differentiability (cf. [**272**]) and then for f now complex valued with $\Box_\epsilon f/\Box t = (\Box_\epsilon f_{\Re}/\Box t) + i(\Box_\epsilon f_{\Im}/\Box t)$ (note the mixing of i terms is not trivial) one has

$$(2.27)\qquad \frac{\Box_\epsilon f}{\Box t} = \frac{\partial f}{\partial t} + \frac{\Box_\epsilon x}{\Box t}\frac{\partial f}{\partial x} + \sum_2^n \frac{1}{j!} a_{\epsilon,j}(t)\frac{\partial^j f}{\partial x^j}\epsilon^{j-1} + o(\epsilon^{1/n})$$

We sketch now the derivation of a SE in the spirit of Nottale but with more mathematical polish. Going to [**272**] one defines (for a nondifferentiable function f)

$$(2.28)\qquad f_\epsilon(t) = \frac{1}{2\epsilon}\int_{t-\epsilon}^{t+\epsilon} f(s)ds;$$

$$f_\epsilon^+(t) = \frac{1}{2\epsilon}\int_t^{t+\epsilon} f(s)ds;\ f_\epsilon^-(t) = \frac{1}{2\epsilon}\int_{t-\epsilon}^t f(s)ds$$

One considers quantum paths à la Feynman so that $lim_{t\to t'}[X(t)-X(t')]^2/(t-t')$ exists. This implies $X(t) \in H^{1/2}$ where H^α means $c\epsilon^\alpha \leq |X(t)-X(t')| \leq C\epsilon^\alpha$ and from Remark 1.2.1 for example this means $dim_H X[a,b] = 1/2$. Next, thinking of classical Lagrangians $L(x,v,t) = (1/2)mv^2 + \mathbf{U}(x,t)$, one defines an operator Q via ($(x,t,v) \sim$ classical variables)

$$(2.29)\quad Q(t) = t;\ Q(x(t)) = X(t);\ Q(v(t)) = \mathcal{V}(t);\ Q\left(\frac{df}{dt}\right) = Q\left(\frac{d}{dt}\right)\cdot Q(f)$$

where $Q(d/dt) = d/dt$ if $Q(f)(t)$ is differentiable and $Q(d/dt) = \Box_\epsilon/\Box t$ where $\epsilon(x,h) > \epsilon > 0$ if $Q(f)(t)$ is nondifferentiable. Note $\mathcal{V}(t) = Q(d/dt)[X(t)]$ so regularity of X determines the form of Q here and for $Q(x) = X \in H^{1/2}$ one has $\mathcal{V} = \Box_\epsilon X/\Box t$. The scalar Euler-Lagrange (EL) equation associated to $\mathcal{L}(X(t),\mathcal{V}(t),t) = Q(L(x(t),v(t),t)$ is

$$(2.30)\qquad \frac{\Box_\epsilon}{\Box t}\left(\frac{\partial\mathcal{L}}{\partial\mathcal{V}}(X(t),\mathcal{V}(t),t)\right) = \frac{\partial\mathcal{L}}{\partial X}(X(t),\mathcal{V}(t),t)$$

Now given a classical $v \sim (1/m)(\partial S/\partial x)$ one gets $\mathcal{V} = (1/m)(\partial\mathfrak{S}/\partial X)$ and $\mathcal{L} = (\partial\mathfrak{S}/\partial t)$ with $\psi(X,t) = exp[i\mathfrak{S}(X,t)/2m\gamma]$. For $\mathcal{L} \sim (1/2)m\mathcal{V}^2 + \mathfrak{U}$ then the quantum (EL) equation is $m(\Box_\epsilon \mathcal{V}/\Box t) = (\partial\mathfrak{U}/\partial X)$ leading to
(2.31)

$$2i\gamma m\left[-\frac{\psi_X^2}{\psi}\left(i\gamma + \frac{a_\epsilon(t)}{2}\right) + \partial_t\psi + \frac{a_\epsilon(t)}{2}\frac{\partial^2\psi}{\partial X^2}\right] = (\mathfrak{U}(X) + \alpha(X))\psi + o(\epsilon^{1/2})$$

where

$$a_\epsilon(t) = \frac{1}{2}\{[(\nabla_+^\epsilon X(t))^2 - (\nabla_-^\epsilon X(t))^2] - i[(\nabla_+^\epsilon X(t))^2 + (\nabla_-^\epsilon X(t))^2]\} \tag{2.32}$$

Then (2.32) is called the generalized SE and the nonlinear character of such equations is discussed in [**192, 223**] for example. In [**272**] one then arrives at a conventional looking SE under the assumption $a_\epsilon = -2i\gamma$, leading to

$$\gamma^2\frac{\partial^2\psi}{\partial X^2} + i\gamma\frac{\partial\psi}{\partial t} = [\mathfrak{U}(X,t) + \alpha(X)]\frac{\psi}{2m} + o(\epsilon^{1/2}) \tag{2.33}$$

One can then always take $\alpha(X) = 0$ and choosing $\gamma = \hbar/2m$ one arrives at $i\hbar\psi_t + (\hbar^2/2m)(\partial^2\psi/\partial t^2) = \mathfrak{U}\psi$. However the requirement $a_\epsilon(t) = -2i\gamma$ seems quite restrictive.

- Note here that the argument using $a_\pm$ is rigorous via [**272**]. $a_\epsilon = -i\hbar/m$ is permissible and in fact can have solutions of $\nabla_\sigma^\epsilon X(t) = constant$ via $X_c(t) = \pm\sqrt{\hbar/2m)}(t - c - (\epsilon/2)) + P_\epsilon(t)$ where $P_\epsilon \in H^{1/2}$ is an arbitrary periodic function.

Referring back to Example 1.2.3 we have $b_\pm(t)(t) \sim \Box_\pm x(t)$ and $V = (/2)(\Box_+ x + \Box_- x)(t)$ with $U = (1/2)(\Box_+ x - \Box_- x)(t)$. The relation between U and the quantum potential Q will formally still hold (cf. also [**273**] on nondifferentiable variational principles) and one can rewrite this as $\sqrt{\rho}U = (\hbar/m)\partial\sqrt{\rho}$; $\sqrt{\rho}Q = -(\hbar^2/2m)\partial^2\sqrt{\rho}$ along with $\partial(\sqrt{\rho}U) = -(2/\hbar)\sqrt{\rho}Q$. If U is not differentiable one could also look at $\sqrt{\rho}U = -(2/\hbar)\int_0^X \sqrt{\rho}QdX' + f(t)$ with $f(t)$ possibly determinable via the term $(\sqrt{\rho}U)(0,t)$.

3. REMARKS ON FRACTAL SPACETIME

There have been a number of articles and books involving fractal methods in spacetime or fractal spacetime itself with impetus coming from quantum physics and relativity. We refer here especially to [**1, 186, 187, 225, 422, 675, 676, 677, 678, 679, 680, 681, 682, 683, 684, 685, 686, 687, 688, 689, 690**] for background to this paper. Many related papers are omitted here and we refer in particular to the journal Chaos, Solitons, and Fractals CSF) for further information. For information on fractals and stochastic processes we refer for example to [**33, 83, 241, 242, 243, 345, 423, 555, 562, 592, 643, 625, 697, 725, 748, 763, 810, 918, 942, 985**]. We discuss here a few background ideas and constructions in order to indicate the ingredients for El Naschie's Cantorian spacetime $\mathfrak{E}^\infty$, whose exact nature is elusive. Suitable references are given but there are many more papers in the journal CSF by El Naschie (and others) based on these fundamental ideas and these are either important in a revolutionary sense or a fascinating refined form of science fiction. In what appears at times to be pure numerology one manages to (rather hastily) produce amazingly close numerical approximations to virtually all the fundamental constants of physics (including string theory). The key concepts revolve around the famous golden ratio $(\sqrt{5}-1)/2$ and a strange Cantorian space $\mathfrak{E}^{\infty}$ which we try to describe below. It is very tempting to want all of these (heuristic) results to be true and the approach seems close enough and universal enough to compel one to think something very important must be involved. Moreover such scope and accuracy cannot be ignored so we try to examine some of the constructions in a didactic manner in order to possibly generate some understanding.

3.1. COMMENTS ON CANTOR SETS.

EXAMPLE 3.1. In the paper [**643**] one discusses random recursive constructions leading to Cantor sets, etc. Associated with each such construction is a universal number α such that almost surely the random object has Hausdorff dimension α (we assume that ideas of Hausdorff and Minkowski-Bouligand (MB) or upper box dimension are known - cf. [**83, 186, 345, 592**]). One construction of a Cantor set goes as follows. Choose x from $[0,1]$ according to the uniform distribution and then choose y from $[x,1]$ according to the uniform distribution on $[x,1]$. Set $J_0=[0,x]$ and $J_1=[y,1]$ and recall the standard 1/3 construction for Cantor sets. Continue this procedure by rescaling to each of the intervals already obtained. With probability one one then obtains a Cantor set S_c^0 with Hausdorff dimension $\alpha=\phi=(\sqrt{5}-1)/2\sim .618$. Note that this is just a particular random Cantor set; there are others with different Hausdorff dimensions (there seems to be some - possibly harmless - confusion on this point in the El Naschie papers). However the golden ratio ϕ is a very interesting number whose importance rivals that of π or e. In particular (cf. [**1**]) ϕ is the hardest number to approximate by rational numbers and could be called the most irrational number. This is because its continued fraction represention involves all $1's$.

EXAMPLE 3.2. From [**676**] the Hausdorff (H) dimension of a traditional triadic Cantor set is $d_c^{(0)}=log(2)/log(3)$. To determine the equivalent to a triadic Cantor set in 2 dimensions one looks for a set which is triadic Cantorian in all directions. The analogue of an area $A=1\times 1$ is a quasi-area $A_c=d_c^{(0)}\times d_c^{(0)}$ and to normalize A_c one uses $\rho_2=(A/A_c)_2=1/(d_c^{(0)})^2$ (for n-dimensions $\rho_n=1/(d_c^{(0)})^{n-1}$). Then the n^{th} Cantor like H dimension $d_c^{(n)}$ will have the form $d_c^{(n)}=\rho_n d_c^{(0)}=1/(d_c^{(0)})^{n-1}$. Note also that the H dimension of a Sierpinski gasket is $d_c^{(n+1)}/d_c^{(n)}=1/d_c^{(0)}=log(3)/log(2)$ and in any event the straight-forward interpretation of $d_c^{(2)}=log(3)/log(2)$ is a scaling of $d_c^{(0)}=log(2)/log(3)$ proportional to the ratio of areas $(A/A_c)_2$. One notes that $d_c^{(4)}=1/(d_c^{(0)})^3=(log(3)/log(2))^3\simeq 3.997\sim 4$ so the 4-dimensional Cantor set is essentially “space filling”.

Another derivation goes as follows. Define probability quotients via $\Omega=dim(subset)/dim(set)$. For a triadic Cantor set in 1-D $\Omega^{(1)}=d_c^{(0)}/d_c^{(1)}=d_c^{(0)}$ $(d_c^{(1)}=1)$. To lift the Cantor set to n-dimensions look at the multiplicative probability law $\Omega^{(n)} = (\Omega^{(1)})^n = (d_c^{(0)})^n$. However since $\Omega^{(1)} = d_c^{(0)}/d_c^{(n)}$ we get

$$d_c^{(0)}/d_c^{(n)} = (d_c^{(0)})^n \Rightarrow d_c^{(n)} = 1/(d_c^{(0)})^{n-1} \tag{3.1}$$

Since $\Omega^{(n-1)}$ is the probability of finding a Cantor point (Cantorian) one can think of the H dimension $d_c^{(n)} = 1/\Omega^{(n-1)}$ as a measure of ignorance. One notes here also that for $d_c^{(0)} = \phi$ (the Cantor set $S_c^{(0)}$ of Example 3.1) one has $d_c^{(4)} = 1/\phi^3 = 4 + \phi^3 \simeq 4.236$ which is surely space filling.

Based on these ideas one proves in [**676, 680, 682**] a number of theorems and we sketch some of this here. One picks a "backbone" Cantor set with H dimension $d_c^{(0)}$ (the choice of $\phi = d_c^{(0)}$ will turn out to be optimal for many arguments). Then one imagines a Cantorian spacetime $\mathfrak{E}^\infty$ built up of an infinite number of spaces of dimension $d_c^{(n)}$ $(-\infty \leq n < \infty)$. The exact form of embedding etc. here is not specified so one imagines e.g. $\mathfrak{E}^\infty = \cup\mathfrak{E}^{(n)}$ (with unions and intersections) in some amorphous sense. There are some connections of this to vonNeumann's continuous geometries indicated in [**684**]. In this connection we remark that only $\mathfrak{E}^{(-\infty)}$ is the completely empty set ($\mathfrak{E}^{(-1)}$ is not empty). First we note that $\phi^2 + \phi - 1 = 0$ leading to

$$1 + \phi = 1/\phi,\ \phi^3 = (2+\phi)/\phi,\ (1+\phi)/(1-\phi) = 1/\phi(1-\phi) = 4 + \phi^3 = 1/\phi^3 \tag{3.2}$$

(a very interesting number indeed). Then one asserts that

THEOREM 3.1. Let $(\Omega^{(1)})^n$ be a geometrical measure in n-dimensional space of a multiplicative point set process and $\Omega^{(1)}$ be the Hausdorff dimension of the backbone (generating) set $d_c^{(0)}$. Then $< d >= 1/d_c^{(0)}(1 - d_c^{(0)})$ (called curiously an average Hausdorff dimension) will be exactly equal to the average space dimension $\tilde{} < n >= (1 + d_c^{(0)})(1 - d_c^{(0)})$ and equivalent to a 4-dimensional Cantor set with H-dimension $d_c^{(4)} = 1/(d_c^{(0)})^3$ if and only if $d_c^{(0)} = \phi$.

To see this take $\Omega^{(n)} = (\Omega^{(1)})^n$ again and consider the total probability of the additive set described by the $\Omega^{(n)}$, namely $Z_0 = \sum_0^\infty (\Omega^{(1)})^n = 1/(1 - \Omega^{(1)})$. It is conceptually easier here to regard this as a sum of weighted dimensions (since $d_c^{(n)} = 1/(d_c^{(0)})^{n-1}$) and consider $w_n = n(d_c^{(0)})^n$. Then the expectation of n becomes (note $d_c^{(n)} \sim 1/(d_c^{(0)})^{n-1} \sim 1/\Omega^{(n-1)}$ so $n(d_c^{(0)})^{n-1} \sim n/d_c^{(n)}$)

$$E(n) = \frac{\sum_1^\infty n^2 (d_c^{(0)})^{n-1}}{\sum_1^\infty n (d_c^{(0)})^{n-1}} = \tilde{} < n >= \frac{1 + d_c^{(0)}}{1 - d_c^{(0)}} \tag{3.3}$$

Another average here is defined via (blackbody gamma distribution)

$$< n >= \frac{\int_0^\infty n^2 (\Omega^{(1)})^n dn}{\int_0^\infty n (\Omega^{(1)})^n dn} = \frac{-2}{log(\Omega^{(1)})} \tag{3.4}$$

which corresponds to $\tilde{}< n >$ after expanding the logarithm and omitting higher order terms. However $\tilde{}< n >$ seems to be the more valid calculation here. Similarly one defines (somewhat ambiguously) an expected value for $d_c^{(n)}$ via

$$< d >= \frac{\sum_1^\infty n(d_c^{(0)})^{n-1}}{\sum_1^\infty (d_c^{(0)})^n} = \frac{1}{d_c^{(0)}(1-d_c^{(0)})} \tag{3.5}$$

This is contrived of course (and cannot represent $E(d_c^{(n)})$ since one is computing reciprocals $\sum(n/d_c^{(n)})$ but we could think of computing an expected ignorance and identifying this with the reciprocal of dimension. Thus the label $< d >$ does not seem to represent an expected dimension but if we accept it as a symbol then for $d_c^{(0)} = \phi$ one has

$$\tilde{}< n >= \frac{1+\phi}{1-\phi} =< d >= \frac{1}{\phi(1-\phi)} = d_c^{(4)} = 4+\phi^3 = \frac{1}{\phi^3} \sim 4.236 \tag{3.6}$$

REMARK 1.3.1. We note that the normalized probability $N = \Omega^{(1)}/Z_0 = \Omega^{(1)}(1-\Omega^{(1)}) = 1/ < d >$ for any $d_c^{(0)}$. Further if $< d >= 4 = 1/d_c^{(0)}(1-d_c^{(0)})$ one has $d_c^{(0)} = 1/2$ while $\tilde{}< n >= 3 < 4 =< d >$. One sees also that $d_c^{(0)} = 1/2$ is the minimum (where $d < d > /d(d_c^{(0)}) = 0$).

REMARK 1.3.2. The results of Theorem 3.1 should really be phrased in terms of $\mathfrak{E}^\infty$ (cf. [**685**]). thus ($H \sim$ Hausdorff dimension and $T \sim$ topological dimension)

$$dim_H \mathfrak{E}^{(n)} = d_c^{(n)} = \frac{1}{(d_c^{(0)})^{n-1}}; \tag{3.7}$$

$$< d >= \frac{1}{d_c^{(0)}(1-d_c^{(0)})}; \tilde{}< dim_T \mathfrak{E}^\infty >= \frac{1+d_c^{(0)}}{1-d_c^{(0)}} = \tilde{}< n >$$

In any event $\mathfrak{E}^\infty$ is formally infinite dimensional but effectively it is $4\pm$ dimensional with an infinite number of internal dimensions. We emphasize that $\mathfrak{E}^\infty$ appears to be constructed from a fixed backbone Cantor set with H dimension $1/2 \leq d_c^{(0)} < 1$; thus each such $d_c^{(0)}$ generates an $\mathfrak{E}^\infty$ space. Note that in [**685**] $\mathfrak{E}^\infty$ is looked upon as a transfinite discretum underpinning the continuum (whatever that means).

REMARK 1.3.3. An interesting argument from [**684**] goes as follows. Thinking of $d_c^{(0)}$ as a geometrical probability one could say that the spatial (3-dimensional) probability of finding a Cantorian "point" in $\mathfrak{E}^\infty$ must be given by the intersection probability $P = (d_c^{(0)})^3$ where $3 \sim 3$ topological spatial dimension. P could then be regarded as a Hurst exponent (cf. [**1, 715, 985**]) and the Hausdorff dimension of the fractal path of a Cantorian would be $d_{path} = 1/H = 1/P = 1/(d_c^{(0)})^3$. Given $d_c^{(0)} = \phi$ this means $d_{path} = 4+\phi^3 \sim 4^+$ so a Cantorian in 3-D would sweep out a 4-D world sheet; i.e. the time dimension is created by the Cantorian space $\mathfrak{E}^\infty$ (! - ?). Conjecturing further (wildly) one could say that perhaps space (and gravity) is created by the fractality of time. This is a typical form of conjecture to be found in the El Naschie papers - extremely thought provoking but ultimately heuristic. Regarding the Hurst exponent one recalls that for Feynmann trajectories in $1+1$ dimensions $d_{path} = 1/H = 1/d_c^{(0)} = d_c^{(2)}$. Thus we are concerned with relating the two determinations of d_{path} (among other matters). Note that path dimension is often thought of as a fractal dimension (M-B or box dimension), which is not necessarily the same as the Hausdorff dimension. However in [**29**] one shows that quantum mechanical free motion produces fractal paths of Hausdorff dimension 2 (cf. also [**583**]).

REMARK 1.3.4. Following [**226**] let $S_c^{(0)}$ correspond to the set with dimension $d_c^{(0)} = \phi$. Then the complementary dimension is $\tilde{d}_c^{(0)} = 1 - \phi = \phi^2$. The path dimension is given as in Remark 1.3.3 by $d_{path} = d_c^{(2)} = 1/\phi = 1 + \phi$ and $\tilde{d}_{path} = \tilde{d}_c^{(2)} = 1/(1-\phi) = 1/\phi^2 = (1+\phi)^2$. Following El Naschie for an equivalence between unions and intersections in a given space one requires (in the present situation) that

(3.8)
$$d_{crit} = d_c^{(2)} + \tilde{d}_c^{(2)} = \frac{1}{\phi} + \frac{1}{\phi^2} = \frac{\phi(1+\phi)}{\phi^3} = \frac{1}{\phi^3} = \frac{1}{\phi} \cdot \frac{1}{\phi^2} = d_c^{(2)} \cdot \tilde{d}_c^{(2)} = 4 + \phi^3$$

where $d_{crit} = 4 + \phi^3 = d_c^{(4)} \sim 4.236$. Thus the critical dimension coincides with the Hausdorff dimension of $S_c^{(4)}$ which is embedded densely into a smooth space of topological dimension 4. On the other hand the backbone set of dimension $d_c^{(0)} = \phi$ is embedded densely into a set of topological dimension zero (a point). Thus one thinks in general of $d_c^{(n)}$ as the H dimension of a Cantor set of dimension ϕ embedded into a smooth space of integer topological dimension n.

REMARK 1.3.5. In [**226**] it is also shown that realization of the spaces $\mathfrak{E}^{(n)}$ comprising $\mathfrak{E}^{\infty}$ can be expressed via the fractal sprays of Lapidus-van Frankenhuysen (cf. [**592**]). Thus we refer to [**592**] for graphics and details and simply sketch some ideas here (with apologies to M. Lapidus). A fractal string is a bounded open subset of $\mathbf{R}$ which is a disjoint union of an infinite number of open intervals $\mathfrak{L} = \ell_1, \ell_2, \cdots$. The geometric zeta function of $\mathfrak{L}$ is $\zeta_{\mathfrak{L}}(s) = \sum_1^{\infty} \ell_j^{-s}$. One assumes a suitable meromorphic extension of $\zeta_{\mathfrak{L}}$ and the complex dimensions of $\mathfrak{L}$ are defined as the poles of this meromorphic extension. The spectrum of $\mathfrak{L}$ is the sequence of frequencies $f = k \cdot \ell_j^{-1}$ $(k = 1, 2, \cdots)$ and the spectral zeta function of $\mathfrak{L}$ is defined as $\zeta_{\nu}(s) = \sum_f f^{-s}$ where in fact $\zeta_{\nu}(s) = \zeta_{\mathfrak{L}}(s)\zeta(s)$ (with $\zeta(s)$ the classical Riemann zeta function). Fractal sprays are higher dimensional generalizations of fractal strings. As an example consider the spray Ω obtained by scaling an open square B of size 1 by the lengths of the standard triadic Cantor string CS. Thus Ω consists of one open square of size 1/3, 2 open squares of size 1/9, 4 open squares of size 1/27, etc. (see [**592**] for pictures and explanations). Then the spectral zeta function for the Dirichlet Laplacian on the square is $\zeta_B(s) = \sum_{n_1,n_2=1}^{\infty} (n_1^2 + n_2^2)^{s/2}$ and the spectral zeta function of the spray is $\zeta_{\nu}(s) = \zeta_{CS}(s) \cdot \zeta_B(s)$. Now $\mathfrak{E}^{\infty}$ is composed of an infinite hierarchy of sets $\mathfrak{E}^{(j)}$ with dimension $(1+\phi)^{j-1} = 1/\phi^{j-1}$ $(j = 0, \pm 1, \pm 2, \cdots)$ and these sets correspond to a special case of boundaries $\partial\Omega$ for fractal sprays Ω whose scaling ratios are suitable binary powers of $2^{-\phi^{j-1}}$. Indeed for $n = 2$ the spectral zeta function of the fractal golden spray indicated above is $\zeta_\nu(s) = (1/(1 - 2 \cdot 2^{s\phi})\zeta_B(s)$. The poles of $\zeta_B(s)$ do not coincide with the zeros of the denominator $1 - 2 \cdot 2^{-s\phi}$ so the (complex) dimensions of the spray correspond to those of the boundary $\partial\Omega$ of Ω. One finds that the real part $\Re s$ of the complex dimensions coincides with $dim\,\mathfrak{E}^{(2)} = 1 + \phi = 1/\phi^2$ and one identifies then $\partial\Omega$ with $\mathfrak{E}^{(2)}$. The procedure generalizes to higher dimensions (with some stipulations) and for dimension n there results $\Re s = 1/\phi^{n-1} = dim\,\mathfrak{E}^{(n)}$. This produces a physical model of the Cantorian fractal space from the boundaries of fractal sprays (see [**226**] for further details and [**592**] for precision). Other (putative) geometric realizations of $\mathfrak{E}^\infty$ are indicated in [**688**] in terms of wild topologies, etc.

3.2. COMMENTS ON HYDRODYNAMICS. We sketch first some material from [**15**] (see also [**294, 715, 718, 720**] and Sections 1-2 for background). Thus let ψ be the wave function of a test particle of mass m_0 in a force field $U(r,t)$ determined via $i\hbar\partial_t\psi = U\psi - (\hbar^2/2m)\nabla^2\psi$ where $\nabla^2 = \Delta$. One writes $\psi(r,t) = R(r,t)exp(iS(r,t))$ with $v = (\hbar/2m)\nabla S$ and $\rho = R \cdot R$ (one assumes $\rho \neq 0$ for physical meaning). Thus the field equations of QM in the hydrodynamic picture are

$$d_t(m_0\rho v) = \partial_t(m_0\rho v) + \nabla(m_0\rho v) = -\rho\nabla(U + Q);\ \ \partial_t\rho + \nabla\cdot(\rho v) = 0 \tag{3.9}$$

where $Q = -(\hbar^2/2m_0)(\Delta\sqrt{\rho}/\sqrt{\rho})$ is the quantum potential (or interior potential). Now because of the nondifferentiability of spacetime an infinity of geodesics will exist between any couple of points A and B. The ensemble will define the probability amplitude (this is a nice assumption but geodesics should be defined here). At each intermediate point C one can consider the family of incoming (backward) and outgoing (forward) geodesics and define average velocities $b_+(C)$ and $b_-(C)$ on these families. These will be different in general and following Nottale this doubling of the velocity vector is at the origin of the complex nature of QM. Even though Nottale reformulates Nelson's stochastic QM the former's interpretation is profoundly different. While Nelson (cf. [**698**]) assumes an underlying Brownian motion of unknown origin which acts on particles in Minkowskian spacetime, and then introduces nondifferentiability as a byproduct of this hypothesis, Nottale assumes as a fundamental and universal principle that spacetime itself is no longer Minkowskian nor differentiable. An interesting comment here from [**15**] is that with Nelson's Browian motion hypothesis, nondifferentiability is but an approximation which expected to break down at the scale of the underlying collisions, where a new physics should be introduced, while Nottale's hypothesis of nondifferentiability is essential and should hold down to the smallest possible length scales. Following Nelson one defines now the mean forward and backward derivatives

$$\frac{d_\pm}{dt}y(t) = lim_{\Delta t\to 0_\pm}\left\langle\frac{y(t+\Delta t) - y(t)}{\Delta t}\right\rangle \tag{3.10}$$

This gives forward and backward mean velocities $(d_+/dt)x(t) = b_+$ and $(d_-/dt)x(t) = b_-$ for a position vector x. Now in Nelson's stochastic mechanics one writes two systems of equations for the forward and backward processes and combines them in the end in a complex equation, Nottale works from the beginning with a complex derivative operator

$$\frac{\delta}{dt} = \frac{(d_+ + d_-) - i(d_+ - d_-)}{2dt} \tag{3.11}$$

leading to $V = (\delta/dt)x(t) = v - iu = (1/2)(b_+ + b_-) - (i/2)(b_+ - b_-)$. One defines also $(d_v/dt) = (1/2)(d_+ + d_-)/dt$ and $(d_u/dt) = (1/2)(d_+ - d_-)/dt$ so that $d_vx/dt = v$ and $d_ux/dt = u$. Here v generalizes the classical velocity while u is a new quantity arising from nondifferentiability. This leads to a stochastic process satisfying (respectively for the forward ($dt > 0$) and backward ($dt < 0$) processes) $dx(t) = b_+[x(t)] + d\xi_+(t) = b_-[x(t)] + d\xi_-(t)$. The $d\xi(t)$ terms can be seen as fractal functions and they amount to a Wiener process when the fractal dimension $D = 2$. Then the $d\xi(t)$ are Gaussian with mean zero, mutually independent, and satisfy $< d\xi_{\pm i}d\xi_{\pm j} >= \pm 2D\delta_{ij}dt$ where $\mathcal{D}$ is a diffusion coefficient determined as $\mathcal{D} = \hbar/2m_0$ when $\tau_0 = \hbar/(m_0c^2)$ (deBroglie time scale in the rest frame (cf. [**15**]). This allows one to give a general expression for the complex time derivative, namely

$$df = \frac{\partial f}{\partial t} + \nabla f \cdot dx + \frac{1}{2}\frac{\partial^2 f}{\partial x_i \partial x_j}dx_i dx_j \tag{3.12}$$

Next compute the forward and backward derivatives of f; then one will arrive at $< dx_idx_j > \to < d\xi_{\pm i}d\xi_{\pm j} >$ so the last term in (3.12) amounts to a Laplacian and one obtains $(d_\pm f/dt) = [\partial_t + b_\pm \cdot \nabla \pm \mathcal{D}\Delta]f$ which is an important result. Thus assume the fractal dimension is not 2 in which case there is no longer a cancellation of the scale dependent terms in (3.12) and instead of $\mathcal{D}\Delta f$ one would obtain an explicitly scale dependent behavior proportional to $\delta t^{(2/D)-1}\Delta f$. In other words the value $D = 2$ for the fractal dimension implies that the scale symmetry becomes hidden in the operator formalism. One obtains the complex time derivative operator in the form $(\delta/dt) = \partial_t + V \cdot \nabla - iD\Delta$ (V as above). Nottale's prescription is then to replace d/dt by δ/dt. In this spirit one can write now $\psi = exp(i(\mathfrak{S}/2m_0\mathcal{D}))$ so that $V = -2i\mathcal{D}\nabla(log(\psi))$ and then the generalized Newton equation $-\nabla U = m_0(\delta/dt)V$ reduces to the SE $(L = (1/2)mv^2 - U)$.

Now assume the velocity field from the hydrodynamic model agrees with the real part v of the complex velocity V and equate the wave functions from the two models $\psi = exp(i\mathfrak{S}/2m_0\mathcal{D})$ and $\psi = Rexp(iS)$ with $m = m_0$; one obtains for $\mathfrak{S} = s + i\sigma$ the formulas $s = 2m_0\mathcal{D}S$, $\mathcal{D} = (\hbar/2m_0)$, and $\sigma = -m_0\mathcal{D}log(\rho)$. Using the definition $V = (1/m_0)\nabla\mathfrak{S} = (1/m_0)\nabla s + (i/m_0)\nabla\sigma = v - iu$ (which results from the above equations) we get

$$v = (1/m_0)\nabla s = 2D\nabla S;\ u = -(1/m_0)\nabla\sigma = \mathcal{D}\nabla log(\rho) \tag{3.13}$$

Note that the imaginary part of the complex velocity coincides with Nottale. Dividing the time dependent SE $i\hbar\psi_t = U\psi - (\hbar^2/2m_0)\Delta\psi$ by $2m_0$ and taking the gradient gives $\nabla U/m_0 = 2\mathcal{D}\nabla[i\partial_t log(\psi) + \mathcal{D}(\Delta\psi/\psi)]$ where $\hbar/2m_0$ has been replaced by $\mathcal{D}$. Then consider the identities

$$\Delta\nabla = \nabla\Delta;\ (\nabla f \cdot \nabla)(\nabla f) = (1/2)\nabla(\nabla f)^2;\ \frac{\Delta f}{f} = \Delta log(f) + (\nabla log(f))^2 \tag{3.14}$$

Then the second term in the right of the equation for $\nabla U/m_0$ becomes $\nabla(\Delta\psi/\psi) = \Delta(\nabla log(\psi)) + 2(\nabla log(\psi)\cdot\nabla)(\nabla log(\psi))$ so we obtain

$$\nabla U = 2iDm_0[\partial_t \nabla log(\psi) - i\mathcal{D}\Delta(\nabla log(\psi) - 2i\mathcal{D}(\nabla log(\psi)\cdot\nabla)(\nabla log(\psi))] \tag{3.15}$$

One can show that this is nothing but the generalized Newton equation $-\nabla U = m_0(\delta/dt)V$. Now replacing the complex velocity $V = -2i\mathcal{D}\nabla log(\psi)$ and taking into account the form of V, we get

$$-\nabla U = m_0\{\partial_t(v - i\mathcal{D}\nabla log(\rho) + [i(v - i\mathcal{D}\nabla log(\rho)\cdot\nabla](v - i\mathcal{D}\nabla log(\rho)) - \\ -i\mathcal{D}\Delta(v - i\mathcal{D}\nabla log(\rho))\} \tag{3.16}$$

Equation (3.16) is a complex differential equation and reduces to

$$m_0[\partial_t v + (v\cdot\nabla)v] = -\nabla\left(U - 2m_0\mathcal{D}^2\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}}\right);\ \nabla\left\{\frac{1}{\rho}\left[\partial_t\rho + \nabla\cdot(\rho v)\right]\right\} \tag{3.17}$$

The last equation in (3.17) reduces to the continuity equation up to a phase factor $\alpha(t)$ which can be set equal to zero (note again that $\rho \neq 0$ is posited). Thus (3.17) is nothing but the fundamental equations (3.9) of the hydrodynamic model. Further combining the imaginary part of the complex velocity with the quantum potential, and using (3.14), one gets $Q = -m_0\mathcal{D}\nabla\cdot u - (1/2)m_0u^2$ (as indicated in Remark 1.2.2). Since u arises from nondifferentiability according to our nondifferentiable space model of QM it follows that the quantum potential comes from the nondifferentiability of the quantum spacetime (note that the x derivatives should be clarified and $\mathfrak{E}^\infty$ has not been utilized).

Putting $U = 0$ in the first equation of (3.17), multiplying by ρ, and taking the second equation into account yields

$$\partial_t(m_0\rho\nu_k) + \frac{\partial}{\partial x_i}(m_0\rho\nu_i\nu_k) = -\rho\frac{\partial}{x_k}\left[2m_0\mathcal{D}^2\frac{1}{\sqrt{\rho}}\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_i}(\sqrt{\rho})\right] \tag{3.18}$$

(here $\nu_k \sim v_k$ seems indicated). Now set $\Pi_{ik} = m_0\rho\nu_i\nu_k - \sigma_{ik}$ along with $\sigma_{ik} = m_0\rho D^2(\partial/\partial x_i)(\partial/\partial x_k)(log(\rho))$. Then (3.18) takes the simple form

$$\partial_t(m_0\rho\nu_k) = -\partial\Pi_{ik}/\partial x_i \tag{3.19}$$

The analogy with classical fluid mechanics works well if one introduces the kinematic $\mu = D/2$ and dynamic $\eta = (1/2)m_0\mathcal{D}\rho$ viscosities. Then Π_{ik} defines the momentum flux density tensor and σ_{ik} the internal stress tensor $\sigma_{ik} = \eta[(\partial u_i/\partial x_k) + (\partial u_k/\partial x_i)]$. One can see that the internal stress tensor is build up using the quantum potential while the equations (3.18) or (3.19) are nothing but systems of Navier-Stokes type for the motion where the quantum potential plays the role of an internal stress tensor. In other words the nondifferentiability of the quantum spacetime manifests itself like an internal stress tensor. For clarity in understanding (3.19) we put this in one dimensional form so (3.18) becomes

$$\partial_t(m_0\rho v) + \partial_x(m_0\rho v^2) = -\rho\partial\left(2m_0\mathcal{D}^2\frac{1}{\sqrt{\rho}}\partial^2\sqrt{\rho}\right) = \rho\partial Q \tag{3.20}$$

and $\Pi = m_0\rho v^2 - \sigma$ with $\sigma = m_0\rho D^2\partial^2 log(\rho)$ which agrees with standard formulas. Now note $\partial\sqrt{\rho} = (1/2)\rho^{-1/2}\rho'$ and $\partial^2\sqrt{\rho} = (1/2)[-(1/2)\rho^{-3/2}(\rho')^2 + \rho^{-1/2}\rho'']$ with $\partial^2 log(\rho) = \partial(\rho'/\rho) = (\rho''/\rho) - (\rho'/\rho)^2$ while

(3.21)

$$-\rho\partial\left[2m_0\mathcal{D}^2\frac{1}{\sqrt{\rho}}\left(\partial^2\sqrt{\rho}\right)\right] = -2m_0\mathcal{D}^2\rho\partial\left[\frac{1}{2\sqrt{\rho}}\left(-\frac{1}{2}\rho^{-3/2}(\rho')^2 + \rho^{-1/2}\rho''\right)\right] =$$

$$= -2m_0\mathcal{D}^2\rho\partial\left[\frac{\rho''}{2\rho} - \frac{1}{4}\left(\frac{(\rho'}{\rho}\right)^2\right] = -m_0\mathcal{D}^2\rho\partial\left[\frac{\rho''}{\rho} - \frac{1}{2}\left(\frac{\rho'}{\rho}\right)^2\right]$$

One wants to show then that (3.19) holds or equivalently $-\partial\sigma = (3.21)$. However

(3.22)

$$-\partial\sigma = -\partial[m_0\rho D^2\partial^2 log(\rho)] = -m_0\mathcal{D}^2\left[\rho'\left(\frac{\rho''}{\rho} - \left(\frac{\rho'}{\rho}\right)^2\right) + \rho\partial\left(\frac{\rho''}{\rho} - \frac{(\rho')^2}{\rho}\right)\right]$$

so we want (3.22) = (3.21) which is easily verified.

4. REMARKS ON FRACTAL CALCULUS

We sketch first (in summary form) from [**748**] where a calculus based on fractal subsets of the real line is formulated. A local calculus based on renormalizing fractional derivatives à la [**562**] is subsumed and embellished. Consider first the concept of content or α-mass for a (generally fractal) subset $F \subset [a,b]$ (in what follows $0 < \alpha \leq 1$). Then define the flag function for a set F and a closed interval I as $\theta(F,I) = 1$ ($F \cap I \neq \emptyset$ and otherwise $\theta = 0$. Then a subdivision $P_{[a,b]} \sim P$ of $[a,b]$ $(a < b)$ is a finite set of points $\{a = x_0, x_1, \cdots, x_n = b\}$ with $x_i < x_{i+1}$. If Q is any subdivision with $P \subset Q$ it is called a refinement and if $a = b$ the set $\{a\}$ is the only subdivision. Define then

$$\sigma^\alpha[F,p] = \sum_0^{n-1}\frac{(x_{i+1}-x_i)^\alpha}{\Gamma(\alpha+1)}\theta(F,[x_i,x_{i+1}]) \tag{4.1}$$

For $a = b$ one defines $\sigma^\alpha[F,P] = 0$. Next given $\delta > 0$ and $a \leq b$ the coarse grained mass $\gamma^\alpha_\delta(F,a,b)$ of $F \cap [a,b]$ is given via

$$\gamma^\alpha_\delta(F,a,b) = inf_{|P|\leq\delta}\sigma^\alpha[F,P] \;\; (|P| = max_{0\leq i\leq n-1}(x_{i+1}-x_i)) \tag{4.2}$$

where the infimum is over P such that $|P| \leq \delta$. Various more or less straightforward properties are:

- For $a \leq b$ and $\delta_1 < \delta_2$ one has $\gamma^\alpha_{\delta_1}(F,a,b) \geq \gamma^\alpha_{\delta_2}(F,a,b)$.
- For $\delta > 0$ and $a < b < c$ one has $\gamma^\alpha_\delta(F,a,b) \leq \gamma^\alpha_\delta(F,a,c)$ and $\gamma^\alpha_\delta(F,b,c) \leq \gamma^\alpha_\delta(F,a,c)$.
- γ^α_δ is continuous in b and a.

Now define the mass function $\gamma^\alpha(F,a,b)$ via $\gamma^\alpha(F,a,b) = lim_{\delta\to 0}\gamma^\alpha_\delta(F,a,b)$. The following results are proved

(1) If $F \cap (a,b) = \emptyset$ then $\gamma^\alpha(F,a,b) = 0$.
(2) Let $a < b < c$ and $\gamma^\alpha(F,a,c) < \infty$. Then $\gamma^\alpha(F,a,c) = \gamma^\alpha(F,a,b) + \gamma^\alpha(F,b,c)$. Hence $\gamma^\alpha(F,a,b)$ is increasing in b and decreasing in a.

(3) Let $a < b$ and $\gamma^\alpha(F,a,b) \neq 0$ be finite. If $0 < y < \gamma^\alpha(F,a,b)$ then there exists c, $a < c < b$ such that $\gamma^\alpha(F,a,c) = y$. Further if $\gamma^\alpha(F,a,b)$ is finite then $\gamma^\alpha(F,a,x)$ is continuous for $x \in (a,b)$.
(4) For $F \subset \mathbf{R}$ and $\lambda \in \mathbf{R}$ let $F + \lambda = \{x + \lambda;\ x \in F\}$. Then $\gamma^\alpha(F+\lambda, a+\lambda, b+\lambda) = \gamma^\alpha(F,a,b)$ and $\gamma^\alpha(\lambda F, \lambda a, \lambda b) = \lambda^\alpha \gamma^\alpha(F,a,b)$.

Now for a_0 an arbitrary fixed real number one defines the integral staircase function of order α for F is

$$S_F^\alpha(x) = \begin{cases} \gamma^\alpha(F,a_0,x) & x \geq a_0 \\ -\gamma^\alpha(F,x,a_0) & otherwise \end{cases} \tag{4.3}$$

The following properties of S_F are restatements of properties for γ^α. thus

- $S_F^\alpha(x)$ is increasing in x.
- If $F \cap (x,y) = 0$ then S_F^α is constant in $[x,y]$.
- $S_F^\alpha(y) - S_F^\alpha(x) = \gamma^\alpha(F,x,y)$.
- S_F^α is continuous on (a,b).

Now one considers the sets F for which the mass function $\gamma^\alpha(F,a,b)$ gives the most useful information. Indeed one can use the mass function to define a fractal dimension. If $0 < \alpha < \beta \leq 1$ one writes
(4.4)

$$\sigma^\beta[F,P] \leq |P|^{\beta-\alpha}\sigma^\alpha[F,P]\frac{\Gamma(\alpha+1)}{\Gamma(\beta+1)};\ \gamma_\delta^\beta(F,a,b) \leq \delta^{\beta-\alpha}\gamma_\delta^\alpha(F,a,b)\frac{\Gamma(\alpha+1)}{\Gamma(\beta+1)}$$

Thus in the limit $\delta \to 0$ one gets $\gamma^\beta(F,a,b) = 0$ provided $\gamma^\alpha(F,a,b) < \infty$ and $\alpha < \beta$. It follows that $\gamma^\alpha(F,a,b)$ is infinite up to a certain value α_0 and then jumps down to zero for $\alpha > \alpha_0$ (if $\alpha_0 < 1$). This number is called the γ-dimension of F; $\gamma^{\alpha_0}(F,a,b)$ may itself be zero, finite, or infinite. To make the definition precise one says that the γ-dimension of $F \cap [a,b]$, denoted by $dim_\gamma(F \cap [a,b])$, is

$$dim_\gamma(F \cap [a,b]) = \begin{cases} inf\{\alpha;\ \gamma^\alpha(F,a,b) = 0\} \\ sup\{\alpha;\ \gamma^\alpha(F,a,b) = \infty\} \end{cases} \tag{4.5}$$

One shows that $dim_H(F \cap [a,b]) \leq dim_\gamma(F \cap [a,b])$ where dim_H denotes Hausdorff dimension. Further $dim_\gamma(F \cap [a,b]) \leq dim_B(F \cap [a,b])$ where dim_B is the box dimension. Some further analysis shows that for $F \subset \mathbf{R}$ compact $dim_\gamma F = dim_H F$.

Next one notes that the correspondence $F \to S_F^\alpha$ is many to one (examples from Cantor sets) and one calls the sets giving rise to the same staircase function "staircasewise congruent". The equivalence class of congruent sets containing F is denoted by $\mathcal{E}_F$; thus if $G \in \mathcal{E}_F$ it follows that $S_G^\alpha = S_F^\alpha$ and $\mathcal{E}_G^\alpha = \mathcal{E}_F^\alpha$. One says that a point x is a point of change of f if f is not constant over any open interval (c,d) containing x. The set of all points of change of f is denoted by $Sch(f)$. In particular if $G \in \mathcal{E}_F^\alpha$ then $S_G^\alpha(x) = S_F^\alpha(x)$ so $Sch(S_G^\alpha) = Sch(S_F^\alpha)$. Thus if $F \subset \mathbf{R}$ is such that $S_F^\alpha(x)$ is finite for all x ($\alpha = dim_\gamma F$) then $H = Sch(S_F^\alpha) \in \mathcal{E}_F^\alpha$. This takes some proving which we omit (cf. [**748**]). As a consequence let $F \subset \mathbf{R}$ be such that $S_F^\alpha(x)$ is finite for all $x \in \mathbf{R}$ ($\alpha = dim_\gamma F$). Then the set $H = Sch(S_F^\alpha)$ is perfect (i.e. H is closed and every point is a limit point). Hence given $S_F^\alpha(x)$ finite for all x ($\alpha = dim_\gamma F$) one calls $Sch(S_F^\alpha)$ the α-perfect representative of $\mathcal{E}_F^\alpha$ and one proves that it is the minimal closed set in $\mathcal{E}_F^\alpha$. Indeed an α-perfect set in $\mathcal{E}_F^\alpha$ is the intersection of all closed sets G in $\mathcal{E}_F^\alpha$. One can also say that if $F \subset \mathbf{R}$ is α-perfect and $x \in F$ then for $y < x < z$ either $S_F^\alpha(y) < S_F^\alpha(x)$ or $S_F^\alpha(x) < S_F^\alpha(z)$ (or both). Thus for an α-perfect set it is assured that the values of $S_F^\alpha(y)$ must be different from $S_F^\alpha(x)$ at all points y on at least one side of x. As an example one shows that the middle third Cantor set $C = E_{1/3}$ is α-perfect for $\alpha = log(2)/log(3) = d_H(C)$ so $C = Sch(S_C^\alpha)$.

Now look at F with the induced topology from $\mathbf{R}$ and consider the idea of F-continuity.

DEFINITION 4.1. Let $F \subset \mathbf{R}$ and $f : \mathbf{R} \to \mathbf{R}$ with $x \in F$. A number ℓ is said to be the limit of f through the points of F, or simply F-limit, as $y \to x$ if given $\epsilon > 0$ there exists $\delta > 0$ such that $y \in F$ and $|y - x| < \delta \Rightarrow |f(y) - \ell| < \epsilon$. In such a case one writes $\ell = F - limit_{y \to x} f(y)$. A function f is F-continuous at $x \in F$ if $f(x) = F - limit_{y \to x} f(y)$ and uniformly F-continuuous on $E \subset F$ if for $\epsilon > 0$ there exists $\delta > 0$ such that $x \in F$, $y \in E$ and $|y - x| < \delta \Rightarrow |f(y) - f(x)| < \epsilon$. One sees that if f is F-continuous on a compact set $E \subset F$ then it is uniformly F-continuous on E.

DEFINITION 4.2. The class of functions $f : \mathbf{R} \to \mathbf{R}$ which are bounded on F is denoted by $B(F)$. Define for $f \in B(F)$ and I a closed interval

$$M[f, F, I] = \begin{cases} sup_{x \in F \cap I} f(x) & F \cap I \neq \emptyset \\ 0 & otherwise \end{cases} \tag{4.6}$$

$$m[f, F, I] = \begin{cases} inf_{x \in F \cap I} f(x) & F \cap I \neq \emptyset \\ 0 & otherwise \end{cases}$$

DEFINITION 4.3. Let $S_F^\alpha(x)$ be finite for $x \in [a, b]$ and P be a subdivision with points $x_0, \cdots, x_n$. The upper F^α and lower F^α sums over P are given respectively by

$$U^\alpha[f, F, P] = \sum_0^{n-1} M[f, F, [x_i, x_{i+1}]](S_F^\alpha(x_{i+1}) - S_F^\alpha(x_i)); \tag{4.7}$$

$$L^\alpha[f, F, P] = \sum_0^{n-1} m[f, F, [x_i, x_{i+1}]](S_F^\alpha(x_{i+1}) - S_F^\alpha(x_i))$$

This is sort of like Riemann-Stieltjes integration and in fact one shows that if Q is a refinement of P then $U^\alpha[f, F, Q] \leq U^\alpha[f, F, P]$ and $L^\alpha[f, F, Q] \geq L^\alpha[f, F, P]$. Further $U^\alpha[f, F, P] \geq L^\alpha[f, F, Q]$ for any subdivisions of $[a, b]$ and this leads to the idea of F-integrability. Thus assume S_F^α is finite on $[a, b]$ and for $f \in B(F)$ one defines lower and upper F^α-integrals via

$$\underline{\int_a^b} f(x) d_F^\alpha x = sup_P L^\alpha[f, F, P]; \quad \overline{\int_a^b} f(x) d_F^\alpha x = inf_P U^\alpha[f, F, P] \tag{4.8}$$

One then says that f is F^α-integrable if (**D15**) $\underline{\int_a^b} f(x) d_F^\alpha x = \overline{\int_a^b} f(x) d_F^\alpha x = \int_a^b f(x) d_F^\alpha x$.

One shows then

(1) $f \in B(F)$ is F^α-integrable on $[a,b]$ if and only if for any $\epsilon > 0$ there is a subdivision P of $[a,b]$ such that $U^\alpha[f,F,P] < L^\alpha[f,F,P] + \epsilon$.
(2) Let $F \cap [a,b]$ be compact with S_F^α finite on $[a,b]$. Let $f \in B(F)$ and $a < b$; then if f is F-continuous on $F \cap [a,b]$ it follows that f is F^α-integrable on $[a,b]$.
(3) Let $a < b$ and f be F^α-integrable on $[a,b]$ with $c \in (a,b)$. Then f is F^α-integrable on $[a,c]$ and $[c,b]$ with $\int_a^b f(x)d_F^\alpha x = \int_a^c f(x)d_F^\alpha x + \int_c^b f(x)d_F^\alpha x$.
(4) If f is F^α-integrable then $\int_a^b \lambda f(x)d_F^\alpha x = \lambda \int_a^b f(x)d_F^\alpha x$ and, for g also F^α-integrable, $\int_a^b (f(x)+g(x))d_F^\alpha x = \int_a^b f(x)d_F^\alpha x + \int_a^b g(x)d_F^\alpha x$.
(5) If f,g are F^α-integrable and $f(x) \geq g(x)$ for $x \in F \cap [a,b]$ then $\int_a^b f(x)d_F^\alpha x \geq \int_a^b g(x)d_F^\alpha x$.

One specifies also $\int_b^a f(x)d_F^\alpha x = -\int_a^b f(x)d_F^\alpha x$ and it is easily shown that if $\chi_F(x)$ is the characteristic function of F then $\int_a^b \chi_F(x)d_F^\alpha x = S_F^\alpha(b) - S_F^\alpha(a)$. Now for differentiation one writes

$$\mathcal{D}_F^\alpha f(x) = \begin{cases} F-lim_{y\to x}\frac{f(y)-f(x)}{S_F^\alpha(y)-S_F^\alpha(x)} & x \in F \\ 0 & otherwise \end{cases} \tag{4.9}$$

if the limit exists. One shows then

(1) If $\mathcal{D}_F^\alpha f(x)$ exists for all $x \in (a,b)$ then $f(x)$ is F-continuous in (a,b).
(2) With obvious hypotheses $\mathcal{D}_F^\alpha(\lambda f(x)) = \lambda \mathcal{D}_F^\alpha f(x)$ and $\mathfrak{D}_F^\alpha(f+g)(x) = \mathcal{D}_F^\alpha f(x) + \mathcal{D}_F^\alpha g(x)$. Further if f is constant then $\mathcal{D}_F^\alpha f = 0$.
(3) $\mathfrak{D}_F^\alpha(S_F^\alpha(x)) = \chi_F(x)$.
(4) (Rolle's theorem) Let $f : \mathbf{R} \to \mathbf{R}$ be continuous with $Sch(f) \subset F$ where F is α-perfect and assume $\mathcal{D}_F^\alpha f(x)$ is defined for all $x \in [a,b]$ with $f(a) = f(b) = 0$. Then there is a point $c \in F \cap [a,b]$ such that $\mathcal{D}_F^\alpha f(c) \geq 0$ and a point $d \in F \cap [a,b]$ where $\mathcal{D}_F^\alpha f(d) \leq 0$.

EXAMPLE 4.1. This is the best that can be done with Rolle's theorem since for C the Cantor set $E_{1/3}$ take $f(x) = S_C^\alpha(x)$ for $0 \leq x \leq 1/2$ and $f(x) = 1 - S_C^\alpha(x)$ for $1/2 < x \leq 1$. This function is continuous with $f(0) = f(1) = 0$ and the set of change $(Sch(f))$ is C. The C^α-derivative is given by $\mathcal{D}_C^\alpha f(x) = \chi_C(x)$ for $0 \leq x \leq 1/2$ and by $-\chi_C(x)$ for $1/2 < x \leq 1$. Thus $x \in C$ which implies $\mathcal{D}_C^\alpha f(x) = \pm 1 \neq 0$.

As a corollary one has the following result: Let f be continuous with $Sch(f) \subset F$ where F is α-perfect; assume $\mathcal{D}_F^\alpha f(s)$ exists at all points of $[a,b]$ and that $S_F^\alpha(b) \neq S_F^\alpha(a)$. Then there are points $c,d \in F$ such that

$$\mathcal{D}_F^\alpha f(c) \geq \frac{f(b)-f(a)}{S_F^\alpha(b) - S_F^\alpha(b)};\ \mathcal{D}_F^\alpha f(d) \leq \frac{f(b)-f(a)}{S_F^\alpha(b) - S_F^\alpha(a)} \tag{4.10}$$

Similarly if f is continuous with $Sch(f) \subset F$ and $\mathcal{D}_F^\alpha f(x) = 0\ \forall x \in [a,b]$ then $f(x)$ is constant on $[a,b]$. There are also other fundamental theorems as follows

(1) (Leibniz rule) If $u,v : \mathbf{R} \to \mathbf{R}$ are F^α-differentiable then $\mathcal{D}_F^\alpha(uv)(x) = (\mathcal{D}_F^\alpha u(x))v(x) + u(x)\mathcal{D}_F^\alpha v(x)$.

(2) Let $F \subset \mathbf{R}$ be α-perfect. If $f \in B(F)$ is F-continuous on $F \cap [a,b]$ with $g(x) = \int_a^x f(y) d_F^\alpha y$ for all $x \in [a,b]$ then $\mathcal{D}_F^\alpha g(x) = f(x)\chi_F(x)$.

(3) Let $f : \mathbf{R} \to \mathbf{R}$ be continuous and F^α-differentiable with $Sch(f)$ contained in an α-perfect set F; let also $h : \mathbf{R} \to \mathbf{R}$ be F-continuous such that $h(x)\chi_F(x) = \mathcal{D}_F^\alpha f(x)$. Then $\int_a^b h(x) d_F^\alpha x = f(b) - f(a)$.

(4) (Integration by parts) Assume: (i) u is continuous on $[a,b]$ and $Sch(u) \subset F$. (ii) $\mathcal{D}_F^\alpha u(x)$ exists and is F-continuous on $[a,b]$. (iii) v is F-continuous on $[a,b]$. Then

$$\int_a^b uv d_F^\alpha x = \left[u(x) \int_a^x v(x') d_F^\alpha x'\right]\Big|_a^b - \int_a^b \mathcal{D}_F^\alpha u(x) \int_a^x v(x') d_F^\alpha x' d_F^\alpha x \tag{4.11}$$

Some examples are given relative to applications and we mention e.g.

EXAMPLE 4.2. Following [**562**] one has a local fractal diffusion equation

$$\mathcal{D}_{F,t}^\alpha (W(x,t)) = \frac{\chi_F(t)}{2} \frac{\partial^2}{\partial x^2} W(x,t) \tag{4.12}$$

with solution

$$W(x,t) = \frac{1}{(2\pi S_F^\alpha(t))^{1/2}} exp\left(\frac{-x^2}{2S_F^\alpha(t)}\right) \tag{4.13}$$

The appendix to [**748**] also gives some formulas for repeated integration and differentiation. For example it is shown that

$$(\mathcal{D}_F^\alpha)^2 (S_F^\alpha(x))^2 = 2\chi_F(x); \quad \int_a^{x'} (S_F^\alpha(x))^n d_F^\alpha x = \frac{1}{n+1} (S_F^\alpha(x'))^{n+1} \tag{4.14}$$

We refer to [**562, 748**] for other interesting material.

5. A BOHMIAN APPROACH TO QUANTUM FRACTALS

The powerful exact uncertainty method of Hall and Reginatto for passing from classical to quantum mechanics has been further embellished and deepened in recent years (see e.g. [**186, 187, 189, 203, 396, 444, 445, 446, 447, 448, 449, 450, 749, 805, 806, 807, 844, 845**] and Sections 1.1, 3.1, and 4.7. In [**445**] one finds an apparent incompleteness in the traditional trajectory based Bohmian mechanics when dealing with a quantum particle in a box. It turns out that there is no suitable HJ equation for describing the motion which in fact has a fractal character. After reviewing the material on scale relativity in Section 1.2 for example it is not surprising to encounter such situations and in [**844**] the Bohmian point of view is reinstated for fractal trajectories. One should also remark in passing that there is much material available on weak or distribution solutions of HJ type equations and some of this should come into play here (cf. [**211**]). The main issue here however is that in order to treat wave functions displaying fractal features (quantum fractals) one needs to enlarge the picture via limiting processes. One derives the quantum trajectories by means of limiting procedures that involve the expansion of the wave function in a series of eigenvectors of the Hamiltonian.

Consider first the quantum analogue of the Weierstrass function

$$W(x) = \sum_{0}^{\infty} b^r Sin(a^r x);\ \ a > 1 > b > 0;\ \ ab \geq 1 \tag{5.1}$$

Then in the problem of a particle in a 1-dimensional box of length L (with $0 < x < L$) one can construct wave functions of the form

$$\Phi_t(x; R) = A \sum_{r=0}^{R} n^{r(s-2)} Sin(p_{n,r} x/\hbar) e^{-iE_{n,r}t/\hbar} \tag{5.2}$$

with $2 > s > 0$ and $n \geq 2$. Here $p_{n,r} = n^r \pi\hbar/L$ is the quantized momentum (with integer quantum number given by $n' = n^r$). $E_{n,r} = p_{n,r}^2/2m$ is the eigenenergy and a is a normalization constant. This wave function, which is a solution of the time dependent SE, is continuous and differentiable everywhere. However the wave function resulting in the limit, namely $\Phi_t(x) = lim_{R\to\infty}\Phi_t(x; R)$ is a fractal object in both space and time (cf. [**626**]). This method for generating quantum fractals basically involves (given s) choosing a quantum number, say n, and then considering the series that contains its powers $n' = n^r$. There is also another related method (cf. [**142**]) of generating quantum fractals based on the presence of discontinuities in the wave function. The emergence of fractal features arises from the perturbations that such discontinuities cause in the wave function during propagation. This generating process can be easily understood by considering a wave function initially uniform along a certain interval $\ell = x_2 - x_1 \leq L$ inside the box

$$\Psi_0(x) = \begin{cases} \frac{1}{\sqrt{\ell}} & x_1 < x < x_2 \\ 0 & otherwise \end{cases} \tag{5.3}$$

The Fourier decomposition of this wave function is

$$\Psi_0(x) = \frac{2}{\pi\sqrt{\ell}} \sum_{1}^{\infty} \frac{1}{n} [Cos(p_n x_1/\hbar) - Cos(p_n x_2/\hbar)] Sin(p_n x/\hbar) \tag{5.4}$$

whose time evolved form is

$$\Psi_t(x) = \frac{2}{\pi\sqrt{\ell}} \sum_{1}^{\infty} \frac{1}{n} [Cos(p_n x_1/\hbar) - Cos(p_n x_2/\hbar)] Sin(p_n x/\hbar) e^{-iE_n t/\hbar} \tag{5.5}$$

It is equivalent to consider $r = R = 1$ in (5.2) and sum over n from 1 to N; the quantum fractal is then obtained in the limit $N \to \infty$. This equivalence is based on the fact that the Fourier decomposition of Ψ_0 gives precisely its expansion in terms of the eigenvectors of the Hamiltonian in the problem of a particle in a box (this is not a general situation).

EXAMPLE 5.1. The fractality of wave functions like $\Phi_t(x)$ or $\Psi_t(x)$ can be analytically estimated (cf. [**142**]) by taking advantage of a result for Fourier series. Thus given an arbitrary function $f(x) = \sum_1^N a_n exp(-inx)$ its real and imaginary pars are fractals (and also $|f(x)|^2$) with dimension $D_f = (5-\beta)/2$ if its power spectrum has the asymptotic form $|a_n|^2 \sim n^{-\beta}$ for $N \to \infty$ with $1 < \beta \leq 3$. Alternatively the fractality of $f(x)$ can also be calculated by measuring the length

$\mathfrak{L}$ of its real and imaginary parts (or $|f(x)|^2$) as a function of the number of terms N considered in the generating sries. Asymptotically the relation bewteen $\mathfrak{L}$ and N can be expressed as $\mathfrak{L}(N) \propto N^{D_f-1}$ which diverges if $f(x)$ is a fractal object. One notes that to increase the number of terms contributing to $f(x)$ is analogous to measuring its length with more precision.

It is known that for quantum fractals the corresponding expected value of the energy $<\hat{H}>$ becomes infinite. This is related to the fact that the familiar form of the SE

$$i\hbar\partial_t\Psi_t(x) = \hat{H}\Psi_t(x) \tag{5.6}$$

does not hold in general (cf. [**445, 1003**]). In this case neither the left side of (5.6) nor the right side belong to the Hilbert space; however the identity

$$[\hat{H} - i\hbar\partial_t]\Psi_t(x) = 0 \tag{5.7}$$

still remains valid. In this situation one says that $\Psi_t(x)$ is a weak solution of the SE (note weak solutions have many meanings and have been extensively studied in PDE - cf. [**211**]).

The formal basis of Bohmian mechanics (BM) is usually established via

$$\Psi_t(x) = \rho_t^{1/2}(x)e^{iS_t(x)/\hbar};\ \frac{\partial\rho_t}{\partial t} + \nabla\cdot\left(\rho_t\frac{\nabla S_t}{m}\right) = 0; \tag{5.8}$$

$$\frac{\partial S_t}{\partial t} + \frac{(\nabla S_t)^2}{2m} + V + Q_t = 0;\ Q_t = -\frac{\hbar^2}{2m}\frac{\nabla^2\rho_t^{1/2}}{\rho_t^{1/2}}$$

One postulates also the trajectory velocity as

$$\dot{x} = \frac{\nabla S_t}{m} = \frac{\hbar}{m}\Im[\Psi_t^{-1}\nabla\Psi_t] \tag{5.9}$$

Now Q_t in (5.8) is well defined provided that the quantum state is also well defined (i.e. continuous and differentiable). However this is not the case for quantum fractals and the theory seems incomplete; the solution is to take into account the decomposition of the quantum fractal in terms of differentiable eigenvectors and redefining Q_t in (5.8). Thus any wave function Ψ_t is expressible as

$$\Psi_t(x:N) = \sum_1^N c_n\xi_n(x)e^{-iE_nt/\hbar} \tag{5.10}$$

in the limit $N \to \infty$ (cf. Φ_t above and (5.5)) where the $\xi_n(x)$ are eigenvectors with eigenvalues E_n of the corresponding Hamiltonian. One can then define the quantum trajectories evolving under the guidance of this wave as

$$x_t = lim_{N\to\infty}x_N(t);\ \dot{x}_N = \frac{\hbar}{m}\Im\left[\Psi_t^{-1}(x;N)\frac{\partial\Psi_t(x;N)}{\partial x}\right] \tag{5.11}$$

Note the calculation of trajectories is not based on S_t, which has no trivial decomposition in a series of nice functions, but this kind of velocity formulation is common in e.g. [**324, 325, 326, 327, 328, 329, 415, 416, 418, 419, 420**] where one modern version of BM is being developed.

EXAMPLE 5.2. A numerical example is given in [**844**] and we only mention a few features here. Thus one considers a highly delocalized particle in a box with wave function (5.4) and $x_1 = 0$ with $x_2 = L$. Then (5.5) becomes

$$\Psi_t(x) = \frac{4}{\pi\sqrt{L}} e^{-iE_1 t/\hbar} \sum_{n\ odd} \frac{1}{n} Sin(p_n x/\hbar) e^{i\omega_{n,1} t} \qquad (5.12)$$

where $\lambda_{n,1} = (E_n - E_1)/\hbar$ (in the numerical calculations one uses $L = m = \hbar = 1$). Here the probability density ρ_t is periodic in time but the wave function is not periodic (this does not affect (5.11)). Various features are observed (e.g. Cantor set structures, Gibbs phenomena, etc.) and graphs are displayed - we omit any further discussion here.

In summary, although the SE is not satisfied by quantum fractals as a whole, it is when one considers its decomposition in terms of the eigenvectors of the Hamiltonian. The contributing eigenvectors are continuous and differentiable and any wave function (regular or not) admits a decomposition in terms of eigenvectors. Correspondingly the Bohmian equation of motion must be reformulated in terms of such decompositions via (5.11) and this can be regarded also as a generalization of (5.8). We mention in passing that from time to time there are papers claiming contradictions between BM and QM and we refer here to [**436, 629, 712**] for some refutations.

REMARK 5.3. Let us mention here a suggestion of 't Hooft [**475**] about establishing the physical link between classical and quantum mechanics by employing the underlying equations of classical mechanics and including into them a specially chosen dissipative function. The wave like QM turns out to follow from the particle like classical mechanics due to embedding in the latter a dissipation "device" responsible for loss of information. Thus the initial precise information about the classical trajectory is lost in QM due to the "dissipative spread" of the trajectory and its transformation into a fuzzy object such as the fractal Hausdorff path of dimension 2 in a simple case of a spinless particle. Some rough calculations in this direction appear in [**426**]. and we refer also to [**122, 427, 749**].

CHAPTER 2

DEBROGLIE-BOHM IN VARIOUS CONTEXTS

The quantum potential arises in various forms, some of which were discussed in Sections 1.1 and 1.2. We return to this now in a somewhat more systematic manner. The original theory goes back to deBroglie and D. Bohm (see e.g. [**94, 95, 128, 129, 154, 471, 472, 532**]) and in its modern version the dominant themes seem to be contained in [**88, 102, 288, 295, 324, 325, 326, 327, 328, 329, 387, 402, 414, 415, 927, 948**] with variations as in [**110, 186, 187, 188, 189, 191, 194, 195, 196, 197, 198, 346, 347, 373, 374, 375**] based on work of Bertoldi, Faraggi, and Matone (cf. also [**68, 138, 148, 165, 236, 305, 520, 574, 575, 576, 873, 881**]) and cosmology following [**123, 188, 189, 219, 498, 499, 500, 501, 571, 709, 710, 711, 840, 841, 871, 872, 873, 875, 876, 895, 989, 990**]. In any event the quantum potential does enter into any trajectory theory of deBroglie-Bohm (dBB) type. The history is discussed for example in [**471**] (cf. also [**68, 126, 127, 129, 154**]) and we have seen how this quantum potential idea can be formulated in various ways in terms of statistical mechanics, hydrodynamics, information and entropy, etc. when dealing with different versions and origins of the SE. Given the existence of particles we finds the pilot wave of thinking very attractive, with the wave function serving to choreograph the particle motion (or perhaps to "create" particles and/or spacetime paths). However the existence of particles itself is not such an assured matter and in field theory approaches for example one will deal with particle currents (cf. [**701**] and see also e.g. [**94, 95, 326, 402, 948**]). The whole idea of quantum particle path seems in any case to be either fractal (cf. [**1, 3, 14, 15, 186, 223, 232, 273, 676, 715, 717, 720, 733, 734, 735, 736**], stochastic (see e.g. [**68, 148, 186, 381, 382, 446, 447, 448, 449, 534, 536, 671, 674, 698, 805, 806**], or field theoretic (cf. [**94, 95, 326, 402, 701, 702, 703, 704, 705, 706, 707, 948**]. The fractal approach sometimes imagines an underlying micro-spacetime where paths are perhaps fractals with jumps, etc. and one possible advantage of a field theoretic approach would be to let the fields sense the ripples, which as e.g. operator valued Schwartz type distributions, they could well accomplish. In fact what comes into question here is the structure of the vacuum and/or of spacetime itself. One can envision microstructures as in [**186, 422, 676, 690**] for example, textures (topological defects) as in [**71, 74, 75, 170, 978**], Planck scale structure and QFT, along with space-time uncertainty relations as in [**71, 316, 317, 604, 1008**], vacuum structures and conformal invariance as in [**668, 669, 835, 831, 837, 838**], pilot wave cosmology as in [**834, 881**], ether theories as in [**851, 919**], etc. Generally there seems to be a sense in which particles cannot be measured as such and hence the idea of particle currents (perhaps corresponding to fuzzy particles or ergodic clumps) should prevail perhaps along with the idea of probability packets. A number of arguments work with a (representative) trajectory as if it were a single particle but there is no reason to take this too seriously; it could be thought of perhaps as a "typical" particle in a cloud but conclusions should perhaps always be constructed from an ensemble point of view. We will try to develop some of this below. The sticky point as we see it now goes as follows. Even though one can write stochastic equations for (typical) particle motion as in the Nelson theory for example one runs into the problem of ever actually being able to localize a particle. Indeed as indicated in [**316, 317**] (working in a relativistic context but this should hold in general) one expects space time uncertainty relations even at a semiclassical level since any localization experiment will generate a gravitational field and deform spacetime. Thus there are relations $[q_\mu, q_\nu] = i\lambda_P^2 Q_{\mu\nu}$ where λ_P is the Planck length and the picture of spacetime as a local Minkowski manifold should break down at distances of order λ_P. One wants the localization experiment to avoid creating a black hole (putting the object out of "reach") for example and this suggests $\Delta x_0(\sum_1^3 \Delta x_i) \gtrsim \lambda_P^2$ with $\Delta x_1 \Delta x_2 + \Delta x_2 \Delta x_3 + \Delta x_3 \Delta x_1 \gtrsim \lambda_P^2$ (cf. [**316, 317**]). On the other hand in [**701**] it is shown that in a relativistic bosonic field theory for example one can speak of currents and n-particle wave functions can have particles attributed to them with well defined trajectories, even though the probability of their experimental detection is zero. Thus one enters an arena of perfectly respectible but undetectible particle trajectories. The discussion in [**256, 326, 920, 953, 961**] is also relevant here; some recourse to the idea of beables, reality, and observables as beables, etc. is also involved (cf. [**94, 95, 256, 961**]). We will have something to say about all these matters.

The dominant approach as in [**324, 325, 326, 327, 402, 948**] will be discussed as needed (a thorough discussion would take a book in itself) and we only note here that one is obliged to use the form $\psi = Rexp(iS/\hbar)$ to make sense out of the constructions (this is no problem with suitable provisos, e.g. that S is not constant - cf. [**110, 191, 346, 347, 373, 374**] and comments later). This leads to

$$(\bigstar)\ S_t + \frac{(S')^2}{2m} - \left(\frac{\hbar^2}{2m}\right)\left(\frac{R''}{R}\right) + V = 0;\ \partial_t R^2 + \partial\left(\frac{R^2 S'}{m}\right) = 0$$

(cf. (1.1.1)) where $Q = -\hbar^2 R''/2mR$ arising from a SE $i\hbar\partial_t\psi = -(\hbar^2/2m)\psi_{xx} + V\psi$ (we use 1-D for simplicity here). In [**324**] one emphasizes configurations based on coordinates whose motion is choreographed by the SE according to the rule

$$(\bigstar\bigstar)\ \dot{q} = v = \frac{\hbar}{m}\Im\frac{\psi^*\psi'}{|\psi|^2} = \frac{\hbar}{m}\Im\left(\frac{\psi'}{\psi}\right)$$

The argument for ($\bigstar\bigstar$) is based on obtaining the simplest Galilean and time reversal invariant form for velocity, transforming correctly under velocity boosts. This leads directly to ($\bigstar\bigstar$) so that Bohmian mechanics (BM) is governed by ($\bigstar\bigstar$) and the SE. It's a fairly convincing argument and no recourse to Floydian need be involved (cf. [**110, 191, 347, 373, 374**]). Note however that if $S = c$ then $\dot{q} = v = (\hbar/m)\Im(R'/R) = 0$ while $p = S' = 0$ so this formulation seems to avoid the $S = constant$ problems indicated in [**110, 191, 347, 373, 374**].

What makes the constant $\hbar/m$ in (★★) important here is that with this value the probability density $|\psi|^2$ on configuration space is equivariant. This means that via the evolution of probability densities $\rho_t + div(v\rho) = 0$ (as in (1.1.5)) the density $\rho = |\psi|^2$ is stationary relative to ψ, i.e. $\rho(t)$ retains the form $|\psi(q,t)|^2$. One calls $\rho = |\psi|^2$ the quantum equilibrium density (QEDY) and says that a system is in quantum equilibrium when its coordinates are randomly distributed according to the QEDY. The quantum equilibrium hypothesis (QEHP) is the assertion that when a system has wave function ψ the distribution ρ of its coordinates satisfies $\rho = |\psi|^2$.

1. THE KLEIN-GORDON AND DIRAC EQUATIONS

Before embarking on further discussion of QM it is necessary to describe some aspects of quantum field theory (QFT) and in particular to give some foundation for the Klein-Gordon (KG) and Dirac equations. For QFT we rely on [**120, 457, 528, 764, 827, 1015**] and concentrate on aspects of general quantum theory that are expressed through such equations. We alternate between signature $(-,+,+,+)$ and $(+,-,-,-)$ in Minkowski space, depending on the source. It is hard to avoid using units $\hbar = c = 1$ when sketching theoretical matters (which is personally repugnant) but we will set $\hbar = c = 1$ and shift to the general notation whenever any real meaning is desired. Thus $|length| \sim |time| \sim |energy|^{-1} \sim |mass|^{-1}$ and $m =$ the inverse Compton wavelength ($mc/\hbar = \ell_C^{-1}$). The approaches in [**457, 764**] seem best adapted to our needs and in particular [**457**] gives a nice discussion motivating second quantization of a nonrelativistic SE (cf. [**701**] for first quantization). The resulting second quantization would be Galiean invariant but not Lorentz invariant so we go directly to the KG equation as follows. Note that there are often notational differences in various treatments of QFT and we use that of [**457**] in general. Start now from $E^2 = p^2 + m^2$ (which is the relativistic form of $E = p^2/2m$) to arrive, via $E \to i\partial_t$ and $p_j \to -i\partial_j = -i\partial/\partial x^j$, at the KG equation

$$(\partial_t^2 - \nabla^2)\phi + m^2\phi = 0 \tag{1.1}$$

where $\phi = \phi(\mathbf{x}, t)$ is a scalar wave function. This can also be derived from an action

$$S(\phi) = \int d^4x \mathfrak{L}(\phi, \partial_\mu\phi) = \frac{1}{2}\int d^4x(\partial^\mu\partial_\mu\phi - m^2\phi) \tag{1.2}$$

$(x^0 = t,\ x = (\mathbf{x}, t))$, provided ϕ transforms as a Lorentz scalar (required also in (1.1)). The first problems arise from negative energy solutions (e.g. $exp[i(\mathbf{k}\cdot\mathbf{x}+\omega t)]$ is a solution of (1.1) with $E = -\omega = -(\mathbf{k}^2 + m^2)^{1/2}$). Secondly the energy spectrum is not bounded below (i.e. one could extract an arbitrary amount of energy from a single particle system). Further, using a positve square root of $E^2 = p^2 + m^2$ would involve a square root of a differential operator and nonlocal terms. Next observe that conserved currents j_μ (with $\partial^\mu j_\mu = 0$) arise à la E. Noether in the form

$$j_0 = \rho = \frac{i}{2m}(\phi^*\phi_t - \phi_t^*\phi);\ j_i = \frac{1}{2im}(\phi^*\partial_i\phi - (\partial_i\phi^*)\phi) \tag{1.3}$$

(where normal ordering is implicit here in order to avoid dealing with a vacuum energy term - to be discussed later). We note that for the plane wave solution above $\rho = -\omega/m = -(1/m)(\mathbf{k}^2 + m^2)^{1/2}$ and this is not a good probability density. The difficulties are resolved by giving up the idea of a one particle theory; it is not compatible with Lorentz invariance and the solution is to quantize the field ϕ.

Thus take $S(\phi)$ as in (1.2) with $\pi = \partial\mathfrak{L}/\partial(\partial_\mu\phi) = \partial_t\phi = \dot{\phi}$ and construct a Hamiltonian

$$H = \frac{1}{2}\int d^3x[\pi^2(x) + |\nabla\phi(x)|^2 + m^2\phi^2(x)] \tag{1.4}$$

In analogy with QM where $[x,p] = i$ one stipulates

$$[\phi(\mathbf{x},t),\pi(\mathbf{y},t)] = i\delta(\mathbf{x}-\mathbf{y});\ [\phi(\mathbf{x},t),\phi(\mathbf{y},t)] = [\pi(\mathbf{x},t),\pi(\mathbf{y},t)] = 0 \tag{1.5}$$

The operator equation $\dot{\phi} = i[H,\phi]$ then yields $\pi = \dot{\phi}$ and $\dot{\pi} = i[H,\pi]$ reproduces the KG equation. This is now a quantum field theory and for a particle interpretation one expands $\phi(\mathbf{x},t)$ in terms of classical solutions of the KG equation via

$$\phi(\mathbf{x},t) = \sum a(\mathbf{k})\phi_k^+(x) + b(\mathbf{k})\phi_k^-(x) = \tag{1.6}$$

$$= \int \frac{d^3\mathbf{k}}{(2\pi)^3}\frac{1}{2\omega_k}(a(\mathbf{k})e^{-i[\omega_k t - \mathbf{k}\cdot\mathbf{x}]} + b(\mathbf{k})e^{i[\omega_k t - \mathbf{k}\cdot\mathbf{x}]})$$

where $\omega_k = (\mathbf{k}^2 + m^2)^{1/2}$ and $\phi_k^\pm$ denotes a classical positive (resp. negative) energy plane wave solution of (1.1) ($k\cdot x = k_0x_0 - \mathbf{k}\cdot\mathbf{x} = \omega_k t - \mathbf{k}\cdot\mathbf{x}$). With $\phi(\mathbf{x},t)$ an operator one has operators $a(\mathbf{k})$ and $b(\mathbf{k})$; further since $\phi(\mathbf{x},t)$ is classically a real field we must have a Hermitian operator here and hence $b(\mathbf{k}) = a^\dagger(\mathbf{k})$. The normalization factor $1/2\omega_k$ is chosen for Lorentz invariance (cf. [**457**] for details). It follows immediately from $\pi = \partial_t\phi$ that

$$\pi(\mathbf{x},t) = \int \frac{k^3k}{(2\pi)^3}\frac{1}{2\omega_k}(-i\omega_k a(\mathbf{k})e^{-ik\cdot x} + i\omega_k a^\dagger(\mathbf{k})e^{ik\cdot x}) \tag{1.7}$$

Some calculation (via Fourier formulas) leads then to

$$a(\mathbf{k}) = \int d^3xe^{ik\cdot x}[\omega_k\phi(\mathbf{x},t) + i\pi(\mathbf{x},t)] \tag{1.8}$$

and the algebra of a, $a^\dagger$ is then determined by $[a(\mathbf{k}), a^\dagger(\mathbf{k}')] = (2\pi)^3 2\omega_k\delta^3(\mathbf{k}-\mathbf{k}')$. The Hamiltonian (1.4) yields

$$H = \frac{1}{2}\int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}\omega_k[a^\dagger(\mathbf{k})a(\mathbf{k}) - a(\mathbf{k})a^\dagger(\mathbf{k})] \tag{1.9}$$

There is a bit of hocus-pocus here since the calculation gives $a^\dagger a + (1/2)[a,a^\dagger]$ ($= (1/2)(a^\dagger a + aa^\dagger)$ formally) but $[a,a^\dagger] \sim \delta(0)$ corresponds to the sum oveer all modes of zero point energies $\omega_k/2$. This infinite energy cannot be detected experimaentally since experiments only measure differences from the ground state of H. In any event the zero point field (ZPF) will be discussed in some detail later.

Now the ground state is defined via $a(\mathbf{k})|0>= 0$ with $<0|0>= 1$ ($a^\dagger(\mathbf{k})|0>$ is a one particle state with energy ω_k and momentum $\mathbf{k}$ while $(a^\dagger(\mathbf{k}))^2|0>$ contains two such particles, etc.). One notes however that the state $a^\dagger(\mathbf{k})|0>$ is not normalizable since $< 0|a(\mathbf{k})a^\dagger(\mathbf{k})|0 >= \delta(0)$ is not normalizable. This is not surprising since $a^\dagger(\mathbf{k})$ creates a particle of definite energy and momentum and by the uncertainty principle its location is unknown. Thus its wave function is a plane wave and such states are not normalizable. In fact $a^\dagger(\mathbf{k})$ is an operator valued distribution and one can do calculations by "smearing" and considering states $\int d^3k f(\mathbf{k})a^\dagger|0>$ for functions f such that $\int d^3k|f(\mathbf{k})|^2 < \infty$ for example. One sees also that the bare vacuum $|0>$ is an eigenstate of the Hamiltonian but its energy is divergent via $< 0|H|0 >= (1/2)\int d^3k\omega_k\delta^3(0)$ (where $(2\pi)^3\delta^3(0) \sim \int d^3x$). To deal with such infinities one subtracts them away, i.e. $H \to H- < 0|H|0 >$ and this corresponds to normal ordering the Hamiltonian via $: aa^\dagger :=: a^\dagger a := a^\dagger a$ leading to

$$:H := \int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}\omega_k a^\dagger(\mathbf{k})a(\mathbf{k}) \tag{1.10}$$

with vanishing vacuum expectation.

REMARK 2.1.1. Regarding Lorentz invariance one recalls that the Lorentz group $O(3,1)$ is the set of 4×4 matrices leaving the form $s^2 = (x^0)^2 - \sum(x^i)^2 = x^\mu g_{\mu\nu}x^\nu$ invariant. One writes $(x')^\mu = \Lambda^\mu_\nu x^\nu$ and notes that $g_\mu = \Lambda^\rho_\mu g_{\rho\sigma}\Lambda^\sigma_\nu \sim g = \Lambda^T g\Lambda$. Since s^2 can be plus or minus there is a splitting into regions $(x-y)^2 > 0$ (time-like), $(x-y)^2 < 0$ (space-like), and $(x-y)^2 = 0$ (light-like). A standard parametrization for Lorentz boosts involves $(x^0 = ct)$

$$x' = \frac{x+vt}{\sqrt{1-(v/c)^2}};\ y' = y;\ z' = z;\ t' = \frac{t + (vx/c^2)}{\sqrt{1-(v/c)^2}} \tag{1.11}$$

One writes e.g. $\gamma = 1/\sqrt{1-(v/c)^2} = cosh(\phi)$ with $sinh(\phi) = \beta\gamma = v\gamma/c$.

REMARK 2.1.2. The total 4-momentum operator is

$$P^\mu = \int \frac{d^3k}{(2\pi)^3}\frac{1}{2\omega_k}k^\mu a^\dagger(\mathbf{k})a(\mathbf{k}) \tag{1.12}$$

and the total angular momentum operator is

$$M^{\mu\nu} = \int d^3x(x^\mu p^\nu - x^\nu p^\mu) \tag{1.13}$$

The Lorentz algebra (for infinitesimal Lorentz transformations) is

$$[M^{\mu\nu}, M^{\lambda\sigma}] = i(\eta^{\mu\lambda}M^{\nu\sigma} - \eta^{\nu\lambda}M^{\mu\sigma} - \eta^{\mu\sigma}M^{\nu\lambda} + \eta^{\nu\sigma}M^{\mu\lambda}) \tag{1.14}$$

where $\eta^{\mu\nu} = diag(1,-1,-1,-1)$.

REMARK 2.1.3. The commutator rules (1.5) are not manifestly Lorentz covariant. However one can verify that the same quantum theory is obtained regardless of what Lorentz frame is chosen; to do this one shows that the QM operator forms of the Lorentz generators satisfy the Lorentz algebra after quantization (this is given as an exercise in [**457**]).

EXAMPLE 1.1. Quantum fields are also discussed briefly in [**471**] and we extract here from this source. The approach follows [**128**] and one takes $\mathcal{L} = (1/2)\partial_\mu\psi\partial^\mu\psi = (1/2)[\dot{\psi}^2 - (\nabla\psi)^2]$ as Lagrangian where $\dot{\psi} = \partial_t\psi$ and variational technique yields the wave equation $\Box\psi = 0$ ($\hbar = c = 1$). Define conjugate momentum as $\pi = \partial\mathcal{L}/\partial\dot{\psi}$, the Hamiltonian via $\mathcal{H} = \pi\dot{\psi} - \mathcal{L} = (1/2)[\pi^2 + (\nabla\psi))^2]$, and the field Hamiltonian by $\mathfrak{H} = \int \mathcal{H} d^3x$. Replacing π by $\delta S/\delta\psi$ where $S[\psi]$ is a functional the classical HJ equation of the field $\partial_t S + H = 0$ becomes

$$\frac{\partial S}{\partial t} + \frac{1}{2}\int d^3x \left[\left(\frac{\delta S}{\delta\psi}\right)^2 + (\nabla S)^2\right] = 0 \tag{1.15}$$

The term $(1/2)\int d^3x(\nabla\psi)^2$ plays the role of an external potential. To quantize the system one treats $\psi(\mathbf{x})$ and $\pi(\mathbf{x})$ as Schrödinger operators with $[\psi(\mathbf{x}), \psi(\mathbf{x}')] = [\pi(\mathbf{x}), \pi(\mathbf{x}')] = 0$ and $[\psi(\mathbf{x}), \pi(\mathbf{x}')] = i\delta(\mathbf{x}-\mathbf{x}')$. Then one works in a representation $|\psi(\mathbf{x}) >$ in which the Hermitian operator $\psi(\mathbf{x})$ is diagonal. The Hamiltonian becomes an operator $\hat{H}$ acting on a wavefunction $\Psi[\psi(\mathbf{x}), t) =< \psi(\mathbf{x})|\Psi(t) >$ which is a functional of the real field ψ and a function of t. This is not a point function of $\mathbf{x}$ since Ψ depends on the variable ψ for all $\mathbf{x}$. Now the SE for the field is $i\partial_t\Psi = \hat{H}\Psi$ or explicitly

$$i\frac{\partial\Psi}{\partial t} = \int d^3x \frac{1}{2}\left[-\frac{\delta^2}{\delta\psi^2} + (\nabla\psi)^2\right]\Psi \tag{1.16}$$

Thus ψ is playing the role of the space variable $\mathbf{x}$ in the particle SE and the continuous index $\mathbf{x}$ here is analogous to a discrete index n in the many particle theory. To arrive at a causal interpretation now one writes $\Psi = Rexp(iS)$ for $R, S[\psi, t]$ real functionals and decomposes (1.16) as

$$\frac{\partial S}{\partial t} + \frac{1}{2}\int d^3x \left[\left(\frac{\delta S}{\delta\psi}\right)^2 + (\nabla\psi)^2\right] + Q = 0;\ \ \frac{\partial R^2}{\partial t} + \int d^3x \frac{\delta}{\delta\psi}\left(R^2\frac{\delta S}{\delta\psi}\right) = 0 \tag{1.17}$$

where the quantum potential is now $Q[\psi, t] = -(1/2R)\int d^3x(\delta^2 R/\delta\psi^2)$. (1.17) now gives a conservation law wherein, at time t, $R^2 D\psi$ is the probability for the field to lie in an element of volume $D\psi$ around ψ, where $D\psi$ means roughly $\prod_{\mathbf{x}} d\psi$ and there is a normalization $\int |\Psi|^2 D\psi = 1$. Now introduce the assumption that at each instant t the field ψ has a well defined value for all $\mathbf{x}$ as in classical field theory, whatever the state Ψ. Then the time evolution is obtained from the solution of the "guidance" formula

$$\frac{\partial\psi(\mathbf{x}, t)}{\partial t} = \left.\frac{\delta S[\psi(\mathbf{x}), t]}{\delta\psi(\mathbf{x})}\right|_{\psi(\mathbf{x})=\psi(\mathbf{x},t)} \tag{1.18}$$

(analogous to $m\ddot{\mathbf{x}} = \nabla S$) once one has specified the initial function $\psi_0(\mathbf{x})$ in the HJ formalism. To find the equation of motion for the field coordinates apply $\delta/\delta\psi$ to the HJ equation (1.17) to get formally ($\dot{\psi} \sim \delta S/\delta\psi$)

$$\frac{d}{dt}\dot{\psi} = -\frac{\delta}{\delta\psi}\left[Q + \frac{1}{2}\int d^3x(\nabla\psi)^2\right];\ \ \frac{d}{dt} = \frac{\partial}{\partial t} + \int d^3x \frac{\partial\psi}{\partial t}\frac{\delta}{\delta\psi} \tag{1.19}$$

This is analogous to $m\ddot{\mathbf{x}} = -\nabla(V+Q)$ and, noting that $d\dot{\psi}/dt = \partial\dot{\psi}/dt$ and taking the classical external force term to the right one arrives, via standard variational methods, at

$$\Box\psi(\mathbf{x}),t) = -\left.\frac{\delta Q[\psi(\mathbf{x},t]}{\delta\psi(\mathbf{x})}\right|_{\psi(\mathbf{x})=\psi(\mathbf{x},t)} \tag{1.20}$$

(note $(\delta/\delta\psi)\int d^3x(\nabla\psi)^2 \sim -2\Delta\psi$ and $(\delta/\delta\psi 0\partial_t\psi = \partial_t(\delta/\delta\psi)\psi = 0)$. The quantum force term on the right side is responsible for all the characteristic effects of QFT. In particular comparing to a classical massive KG equation $\Box\psi + m^2\psi = 0$ with suitable initial conditions one can argue that the quantum force generates mass in the sense that the massless quantum field acts as if it were a classical field with mass given via the quantum potential (cf. Remark 2.2.1 below).

1.1. ELECTROMAGNETISM AND THE DIRAC EQUATION. It will be useful to have a differential form discription of EM fields and we supply this via [**723**]. Thus one thinks of tensors $T = T^{\sigma}_{\mu\nu}\partial_\sigma \otimes dx^\mu \otimes dx^\nu$ with contractions of the form $T(dx^\sigma, \partial_\sigma) \sim T_\nu dx^\nu$. For $\eta = \eta_{\mu\nu}dx^\mu \otimes dx^\nu$ one has $\eta^{-1} = \eta^{\mu\nu}\partial_\mu \otimes \partial_\nu$ and $\eta\eta^{-1} = 1 \sim diag(\delta^\mu_\mu)$. Note also e.g.

$$\eta_{\mu\nu}dx^\mu \otimes dx^\nu(\mathbf{u},\mathbf{w}) = \eta_{\mu\nu}dx^\mu(\mathbf{u})dx^\nu(\mathbf{w}) = \tag{1.21}$$
$$= \eta_{\mu\nu}dx^\mu(u^\alpha\partial_\alpha)dx^\nu(w^\tau\partial_\tau) = \eta_{\mu\nu}u^\mu w^\nu$$

$$\eta(\mathbf{u}) = \eta_{\mu\nu}dx^\mu \otimes dx^\nu(\mathbf{u}) = \eta_{\mu\nu}dx^\mu(\mathbf{u})dx^\nu = \tag{1.22}$$
$$= \eta_{\mu\nu}dx^\mu(u^\alpha\partial_\alpha)dx^\nu = \eta_{\mu\nu}u^\mu dx^\nu = u_\nu dx^\nu$$

for a metric η. Recall $\alpha \wedge \beta = \alpha \otimes \beta - \beta \otimes \alpha$ and

$$\alpha \wedge \beta = \alpha_\mu dx^\mu \wedge \beta_\nu dx^\nu = (1/2)(\alpha_\mu\beta_\nu - \alpha_\nu\beta_\mu)dx^\mu \wedge dx^\nu \tag{1.23}$$

The EM field tensor is $F = (1/2)F_{\mu\nu}dx^\mu \wedge dx^\nu$ where

$$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}; \tag{1.24}$$

$$F = E_x dx^0 \wedge dx^1 + E_y dx^0 \wedge dx^2 + E_z dx^0 \wedge dx^3 - B_z dx^1 \wedge dx^2 +$$
$$+ B_y dx^1 \wedge dx^3 - B_x dx^2 \wedge dx^3$$

The equations of motion of an electric charge is then $d\mathbf{p}/d\tau = (e/m)\mathbf{F}(\mathbf{p})$ where $\mathbf{p} = p^\mu\partial_\mu$. There is only one 4-form, namely $\epsilon = dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 = (1/4!)\epsilon_{\mu\nu\sigma\tau}dx^\mu \wedge dx^\nu \wedge dx^\sigma \wedge dx^\tau$ where $\epsilon_{\mu\nu\sigma\tau}$ is totally antisymmetric. Recall also for $\alpha = \alpha_{\mu\nu\cdots}dx^\mu \wedge dx^\nu \cdots$ one has $d\alpha = d\alpha_{\mu\nu\cdots} \wedge dx^\mu \wedge dx^\nu \cdots = \partial_\alpha\alpha_{\mu\nu\cdots}dx^\sigma \wedge dx^\mu \wedge dx^\nu \cdots$ and $dd\alpha = 0$. Define also the Hodge star operator on F and j via ${}^*F = (1/4)\epsilon_{\mu\nu\sigma\tau}F^{\sigma\tau}dx^\mu \wedge dx^\nu$ and ${}^*j = (1/3!)\epsilon_{\mu\nu\sigma\tau}j^\tau dx^\mu \wedge dx^\nu \wedge dx^\sigma$; these are called dual tensors. Now the Maxwell equations are

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c}j^\nu;\ \partial^\alpha F^{\mu\nu} + \partial^\mu F^{\nu\alpha} + \partial^\nu F^{\alpha\mu} = 0 \tag{1.25}$$

and this can now be written in the form

$$dF = 0;\ d^*F = \frac{4\pi}{c} * j \tag{1.26}$$

and $0 = d^*j = 0$ is automatic. In terms of $A = A_\mu dx^\mu$ where $F = dA$ the relation $dF = 0$ is an identity $ddA = 0$.

A few remarks about the tensor nature of j^μ and $F^{\mu\nu}$ are in order and we write $n = n(x)$ and $\mathbf{v} = \mathbf{v}(x)$ for number density and velocity with charge density $\rho(x) = qn(x)$ and current density $\mathbf{j} = qn(x)\mathbf{v}(x)$. The conservation of particle number leads to $\nabla \cdot \mathbf{j} + \rho_t = 0$ and one writes

$$(1.27) \quad j^\nu = (c\rho, j_x, j_y, j_z) = (c\rho n, qnv_x, qnv_y, qnv_z) \equiv j^\nu = n_0 q u^\nu \equiv j^\nu = \rho_0 u^\nu$$

where $n_0 = n\sqrt{1-(v^2/c^2)}$ and $\rho_0 = qn_0$ (ρ_0 here is charge density). Since j^ν consists of u^ν multiplied by a scalar it must have the transformation law of a 4-vector $j^{'\beta} = a^\beta_{\ \nu} j^\nu$ under Lorentz transformations ($a^\beta_\nu \sim \Lambda^\beta_\nu$). Then the conservation law can be written as $\partial_\nu j^\nu = 0$ with obvious Lorentz invariance. After some argument one shows also that $F^{\mu\nu} = a_\beta^{\ \nu} a_\alpha^{\ \mu} F^{'\alpha\beta}$ under Lorentz transformations so $F^{\mu\nu}$ is indeed a tensor. The equation of motion for a charged particle can be written now as

$$(1.28) \qquad (d\mathbf{p}/dt) = q\mathbf{E} + (q/c)\mathbf{v}\times\mathbf{B};\ \mathbf{p} = m\mathbf{v}/\sqrt{1-(v^2/c^2)}$$

This is equivalent to $dp^\mu/dt = (q/m)p_\nu F^{\mu\nu}$ with obvious Lorentz invariance. The energy momentum tensor of the EM field is

$$(1.29) \qquad T^{\mu\nu} = -(1/4\pi)[F^{\mu\alpha}F^\nu_{\ \alpha} - (1/4)\eta^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}]$$

(cf. [**723**] for details) and in particular $T^{00} = (1/8\pi)(\mathbf{E}^2+\mathbf{B}^2)$ while the Poynting vector is $T^{0k} = (1/4\pi)(\mathbf{E}\times\mathbf{B})^k$.

One can equally well work in a curved space where e.g. covariant derivatives are defined via $\nabla_n T = lim_{d\lambda\to 0}[(T(\lambda+d\lambda)-T(\lambda)-\delta T]/d\lambda$ where δT is the change in T produced by parallel transport. One has then the usual rules $\nabla_u(T\otimes R) = \nabla_u T\otimes R + T\otimes\nabla_u R$ and for $\mathbf{v} = v^\nu\partial_\nu$ one finds $\nabla_\mu\mathbf{v} = \partial_\mu v^\nu\partial_\nu + v^\nu\nabla_\mu\partial_\nu$. Now if $\mathbf{v}$ was constructed by parallel transport its covariant derivative is zero so, acting with the dual vector dx^α gives

$$(1.30) \qquad \frac{\partial x^\nu}{\partial x^\mu}dx^\alpha(\partial_\nu) + v^\nu dx^\alpha(\nabla_\mu\partial_\nu) = 0 \equiv \partial_\mu v^\alpha + v^\nu dx^\alpha(\nabla_\mu\partial_\nu) = 0$$

Comparing this with the standard $\partial_\mu v^\alpha + \Gamma^\alpha_{\ \mu\nu}v^\nu = 0$ gives $dx^\alpha(\nabla_\mu\partial_\nu) = \Gamma^\alpha_{\ \mu\nu}$. One can show also for vectors u, v, w (boldface omitted) and a 1-form α

$$(1.31) \qquad (\nabla_u\nabla_v - \nabla_v\nabla_u - uv + vu)\alpha(w) = R(\alpha,u,v,w);$$
$$R = F^\sigma_{\ \beta\mu\nu}\partial_\sigma\otimes dx^\beta\otimes dx^\mu\otimes dx^\nu$$

so R represents the Riemann tensor.

For the nonrelativistic theory we recall from [**649**] that one can define a transverse and longitudinal component of a field F via

$$(1.32) \qquad F^{||}(r) = -\frac{1}{4\pi}\int d^3r'\frac{\nabla'\cdot F(r')}{|r-r'|};\ F^\perp(r) = \frac{1}{4\pi}\nabla\times\nabla\times\int d^3r'\frac{F(r')}{|r-r'|}$$

For a point particle of mass m and charge e in a field with potentials A and ϕ one has nonrelativistic equations $m\ddot{x} = eE + (e/c)v\times B$ (boldface is suppressed here) where one recalls $B = \nabla \times A$, $v = \dot{x}$, and $E = -\nabla\phi - (1/c)A_t$ with $H = (1/2m)(p - (e/c)A)^2 + e\phi$ leading to

$$\dot{x} = \frac{1}{2m}\left(p - \frac{e}{c}A\right);\ \dot{p} = \frac{e}{c}[v \times B + (v \cdot \nabla)A] - e\nabla\phi \tag{1.33}$$

Recall here also

$$B = \nabla \times A;\ \nabla \cdot E = 0;\ \nabla \cdot B = 0;\ \nabla \times E = -(1/c)B_t; \tag{1.34}$$
$$\nabla \times B = (1/c)E_t;\ E = -(1/c)A_t - \nabla\phi$$

(the Coulomb gauge $\nabla \cdot A = 0$ is used here). One has now $E = E^{\perp} + E^{||} \sim E^T + E^L$ with $\nabla \cdot E^{\perp} = 0$ and $\nabla \times E^{||} = 0$ and in Coulomb gauge $E^{\perp} = -(1/c)A_t$ and $E^{||} = -\nabla\phi$. Further

$$H \sim \frac{1}{2m}\left(p - \frac{e}{c}A\right)^2 + e\phi + \frac{1}{8\pi}\int d^3r((E^{\perp})^2 + B^2) \tag{1.35}$$

(covering time evolution of both particle and fields).

For the relativistic theory one goes to the Dirac equation

$$i(\partial_t + \alpha \cdot \nabla)\psi = \beta m\psi \tag{1.36}$$

which, to satisfy $E^2 = \mathbf{p}^2 + m^2$ with $E \sim i\partial_t$ and $\mathbf{p} \sim -i\nabla$, implies $-\partial_t^2\psi = (-i\alpha \cdot \nabla + \beta m)^2\psi$ and ψ will satisfy the KG equation if $\beta^2 = 1$, $\alpha_i\beta + \beta\alpha_i \equiv \{\alpha_i, \beta\} = 0$, and $\{\alpha_i, \alpha_j\} = 2\delta_{ij}$ (note $c = \hbar = 1$ here with $\alpha \cdot \nabla \sim \sum \alpha_\mu \partial_\mu$ and cf. [**647, 650**] for notations and background). This leads to matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix};\ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \tag{1.37}$$
$$\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix};\ \alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix};\ \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

where α_i and β are 4×4 matrices. Then for convenience take $\gamma^0 = \beta$ and $\gamma^i = \beta\alpha_i$ which satisfy $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ (Lorentz metric) with $(\gamma^i)^\dagger = -\gamma^i$, $(\gamma^i)^2 = -1$, $(\gamma^0)^\dagger = \gamma^0$, and $(\gamma^0)^2 = 1$. The Dirac equation for a free particle can now be written

$$\left(i\gamma^\mu \frac{\partial}{\partial x^\mu} - m\right)\psi = 0 \equiv (i\not{\partial} - m)\psi = 0 \tag{1.38}$$

where $\not{A} = g_{\mu\nu}\gamma^\mu A^\nu = \gamma^\mu A_\mu$ and $\not{\partial} = \gamma^\mu\partial_\mu$. Taking Hermitian conjugates in (1.36), noting that α and β are Hermitian, one gets $\bar{\psi}(i\overleftarrow{\not{\partial}} + m) = 0$ where $\bar{\psi} = \psi^\dagger\beta$. To define a conserved current one has an equation $\bar{\psi}\gamma^\mu\partial_\mu\psi + \gamma^\mu\bar{\psi}_\mu\psi = \partial_\mu(\bar{\psi}\gamma^\mu\psi) = 0$ leading to the conserved current $j^\mu = \bar{\psi}\gamma^\mu\psi = (\psi^\dagger\psi, \psi^\dagger\alpha\psi)$ (this means $\rho = \psi^\dagger\psi$ and $\mathbf{j} = \psi^\dagger\alpha\psi$ with $\partial_t\rho + \nabla \cdot \mathbf{j} = 0$). The Dirac equation has the Hamiltonian form

$$i\partial_t\psi = -i\alpha \cdot \nabla\psi + \beta m\psi = (\alpha \cdot \mathbf{p} + \beta m)\psi \equiv H\psi \tag{1.39}$$

$(\alpha \cdot \mathbf{p} \sim \sum \alpha_\mu p_\mu)$. To obtain a Dirac equation for an electron coupled to a prescribed external EM field with vector and scalar potentials A and ϕ one substitutes $p^\mu \to p^\mu - eA^\mu$, i.e. $\mathbf{p} \to \mathbf{p} - e\mathbf{A}$ and $p^0 = i\partial_t \to i\partial_t - e\Phi$, to obtain

$$i\partial_t\psi = [\alpha \cdot (\mathbf{p} - e\mathbf{A}) + e\Phi + \beta m]\psi \tag{1.40}$$

This identifies the Hamiltonian as $H = \alpha\cdot(\mathbf{p}-e\mathbf{A})+e\Phi+\beta m = \alpha\cdot\mathbf{p}+\beta m+H_{int}$ where $H_{int} = -e\alpha\cdot\mathbf{A}+e\Phi$, suggesting α as the operator corresponding to the velocity v/c; this is strengthened by the Heisenberg equations of motion

$$\dot{\mathbf{r}} = \left(\frac{1}{i\hbar}\right)[\mathbf{r},H] = \alpha;\ \dot{\pi} = \left(\frac{1}{i\hbar}\right)[\pi,H] = e(\mathbf{E}+\alpha\times\mathbf{B}) \tag{1.41}$$

Another bit of notation now from [**650**] is useful. Thus (again with $c = \hbar = 1$) one can define e.g.

$$\sigma_z = -i\alpha_x\alpha_y;\ \sigma_x = -i\alpha_y\alpha_z;\ \sigma_y = -i\alpha_z\alpha_x;\ \rho_3 = \beta; \tag{1.42}$$
$$\rho_1 = \sigma_z\alpha_z = -i\alpha_x\alpha_y\alpha_z;\ \rho_2 = i\rho_1\rho_3 = \beta\alpha_x\alpha_y\alpha_z$$

so that $\beta = \rho_3$ and $\alpha^k = \rho_1\sigma^k$. Recall also that the angular momentum $\vec{\ell}$ of a particle is $\vec{\ell} = \mathbf{r}\times\mathbf{p}$ ($\sim (-i)\mathbf{r}\times\nabla$) with components ℓ_k satisfying $[\ell_x,\ell_y] = i\ell_z,\ [\ell_y,\ell_z] = i\ell_x$, and $[\ell_z,\ell_x] = i\ell_y$. Any vector operator L satisfying such relations is called an angular momentum. Next one defines $\sigma_{\mu\nu} = (1/2)i[\gamma_\mu,\gamma_\nu] = i\gamma_\mu\gamma_\nu$ ($\mu\neq\nu$) and $S_{\alpha\beta} = (1/2)\sigma_{\alpha\beta}$. Then the 6 components $S_{\alpha\beta}$ satisfy

$$S_{10} = (i/2)\alpha_x;\ S_{20} = (i/2)\alpha_y;\ S_{30} = (i/1)\alpha_z; \tag{1.43}$$
$$S_{23} = (1/2)\sigma_x;\ ,\ S_{31} = (1/2)\sigma_y;\ S_{12} = (1/2)\sigma_z$$

The $S_{\alpha\beta}$ arise in representing infinitesimal rotations for the orthochronous Lorentz group via matrices $I+i\epsilon S_{\alpha\beta}$. Further one can represent total angular momentum J in the form $J = L+S$ where $L = \mathbf{r}\times\mathbf{p}$ and $S = (1/2)\sigma$ (L is orbital angular momentum and S represents spin). We recall that the gamma matrices are given via $\gamma = \beta\alpha$. Finally $[(i\partial_t - e\phi) - \alpha\cdot(-i\nabla - e\mathbf{A}) - \beta m]\psi = 0$ (cf. (1.40)) and one gets

$$[i\gamma^\mu D_\mu - m)\psi = [\gamma^\mu(i\partial_\mu - eA_\mu) - m]\psi = 0 \tag{1.44}$$
$$D_\mu = \partial_\mu + ieA_\mu \equiv (\partial_0 + ie\phi, \nabla - ie\mathbf{A})$$

Working on the left with $(-i\gamma^\lambda D_\lambda - m)$ gives then $[\gamma^\lambda\gamma^\mu D_\lambda D_\mu + m^2]\psi = 0$ where $\gamma^\lambda\gamma^\mu = g^{\lambda\mu} + (1/2)[\gamma^\lambda,\gamma^\mu]$. By renaming the dummy indices one obtains $[\gamma^\lambda,\gamma^\mu]D_\lambda D_\mu = -[\gamma^\lambda,\gamma^\mu]D_\mu D_\lambda = (1/2)[\gamma^\lambda,\gamma^\mu][D_\lambda,D_\mu]$ leading to

$$[D_\lambda,D_\mu] = ie[\partial_\lambda,A_\mu] + ie[A_\lambda,\partial_\mu] = ie(\partial_\lambda A_\mu - \partial_\mu A_\lambda) = ieF_{\lambda\mu} \tag{1.45}$$

This yields then $\gamma^\lambda\gamma^\mu D_\lambda D_\mu = D_\mu D_\lambda + eS^{\lambda\mu}F_{\lambda\mu}$ where $S^{\lambda\mu}$ represents the spin of the particle. Therefore one can write $[D_\mu D^\mu + eS^{\lambda\mu}F_{\lambda\mu} + m^2]\psi = 0$. Comparing with the standard form of the KG equation we see that this differs by the term $eS^{\lambda\mu}F_{\lambda\mu}$ which is the spin coupling of the particle to the EM field and has no classical analogue.

2. BERTOLDI-FARAGGI-MATONE THEORY

The equivalence principle (EP) of Faraggi-Matone (cf. [**110, 191, 193, 198, 347, 641**]) is based on the idea that all physical systems can be connected by a coordinate transformation to the free situation with vanishing energy (i.e. all potentials are equivalent under coordinate transformations). This automatically leads to the quantum stationary Hamilton-Jacobi equation (QSHJE) which is a third order nonlinear differential equation providing a trajectory representation of quantum mechanics (QM). The theory transcends in several respects the Bohm theory and in particular utilizes a Floydian time (cf. [**373, 374**]) leading to $\dot{q} = p/m_Q \neq p/m$ where $m_Q = m(1 - \partial_E Q)$ is the "quantum mass" and Q the "quantum potential" (cf. also Section 7.4). Thus the EP is reminscient of the Einstein equivalence of relativity theory. This latter served as a midwife to the birth of relativity but was somewhat inaccurate in its original form. It is better put as saying that all laws of physics should be invariant under general coordinate transformations (cf. [**723**]). This demands that not only the form but also the content of the equations be unchanged. More precisely the equations should be covariant and all absolute constants in the equations are to be left unchanged (e.g. c, $\hbar$, e, m and $\eta_{\mu\nu}$ = Minkowski tensor). Now for the EP, the classical picture with $S^{cl}(q, Q^0, t)$ the Hamilton principal function ($p = \partial S^{cl}/\partial q$) and P^0, Q^0 playing the role of initial conditions involves the classical HJ equation (CHJE) $H(q, p = (\partial S^{cl}/\partial q), t) + (\partial S^{cl}/\partial t) = 0$. For time independent V one writes $S^{cl} = S_0^{cl}(q, Q^0) - Et$ and arrives at the classical stationary HJ equation (CSHJE) $(1/2m)(\partial S_0^{cl}/\partial q)^2 + \mathfrak{W} = 0$ where $\mathfrak{W} = V(q) - E$. In the Bohm theory one looked at Schrödinger equations $i\hbar\psi_t = -(\hbar^2/2m)\psi'' + V\psi$ with $\psi = \psi(q)exp(-iEt/\hbar)$ and $\psi(q) = R(qexp(i\hat{W}/\hbar)$ leading to

$$(2.1) \qquad \left(\frac{1}{2m}\right)(\hat{W}')^2 + V - E - \frac{\hbar^2 R''}{2mR} = 0;\ (R^2\hat{W}')' = 0$$

where $\hat{Q} = -\hbar^2 R''/2mR$ was called the quantum potential; this can be written in the Schwartzian form $\hat{Q} = (\hbar^2/4m)\{\hat{W}; q\}$ (via $R^2\hat{W}' = c$). Here $\{f; q\} = (f'''/f') - (3/2)(f''/f')^2$. Writing $\mathfrak{W} = V(q) - E$ as in above we have the quantum stationary HJ equation (QSHJE)

$$(2.2) \quad (1/2m)(\partial\hat{W}'/\partial q)^2 + \mathfrak{W}(q) + \hat{Q}(q) = 0 \equiv \mathfrak{W} = -(\hbar^2/4m)\{exp(2iS_0/\hbar); q\}$$

This was worked out in the Bohm school (without the Schwarzian connections) but $\psi = Rexp(i\hat{W}/\hbar)$ is not appropriate for all situations and care must be taken ($\hat{W} = constant$ must be excluded for example - cf. [**347, 373, 374**]). The technique of Faraggi-Matone (FM) is completely general and with only the EP as guide one exploits the relations between Schwarzians, Legendre duality, and the geometry of a second order differential operator $D_x^2 + V(x)$ (Möbius transformations play an important role here) to arrive at the QSHJE in the form

$$(2.3) \qquad \frac{1}{2m}\left(\frac{\partial S_0^v(q^v)}{\partial q^v}\right)^2 + \mathfrak{W}(q^v) + \mathfrak{Q}^v(q^v) = 0$$

where $v : q \to q^v$ represents an arbitrary locally invertible coordinate transformation. Note in this direction for example that the Schwarzian derivative of the the ratio of two linearly independent elements in $ker(D_x^2 + V(x))$ is twice $V(x)$. In particular given an arbitrary system with coordinate q and reduced action $S_0(q)$ the system with coordinate q^0 corresponding to $V - E = 0$ involves $\mathfrak{W}(q) = (q^0; q)$ where (q^0, q) is a cocycle term which has the form $(q^a; q^b) = -(\hbar^2/4m)\{q^a; q^b\}$. In fact it can be said that the essence of the EP is the cocycle condition

$$(2.4) \qquad (q^a; q^c) = (\partial_{q^c} q^b)^2[(q^a; q^b) - (q^c; q^b)]$$

In addition FM developed a theory of (x, ψ) duality (cf. [**346**])) which related the space coordinate and the wave function via a prepotential (free energy) in the form $\mathfrak{F} = (1/2)\psi\bar{\psi} + iX/\epsilon$ for example. A number of interesting philosophical points arise (e.g. the emergence of space from the wave function) and we connected this to various features of dispersionless KdV in [**191, 198**] in a sort of extended WKB spirit (cf. also Section 7.3). One should note here that although a form $\psi = Rexp(i\hat{W}/\hbar)$ is not generally appropriate it is correct when one is dealing with two independent solutions of the Schrödinger equation ψ and $\bar{\psi}$ which are not proportional. In this context we utilized some interplay between various geometric properties of KdV which involve the Lax operator $L^2 = D_x^2 + V(x)$ and of course this is all related to Schwartzians, Virasoro algebras, and vector fields on S^1 (see e.g. [**191, 192, 198, 200, 201**]). Thus the simple presence of the Schrödinger equation (SE) in QM automatically incorporates a host of geometrical properties of $D_x = d/dx$ and the circle S^1. In fact since the FM theory exhibits the fundamental nature of the SE via its geometrical properties connected to the QSHJE one could speculate about trivializing QM (for 1-D) to a study of S^1 and ∂_x.

We import here some comments based on [**110**] concerning the Klein-Gordon (KG) equation and the equivalence principle (EP) (details are in [**110**] and cf. also [**164, 165, 166, 298, 472, 474, 473, 478, 479, 480, 666, 667**] for the KG equation which is treated in some detail later at several places in this book). One starts with the relativistic classical Hamilton-Jacobi equation (RCHJE) with a potential $V(q,t)$ given as

$$\frac{1}{2m}\sum_1^D(\partial_k S^{cl}(q,t))^2 + \mathfrak{W}_{rel}(q,t) = 0; \tag{2.5}$$

$$\mathfrak{W}_{rel}(q,t) = \frac{1}{2mc^2}[m^2c^4 - (V(q,t) + \partial_t S^{cl}(q,t))^2]$$

In the time-independent case one has $S^{cl}(q,t) = S_0^{cl}(q) - Et$ and (2.3) becomes

$$\frac{1}{2m}\sum_1^D(\partial_k S_0^{cl})^2 + \mathfrak{W}_{rel} = 0;\ \mathfrak{W}_{rel}(q) = \frac{1}{2mc^2}[m^2c^4 - (V(q) - E)^2] \tag{2.6}$$

In the latter case one can go through the same steps as in the nonrelativistic case and the relativistic quantum HJ equation (RQHJE) becomes

$$(1/2m)(\nabla S_0)^2 + \mathfrak{W}_{rel} - (\hbar^2/2m)(\Delta R/R) = 0;\ \nabla\cdot(R^2\nabla S_0) = 0 \tag{2.7}$$

these equations imply the stationary KG equation

$$-\hbar^2c^2\Delta\psi + (m^2c^4 - V^2 + 2EV - E^2)\psi = 0 \tag{2.8}$$

where $\psi = Rexp(iS_0/\hbar)$. Now in the time dependent case the (D+1)-dimensional RCHJE is ($\eta^{\mu\nu} = diag(-1, 1, \cdots, 1)$)

$$(1/2m)\eta^{\mu\nu}\partial_\mu S^{cl}\partial_\nu S^{cl} + \mathfrak{W}'_{rel} = 0; \tag{2.9}$$

$$\mathfrak{W}'_{rel} = (1/2mc^2)[m^2c^4 - V^2(q) - 2cV(q)\partial_0 S^{cl}(q)]$$

with $q = (ct, q_1, \cdots, q_D)$. Thus (2.9) has the same structure as (2.6) with Euclidean metric replaced by the Minkowskian one. We know how to implement the EP by adding Q via $(1/2m)(\partial S)^2 + \mathfrak{W}_{rel} + Q = 0$ (cf. [**347**] and remarks above). Note now that $\mathfrak{W}'_{rel}$ depends on S^{cl} requires an identification

$$\mathfrak{W}_{rel} = (1/2mc^2)[m^2c^4 - V^2(q) - 2cV(q)\partial_0 S(q)] \tag{2.10}$$

(S replacing S^{cl}) and implementation of the EP requires that for an arbitrary $\mathfrak{W}^a$ state ($q \sim q^a$) one must have

$$\mathfrak{W}^b_{rel}(q^b) = (p^b|p^a)\mathfrak{W}^a_{rel}(q^a) + (q^q; q^b);\ Q^b(q^b) = (p^b|p^a)Q(q^a) - (q^a; q^b) \tag{2.11}$$

where

$$(p^b|p) = [\eta^{\mu\nu}p^b_\mu p^b_\nu/\eta^{\mu\nu}p_\mu p_\nu] = p^T J\eta J^T p/p^T\eta p;\ J^\mu_\nu = \partial q^\mu/\partial q^{b^\nu} \tag{2.12}$$

(J is a Jacobian and these formulas are the natural multidimensional generalization - see [**110**] for details). Furthermore there is a cocycle condition $(q^a; q^c) = (p^c|p^b)[(q^a; q^b) - (q^c; q^b)]$.

Next one shows that $\mathfrak{W}_{rel} = (\hbar^2/2m)[\Box(Rexp(iS/\hbar))/Rexp(iS/\hbar)]$ and hence the corresponding quantum potential is $Q_{rel} = -(\hbar^2/2m)[\Box R/R]$. Then the RQHJE becomes $(1/2m)(\partial S)^2 + \mathfrak{W}_{rel} + Q = 0$ with $\partial \cdot (R^2 \partial S) = 0$ (here $\Box R = \partial_\mu \partial^\mu R$) and this reduces to the standard SE in the classical limit $c \to \infty$ (note $\partial \sim (\partial_0, \partial_1, \cdots, \partial_D)$ with $q_0 = ct$, etc. - cf. (2.9)). To see how the EP is simply implemented one considers the so called minimal coupling prescription for an interaction with an electromagnetic four vector A_μ. Thus set $P^{cl}_\mu = p^{cl}_\mu + eA_\mu$ where p^{cl}_μ is a particle momentum and $P^{cl}_\mu = \partial_\mu S^{cl}$ is the generalized momentum. Then the RCHJE reads as $(1/2m)(\partial S^{cl} - eA)^2 + (1/2)mc^2 = 0$ where $A_0 = -V/ec$. Then $\mathfrak{W} = (1/2)mc^2$ and the critical case $\mathfrak{W} = 0$ corresponds to the limit situation where $m = 0$. One adds the standard Q correction for implementation of the EP to get $(1/2m)(\partial S - eA)^2 + (1/2)mc^2 + Q = 0$ and there are transformation properties (here $(\partial S - eA)^2 \sim \sum(\partial_\mu S - eA_\mu)^2$)

$$\mathfrak{W}(q^b) = (p^b|p^a)\mathfrak{W}^a(q^a) + (q^a; q^b);\ Q^b(q^b) = (p^q|p^a)Q^a(q^a) - (q^a; q^b) \tag{2.13}$$

$$(p^b|p) = \frac{(p^b - eA^b)^2}{(p - eA)^2} = \frac{(p - eA)^T J\eta J^T (p - eA)}{(p - eA)^T \eta (p - eA)}$$

Here J is a Jacobian $J^\mu_\nu = \partial q^\mu/\partial q^{b^\nu}$ and this all implies the cocycle condition again. One finds now that (recall $\partial \cdot (R^2(\partial S - eA)) = 0$ - continuity equation)

$$(\partial S - eA)^2 = \hbar^2 \left(\frac{\Box R}{R} - \frac{D^2(Re^{iS/\hbar})}{Re^{iS/\hbar}}\right);\ D_\mu = \partial_\mu - \frac{i}{\hbar}eA_\mu \tag{2.14}$$

and it follows that

$$\mathfrak{W} = \frac{\hbar^2}{2m}\frac{D^2(Re^{iS/\hbar})}{Re^{iS/\hbar}};\ Q = -\frac{\hbar^2}{2m}\frac{\Box R}{R};\ D^2 = \Box - \frac{2ieA\partial}{\hbar} - \frac{e^2A^2}{\hbar^2} - \frac{ie\partial A}{\hbar} \tag{2.15}$$

$$(\partial S - eA)^2 + m^2c^2 - \hbar^2\frac{\Box R}{R} = 0;\ \partial \cdot (R^2(\partial S - eA)) = 0 \tag{2.16}$$

Note also that (2.9) agrees with $(1/2m)(\partial S^{cl} - eA)^2 + (1/2)mc^2 = 0$ after setting $\mathfrak{W}_{rel} = mc^2/2$ and replacing $\partial_\mu S^{cl}$ by $\partial_\mu S^{cl} - eA_\mu$. One can check that (2.16) implies the KG equation $(i\hbar\partial + eA)^2\psi + m^2c^2\psi = 0$ with $\psi = Rexp(iS/\hbar)$.

REMARK 2.2.1. We extract now a remark about mass generation and the EP from [**110**]. Thus a special property of the EP is that it cannot be implemented in classical mechanics (CM) because of the fixed point corresponding to $\mathfrak{W} = 0$. One is forced to introduce a uniquely determined piece to the classical HJ equation (namely a quantum potential Q). In the case of the RCHJE the fixed point $\mathfrak{W}(q^0) = 0$ corresponds to $m = 0$ and the EP then implies that all the other masses can be generated by a coordinate transformation. Consequently one concludes that masses correspond to the inhomogeneous term in the transformation properties of the $\mathfrak{W}^0$ state, i.e. $(1/2)mc^2 = (q^0; q)$. Furthermore by (2.13) masses are expressed in terms of the quantum potential $(1/2)mc^2 = (p|p^0)Q^0(q^0) - Q(q)$. In particular in [**347**] the role of the quantum potential was seen as a sort of intrinsic self energy which is reminiscent of the relativistic self energy and this provides a more explicit evidence of such an interpretation.

REMARK 2.2.2. In a previous paper [**194**] (working with stationary states and ψ satisfying the Schrödinger equation (SE) $-(\hbar^2/2m)\psi'' + V\psi = E\psi$) we suggested that the notion of uncertainty in quantum mechanics (QM) could be phrased as incomplete information. The background theory here is taken to be the trajectory theory of Bertoldi-Faraggi-Matone (and Floyd) as above and the idea in [**194**] goes as follows. First recall that microstates satisfy a third order quantum stationary Hamilton-Jacobi equation (QSHJE)

$$\frac{1}{2m}(S_0')^2 + \mathfrak{W}(q) + Q(q) = 0;\ Q(q) = \frac{\hbar^2}{4m}\{S_0; q\}; \tag{2.17}$$

$$\mathfrak{W}(q) = -\frac{\hbar^2}{4m}\{exp(2iS_0/\hbar); q\} \sim V(q) - E$$

where $\{f; q\} = (f'''/f') - (3/2)(f''/f')^2$ is the Schwarzian and S_0 is the Hamilton principle function. Also one recalls that the EP of Faraggi-Matone can only be implemented when $S_0 \neq const$; thus consider $\psi = Rexp(iS_0/\hbar)$ with $Q = -\hbar^2 R''/2mR$ and $(R^2 S_0')' = 0$ where $S_0' = p$ and $m_Q \dot{q} = p$ with $m_Q = m(1 - \partial_E Q)$ and $t \sim \partial_E S_0$ (Q in (2.17) is the definitive form - cf. [**349**]). Thus microstates require three initial or boundary conditions in general to determine S_0 whereas the SE involves only two such conditions (cf. also Section 7.4 and [**138, 140, 139, 305, 306, 307, 308, 309, 347, 348, 349, 373, 374, 375, 520**]). Hence in dealing with the SE in the standard QM Hilbert space formulation one is not using complete information about the "particles" described by microstate trajectories. The price of underdetermination is then uncertainty in q, p, t for example. In the present note we will make this more precise and add further discussion. Following [**197**] we now make this more precise and add further discussion. For the stationary SE $-(\hbar^2/2m)\psi'' + V\psi = E\psi$ it is shown in [**347**] that one has a general formula

$$e^{2iS_0(\delta)/\hbar} = e^{i\alpha}\frac{w + i\bar{\ell}}{w - i\ell} \tag{2.18}$$

($\delta \sim (\alpha, \ell)$) with three integration constants, α, ℓ_1, ℓ_2 where $\ell = \ell_1 + i\ell_2$ and $w \sim \psi^D/\psi \in \mathbf{R}$. Note ψ and ψ^D are linearly independent solutions of the SE and one can arrange that $\psi^D/\psi \in \mathbf{R}$ in describing any situation. Here p is determined by the two constants in ℓ and has a form

$$p = \frac{\pm\hbar\Omega\ell_1}{|\psi^D - i\ell\psi|^2} \tag{2.19}$$

(where $w \sim \psi^D/\psi$ above and $\Omega = \psi'\psi^D - \psi(\psi^D)'$). Now let p be determined exactly with $p = p(q, E)$ via the Schrödinger equation and S_0'. Then $\dot{q} = (\partial_E p)^{-1}$ is also exact so $\Delta q = (\partial_E p)^{-1}(\tau)\Delta t$ for some τ with $0 \leq \tau \leq t$ is exact (up to knowledge of τ). Thus given the wave function ψ satisfying the stationary SE with two boundary conditions at $q = 0$ say to fix uniqueness, one can create a probability density $|\psi|^2(q, E)$ and the function S_0'. This determines p uniquely and hence $\dot{q}$. The additional constant needed for S_0 appears in (2.18) and we can write $S_0 = S_0(\alpha, q, E)$ since from (2.18) one has

$$S_0 - (\hbar/2)\alpha = -(i\hbar/2)log(\beta) \tag{2.20}$$

and $\beta = (w + i\bar{\ell})/(w - i\ell)$ with $w = \psi^D/\psi$ is to be considered as known via a determination of suitable ψ, ψ^D. Hence $\partial_\alpha S_0 = -\hbar/2$ and consequently $\Delta S_0 \sim \partial_\alpha S_0 \delta\alpha = -(\hbar/2)\Delta\alpha$ measures the indeterminacy or uncertainty in S_0.

Let us expand upon this as follows. Note first that the determination of constants necessary to fix S_0 from the QSHJE is not usually the same as that involved in fixing ℓ, $\bar{\ell}$ in (2.18). In paricular differentiating in q one gets

$$S_0' = -\frac{i\hbar\beta'}{\beta};\ \beta' = -\frac{2i\Re\ell w'}{(w - i\ell)^2} \tag{2.21}$$

Since $w' = -\Omega/\psi^2$ where $\Omega = \psi'\psi^D - \psi(\psi^D)'$ we get $\beta' = -2i\ell_1\Omega/(\psi^D - i\ell\psi)^2$ and consequently

$$S_0' = -\frac{\hbar\ell_1\Omega}{|\psi^D - i\ell\psi|^2} \tag{2.22}$$

which agrees with p in (2.19) ($\pm\hbar$ simply indicates direction). We see that e.g. $S_0(x_0) = i\hbar\ell_1\Omega/|\psi^D(x_0) - i\ell\psi(x_0)|^2 = f(\ell_1, \ell_2, x_0)$ and $S_0'' = g(\ell_1, \ell_2, x_0)$ determine the relation between $(p(x_0), p'(x_0))$ and $(\ell_1,\ \ell_2)$ but they are generally different numbers. In any case, taking α to be the arbitrary unknown constant in the determination of S_0, we have $S_0 = S_0(q, E, \alpha)$ with $q = q(S_0, E, \alpha)$ and $t = t(S_0, E, \alpha) = \partial_E S_0$ (emergence of time from the wave function). One can then write e.g.

$$\Delta q = (\partial q/\partial S_0)(\hat{S}_0, E, \alpha)\Delta S_0 = (1/p)(\hat{q}, E)\Delta S_0 = -(1/p)(\hat{q}, E)(\hbar/2)\Delta\alpha \tag{2.23}$$

(for intermediate values $(\hat{S}_0,\ \hat{q})$) leading to

THEOREM 2.1. With p determined uniquely by two "initial" conditions so that Δp is determined and q given via (2.18) we have from (2.23) the inequality $\Delta p \Delta q = O(\hbar)$ which resembles the Heisenberg uncertainty relation.

COROLLARY 2.1. Similarly $\Delta t = (\partial t/\partial S_0)(\hat{S}_0, E, \alpha)\Delta S_0$ for some intermediate value $\hat{S}_0$ and hence as before $\Delta E \Delta t = O(\hbar)$ (ΔE being precise).

Note that there is no physical argument here; one is simply looking at the number of conditions necessary to fix solutions of a differential equation. In fact (based on some corresondence with E. Floyd) it seems somewhat difficult to produce a viable physical argument. We refer also to Remark 3.1.2 for additional discussion.

REMARK 2.2.3. In order to get at the time dependent SE from the BFM (Bertoldi-Faraggi-Matone) theory we proceed as follows. From the previous discussion on the KG equation one sees that (dropping the EM terms) in the time independent case one has $S^{cl}(q,t) = S_0^{cl}(q) - Et$

(2.24)
$$(1/2m)\sum_1^D(\partial_k S_0^{cl})^2 + \mathfrak{W}_{rel} = 0;\ \mathfrak{W}_{rel}(q) = (1/2mc^2)[m^2c^4 - (V(q)-E)^2]$$

leading to a stationary RQHJE

(2.25)
$$(1/2m)(\nabla S_0)^2 + \mathfrak{W}_{rel} - (\hbar^2/2m)(\Delta R/R) = 0;\ \nabla\cdot(R^2\nabla S_0) = 0$$

This implies also the stationary KG equation

(2.26)
$$-\hbar^2c^2\Delta\psi + (m^2c^4 - V^2 + 2VE - E^2)\psi = 0$$

Now in the time dependent case one can write $(1/2m)\eta^{\mu\nu}\partial_\mu S^{cl}\partial_\nu S^{cl} + \mathfrak{W}'_{rel} = 0$ where $\eta \sim diag(-1,1,\cdots,1)$ and

(2.27)
$$\mathfrak{W}'_{rel}(q) = (1/2mc^2)[m^2c^4 - V^2(q) - 2cV(q)\partial_0 S^{cl}(q)]$$

with $q \equiv (ct, q_1,\cdots,q_D)$. Thus we have the same structure as (2.24) with Euclidean metric replaced by a Minkowskian one. To implement the EP we have to modify the classical equation by adding a function to be determined, namely $(1/2m)(\partial S)^2 + \mathfrak{W}_{rel} + Q = 0$ ($(\partial S)^2 \sim \sum(\partial_\mu S)^2$ etc.). Observe that since $\mathfrak{W}'_{rel}$ depends on S^{cl} we have to make the identification $\mathfrak{W}_{rel} = (1/2mc^2)[m^2c^4 - V^2(q) - 2cV(q)\partial_0 S(q)]$ which differs from $\mathfrak{W}'_{rel}$ since S now appears instead of S^{cl}. Implementation of the EP requires that for an arbitrary $\mathfrak{W}^a$ state

(2.28)
$$\mathfrak{W}^b_{rel}(q^b) = (p^b|p^a)\mathfrak{W}^a_{rel}(q^a) + (q^q;q^b);\ Q^b(q^b) = (p^b|p^a)Q^a(q^a) - (q^a;q^b)$$

where now $(p^b|p) = \eta^{\mu\nu}p^b_\mu p^b_\nu/\eta^{\mu\nu}p_\mu p_\nu = p^TJ\eta J^Tp/p^T\eta p$ and $J^\mu_\nu = \partial q^\mu/\partial(q^b)^\nu$. This leads to the cocycle condition $(q^a;q^c) = (p^c|p^b)[(q^q;q^b) - (q^c;q^b)]$ as before. Now consider the identity

(2.29)
$$\alpha^2(\partial S)^2 = \Box(Rexp(\alpha S))/Rexp(\alpha S) - (\Box R/R) - (\alpha\partial\cdot(R^2\partial S)/R^2)$$

and if R satisfies the continuity equation $\partial\cdot(R^2\partial S) = 0$ one sets $\alpha = i/\hbar$ to obtain

(2.30)
$$\frac{1}{2m}(\partial S)^2 = -\frac{\hbar^2}{2m}\frac{\Box(Re^{iS/\hbar})}{Re^{iS/\hbar}} + \frac{\hbar^2}{2m}\frac{\Box R}{R}$$

Then it is shown that $\mathfrak{W}_{rel} = (\hbar^2/2m)(\Box(Rexp(iS/\hbar))/Rexp(iS/\hbar)$ so $Q_{rel} = -(\hbar^2/2m)(\Box R/R)$. Thus the RQHJE has the form (cf. (2.14) - (2.16))

(2.31)
$$\frac{1}{2m}(\partial S)^2 + \mathfrak{W}_{rel} - \frac{\hbar^2}{2m}\frac{\Box R}{R} = 0;\ \partial\cdot(R^2\partial S) = 0$$

Now for the time dependent SE one takes the nonrelativistic limit of the RQHJE. For the classical limit one makes the usual substitution $S = S' - mc^2t$ so as $c \to \infty$ $\mathfrak{W}_{rel} \to (1/2)mc^2 + V$ and $-(1/2m)(\partial_0 S)^2 \to \partial_t S' - (1/2)mc^2$ with $\partial(R^2\partial S) = 0 \to m\partial_t(R')^2 + \nabla\cdot((R')^2\nabla S') = 0$. Therefore (removing the primes) (2.31) becomes $(1/2m)(\nabla S)^2 + V + \partial_t S - (\hbar^2/2m)(\Delta R/R) = 0$ with the time dependent nonrelativistic continuity equation being $m\partial_t R^2 + \nabla\cdot(R^2\nabla S) = 0$. This leads then (for $\psi \sim Rexp(iS/\hbar)$) to the SE

$$i\hbar\partial_t\psi = \left(-\frac{\hbar^2}{2m}\Delta + V\right)\psi \tag{2.32}$$

One sees from all this that the BFM theory is profoundly governed by the equivalence principle and produces a usable framework for computation. It is surprising that it has not attracted more adherents.

3. FIELD THEORY MODELS

In trying to imagine particle trajectories of a fractal nature or in a fractal medium we are tempted to abandon (or rather relax) the particle idea and switch to quantum fields (QF). Let the fields sense the bumps and fractality; if one can think of fields as operator valued distributions for example then fractal supports for example are quite reasonable. There are other reasons of course since the notion of particle in quantum field theory (QFT) has a rather fuzzy nature anyway. Then of course there are problems with QFT itself (cf. [**973**]) as well as arguments that there is no first quantization (except perhaps in the Bohm theory - cf. [**701, 1016**]). We review here some aspects of particles arising from QF and QFT methods, especially in a Bohmian spirit (cf. [**77, 110, 256, 324, 325, 326, 454, 472, 478, 479, 480, 494, 532, 634, 701, 702, 703, 704, 705, 706, 707, 708, 709, 710, 711, 984**]). We refer to [**454, 973**] for interesting philosophical discussion about particles and localized objects in a QFT and will extract here from [**77, 256, 326, 703, 704**]; for QFT we refer to [**457, 912, 935, 1015**]. Many details are omitted and standard QFT techniques are assumed to be known and we will concentrate here on derivations of KG type equations and the nature of the quantum potential (the Dirac equation will be treated later).

3.1. EMERGENCE OF PARTICLES. The papers [**704**] are impressive in producing a local operator describing the particle density current for scalar and spinor fields in an arbitrary gravitational and electromagnetic background. This enables one to describe particles in a local, general covariant, and gauge invariant manner. The current depends on the choice of a 2-point Wightman function and a most natural choice based on the Green's function à la Schwinger- deWitt leads to local conservation of the current provided that interaction with quantum fields is absent. Interactions lead to local nonconservation of current which describes local particle production consistent with the usual global description based on the interaction picture. The material is quite technical but we feel it is important and will sketch some of the main points; the discussion should provide a good exercise in field theoretic technique. The notation is indicated as we proceed and we make no attempt to be consistent with other notation. Thus let $g_{\mu\nu}$ be a classical background metric, g the determinant, and R the curvature. The action of a Hermitian scalar field ϕ can be written as

$$S = \frac{1}{2}\int d^4x|g|^{1/2}[g^{\mu\nu}(\partial_\mu\phi)(\partial_\nu\phi) - m^2\phi^2 - \xi R\phi^2] \tag{3.1}$$

where ξ is a coupling constant. Writing this as $S = \int d^4x|g|^{1/2}\mathcal{L}$ the canonical mommentum vector is $\pi_\mu = [\partial\mathcal{L}/\partial(\partial^\mu\phi)] = \partial_\mu\phi$ (standard $g_{\mu\nu}$). The corresponding equation of motion is $(\nabla^\mu\partial_\mu + m^2 + \xi R)\phi = 0$ where ∇^μ is the covariant derivative. Let Σ be a spacelike Cauchy hypersurface with unit normal vector n^μ; the canonical momentum scalar is defined as $\pi = n^\mu\pi_\mu$ and the volume element on Σ is $d\Sigma^\mu = d^3x|g^{(3)}|^{1/2}n^\mu$ with scalar product $(\phi_1,\phi_2) = i\int_\Sigma d\Sigma^\mu\phi_1^*\overleftrightarrow{\partial_\mu}\phi_2$ where $a\overleftrightarrow{\partial_\mu}b = a\partial_\mu g - (\partial_\mu a)b$. If ϕ_i are solutions of the equation of motion then the scalar product does not depend on Σ. One chooses coordinates (t,x) such that $t = c$ on Σ so that $n^\mu = g_0^\mu/\sqrt{g_{00}}$ and the canonical commutation relations become

$$[\phi(x),\phi(x')]_\Sigma = [\pi(x),\pi(x')]_\Sigma = 0;\ [\phi(x),\pi(x')]_\Sigma = |g^{(3)}|^{-1/2}i\delta^3(x-x') \tag{3.2}$$

(here x, x' lie on Σ). This can be written in a manifestly covariant form via

$$\int_\Sigma d\Sigma'^\mu[\phi(x),\partial'_\mu\phi(x')]\chi(x') = \int_\Sigma d\Sigma'^\mu[\phi(x'),\partial_\mu\phi(x)]\chi(x') = i\chi(x) \tag{3.3}$$

for an arbitrary test function χ. For practical reasons one writes $\tilde{n}^\mu = |g^{(3)}|^{1/2}n^\mu$ where the tilde indicates that it is not a vector. Then $\nabla_\mu\tilde{n}_\nu = 0$ and in fact $\tilde{n}^\mu = (|g^{(3)}|^{1/2}/\sqrt{g_{00}}, 0, 0, 0)$. It follows that $d\Sigma^\mu = d^3x\tilde{n}^\mu$ while (2.11) can be written as $\tilde{n}^0(x')[\phi(x),\partial'_0\phi(x')]_\Sigma = i\delta^3(x-x')$. Consequently

$$[\phi(x),\tilde{\pi}(x')]_\Sigma = i\delta^3(x-x');\ \tilde{\pi} = |g^{(3)}|^{1/2}\pi \tag{3.4}$$

Now choose a particular complete orthonormal set of solutions $\{f_k(x)\}$ of the equation of motion satisfying therefore

$$(f_k, f_{k'}) = -(f_k^*, f_{k'}^*) = \delta_{kk'};\ (f_k^*, f_{k'}) = (f_k, f_{k'}^*) = 0 \tag{3.5}$$

One can then write $\phi(x) = \sum_k a_k f_k(x) + a_k^\dagger f_k^*(x)$ from which we deduce that $a_k = (f_k,\phi)$ and $a_k^\dagger = -(f_k^*,\phi)$ while $[a_k, a_{k'}^\dagger] = \delta_{kk'}$ and $[a_k, a_{k'}] = [a_k^\dagger, a_{k'}^\dagger] = 0$. The lowering and raising operators a_k and $a_k^\dagger$ induce the representation of the field algebra in the usual manner and $a_k|0>= 0$. The number operator is $N = \sum a_k^\dagger a_k$ and one defines a two point function $W(x,x') = \sum f_k(x)f_k^*(x')$ (different definitions appear later). Using the equation of motion one finds that W is a Wightman function $W(x,x') =< 0|\phi(x)\phi(x')|0 >$ and one has $W^*(x,x') = W(x',x)$. Further, via the equation of motion, for f_k, f_k^* one has

$$(\nabla^\mu\partial_\mu + m^2 + \xi R(x))W(x,x') = 0 = (\nabla'^\mu\partial'_\mu + m^2 + \xi R(x'))W(x,x') \tag{3.6}$$

From the form of W and the commutation relations there results also

$$W(x,x')|_\Sigma = W(x',x)|_\Sigma;\ \partial_0\partial'_0 W(x,x')|_\Sigma = \partial_0\partial'_0 W(x',x)|_\Sigma; \tag{3.7}$$
$$\tilde{n}^0\partial'_0[W(x,x') - W(x',x)]_\Sigma = i\delta^3(x-x')$$

The number operator given by $N = \sum a_k^\dagger a_k$ is a global quantity. However a new way of looking into the concept of particles emerges when $a_k = (f_k, \phi)$, etc. is put into N; using the scalar product along with the expression for W leads to

$$N = \int_\Sigma d\Sigma^\mu \int_\Sigma d\Sigma'^\nu W(x, s') \overleftrightarrow{\partial_\mu} \overleftrightarrow{\partial'_\nu} \phi(x)\phi(x') \tag{3.8}$$

By interchanging the names of the coordinates x, x' and the names of the indices $\mu\,\nu$ this can be written as a sum of two equal terms

$$N = \frac{1}{2}\int_\Sigma d\Sigma^\mu \int_\Sigma d\Sigma'^\nu W(x, x') \overleftrightarrow{\partial_\mu} \overleftrightarrow{\partial'_\nu} \phi(x)\phi(x') + \tag{3.9}$$
$$\frac{1}{2}\int_\Sigma d\Sigma^\mu \int_\Sigma d\Sigma'^\nu W(x', x) \overleftrightarrow{\partial_\mu} \overleftrightarrow{\partial'_\nu} \phi(x')\phi(x)$$

Using also $W^*(x, x') = W(x', x)$ one sees that (3.9) can be written as $N = \int_\Sigma d\Sigma^\mu j_\mu(x)$ where

$$j_\mu(x) = (1/2)\int_\Sigma d\Sigma'^\nu \{W(x, x') \overleftrightarrow{\partial_\mu} \overleftrightarrow{\partial'_\nu} \phi(x)\phi(x') + h.c.\} \tag{3.10}$$

(where $h.c.$ denotes hermitian conjugate). Evidently the vector $j_\mu(x)$ should be interpreted as the local current of particle density. This representation has three advantages over $N = \sum a_k^\dagger a_k$: (i) It avoids the use of a_k, $a_k^\dagger$ related to a particular choice of modes $f_k(x)$. (ii) It is manifestly covariant. (iii) The local current $j_\mu(x)$ allows one to view the concept of particles in a local manner. If now one puts all this together with the antisymmetry of $\overleftrightarrow{\partial_\mu}$ we find

$$j_\mu = i\sum_{k,k'} f_k^* \overleftrightarrow{\partial_\mu} f_{k'} a_k^\dagger a_{k'} \tag{3.11}$$

From this we see that j_μ is automatically normally ordered and has the property $j_\mu|0>= 0$ (not surprising since $N = \sum a_k^\dagger a_k$ is normally ordered). Further one finds $\nabla^\mu j_\mu = 0$ (covariant conservation law) so the background gravitational field does not produce particles provided that a unique vacuum defined by $a_k|0>= 0$ exists. This also implies global conservation since it provides that $N = \int_\Sigma d\Sigma^\mu j_\mu(x)$ does not depend on time. The extra terms in $\nabla^\mu j_\mu = 0$ originating from the fact that $\nabla_\mu \neq \partial_\mu$ are compensated by the extra terms in $N = \int_\Sigma d\Sigma^\mu j_\mu$ that originate from the fact that $d\Sigma^\mu$ is not written in "flat" coordinates. The choice of vacuum is related to the choice of $W(x, x')$. Note that although $j_\mu(x)$ is a local operator some nonlocal features of the particle concept still remain because (3.10) involves an integration over Σ on which x lies. Since $\phi(x')$ satisfies the equation of motion and $W(x, x')$ satisfies (3.6) this integral does not depend on Σ. However it does depend on the choice of $W(x, x')$. Note also the separation between x and x' in (3.10) is spacelike which softens the nonlocal features because $W(x, x')$ decreases rapidly with spacelike separation - in fact it is negligible when the space like separation is much larger than the Compton wavelength.

We pick this up now in [**702**]. Thus consider a scalar Hermitian field $\phi(x)$ in a curved background satisfying the equation of motion and choose a particular complete orthonormal set $\{f_k(x)\}$ having relations and scalar product as before. The

field ϕ can be expanded as $\phi(x) = \phi^+(x) + \phi^-(x)$ where $\phi^+(x) = \sum a_k f_k(x)$ and $\phi^-(x) = \sum a_k^\dagger f_k^*(x)$. Introducing the two point function $W^+(x,x') = \sum f_k(x) f_k^*(x')$ with $W^-(x,x') = \sum f_k^*(x) f_k(x')$ one finds the remarkable result that
(3.12)

$$\phi^+(x) = i\int_\Sigma d\Sigma'^\nu W^+(x,x')\overleftrightarrow{\partial}'_\nu \phi(x');\ \phi^-(x) = -i\int_\Sigma d\Sigma'^\nu W^-(x,x')\overleftrightarrow{\partial}'_\nu \phi(x')$$

We see that the extraction of $\phi^\pm(x)$ from $\phi(x)$ is a nonlocal procedure. Note however that the integrals in (3.12) do not depend on the choice of the timelike Cauchy hypersurface Σ because $W^\pm(x,x')$ satisfies the equation of motion with respect to x' just as $\phi(x')$ does. However these integrals do depend on the choice of $W^\pm(x,x')$, i.e. on the choice of the set $\{f_k(x)\}$. Now define normal ordering in the usual way, putting ϕ^- on the left, explicitly $:\phi^+\phi^-: = \phi^-\phi^+$ while the ordering of the combinations $\phi^-\phi^+$, $\phi^+\phi^+$, and $\phi^-\phi^-$ leaves these combinations unchanged. Generalize this now by introducing 4 different orderings $N_{(\pm)}$ and $A_{(\pm)}$ defined via

$$N_+\phi^+\phi^- = \phi^-\phi^+;\ N_-\phi^+\phi^- = -\phi^-\phi^+; \tag{3.13}$$

$$A_+\phi^-\phi^+ = \phi^+\phi^-;\ A_-\phi^-\phi^+ = -\phi^+\phi^-$$

Thus N_+ is normal ordering, N_- will be useful, and the antinormal orderings $A_\pm$ can be used via symmetric orderings $S_+ = (1/2)[N_+ + A_+]$ and $S_- = (1/2)[N_- + A_-]$. When S_+ acts on a bilinear combination of fields it acts as the default ordering, i.e. $S_+\phi\phi = \phi\phi$.

Now the particle current for scalar Hermitian fields can be written as (cf. (3.10))
(3.14)

$$j_\mu(x) = \frac{1}{2}\int_\Sigma d\Sigma'^\nu \left[W^+(x,x')\overleftrightarrow{\partial_\mu}\overleftrightarrow{\partial}'_\nu \phi(x)\phi(x;) + W^-(x,x')\overleftrightarrow{\partial_\mu}\overleftrightarrow{\partial}'_\nu \phi(x')\phi(x)\right]$$

(3.14) can be written in a local form as $j_\mu(x) = (i/2)[\phi(x)\overleftrightarrow{\partial_\mu}\phi^+(x) + \phi^-(x)\overleftrightarrow{\partial_\mu}\phi(x)]$ (via (3.12)). Using the identities $\phi^+\overleftrightarrow{\partial_\mu}\phi^+ = \phi^-\overleftrightarrow{\partial_\mu}\phi^-$ this can be written in the elegant form $j_\mu = i\phi^-\overleftrightarrow{\partial_\mu}\phi^+$. Similarly using (3.13) this can be written in another elegant form without explicit use of $\phi^\pm$, namely $j_\mu = (i/2)N_-\phi\overleftrightarrow{\partial_\mu}\phi$. Note that the expression on the right here without the ordering N_- vanishes identically - this peculiar feature may explain why the particle current was not previously discovered. The normal ordering N_- provides that $j_\mu|0>=0$ which is related to the fact that the total number of particles is $N = \int_\Sigma d\Sigma^\mu j_\mu = \sum a_k^\dagger a_k$. Alternatively one can choose the symmetric ordering S_- and define the particle current as $j_\mu = (i/2)S_-\phi\overleftrightarrow{\partial_\mu}\phi$. This leads to the total number of particles $N = (1/2)\sum(a_k^\dagger a_k + a_k a_k^\dagger) = \sum[a_k^\dagger a_k + (1/2)]$.

When the gravitational background is time dependent one can introduce a new set of solutions $u_k(x)$ for each time t, such that the $u_k(x)$ are positive frequency modes at that time. This leads to functions with an extra time dependence $u_k(x,t)$

that do not satisfy the equation of motion (cf. [**704**]). Define $\phi^{\pm}$ as in (3.12) but with the two point functions

$$W^+(x,x') = \sum u_k(x,t)u_k^*(x',t');\ W^-(x,x') = \sum u_k^*(x,t)u_k(x',t') \tag{3.15}$$

As shown in [**704**] such a choice leads to a local description of particle creation consistent with the conventional global description based on the Bogoliubov transformation. Putting $\phi(x) = \sum a_k f_k(x) + a_k^\dagger f_k^*(x)$ in (3.12) with (3.15) yields $\phi^+(x) = \sum A_k(t)u_k(x,t)$ and $\phi^-(x) = \sum A_k^\dagger(t)u_k^*(x)$ where

$$A_k(t) = \sum \alpha_{kj}^*(t)a_j - \beta_{kj}^*(t)a_k^\dagger;\ \alpha_{jk} = (f_j,u_k);\ \beta_{jk}(t) = -(f_j^*,u_k) \tag{3.16}$$

Putting these $\phi^{\pm}$ in $j_\mu = i\phi^- \overleftarrow{\partial}\phi^+$ one finds

$$j_\mu(x) = i\sum_{k,k'} A_k^\dagger(t)u_k^*(x,t)\overleftrightarrow{\partial_\mu}A_{k'}(t)u_{k'}(x,t) \tag{3.17}$$

Note that because of the extra time dependence the fields $\phi^{\pm}$ do not satisy the equation of motion $(\nabla^\mu\partial_\mu + m^2 + \xi R)\phi = 0$ and hence the current (3.17) is not conserved, i.e. $\nabla^\mu j_\mu$ is a nonvanishing local scalar function describing the creation of particles in a local and invariant manner as in [**704**]. In [**702**] there follows a discussion about where and when particles are created with conclusion that this happens at the spacetime points where the metric is time dependent. Hawking radiation is then cited as an example. Generally the choice of the 2-point function (3.15) depends on the choice of time coordinate. Therefore in general a natural choice of the 2-point function (3.15) does not exist. In [**704**] an alternative choice is introduced via $W^{\pm}(x,x') = G^{\pm}(x,x')$ where $G^{\pm}(x,x')$ is determined by the Schwinger- deWitt function. As argued in [**704**] this choice seems to be the most natural since the $G^{\pm}$ satisfy the equation of motion and hence the particle current in which $\phi^{\pm}$ are calculated by putting $:\phi^+\phi^-: = \phi^-\phi^+$ in (3.12) is conserved; this suggests that classical gravitational backgrounds do not create particles (see below).

A complex scalar field $\phi(x)$ and its Hermitian conjugate $\phi^\dagger$ in an arbitrary gravitational background can be expanded as

$$\phi = \phi^{P+} + \phi^{A-};\ \phi^\dagger = \phi^{P-} + \phi^{A+};\ \phi^{P+} = \sum a_k f_k(x); \tag{3.18}$$

$$\phi^{P-} = \sum a_k^\dagger f_k^*;\ \phi^{A+} = \sum b_k f_k(x);\ \phi^{A-} = \sum b_k^\dagger f_k^*(x)$$

In a similar manner to the preceeding one finds also

(3.19)

$$\phi^{P+} = i\int d\Sigma'^\nu W^+(x,x')\overleftrightarrow{\partial'_\nu}\phi(x');\ \phi^{A+} = i\int_\Sigma d\Sigma'^\nu W^+(x,x')\overleftrightarrow{\partial'_\nu}\phi^\dagger(x');$$

$$\phi^{P-} = -i\int_\Sigma d\Sigma'^\nu W^-(x,x')\overleftrightarrow{\partial'_\nu}\phi^\dagger(x');\ \phi^{A-} = -i\int_\Sigma d\Sigma'^\nu W^-(x,x')\overleftrightarrow{\partial'_\nu}\phi(x')$$

The particle current $j^P_\mu(x)$ and the antiparticle current $j^A_\mu(x)$ are then (cf. [**704**])
(3.20)

$$j^P_\mu(x) = \frac{1}{2}\int_\Sigma d\Sigma'^\nu \left[W^+(x,x')\overleftrightarrow{\partial_\mu}\overleftrightarrow{\partial'_\nu}\phi^\dagger(x)\phi(x') + W^-(x,x')\overleftrightarrow{\partial_\mu}\overleftrightarrow{\partial'_\nu}\phi^\dagger(x')\phi(x)\right];$$

$$j^A_\mu(x) = \frac{1}{2}\int_\Sigma d\Sigma'^\nu \left[W^+(x,x')\overleftrightarrow{\partial_\mu}\overleftrightarrow{\partial'_\nu}\phi(x)\phi^\dagger(x') + W^-(x,x')\overleftrightarrow{\partial_\mu}\overleftrightarrow{\partial'_\nu}\phi(x')\phi^\dagger(x)\right]$$

Consequently they can be written in a purely local form as

$$j^P_\mu = i\phi^{P-}\overleftrightarrow{\partial_\mu}\phi^{P+} + j^{mix}_\mu;\ j^A_\mu = i\phi^{A-}\overleftrightarrow{\partial_\mu}\phi^{A+} - j^{mix}_\mu; \tag{3.21}$$

$$j^{mix}_u = \frac{i}{2}\left[\phi^{P-}\overleftrightarrow{\partial_\mu}\phi^{A-} - \phi^{P+}\overleftrightarrow{\partial_\mu}\phi^{A+}\right]$$

The current of charge j^-_μ has the form $j^-_\mu = j^P_\mu - j^A_\mu$ which can be written as (cf. [**704**]) $j^-_\mu =: i\phi^\dagger\overleftrightarrow{\partial_\mu}\phi := \frac{i}{2}\left[\phi^\dagger\overleftrightarrow{\partial_\mu}\phi - \phi\overleftrightarrow{\partial_\mu}\phi^\dagger\right]$. Using (3.13) this can also be written as

$$j^-_\mu = N_+ i\phi^\dagger\overleftrightarrow{\partial_\mu}\phi = (i/2)N_+[\phi^\dagger\overleftrightarrow{\partial_\mu}\phi - \phi\overleftrightarrow{\partial_\mu}\phi^\dagger] \tag{3.22}$$

The current of total number of particles is now defined as $j^+_\mu = j^P_\mu + j^A_\mu$ and it is shown in [**704**] that j^+_μ can be written as $j_\mu = j^1_\mu + j^2_\mu$ where $\phi = (1/\sqrt{2})(\phi_1 + i\phi_2)$ (j^i_μ are two currents of the form (3.14). Therefore using $j_\mu = (i/2)N_-\phi\overleftrightarrow{\partial_\mu}\phi$ one can write j_μ as $j^+_\mu = (i/2)N_-[\phi_1\overleftrightarrow{\partial_\mu}\phi_1 + \phi_2\overleftrightarrow{\partial_\mu}\phi_2]$. Finally one shows that this can be written in a form analogous to (3.22) as $j^+_\mu = (i/2)N_-[\phi^\dagger\overleftrightarrow{\partial_\mu}\phi + \phi\overleftrightarrow{\partial_\mu}\phi^\dagger]$. This can be summarized by defining currents $q^\pm_\mu = (1/2)[\phi^\dagger\overleftrightarrow{\partial_\mu}\phi \pm \phi\overleftrightarrow{\partial_\mu}\phi^\dagger]$ leading to $j^\pm_\mu = N_\mu q^\pm_\mu$. The current q^+_μ vanishes but $N_- q^+_\mu$ does not vanish. These results can be easily generalized to the case where the field interacts with a backgound EM field (as in [**704**]). The equations are essentially the same but the derivatives ∂_μ are replaced by the corresponding gauge covariant derivatives and the particle 2-point functions $W^{P\pm}$ are not equal to the antiparticle 2-point functions $W^{A\pm}$. As in the gravitational case in the case of interaction with an EM background three different choices for the 2-point functions exist and we refer to [**704**] for details.

REMARK 2.3.1 In a classical field theory the energy-momentum tensor (EMT) of a real scalar field is

$$T_{\mu\nu} = (\partial_\mu\phi)(\partial_\nu\phi) - (1/2)g_{\mu\nu}[g^{\alpha\beta}(\partial_\alpha\phi)(\partial_\beta\phi) - m^2\phi^2] \tag{3.23}$$

Contrary to the conventional idea of particles in QFT the EMT is a local quantity. Therefore the relation between the definition of particles and that of EMT is not clear in the conventional approach to QFT in curved spacetime. Here one can exploit the local and covariant description of particles to find a clear relation between particles and EMT. One has to choose some ordering of the operators in (3.23) just as a choice of ordering is needed in order to define the particle current. Although the choice is not obvious it seems natural that the choice for one quantity should determine the choice for the other. Thus if the quantum EMT is defined via : $T_{\mu\nu} := N_+ T_{\mu\nu}$ then the particle current should be defined as $N_- i\phi\overleftrightarrow{\partial_\mu}\phi$. The nonlocalities related to the extraction of ϕ^+ and ϕ^- from ϕ needed for the definitions of the normal orderings N_+ and N_- appear both in the EMT and in the particle current. Similarly if $W^{\pm}$ is chosen as in $W^+(x,x') = \sum f_k(x)f_k^*(x')$ for one quantity then it should be chosen in the same way for the other. The choices as above lead to a consistent picture in which both the energy and the number of particles vanish in the vacuum $|0>$ defined by $a_k|0>=0$. Alternatively if $W^{\pm}$ is chosen as in (3.15) for the definition of particles it should be chosen in the same way for the definition of the EMT. Assume for simplicity that spacetime is flat at some late time t. Then the normally ordered operator of the total number of particles at t is $N(t) = \sum_q A_q^\dagger(t)A_q(t)$ (cf. (3.16)) while the normally ordered operator of energy is $H(t) = \sum_q \omega_q A^\dagger(t)A_q(t)$ (note here $q \sim \mathbf{q}$). Owing to the extra time dependence it is clear that both the particle current and the EMT are not conserved in this case. Thus it is clear that the produced energy exactly corresponds to the produced particles. A similar analysis can be caried out for the particle-antiparticle pair creation caused by a classical EM background. Since the energy should be conserved this suggests that $W^{\pm}$ should not be chosen as in (3.15), i.e. that classical backgrounds do not cause particle creation (see [**702**] for more discussion). The main point in all this is that particle currents as developed above can be written in a purely local form. The nonlocalities are hidden in the extraction of $\phi^{\pm}$ from ϕ. The formalism also reveals a relation between EM and particles suggesting that it might not be consistent to use semiclassical methods to describe particle creation; it also suggests that the vacuum energy might contribute to dark matter that does not form structures, instead of contributing to the cosmological constant.

3.2. BOSONIC BOHMIAN THEORY. We follow here [**703**] concerning Bohmian particle trajectories in relativistic bosonic and fermionic QFT. First we recall that there is no objection to a Bohmian type theory for QFT and no contradiction to Bell's theorems etc. (see e.g. [**77, 126, 256, 326**]). Without discussing all the objections to such a theory we simply construct one following Nikolic (cf. also [**180, 453, 588**] for related information). Thus consider first particle trajectories in relativistic QM and posit a real scalar field $\phi(x)$ satisfying the Klein-Gordon equation in a Minkowski metric $\eta_{\mu\nu} = diag(1,-1,-1,-1)$ written as $(\partial_0^2 - \nabla^2 + m^2)\phi = 0$. Let $\psi = \phi^+$ with $\psi^* = \phi^-$ correspond to positive and negative frequency parts of $\phi = \phi^+ + \phi^-$. The particle current is $j_\mu = i\psi^* \overleftrightarrow{\partial_\mu}\psi$ and $N = \int d^3x j_0$ is the positive definite number of particles (not the charge). This is most easily seen from the plane wave expansion $\phi^+(x) = \int d^3k a(\kappa) exp(-ikx)/\sqrt{(2\pi)^3 2k_0}$ since then $N = \int d^3k a^\dagger(\kappa)a(\kappa)$ (see above and [**702, 704**] where it is shown that the particle current and the decomposition $\phi = \phi^+ + \phi^-$ make sense even when a background gravitational field or some other potential is present). One can write also $j_0 = i(\phi^-\pi^+ - \phi^+\pi^-)$ where $\pi = \pi^+ + \pi^-$ is the canonical momentum (cf. [**471**]). Alternatively ϕ may be interpreted not as a field containg an arbitrary number of particles but rather as a one particle wave function. Here we note that contrary to a field a wave function is not an observable and so doing we normalize ϕ here so that $N = 1$. The current j_μ is conserved via $\partial_\mu j^\mu = 0$ which implies that $N = \int d^3x j_0$ is also conserved, i.e. $dN/dt = 0$. In the causal interpretation one postulates that the particle has the trajectory determined by $dx^\mu/d\tau = j^\mu/2m\psi^*\psi$. The affine parameter τ can be eliminated by writing the trajectory equation as $d\mathbf{x}/dt = \mathbf{j}(t,\mathbf{x})/j_0(t,\mathbf{x})$ where $t = x^0,\ \mathbf{x} = (x^1,x^2,x^3)$ and $\mathbf{j} = (j^1,j^2,j^3)$. By writing $\psi = Rexp(iS)$ where R, S) are real one arrives at a Hamilton-Jacobi (HJ) form $dx^\mu/d\tau = -(1/m)\partial^{mu}S$ and the KG equation is equivalent to

$$\partial^\mu(R^2\partial_\mu S) = 0;\ \frac{(\partial^\mu S)(\partial_\mu S)}{2m} - \frac{m}{2} + Q = 0 \tag{3.24}$$

Here $Q = -(1/2m)(\partial^\mu\partial_\mu R/R$ is the quantum potential. One has put here $c = \hbar = 1$ and reinserted we would have

$$\frac{(\partial^\mu S)(\partial_\mu S)}{2m} - \frac{c^2 m}{2} - \frac{\hbar^2}{2m}\frac{\partial^\mu\partial_\mu R}{R} = 0 \tag{3.25}$$

From the HJ form and (3.24) plus the identity $d/d\tau = (dx^\mu/dt)\partial_\mu$ one arrives at the equations of motion $m(d^2x^\mu/d\tau^2) = \partial^\mu Q$. A typical trajectory arising from $d\mathbf{x}/dt = \mathbf{j}/j_0$ could be imagined as an S shaped curve in the $t - x$ plane (with t horizontal) and cut with a vertical line through the middle of the S. The velocity may be superluminal and may move backwards in time (at points where $j_0 < 0$). There is no paradox with backwards in time motion since it is physically indistinguishable from a motion forwards with negative energy. One introduces a physical number of particles via $N_{phys} = \int d^3x|j_0|$. Contrary to $N = \int d^3xj_0$ the physical number of particles is not conserved. A pair of particles one with positive and the other with negative energy may be created or annihilated; this resembles the behavior of virtual particles in convential QFT.

Now go to relativistic QFT where in the Heisenberg picture the Hermitian field operator $\hat{\phi}(x)$ satisfies

$$(\partial_0^2 - \nabla^2 + m^2)\hat{\phi} = J(\hat{\phi}) \tag{3.26}$$

where J is a nonlinear function describing the interaction. In the Schrödinger picture the time evolution is determined via the Schrödinger equation (SE) in the form $H[\phi, -i\delta/\delta\phi]\Psi[\phi,t] = i\partial_t\Psi[\phi,t]$ where Ψ is a functional with respect to $\phi(\mathbf{x})$ and a function of t. A normalized solution of this can be expanded as $\Psi[\phi,t] = \sum_{-\infty}^{\infty}\tilde{\Psi}_n[\phi,t]$ where the $\tilde{\Psi}_n$ are unnormalized n-particle wave functionals. Since any (reasonable) $\phi(\mathbf{x})$ can be Fourier expanded one can write

$$\tilde{\Psi}_n[\phi,t] = \int d^3k_1\cdots d^3k_n c_n(\mathbf{k}^{(n)},t)\Psi_{n,\mathbf{k}^{(n)}}[\phi] \tag{3.27}$$

where $\mathbf{k}^{(n)} = \{\mathbf{k}_1,\cdots,\mathbf{k}_n\}$. These functionals in (3.27) constitute a complete orthonormal basis which generalizes the basis of Hermite functions and they satisfy

$$\int \mathcal{D}\phi\Psi_0^*[\phi]\phi(\mathbf{x}_1)\cdots\phi(\mathbf{x}_{n'})\Psi_{n,\mathbf{k}^{(n)}}[\phi] = 0\ \ (n \neq n) \tag{3.28}$$

For free fields (i.e. when $J = 0$ in (3.26) one has

$$c_n(\mathbf{k}^{(n)},t) = c_n(\mathbf{k}^{(n)})e^{-i\omega_n(\mathbf{k}^{(n)})t};\ \omega_n = E_0 + \sum_1^n\sqrt{\mathbf{k}_j^2 + m^2} \tag{3.29}$$

where E_0 is the vacuum energy. In this case the quantities $|c_n(\mathbf{k}^{(n)},t)|^2$ do not depend on time so the number of particles (corresponding to the quantized version of $N = \int d^3xj_0$) is conserved. In a more general situation with interactions the SE leads to a more complicated time depenedence of the coefficients c_n and the number of particles is not conserved. Now the n-particle wave function is

$$\psi_n(\mathbf{x}^{(n)},t) =< 0|\hat{\phi}(t,\mathbf{x}_1)\cdots\hat{\phi}(t,\mathbf{x}_n)|\Psi > \tag{3.30}$$

(the multiplication of the right side by $(n!)^{-1/2}$ would lead to a normalized wave function only if $\Psi = \tilde{\Psi}_n$). The generalization of (3.30) to the interacting case is not trivial because with an unstable vacuum it is not clear what is the analogue of $< 0|$. Here the Schrödinger picture is more convenient where (3.30) becomes

$$\psi_n(\mathbf{x}^{(n)},t) = \int \mathcal{D}\phi\Psi_0^*[\phi]exp(-i\phi_0(t))\phi(\mathbf{x}_1)\cdots\phi(\mathbf{x}_n)\Psi[\phi,t] \tag{3.31}$$

where $\phi_0(t) = -E_0t$. For the interacting case one uses a different phase $\phi_0(t)$ defined via an expansion, namely

$$\hat{U}(t)\Psi_0[\phi] = r_0(t)exp(i\phi_0(t))\Psi_0[\phi] + \sum_1^\infty \cdots \tag{3.32}$$

where $r_0(t) \geq 0$ and $\hat{U}(t) = U(\phi, -i\delta/\delta\phi, t]$ is the unitary time evolution operator. One sees that even in the interacting case only the $\tilde{\Psi}_n$ part of Ψ contributes to (3.31) so $\tilde{\Psi}_n$ can be called the n-particle wave functional. The wave function (3.30) can also be generalized to a nonequaltime wave function $\psi_n(x^{(n)}) = S_{\{x_j\}} < 0|\hat{\phi}(x_1)\cdots\hat{\phi}(x_n)|\Psi >$ (here $S_{\{x_j\}}$ denotes symmetrization over all x_j which is needed because the field operators do not commute for nonequal times. For the interacting case the nonequaltime wave function is defined as a generalization of (3.30) with the replacements

$$\phi(\mathbf{x}_j) \to \hat{U}^\dagger(t_j)\phi(\mathbf{x}_j)\hat{U}(t_j);\ \Psi[\phi,t] \to \hat{U}^\dagger(t)\Psi[\phi,t] = \Psi[\phi]; \tag{3.33}$$
$$e^{-i\phi_0(t)} \to e^{-i\phi_0(t_1)}\hat{U}(t_1)$$

followed by symmetrization.

In the deBroglie-Bohm (dBB) interpretation the field $\phi(x)$ has a causal evolution determined by

$$(\partial_0^2 - \nabla^2 + m^2)\phi(x) = J(\phi(x)) - \left(\frac{\delta Q[\phi,t]}{\delta\phi(\mathbf{x})}\right)_{\phi(\mathbf{x})=\phi(x)}; \tag{3.34}$$
$$Q = -\frac{1}{2|\Psi|}\int d^3x\frac{\delta^2|\Psi|}{\delta\phi^2(\mathbf{x})}$$

where Q is the quantum potential again. However the n particles attributed to the wave function ψ_n also have causal trajectories determined by a generalization of $d\mathbf{x}/dt = \mathbf{j}/j_0$ as

$$\frac{d\mathbf{x}_{n,j}}{dt} = \left(\frac{\psi_n^*(x^{(n)})\overleftrightarrow{\nabla}_j\psi_n(x^{(n)})}{\psi_n^*(x^{(n)})\overleftrightarrow{\partial}_{t_j}\psi_n(x^{(n)})}\right)_{t_1=\cdots=t_n=t} \tag{3.35}$$

These n-particles have well defined trajectories even when the probability (in the conventional interpretation of QFT) of the experimental detection is equal to zero. In the dBB interpretation of QFT we can introduce a new causally evolving "effectivity" parameter $e_n[\phi, t]$ defined as

$$e_n[\phi, t] = |\tilde{\Psi}_n[\phi, t]|^2 / \sum_{n'}^{\infty} |\tilde{\Psi}_{n'}[\phi, t]|^2 \tag{3.36}$$

The evolution of this parameter is determined by the evolution of ϕ given via (3.34) and by the solution $\Psi = \sum \tilde{\Psi}$ of the SE. This parameter might be interpreted as a probability that there are n particles in the system at time t if the field is equal (but not measured!) to be $\phi(\mathbf{x})$ at that time. However in the dBB theory one does not want a stochastic interpretation. **Hence assume that e_n is an actual property of the particles guided by the wave function ψ_n and call it the effectivity of these n particles.** This is a nonlocal hidden variable attributed to the particles and it is introduced to provide a deterministic description of the creation and destruction of particles. One postulates that the effective mass of a particles guided by ψ_n is $m_{eff} = e_n m$ and similarly for the energy, momentum, charge, etc. This is achieved by postulating that the mass density is $\rho_{mass}(\mathbf{x}, t) = m\sum_1^{\infty} e_n \sum_1^n \delta^3(\mathbf{x} - \mathbf{x}_{n,j}(t))$ and similarly for other quantities. Thus if $e_n = 0$ such particles are ineffective, i.e. their effect is the same as if they didn't exist while if $e_n = 1$ they exist in the usual sense. However the trajectories are defined even for the particles for which $e_n = 0$ and QFT is a theory of an infinite number of particles although some of them may be ineffective (conventionally one would say they are virtual). We will say more about this later.

3.3. FERMIONIC THEORY. This extraction from [**701**] (cf. also [**325**]) becomes even more technical but a sketch should be rewarding; there is more detail and discussion in [**701**]. The Dirac equation in Minkowski space $\eta_{\mu\nu} = diag(1, -1, -1, -1)$ is $i\gamma^\mu\partial_\mu - m)\psi(x) = 0$ where $x = (x^i) = (t, \mathbf{x})$ with $\mathbf{x} \in \mathbf{R}^3$ (cf. Section 2.1.1). A general solution can be written as $\psi(x) = \psi^P(x) + \psi^A(x)$ where the particle and antiparticle parts can be expanded as $\psi^P = \sum b_k u_k(x)$ and $\psi^A = \sum d_k^* v_k(x)$. Here u_k (resp. v_k) are positive (resp. negative) frequency 4-spinors that, together, form a complete orthonormal set of solutions to the Dirac equation. The label k means $(\mathbf{k}, s)$ where $s = \pm 1/2$ is the spin label. Writing $\Omega^P(x, x') = \sum u_k(x)u_k^\dagger(x')$ and $\Omega^A(x, x') = \sum v_k(x)v_k^\dagger(x')$ one can write

$$\psi^P = \int d^3x'\Omega^P(x, x')\psi(x'); \;\; \psi^A(x) = \int d^3x'\Omega^A(x, x')\psi(x') \tag{3.37}$$

where $t = t'$. The particle and antiparticle currents are $j_\mu^P = \bar{\psi}^P\gamma_\mu\psi^P$ and $j_\mu^A = \bar{\psi}^A\gamma_\mu\psi^A$ where $\bar{\psi} = \psi^\dagger\gamma_0$. Since ψ^P and ψ^A satisfy the Dirac equation the currents j_μ^P, j_μ^A are separately conserved, i.e. $\partial^\mu j_\mu^P = \partial^\mu j_\mu^A = 0$. One postulates then trajectories of the form

$$\frac{d\mathbf{x}^P}{dt} = \frac{\mathbf{j}^P(t, \mathbf{x}^P)}{j_0^P(t, \mathbf{x})}; \;\; \frac{d\mathbf{x}^A}{dt} = \frac{\mathbf{j}^A(t, \mathbf{x}^A)}{j_0^A(t, \mathbf{x}^A)} \tag{3.38}$$

where $\mathbf{j} = (j^1, j^2, j^3)$ for a causal interpretation of the Dirac equation. Now in QFT the coefficients b_k and d_k^* become anticommuting operators with $\hat{b}_k^\dagger$ and $\hat{d}_k^\dagger$ creating particles and antiparticles while $\hat{b}_k$ and $\hat{d}_k$ annihilate them. In the Schrödinger picture the field opperators $\hat{\psi}(\mathbf{x})$ and $\hat{\psi}^\dagger(\mathbf{x})$ satisfy the commutation relations $\{\hat{\psi}_a(\mathbf{x}), \hat{\psi}^\dagger_{a'}(\mathbf{x}')\} = \delta_{aa'}\delta^3(\mathbf{x}-\mathbf{x}')$ while other commutators vanish (a is the spinor index). These relations can be represented via

$$\hat{\psi}_a(\mathbf{x}) = \frac{1}{\sqrt{2}}\left[\eta_a(\mathbf{x}) + \frac{\delta}{\delta\eta_a^*(\mathbf{x})}\right];\ \hat{\psi}_a^\dagger(\mathbf{x}) = \frac{1}{\sqrt{2}}\left[\eta_a^*(\mathbf{x}) + \frac{\delta}{\delta\eta_z(\mathbf{x})}\right] \tag{3.39}$$

where $\eta_a,\ \eta_a^*$ are anticommuting Grassmann numbers satisfying $\{\eta_a(\mathbf{x}), \eta_{a'}(\mathbf{x}')\} = \{\eta_a^*(\mathbf{x}), \eta_{a'}^*(\mathbf{x}')\} = \{\eta_a(\mathbf{x}), \eta_{a'}^*(\mathbf{x}')\} = 0$. Next introduce a complete orthonormal set of spinors $u_k(\mathbf{x})$ and $v_k(\mathbf{x})$ which are equal to the spinors $u_k(x)$ and $v_k(x)$ at $t = 0$. An arbitrary quantum state may then be obtained by acting with creation operators

$$b_k^\dagger = \int d^3x \hat{\psi}^\dagger(\mathbf{x}) u_k(\mathbf{x});\ \hat{d}_k^\dagger = \int d^3x v_k^\dagger(\mathbf{x})\hat{\psi}(\mathbf{x}) \tag{3.40}$$

on the vacuum $|0>=|\Psi_0>$ represented by

$$\Psi_0[\eta, \eta^\dagger] = Nexp\{\int d^3x \int d^3x' \eta^\dagger(\mathbf{x})\Omega(\mathbf{x},\mathbf{x}')\eta(\mathbf{x}')\} \tag{3.41}$$

Here $\Omega(\mathbf{x},\mathbf{x}') = (\Omega^A - \Omega^P)(\mathbf{x},\mathbf{x}')$, N is a constant such that $<\Psi_0|\Psi_0>=1$ and the scalar product is $<\Psi|\Psi>=\int \mathcal{D}^2\eta\Psi^*[\eta,\eta^\dagger\Psi'[\eta,\eta^\dagger]$; also $\mathcal{D}^2 = \mathcal{D}\eta\mathcal{D}\eta^\dagger$ and Ψ^* is dual (not simply the complex conjugate) to Ψ. The vacuum is chosen such that $\hat{b}_k\Psi_0 = \hat{d}_k\Psi_0 = 0$. A functional $\Psi[\eta,\eta^\dagger]$ can be expanded as $\Psi[\eta,\eta^\dagger] = \sum c_K\Psi_K[\eta,\eta^\dagger]$ where the set $\{\psi_K\}$ is a complete orthonormal set of Grassmann valued functionals. This is chosen so that each Ψ_K is proportional to a functional of the form $\hat{b}^\dagger_{k_1}\cdots\hat{b}^\dagger_{k_{n_P}}\hat{d}^\dagger_{k'_1}\cdots\hat{d}^\dagger_{k'_{n_A}}\Psi_0$ which means that each Ψ_K has a definite number n_P of particles and n_A of antiparticles. Therefore one can write $\Psi[\eta,\eta^\dagger] = \sum_{n_P,n_A=0}^\infty \tilde{\Psi}_{n_P,n_A}[\eta,\eta^\dagger]$ where the tilde denotes that these functionals, in contrast to Ψ and Ψ_K, do not have unit norm. Time dependent states $\Psi[\eta,\eta^\dagger,t]$ can be expanded as

$$\Psi[\eta,\eta^\dagger,t] = \sum_K c_K(t)\Psi_K[\eta,\eta^\dagger] = \sum_{n_p,n_A=0}^\infty \tilde{\Psi}_{n_P,n_A}[\eta,\eta^\dagger,t] \tag{3.42}$$

The time dependence of the c-number coefficients $c_K(t)$ is governed by the functional SE

$$H[\hat{\psi},\hat{\psi}^\dagger]\Psi[\eta,\eta^\dagger,t] = i\partial_t\Psi[\eta,\eta^\dagger,t] \tag{3.43}$$

Since the Hamiltonian H is a Hermitian operator the norms $<\Psi(t)|\Psi(t)>=\sum|c_K(t)|^2$ do not depend on time. In particular if H is the free Hamiltonian (i.e. the Hamiltonian that generates the second quantized free Dirac equation) then the quantities $|c_K(t)|$ do not depend on time, which means that the average number of particles and antiparticles does not change with time when there are no interactions.

Next introduce the wave function of n_P particles and n_A antiparticles via

$$\psi_{n_P,n_A} \equiv \psi_{b_1\cdots b_{n_P}d_1\cdots d_{n_A}}(\mathbf{x}_1,\cdots,\mathbf{x}_{n_P},\mathbf{y}_1\cdots,\mathbf{y}_{n_A},t) \tag{3.44}$$

It has n_P+n_A spinor indices and for free fields the (unnormalized) wave function can be calculated using the Heisenberg picture as

$$\psi_{n_P,n_A} =< 0|\hat{\psi}^P_{b_1}(t,\mathbf{x}_1)\cdots\hat{\psi}^{A\dagger}_{d_{n_A}}(t,\mathbf{y}_{n_A})|\Psi > \tag{3.45}$$

where $\hat{\psi}^P$ and $\hat{\psi}^A$ are extracted from $\hat{\psi}$ using (3.37). In the general interacting case the wave function can be calculated using the Schrödinger picture as

$$\psi_{n_P,n_A} = \int \mathcal{D}^2\eta\Psi^*_0[\eta,\eta^\dagger]e^{-i\phi_0(t)}\hat{\psi}^P_{b_1}(\mathbf{x}_1)\cdots\hat{\psi}^{A\dagger}_{d_{n_A}}(y_{n_A})\Psi[\eta,\eta^\dagger,t] \tag{3.46}$$

Here the phase $\phi_0(t)$ is defined by an expansion as in (3.42), namely

$$\hat{U}(t)\Psi_0[\eta,\eta^\dagger] = r_0(t)exp(i\phi_0(t))\Psi_0[\eta,\eta^\dagger] + \sum_{(n_P,n_A)\neq(0,0)}\cdots \tag{3.47}$$

where $r_0(t)\geq 0$ and $\hat{U}(t) = U[\hat{\psi},\hat{\psi}^\dagger,t]$ is the unitary time evolution operator that satisfies the SE (3.43). The current attributed to the i^{th} corpuscle (particle or antiparticle) in the wave function ψ_{n_P,n_A} is $j_{\mu(i)} = \bar{\psi}_{n_P,n_A}\gamma_{\mu(i)}\psi_{n_P,n_A}$ where one writes

$$\bar{\psi}\Gamma_i\psi = \bar{\psi}_{a_1\cdots a_i\cdots a_n}(\Gamma)_{a_i a_i'}\psi_{a_1\cdots a_i'\cdots a_n}; \tag{3.48}$$
$$\bar{\psi}_{a_1\cdots a_n} = \psi^*_{a_1'\cdots a_n'}(\gamma_0)_{a_1'a_1}\cdots(\gamma_0)_{a_n'a_n}$$

Hence the trajectory of the i^{th} corpuscle guided by the wave function ψ_{n_P,n_A} is given by the generalization of (3.38), namely $d\mathbf{x}_i/dt = \mathbf{j}_i/j_{0(i)}$.

We now need a causal interpretation of the processes of creation and destruction of particles and antiparticles. For bosonic fields this was achieved by introducing the effectivity parameter in Section 1.3.2 but this cannot be done for the Grassmann fields $\eta,\ \eta^\dagger$ because $\Psi^*[\eta,\eta^\dagger,t]\Psi[\eta,\eta^\dagger,t]$ is Grassmann valued and cannot be interpreted as a probability density. Hence another formulation of fermionic states is developed here, more similar to the bosonic states. First the notion of the scalar product can be generalized in such a way that it may be Grassmann valued which allows one to write $\Psi[\eta,\eta^\dagger,t] =< \eta,\eta^\dagger|\Psi(t) >$ and $1 = \int\mathcal{D}^2\eta|\eta,\eta^\dagger >< \eta,\eta^\dagger|$ (cf. [**457**]). We can also introduce

$$< \phi,\phi^\dagger|\eta,\eta^\dagger >= \sum_K < \phi,\phi^\dagger|\Psi_K >< \Psi_K|\eta,\eta^\dagger >= \sum_K \Psi_K[\phi,\phi^\dagger]\Psi^*_K[\eta,\eta^\dagger] \tag{3.49}$$

so one sees that the sets $\{\Psi_K[\eta,\eta^\dagger]\}$ and $\{\Psi_K[\phi,\phi^\dagger]\}$ are two representations of the same orthonormal basis $\{|\Psi_K >\}$ for the same Hilbert space of fermionic states. In other words the state $|\Psi(t) >$ can be represented as $\Psi[\phi,\phi^\dagger,t] =< \phi,\phi^\dagger|\Psi(t) >$ which can be expanded as

$$\Psi[\phi,\phi^\dagger,t] = \sum_K c_K(t)\Psi_K[\phi,\phi^\dagger] = \sum_{n_P,n_A=0}^{\infty}\tilde{\psi}_{n_P,n_A}[\phi,\phi^\dagger,t] \tag{3.50}$$

Putting the unit operator $1 = \int \mathcal{D}^2\phi|\phi,\phi^\dagger >< \phi,\phi^\dagger|$ in the expression for $< \Psi(t),\Psi(t) >$ we see that the time independent norm can be written as

$$< \Psi(t)|\Psi(t) >= \int \mathcal{D}^2\phi\Psi^*[\phi,\phi^\dagger,t)]\Psi[\phi,\phi^\dagger,t] \tag{3.51}$$

Therefore the quantity $\rho[\phi,\phi^\dagger,t] = \Psi^*[\phi,\phi^\dagger,t]\Psi[\phi,\phi^\dagger,t]$ can be interpreted as a positive definite probability density for spinors ϕ, $\phi^\dagger$ to have space dependence $\phi(\mathbf{x})$ and $\phi^\dagger(\mathbf{x})$ respectively at time t. The SE (3.43) can also be written in the ϕ-representation as $\hat{H}_\phi\Psi[\phi,\phi^\dagger,t] = i\partial_t\Psi[\phi,\phi^\dagger,t]$ where the Hamiltonian $\hat{H}_\phi$ is defined by its action on wave functionals $\Psi[\phi,\phi^\dagger,t]$ determined via

(3.52)
$$\hat{H}_\phi[\phi,\phi^\dagger,t] = \int \mathcal{D}^2\eta \int \mathcal{D}^2\phi' < \phi,\phi^\dagger|\eta,\eta^\dagger > \hat{H} < \eta,\eta^\dagger|\phi',\phi^{'\dagger} > \Psi[\phi',\phi^{'\dagger},t]$$

where $\hat{H} = H[\hat{\psi},\hat{\psi}^\dagger]$ is the Hamiltonian of (3.43).

One can now obtain a causal interpretation of a quantum system described by a c-number valued wave function satisfying a SE. The material is written for an n-dimensional vector $\vec{\phi}$ but in a form that generalizes to infinite dimensions. The wave function $\psi(\vec{\phi},t)$ satisfies the SE $\hat{H}\psi = i\partial_t\psi$ where $\hat{H}$ is an arbitrary Hermitian Hamiltonian written in the $\vec{\phi}$ representation. The quantity $\rho = \psi^*\psi$ is the probability density for the variables $\vec{\phi}$ and the average velocity is

$$d < \vec{\phi} > (t)/dt = \int d^n\phi\rho(\vec{\phi},t)\vec{u}(\vec{\phi},t);\ \vec{u} = i\psi^*[\hat{H},\vec{\phi}]\psi/\psi^*\psi \tag{3.53}$$

Introduce a source J via $J = (\partial\rho/\partial t)+\vec{\nabla}(\rho\vec{u})$ (note e.g. for the example of (3.52) J does not vanish even though it frequently will vanish). One wants to find a quantity $\vec{v}(\vec{\phi},t)$ that has the property (3.53) in the form $d < \vec{\phi} > (t)/dt = \int d^n\phi\rho(\vec{\phi},t)\vec{v}(\vec{\phi},t)$ but at the same time satisfies the equivariance property $\partial_t\rho+\vec{\nabla}(\rho\vec{v}) = 0$. These two properties allow one to postulate a consistent causal interpretation of QM in which $\vec{\phi}$ has definite values at each time t determined via $d\vec{\phi}/dt = \vec{v}(\vec{\phi},t)$. In particular the equivariance provides that the statistical distribution of the variables $\vec{\phi}$ is given by ρ for any time t provided that it is given by ρ for some initial time t_0. When $J = 0$ then $\vec{v} = \vec{u}$ which corresponds to the dBB interpretation. The aim now is to generalize this to the general case of $\vec{v}$ in the form $\vec{v} = \vec{u}+\rho^{-1}\vec{\mathcal{E}}$ where $\vec{\mathcal{E}}(\vec{\phi},t)$ is the quantity to be determined. From $\partial_t\rho+\vec{\nabla}(\rho\vec{v}) = 0$ we see that $\vec{\mathcal{E}}$ must be a solution of the equation $\vec{\nabla}\vec{\mathcal{E}} = -J$. Now let $\vec{\mathcal{E}}$ be some particular solution of this equation; then $\vec{\mathcal{E}}(\vec{\phi},t) = \vec{e}(t) + \vec{E}(\vec{\phi},t)$ is also a solution for an arbitrary $\vec{\phi}$ independent function $\vec{e}(t)$. Comparing with (3.53) one sees that $\int d^n\phi\vec{\mathcal{E}} = 0$ is required. This fixes the function $\vec{e}$ to be $\vec{e}(t) = -V^{-1}\int d^n\phi\vec{E}(\vec{\phi},t)$ where $V = \int d^n\phi$. Thus it remains to choose $\vec{E}$ and in [**703**] one takes $\vec{E}$ such that $\vec{E} = 0$ when $J = 0$ so that $\vec{\mathcal{E}} = 0$ when $J = 0$ as well; thus $\vec{v} = \vec{u}$ when $J = 0$. There is still some arbitrariness in $\vec{E}$ so take $\vec{E} = \vec{\nabla}\Phi$ where $\vec{\nabla}^2\Phi = -J$, which is solved via $\Phi(\vec{\phi},t)\int d^n\phi' G(\vec{\phi},\vec{\phi}')J(\vec{\phi}',t)$, so that $\vec{\nabla}^2 G(\vec{\phi},\vec{\phi}') = -\delta^n(\vec{\phi}-\vec{\phi}')$. The solution can be expressed as a Fourier transform $G(\vec{\phi},\vec{\phi}') = \int(d^nk/(2\pi)^n)exp[i\vec{k}(\vec{\phi}-\vec{\phi}')]/\vec{k}^2$.

To eleminate the factor $1/(2\pi)^n$ one uses a new integration variable $\vec{\chi} = \vec{k}/2\pi$ and we obtain

$$\Phi(\vec{\phi},t) = \int d^n\chi \int d^n\phi'[exp(2i\pi\vec{\chi}(\vec{\phi}-\vec{\phi}'))]J(\vec{\phi}',t)/(2\pi)^2\vec{\chi}^2] \tag{3.54}$$

Now for a causal interpretation of fermionic QFT one writes first for simplicity $A[\mathbf{x}]$ for functionals of the form $A[\phi,\phi^\dagger,t,\mathbf{x}]$ and introduces

$$u_a[\mathbf{x}] = i\frac{\Psi^*[\hat{H}_\phi,\phi_a(\mathbf{x})]\Psi}{\Psi^*\Psi};\ u_a^*[\mathbf{x}] = i\frac{\Psi^*[\hat{H}_\phi,\phi_a^*(\mathbf{x})]\Psi}{\Psi^*\Psi} \tag{3.55}$$

where $\Psi = \Psi[\phi,\phi^\dagger,t]$. Next introduce the source

$$J = \frac{\partial\rho}{\partial t} + \sum_a \int d^3x \left[\frac{\delta(\rho u_a[\mathbf{x}])}{\delta\phi_z(\mathbf{x})} + \frac{\delta(\rho u_a^*[\mathbf{x}])}{\delta\phi_a^*(\mathbf{x})}\right] \tag{3.56}$$

where $\rho = \Psi^*\Psi$. Introduce now the notation $\alpha\cdot\beta = \sum_a \int d^3x[\alpha_a(\mathbf{x})\beta_a(\mathbf{x}) + \alpha_a^*(\mathbf{x})\beta_a^*(\mathbf{x})]$ and (3.51) generalizes to

$$\Phi[\phi,\phi^\dagger,t] = \int \mathcal{D}^2\chi \int \mathcal{D}^2\phi' \frac{e^{2\pi i\chi\cdot(\phi-\phi')}}{(2\pi)^2\chi\cdot\chi} J[\phi',\phi'^\dagger,t] \tag{3.57}$$

Then write for $V = \int \mathcal{D}^2\phi$

$$E_a[\mathbf{x}] = \frac{\delta\Phi}{\delta\phi_a(\mathbf{x})};\ E_a^*[\mathbf{x}] = \frac{\delta\Phi}{\delta\phi_a^*(\mathbf{x})}; \tag{3.58}$$

$$e_a(t,\mathbf{x}) = -V^{-1}\int \mathcal{D}^2\phi E_a[\phi,\phi^\dagger,t,\mathbf{x}];\ e_a^*(t,\mathbf{x}) = -V^{-1}\int \mathcal{D}^2\phi E_a^*[\phi,\phi^\dagger,t,\mathbf{x}]$$

The corresponding velocities are then

$$v_a[\mathbf{x}] = u_a[\mathbf{x}] + \rho^{-1}(e_a(t,\mathbf{x}) + E_a[\mathbf{x}]);\ v_a^*[\mathbf{x}] = u_a^*[\mathbf{x}] + \rho^{-1}(e_a^*(t,\mathbf{x}) + E_a^*[\mathbf{x}]) \tag{3.59}$$

Next introduce hidden variables $\phi(t,\mathbf{x})$ and $\phi^\dagger(t,\mathbf{x})$ with causal evolution given then by

$$\frac{\partial\phi_a(t,\mathbf{x})}{\partial t} = v_a[\phi,\phi^\dagger,t,\mathbf{x}];\ \frac{\partial\phi_a^*(t,\mathbf{x})}{\partial t} = v_a^*[\phi,\phi^\dagger,t,\mathbf{x}] \tag{3.60}$$

where it is understood that the right sides are calculated at $\phi(\mathbf{x}) = \phi(t,\mathbf{x})$ etc. In analogy with the bosonic fields treated earlier one introduces effectivity parameters guided by the wave function ψ_{n_P,n_A} given by

$$e_{n_P,n_A}[\phi,\phi^\dagger,t] = \frac{|\tilde{\Psi}_{n_P,n_A}[\phi,\phi^\dagger,t]|^2}{\sum_{n'_P,n'_A}|\tilde{\Psi}_{n'_P,n'_A}[\phi,\phi^\dagger,t]|^2} \tag{3.61}$$

REMARK 2.3.2. Concerning the nature of the effectivity parameter we extract from [**701**] as follows. In the bosonic theory the analogue of (3.61) is

$$e_n[\{\phi\},t] = \frac{|\tilde{\Psi}_n[\{\phi\}],t|^2}{\sum_{n'}|\tilde{\Psi}_{n'}[\{\phi\}],t|^2} \tag{3.62}$$

$\{\phi\} = \{\phi_1,\cdots,\phi_{N_s}\}$ where N_s is the number of different particle species. Now the measured effectivity can be any number between 0 and 1 and this is no contradiction since if different $\tilde{\Psi}_n$ in the expansion do not overlap in the ϕ space then they represent a set of nonoverlapping "channels" for the causally evolving field ϕ. The field necessarily enters one and only one of the channels and one sees that $e_n = 1$ for the nonempty channel with $e_{n'} = 0$ for all empty channels. The effect is the same as if the wave functional Ψ "collapsed" into one of the states $\tilde{\psi}_n$ with a definite number of particles. In a more general situation different $\tilde{\Psi}_n$ of the measured particles may overlap. However the general theory of ideal quantum measurements (cf. [**126**]) provides that the total wave functional can be written again as a sum of nonoverlappiing wave functionals in the $\{\phi\}$ space, where one of the fields represents the measured field, while the others represent fields of the measuring apparatus. Thus only one of the $\tilde{\Psi}$ in (3.62) becomes nonempty with the corresponding $e_n = 1$ while all the other $e_{n'} = 0$. The essential point is that from the point of view of an observer who does not know the actual field configuration the probability for such an effective collapse of the wave functional is exactly equal to the usual quantum mechanical probability for such a collapse. Hence the theory has the same statistical properties as the usual theory. In the case when all the effectivities are less than 1 (i.e. the wave functional has not collapsed) the theory does not agree nor disagree with standard theory; effectivity is a hidden variable. This agrees with the Bohmian particle positions which agree with the standard quantum theory only when the wave function effectively collapses into a state with a definite particle position. Similar comments apply to the fermionic picture. In an ideal experiment in which the number of particles is measured, different Ψ_{n_P,n_A} do not overlap in the $(\phi, \phi^\dagger)$ space and the fields ϕ, $\phi^\dagger$ necessarily enter into a unique "channel" $\tilde{\Psi}_{n_P,n_A}$, etc.

REMARK 2.3.3. In [**711**] one addresses the question of statistical transparency. Thus the probabilitistic interpretation of the nonrelativistic SE does not work for the relativistic KG equation $(\partial^\mu\partial_\mu + m^2)\psi = 0$ (where $x = (\mathbf{x}, t)$ and $\hbar = c = 1$) since $|\psi|^2$ does not correspond to a probability density. There is a conserved current $j^\mu = i\psi^* \overleftrightarrow{\partial^\mu}\psi$ (where $a\overleftrightarrow{\partial^\mu}b = a\partial^\mu b - b\partial^\mu a$) but the time component j^0 is not positive definite. In [**701, 703**] the equations that determine the Bohmain trajectories of relativistic quantum particles described by many particle wave functions were written in a form requiring a preferred time coordinate. However a preferred Lorentz frame is not necessare (cf. [**105**]) and this is developed in [**711**] following [**105, 703**]. First note that as in [**105, 703**] it appears that particles may be superluminal and the principle of Lorentz covariance does not forbid superluminal velocities and conversly superluminal velocities do not lead to causal paradoxes (cf. [**105, 711**]). As noted in [**105**] the Lorentz-covariant Bohmian interprtation of the many particle KG equation is not statistically transparent. This means that the statistical distribution of particle positions cannot be calculated in a simple way from the wave function alone without the knowledge of particle trajetories. One knows that classcal QM is statistically transparent of course and this perhaps helps to explain why Bohmian mechanics has not attracted more attention. However statistical transparency (ST) may not be a fundamental property of nature as the following facts suggest:

- Classical mechanics, relativistic or nonrelativistic, is not ST.

- Relativistic QM based on the KG equation (or some of its generalizations) is not ST.
- The relativistic Dirac equation is ST but its many particle relativistic generalization is not (unless a preferred time coordinate is determined in an as yet unknown dynamical manner).
- Nonrelativistic QM is ST but not completely so since it distinguishes the time variable (e.g. $\rho(x^1, x^2, t)$ is not a probability density).
- The background independent quantum gravity based on the Wheeler-DeWitt (WDW) equation lacks the notion of time and is not ST.

The upshot is that since statistical probabilities can be calculated via Bohmian trajectories that theory is more powerful than other interpretations of general QM (see [**711**] for discussion on this). Now let $\hat{\phi}(x)$ be a scalar field operator satisfying the KG equation (an Hermitian uncharged field for simplicity so that negative values of the time component of the current cannot be interpreted as negatively charged particles). The corresponding n-particle wave function is (cf. [**703**])

$$\psi(x_1, \cdots, x_n) = (n!)^{-1/2} S_{\{x_a\}} < 0|\hat{\phi}(x_1) \cdots \hat{\phi}(x_n)|n > \tag{3.63}$$

Here $S_{\{x_a\}}$ $(a = 1, \cdots, n)$ denotes the symmetrization over all x_a which is needed because the field operators do not commute for nonequal times. The wave function ψ satisfies n KG equations

$$(\partial_a^\mu \partial_{a\mu} + m^2)\psi(x_1, \cdots, x_n) = 0 \tag{3.64}$$

Although the operator $\hat{\phi}$ is Hermitian the nondiagonal matrix element ψ defined by (3.63) is complex and one can introduce n real 4-currrents $j_a^\mu = i\psi^* \overleftrightarrow{\partial_a^\mu} \psi$ each of which is separately conserved via $\partial_a^\mu j_{a\mu} = 0$. Equation (3.64) also implies

$$\left(\sum_a \partial_a^\mu \partial_{a\mu} + nm^2\right) \psi(x_1, \cdots, x_n) = 0 \tag{3.65}$$

and the separate conservation equations imply that $\sum_a \partial_a^\mu j_{a\mu} = 0$. Now write $\psi = Rexp(iS)$ with R and S real. Then (3.65) is equivalent to a set of two equations

$$\sum_a \partial_a^\mu (^2 \partial_{a\mu} S) = 0; \; -\frac{\sum_a (\partial_a^\mu S)(\partial_{a\mu} S)}{2m} + \frac{nm}{2} + Q = 0; \; Q = \frac{1}{2m} \frac{\sum_a \partial_a^\mu \partial_{a\mu} R}{2mR} \tag{3.66}$$

where Q is the quantum potential. The first equation is equivalent to a current conservation equation while the second is the quantum analogue of the relativistic HJ equation for n particles. The Bohmian interpretation consistists in postulating the existence of particle trajectories $x_a^\mu(s)$ satisfying $dx_a^\mu/ds = -(1/m)\partial_a^\mu s$ where s is an affine parameter along the n curves in the 4-dimensional Minkowski space. This equation has a form identical to the corresponding classical relativistic equation and can also be written as $dx_a^\mu/ds = j_a^\mu/2m\psi^*\psi$. Hence using $d/ds = \sum_a (dx_z^\mu/ds)\partial_{a\mu}$ one finds the equations of motion

$$m\frac{d^2 x_a^\mu}{ds^2} = \partial_a^\mu Q \tag{3.67}$$

Note that the equations above for the particle trajectories are nonlocal but still Lorentz covariant. The Lorentz covariance is a consequence of the fact that the trajectories in spacetime do not depend on the choice of affine parameter s (cf. [**105**]). Instead, by choosing n "initial" spacetime positions x_a, the n trajectories are uniquely determined by the vector fields j_a^μ or $-\partial_a^\mu S$ (i.e. the trajectories are integral curves of these vector fields). The nonlocality is encoded in the fact that the right hand side of (3.67) depends not only on x_a but also on all the other $x_{a'}$. This is a consequence of the fact that $Q(x_1,\cdots,x_n)$ in (3.66) is not of the form $\sum_a Q_a(x_1,\cdots,x_n)$, which in turn is related to the fact that $S(x_1,\cdots,x_n)$ is not of the form $\sum_a S(x_a)$. Note also that the fact that we parametrize all trajectories with the same parameter s is not directly related to the nonlocality, because such a parametrization can be used even in local classical physics. When the interactions are local then one can even use another parameter s_a for each curve but when the interactions are not local one must use a single parameter s; new separate parameters could only be used after the equations are solved. In the nonrelativistic limit all wave function frequencies are (approximately) equal to m so from $j_a^\mu \psi^* \overleftrightarrow{\partial_a^\mu}\psi$ all time components are equal and given by $j_a^0 = 2m\psi^*\psi = \tilde{\rho}$ which does not depend on a. Writing then $\rho(\mathbf{x}_1,\cdots,\mathbf{x}_n) = \tilde{\rho}(x_1,\cdots,x_n)|_{t_1=\cdots=t_n=t}$ one obtains $\partial_t\rho + \sum_a \partial_a^i j_{ai} = 0$ and this implies that ρ can be interpreted as a probability density. In the full relativistic there is generally no analogue of such a function ρ. We refer to [**711**] for more discussion.

4. DeDONDER, WEYL, AND BOHM

We go here to a fascinating paper [**708**] which gives a manifestly covariant canonical method of field quantization based on the classical DeDonder-Weyl (DW) formulation of field theory (cf. also Appendix A for some background on DW theory following [**586**]). The Bohmian formulation is not postulated for intepretational purposes here but derived from purely technical requirements, namely covariance and consistency with standard QM. It arises automatically as a part of the formalism without which the theory cannot be formulated consistently. This together with the results of [**701, 711**] suggest that it is Bohmian mechanics that might be the missing bridge between QM and relativity; further (as will be seen later) it should play an important role in cosmology. The classical covariant canonical DeDonder-Weyl formalism is given first following [**586**] and for simplicity one real scalar field in Minkowski spacetime is used. Thus let $\phi(x)$ be a real scalar field described by

$$\mathfrak{A} = \int d^4x\mathfrak{L};\ \mathfrak{L} = \frac{1}{2}(\partial^\mu\phi)(\partial_\mu\phi) - V(\phi) \tag{4.1}$$

As usual one has

$$\pi^\mu = \frac{\partial\mathfrak{L}}{\partial(\partial_\mu\phi)} = \partial^\mu\phi;\ \partial_\mu\phi = \frac{\partial\mathfrak{H}}{\partial\pi^\mu};\ \partial_\mu\pi^\mu = -\frac{\partial\mathfrak{H}}{\partial\phi} \tag{4.2}$$

where the scalar DeDonder-Weyl (DDW) Hamilonian (not related to the energy density) is given by the Legendre transform $\mathfrak{H}(\pi^\mu,\phi) = \pi^\mu\partial_\mu\phi - \mathfrak{L} = (1/2)\pi^\mu\pi_\mu + V$. The equations (4.2) are equivalent to the standard Euler-Lagrange (EL) equations and by introducing the local vector $S^\mu(\phi(x),x)$ the dynamics can also be described by the covariant DDW HJ equation and equations of motion

$$\mathfrak{H}\left(\frac{\partial S^{\alpha}}{\partial\phi},\phi\right)+\partial_{\mu}S^{\mu}=0;\ \partial^{\mu}\phi=\pi^{\mu}=\frac{\partial S^{\mu}}{\partial\phi} \tag{4.3}$$

Note here ∂_μ is the partial derivative acting only on the second argument of $S^\mu(\phi(x),x)$; the corresonding total derivative is $d_\mu = \partial_\mu + (\partial_\mu\phi)(\partial/\partial\phi)$. Note that the first equation in (4.3) is a single equation for four quantities S^μ so there is a lot of freedom in finding solutions. Nevertheless the theory is equivalent to other formulations of classical field theory. Now following [**533**] one considers the relation between the covariant HJ equation and the conventional HJ equation; the latter can be derived from the former as follows. Using (4.2), (4.3) takes the form $(1/2)\partial_\phi S_\mu \partial_\phi S^\mu + V + \partial_\mu S^\mu = 0$. Then using the equation of motion in (4.3) write the first term as

$$\frac{1}{2}\frac{\partial S_{\mu}}{\partial\phi}\frac{\partial S^{\mu}}{\partial\phi}=\frac{1}{2}\frac{\partial S^{0}}{\partial\phi}\frac{\partial S^{0}}{\partial\phi}+\frac{1}{2}(\partial_{i}\phi)(\partial^{i}\phi) \tag{4.4}$$

Similarly using (4.3) the last term is $\partial_\mu S^\mu = \partial_0 S^0 + d_i S^i - (\partial_i\phi)(\partial^i\phi)$. Now introduce the quantity $\mathfrak{S} = \int d^3x S^0$ so $[\partial S^0(\phi(x),x)/\partial\phi(x)] = [\delta\mathfrak{S}([\phi(\mathbf{x},t)],t)/\delta\phi(\mathbf{x},t)]$ where $\delta/\delta\phi(\mathbf{x},t) \equiv [\delta/\delta\phi(x)]_{\phi(x)=\phi(\mathbf{x},t)}$ is the space functional derivative. Putting this together gives then

$$\int d^{3}x\left[\frac{1}{2}\left(\frac{\delta\mathfrak{S}}{\delta\phi(\mathbf{x},t)}\right)^{2}+\frac{1}{2}(\nabla\phi)^{2}+V(\phi)\right]+\partial_{t}\mathfrak{S}=0 \tag{4.5}$$

which is the standard noncovariant HJ equation. The time evolution of $\phi(\mathbf{x},t)$ is given by $\partial_t\phi(\mathbf{x},t) = \delta\mathfrak{S}/\delta\phi(\mathbf{x},t)$ which arises from the time component of (4.3). Note that in deriving (4.5) it was necessary to use the space part of the equations of motion (4.3) (this does not play an important role in classical physics but is important here). Now for the Bohmian formulation look at the SE $\hat{H}\Psi = i\hbar\partial_t\Psi$ where we write

$$\hat{H}=\int d^{3}x\left[-\frac{\hbar^{2}}{2}\left(\frac{\delta}{\delta\phi(\mathbf{x})}\right)^{2}+\frac{1}{2}(\nabla\phi)^{2}+V(\phi)\right]; \tag{4.6}$$

$$\Psi([\phi(\mathbf{x})],t)=\mathfrak{R}([\phi(\mathbf{x})],t)e^{i\mathfrak{S}(([\phi(\mathbf{x})],t)/\hbar}$$

Then the complex SE equation is equivalent to two real equations

$$\int d^{3}x\left[\frac{1}{2}\left(\frac{\delta\mathfrak{S}}{\delta\phi(\mathbf{x})}\right)^{2}+\frac{1}{2}(\nabla\phi)^{2}+V(\phi)+Q\right]+\partial_{t}S=0; \tag{4.7}$$

$$\int d^{3}x\left[\frac{\delta\mathfrak{R}}{\delta\phi(\mathbf{x})}\frac{\delta\mathfrak{S}}{\delta\phi(\mathbf{x})}+J\right]+\partial_{t}\mathfrak{R}=0;\ Q=-\frac{\hbar^{2}}{2\mathfrak{R}}\frac{\delta^{2}\mathfrak{R}}{\delta\phi^{2}(\mathbf{x})};\ J=\frac{\mathfrak{R}}{2}\frac{\delta^{2}\mathfrak{S}}{\delta\phi^{2}(\mathbf{x})}$$

The second equation is also equivalent to

$$\partial_{t}\mathfrak{R}^{2}+\int d^{3}x\frac{\delta}{\delta\phi(\mathbf{x})}\left(\mathfrak{R}^{2}\frac{\delta\mathfrak{S}}{\delta\phi(\mathbf{x})}\right)=0 \tag{4.8}$$

and this exhibits the unitarity of the theory because it provides that the norm $\int[d\phi(\mathbf{x})]^2\Psi^*\Psi = \int[d\phi(\mathbf{x})]\mathfrak{R}^2$ does not depend on time. The quantity $\mathfrak{R}^2([\phi(\mathbf{x})],t)$ represents the probability density for fields to have the configuration $\phi(\mathbf{x})$ at time t.

One can take (4.7) as the starting point for quantization of fields (note $exp(i\mathfrak{S}/\hbar)$ should be single valued). Equations (4.7) and (4.8) suggest a Bohmian interpretation with deterministic time evolution given via $\partial_t \phi$. Remarkably the statistical predictions of this deterministic interpretation are equivalent to those of the conventional interpretation. All quantum uncertainties are a consequence of the ignorance of the actual initial field configuration $\phi(\mathbf{x}, t_0)$. The main reason for the consistency of this interpretation is the fact that (4.8) with $\partial_t \phi$ as above represents the continuity equation which provides that the statistical distribution $\rho([\phi(\mathbf{x})], t)$ of field configurations $\phi(\mathbf{x})$ is given by the quantum distribution $\rho = \mathfrak{R}^2$ at any time t, provided that ρ is given by $\mathfrak{R}^2$ at some initial time. The initial distribution is arbitrary in principle but a quantum H theorem explains why the quantum distribution is the most probable (cf. [**954**]). Comparing (4.7) with (4.5) we see that the quantum field satisfies an equation similar to the classical one, with the addition of a term resulting from the nonlocal quantum potential Q. The quantum equation of motion then turns out to be

$$\partial^\mu \partial_\mu \phi + \frac{\partial V(\phi)}{\partial \phi} + \frac{\delta \mathfrak{Q}}{\delta \phi(\mathbf{x}; t)} = 0 \tag{4.9}$$

where $\mathfrak{Q} = \int d^3x Q$. A priori perhaps the main unattractive feature of the Bohmian formulation appears to be the lack of covariance, i.e. a preferred Lorentz frame is needed and this can be remedied with the DDW presentation to follow.

Thus one wants a quantum substitute for the classical covariant DDW HJ equation $(1/2)\partial_\phi S_\mu \partial_\phi S^\mu + V + \partial_\mu S^\mu = 0$. Define then the derivative

$$\frac{dA([\phi], x)}{d\phi(x)} = \int d^4x' \frac{\delta A([\phi], x')}{\delta \phi(x)} \tag{4.10}$$

where $\delta/\delta\phi(x)$ is the spacetime functional derivative (not the space functional derivative used before in (4.5)). In particular if $A([\phi], x)$ is a local functional, i.e. if $A([\phi], x) = A(\phi(x), x)$ then

$$\frac{dA(\phi(x), x)}{d\phi(x)} = \int d^4x' \frac{\delta A(\phi(x'), x')}{\delta \phi(x)} = \frac{\partial A(\phi(x), x)}{\partial \phi(x)} \tag{4.11}$$

Thus $d/d\phi$ is a generalization of $\partial/\partial\phi$ such that its action on nonlocal functionals is also well defined. An example of interest is a functional nonlocal in space but local in time so that

$$\frac{\delta A([\phi], x')}{\delta \phi(x)} = \frac{\delta A([\phi], x')}{\delta \phi(\mathbf{x}), x^0)} \delta((x')^0 - x^0) \Rightarrow \tag{4.12}$$
$$\Rightarrow \frac{dA([\phi], x)}{d\phi(x)} = \frac{\delta}{\delta \phi(\mathbf{x}, x^0)} \int d^3x' A([\phi], \mathbf{x}', x^0)$$

Now the first equation in (4.3) and the equations of motion become

$$\frac{1}{2} \frac{dS_\mu}{d\phi} \frac{dS^\mu}{d\phi} + V + \partial_\mu S^\mu = 0; \;\; \partial^\mu \phi = \frac{dS^\mu}{d\phi} \tag{4.13}$$

which is appropriate for the quantum modification. Next one proposes a method of quantization that combines the classical covariant canonical DDW formalism with the standard specetime asymmetric canonical quantization of fields. The starting point is the relation between the noncovariant classical HJ equation (4.5) and its quantum analogue (4.7). Suppressing the time dependence of the field in (4.5) we see that they differ only in the existence of the Q term in the quantum case. This suggests the following quantum analogue of the classical covariant equation (4.13)

$$\frac{1}{2}\frac{dS_\mu}{d\phi}\frac{dS^\mu}{d\phi}+V+Q+\partial_\mu S^\mu=0 \tag{4.14}$$

Here $S^\mu = S^\mu([\phi],x)$ is a functional of $\phi(x)$ so S^μ at x may depend on the field $\phi(x')$ at all points x'. One can also allow for time nonlocalities (cf. [**711**]). Thus (4.15) is manifestly covariant provided that Q given by (4.7) can be written in a covariant form. The quantum equation (4.14) must be consistent with the conventional quantum equation (4.7); indeed by using a similar procedure to that used in showing that (4.3) implies (4.5) one can show that (4.14) implies (4.7) provided that some additional conditions are fulfilled. First S^0 must be local in time so that (4.12) can be used. Second S^i must be completely local so that $dS^i/d\phi=\partial S^i/\partial\phi$, which implies

$$d_iS^i=\partial_iS^i+(\partial_i\phi)\frac{dS^i}{d\phi} \tag{4.15}$$

However just as in the classical case in this procedure it is necessary to use the space part of the equations of motion (4.3). Therefore these classical equations of motion must be valid even in the quantum case. Since we want a covariant theory in which space and time play equal roles the validity of the space part of the (4.3) implies that its time part should also be valid. Consequently in the covariant quantum theory based on the DDW formalism one must require the validity of the second equation in (4.13). This requirement is nothing but a covariant version of the Bohmian equation of motion written for an arbitrarily nonlocal S^μ (this clarifies and generalizes results in [**533**]). The next step is to find a covariant substitute for the second equation in (4.7). One introduces a vector $R^\mu([\phi],x)$ which will generate a preferred foliation of spacetime such that the vector R^μ is normal to the leaves of the foliation. Then define

$$\mathfrak{R}([\phi],\Sigma)=\int_\Sigma d\Sigma_\mu R^\mu;\ \ \mathfrak{S}([\phi],x)=\int_\Sigma d\Sigma_\mu S^\mu \tag{4.16}$$

where Σ is a leaf (a 3-dimensional hypersurface) generated by R^μ. Hence the covariant version of $\Psi=\mathfrak{R}exp(i\mathfrak{S})$ is $\Psi([\phi],\Sigma)=\mathfrak{R}([\phi],\Sigma)exp(i\mathfrak{S}([\phi],\Sigma)/\hbar)$. For R^μ one postulates the equation

$$\frac{dR^\mu}{d\phi}\frac{dS^\mu}{d\phi}+J+\partial_\mu R^\mu=0 \tag{4.17}$$

In this way a preferred foliation emerges dynamically as a foliation generated by the solution R^μ of the equations (4.17) and (4.14). Note that R^μ does not play any role in classical physics so the existence of a preferred foliation is a purely quantum effect. Now the relation between (4.17) and (4.7) is obtained by assuming that nature has chosen a solution of the form $R^\mu=(R^0,0,0,0)$ where R^0 is local in time. Then integrating (4.17) over d^3x and assuming again that S^0 is local in time one obtains (4.7). Thus (4.17) is a covariant substitute for the second equation in (4.7). It remains to write covariant versions for Q and J and these are

$$Q = -\frac{\hbar^2}{2\mathfrak{R}}\frac{\delta^2\mathfrak{R}}{\delta_\Sigma\phi^2(x)};\ J = \frac{\mathfrak{R}}{2}\frac{\delta^2\mathfrak{S}}{\delta_\Sigma\phi^2(x)} \tag{4.18}$$

where $\delta/\delta_\Sigma\phi(x)$ is a version of the space functional derivative in which Σ is generated by R^μ. Thus (4.17) and (4.14) with (4.18) represent a covariant substitute for the functional SE equivalent to (4.8). The covariant Bohmain equations (4.13) imply a covariant version of (4.9), namely

$$\partial^\mu\partial_\mu\phi + \frac{\partial V}{\partial\phi} + \frac{dQ}{d\phi} = 0 \tag{4.19}$$

Since the last term can also be written as $\delta(\int d^4xQ)/\delta\phi(x)$ the equation of motion (4.19) can be obtained by varying the quantum action

$$\mathfrak{A}_Q = \int d^4x\mathfrak{L}_Q = \int d^4x(\mathfrak{L} - Q) \tag{4.20}$$

Thus in summary the covariant canonical quantization of fields is given by equations (4.13), (4.14), (4.17), and (4.18). The conventional functional SE corresponds to a special class of solutions for which $R^i = 0$, S^i are local, while R^0 and S^0 are local in time. In [**708**] a multifield generalization is also spelled out, a toy model is considered, and applications to quantum gravity are treated. The main result is that a manifestly covariant method of field quantization based on the DDW formalism is developed which treats space and time on an equal footing. Unlike the conventional canonical quantization it is not formulated in terms of a single complex SE but in terms of two coupled real equations. The need for a Bohmian formulation emerges from the requirement that the covariant method should be consistent with the conventional noncovariant method. This suggests that Bohmian mechanics (BM) might be a part of the formalism without which the covariant quantum theory cannot be formulated consistently.

5. QFT AND STOCHASTIC JUMPS

The most extensive treatment of Bohmian theory is due to a group based in Germany, Italy, and the USA consisting of V. Allori, A. Barut, K. Berndl, M. Daumer, D. Dürr, H. Georgi, S. Goldstein, J. Lebowitz, S. Teufel, R. Tumulka, and N. Zanghi (cf. [**1, 26, 88, 102, 103, 104, 105, 288, 324, 325, 326, 327, 328, 329, 330, 414, 415, 437, 417, 418, 415, 416, 417, 418, 419, 927, 928, 948**]). There is also of course the pioneering work of deBroglie and Bohm (see e.g. [**154, 126, 127, 128, 129, 154**]) as well as important work of many other people (cf. [**68, 94, 95, 110, 138, 148, 164, 165, 166, 186, 187, 188, 189, 191, 197, 198, 236, 277, 295, 298, 305, 306, 346, 347, 373, 374, 375, 438, 472, 471, 474, 478, 479, 480, 873, 905, 953, 961**]). We make no attempt to survey the philosophy of Bohmian mechanics (BM), or better deBroglie-Bohm theory (dBB theory), here. This involves many issues, some of them delicate, which are discussed at length in the references cited. The book [**111**] by Holland provides a good beginning and in view of recent work perhaps another book on this subject alone would be welcome. There is a lot of associated "philosophy", involving hidden variables, nonlocality, EPR ideas, wave function collapse, pilot waves, implicate order, measurement problems, decoherence, etc., much of which has been resolved or might well be forgotten. Many matters are indeed clarified already in the literature above (cf. in particular [**94, 95, 330, 415, 471**]) and we will not belabor philosophical matters. It may well be that a completely unified mathematical theory is beyond reach at the moment but thre are already quite accurate and workable models available and the philosophy of dBB theory as developed by the American-German-Italian school mentioned is quite sophisticated and convincing.

Basically, following [**415**], for the nonrelativistic theory, GM for N particles is determined by the two equations

$$i\hbar\psi_t = H\psi;\ \frac{dq_k}{dt} = \frac{\hbar}{2m_k}\Im\left[\frac{\psi^*\partial_k\psi}{\psi^*\psi}\right] \tag{5.1}$$

The latter equation is called the guidance or pilot equation which choreographs the motion of the particles. If ψ is spinor valued the products in the numerator and denominator are scalar products and if external magnetic fields are present the gradient $\nabla \sim (\partial_k)$ should be understood as the covariant derivative involving the vector potential (thus accomodating some versions of field theory - more on this later). Since the denominator vanishes at nodes of ψ existence and uniqueness of solutions for Bohmian dynamics is nontrivial but this is proved in [**104, 106**]. This formula extends to spin and the right side corresponds to J/ρ which is the ratio of the quantum probability current to the quantum probability density. Further from the quantum continuity condition $\partial_t\rho + div(J) = 0$ (derivable from the SE) it follows that if the configuration of particles is random at the initial time t_0 with probability distribution $\psi^*\psi$ then this remains true for all times (assuming no interaction with the environment). Upon setting $\psi = Rexp(iS/\hbar)$ one identifies $p_k = m_k v_k$ with $\partial_k S$ (which is equivalent to the guiding equation for particles without spin) and this corresponds to particles being acted upon by the force $\partial_k Q$ generated by the quantum potential (in addition to any "classical" forces).

REMARK 2.5.1. Recall from Section 2.3.2 that in the BFM theory of Bohmian type $\dot{q} = p/m_Q \neq p/m$ where $m_Q = m(1-\partial_E Q)$ in stationary situations with energy E. Here one is using a Floydian time and there has been a great deal of discussion, involving e.g. tunneling times (see e.g. [**138, 139, 140, 191, 194, 197, 198, 305, 306, 307, 296, 309, 347, 373, 374, 375, 376, 520**]). We do not attempt to resolve any issues here and refer to the references for up to date information.

In any event we proceed with BM or dBB theory in full confidence not only that it works but that it is probably the best way to look at QM. We regard the quantum potential Q as being a quantization vehicle which expresses the influence of quantum fluctuations (cf. Chapters 1,4,5); it also arises in describing Weyl curvature (cf. Chapters 4,5) and thus we regard it as perhaps the fundamental object of QM. Returning now to [**415**] one notes that the predictions of

BM for measurements must agree with those of standard QM provided configurations are random with distributions given by the quantum equilibrium distribution $|\psi|^2$. Then a probability distribution ρ^ψ depending on ψ is called equivariant if $(\rho^\psi)_t = \rho^{\psi_t}$ where the right side comes from the SE and the left from the guiding equation (since $\rho_t + div(J) = 0$ with $v = J/\rho$ arises in (5.1)). This has been studied in detail and we summarize some results below. Further BM can handle spin via (5.1) (as mentioned above) and nonlocality is no problem; however Lorentz invariance, even for standard QM, is tricky and one views it as an emergent symmetry. Further QFT with particle creation and annihilation is a current topic of research (cf. Sections 2.3 and 2.4) and some additional remarks in this direction will follow. The papers [**324, 325**] are mainly about quantum equilibrium, absolute uncertainty, and the nature of operators. There are two long papers here (75 and 77 pages) and an earlier paper of 35 pages so we make no attempt to cover this here. We mention briefly some results of the two more recent papers however. Thus from the abstract to the second paper of [**324**] the quantum formalism is treated as a measurement formalism, i.e. a phenomenological formalism describing certain macroscopic regularities. One argues that it can be regarded and best be understood as arising from Bohmian mechanics, which is what emerges from the SE for a system of particles when one merely insists that "particles' means particles. BM is a fully deterministic theory of particles in motion, a motion choreographed by the wave function. One finds that a Bohmian universe, although deterministic, evolves in such a manner that an appearance of randomness emerges, precisely as described by the quantum formalism and given by $\rho = |\psi|^2$. A crucial ingredient in the analysis of the origin of this randomness is the notion of the effective wave function of a subsystem. When the quantum formalism is regarded as arising in this way the paradoxes and perplexities so often associated with (nonrelativistic) quantum theory evaporate. A fundamental fact here is that given a SE $i\hbar\psi_t = -(\hbar^2/2)\sum(\Delta_k\psi/m_k)+V\psi$ one can derive a velocity formula $v_k^\psi = (\hbar/m_k)\Im(\nabla_k\psi/\psi)$ by general arguments based on symmetry considerations and this yields (5.1) without any recourse to a formula $\psi = Rexp(iS/\hbar)$. Further the continuity equation $\rho_t + div(\rho v^\psi) = 0$ holds and this implies the equivariance $\rho(q,t) = |\psi(q,t)|^2$ provided this is true at (t_0, q_0). The distribution $\rho = |\psi|^2$ is called the quantum equilibrium distribution (QELD) and a system is in quantum equilibrium when the QELD is appropriate for its description. The quantum equilibrium hypothesis (QEH) is that if a system has wave function ψ then $\rho = |\psi|^2$. It is necessary to discuss wave functions of systems and subsystems at some length and it is argued that in a universe governed by BM it is impossible to know more about the configuration of any subsystem than what is expressed via $\rho = |\psi|^2$ (despite the fact that for BM the actual configuration is an objective property, beyond the wave function). Moreover, this uncertainty, of an absolute and precise character, emerges with complete ease, the structure of BM being such that it allows for the formulation and clean demonstration of statistical statements of a purely objective character which nontheless imply the claims concerning the irreducible limitations on possible knowledge, whatever this knowledge may precisely mean and however one might attempt to obtain this knowledge, provided it is consistent with BM. This limitation on what can be known is called absolute uncertainty. One proceeds by analysis of systems and subsystems and we refer to [**324**] for details. In [**325**] one shows how the entire quantum formalism, operators as observables, etc. naturally emerges in BM from the analysis of measurments. It is however quite technical, with considerable important and delicate reasoning, and we cannot possibly deal with it in a reasonable number of pages.

We go to [**326**] now where a comprehensive theory is developed for Bohmian mechanics and QFT (cf. also [**330**]). Bohm and subsequently Bell had proposed such models and the latter is model is modified and expanded in [**326, 330**] in the context of what are called Bell models. One will treat the configuration space variables in terms of Markov processes with jumps (which is reminiscent of the diffusion picture in [**673, 674**] (cf. also [**186**]). Roughly one thinks of world lines involving particle creation and annihilation, hence jumps, and writes $\mathfrak{Q} = \cup_0^\infty \mathfrak{Q}^n$ where, taking identical particles, the sector $\mathfrak{Q}^n$ is best defined as $\mathbf{R}^{3n}/S_n$ where $S \sim$ permutations. For several particle species one forms several copies of $\mathfrak{Q}$, one for each species, and obtains a union of sectors $\mathfrak{Q}^{(n)}$ where now $n \sim (n_1, \ldots, n_k)$ for the k species of particles. Note that a path $Q(t)$ will typically have discontinuities, even if there is nothing discontinuous in the world line pattern, because it jumps to a different sector at every creation or annihilation event. One can think of the bosonic Fock space as a space of L^2 functions on $\cup_n \mathbf{R}^{3n}/S_n$ with the fermionic Fock space being L^2 functions on $\cup_n \mathbf{R}^{3n}$, antisymmetric under permutation. A Bell type QFT specifies such world line patterns or histories in configuration space by specifying three sorts of “laws of motion”: when to jump, where to jump, and how to move between jumps. One consequence of these laws (to be enumerated) is the property of preservation of $|\Psi_0|^2$ at time t_0 to be equal to $|\Psi_t|^2$ at time t; this is called equivariance (see above and cf. [**325, 327**] for more detail on equivariance for Bohmian mechanics - the same sort of reasoning will apply here). One will use the quantum state vector Ψ to determine the laws of motion and here a state described by the pair (Ψ_t, Q_t) where Ψ evolves according to the SE $i\hbar\partial_t\Psi_t = H\Psi$. Typically $H = H_0 + H_I$ and it is important to note that although there is an aactual particle number $N(t) = \#Q(t)$ or $Q(t) \in \mathfrak{Q}^{N(t)}$, Ψ need not be a number eigenstate (i.e. concentrated in one sector). This is similar to the usual double-slit experiment in which the particle passes through only one slit although the wavefunction passes through both. As with this experiment, the part of the wave function that passes through another sector of $\mathfrak{Q}$ (or another slit) may well influence the behavior of $Q(t)$ at a later time. The laws of motion of Q_t depend on Ψ_t (and on H) and the continuous part of the motion is governed by

$$\text{(5.2)} \qquad \frac{dQ_t}{dt} = v^{\Psi_t}(Q_t) = \Re\frac{\Psi_t^*(Q_t)(\dot{\hat{q}}\Psi_t)(Q_t)}{\Psi_t^*(Q_t)\Psi_t(Q_t)};\ \dot{\hat{q}} = \frac{d}{d\tau}e^{iH_0\tau/\hbar}\Big|_{\tau=0} = \frac{i}{\hbar}[H_0, \hat{q}]$$

Here $\dot{\hat{q}}$ is the time derivative of the $\mathfrak{Q}$ valued Heisenberg position operator $\hat{q}$ evolved with H_0 alone. One should understand this as saying that for any smooth function $f : \mathfrak{Q} \to \mathbf{R}$

$$\text{(5.3)} \qquad \frac{df(Q_t)}{dt} = \Re\frac{\Psi_t^*(Q_t)(i/\hbar)[H_0, \hat{f}]\Psi_t)(Q_t)}{\Psi_t^*(Q_t)\Psi_t(Q_t)}$$

This expression is of the form $v^\Psi \cdot \nabla f(Q_t)$ (as it must be for defining a dynamics for Q_t) if the free Hamiltonian is a differential operator of up to second order (more on this later). **Note that the KG equation is not covered by** (5.2) **or** (5.3). The numerator and denominators above involve, when appropriate, scalar products in spin space. One may view v as a vector field on $\mathfrak{Q}$ and thus as consisting of one vector field v^n on every manifold $\mathfrak{Q}^n$; it is then $v^{N(t)}$ that governs the motion of $Q(t)$ in (5.2). If H_0 were the Schrödinger operator $-\sum_1^n(\hbar^2/2m)\Delta_i + V$ (5.2) yields the Bohm velocities $v_i^\Psi = (\hbar/m_i)\Im[\Psi^*\nabla_i\Psi/\Psi^*\Psi]$. When H_0 is the "second quantization" of a 1-particle Schrödinger operator (5.2) involves equal masses in every sector $\mathfrak{Q}^n$. Similarly in case H_0 is the second quantization of the Dirac opertor $-ic\hbar\vec{\alpha}\cdot\nabla + \beta mc^2$ (5.2) says that a configuration $Q(t)$ (with N particles) moves according to (the N-particle version of) the known variant of Bohm's velocity formula for Dirac wavefunctions $v^\Psi = (\Psi^*\alpha\Psi/\Psi^*\Psi)c$ (cf. [**127**]). The jumps now are stochastic in nature, i.e. they occur at random times and lead to random destinations. In Bell type QFT God does play dice. There are no hidden variables which would fully determine the time and destination of a jump (cf. here Section 2.3 and the effectivity parameters). The probability of jumping, with the next dt seconds to the volume dq in $\mathfrak{Q}$ is $\sigma^\Psi(dq|Q_t)dt$ with

$$(5.4)\qquad \sigma^\Psi(dq|q') = \frac{2}{\hbar}\frac{[\Im\Psi^*(q) < q|H_I|q' > \Psi(q')]^+}{\Psi^*(q')\Psi(q')}dq$$

where $x^+ = max(x,0)$. Thus the jump rate σ^Ψ depends on the present configuration Q_t, on the state vector Ψ_t which has a guiding role similar to that in the Bohm theory, and of course on the overall setup of the QFT as encoded in the interaction Hamiltonian H_I (cf. [**326**] for a simple example). There is a striking similarity between (5.4) and (5.2) in that they are both cases of "minimal" Markov processes associated with a given Hamiltonian (more on this below). When H_0 is replaced by H_I in the right side of (5.3) one obtains an operator on functions $f(q)$ that is naturally associated with the process defined by the jump rates (5.4).

The field operators (operator valued fields on spacetime) provide a connection, the only connection in fact, between spacetime and the abstract Hilbert space containing the quantum states $|\Psi>$, which are usually regarded not as functions but as abstract vectors. What is crucial now is that (i) The field operators naturally correspond to the spatial structure provided by a projection valued (PV) measure on configuration space $\mathfrak{Q}$, and (ii) The process defined here can be efficiently expressed in terms of a PV measure. Thus consider a PV measure P on $\mathfrak{Q}$ acting on $\mathcal{H}$ where for $B \subset \mathfrak{Q}$, $P(B)$ means the projection to the space of states localized in B. Then one can rewrite the formulas above in terms of P and $|\Psi>$ and we get

$$(5.5)\qquad \frac{df(Q_t)}{dt} = \Re\frac{< \Psi|P(dq)\frac{i}{\hbar}[H_0,\hat{f}]|\Psi >}{< \Psi|P(dq)|\Psi >}\Bigg|_{q=Q_t};\ \hat{f} = \int_{q\in\mathfrak{Q}} f(q)P(dq)$$

(for smooth functions $f : \mathfrak{Q} \to \mathbf{R}$) and

$$(5.6)\qquad \sigma^\Psi(dq|q') = \frac{2}{\hbar}\frac{\Im < \Psi|P(dq)H_IP(dq')|\Psi >]^+}{< \Psi|P(dq')|\Psi >}$$

Note that $< \Psi|P(dq)|\Psi >$ is the probability distribution analogous to the standard $|\Psi(q)|^2 dq$. The next question is how to obtain the PV measure P from the field operators. Such a measure is equivalent to a system of number operators (more on this below); thus an additive operator valued set function $N(R),\ R \in \mathbf{R}^3$ such that the $N(R)$ commute pairwise and have spectra in the nonnegative integers. By virtue of the canonical commutation and anticommutation relations for the field operators $\phi(\mathbf{x})$ the easiest way to obtain such a system of number operators is via $N(R) = \int_R \phi^*(\mathbf{x})\phi(\mathbf{x})d^3\mathbf{x}$. Thus what one needs from a QFT in order to construct trajectories are: (i) a Hilbert space $\mathcal{H}$ (ii) a Hamiltonian $H = H_0 + H_I$ (iii) a configuration space $\mathfrak{Q}$ (or measurable space), and (iv) a PV measure on $\mathfrak{Q}$ acting on $\mathcal{H}$. This will be done below following [**326**].

We go now to the last paper in [**326**] which is titled quantum Hamiltonians and stochastic jumps. The idea is that for the Hamiltonian of a QFT there is associated a $|\Psi|^2$ distributed Markov process, typically a jump process, on the configuration space of a variable number of particles. A theory is developed generalizing work of J. Bell and the authors of [**326**]. The central formula of the paper is

$$\sigma(dq|q') = \frac{[(2/\hbar)\Im < \Psi|P(dq)HP(dq')|\Psi >]^+}{< \Psi|P(dq')|\Psi >} \tag{5.7}$$

It plays a role similar to that of Bohm's equation of motion

$$\frac{dQ}{dt} = v(Q);\ v = \hbar\Im\frac{\Psi^*\nabla\Psi}{\Psi^*\Psi} \tag{5.8}$$

Together these two equations make possible a formulation of QFT that makes no reference to observers or measurements, while implying that observers, when making measurements, will arrive at precisely the results that QFT is known to predict. This formulation takes up ideas from the seminal papers of J. Bell [**94, 95**] and such theories will be referred to as Bell-type QFT's. The aim is to present methods for constructing a canonical Bell type model for more of less any regularized QFT. One assumes a well defined Hamiltonian as given (with cutoffs included if needed). The primary variables of such theories are particle positions and Bell suggested a dynamical law governing the motion of the particles in which the Hamiltonian H and the state vector Ψ determine the jump rates σ. These rates are in a sense the smallest choice possible (explained below) and are called minimal jump rates; they preserve the $|\Psi|^2$ distribution. Bell type QFT's can also be regarded as extensions of Bohmian mechanics which cover particle creation and annihilation; the quantum equilibrium distribution more or less dictates that creation of a particle occurs in a stochastic manner as in the Bell model. We recall that for Bohmian mechanics in addition to (5.8) one has an evolution equation $i\hbar\partial_t\Psi = H\Psi$ for the wave function with $H = -(\hbar^2/2\Delta + V$ for spinless particles $(\Delta = div\nabla)$. For particles with spin Ψ takes values in the appropriate spin space $\mathbf{C}^k$, V may be matrix valued, and inner products in (1.66) are understood as involving inner products in spin spaces. The success of the Bohmian method is based on the preservation of $|\Psi|^2$, called equivariance and this follows immediately from comparing the continuity equation for a probability distribution ρ associated with (5.8), namely $\partial_t\rho = -div(\rho v)$, with the equation for $|\Psi|^2$ following from the

SE, namely

$$\partial_t|\Psi|^2(q,t) = (2/\hbar)\Im[\Psi^*(q,t)(H\Psi)(q,t)] \tag{5.9}$$

In fact it follows from the continuity equation that

$$(2/\hbar)\Im[\Psi^*(q,t)(H\Psi)(q,t)] = -div[\hbar\Im\Psi^*(q,t)\nabla\Psi(q,t)] \tag{5.10}$$

so recalling (5.8), one has $\partial_t|\Psi|^2 = -div(|\Psi|^2 v)$, and hence if $\rho+|\Psi_t|^2$ as some time t there results $\rho = |\psi_t|^2$ for all times. One is led naturaly to the consideration of Markov processes as candidates for the equivariant motion of the configuration Q for more general Hamiltonians (see e.g. [**506, 674, 810, 815**] for Markov processes - [**674**] is especially good for Markov processes with jumps and dynamics but we follow [**506, 815**] for background since the ideas are more or less clearly stated without a deathly deluge of definitions and notation - of course for a good theory much of the verbiage is actually important).

DEFINITION 5.1. Let ($\mathfrak{E}$ be a Borel σ-algebra of subsets of E. For Ω generally a path space (e.g. $\Omega \sim C(\mathbf{R}^+, E)$ with $X_t(\omega) = X(t,\omega) = \omega(t)$ and $\mathfrak{F}_t = \sigma(X_s(\omega),\, s \le t)$ a filtration by Borel sub σ-algebras) a Markov process (**C11**) $X = (\Omega, \{\mathfrak{F}_t\}, \{X_t\}, \{P_t\}, \{P^x,\, x\}E\})$ with $t \ge 0$ and state space $(E, \mathfrak{E})$, is an E valued stochastic process adapted to $\{\mathfrak{F}_t\}$ such that for $0 \le s \le t$, $f \in b\mathfrak{E}$ ($b\mathfrak{E}$ means bounded $\mathfrak{E}$ measurable functions), and $x \in E$, $E^x[f(X_{s+t})|\mathfrak{F}_t] = (P_t f)(X_s)$, $P^x ae$ (ae means almost everywhere). Here $\{P_t\}$ is a transition function on $(E, \mathfrak{E})$, i.e. a family of kernels $P_t : E \times \mathfrak{E} \to [0,1]$ such that

1. For $t \ge 0$ and $x \in E$, $P_t(x, \cdot)$ is a measure on $\mathfrak{E}$ with $P_t(x, E) \le 1$
2. For $t \ge 0$ and $\Gamma \in \mathfrak{E}$ $P_t(\cdot, \Gamma)$ is $\mathfrak{E}$ measurable
3. For $x, t \ge 0$, $x \in E$, and $\Gamma \in \mathfrak{E}$ one has $P_{t+s}(x,\Gamma) = \int_E P_s(x,dy)P_t(y,\Gamma)$

The equation in #3 is called the Chapman-Kolmogorov (CK) equation and, thinking of the transition functions as inducing a family $\{P_t\}$ of positive bounded operators or norm less than or equal to 1 on $b\mathfrak{E}$ one has $P_t f(x) = (P_t f)(x) = \int_E P_t(x,dy)f(y)$ in which case the CK equation has the semigroup property $P_s P_t = P_{s+t}$ for $s, t \ge 0$.

Under mild regularity conditions if a transition semigroup $\{P_t\}$ is given there will exist on some probability space a Markov process X with suitable paths such that the strong Markov property holds, i.e. $E^x[f(X_{S+t})]|\mathfrak{F}_x] = (P_t f)(X_S)$ P^x ae whenever S is a finite stopping time (here $S : \Omega \to [0,\infty]$ is a $\mathfrak{E}$ stopping time if $\{S \le t\} = \{\omega;\, S(\omega) \le t\} \in \mathfrak{E}_t$ for every $t < \infty$).

EXAMPLE 5.1. A Markov process with countable state space is called a Markov chain. One writes $p_{ij}(t) = P_t(i, \{j\})$ with $P(t) = \{p_{ij}(t);\, i,j \in E\}$. Assume $P_t(i,E) = 1$ and $p_{ij}(t) \to \delta_{ij}$ as $t \downarrow 0$. This will imply that in fact $p'_{ij}(0) = q_{ij}$ exists and the matrix $Q = (q_{ij})$ is called an infinitesimal generator of $\{P_t\}$ with $q_{ij} \ge 0$ $(i \ne j)$ and $\sum_j q_{ik} = 0$ $(i \in E)$. This illustrates some important structure for Markov processes. Thus when $P'(0) = Q$ exists one can write $P'(t) = lim_{\epsilon\to 0}\epsilon^{-1}[P(t+\epsilon) - P(t)] = lim_{\epsilon\to 0}P(t)[P(\epsilon) - I] = P(t)Q$. Then solving this equation one has $P(t) = exp(tQ$ as a semigroup generated by Q. The resolvent is defined via $R_\lambda = \int_0^\infty exp(-\lambda t)P_t dt$ and one can regard it as

$$(5.11) \qquad (\lambda R_\lambda)_{ij} = \int_0^\infty \lambda exp(-\lambda t)p_{ij}(t)dt = \mathbf{P}(X_{\mathbf{T}} = j|X_0 = i)$$

where $\mathbf{T}$ is a random variable independent of X with the exponential distribution of rate λ. It follows then that $R_\lambda = (\lambda - Q)^{-1}$ and $R_\lambda - R_\mu = (\mu - \lambda)R_\lambda R_\mu$. The whole subject is full of pathological situations however and we make no attempt to describe this.

REMARK 2.5.2. [**674**] is oriented toward diffusion processes and departs from the concept that the kinematics of quantum particles is stochastic calculus (in particular Markov processes) while the kinematics of classical particles is classical differential calculus. The relation between these two calculi must be established. Thus classically $x(t) = x(a) + \int_a^t v(s, x(s))ds$ while for a particle with say Brownian noise B_t and a drift field $a(t, x(t))$ one has

$$(5.12) \qquad X_t = X_a + \int_a^t a(s, X_s)ds + \int_a^t \sigma(s, X_s)dB_s$$

We recall $dB_t^2 \sim dt$ so X_t has no velocity and the drift field $a(t, x)$ is not an average speed. However $P[X] = \int_\Omega X dP$ is the expectation (since $P[\sigma(t, X_t)dB_t] = 0$). Now the notation for Markov processes involves nonnegative transition functions $P(s, x; t, B)$ with $a \leq s \leq t \leq b,\ x \in \mathbf{R}^d$, and $B \in \mathfrak{B}(\mathbf{R}^d)$ which are measures in B, measurable in x, and satisfy the CK equation

$$(5.13) \qquad P(s, x; t, B) = \int_{\mathbf{R}^d} P(s, x; r, dy)P(r, y; t, B);\ P(s, x; t, \mathbf{R}^d) = 1$$

If there is a measurable function p such that $P(s, x; t, B) = \int_B p(s, x; t, y)dy$ $(t - s > 0)$ then p is a transition density. One defines a probability measure P on a path space $\Omega = (\mathbf{R}^d)^{[a,b]}$ via finite dimensional distributions

(5.14)

$$P[f(X_a, X_{t_1}, \cdots, X_{t_n}, X_b)] = \int \mu_a(dx_0)P(a, x_0; t_1, dx_1)P(t_1, x_1; t_2, dx_2) \cdots \times$$
$$\times \cdots P(t_{n-1}, x_{n-1}; b, dx_n)f(x_0, \cdots, x_n)$$

Moreover one defines a family $\{X_t;\ t \in [a, b]\}$ on Ω via $X_t(\omega) = \omega(t),\ \omega \in \Omega$. Note one assumes the right continuity of $X_t(\omega)$ ae. This representation can be written as $P = [\mu_a P >>$ and is called the Kolmogorov representation of P. Let now $\{\mathfrak{F}_s^t\}$ be a filtration as before, i.e. a family of σ-fields generated by $\{X_r(\omega);\ s \leq r \leq t\}$. Then we have a Markov process $\{X_t, t \in [a, b], \mathfrak{F}_s^t, P\}$. Replacing μ_a by δ_x and a by s with $s < t_1 < \cdots < t_{n-1} < t_n \leq b$ one defines probability measures $P_{(s,x)},\ (s, x) \in [a, b] \times \mathbf{R}^d$ from (1.67) via

$$(5.15) \qquad P_{(s,x)}[f(X_{t_1}, \cdots, X_{t_{n-1}}, X_{t_n})] =$$
$$= \int P(s, x; t_1, dx_1) \cdots P(t_{n-1}, x_{n-1}; t_n, dx_n)f(x_1, \cdots, x_n)$$

As a special case one has $P_{(s,x)}[f(t, X_t)] = \int P(s, x; t, dy)f(t, y)$ and one can also prove that $P[GF] = P[GP_{(s,X_s)}[F]]$ for any bounded $\mathfrak{F}_a^s$ measurable G and any bounded $\mathfrak{F}_s^b$ measurable F. This is the time inhomogeneous Markov property which can be written in terms of conditional expectations as $P[F|\mathfrak{F}_a^s] = P_{(s,X_s)}[F],\ P\ ae.$ There is a great deal of material in [**674**] about Markov processes with jumps but we prefer to stay here with [**326**] for notational convenience.

Going back to [**326**] we consider a Markov process Q_t on configuration space with transition probabilities characterized by the backward generator L_t, a time dependent linear operator acting on functions f via $L_t f(q) = (d/ds)E(f(Q_{t+s}|Q_t = q)$ where d/ds means the right derivative at $s = 0$ and $E(\cdot|\cdot)$ is conditional expectation. Equivalently the transition probabilities are characterized by the forward generator $\mathcal{L}_t$ (or simply generator) which is also a linear operator but acts on (signed) measures on the configuration space. Its defining property is that for every process Q_t with the given transition properties $\partial_t\rho_t = \mathcal{L}_t\rho_t$. Thus $\mathcal{L}$ is dual to L_t in the sense

$$\int f(q)\mathcal{L}_t\rho(dq) = \int L_t f(q)\rho(dq) \tag{5.16}$$

Given equivariance for $|\Psi|^2$, one says that the corresponding transition probabilities are equivariant and this is equivalent to $\mathcal{L}_t|\Psi|^2 = \partial_y|\Psi|^2$ for all t; when this holds one says that $\mathcal{L}_t$ is an equivariant generator (with respect to Ψ_t and H). One says that a Markov process is Q equivariant if and only if for every t the distribution ρ_t of Q_t equals $|\Psi_t|^2$. For this equivariant transition probabilities are necessary but not sufficient; however for equivariant transition probabilities there is a unique equivariant Markov process. The crucial idea here for construction of an equivariant Markov process is to note that (5.9) is completely general and to find a generator $\mathcal{L}_t$ such that the right side of (5.9) can be read as the action of $\mathcal{L}$ on $\rho = |\Psi|^2$ means $(2/\hbar)\Im\Psi^* H\Psi = \mathcal{L}|\Psi|^2$. This will be implemented later. For H of the form $-(\hbar^2/2)\Delta + V$ one has (5.10) and hence

$$\frac{2}{\hbar}\Im\Psi^* H\Psi = -div(\hbar\Im\Psi^*\nabla\Psi) = -div\left(|\Psi|^2\hbar\Im\frac{\Psi^*\nabla\Psi}{|\Psi|^2}\right) \tag{5.17}$$

Since the generator of the (deterministic) Markov process corresponding to the dynamical system $dQ/dt = v(Q)$ is given by a velocity vector field is $\mathcal{L}\rho = -div(\rho v)$ we may recognize the last term of (1.67) as $\mathcal{L}|\Psi|^2$ with $\mathcal{L}$ the generator of the deterministic process defined by (5.8). Thus Bohmian mechanics arises as the natural equivariant process on configuration space associated with H and Ψ. One notes that Bohmian mechanics is not the only solution of $(2/\hbar)\Im\Psi^* H\Psi - \mathfrak{L}|\Psi|^2$; there are alternatives such as Nelson's stochastic mechanics (and hence Nagasawa's theory of [**672, 674**]) and other velocity formulas (cf. [**295**]).

For equivariant jump processes one says that a (pure) jump process is a Markov process on $\mathfrak{Q}$ for which the only motion that occurs is via jumps. Given that $Q_t = q$ the probability for a jump to q' (i.e. into the infinitesimal volume dq' around q') by time $t + dt$ is $\sigma_t(d'q|q)dt$ where σ is called the jump rate. Here σ is a finite measure in the first variable; $\sigma(B|q)$ is the rate (i.e. the probability per unit time) of jumping to somewhere in the set $B \subset \mathfrak{Q}$ given that the present location is q. The overall jump rate is $\sigma(\mathfrak{Q}|q)$ (sometimes one writes $\rho(dq) = \rho(q)dq$). A jump first occurs when a random waiting time T has elapsed, after the time t_0 at which the process was started or at which the most recent previous jump has occured. For purposes of simulating or constructing the process, the destination q' can be chosen at the time of jumping, t_0+T, with probability distribution $\sigma_{t_0+T}(\mathfrak{Q}|q)^{-1}\sigma_{t_0+T}(\cdot|q)$. In case the overall jump rate is time independent T is exponentially distributed with mean of $\sigma(\mathfrak{Q}|q)^{-1}$. When the rates are time dependent (as they will typically in what follows) the waiting time remains such that $\int_{t_0}^{t_0+T}\sigma_t(\mathfrak{Q}|q)dt$ is exponentially distributed with mean 1, i.e. T becomes exponential after a suitable (time dependent) rescaling of time. The generator of a pure jump process can be expressed in terms of the rates

$$\mathcal{L}\rho(dq)=\int_{q'\in\mathfrak{Q}}(\sigma(dq|q')\rho(dq')-\sigma(dq')|q)\rho(dq)) \tag{5.18}$$

which is a balance or master equation expressing $\partial_t\rho$ as the gain due to jumps to dq minus the loss due to jumps away from dq. One says the jump rates are equivariant if $\mathcal{L}_\sigma$ is an equivariant generator.

Given a Hamiltonian $H=H_0+H_I$ one obtains

$$(2/\hbar)\Im\Psi^*H_0\Psi+(2/\hbar)\Im\Psi^*H_I\Psi-\mathcal{L}|\Psi|^2 \tag{5.19}$$

This opens the possibility of finding a generator $\mathcal{L}=\mathcal{L}_0+\mathcal{L}_I$ given $(2/\hbar)\Im\Psi^*H_0\Psi=\mathcal{L}_0|\Psi|^2$ and $(2/\hbar)\Im\Psi^*H_I\Psi=\mathcal{L}_I|\Psi|^2$; this will be called process additivity and correspondingly $L=L_0+L_I$. If one has two deterministic processes of the form $\mathcal{L}\rho=-div(\rho v)$ then adding generators corresponds to $v=v_+v_2$. For a pure jump process adding generators corresponds to adding rates σ_i which is equivalent to saying there are two kinds of jumps. Now add generators for a deterministic and a jump process via

$$\mathcal{L}\rho(q)=-div(\rho v)(q)+\int_{q'\in\mathfrak{Q}}(\sigma(q|q')\rho(q')-\sigma(q'|q)\rho(q))\,dq' \tag{5.20}$$

This process moves with velocity $v(q)$ until it jumps to q' where it continues moving with velocity $v(q')$. One can understand (5.20) in terms of gain or loss of probability density due to motion and jumps; the process is piecewise deterministic with random intervals between jumps and random destinations. Note that for a Wiener process the generator is the Laplacian and adding to it the generator of a deterministic process means introducing a drift.

Now consider H_I and note that in QFT's with cutoffs it is usually the case that H_I is an integral operator. Hence one writes here $H\sim H_I$ and thinks of it as an integral operator with $\mathfrak{Q}\sim\mathbf{R}^n$. What characterizes jump processes is that some amount of probability that vanishes at $q\in\mathfrak{Q}$ can reappear in an entirely different region say at $q'\in\mathfrak{Q}$. This suggests that the Hamiltonians for which the expression (5.9) for $\partial_t|\Psi|^2$ is naturally an integral over q' correspond to pure jump processes. Thus when is the left side of $(2/\hbar)\Im|psi^*H\Psi=\mathfrak{L}|\Psi|^2$ an integral over q' or $(H\Psi)(q)=\int dq'<q|H|q'>\Psi(q')$. In this case one should choose the jump rates so that when $\rho=|\Psi|^2$ one has

$$\sigma(q|q')\rho(q')-\sigma(q'|q)\rho(q)=(2/\hbar)\Im\Psi^*(q)<q|H|q'>\Psi(q') \tag{5.21}$$

This suggests, since jump rates are nonnegative and the right side of (5.21) is antisymmetric) that $\sigma(q|q')\rho(q') = [(2/\hbar)\Im\Psi^*(q) < q|H|q' > \Psi(q')]^+$ or

$$\sigma(q|q') = \frac{(2/\hbar)\Im\Psi^*(q) < q|H|q' > \Psi(q')]^+}{\Psi^*(q')\Psi(q')} \tag{5.22}$$

These rates are an instance of what can be called minimal jump rates associated with H (and Ψ). They are actually the minimal possible values given (5.21) and this is discussed further in [**326**]. Minimality entails that at any time t one of the transitions $q_1 \to q_2$ or $q_2 \to q_1$ is forbidden and this will be called a minimal jump process. One summarizes motions via

H	*motion*
integral operator	*jumps*
differential operator	*deterministic continuous motion*
multiplication operator	*no motion* ($\mathcal{L} = 0$)

The reasoning above applies to the more general setting of arbitrary configuration spaces $\mathfrak{Q}$ and generalized observables - POVM's - defining what the "position" representation is to be. One takes the following ingredients from QFT

(1) A Hilbert space $\mathcal{H}$ with scalar product $< \Psi|\Phi >$.
(2) A unitary one parameter group U_t in $\mathcal{H}$ with Hamiltonian H, i.e. $U_t = exp[-(i/\hbar)tH]$, so that in the Schrödinger picture the state Ψ evolves via $i\hbar\partial_t\Psi = H\Psi$. U_t could be part of a representation of the Poincaré group.
(3) A positive operator valued measure (POVM) $P(dq)$ on $\mathfrak{Q}$ acting on $\mathcal{H}$ so that the probability that the system in the state Ψ is localized in dq at time t is $\mathbf{P}_t(dq) =< \Psi_t|P(dq)|\Psi_t >$.

Mathematically a POVM on $\mathfrak{Q}$ is a countably additive set function (measure) defined on measurable subsets of $\mathfrak{Q}$ with values in the positive (bounded self adjoint) operators on a Hilbert space $\mathcal{H}$ such that $P(\mathfrak{Q}) = Id$. Physically for purposes here $P(\cdot)$ represents the (generalized) position observable, with values in $\mathfrak{Q}$. The notion of POVM generalizes the more familiar situation of observables given by a set of commuting self adjoint operators, corresponding by means of the spectral theorm to a projection valued measure (PVM) - the case where the positive operators are projection operators (see [**326**] for discussion). The goal now is to specify equivariant jump rates $\sigma = \sigma^{\Psi,H,P}$ so that $\mathcal{L}_\sigma\mathbf{P} = d\mathbf{P}/dt$. To this end one could take the following steps.

(1) Note that $(d\mathbf{P}_t(dq)/dt) = (2/\hbar)\Im < \Psi_t|P(dq)H|\Psi_t >$.
(2) Insert the resolution of the identity $I = \int_{q'\in\mathfrak{Q}} P(dq')$ and obtain

$$(d\mathbf{P}_t(dq)/dt) = \int_{q'\in\mathfrak{Q}} \mathbf{J}_t(dq, dq'); \tag{5.23}$$

$$\mathbf{J}_t(dq, dq') = (2/\hbar)\Im < \Psi_t|P(dq)HP(dq')|\Psi_t >$$

(3) Observe that $\mathbf{J}$is antisymmetric so since $x = x^+ - (-x)^+$ one has

$$\mathbf{J}(dq, dq') = [(2/\hbar)\Im < \Psi|P(dq)HP(dq')|\Psi]^+ - \tag{5.24}$$

$$-[(2/\hbar)\Im < \Psi|P(dq')HP(dq)|\Psi >]^+$$

(4) Multiply and divide both terms by $\mathbf{P}(\cdot)$ obtaining

$$\int_{q'\in\mathfrak{Q}} \mathbf{J}(dq,dq') = \int_{q'\in\mathfrak{Q}} \left(\frac{(2/\hbar)\Im < \Psi|P(ddq)HP(dq')|\Psi >]^+}{< \Psi|P(dq')|\Psi >}\mathbf{P}(dq') - \right. \tag{5.25}$$

$$\left. - \frac{[(2/\hbar)\Im < \Psi|P(dq')HP(dq)|\Psi >]^+}{< \Psi|P(dq)|\Psi >}\mathbf{P}(dq) \right)$$

(5) By comparison with (5.18) recognize the right side of the above equation as $\mathcal{L}_\sigma \mathbf{P}$ with $\mathcal{L}_\sigma$ the generator of a Markov jump process with jump rates (5.7) (minimal jump rates).

Note the right side of (5.7) should be understood as a density (Radon-Nikodym derivative).

When H_0 is made of differential operators of up to second order one can characterize the process associated with H_0 in a particularly succinct manner as follows. Define for any H, P, Ψ an operator L acting on functions $f : \mathfrak{Q} \to \mathbf{R}$ which may or may not be the backward generator of a process via

$$Lf(q) = \Re \frac{< \Psi|P(dq)\hat{L}\hat{f}|\Psi >}{< \Psi|P(dq)\Psi >} = \Re \frac{< \Psi|P(dq)(i/\hbar)[H,\hat{f}]|\Psi >}{< \Psi|P(dq)|\Psi >} \tag{5.26}$$

where [,] means the commutator and $\hat{f} = \int_{q\in\mathfrak{Q}} f(q)P(dq)$ with $\hat{L}$ the generator of the (Heisenberg) evolution $\hat{f}$,

$$\hat{L}\hat{f} = (d/d\tau)exp(iH\tau/\hbar)\hat{f}exp(-iH\tau/\hbar)|_{\tau=0} = (i/\hbar)[H,\hat{f}] \tag{5.27}$$

Note if P is a PVM then $\hat{f} = f(\hat{q})$. (5.26) could be guessed in the following manner: Since Lf is in a certain sense the time derivative of f it might be expected to be related to $\hat{L}\hat{f}$ which is in a certain sense (cf. (5.27)) the time derivative of $\hat{f}$. As a way of turning the operator $\hat{L}\hat{f}$ into a function $Lf(q)$ the middle term in (5.26) is an obvious possibility. Note also that this way of arriving at (5.26) does not make use of equivariance. The formula for the forward generator equivalent to (5.26) reads

$$\mathcal{L}\rho(dq) = \Re < \Psi|(\widehat{d\rho/d\mathbf{P}})(i/\hbar)[H,P(dq)]|\Psi > \tag{5.28}$$

Whenever L is indeed a backward generator we call it the minimal free (backward) generator associated with Ψ, H, P. Then the corresponding process is equivariant and this is the case if (and there is reason to expect, only if) P is a PVM and H is a differential operator of up to second order in the position representation, in which P is diagonal. In that case the process is deterministic and the backward generator has the form $L = v\cdot\nabla$ where v is the velocity field; thus (5.26) directly specifies the velocity in the form of a first order differential operator $v\cdot\nabla$. In case H is the N-particle Schrödinger operator with or without spin (5.26) yields the Bohmian velocity (5.8) and if H is the Dirac operator the Bohm-Dirac velocity emerges. Thus in some cases (5.26) leads to just the right backward generator. In [**326**] there are many examples and mathematical sections designed to prove various assertions but we omit this here..

6. BOHMIAN MECHANICS IN QFT

We extract here from a fascinating paper [**713**] by H. Nikolić. Quantum field theory (QFT) can be formulated in the Schrödinger picture by using a functional time dependent SE but this requires a choice of time coordinate and the corresponding choice of a preferred foliation of spacetime producing a relativistically noncovariant theory. The problem of noncovariance can be solved by replacing the usual time dependent SE with the many fingered time (MFT) Tomonaga-Schwinger equation, which does not require a preferred foliation and the quantum state is a functional of an arbitrary timelike hypersurface. In a manifestly covariant formulation introduced in [**316**] the hypersurface does not even have to be timelike. The paper [**713**] develops a Bohmian interpretation for the MFT theory for QFT and refers to [**77, 108, 255, 473, 478, 587, 711, 708, 769, 772, 774, 819, 820, 876, 910**] for background and related information. Thus let $x = \{x^\mu\} = (x^0, \mathbf{x})$ be spacetime coordinates. A timelike Cauchy hypersurface Σ can be defined via $x^0 = T(\mathbf{x})$ with $\mathbf{x}$ denoting coordinates on Σ. Let $\phi(\mathbf{x})$ be a dynamical field on Σ (a real scalar field for convenience) and write T, ϕ without an argument for the functions themselves with $\phi = \phi|_\Sigma$ etc. Let $\hat{\mathfrak{H}}(\mathbf{x})$ be the Hamiltonian density operator and then the dynamics of a field ϕ is described by the MFT Tomonaga-Schwinger equation

$$\hat{\mathfrak{H}}\Psi[\phi, T] = i\frac{\delta\Psi[\phi, T]}{\delta T(\mathbf{x})} \tag{6.1}$$

Note $\delta T(\mathbf{x})$ denotes an infinitesimal change of the hypersurface Σ. The quantity $\rho[\phi, T] = |\Psi[\phi, T]|^2$ represents the probability density for the field to have a value ϕ on Σ or equivalently the probability density for the field to have a value ϕ at time T. One can say that ϕ has a definite value φ at some time T_0 if

$$\Psi[\phi, T_0] = \delta(\phi - \varphi) = \prod_{\mathbf{x}\in\Sigma} \delta(\phi(\mathbf{x}) - \varphi(\mathbf{x})) \tag{6.2}$$

[**713**] then provides an important discussion of measurement and contextuality in QM which we largly omit here in order to go directly to the Bohmian formulation.

For simplicity take a free scalar field with

$$\hat{\mathfrak{H}}(\mathbf{x}) = -\frac{1}{2}\frac{\delta^2}{\delta\phi^2(\mathbf{x})} + \frac{1}{2}[(\nabla\phi(\mathbf{x}))^2 + m^2\phi^2(\mathbf{x})] \tag{6.3}$$

Writing $\Psi = Rexp(iS)$ with R and S real functionals the complex equation (6.1) is equivalent to two real equations with

$$\frac{1}{2}\left(\frac{\delta S}{\delta\phi(\mathbf{x})}\right)^2 + \frac{1}{2}[(\nabla\phi(\mathbf{x}))^2 + m^2\phi^2(\mathbf{x})] + \mathfrak{Q}(\mathbf{x}, \phi, T] + \frac{\delta S}{\delta T(\mathbf{x})} = 0; \tag{6.4}$$

$$\frac{\delta\rho}{\delta T(\mathbf{x})} + \frac{\delta}{\delta\phi(\mathbf{x})}\left(\rho\frac{\delta S}{\delta T(\mathbf{x})}\right) = 0;\ \mathfrak{Q}(\mathbf{x}, \phi, T] = -\frac{1}{2R}\frac{\delta^2 R}{\delta\phi^2(\mathbf{x})}$$

The conservation equation shows that it is consistent to interpret $\rho[\phi, T]$ as the probability density for the field to have the value ϕ at the hypersurface determined by the time T. Now let σ_x be a small region around $\mathbf{x}$ and define the derivative

$$\frac{\partial}{\partial T(\mathbf{x})} = \lim_{\sigma_x \to 0} \int_{\sigma_x} d^3x \frac{\delta}{\delta T(\mathbf{x})} \tag{6.5}$$

where $\sigma_x \to 0$ means that the 3-volume goes to zero (note $\partial T(\mathbf{y})/\partial T(\mathbf{x}) = \delta_{xy}$). It is convenient to integrate (6.4) inside a small σ_x leading to

$$\frac{\partial \rho}{\partial T(\mathbf{x})} + \frac{\partial}{\partial \phi(\mathbf{x})}\left(\rho \frac{\delta S}{\delta \phi(\mathbf{x})}\right) = 0 \tag{6.6}$$

where $\partial/\partial\phi(\mathbf{x})$ is defined as in (6.5). The Bohmian interpretation consists now in introducing a deterministic time dependent hidden variable such that the time evolution of this variable is consistent with the probabilistic interpretation of ρ. From (6.6) one sees that this is naturally achieved by introducing a MFT field $\Phi(x, T]$ that satisfies the MFT Bohmian equations of motion

$$\frac{\partial \Phi(\mathbf{x}, T]}{\partial T(\mathbf{x})} = \left.\frac{\delta S}{\delta \phi(\mathbf{x})}\right|_{\phi=\Phi} \tag{6.7}$$

From (6.7) and the quantum MFT HJ equation (6.4) results

$$\left[\left(\frac{\partial}{\partial T(\mathbf{x})}\right)^2 - \nabla_x^2 + m^2\right] \Phi(\mathbf{x}, T] = -\left.\frac{\partial \mathfrak{Q}(\mathbf{x}, \phi, T]}{\partial \psi(\mathbf{x})}\right|_{\phi=\Phi} \tag{6.8}$$

This can be viewed as a MFT KG equation modified with a nonlocal quantum term on the right side. The general solution of (6.7) has the form

$$\Phi_{gen}(\mathbf{x}), T] = F(\mathbf{x}, c(\mathbf{x}, T]; T] \tag{6.9}$$

where F is a function(al) that depends on the right side of (6.7) and $c(\mathbf{x}, T]$ is an arbitrary function(al) with the property

$$\frac{\delta c(\mathbf{x}, T]}{\delta T(\mathbf{x})} = 0 \tag{6.10}$$

This quantity can be viewed as an arbitrary MFT integration constant - it is constant in the sense that it does not depend on $T(\mathbf{x})$, but it may depend on T at other points $\mathbf{x}' \neq \mathbf{x}$. To provide the correct classical limit (indicated below) one restricts $c(\mathbf{x}, T]$ to satisfy

$$c(\mathbf{x}, T] = c(\mathbf{x}) \tag{6.11}$$

where $c(\mathbf{x})$ is an arbitrary function. Here it is essential to realize that $\Phi(\mathbf{x}, T]$ is a function of $\mathbf{x}$ but a functional of T; the field Φ depends not only on $(\mathbf{x}, T(\mathbf{x})) \equiv (\mathbf{x}, x^0) \equiv x$ but also on the choice of the whole hypersurface Σ that contains the point $\mathbf{x}$. Consequently the MFT Bohmian interpretation does not in general assign a value of the field at the point x unless the whole hypersurface containing x is specified. On the other hand if e.g. $\delta S/\delta\phi(\mathbf{x})$ on the right side in (6.7) is a local functional, i.e. of the form $V(\mathbf{x}, \phi(\underline{x}), T(\mathbf{x})$, then the solution of (6.7) is a local functional of the form

$$\Phi(\mathbf{x}, T(\mathbf{x}) = \Phi(\mathbf{x}, x^0) = \Phi(x) \tag{6.12}$$

This occurs for example when the wave functional is a local product $\Psi[\phi, T] = \prod_{\mathbf{x}} \psi_{\mathbf{x}}(\phi(\mathbf{x}, T(\mathbf{x})$. Interactions with the measuring apparatus can also produce locality. As for the classical limit one can formulate the classical HJ equation as a MFT theory (cf. [**819, 820**]) without of course the $\mathfrak{Q}$ term. Hence by imposing a restriction similar to (6.11) the solution $S[\phi, T]$ can be chosen so that $\delta S/\delta\phi(\mathbf{x})$ is a local functional; the restriction (6.11) again implies that the classical solution Φ is also a local functional.

The MFT formalism was introduced by Tomonaga and Schwinger to provide the manifest covariance of QFT in the interaction picture. The picture here is so far not manifestly covariant since time is not treated on an equal footing with space. However the MFT formalism can be here also in a manifestly covariant manner via [**316, 819, 820**]. One starts by introducing a set of 3 real parameters $\{s^1, s^2, s^3\} \equiv \mathbf{s}$ to serve as coordinates on a 3-dimensional manifold (a priori $\mathbf{s}$ is not related to $\mathbf{x}$). The 3-dimensional manifold Σ can be embedded in the 4-dimensional spacetime by introducing 4 functions $X^\mu(\mathbf{s})$ and a 3-dimensional hypersurface is given via $x^\mu = X^\mu(\mathbf{s})$. The 3 parameters s^i can be eliminated leading to an equation of the form $f(x^0, x^1, x^2, x^3) = 0$ and assuming that the background spacetime metric $g_{\mu\nu}(x)$ is given the induced metric $q_{ij}(\mathbf{s})$ on this hypersurface is

$$q_{ij}(\mathbf{s}) = g_{\mu\nu}(X(\mathbf{s}))\frac{\partial X^\mu(\mathbf{s})}{\partial s^i}\frac{\partial X^\nu(\mathbf{s})}{\partial s^j} \tag{6.13}$$

Similarly a normal to the surface is

$$\tilde{n}(\mathbf{s}) = \epsilon_{\mu\alpha\beta\gamma}\frac{\partial X^\alpha}{\partial s^1}\frac{\partial X^\beta}{\partial s^2}\frac{\partial X^\gamma}{\partial s^3} \tag{6.14}$$

and the unit normal transforming as a spacetime vector is

$$n^\mu(\mathbf{s}) = \frac{g^{\mu\nu}\tilde{n}_\nu}{\sqrt{|g^{\alpha\beta}\tilde{n}_\alpha\tilde{n}_\beta|}} \tag{6.15}$$

Now some of the original equations above can be written in a covariant form by making the replacements

$$\mathbf{x} \to \mathbf{s};\ \frac{\delta}{\delta T(\mathbf{x})} \to n^\mu(\mathbf{s})\frac{\delta}{\delta X^\mu(\mathbf{s})} \tag{6.16}$$

The Tomonaga-Schwinger equation (6.1) becomes

$$\hat{\mathfrak{H}}(\mathbf{s})\Psi[\phi, X] = i n^\mu(\mathbf{s})\frac{\delta\Psi[\phi, X]}{\delta X^\mu(\mathbf{s})} \tag{6.17}$$

For free fields the Hamiltonian density operator in curved spacetime is

$$\hat{\mathfrak{H}} = \frac{-1}{2|q|^{1/2}}\frac{\delta^2}{\delta\phi^2(\mathbf{s})} + \frac{|q|^{1/2}}{2}[-q^{ij}(\partial_i\phi)(\partial_j\phi) + m^2\phi^2] \tag{6.18}$$

The Bohmian equations of motion (6.7) becomes

$$\frac{\partial\Phi(\mathbf{s}, T]}{\partial\tau(\mathbf{s})} = \frac{1}{|q(\mathbf{s})|^{1/2}}\left.\frac{\delta S}{\delta\phi(\mathbf{s})}\right|_{\phi=\Phi};\ \frac{\partial}{\partial\tau(\mathbf{s})} \equiv lim_{\sigma_x\to 0}\int_{\sigma_x} d^3s n^\mu(\mathbf{s})\frac{\delta}{\delta X^\mu(\mathbf{s})} \tag{6.19}$$

Similarly (6.8) becomes

$$\left[\left(\frac{\partial}{\partial\tau(\mathbf{s})}\right)^2 + \nabla^i\nabla_i + m^2\right]\Phi(\mathbf{s},X] = -\frac{1}{|q(\mathbf{s})|^{1/2}}\frac{\partial\mathfrak{Q}(\mathbf{s},\phi,X]}{\partial\phi\mathbf{s})}\Bigg|_{\phi=\Phi} \tag{6.20}$$

where ∇_i is the covariant derivative with respect to s^i and

$$\mathfrak{Q}(\mathbf{s},\phi,X] = -\frac{1}{|q(\mathbf{s})|^{1/2}}\frac{1}{2R}\frac{\delta^2 R}{\delta\phi^2(\mathbf{s})} \tag{6.21}$$

corresponding to a quantum potential. The same hypersurface Σ can be parametrized by different sets of 4 functions $X^\mu(\mathbf{s})$ of course but quantities such as $\Psi[\phi,X]$ and $\Phi(\mathbf{s},X]$ depend on Σ, not on the parametrization. The freedom in choosing functions $X^\mu(\mathbf{s})$ is sort of a gauge freedom related to the covariance. Now to find a solution of the covariant equations above it is convenient to fix a gauge and for a timelike surface the simplest choice is $X^i(\mathbf{s}) = s^i$. This implies $\delta X(\mathbf{s}) = 0$ which leads to equations similar to those obtained previously. For example (6.19) becomes

$$(g^{00}(\mathbf{x}))^{1/2}\frac{\partial\Phi(\mathbf{x},X^0]}{\partial X^0(\mathbf{x})} = \frac{1}{|q(\mathbf{x})|^{1/2}}\frac{\delta S}{\delta\phi(\mathbf{x})}\Bigg|_{\phi=\Phi} \tag{6.22}$$

which is the curved spacetime version of (6.7).

CHAPTER 3

GRAVITY AND THE QUANTUM POTENTIAL

Just as we plunged into QM in Chapters 1 and 2 we plunge again into general relativity (GR), Weyl geometry, Dirac-Weyl (DW) theory, and deBroglie-Bohm-Weyl (dBBW) theory. There are many good books available for background in general relativity, especially [**69**] (marvelous for conceptual purposes and for a modern perspective) and [**12**] (a classic masterpiece with all the indices in their place). In addition we mention some excellent books and papers which will arise in references later, namely [**52, 121, 351, 458, 498, 551, 657, 715, 723, 819, 910, 972**]. To develop all the background differential geometry requires a book in itself and the presentation adopted here will in fact include all this implicitly since the topics range over a fairly wide field (see also Chapter 5 where cosmology plays a more central role).

1. INTRODUCTION

A complete description of necessary geometric ideas appears in [**657**] for example and we only make some definitions and express some relations here, usng the venerable tensor notation of indices, etc., since even today much of the physics literature appears in this form. For differential geometry one can refer to [**134, 276, 998**]. First we give some background on Weyl geometry and Brans-Dicke theory following [**12**]; for differential geometry we use the tensor notation of [**12**] and refer to e.g. [**121, 358, 458, 498, 723, 731, 972, 998**] for other notation (see also [**990**] for an interesting variation). One thinks of a differential manifold $M = \{U_i, \phi_i\}$ with $\phi : U_i \to \mathbf{R}^4$ and metric $g \sim g_{ij}dx^i dx^j$ satisfying $g(\partial_k, \partial_\ell) = g_{k\ell} =< \partial_k, \partial_\ell >= g_{\ell k}$. This is for the bare essentials; one can also imagine tangent vectors $X_i \sim \partial_i$ and dual cotangent vectors $\theta^i \sim dx^i$, etc. Given a coordinate change $\tilde{x}^i = \tilde{x}^i(x^j)$ a vector ξ^i transforming via $\tilde{\xi}^i = \sum \partial_i \tilde{x}^j \xi^j$ is called contravariant (e.g. $d\tilde{x}^i = \sum \partial_j \tilde{x}^i dx^j$). On the other hand $\partial\phi/\partial\tilde{x}^i = \sum(\partial\phi/\partial x^j)(\partial x^j/\partial\tilde{x}^i$ leads to the idea of covariant vectors $A_j \sim \partial\phi/\partial x^j$ transforming via $\tilde{A}_i = \sum(\partial x^j/\partial\tilde{x}^i)A_j$ (i.e. $\partial/\partial\tilde{x}^i \sim (\partial x^j/\partial\tilde{x}^j)\partial/\partial x^j$). Now define connection coefficients or Christoffel symbols via (strictly one writes $T^\gamma{}_\alpha = g_{\alpha\beta}T^{\gamma\beta}$ and $T_\alpha{}^\gamma = g_{\alpha\beta}T^{\beta\gamma}$ which are generally different - we use that notation here but it is sometimes not used later when it is unnecessary due to symmetries, etc.)

$$\Gamma^r{}_{ki} = -\left\{ \begin{matrix} r \\ k \;\; i \end{matrix} \right\} = -\frac{1}{2}\sum(\partial_i g_{k\ell} + \partial_k g_{\ell i} - \partial_\ell g_{ik})g^{\ell r} = \Gamma^r{}_{ik} \tag{1.1}$$

(note this differs by a minus sign from some other authors). Note also that (1.1) follows from equations

$$\partial_\ell g_{ik} + g_{rk}\Gamma^r{}_{i\ell} + g_{ir}\Gamma^r{}_{\ell k} = 0 \tag{1.2}$$

and cyclic permutation; the basic definition of $\Gamma^i{}_{mj}$ is found in the transplantation law $d\xi^i = \Gamma^i{}_{mj}dx^m\xi^j$. Next for tensors $T^\alpha_{\beta\gamma}$ define derivatives $T^\alpha_{\beta\gamma|k} = \partial_k T^\alpha_{\beta\gamma}$ and

$$T^\alpha_{\beta\gamma||\ell} = \partial_\ell T^\alpha_{\beta\gamma} - \Gamma^\alpha_{\ell s}T^s_{\beta\gamma} + \Gamma^s{}_{\ell\beta}T^\alpha_{s\gamma} + \Gamma^s{}_{\ell\gamma}T^\alpha_{\beta s} \tag{1.3}$$

In particular covariant derivatives for contravariant and covariant vectors respectively are defined via

$$\xi^i_{||k} = \partial_k\xi^i - \Gamma^i{}_{k\ell}\xi^\ell = \nabla_k\xi^i;\ \eta_{m||\ell} = \partial_\ell\eta_m + \Gamma^r{}_{m\ell}\eta_r = \nabla_\ell\eta_m \tag{1.4}$$

respectively. Now to describe Weyl geometry one notes first that for Riemannian geometry transplantation holds along with

$$\ell^2 = \|\xi\|^2 = g_{\alpha\beta}\xi^\alpha\xi^\beta;\ \xi^\alpha\eta_\alpha = g_{\alpha\beta}\xi^\alpha\eta^\beta \tag{1.5}$$

For Weyl geometry however one does not demand conservation of lengths and scalar products under affine transplantation as above. Thus assume $d\ell = (\phi_\beta dx^\beta)\ell$ where the covariant vector ϕ_β plays a role analogous to $\Gamma^\alpha_{\beta\gamma}$ and one obtains

$$d\ell^2 = 2\ell^2(\phi_\beta dx^\beta) = d(g_{\alpha\beta}\xi^\alpha\xi^\beta) = \tag{1.6}$$

$$= g_{\alpha\beta|\gamma}\xi^\alpha\xi^\beta dx^\gamma + g_{\alpha\beta}\Gamma^\alpha_{\rho\gamma}\xi^\rho\xi^\beta dx^\gamma + g_{\alpha\beta}\Gamma^\beta_{\rho\gamma}\xi^\alpha\xi^\rho dx^\gamma$$

Rearranging etc. and using (1.5) again gives

$$(g_{\alpha\beta|\gamma} - 2g_{\alpha\beta}\phi_\gamma) + g_{\sigma\beta}\Gamma^\sigma{}_{\alpha\gamma} + g_{\sigma\alpha}\Gamma^\sigma{}_{\beta\gamma} = 0; \tag{1.7}$$

$$\Gamma^\alpha_{\beta\gamma} = -\left\{\begin{array}{c}\alpha\\ \beta\ \gamma\end{array}\right\} + g^{\sigma\alpha}[g_{\sigma\beta}\phi_\gamma + g_{\sigma\gamma}\phi_\beta - g_{\beta\gamma}\phi_\sigma]$$

Thus we can prescribe the metric $g_{\alpha\beta}$ and the covariant vector field ϕ_γ and determine by (1.7) the field of connection coefficients $\Gamma^\alpha_{\beta\gamma}$ which admits the affine transplantation law as above. If one takes $\phi_\gamma = 0$ the Weyl geometry reduces to Riemannian geometry. This leads one to consider new metric tensors via a metric change $\hat{g}_{\alpha\beta} = f(x^\lambda)g_{\alpha\beta}$ and it turns out that $(1/2)\partial log(f)/\partial x^\lambda$ plays the role of ϕ_λ. Here the metric change is called a gauge transformation and the ordinary connections $\left\{\begin{array}{c}\alpha\\ \beta\ \gamma\end{array}\right\}$ constructed from $g_{\alpha\beta}$ are equal to the more general connections $\hat{\Gamma}^\alpha_{\beta\gamma}$ constructed according to (1.7) from $\hat{g}_{\alpha\beta}$ and $\hat{\phi}_\lambda = (1/2)\partial log(f)/\partial x^\lambda$. The generalized differential geometry is conformal in that the ratio

$$\frac{\xi^\alpha\eta_\alpha}{\|\xi\|\|\eta\|} = \frac{g_{\alpha\beta}\xi^\alpha\eta^\beta}{[(g_{\alpha\beta}\xi^\alpha\xi^\beta)(g_{\alpha\beta}\eta^\alpha\eta^\beta)]^{1/2}} \tag{1.8}$$

does not change under the gauge transformation $\hat{g}_{\alpha\beta} \to f(x^\lambda)g_{\alpha\beta}$. Again if one has a Weyl geometry characterized by $g_{\alpha\beta}$ and ϕ_α with connections determined by (1.7) one may replace the geometric quantities by use of a scalar field f with

$$\hat{g}_{\alpha\beta} = f(x^\lambda)g_{\alpha\beta},\ \hat{\phi}_\alpha = \phi_\alpha + (1/2)(log(f)_{|\alpha};\ \hat{\Gamma}^\alpha_{\beta\gamma} = \Gamma^\alpha_{\beta\gamma} \tag{1.9}$$

without changing the intrinsic geometric properties of vector fields; the only change is that of local lengths of a vector via $\hat{\ell}^2 = f(x^\lambda)\ell^2$. Note that one can reduce

$\hat{\phi}_\alpha$ to the zero vector field if and only if ϕ_α is a gradient field, namely $F_{\alpha\beta} = \phi_{\alpha|\beta} - \phi_{\beta|\alpha} = 0$ (i.e. $\phi_\alpha = (1/2)\partial_a log(f) \equiv \partial_\beta\phi_\alpha = \partial_\alpha\phi_\beta$). In this case one has length preservation after transplantation around an arbitrary closed curve and the vanishing of $F_{\alpha\beta}$ guarantees a choice of metric in which the Weyl geometry becomes Riemannian; thus $F_{\alpha\beta}$ is an intrinsic geometric quantity for Weyl geometry; note $F_{\alpha\beta} = -F_{\beta\alpha}$ and

$$\{F_{\alpha\beta|\gamma}\} = 0;\ \{F_{\mu\nu|\lambda}\} = F_{\mu\nu|\lambda} + F_{\lambda\mu|\nu} + F_{\nu\lambda|\mu} \tag{1.10}$$

Similarly the concept of covariant differentiation depends only on the idea of vector transplantation. Indeed one can define covariant derivatives via

$$\xi^\alpha_{||\beta} = \xi^\alpha_{|\beta} - \Gamma^\alpha_{\ \beta\gamma}\xi^\gamma \tag{1.11}$$

In Riemannian geometry the curvature tensor is

$$\xi^\alpha_{||\beta|\gamma} - \xi^\alpha_{||\gamma|\beta} = R^\alpha_{\eta\beta\gamma}\xi^\eta;\ R^\alpha_{\ \beta\gamma\delta} = -\Gamma^\alpha_{\ \beta\gamma|\delta} + \Gamma^\alpha_{\ \beta\delta|\gamma} + \Gamma^\alpha_{\ \tau\delta}\Gamma^\tau_{\ \beta\gamma} - \Gamma^\alpha_{\ \tau\gamma}\Gamma^\tau_{\ \beta\delta} \tag{1.12}$$

Using (1.8) one then can express this in terms of $g_{\alpha\beta}$ and ϕ_α but this is complicated. Equations for $R_{\beta\delta} = R^\alpha_{\ \beta\alpha\delta}$ and $R = g^{\beta\delta}R_{\beta\delta}$ are however given in [**12**]. One notes that in Weyl geometry if a vector ξ^α is given, independent of the metric, then $\xi_\alpha = g_{\alpha\beta}\xi^\beta$ will depend on the metric and under a gauge transformation one has $\hat{\xi}_\alpha = f(x^\lambda)\xi_\alpha$. Hence the covariant form of a gauge invariant contravariant vector becomes gauge dependent and one says that a tensor is of weight n if, under a gauge transformation, $\hat{T}^{\alpha\cdots}_{\beta\cdots} = f(x^\lambda)^n T^{\alpha\cdots}_{\beta\cdots}$. Note ϕ_α plays a singular role in (1.9) and has no weight. Similarly $\sqrt{-\hat{g}} = f^2\sqrt{-g}$ (weight 2) and $F^{\alpha\beta} = g^{\alpha\mu}g^{\beta\nu}F_{\mu\nu}$ has weight -2 while $\mathfrak{F}^{\alpha\beta} = F^{\alpha\beta}\sqrt{-g}$ has weight 0 and is gauge invariant. Similarly $F_{\alpha\beta}F^{\alpha\beta}\sqrt{-g}$ is gauge invariant. Now for Weyl's theory of electromagnetism one wants to interpret ϕ_α as an EM potential and one has automatically the Maxwell equations

$$\{F_{\alpha\beta|\gamma}\} = 0;\ \mathfrak{F}^{\alpha\beta}_{|\beta} = \mathfrak{s}^\alpha \tag{1.13}$$

(the latter equation being gauge invariant source equations). These equations are gauge invariant as a natural consequence of the geometric interpretation of the EM field. For the interaction between the EM and gravitational fields one sets up some field equations as indicated in [**12**] and the interaction between the metric quantities and the EM fields is exhibited there (there is much more on EM theory later and see also Section 2.1.1).

REMARK 3.1.1. As indicated earlier in [**12**] $R^i_{\ jk}$ is defined with a minus sign compared with e.g. [**723, 998**] for example. There is also a difference in definition of the Ricci tensor which is taken to be $G^{\beta\delta} = R^{\beta\delta} - (1/2)g^{\beta\delta}R$ in [**12**] with $R = R^\delta_\delta$ so that $G_{\mu\gamma} = g_{\mu\beta}g_{\gamma\delta}G^{\beta\delta} = R_{\mu\gamma} - (1/2)g_{\mu\gamma}R$ with $G^\gamma_\eta = R^\eta_\eta - 2R \Rightarrow G^\eta_\eta = -R$ (recall $n = 4$). In [**723**] the Ricci tensor is simply $R_{\beta\mu} = R^\alpha_{\beta\mu\alpha}$ where $R^\alpha_{\beta\mu\nu}$ is the Riemann curvature tensor and $R = R^\eta_\eta$ again. This is similar to [**998**] where the Ricci tensor is defined as $\rho_{j\ell} = R^i_{ji\ell}$. To clarify all this we note that

$$R_{\eta\gamma} = R^\alpha_{\ \eta\alpha\gamma} = g^{\alpha\beta}R_{\beta\eta\alpha\gamma} = -g^{\alpha\beta}R_{\beta\eta\gamma\alpha} = -R^\alpha_{\ \eta\gamma\alpha} \tag{1.14}$$

which reveals the minus sign difference.

2. SKETCH OF DEBROGLIE-BOHM-WEYL THEORY

From Chapters 1 and 2 we know something about Bohmian mechanics and the quantum potential and we go now to the papers [**869, 870, 871, 872, 873, 874, 875, 876**] by A. and F. Shojai to begin the present discussion (cf. also [**8, 117, 118, 284, 668, 669, 831, 832, 834, 835, 836, 837, 838, 864, 865, 866, 867, 868, 881**] for related work from the Tehran school and [**189, 219, 611, 731, 840, 841, 872**] for linking of dBB theory with Weyl geometry). In nonrelativistic deBroglie-Bohm theory the quantum potential is $Q = -(\hbar^2/2m)(\nabla^2|\Psi|/|\Psi|)$. The particles trajectory can be derived from Newton's law of motion in which the quantum force $-\nabla Q$ is present in addition to the classical force $-\nabla V$. The enigmatic quantum behavior is attributed here to the quantum force or quantum potential (with Ψ determining a "pilot wave" which guides the particle motion). Setting $\Psi = \sqrt{\rho}exp[iS/\hbar]$ one has

$$\frac{\partial S}{\partial t} + \frac{|\nabla S|^2}{2m} + V + Q = 0;\ \frac{\partial \rho}{\partial t} + \nabla \cdot \left(\rho \frac{\nabla S}{m}\right) = 0 \tag{2.1}$$

The first equation in (2.1) is a Hamilton-Jacobi (HJ) equation which is identical to Newton's law and represents an energy condition $E = (|p|^2/2m) + V + Q$ (recall from HJ theory $-(\partial S/\partial t) = E(= H)$ and $\nabla S = p$. The second equation represents a continuity equation for a hypothetical ensemble related to the particle in question. For the relativistic extension one could simply try to generalize the relativistic energy equation $\eta_{\mu\nu}P^\mu P^\nu = m^2c^2$ to the form

$$\eta_{\mu\nu}P^\mu P^\nu = m^2c^2(1+\mathcal{Q}) = \mathcal{M}^2c^2;\ \mathcal{Q} = (\hbar^2/m^2c^2)(\Box|\Psi|/|\Psi|) \tag{2.2}$$

$$\mathcal{M}^2 = m^2\left(1 + \alpha\frac{\Box|\Psi|}{|\Psi|}\right);\ \ \alpha = \frac{\hbar^2}{m^2c^2} \tag{2.3}$$

This could be derived e.g. by setting $\Psi = \sqrt{\rho}exp(iS/\hbar)$ in the Klein-Gordon (KG) equation and separating the real and imaginary parts, leading to the relativistic HJ equation $\eta_{\mu\nu}\partial^\mu S\partial^\nu S = \mathfrak{M}^2c^2$ (as in (2.1) - note $P^\mu = -\partial^\mu S$) and the continuity equation is $\partial_\mu(\rho\partial^\mu S) = 0$. The problem of $\mathcal{M}^2$ not being positive definite here (i.e. tachyons) is serious however and in fact (2.2) is not the correct equation (see e.g. [**871, 873, 876**]). One must use the covariant derivatives ∇_μ in place of ∂_μ and for spin zero in a curved background there results ($\mathcal{Q}$ as above)

$$\nabla_\mu(\rho\nabla^\mu S) = 0;\ g^{\mu\nu}\nabla_\mu S\nabla_\nu S = \mathfrak{M}^2c^2;\ \mathfrak{M}^2 = m^2e^{\mathcal{Q}} \tag{2.4}$$

To see this one must require that a correct relativistic equation of motion should not only be Poincaré invariant but also it should have the correct nonrelativistic limit. Thus for a relativistic particle of mass $\mathfrak{M}$ (which is a Lorentz invariant quantity) $\mathfrak{A} = \int d\lambda(1/2)\mathfrak{M}(r)(dr_\mu/d\lambda)(dr^\nu/d\lambda)$ is the action functional where λ is any scalar parameter parametrizing the path $r_\mu(\lambda)$ (it could e.g. be the proper time τ). Varying the path via $r_\mu \to r'_\mu = r_\mu + \epsilon_\mu$ one gets (cf. [**871**])

$$\mathfrak{A} \to \mathfrak{A}' = \mathfrak{A} + \delta\mathfrak{A} = \mathfrak{A} + \int d\lambda\left[\mathfrak{M}\frac{dr_\mu}{d\lambda}\frac{d\epsilon^\mu}{d\lambda} + \frac{1}{2}\frac{dr_\mu}{d\lambda}\frac{dr^\mu}{d\lambda}\epsilon_\nu\partial^\nu\mathfrak{M}\right] \tag{2.5}$$

By least action the correct path satisfies $\delta\mathfrak{A} = 0$ with fixed boundaries so the equation of motion is

$$(d/d\lambda)(\mathfrak{M}u_\mu) = (1/2)u_\nu u^\nu\partial_\mu\mathfrak{M}; \tag{2.6}$$
$$\mathfrak{M}(du_\mu/d\lambda) = ((1/2)\eta_{\mu\nu}u_\alpha u^\alpha - u_\mu u_\nu)\partial^\nu\mathfrak{M}$$

where $u_\mu = dr_\mu/d\lambda$. Now look at the symmetries of the action functional via $\lambda \to \lambda+\delta$. The conserved current is then the Hamiltonian $\mathfrak{H} = -\mathfrak{L}+u_\mu(\partial\mathfrak{L}/\partial u_\mu) = (1/2)\mathfrak{M}u_\mu u^\mu = E$. This can be seen by setting $\delta\mathfrak{A} = 0$ where

$$0 = \delta\mathfrak{A} = \mathfrak{A}' - \mathfrak{A} = \int d\lambda\left[\frac{1}{2}u_\mu u^\mu u^\nu\partial_\nu\mathfrak{M} + \mathfrak{M}u_\mu\frac{du^\mu}{d\lambda}\right]\delta \tag{2.7}$$

which means that the integrand is zero, i.e. $(d/d\lambda)[(1/2)\mathfrak{M}u_\mu u^\mu] = 0$. Since the proper time is defined as $c^2d\tau^2 = dr_\mu dr^\mu$ this leads to $(d\tau/d\lambda) = \sqrt{(2E/\mathfrak{M}c^2)}$ and the equation of motion becomes

$$\mathfrak{M}(dv_\mu/d\tau) = (1/2)(c^2\eta_{\mu\nu} - v_\mu v_\nu)\partial^\nu\mathfrak{M} \tag{2.8}$$

where $v_\mu = dr_\mu/d\tau$. The nonrelativistic limit can be derived by letting the particles velocity be ignorable with respect to light velocity. In this limit the proper time is identical to the time coordinate $\tau = t$ and the result is that the $\mu = 0$ component is satisfied identically via $(r \sim \vec{r})$

$$\mathfrak{M}\frac{d^2r}{dt^2} = -\frac{1}{2}c^2\nabla\mathfrak{M} \Rightarrow m\left(\frac{d^2r}{dt^2}\right) = -\nabla\left[\frac{mc^2}{2}log\left(\frac{\mathfrak{M}}{\mu}\right)\right] \tag{2.9}$$

where μ is an arbitrary mass scale. In order to have the correct limit the term in parenthesis on the right side should be equal to the quantum potential so $(mc^2/2)log(\mathfrak{M}/\mu) = (\hbar^2/2m)(\nabla^2|\psi|/|\psi|)$ and hence

$$\mathfrak{M} = \mu exp[-(\hbar^2/m^2c^2)(\nabla^2|\Psi|/|\Psi|)] \tag{2.10}$$

One infers that the relativistic quantum mass field is $\mathfrak{M} = \mu exp[(\hbar^2/2m)(\Box|\Psi|/|\Psi|)]$ (manifestly invariant) and setting $\mu = m$ we get (cf. also (2.12) below)

$$\mathfrak{M} = mexp[(\hbar^2/m^2c^2)(\Box|\Psi|/|\Psi|)] \tag{2.11}$$

If one starts with the standard relativistic theory and goes to the nonrelativistic limit one does not get the correct nonrelativistic equations; this is a result of an improper decomposition of the wave function into its phase and norm in the KG equation (cf. also [**110**] for related procedures). One notes here also that (2.11) leads to a positive definite mass squared. Also from [**871**] this can be extended to a many particle version and to a curved spacetime. However, for a particle in a curved background we will take (cf. [**873**] which we follow for the rest of this section)

$$\nabla_\mu(\rho\nabla^\mu S) = 0;\ g^{\mu\nu}\nabla_\mu S\nabla_\nu S = \mathfrak{M}^2c^2;\ \mathfrak{M}^2 = m^2e^{\mathfrak{Q}};\ \mathfrak{Q} = \frac{\hbar^2}{m^2c^2}\frac{\Box_g|\Psi|}{|\Psi|} \tag{2.12}$$

((2.11) suggests that $\mathfrak{M}^2 = m^2exp(2\mathfrak{Q})$ but (2.12) is used for compatibility with the KG approach, etc., where $exp(\mathfrak{Q}) \sim 1 + \mathfrak{Q}$ - cf. remarks after (2.28) below - in any case the qualitative features are close here for either formula). Since, following deBroglie, the quantum HJ equation (QHJE) in (2.12) can be written in the form $(m^2/\mathfrak{M}^2)g^{\mu\nu}\nabla_\mu S\nabla_\nu S = m^2c^2$, **the quantum effects are identical to a change of spacetime metric**

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} = (\mathfrak{M}^2/m^2)g_{\mu\nu} \tag{2.13}$$

which is a conformal transformation. The QHJE becomes then $\tilde{g}^{\mu\nu}\tilde{\nabla}_\mu S\tilde{\nabla}_\nu S = m^2c^2$ where $\tilde{\nabla}_\mu$ represents covariant differentiation with respect to the metric $\tilde{g}_{\mu\nu}$ and the continuity equation is then $\tilde{g}_{\mu\nu}\tilde{\nabla}_\mu(\rho\tilde{\nabla}_\nu S) = 0$. The important conclusion here is that the presence of the quantum potential is equivalent to a curved spacetime with its metric given by (2.13). This is a geometrization of the quantum aspects of matter and it seems that there is a dual aspect to the role of geometry in physics. The spacetime geometry sometimes looks like "gravity" and sometimes reveals quantum behavior. The curvature due to the quantum potential may have a large influence on the classical contribution to the curvature of spacetime. The particle trajectory can now be derived from the guidance relation via differentiation of (2.12) leading to the Newton equations of motion

$$\mathfrak{M}\frac{d^2x^\mu}{d\tau^2} + \mathfrak{M}\Gamma^\mu_{\nu\kappa}u^\nu u^\kappa = (c^2g^{\mu\nu} - u^\mu u^\nu)\nabla_\nu\mathfrak{M} \tag{2.14}$$

Using the conformal transformation above (2.14) reduces to the standard geodesic equation.

Now a general "canonical" relativistic system consisting of gravity and classical matter (no quantum effects) is determined by the action

$$\mathcal{A} = \frac{1}{2\kappa}\int d^4x\sqrt{-g}\mathcal{R} + \int d^4x\sqrt{-g}\frac{\hbar^2}{2m}\left(\frac{\rho}{\hbar^2}\mathcal{D}_\mu S\mathcal{D}^\mu S - \frac{m^2}{\hbar^2}\rho\right) \tag{2.15}$$

where $\kappa = 8\pi G$ and $c = 1$ for convenience. It was seen above that via deBroglie the introduction of a quantum potential is equivalent to introducing a conformal factor $\Omega^2 = \mathfrak{M}^2/m^2$ in the metric. Hence in order to introduce quantum effects of matter into the action (2.15) one uses this conformal transformation to get $(1 + Q \sim exp(Q))$

$$\mathfrak{A} = \frac{1}{2\kappa}\int d^4x\sqrt{-\bar{g}}(\bar{\mathcal{R}}\Omega^2 - 6\bar{\nabla}_\mu\Omega\bar{\nabla}^\mu\Omega)+ \tag{2.16}$$

$$+\int d^4x\sqrt{-\bar{g}}\left(\frac{\rho}{m}\Omega^2\bar{\nabla}_\mu S\bar{\nabla}^\mu S - m\rho\Omega^4\right) + \int d^4x\sqrt{-\bar{g}}\lambda\left[\Omega^2 - \left(1 + \frac{\hbar^2}{m^2}\frac{\Box\sqrt{\rho}}{\sqrt{\rho}}\right)\right]$$

where a bar over any quantity means that it corresponds to the nonquantum regime. Here only the first two terms of the expansion of $\mathfrak{M}^2 = m^2exp(\mathfrak{Q})$ in (2.12) have been used, namely $\mathfrak{M}^2 \sim m^2(1 + \mathfrak{Q})$. No physical change is involved in considering all the terms. λ is a Lagrange multiplier introduced to identify the conformal factor with its Bohmian value. One uses here $\bar{g}_{\mu\nu}$ to raise of lower indices and to evaluate the covariant derivatives; the physical metric (containing the quantum effects of matter) is $g_{\mu\nu} = \Omega^2\bar{g}_{\mu\nu}$. By variation of the action with respect to $\bar{g}_{\mu\nu}$, Ω, ρ, S, and λ one arrives at the following quantum equations of motion:

(1) The equation of motion for Ω

$$\bar{\mathcal{R}}\Omega + 6\Box\Omega + \frac{2\kappa}{m}\rho\Omega(\bar{\nabla}_\mu S\bar{\nabla}^\mu S - 2m^2\Omega^2) + 2\kappa\lambda\Omega = 0 \tag{2.17}$$

(2) The continuity equation for particles $\bar{\nabla}_\mu(\rho\Omega^2\bar{\nabla}^\mu S) = 0$

(3) The equations of motion for particles (here $a' \equiv \bar{a}$)

$$(\bar{\nabla}_\mu S\bar{\nabla}^\mu S - m^2\Omega^2)\Omega^2\sqrt{\rho} + \frac{\hbar^2}{2m}\left[\Box'\left(\frac{\lambda}{\sqrt{\rho}}\right) - \lambda\frac{\Box'\sqrt{\rho}}{\rho}\right] = 0 \tag{2.18}$$

(4) The modified Einstein equations for $\bar{g}_{\mu\nu}$

(2.19)

$$\Omega^2\left[\bar{\mathcal{R}}_{\mu\nu} - \tfrac{1}{2}\bar{g}_{\mu\nu}\bar{\mathcal{R}}\right] - [\bar{g}_{\mu\nu}\Box' - \bar{\nabla}_\mu\bar{\nabla}_\nu]\Omega^2 - 6\bar{\nabla}_\mu\Omega\bar{\nabla}_\nu\Omega + 3\bar{g}_{\mu\nu}\bar{\nabla}_\alpha\Omega\bar{\nabla}^\alpha\Omega +$$
$$+\frac{2\kappa}{m}\rho\Omega^2\bar{\nabla}_\mu S\bar{\nabla}_\nu S - \frac{\kappa}{m}\rho\Omega^2\bar{g}_{\mu\nu}\bar{\nabla}_\alpha S\bar{\nabla}^\alpha S + \kappa m\rho\Omega^4\bar{g}_{\mu\nu} +$$
$$+\frac{\kappa\hbar^2}{m^2}\left[\bar{\nabla}_\mu\sqrt{\rho}\bar{\nabla}_\nu\left(\frac{\lambda}{\sqrt{\rho}}\right) + \bar{\nabla}_\nu\sqrt{\rho}\bar{\nabla}_\mu\left(\frac{\lambda}{\sqrt{\rho}}\right)\right] - \frac{\kappa\hbar^2}{m^2}\bar{g}_{\mu\nu}\bar{\nabla}_\alpha\left[\lambda\frac{\bar{\nabla}^\alpha\sqrt{\rho}}{\sqrt{\rho}}\right] = 0$$

(5) The constraint equation $\Omega^2 = 1 + (\hbar^2/m^2)[(\Box\sqrt{\rho})/\sqrt{\rho}]$

Thus the back reaction effects of the quantum factor on the background metric are contained in these highly coupled equations (cf. also [**27**]). A simpler form of (2.17) can be obtained by taking the trace of (2.19) and using (2.17) which produces $\lambda = (\hbar^2/m^2)\bar{\nabla}_\mu[\lambda(\bar{\nabla}^\mu\sqrt{\rho})/\sqrt{\rho}]$. A solution of this via perturbation methods using the small parameter $\alpha = \hbar^2/m^2$ yields the trivial solution $\lambda = 0$ so the above equations reduce to

$$\bar{\nabla}_\mu(\rho\Omega^2\bar{\nabla}^\mu S) = 0;\ \bar{\nabla}_\mu S\bar{\nabla}^\mu S = m^2\Omega^2;\ \mathfrak{G}_{\mu\nu} = -\kappa\mathfrak{T}^{(m)}_{\mu\nu} - \kappa\mathfrak{T}^{(\Omega)}_{\mu\nu} \tag{2.20}$$

where $\mathfrak{T}^{(m)}_{\mu\nu}$ is the matter energy-momentum (EM) tensor and

$$\kappa\mathfrak{T}^{(\Omega)}_{\mu\nu} = \frac{[g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu]\Omega^2}{\Omega^2} + 6\frac{\nabla_\mu\Omega\nabla_\nu\Omega}{\omega^2} - 2g_{\mu\nu}\frac{\nabla_\alpha\Omega\nabla^\alpha\Omega}{\Omega^2} \tag{2.21}$$

with $\Omega^2 = 1+\alpha(\Box\sqrt{\rho}/\sqrt{\rho})$. Note that the second relation in (2.20) is the Bohmian equation of motion and written in terms of $g_{\mu\nu}$ it becomes $\nabla_\mu S\nabla^\mu S = m^2c^2$.

In the preceeding one has tacitly assumed that there is an ensemble of quantum particles so what about a single particle? One translates now the quantum potential into purely geometrical terms without reference to matter parameters so that the original form of the quantum potential can only be deduced after using the field equations. Thus the theory will work for a single particle or an ensemble and in this connection we make

REMARK 3.2.1. One notes that the use of $\psi\psi^*$ automatically suggests or involves an ensemble if it is to be interpreted as a probability density. Thus the idea that a particle has only a probability of being at or near x seems to mean that some paths take it there but others don't and this is consistent with Feynman's use of path integrals for example. This seems also to say that there is no such thing as a particle, only a collection of versions or cloud connected to the particle idea. Bohmian theory on the other hand for a fixed energy gives a one parameter family of trajectories associated to ψ (see here Section 2.2 and [**197**] for details). This is because the trajectory arises from a third order differential while fixing the solution ψ of the second order stationary Schrödinger equation involves only two "boundary" conditions. As was shown in [**197**] this automatically generates a Heisenberg inequality $\Delta x \Delta p \geq c\hbar$; i.e. the uncertainty is built in when using the wave function ψ and amazingly can be expressed by the operator theoretical framework of quantum mechanics. Thus a one parameter family of paths can be associated with the use of $\psi\psi^*$ and this generates the cloud or ensemble automatically associated with the use of ψ. In fact, based on Remark 2.2.2, one might conjecture that upon using a wave function discription of quantum particle motion, one opens the door to a cloud of particles, all of whose motions are incompletely governed by the SE, since one determining condition for particle motion is ignored. Thus automatically the quantum potential will give rise to a force acting on any such particular trajectory and the "ensemble" idea naturally applies to a cloud of identical particles (cf. also Theorem 1.2.1 and Corollary 1.2.1).

Now first ignore gravity and look at the geometrical properties of the conformal factor given via

$$g_{\mu\nu} = e^{4\Sigma}\eta_{\mu\nu};\ e^{4\Sigma} = \frac{\mathfrak{M}^2}{m^2} = exp\left(\alpha\frac{\Box_\eta\sqrt{\rho}}{\sqrt{\rho}}\right) = exp\left(\alpha\frac{\Box_\eta\sqrt{|\mathfrak{T}|}}{\sqrt{|\mathfrak{T}|}}\right) \tag{2.22}$$

where $\mathfrak{T}$ is the trace of the EM tensor and is substituted for ρ (true for dust). The Einstein tensor for this metric is

$$\mathfrak{G}_{\mu\nu} = 4g_{\mu\nu}\Box_\eta exp(-\Sigma) + 2exp(-2\Sigma)\partial_\mu\partial_\nu exp(2\Sigma) \tag{2.23}$$

Hence as an Ansatz one can suppose that in the presence of gravitational effects the field equation would have a form

$$\mathcal{R}_{\mu\nu} - \frac{1}{2}\mathcal{R}g_{\mu\nu} = \kappa\mathfrak{T}_{\mu\nu} + 4g_{\mu\nu}e^{\Sigma}\Box e^{-\Sigma} + 2e^{-2\Sigma}\nabla_\mu\nabla_\nu e^{2\Sigma} \tag{2.24}$$

This is written in a manner such that in the limit $\mathfrak{T}_{\mu\nu} \to 0$ one will obtain (2.22). Taking the trace of the last equation one gets $-\mathcal{R} = \kappa\mathfrak{T} - 12\Box\Sigma + 24(\nabla\Sigma)^2$ which has the iterative solution $\kappa\mathfrak{T} = -\mathcal{R} + 12\alpha\Box[(\Box\sqrt{\mathcal{R}})/\sqrt{\mathcal{R}}]$ leading to

$$\Sigma = \alpha[(\Box\sqrt{|\mathfrak{T}|}/\sqrt{|\mathfrak{T}|})] \simeq \alpha[(\Box\sqrt{|\mathcal{R}|})/\sqrt{|\mathcal{R}|})] \tag{2.25}$$

to first order in α.

One goes now to the field equations for a toy model. First from the above one sees that $\mathfrak{T}$ can be replaced by $\mathcal{R}$ in the expression for the quantum potential or for the conformal factor of the metric. This is important since the explicit reference to ensemble density is removed and the theory works for a single particle or an ensemble. So from (2.24) for a toy quantum gravity theory one assumes the following field equations

$$\mathfrak{G}_{\mu\nu} - \kappa\mathfrak{T}_{\mu\nu} - \mathfrak{Z}_{\mu\nu\alpha\beta}exp\left(\frac{\alpha}{2}\Phi\right)\nabla^\alpha\nabla^\beta exp\left(-\frac{\alpha}{2}\Phi\right) = 0 \tag{2.26}$$

where $\mathfrak{Z}_{\mu\nu\alpha\beta} = 2[g_{\mu\nu}g_{\alpha\beta} - g_{\mu\alpha}g_{\nu\beta}]$ and $\Phi = (\Box\sqrt{|\mathcal{R}|}/\sqrt{|\mathcal{R}|})$. The number 2 and the minus sign of the second term are chosen so that the energy equation derived later will be correct. Note that the trace of (2.26) is

$$\mathcal{R} + \kappa\mathfrak{T} + 6exp(\alpha\Phi/2)\Box exp(-\alpha\Phi/2) = 0 \tag{2.27}$$

and this represents the connection of the Ricci scalar curvature of space time and the trace of the matter EM tensor. If a perturbative solution is admitted one can expand in powers of α to find $\mathcal{R}^{(0)} = -\kappa\mathfrak{T}$ and $\mathcal{R}^{(1)} = -\kappa\mathfrak{T} - 6exp(\alpha\Phi^0/2)\Box exp(-\alpha\Phi^0/2)$ where $\Phi^{(0)} = \Box\sqrt{|\mathfrak{T}|}/\sqrt{|\mathfrak{T}|}$. The energy relation can be obtained by taking the four divergence of the field equations and since the divergence of the Einstein tensor is zero one obtains

$$\kappa\nabla^\nu\mathfrak{T}_{\mu\nu} = \alpha\mathcal{R}_{\mu\nu}\nabla^\nu\Phi - \frac{\alpha^2}{4}\nabla_\mu(\nabla\Phi)^2 + \frac{\alpha^2}{2}\nabla_\mu\Phi\Box\Phi \tag{2.28}$$

For a dust with $\mathfrak{T}_{\mu\nu} = \rho u_\mu u_\nu$ and u_μ the velocity field, the conservation of mass law is $\nabla^\nu(\rho\mathfrak{M}u_\nu) = 0$ so one gets to first order in α $\nabla_\mu\mathfrak{M}/\mathfrak{M} = -(\alpha/2)\nabla_\mu\Phi$ or $\mathfrak{M}^2 = m^2 exp(-\alpha\Phi)$ where m is an integration constant. This is the correct relation of mass and quantum potential.

In [**873**] there is then some discussion about making the conformal factor dynamical via a general scalar tensor action (cf. also [**867**]) and subsequently one makes both the conformal factor and the quantum potential into dynamical fields and creates a scalar tensor theory with two scalar fields. Thus start with a general action

$$\mathfrak{A} = \int d^4x\sqrt{-g}\left[\phi\mathcal{R} - \omega\frac{\nabla_\mu\phi\nabla^\mu\phi}{\phi} - \frac{\nabla_\mu Q\nabla^\mu Q}{\phi} + 2\Lambda\phi + \mathfrak{L}_m\right] \tag{2.29}$$

The cosmological constant generally has an interaction term with the scalar field and here one uses an ad hoc matter Lagrangian

$$\mathfrak{L}_m = \frac{\rho}{m}\phi^a\nabla_\mu S\nabla^\mu S - m\rho\phi^b - \Lambda(1+Q)^c + \alpha\rho(e^{\ell Q} - 1) \tag{2.30}$$

(only the first two terms $1 + Q$ from $exp(Q)$ are used for simplicity in the third term). Here a, b, c are constants to be fixed later and the last term is chosen (heuristically) in such a manner as to have an interaction between the quantum potential field and the ensemble density (via the equations of motion); further the interaction is chosen so that it vanishes in the classical limit but this is ad hoc. Variation of the above action yields

(1) The scalar fields equation of motion

$$\mathcal{R} + \frac{2\omega}{\phi}\Box\phi - \frac{\omega}{\phi^2}\nabla^\mu\phi\nabla_\mu\phi + 2\Lambda + \tag{2.31}$$

$$+\frac{1}{\phi^2}\nabla^\mu Q\nabla_\mu Q + \frac{a}{m}\rho\phi^{a-1}\nabla^\mu S\nabla_\mu S - mb\rho\phi^{b-1} = 0$$

(2) The quantum potential equations of motion

$$(\Box Q/\phi) - (\nabla_\mu Q\nabla^\mu\phi/\phi^2) - \Lambda c(1+Q)^{c-1} + \alpha\ell\rho exp(\ell Q) = 0 \tag{2.32}$$

(3) The generalized Einstein equations

$$(2.33)\qquad \mathfrak{G}^{\mu\nu}-\Lambda g^{\mu\nu}=-\frac{1}{\phi}\mathfrak{T}^{\mu\nu}-\frac{1}{\phi}[\nabla^\mu\nabla^\nu-g^{\mu\nu}\Box]\phi+\frac{\omega}{\phi^2}\nabla^\mu\phi\nabla^\nu\phi-$$
$$-\frac{\omega}{2\phi^2}g^{\mu\nu}\nabla^\alpha\phi\nabla_\alpha\phi+\frac{1}{\phi^2}\nabla^\mu Q\nabla^\nu Q-\frac{1}{2\phi^2}g^{\mu\nu}\nabla^\alpha Q\nabla_\alpha Q$$

(4) The continuity equation $\nabla_\mu(\rho\phi^a\nabla^\mu S)=0$

(5) The quantum Hamilton Jacobi equation

$$(2.34)\qquad \nabla^\mu S\nabla_\mu S=m^2\phi^{b-a}-\alpha m\phi^{-a}(e^{\ell Q}-1)$$

In (2.31) the scalar curvature and the term $\nabla^\mu S\nabla_\mu S$ can be eliminated using (2.33) and (2.34); further on using the matter Lagrangian and the definition of the EM tensor one has

$$(2.35)\qquad (2\omega-3)\Box\phi=(a+1)\rho\alpha(e^{\ell Q}-1)-2\Lambda(1+Q)^c+2\Lambda\phi-\frac{2}{\phi}\nabla_\mu Q\nabla^\mu Q$$

(where $b=a+1$). Solving (2.32) and (2.35) with a perturbation expansion in α one finds

$$(2.36)\qquad Q=Q_0+\alpha Q_1+\cdots;\ \phi=1+\alpha Q_1+\cdots;\ \sqrt{\rho}=\sqrt{\rho_0}+\alpha\sqrt{\rho_1}+\cdots$$

where the conformal factor is chosen to be unity at zeroth order so that as $\alpha\to 0$ (2.34) goes to the classical HJ equation. Further since by (2.34) the quantum mass is $m^2\phi+\cdots$ the first order term in ϕ is chosen to be Q_1 (cf. (2.12)). Also we will see that $Q_1\sim\Box\sqrt{\rho}/\sqrt{\rho}$ plus corrections which is in accord with Q as a quantum potential field. In any case after some computation one obtains $a=2\omega k$, $b=a+1$, and $\ell=(1/4)(2\omega k+1)=(1/4)(a+1)=b/4$ with $Q_0=[1/c(2c-3)]\{[-(2\omega k+1)/2\Lambda]k\sqrt{\rho_0}-(2c^2-c+1)\}$ while ρ_0 can be determined (cf. [**873**] for details). Thus heuristically the quantum potential can be regarded as a dynamical field and perturbatively one gets the correct dependence of quantum potential upon density, modulo some corrective terms.

One goes next to a number of examples and we only consider here the conformally flat solution (cf. also [**869**]). Thus take $g_{\mu\nu}=exp(2\Sigma)\eta_{\mu\nu}$ where $\Sigma<<1$. One obtains from (2.24)

$$(2.37)\qquad \mathcal{R}_{\mu\nu}=\eta_{\mu\nu}\Box\Sigma+2\partial_\mu\partial_\nu\Sigma\Rightarrow\mathfrak{G}_{\mu\nu}=2\partial_\mu\partial_\nu\Sigma-2\eta_{\mu\nu}\Box\Sigma$$

One can solve this iteratively to get

$$(2.38)\qquad \mathcal{R}^{(0)}=-\kappa\mathfrak{T}\Rightarrow\Sigma^{(0)}=-\frac{\kappa}{6}\Box^{-1}\mathfrak{T};$$
$$\mathcal{R}^{(1)}=-\kappa\mathfrak{T}+3\alpha\Box\frac{\Box\sqrt{|\mathfrak{T}|}}{\sqrt{|\mathfrak{T}|}}\Rightarrow\Sigma^{(1)}=-\frac{\kappa}{6}\Box^{-1}\mathfrak{T}+\frac{\alpha}{2}\frac{\Box\sqrt{|\mathfrak{T}|}}{\sqrt{|\mathfrak{T}|}}$$

Consequently

$$(2.39)\qquad \Sigma=-\frac{\kappa}{6}\Box^{-1}\mathfrak{T}+\frac{\alpha}{2}\frac{\Box\sqrt{|\mathfrak{T}|}}{\sqrt{|\mathfrak{T}|}}+\cdots$$

The first term is pure gravity, the second pure quantum, and the remaining terms involve gravity-quantum interactions. Other impressive examples are given (cf. also [**869**]).

One goes now to a generalized equivalence principle. The gravitational effects determine the causal structure of spacetime as long as quantum effects give its conformal structure. This does not mean that quantum effects have nothing to do with the causal structure; they can act on the causal structure through back reaction terms appearing in the metric field equations. The conformal factor of the metric is a function of the quantum potential and the mass of a relativistic particle is a field produced by quantum corrections to the classical mass. One has shown that the presence of the quantum potential is equivalent to a conformal mapping of the metric. Thus in different conformally related frames one feels different quantum masses and different curvatures. In particular there are two frames with one containing the quantum mass field and the classical metric while the other contains the classical mass and the quantum metric. In general frames both the spacetime metric and the mass field have quantum properties so one can state that different conformal frames are identical pictures of the gravitational and quantum phenomena. We feel different quantum forces in different conformal frames. The question then arises of whether the geometrization of quantum effects implies conformal invariance just as gravitational effects imply general coordinate invariance. One sees here that Weyl geometry provides additional degrees of freedom which can be identified with quantum effects and seems to create a unified geometric framework for understanding both gravitational and quantum forces. Some features here are: (i) Quantum effects appear independent of any preferred length scale. (ii) The quantum mass of a particle is a field. (iii) The gravitational constant is also a field depending on the matter distribution via the quantum potential (cf. [**867, 874**]). (iv) A local variation of matter field distribution changes the quantum potential acting on the geometry and alters it globally; the nonlocal character is forced by the quantum potential (cf. [**868**]).

2.1. DIRAC-WEYL ACTION. Next (still following [**873**]) one goes to Weyl geometry based on the Weyl-Dirac action

$$\mathfrak{A} = \int d^4x\sqrt{-g}(F_{\mu\nu}F^{\mu\nu} - \beta^2\ {}^W\mathcal{R} + (\sigma+6)\beta_{;\mu}\beta^{;\mu} + \mathfrak{L}_{matter}) \tag{2.40}$$

Here $F_{\mu\nu}$ is the curl of the Weyl 4-vector ϕ_μ, σ is an arbitrary constant and β is a scalar field of weight -1. The symbol ";" represents a covariant derivative under general coordinate and conformal transformations (Weyl covariant derivative) defined as $X_{;\mu} = {}^W\nabla_\mu X - \mathcal{N}\phi_\mu X$ where $\mathcal{N}$ is the Weyl weight of X. The equations of motion are then

$$\mathfrak{G}^{\mu\nu} = -\frac{8\pi}{\beta^2}(\mathfrak{T}^{\mu\nu} + M^{\mu\nu}) + \frac{2}{\beta}(g^{\mu\nu\,W}\nabla^{\alpha\,W}\nabla_\alpha\beta - {}^W\nabla^{\mu\,W}\nabla^\nu\beta) + \tag{2.41}$$

$$+\frac{1}{\beta^2}(4\nabla^\mu\beta\nabla^\nu\beta - g^{\mu\nu}\nabla^\alpha\beta\nabla_\alpha\beta) + \frac{\sigma}{\beta^2}(\beta^{;\mu}\beta^{;\nu} - \frac{1}{2}g^{\mu\nu}\beta^{;\alpha}\beta_{;\alpha});$$

$${}^W\nabla_\mu F^{\mu\nu} = \frac{1}{2}\sigma(\beta^2\phi^\mu + \beta\nabla^\mu\beta) + 4\pi J^\mu;$$

$$\mathcal{R} = -(\sigma+6)\frac{{}^W\Box\beta}{\beta} + \sigma\phi_\alpha\phi^\alpha - \sigma{}^W\nabla^\alpha\phi_\alpha + \frac{\psi}{2\beta}$$

where

$$M^{\mu\nu} = (1/4\pi)[(1/4)g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta} - F^{\mu}_{\alpha}F^{\nu\alpha}] \tag{2.42}$$

and

$$8\pi\mathfrak{T}^{\mu\nu} = \frac{1}{\sqrt{-g}}\frac{\delta\sqrt{-g}\mathfrak{L}_{matter}}{\delta g_{\mu\nu}};\ 16\pi J^{\mu} = \frac{\delta\mathfrak{L}_{matter}}{\delta\phi_{\mu}};\ \psi = \frac{\delta\mathfrak{L}_{matter}}{\delta\beta} \tag{2.43}$$

For the equations of motion of matter and the trace of the EM tensor one uses invariance of the action under coordinate and gauge transformations, leading to

$$ {}^{W}\nabla_{\nu}\mathfrak{T}^{\mu\nu} - \mathfrak{T}\frac{\nabla^{\mu}\beta}{\beta} = J_{\alpha}\phi^{\alpha\mu} - \left(\phi^{\mu} + \frac{\nabla^{\mu}\beta}{\beta}\right){}^{W}\nabla_{\alpha}J^{\alpha}; \tag{2.44}$$

$$16\pi\mathfrak{T} - 16\pi{}^{W}\nabla_{\mu}J^{\mu} - \beta\psi = 0$$

The first relation is a geometrical identity (Bianchi identity) and the second shows the mutual dependence of the field equations. Note that in the Weyl-Dirac theory the Weyl vector does not couple to spinors so ϕ_{μ} cannot be interpreted as the EM potential; the Weyl vector is used as part of the spacetime geometry and the auxillary field (gauge field) β represents the quantum mass field. The gravity fields $g_{\mu\nu}$ and ϕ_{μ} and the quantum mass field determine the spacetime geometry. Now one constructs a Bohmian quantum gravity which is conformally invariant in the framework of Weyl geometry. If the model has mass this must be a field (since mass has non-zero Weyl weight). The Weyl-Dirac action is a general Weyl invariant action as above and for simplicity now assume the matter Lagrangian does not depend on the Weyl vector so that $J_{\mu} = 0$. The equations of motion are then

$$\mathfrak{G}^{\mu\nu} = -\frac{8\pi}{\beta^2}(\mathfrak{T}^{\mu\nu} + M^{\mu\nu}) + \frac{2}{\beta}(g^{\mu\nu}{}^{W}\nabla^{\alpha}{}^{W}\nabla_{\alpha}\beta - {}^{W}\nabla^{\mu}{}^{W}\nabla^{\nu}\beta) + \tag{2.45}$$

$$+\frac{1}{\beta^2}(4\nabla^{\mu}\beta\nabla^{\nu}\beta - g^{\mu\nu}\nabla^{\alpha}\beta\nabla_{\alpha}\beta) + \frac{\sigma}{\beta^2}\left(\beta^{;\mu}\beta^{;\nu} - \frac{1}{2}g^{\mu\nu}\beta^{;\alpha}\beta_{;\alpha}\right);$$

$${}^{W}\nabla_{\nu}F^{\mu\nu} = \frac{1}{2}\sigma(\beta^2\phi^{\mu} + \beta\nabla^{\mu}\beta);\ \mathcal{R} = -(\sigma+6)\frac{{}^{W}\Box\beta}{\beta} + \sigma\phi_{\alpha}\phi^{\alpha} - \sigma{}^{W}\nabla^{\alpha}\phi_{\alpha} + \frac{\psi}{2\beta}$$

The symmetry conditions are

$${}^{W}\nabla_{\nu}\mathfrak{T}^{\mu\nu} - \mathfrak{T}(\nabla^{\mu}\beta/\beta) = 0;\ 16\pi\mathfrak{T} - \beta\psi = 0 \tag{2.46}$$

(recall $\mathfrak{T} = \mathfrak{T}^{\mu\nu}_{\mu\nu}$). One notes that from (2.45) results ${}^{W}\nabla_{\mu}(\beta^2\phi^{\mu} + \beta\nabla^{\mu}\beta) = 0$ so ϕ_{μ} is not independent of β. To see how this is related to the Bohmian quantum theory one introduces a quantum mass field and shows it is proportional to the Dirac field. Thus using (2.45) and (2.46) one has

$$\Box\beta + \frac{1}{6}\beta\mathcal{R} = \frac{4\pi}{3}\frac{\mathfrak{T}}{\beta} + \sigma\beta\phi_{\alpha}\phi^{\alpha} + 2(\sigma-6)\phi^{\gamma}\nabla_{\gamma}\beta + \frac{\sigma}{\beta}\nabla^{\mu}\beta\nabla_{\mu}\beta \tag{2.47}$$

This can be solved iteratively via

$$\beta^2 = (8\pi\mathfrak{T}/\mathcal{R}) - \{1/[(\mathcal{R}/6) - \sigma\phi_{\alpha}\phi^{\alpha}]\}\beta\Box\beta + \cdots \tag{2.48}$$

Now assuming $\mathfrak{T}^{\mu\nu} = \rho u^\mu u^\nu$ (dust with $\mathfrak{T} = \rho$) we multiply (2.46) by u_μ and sum to get

$$ {}^W\nabla_\nu(\rho u^\nu) - \rho(u_\mu \nabla^\mu \beta/\beta) = 0 \tag{2.49}$$

Then put (2.46) into (2.49) which yields

$$ u^{\nu W}\nabla_\nu u^\mu = (1/\beta)(g^{\mu\nu} - u^\mu u^\nu)\nabla_\nu \beta \tag{2.50}$$

To see this write (assuming $g^{\mu\nu}\nabla_\nu\beta = \nabla^\mu\beta$)

$$ {}^W\nabla_\nu(\rho u^\mu u^\nu) = u^{\mu W}\nabla_\nu \rho u^\mu + \rho u^{\nu W}\nabla_\nu u^\mu \Rightarrow \tag{2.51}$$

$$\Rightarrow u^\mu\left(\frac{u_\mu \nabla^\mu\beta}{\beta}\right) + u^{\nu W}\nabla_\nu u^\mu - \frac{\nabla^\mu\beta}{\beta} = 0 \Rightarrow u^{\nu W}\nabla_\nu u^\mu = (1 - u^\mu u_\mu)\frac{\nabla^\mu\beta}{\beta} =$$

$$(g^{\mu\nu} - u^\mu u_\mu g^{\mu\nu})\frac{\nabla_\nu\beta}{\beta} = (g^{\mu\nu} - u^\mu u^\nu)\frac{\nabla_\nu\beta}{\beta}$$

which is (2.49). Then from (2.48)

$$ \beta^{2(1)} = \frac{8\pi\mathfrak{T}}{\mathcal{R}};\ \beta^{2(2)} = \frac{8\pi\mathfrak{T}}{\mathcal{R}}\left(1 - \frac{1}{(\mathcal{R}/6) - \sigma\phi_\alpha\phi^\alpha}\frac{\Box\sqrt{\mathfrak{T}}}{\sqrt{\mathfrak{T}}}\right);\cdots \tag{2.52}$$

Comparing with (2.14) and (2.3) shows that we have the correct equations for the Bohmian theory provided one identifies

$$ \beta \sim \mathfrak{M};\ \frac{8\pi\mathfrak{T}}{\mathcal{R}} \sim m^2;\ \frac{1}{\sigma\phi_\alpha\phi^\alpha - (\mathcal{R}/6)} \sim \alpha = \frac{\hbar^2}{m^2c^2} \tag{2.53}$$

Thus β is the Bohmian quantum mass field and the coupling constant α (which depends on $\hbar$) is also a field, related to geometrical properties of spacetime. One notes that the quantum effects and the length scale of the spacetime are related. To see this suppose one is in a gauge in which the Dirac field is constant; apply a gauge transformation to change this to a general spacetime dependent function, i.e.

$$ \beta = \beta_0 \to \beta(x) = \beta_0 exp(-\Xi(x));\ \phi_\mu \to \phi_\mu + \partial_\mu\Xi \tag{2.54}$$

Thus the gauge in which the quantum mass is constant (and the quantum force is zero) and the gauge in which the quantum mass is spacetime dependent are related to one another via a scale change. In particular ϕ_μ in the two gauges differ by $-\nabla_\mu(\beta/\beta_0)$ and since ϕ_μ is a part of Weyl geometry and the Dirac field represents the quantum mass one concludes that the quantum effects are geometrized (cf. also (2.45) which shows that ϕ_μ is not independent of β so the Weyl vector is determined by the quantum mass and thus the geometrical aspects of the manifold are related to quantum effects).

2.2. REMARKS ON CONFORMAL GRAVITY. We go here to a series of papers by Arias, Bonal, Cardenas, Gonzalez, Leyva, Martin, and Quiros (cf. [**46, 47, 48, 133, 181, 182, 796, 797, 798, 799**]) and sketch at some length some results concerning Brans-Dicke theory, conformal gravity, and deBroglie-Bohm-Weyl (dBBW) theory (many other topics are also covered in these papers which we omit here - cf. also [**24, 37, 801, 974, 975**]). The presentation in [**188**] of this material is difficult to read and we try here for a smoother development. In [**188**] we started with [**797, 799**] and then gave a later reformulation from [**798**]; we expand upon this now (still in a more or less chronological order [**799, 797, 798, 796**]) and try to make matters clearer. Questions about the physical significance of Riemannian geometry in relativity have been raised in the past (cf. [**150, 301**]) due to the arbitrariness in the metric tensor resulting from the indefiniteness in the choice of units of measure. In fact Brans-Dicke (BD) theory with a changing dimensionless gravitational coupling constant $Gm^2 \sim \phi^{-1}$ (with m the intertial mass of some elementary particle and ϕ the BD field - $\hbar = c = 1$ here) can be formulated in two different ways since either m or G could vary with position in spacetime. The choice $G \sim \phi^{-1}$ with $m = const.$ leads to the Jordan frame (JF) formalism based on the Lagrangian

$$L^{BD}[g,\phi] = \frac{\sqrt{-g}}{16\pi}\left(\phi R - \frac{\omega}{\phi}g^{nm}\nabla_n\phi\nabla_m\phi\right) + L_M[g] \tag{2.55}$$

where R is the curvature scalar, ω is the BD coupling constant, and $L_M[g]$ is the Lagrangian density for ordinary matter minimally coupled to the scalar field. On the other hand the choice $m \sim \phi^{-1/2}$ with G constant leads to the Einstein frame (EF) BD theory based on the Lagrangian

$$\hat{L}^{BD} = \frac{\sqrt{-g}}{16\pi}\left(\hat{R} - \left(\omega + \frac{3}{2}\right)\hat{g}^{nm}\hat{\nabla}_n\hat{\phi}\hat{\nabla}_m\hat{\phi}\right) + \hat{L}_M[\hat{g},\hat{\phi}] \tag{2.56}$$

where now in the EF metric $\hat{g}$ the ordinary matter is nonminimally coupled to the scalar field $\hat{\phi} \equiv log(\phi)$ through the Lagrangian density $\hat{L}_M[\hat{g},\hat{\phi}]$. Both JF and EF formulations of BD gravity are equivalent representations of the same physical situation since they both belong to the same conformal class (cf. [**150**]); in particular $L^{BD}_{EF} \equiv L^{BD}_{JF}$ via a rescaling of spacetime metric $g \to \hat{g} = \phi g$ or $\hat{g}_{ab} = \phi g_{ab}$ where ϕ is smooth and nonvanishing. This rescaling can be interpreted as a particular transformation of the physical units and any dimensionless number (e.g. Gm^2) is invariant; experimental observations are unchanged since spacetime coincidences are not affected. Hence both based formulations (one based on varying G and the other on varying m are indistinguishable) and one has physically equivalent representations of a same physical situation. The same line of reasoning can be applied if minimal and nonminimal coupling to matter are interchanged via

$$\text{(2.57)}\qquad (\mathbf{A})\ L^{GR}[g,\phi] = \frac{\sqrt{-g}}{16\pi}\left(\phi R - \frac{\omega}{\phi}g^{nm}\nabla_n\phi\nabla_m\phi\right) + L_M[g,\phi];$$

$$(\mathbf{B})\ \hat{L}^{GR} = \frac{\sqrt{-g}}{16\pi}\left(\hat{R} - \left(\omega + \frac{3}{2}\right)\hat{g}^{nm}\hat{\nabla}_n\hat{\phi}\hat{\nabla}_m\hat{\phi}\right) + \hat{L}_M[\hat{g}]$$

Both Lagrangians represent equivalent pictures of GR and **(B)** is simply GR with a scalar field as an additional source of gravity (EFGR) and its conformally equivalent Lagrangian **(A)** refers to Jordan frame GR (JFGR). The field equations derivable from Lagrangian **(B)** are

$$\hat{G}_{ab} = 8\pi\hat{T}_{ab} + \left(\omega + \frac{3}{2}\right)\left(\hat{\nabla}_a\hat{\phi}\hat{\nabla}_b\hat{\phi} - \frac{1}{2}\hat{g}_{ab}\hat{g}^{nm}\hat{\nabla}_n\hat{\phi}\hat{\nabla}_m\hat{\phi}\right); \quad (2.58)$$

$$\Box\hat{\phi} = 0;\ \hat{\nabla}_n\hat{T}^{na} = 0;\ \Box = \hat{g}^{nm}\hat{\nabla}_n\hat{\nabla}_m$$

where $\hat{G}_{ab} = \hat{R}_{ab} - (1/2)\hat{g}_{ab}\hat{R}$ and $\hat{T}_{ab} = (2/\sqrt{\hat{g}})(\partial/\partial\hat{g}^{ab})(\sqrt{-\hat{g}}\hat{L}_M)$. Some disadvantages for JFGR historically involve first that the BD scalar field is nonminimally coupled both to scalar curvature and to ordinary matter so the gravitational constant G varies as $G \sim \phi^{-1}$. At the same time the material test particles don't follow the geodesics of the geometry since they are acted on by both the metric field and the scalar field. In particular masses vary from point to point in spacetime so as to preserve a constant Gm^2 (so $m \sim \phi^{1/2}$). The most serious (but illusory) objection is linked with the formulation of the theory in unphysical variables so that the kinetic energy of the scalar field is not positive deffinite (cf. [**351**]). However one shows in [**799**] that the indefiniteness in the sign of the energy density in the Jordan frame is only apparent; in fact once the scalar field energy density is positive definite in the Einstein frame it is also in the Jordan frame.

Usually the JF formulation of BD gravity is linked with Riemannian geometry (cf. [**150**]). This is directly related to the fact that in the JFBD formalism ordinary matter is minimally coupled to the scalar BD field through $L_M[g]$ in (2.55). This means that point particles follow the geodesics of the Riemannian geometry. This geometry is based on the parallel transport law and length preservation law

$$d\xi^a = -\gamma^a_{nm}\xi^m dx^n;\ dg(\xi,\xi) = 0 \quad (2.59)$$

where $g(\xi,\xi) = g_{nm}\xi^n\xi^m$ and γ^a_{nm} are the affine connections of the manifold. These postulates mean that $\gamma^a_{bc} = \Gamma^a_{bc} = (1/2)g^{an}(g_{nb,c} + g_{nc,b} - g_{bc,n})$ (Christoffel symbols). After the rescaling $\hat{g}_{ab} = \phi g_{ab}$ the above parallel transport and length rules become (recall $\hat{\phi} \sim log(\phi)$)

$$d\xi^a = -\hat{\gamma}^a_{nm}\xi^m dx^n;\ d\hat{g}(\xi,\xi) = dx^n\hat{\nabla}_n\hat{\phi}\hat{g}(\xi,\xi); \quad (2.60)$$

$$\hat{\gamma}^a_{bc} = \hat{\Gamma}^a_{bc} - \frac{1}{2}(\hat{\nabla}_b\hat{\phi}\delta^a_c + \hat{\nabla}_c\hat{\phi}\delta^a_b - \hat{\nabla}^a\hat{\phi}\hat{g}_{bc})$$

Thus the affine connections of the manifold don't coincide with the Christoffel symbols of the metric and one has a Weyl type manifold. Thus JF and EF Lagrangians of both BD and GR theories are connected by conformal rescaling of the metric together with scalar field redefinition. This means JF and EF formulations on the one hand and Riemannian and Weyl type geometries on the other form conformal equivalence classes (uniquely defined only after the coupling of matter fields to the metric). In BD theory for example matter minimally couples to the JF so the test particles follow the geodesics of the Riemannian geometry (i.e. JFBD is linked to Riemannian geometry) while EFBD theory (conformal to JFBD) is linked to a Weyl type geometry. Similarly EFGR is linked with Riemannian geometry and JFGR (conformal to EFGR) is linked to a Weyl type geometry. When the matter part of the Lagrangian is absent both BD and GR theories can be interprted on the grounds of either Riemann or Weyl type geometry and one can conclude that BD and GR theory with an extra scalar field coincide.

The field equations of JFGR can be derived, either directly from (2.57) (equation **(A)**) or by conformally mapping (2.58) back to the JF metric according to $\hat{g} = \phi g$, to obtain

$$\Box\phi = 0;\ \nabla_n T^{na} = \frac{1}{2}\phi^{-1}\nabla^a\phi T; \tag{2.61}$$

$$G_{ab} = \frac{8\pi}{\phi}T_{ab} + \frac{\omega}{\phi^2}(\nabla_a\phi\nabla_b\phi - \frac{1}{2}g_{ab}g^{nm}\nabla - n\phi\nabla_m\phi) + \frac{1}{\phi}(\nabla_a\nabla_b\phi - g_{ab}\Box\phi)$$

where $T_{ab} = (2/\sqrt{-g})(\partial/\partial g^{ab})(\sqrt{-g}L_M)$ is the stress energy tensor for ordinary matter in the Jordan frame. The energy is not conserved because the scalar field ϕ exchanges energy with the metric and with the matter fields. The equation of motion of an uncharged spinless mass point that is acted both by the JF metric field g and the scalar field ϕ is

$$\frac{d^2x^a}{ds^2} = -\Gamma^a_{nm}\frac{dx^m}{ds}\frac{dx^n}{ds} - \frac{1}{2}\phi^{-1}\nabla_n\phi\left(\frac{dx^n}{ds}\frac{dx^a}{ds} - g^{an}\right) \tag{2.62}$$

This does not coincide with the geodesic equation of the JF metric and this, together with the more complex structure of (2.61) in comparison to (2.58), introduces additional complications in the dynamics of matter fields. The fact that the Jordan frame does not lead to a well defined energy momentum tensor for the scalar field is perhaps the most serious objection to this representation (cf. [**351**]). Thus the kinetic energy of the JF scalar field is negative definite or indefinite unlike the Einstein frame where for $\omega > -(3/2)$ it is positive definite; this implies no stable ground state and hence unphysical variables (cf. [**351**]). However although the right side of (2.61) does not have a definite sign the scalar field stress energy tensor can be given the canonical form (cf. [**842**] for example). In [**799**] one obtains the same result as in [**842**] by rewriting equation (2.61) in terms of affine magnitudes in the Weyl type manifold. Thus the affine connections of the JF (Weyl type) manifold γ^a_{bc} are related with the Christoffel symbols of the JF metric through $\gamma^a_{bc} = \Gamma^a_{bc} + (1/2)\phi^{-1}(\nabla_b\phi\delta^a_c + \nabla_c\phi\delta^a_b - \nabla^a\phi g_{bc})$ and one can define the affine Einstein tensor ${}^{\gamma}G_{ab}$ via the γ^a_{bc} instead of the Christoffel symbols of Γ^a_{bc} so that (2.61) becomes

$${}^{\gamma}G_{ab} = \frac{8\pi}{\phi}T_{ab} + \frac{[\omega + (3/2)]}{\phi^2}(\nabla_a\phi\nabla_b\phi - (1/2)g_{ab}g^{nm}\nabla - n\phi\nabla_m\phi) \tag{2.63}$$

Now $(\phi/8\pi)$ times the second term in the right side has the canonical form for the stress energy tensor (true stress energy tensor) while $(\phi/8\pi)$ times the sum of the second and third terms in the right side will be called the effective stress energy tensor for the BD scalar field ϕ (cf. (2.58)). Thus once the scalar field energy density is positive definite in the Einstein frame it is also in the Jordan frame. This removes the main physical objection to the Jordan frame formulation of GR.

Another remarkable feature of JFGR is the invariance under the following conformal transformations

$$\text{(A)}\ \tilde{g}_{ab} = \phi^2 g_{ab};\ \tilde{\phi} = \phi^{-1}\ \ \text{(B)}\ \tilde{g}_{ab} = f g_{ab};\ \tilde{\phi} = f^{-1}\phi \tag{2.64}$$

where f is smooth. Also JFBD based on **(A)** in (2.57) is invariant under the more general rescaling (cf. [**350, 796**])

$$\tilde{g}_{ab} = \phi^{2\alpha} g_{ab};\ \tilde{\phi} = \phi^{1-2\alpha};\ \tilde{\omega} = \frac{\omega - 6\alpha(\alpha-1)}{(1-2\alpha)^2} \tag{2.65}$$

for $\alpha \neq 1/2$. The conformal invariance of a given theory of gravitation under a transformation of physical units is very desirable and in particular **(A)** in (2.57) is thus a better candidate for BD theory than classical theories given by (2.55), (2.56), or **(B)** in (2.57) which are not invariant.

We go now to [**798**] where a number of argments from [**797, 799**] are repeated and amplified for greater clarity. It has been demonstrated already that GR with an extra scalar field and its conformal formulation (JFGR) are different but physically equivalent representations of the same theory. The claim is based on the argument that spacetime coincidences (coordinates) are not affected by a conformal rescaling of the spacetime metric ($\bigstar$) $\hat{g}_{ab} = \Omega^2 g_{ab}$ where Ω^2 is a smooth nonvanishing function on the manifold. Thus the experimental observations (measuements) being nothing but verifications of these coincidences are unchanged too by ($\bigstar$). This means that canonical GR and its conformal image are experimentally indistinguishable. Now a possible objection to this claim could be based on the following argument (which will be refuted). **ARGUMENT**: In canonical GR the matter fields couple minimally to the metric $\hat{g}$ that determines metrical relations on a Riemannian spacetime. Hence matter particles follow the geodesics of the metric $\hat{g}$ (in Riemannian geometry) and their masses are constant over the spacetime manifold, i.e. it is the metric which matter "feels" or perhaps the "physical metric". Under the conformal rescaling the matter fields become non-minimally coupled to the conformal metric g and hence matter particles do not follow the geodesics of this last metric. Furthermore, it is not the metric that determines metrical relations on the Riemannian manifold. This line of reasoning leads to the following conclusion. Although canonical GR and its conformal image may be physically equivalent theories, nevertheless, the physical metric is that which determines metrical relations on a Riemannian spacetime and the conformal metric g is not the physical metric. **REFUTATION - to be developed**: Under the conformal rescaling ($\bigstar$) not only the Lagrangian of the theory is mapped into its conformal Lagrangian but the spacetime geometry itself is mapped too into a conformal geometry. In this last geometry metrical relations involve both the conformal metric g and the conformal factor Ω^2 generating ($\bigstar$). Hence in the conformal Lagrangian the matter fields should "feel" both the metric and the scalar function Ω, i.e. the matter particles would not follow the geodesics of the conformal metric alone. The result is that under ($\bigstar$) the "physical" metric of the untransformed geometry is effectively mapped into the "physical" metric of the conformal geometry. This "missing detail" has apparently been a source of long standing confusion and, although details have been sketched already, more will be provided.

Another question regarding metric theories of spacetime is also clarified, namely the physical content of a given theory of spacetime should be contained in the invariants of the group of position dependent transformations of units and coordinate transformations (cf. [**150**]). All known metric theories of spacetime, including GR, BD, and scalar-tensor theories in general fufill the requirement of invariance under the group of coordinate transformations. It is also evident that any consistent formulation of a given effective theory of spacetime must be invariant also under the group of transformations of units of length, time, and mass. This aspect is treated below and one shows that the only consistent formulation of gravity (among those studied here) is the conformal representation of GR.

Now with some repetition one considers various Lagrangians again. First is that for GR with an extra scalar field, namely ($\alpha \geq 0$ is a free parameter)

$$\hat{\mathfrak{L}}_{GR} = \sqrt{-\hat{g}}(\hat{R} - \alpha(\hat{\nabla}\hat{\phi})^2 + 16\pi\sqrt{-\hat{g}}L_M \tag{2.66}$$

(note $(\hat{\nabla}\hat{\phi})^2 = \hat{g}^{mn}\hat{\phi}_{,m}\hat{\phi}_{,n}$). When $\hat{\phi}$ is constant or $\alpha = 0$ one recovers the usual Einstein theory. Under the conformal rescaling ($\bigstar$) with $\Omega^2 = exp(\hat{\phi})$ the Lagrangian in (2.66) is mapped into its conformal Lagrangian (cf. (2.55))

$$\mathfrak{L}_{GR} = \sqrt{-g}e^{\hat{\phi}}(R - (\alpha - (3/2))(\nabla\hat{\phi})^2 + 16\pi\sqrt{-g}e^{2\hat{\phi}}L_M \equiv \tag{2.67}$$

$$\equiv \sqrt{-g}\left(\phi R - \left(\alpha - \frac{3}{2}\right)\frac{(\nabla\phi)^2}{\phi}\right) + 16\pi\sqrt{-g}L_M$$

(the latter expression having a more usual BD form). Due to the minimal coupling of the scalar field $\hat{\phi}$ to the curvature in canonical GR ((2.66)) the effective gravitational constant $\hat{G}$ (set equal to 1 in (2.66)) is a real constant. The minimal coupling of the matter fields to the metric in (2.66) entails that matter particles follow the geodesics of the metric $\hat{g}$. Hence the inertial mass $\hat{m}$ is constant too over spacetime. This implies that the dimensionless gravitational coupling constant $\hat{G}\hat{m}^2$ ($c = \hbar = 1$) is constant in spacetime - unlike BD theory where this evolves as ϕ^{-1}. This is a conformal invariant feature of GR since dimensionless constants do not change under ($\bigstar$); in other words in conformal GR Gm^2 is constant as well. However in this case ((2.67)) the effective gravitational constant varies like $G \sim \phi^{-1}$ and hence the particle masses vary like $m = exp(\hat{\phi}/2)\hat{m} = \phi^{1/2}\hat{m}$. According to [**301**] the conformal transformation ($\bigstar$) (with $\Omega^2 = \phi$) can be interpreted as a transformation of the units of length time and reciprocal mass; in particular there results $ds = \phi^{-1/2}d\hat{s}$ while $m^{-1} = \phi^{-1/2}\hat{m}^{-1}$. A careful look at (2.66) - (2.67) shows that Einstein's laws of gravity derivable from (2.66) change under the units transformation ($\bigstar$) and this seems to be a serious drawback of canonical GR (and BD theory and scalar-tensor theories in general) since in any consistent theory of spacetime the laws of physics must be invariant under a change of the units of length, time, and mass. This will be clarified below where it is shown that ($\bigstar$) with $\Omega^2 = \phi = exp(\hat{\phi})$ cannot be taken properly as a units transformation. It is just a transformation allowing jumping from one formulation to its conformal equivalent.

In [**797, 799**] one claimed that canonical GR ((2.66)) and its conformal Lagrangian (2.67) are physically equivalent theories since they are indistinguishable from the observational point of view. However it is common to believe that only one of the conformally related metrics is the "physical" metric, i.e. that which detrmines metrical relations on the spacetime manifold. The reasoning leading to this conclusion is based on the following analysis. Take for instance GR with an extra scalar field. Due to the minimal coupling of the matter fields to the metric in (2.66) the matter particles follow the geodesics of the metric $\hat{g}$, namely

$$\frac{d^2x^a}{d\hat{s}^2}+\hat{\Gamma}^a_{mn}\frac{dx^m}{d\hat{s}}\frac{dx^n}{d\hat{s}}=0 \tag{2.68}$$

where $\hat{\Gamma}^a_{bc}=(1/2)\hat{g}^{an}(\hat{g}_{bn,c}+\hat{g}_{cn,b}-\hat{g}_{bc,n})$ are the Christoffel symbols of the metric $\hat{g}$. These coincide with the geodesics of the Riemannian geometry where metrical relations are given by $\hat{g}$ via $\hat{g}(\hat{X},\hat{Y})=\hat{g}_{mn}\hat{X}^m\hat{Y}^n$ and the line element $d\hat{s}^2=\hat{g}_{mn}dx^m dx^n$, etc. It is the reason why canonical GR based on (2.66) is naturally linked with Riemannian geometry (it is the same for JFBD since the matter fields couple minimally to the spacetime metric). The units of this geometry ar constant over the manifold. On the other hand since the matter fields are non-minimally coupled to the metric in the conformal GR the matter particles would not follow the geodesics of the conformal metric g but rather curves which are solutions of the equation conformal to (2.68), namely (2.62) where now Γ^a_{bc} are the Christoffel symbols of the metric g conformal to $\hat{g}$. Hence if one assumes that the spacetime geometry is fixed to be Riemannian and that it is unchanged under the conformal rescaling (★) with $\Omega^2=\phi$ one efffectively arrives at the conclusion that $\hat{g}$ is the "physical" metric. **However this assumption is wrong** and is the source of much long standing confusion (to be further clarified below).

REMARK 3.2.2. One notes that conformal Riemannian geometry (corresponding to Weyl geometry here) develops as follows. Let $\lambda(t)$ be a curve with local coordinates $x^a(t)$ and let X with local coordinates $X^a=dx^a/dt$ be a tangent vector to $\lambda(t)$. The covariant derivative of a given vector field $\hat{Y}$ along λ is given by

$$\frac{\hat{D}\hat{Y}^a}{\partial t}=\frac{\partial\hat{Y}^a}{\partial t}+\hat{\gamma}^a_{mn}\hat{Y}^m\frac{dx^n}{dt} \tag{2.69}$$

where $\hat{\gamma}^a_b$ is a symmetric connection. Given a metric $\hat{g}$ on a manifold $\hat{M}$ the Riemannian geometry is fixed by the condition that there is a unique torsion free (symmetric) connction on $\hat{M}$ such that the covariant derivative of $\hat{g}$ is zero; then parallel transport of vectors $\hat{Y}$ $(\hat{D}\hat{Y}^a/\partial t)=0$ and this preserves scalar products, i.e. $d\hat{g}(\hat{Y},\hat{Y})=0$. The laws of parallel transport and length preservation entail that the symmetric connection $\hat{\gamma}^a_{bc}$ coincides with the Christoffel symbols of the metric $\hat{g}$, so $\hat{\gamma}^a_{bc}=\hat{\Gamma}^a_{bc}$. Suppose now that $\hat{Y}$ transform under (★) (with $\Omega^2=\phi$) as $\hat{Y}=h(\phi)Y^a$; then $dg(Y,Y)=-d[log(\phi h^2)]g(Y,Y)$ which resembles the law of length transport in Weyl geometry. Hence given a Riemannian geometry on $\hat{M}$, under (★) with $\Omega^2=\phi$ one arrives at a Weyl geometry on M. The parallel transport law conformal to (2.69) is

$$\frac{DY^a}{\partial t}+\frac{\partial}{\partial t}(log(h)Y^a;\ \frac{DY^a}{\partial t}=\frac{\partial Y^a}{\partial t}+\gamma^a_{mn}Y^m\frac{dx^n}{dt} \tag{2.70}$$

Here γ^a_{bc} is the symmetric connection on the Weyl manifold M related to the Christoffel symbols of the conformal metric g via

$$\gamma^a_{bc}=\Gamma^a_{bc}+\frac{1}{2}\phi^{-1}(\phi_{,b}\delta^a_c+\phi_{,c}\delta^a_b-g_{bc}g^{an}\phi_{,n}) \tag{2.71}$$

There will be particle motions as in (2.62) and in particular Weyl geometry includes units of measure with point dependent length.

REMARK 3.2.3. We go now to transformations of units following [**796, 798**]. Consider two Lagrangians

$$L_1=\sqrt{-g}(R-\alpha(\nabla\phi)^2);\ L_2=\sqrt{-g}\left(\phi R-\left(\alpha-\frac{3}{2}\right)\frac{(\nabla\phi)^2}{\phi}\right) \tag{2.72}$$

with respect to transformations (★) (note L_2 can be obtained from L_1 by rescaling $g\to\phi g$ and $\phi\to log(\phi)$). Consider first conformal transformations $\tilde{g}_{ab}=\phi^\sigma g_{ab}$ (σ arbitrary) leading to

$$\tilde{L}_1=\sqrt{-\tilde{g}}(\phi^\sigma\tilde{R}+[(3\sigma_(3/2)\sigma^2]\phi^{-2-\sigma}-\alpha\phi^\sigma)(\tilde{\nabla}\phi)^2) \tag{2.73}$$

Hence the laws of gravity it describes change under this transformation; in particular in the conformal (tilde) frame the effective gravitational constant depends on ϕ due to the nonminimal coupling between the scalar field and the curvature. On the other hand L_2 is mapped to

$$\tilde{L}_2=\sqrt{-\tilde{g}}\left(\phi^{1-\sigma}\tilde{R}-\frac{(\alpha-(3/2)-3\sigma+(3/2)\sigma^2)}{(1-\sigma)^2}\phi^{\sigma-1}(\tilde{\nabla}\phi^{1-\sigma}\right) \tag{2.74}$$

Consequently if we introduce a new scalar field variable $\tilde{\phi}=\phi^{1-\sigma}$ and redefine the free parameter of the theory via $\tilde{\alpha}=[\alpha+3\sigma(\sigma-2)]/(1-\sigma)^2$ the Lagrangian $\tilde{L}_2$ takes the form

$$\tilde{L}_2=\sqrt{-\tilde{g}}\left(\tilde{\phi}\tilde{R}-\left(\tilde{\alpha}-\frac{3}{2}\right)\frac{(\tilde{\nabla}\tilde{\phi})^2}{\tilde{\phi}}\right) \tag{2.75}$$

Hence the Lagrangian L_2 is invariant in form under the conformal transformation and field transformation indicated. The composition of two such transformations with parameters σ_1 and σ_2 yields a transformation of the same form with parameter $\sigma_3=\sigma_1+\sigma_2-\sigma_1\sigma_2$. The identity transformation involves $\sigma=0$ and the inverse for σ is a transformation with parameter $\bar{\sigma}=-[\sigma/(1-\sigma)]$. Hence excluding the value $\sigma=1$ we have a one parameter Abelian group of transformations ($\sigma_3(\sigma_1,\sigma_2)=\sigma_3(\sigma_2,\sigma_1)$). This all leads to the conclusion that, since any consistent theory of spacetime must be invariant under the one parameter group of transformations of units (length, time, mass), spacetime theores based on the Lagrangian for pure GR of the form L_1 ar not consistent theories while those based on Lagrangians of the form L_2 may in principle be consistent formulations of a spacetime theory. In particular canonical GR and the Einstein frame formulation of BD theory are not consistent formulations.

Consider now, separately, matter Lagrangians

$$(\mathbf{A})\ \sqrt{-g}\phi^2 L_M;\quad (\mathbf{B})\ \sqrt{-g}L_M \tag{2.76}$$

(**(B)** involves minimal coupling of matter to the metric and **(A)** is non-minimal. Under $\tilde{g}_{ab} = \phi^\sigma g_{ab}$ we have $(\mathbf{A}) \to \sqrt{-\tilde{g}}\phi^{2-2\sigma}L_M$ and hence via scalar field redefinition **(A)** is invariant in form under the group of units transformations. However **(B)** with minimal coupling is not invariant and hence JFBD based on $L_{BD} = L_2 + 16\pi\sqrt{-g}L_M$ coupling (L_2 as in (2.72)) is not a consistent theory of spacetime. The only surviving candidate is conformal GR based on (2.67), namely $L_2 + 16\pi\sqrt{-g}\phi^2 L_M$, and this theory provides a consistent formulation of the laws of gravity. Thus the conformal version of GR involving Weyl type geometry is the object to study and this is picked up again in [**133, 796**] along with some connections to Bohmian theory. Various other topics involving cosmology and singularities are also studied in [**796, 797, 798, 799**] but we omit this here.

REMARK 3.2.4. We make a few comments now following [**133, 796**] about Weyl geometry and the quantum potential. First we have seen that Einstein's GR is incomplete and a Weylian form seems preferable. Secondly there seems to be evidence that a Weylian form can solve (or smooth) various problems involving singularities (cf. [**796, 797, 798, 799**] for some information in this direction). One recalls also the arguments emanating from string theory that a dilaton should coupled to gravity in the low energy limit (cf. [**429**]). It is to be noted that Weyl spacetimes conformally linked to Riemannian structure (such as conformal GR) are called Weyl integrable spacetimes (WISP). The terminology arises from the condition $g_{ab;c} = 0$ for a Riemannian space (where the symbol ";" denotes covariant differentiation. Then if $\hat{g} = \Omega^2 g$ with $\Omega^2 = \phi = exp(\psi)$ (note we are switching the roles of g and $\hat{g}$ used earlier) there results $\hat{g}_{ab;c} = \psi_{,c}\hat{g}_{ab}$ (affine covariant derivative involving $\hat{\Gamma}^a_{bc}$) which are the Weyl affine connection coefficients of the conformal manifold. Comparing to the requirement $\hat{g}_{ab;c} = w_c\hat{g}_{ab}$ for Weyl geometries (w_c is the Weyl gauge vector) we see that Weyl structures conformally linked to Riemannian geometry have the property that $w_c = \psi_{,c}$ (ψ here corresponds to the dilaton) and this is the origin of the term integrable since via $d\ell = \ell dx^n\psi_{,n}$ one arrives at $\oint d\ell = 0$ for WISP (which eliminates the second clock effect often used to critisize Weyl spacetime). Note that the equations of motion of a free particle (or geodesic curves) in the WIST are given by (2.62) (with $\phi = log(\psi)$) and setting e.g. $exp(\psi) = 1 + Q$ where Q is the quantum potential one can regard the last term in (2.62) as the quantum force (see here Section 3.2 for a more refined approach). In any event the moral here is that Weyl geometry implicitly contains the quantum effects of matter - it is already a quantum geometry! In particular a free falling test particle would not "feel" any special quantum force since the effect is built into the free fall.

3. THE SCHRÖDINGER EQUATION IN WEYL SPACE

We go now to Santamato [**840**] and derive the SE from classical mechanics in Weyl space (i.e. from Weyl geometry - cf. also [**63, 188, 189, 219, 224, 490, 841, 989**]). The idea is to relate the quantum force (arising from the quantum potential) to geometrical properties of spacetime; the Klein-Gordon (KG) equation is also treated in this spirit in [**219, 841**] and we discuss this later. One wants to show how geometry acts as a guidance field for matter (as in general relativity). Initial positions are assumed random (as in the Madelung approach) and thus the theory is statistical and is really describing the motion of an ensemble. Thus assume that the particle motion is given by some random process $q^i(t,\omega)$ in a manifold M (where ω is the sample space tag) whose probability density $\rho(q,t)$ exists and is properly normalizable. Assume that the process $q^i(t,\omega)$ is the solution of differential equations

$$\dot{q}^i(t,\omega) = (dq^i/dt)(t,\omega) = v^i(q(t,\omega),t) \tag{3.1}$$

with random initial conditions $q^i(t_0,\omega) = q_0^i(\omega)$. Once the joint distribution of the random variables $q_0^i(\omega)$ is given the process $q^i(t,\omega)$ is uniquely determined by (3.1). One knows that in this situation $\partial_t\rho + \partial_i(\rho v^i) = 0$ (continuity equation) with initial Cauchy data $\rho(q,t) = \rho_0(q)$. The natural origin of v^i arises via a least action principle based on a Lagrangian $L(q,\dot{q},t)$ with

$$L^*(q,\dot{q},t) = L(q,\dot{q},t) - \Phi(q,\dot{q},t);\ \Phi = \frac{dS}{dt} = \partial_t S + \dot{q}^i\partial_i S \tag{3.2}$$

Then $v^i(q,t)$ arises by minimizing

$$I(t_0,t_1) = E[\int_{t_0}^{t_1} L^*(q(t,\omega),\dot{q}(t,\omega),t)dt] \tag{3.3}$$

where t_0, t_1 are arbitrary and E denotes the expectation (cf. [**186, 187, 672, 674, 698**] for stochastic ideas). The minimum is to be achieved over the class of all random motions $q^i(t,\omega)$ obeying (3.2) with arbitrarily varied velocity field $v^i(q,t)$ but having common initial values. One proves first

$$\partial_t S + H(q,\nabla S,t) = 0;\ v^i(q,t) = \frac{\partial H}{\partial p_i}(q,\nabla S(q,t),t) \tag{3.4}$$

Thus the value of I in (3.3) along the random curve $q^i(t,q_0(\omega))$ is

$$I(t_1,t_0,\omega) = \int_{t_0}^{t_1} L^*(q(,q_0(\omega)),\dot{q}(t,q_0(\omega)),t)dt \tag{3.5}$$

Let $\mu(q_0)$ denote the joint probability density of the random variables $q_0^i(\omega)$ and then the expectation value of the random integral is

$$I(t_1,t_0) = E[I(t_1,t_0,\omega)] = \int_{\mathbf{R}^n}\int_{t_0}^{t_1} \mu(q_0)L^*(q(t,q_0),\dot{q}(t,q_0),t)d^nq_0dt \tag{3.6}$$

Standard variational methods give then

$$\delta I = \int_{\mathbf{R}^n} d^nq_0\mu(_0)\left[\frac{\partial L^*}{\partial\dot{q}^i}(q(t_1,q_0),\partial_t q(t_1,q_0),t)\delta q^i(t_1,q_0) - \right. \tag{3.7}$$

$$\left. - \int_{t_0}^{t_1} dt\left(\frac{\partial}{\partial t}\frac{\partial L^*}{\partial\dot{q}^i}(q(t,q_0),\partial_t q)t,q_0),t) - \frac{\partial L^*}{\partial q^i}(q(t,q_0),\partial_t q(t,q_0),t)\right)\delta q^i(t,q_0)\right]$$

where one uses the fact that $\mu(q_0)$ is independent of time and $\delta q^i(t_0, q_0) = 0$ (recall common initial data is assumed). Therefore

$$\text{(3.8)} \qquad (\mathbf{A})\ \ (\partial L^*/\partial \dot{q}^i)(q(t,q_0), \partial_t q(t,q_0), t) = 0;$$

$$(\mathbf{B})\ \ \frac{\partial}{\partial t}\frac{\partial L^*}{\partial \dot{q}^i}(q(t,q_0), \partial_t q(t,q_0,t) - \frac{\partial L^*}{\partial q^i}(q(t,q_0), \partial_t q(t,q_0), t) = 0$$

are the necessary conditions for obtaining a minimum of I. Conditions **(B)** are the usual Euler-Lagrange (EL) equations whereas **(A)** is a consequence of the fact that in the most general case one must retain varied motions with $\delta q^i(t_1, q_0)$ different from zero at the final time t_1. Note that since L^* differs from L by a total time derivative one can safely replace L^* by L in **(B)** and putting (3.2) into **(A)** one obtains the classical equations

$$\text{(3.9)} \qquad p_i = (\partial L/\partial \dot{q}^i)(q(t,q_0), \dot{q}(t,q_0), t) = \partial_i S(q(t,q_0), t)$$

It is known now that if $det[(\partial^2 L/\partial \dot{q}^i \partial \dot{q}^j] \neq 0$ then the second equation in (3.4) is a consequence of the gradient condition (3.9) and of the definition of the Hamiltonian function $H(q,p,t) = p_i \dot{q}^i - L$. Moreover **(B)** in (3.8) and (3.9) entrain the HJ equation in (3.4). In order to show that the average action integral (3.6) actually gives a minimum one needs $\delta^2 I > 0$ but this is not necessary for Lagrangians whose Hamiltonian H has the form

$$\text{(3.10)} \qquad H_C(q,p,t) = \frac{1}{2m} g^{ik}(p_i - A_i)(p_k - A_k) + V$$

with arbitrary fields A_i and V (particle of mass m in an EM field A) which is the form for nonrelativistic applications; given positive definite g_{ik} such Hamiltonians involve sufficiency conditions $det[\partial^2 L/\partial \dot{q}^i \partial \dot{q}^k] = mg > 0$. Finally **(B)** in (3.8) with L^* replaced by L) shows that along particle trajectories the EL equations are satisfied, i.e. the particle undergoes a classical motion with probability one. Notice here that in (3.4) no explicit mention of generalized momenta is made; one is dealing with a random motion entirely based on position. Moreover the minimum principle (3.3) defines a 1-1 correspondence between solutions $S(q,t)$ in (3.4) and minimizing random motions $q^i(t,\omega)$. Provided v^i is given via (3.4) the particle undergoes a classical motion with probability one. Thus once the Lagrangian L or equivalently the Hamiltonian H is given, $\partial_t \rho + \partial_i(\rho v^i) = 0$ and (3.4) uniquely determine the stochastic process $q^i(t,\omega)$. Now suppose that some geometric structure is given on M so that the notion of scalar curvature $R(q,t)$ of M is meaningful. Then we assume (ad hoc) that the actual Lagrangian is

$$\text{(3.11)} \qquad L(q,\dot{q},t) = L_C(q,\dot{q},t) + \gamma(\hbar^2/m)R(q,t)$$

where $\gamma = (1/6)(n-2)/(n-1)$ with $n = dim(M)$. Since both L_C and R are independent of $\hbar$ we have $L \to L_C$ as $\hbar \to 0$.

Now for a differential manifold with $ds^2 = g_{ik}(q)dq^i dq^k$ it is standard that in a transplantation $q^i \to q^i + \delta q^i$ one has $\delta A^i = \Gamma^i_{k\ell} A^\ell dq^k$ with $\Gamma^i_{k\ell}$ general affine connection coefficients on M (Riemannian structure is not assumed). In [**840**] it is assumed that for $\ell = (g_{ik}A^i A^k)^{1/2}$ one has $\delta \ell = \ell \phi_k dq^k$ where the ϕ_k are covariant components of an arbitrary vector (Weyl geometry). Then the actual affine connections $\Gamma^i_{k\ell}$ can be found by comparing this with $\delta\ell^2 = \delta(g_{ik}A^iA^k)$ and using $\delta A^i = \Gamma^i_{k\ell}A^\ell dq^k$. A little linear algebra gives then

$$\Gamma^i_{k\ell} = -\left\{ \begin{matrix} i \\ k\ \ell \end{matrix} \right\} + g^{im}(g_{mk}\phi_\ell + g_{m\ell}\phi_k - g_{k\ell}\phi_m) \tag{3.12}$$

Thus we may prescribe the metric tensor g_{ik} and ϕ_i and determine via (3.12) the connection coefficients. Note that $\Gamma^i_{k\ell} = \Gamma^i_{\ell k}$ and for $\phi_i = 0$ one has Riemannian geometry. Covariant derivatives are defined via

$$A^k_{,1} = \partial_i A^k - \Gamma^{k\ell}A^\ell;\ A_{k,i} = \partial_i A_k + \Gamma^\ell_{ki}A_\ell \tag{3.13}$$

for covariant and contravariant vectors respectively (where $S_{,i} = \partial_i S$). Note Ricci's lemma no longer holds (i.e. $g_{ik,\ell} \neq 0$) so covariant differentiation and operations of raising or lowering indices do not commute. The curvature tensor $R^i_{k\ell m}$ in Weyl geometry is introduced via $A^i_{,k,\ell} - A^i_{,\ell,k} = F^i_{mk\ell}A^m$ from which arises the standard formula of Riemannian geometry

$$R^i_{mk\ell} = -\partial_\ell\Gamma^i_{mk} + \partial_k\Gamma^i_{m\ell} + \Gamma^i_{n\ell}\Gamma^n_{mk} - \Gamma^i_{nk}\Gamma^n_{m\ell} \tag{3.14}$$

where (3.12) is used in place of the Christoffel symbols. The tensor $R^i_{mk\ell}$ obeys the same symmetry relations as the curvature tensor of Riemann geometry as well as the Bianchi identity. The Ricci symmetric tensor R_{ik} and the scalar curvature R are defined by the same formulas also, viz. $R_{ik} = R^\ell_{i\ell k}$ and $R = g^{ik}R_{ik}$. For completeness one derives here

$$R = \dot{R} + (n-1)[(n-2)\phi_i\phi^i - 2(1/\sqrt{g})\partial_i(\sqrt{g}\phi^i)] \tag{3.15}$$

where $\dot{R}$ is the Riemannian curvature built by the Christoffel symbols. Thus from (3.12) one obtains

$$g^{k\ell}\Gamma^i_{k\ell} = -g^{k\ell}\left\{ \begin{matrix} i \\ k\ \ell \end{matrix} \right\} - (n-2)\phi^i;\ \Gamma^i_{k\ell} = -\left\{ \begin{matrix} i \\ k\ \ell \end{matrix} \right\} + n\phi_k \tag{3.16}$$

Since the form of a scalar is independent of the coordinate system used one may compute R in a geodesic system where the Christoffel symbols and all $\partial_\ell g_{ik}$ vanish; then (3.12) reduces to $\Gamma^i_{k\ell} = \phi_k\kappa^i_\ell + \phi_\ell\delta^i_k - g_{k\ell}\phi^i$ and hence

$$R = -g^{km}\partial_m\Gamma^i_{k\ell} + \partial_i(g^{k\ell}\Gamma^i_{k\ell}) + g^{\ell m}\Gamma^i_{n\ell}\Gamma^n_{mi} - g^{m\ell}\Gamma^i_{n\ell}\Gamma^n_{m\ell} \tag{3.17}$$

Further one has $g^{\ell m}\Gamma^i_{n\ell}\Gamma^n_{mi} = -(n-2)(\phi_k\phi^k)$ at the point in consideration. Putting all this in (3.17) one arrives at

$$R = \dot{R} + (n-1)(n-2)(\phi_k\phi^k) - 2(n-1)\partial_k\phi^k \tag{3.18}$$

which becomes (3.15) in covariant form. Now the geometry is to be derived from physical principles so the ϕ_i cannot be arbitrary but must be obtained by the same averaged least action principle (3.3) giving the motion of the particle. The minimum in (3.3) is to be evaluated now with respect to the class of all Weyl geometries having arbitrarily varied gauge vectors but fixed metric tensor. Note that once (3.11) is inserted in (3.2) the only term in (3.3) containing the gauge vector is the curvature term. Then observing that $\gamma > 0$ when $n \geq 3$ the minimum principle (3.3) may be reduced to the simpler form $E[R(q(t,\omega),t)] = min$ where only the gauge vectors ϕ_i are varied. Using (3.15) this is easily done. First a little argument shows that $\hat{\rho}(q,t) = \rho(q,t)/\sqrt{g}$ transforms as a scalar in a coordinate change and this will be called the scalar probability density of the random motion of the particle (statistical determination of geometry). Starting from $\partial_t\rho + \partial_i(\rho v^i) = 0$ a manifestly covariant equation for $\hat{\rho}$ is found to be $\partial_t\hat{\rho} + (1/\sqrt{g})\partial_i(\sqrt{g}v^i\hat{\rho}) = 0$. Now return to the minimum problem $E[R(q(t,\omega),t)] = min$; from (3.15) and $\hat{\rho} = \rho/\sqrt{g}$ one obtains

$$E[R(q(t,\omega),t)] = E[\dot{R}(q(t,\omega),t)] + \tag{3.19}$$

$$+(n-1)\int_M [(n-2)\phi_i\phi^i - 2(1/\sqrt{g})\partial_i(\sqrt{g}\phi^i)]\hat{\rho}(q,t)\sqrt{g}d^nq$$

Assuming fields go to 0 rapidly enough on ∂M and integrating by parts one gets then

$$E[R] = E[\dot{R}] - \frac{n-1}{n-2}E[g^{ik}\partial_i(log(\hat{\rho})\partial_k(log(\hat{\rho})] + \tag{3.20}$$

$$+\frac{n-1}{n-2}E\{g^{ik}[(n-2)\phi_i + \partial_i(log(\hat{\rho})][(n-2)\phi_k + \partial_k(log(\hat{\rho})]\}$$

Since the first two terms on the right are independent of the gauge vector and g^{ik} is positive definite $E[R]$ will be a minimum when

$$\phi_i(q,t) = -[1/(n-2)]\partial_i[log(\hat{\rho})(q,t)] \tag{3.21}$$

This shows that the geometric properties of space are indeed affected by the presence of the particle and in turn the alteration of geometry acts on the particle through the quantum force $f_i = \gamma(\hbar^2/m)\partial_i R$ which according to (3.15) depends on the gauge vector and its derivatives. It is this peculiar feedback between the geometry of space and the motion of the particle which produces quantum effects.

In this spirit one goes now to a geometrical derivation of the SE. Thus inserting (3.21) into (3.16) one gets

$$R = \dot{R} + (1/2\gamma\sqrt{\hat{\rho}})[1/\sqrt{g})\partial_i(\sqrt{g}g^{ik}\partial_k\sqrt{\hat{\rho}})] \tag{3.22}$$

where the value $(n-2)/6(n-1)$ for γ is used. On the other hand the HJ equation (3.4) can be written as

$$\partial_t S + H_C(q, \nabla S, t) - \gamma(\hbar^2/m)R = 0 \tag{3.23}$$

where (3.11) has been used. When (3.22) is introduced into (3.23) the HJ equation and the continuity equation $\partial_t\hat{\rho} + (1/\sqrt{g})(\sqrt{g}v^i\hat{\rho}) = 0$, with velocity field given by (3.4), form a set of two nonlinear PDE which are coupled by the curvature of space. Therefore self consistent random motions of the particle (i.e. random motions compatible with (3.17)) are obtained by solving (3.23) and the continuity equation simultaneously. For every pair of solutions $S(q,t,\hat{\rho}(q,t))$ one gets a possible random motion for the particle whose invariant probability density is $\hat{\rho}$. The present approach is so different from traditional QM that a proof of equivalence is needed and this is only done for Hamiltonians of the form (3.10) (which is not very restrictive). The HJ equation corresponding to (3.10) is

$$\partial_t S + \frac{1}{2m} g^{ik}(\partial_i S - A_i)(\partial_k S - A_k) + V - \gamma\frac{\hbar^2}{m} R = 0 \tag{3.24}$$

with R given by (3.22). Moreover using (3.4) as well as (3.10) the continuity equation becomes

$$\partial_t \hat{\rho} + (1/m\sqrt{g})\partial_i[\hat{\rho}\sqrt{g}g^{ik}(\partial_k S - A_k)] = 0 \tag{3.25}$$

Owing to (3.22),(3.24) and (3.25) form a set of two nonlinear PDE which must be solved for the unknown functions S and $\hat{\rho}$. Now a straightforward calculations shows that, setting

$$\psi(q,t) = \sqrt{\hat{\rho}(q,t)} exp](i/\hbar)S(q,t)], \tag{3.26}$$

the quantity ψ obeys a linear PDE (corrected from [**840**])

$$i\hbar\partial_t\psi = \frac{1}{2m}\left\{\left[\frac{i\hbar\partial_i\sqrt{g}}{\sqrt{g}} + A_i\right] g^{ik}(i\hbar\partial_k + A_k)\right\}\psi + \left[V - \gamma\frac{\hbar^2}{m}\dot{R}\right]\psi \tag{3.27}$$

where only the Riemannian curvature $\dot{R}$ is present (any explicit reference to the gauge vector ϕ_i having disappeared). (3.27) is of course the SE in curvilinear coordinates whose invariance under point transformations is well known. Moreover (3.26) shows that $|\psi|^2 = \hat{\rho}(q,t)$ is the invariant probability density of finding the particle in the volume element $d^n q$ at time t. Then following Nelson's arguments that the SE together with the density formula contains QM the present theory is physically equivalent to traditional nonrelativistic QM. One sees also from (3.26) and (3.27) that the time independent SE is obtained via $S = S_0(q) - Et$ with constant E and $\hat{\rho}(q)$. In this case the scalar curvature of space becomes time independent; since starting data at t_0 is meaningless one replaces the continuity equation with a condition $\int_M \hat{\rho}(q)\sqrt{g}d^n q = 1$.

REMARK 3.3.1. We recall (cf. [**188**]) that in the nonrelativistic context the quantum potential has the form $Q = -(\hbar^2/2m)(\partial^2\sqrt{\rho}/\sqrt{\rho})$ ($\rho \sim \hat{\rho}$ here) and in more dimensions this corresponds to $Q = -(\hbar^2/2m)(\Delta\sqrt{\rho}/\sqrt{\rho})$. Here we have a SE involving $\psi = \sqrt{\rho} exp[(i/\hbar)S]$ with corresponding HJ equation (3.24) which corresponds to the flat space 1-D $S_t + (s')^2/2m + V + Q = 0$ with continuity equation $\partial_t\rho + \partial(\rho S'/m) = 0$ (take $A_k = 0$ here). The continuity equation in (3.25) corresponds to $\partial_t\rho + (1/m\sqrt{g})\partial_i[\rho\sqrt{g}g^{ik}(\partial_k S)] = 0$. For $A_k = 0$ (3.24) becomes

$$\partial_t S + (1/2m)g^{ik}\partial_i S\partial_k S + V - \gamma(\hbar^2/m)R = 0 \tag{3.28}$$

This leads to an identification $Q \sim -\gamma(\hbar^2/m)R$ where R is the Ricci scalar in the Weyl geometry (related to the Riemannian curvature built on standard Christoffel symbols via (3.15)). Here $\gamma = (1/6)[(n-2)/(n-2)]$ as above which for $n = 3$ becomes $\gamma = 1/12$; further the Weyl field $\phi_i = -\partial_i log(\rho)$. Consequently (see below).

PROPOSITION 3.1. For the SE (3.27) in Weyl space the quantum potential is $Q = -(\hbar^2/12m)R$ where R is the Weyl-Ricci scalar curvature. For Riemannian flat space $\dot{R} = 0$ this becomes via (3.22)

$$R = \frac{1}{2\gamma\sqrt{\rho}}\partial_i g^{ik}\partial_k\sqrt{\rho} \sim \frac{1}{2\gamma}\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}} \Rightarrow Q = -\frac{\hbar^2}{2m}\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}} \tag{3.29}$$

as it should and the SE (3.27) reduces to the standard SE in the form $i\hbar\partial_t\psi = -(\hbar^2/2m)\Delta\psi + V\psi$ $(A_k = 0)$.

3.1. FISHER INFORMATION REVISITED. Via Remarks 1.1.4, 1.1.5, and 1.1.6 from Chapter 1 (based on [**395, 446, 447, 448, 449, 805, 806**]) we recall the derivation of the SE in Theorem 1.1.1. Thus with some repetition recall first that the classical Fisher information associated with translations of a 1-D observable X with probability density $P(x)$ is

$$F_X = \int dx\, P(x)([log(P(x)]')^2 > 0 \tag{3.30}$$

One has a well known Cramer-Rao inequality $Var(X) \geq F_X^{-1}$ where $Var(X) \sim$ variance of X. A Fisher length for X is defined via $\delta X = F_X^{-1/2}$ and this quantifies the length scale over which $p(x)$ (or better $log(p(x))$) varies appreciably. Then the root mean square deviation ΔX satisfies $\Delta X \geq \delta X$. Let now P be the momentum observable conjugate to X, and P_{cl} a classical momentum observable corresponding to the state ψ given via $p_{cl}(x) = (\hbar/2i)[(\psi'/\psi) - (\bar{\psi}'/\bar{\psi})]$. One has then the identity $< p >_\psi = < p_{cl} >_\psi$ following via integration by parts. Now define the nonclassical momentum by $p_{nc} = p - p_{cl}$ and one shows then

$$\Delta X \Delta p \geq \delta X \Delta p \geq \delta X \Delta p_{nc} = \hbar/2 \tag{3.31}$$

Then consider a classical ensemble of n-dimensional particles of mass m moving under a potential V. The motion can be described via the HJ and continuity equations

$$\frac{\partial s}{\partial t} + \frac{1}{2m}|\nabla s|^2 + V = 0;\ \frac{\partial P}{\partial t} + \nabla\cdot\left[P\frac{\nabla s}{m}\right] = 0 \tag{3.32}$$

for the momentum potential s and the position probability density P (note that there is no quantum potential and this will be supplied by the information term). These equations follow from the variational principle $\delta L = 0$ with Lagrangian $L = \int dt\, d^n x\, P\left[(\partial s/\partial t) + (1/2m)|\nabla s|^2 + V\right]$. It is now assumed that the classical Lagrangian must be modified due to the existence of random momentum fluctuations. The nature of such fluctuations is immaterial and one can assume that the momentum associated with position x is given by $p = \nabla s + N$ where the fluctuation term N vanishes on average at each point x. Thus s changes to being an average momentum potential. It follows that the average kinetic energy $< |\nabla s|^2 > /2m$ appearing in the Lagrangian above should be replaced by $< |\nabla s + N|^2 > /2m$ giving rise to

$$L' = L + (2m)^{-1}\int dt < N\cdot N >= L + (2m)^{-1}\int dt(\Delta N)^2 \tag{3.33}$$

where $\Delta N = < N \cdot N >^{1/2}$ is a measure of the strength of the quantum fluctuations. The additional term is specified uniquely, up to a multiplicative constant, by the three assumptions given in Remark 1.1.4 This leads to the result that

$$(\Delta N)^2 = c \int d^n x \, P |\nabla log(P)|^2 \tag{3.34}$$

where c is a positive universal constant (cf. [**446**]). Further for $\hbar = 2\sqrt{c}$ and $\psi = \sqrt{P} exp(is/\hbar)$ the equations of motion for p and s arising from $\delta L' = 0$ are $i\hbar \frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m} \nabla^2 \psi + V\psi$.

A second derivation is given in Remark 1.1.5. Thus let $P(y^i)$ be a probability density and $P(y^i + \Delta y^i)$ be the density resulting from a small change in the y^i. Calculate the cross entropy via

$$J(P(y^i + \Delta y^i) : P(y^i)) = \int P(y^i + \Delta y^i) log \frac{P(y^i + \Delta y^i)}{P(y^i)} d^n y \simeq \tag{3.35}$$

$$\simeq \left[\frac{1}{2} \int \frac{1}{P(y^i)} \frac{\partial P(y^i)}{\partial y^i} \frac{\partial P(y^i)}{\partial y^k)} d^n y \right] \Delta y^i \Delta y^k = I_{jk} \Delta y^i \Delta y^k$$

The I_{jk} are the elements of the Fisher information matrix. The most general expression has the form

$$I_{jk}(\theta^i) = \frac{1}{2} \int \frac{1}{P(x^i|\theta^i)} \frac{\partial P(x^i|\theta^i)}{\partial \theta^j} \frac{\partial P(x^i|\theta^i)}{\partial \theta^k} d^n x \tag{3.36}$$

where $P(x^i|\theta^i)$ is a probability distribution depending on parameters θ^i in addition to the x^i. For $P(x^i|\theta^i) = P(x^i + \theta^i)$ one recovers (3.35). If P is defined over an n-dimensional manifold with positive inverse metric g^{ik} one obtains a natural definition of the information associated with P via

$$I = g^{ik} I_{ik} = \frac{g^{ik}}{2} \int \frac{1}{P} \frac{\partial P}{\partial y^i} \frac{\partial P}{\partial y^k} d^n y \tag{3.37}$$

Now in the HJ formulation of classical mechanics the equation of motion takes the form

$$\frac{\partial S}{\partial t} + \frac{1}{2} g^{\mu\nu} \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu} + V = 0 \tag{3.38}$$

where $g^{\mu\nu} = diag(1/m, \cdots, 1/m)$. The velocity field u^μ is then $u^\mu = g^{\mu\nu}(\partial S/\partial x^\nu)$. When the exact coordinates are unknown one can describe the system by means of a probability density $P(t, x^\mu)$ with $\int P d^n x = 1$ and

$$(\partial P/\partial t) + (\partial/\partial x^\mu)(P g^{\mu\nu}(\partial S/\partial x^\nu) = 0 \tag{3.39}$$

These equations completely describe the motion and can be derived from the Lagrangian

$$L_{CL} = \int P \left\{ (\partial S/\partial t) + (1/2) g^{\mu\nu} (\partial S/\partial x^\mu)(\partial S/\partial x^\nu) + V \right\} dt d^n x \tag{3.40}$$

using fixed endpoint variation in S and P. Quantization is obtained by adding a term proportional to the information I defined in (3.37). This leads to

(3.41)
$$L_{QM} = L_{CL} + \lambda I = \int P\left\{\frac{\partial S}{\partial t} + \frac{1}{2}g^{\mu\nu}\left[\frac{\partial S}{\partial x^\mu}\frac{\partial S}{\partial x^\nu} + \frac{\lambda}{P^2}\frac{\partial P}{\partial x^\mu}\frac{\partial P}{\partial x^\nu}\right] + V\right\} dtd^nx$$

Fixed endpoint variation in S leads again to (3.39) while variation in P leads to

$$\frac{\partial S}{\partial t} + \frac{1}{2}g^{\mu\nu}\left[\frac{\partial S}{\partial x^\mu}\frac{\partial S}{\partial x^\nu} + \lambda\left(\frac{1}{P^2}\frac{\partial P}{\partial x^\mu}\frac{\partial P}{\partial x^\nu} - \frac{2}{P}\frac{\partial^2 P}{\partial x^\mu \partial x^\nu}\right)\right] + V = 0 \quad (3.42)$$

These equations are equivalent to the SE if $\psi = \sqrt{P}exp(iS/\hbar)$ with $\lambda = (2\hbar)^2$ (recall also Remark 1.1.6 for connections to entropy). Now following ideas in [**219, 223, 715**] we note in (3.41) for $\phi_\mu \sim A_\mu = \partial_\mu log(P)$ (which arises in (3.21)) and $p_\mu = \partial_u S$, a complex velocity can be envisioned leading to (cf. also [**224**])

$$|p_\mu + i\sqrt{\lambda}A_\mu|^2 = p_\mu^2 + \lambda A_\mu^2 \sim g^{\mu\nu}\left(\frac{\partial S}{\partial x^\mu}\frac{\partial S}{\partial x^\nu} + \frac{\lambda}{P^2}\frac{\partial P}{\partial x^\mu}\frac{\partial P}{\partial x^\nu}\right) \quad (3.43)$$

Further I in (3.37) is exactly known from ϕ_μ so one has a direct connection between Fisher information and the Weyl field ϕ_μ, along with motivation for a complex velocity (cf. Sections 1.2 and 1.3).

REMARK 3.3.2. Comparing now with [**189**] and quantum geometry in the form $ds^2 = \sum(dp_j^2/p_j)$ on a space of probability distributions (to be discussed in Chapter 5) we can define (3.37) as a Fisher information metric in the present context. This should be positive definite in view of its relation to $(\Delta N)^2$ in (3.34) for example. Now for $\psi = Rexp(iS/\hbar)$ one has ($\rho \sim \hat{\rho}$ here)

$$-\frac{\hbar^2}{2m}\frac{R''}{R} \equiv -\frac{\hbar^2}{2m}\frac{\partial^2\sqrt{\rho}}{\sqrt{\rho}} = -\frac{\hbar^2}{8m}\left[\frac{2\rho''}{\rho} - \left(\frac{\rho'}{\rho}\right)^2\right] \quad (3.44)$$

in 1-D while in more dimensions we have a form ($\rho \sim P$)

$$Q \sim -2\hbar^2 g^{\mu\nu}\left[\frac{1}{P^2}\frac{\partial P}{\partial x^\mu}\frac{\partial P}{\partial x^\nu} - \frac{2}{P}\frac{\partial^2 P}{\partial x^\mu \partial x^\nu}\right] \quad (3.45)$$

as in (3.44) (arising from the Fisher metric I of (3.37) upon variation in P in the Lagrangian). It can also be related to an osmotic velocity field $u = D\nabla log(\rho)$ via $Q = (1/2)u^2 + D\nabla \cdot u$ connected to Brownian motion where D is a diffusion coefficient (cf. [**223, 395, 715**]). For $\phi_\mu = -\partial_\mu log(P)$ we have then $\mathbf{u} = -D\phi$ with $Q = D^2((1/2)(|\phi|^2 - \nabla\cdot\phi)$, expressing Q directly in terms of the Weyl vector. This enforces the idea that QM is built into Weyl geometry!

3.2. THE KG EQUATION. The formulation above from [**840**] was modified in [**841**] to a derivation of the Klein-Gordon (KG) equation via an average action principle. The spacetime geometry was then obtained from the average action principle to obtain Weyl connections with a gauge field ϕ_μ (thus the geometry has a statistical origin). The Riemann scalar curvature $\dot{R}$ is then related to the Weyl scalar curvature R via an equation

$$R = \dot{R} - 3[(1/2)g^{\mu\nu}\phi_\mu\phi_\nu + (1/\sqrt{-g})\partial_\mu(\sqrt{-g}g^{\mu\nu}\phi_\nu)] \quad (3.46)$$

Explicit reference to the underlying Weyl structure disappears in the resulting SE (as in (3.27)). The HJ equation in [**841**] has this form (for $A_\mu = 0$ and $V = 0$) $g^{\mu\nu}\partial_\mu S\partial_\nu S = m^2 - (R/6)$ so in some sense (recall here $\hbar = c = 1$) $m^2 - (R/6) \sim \mathfrak{M}^2$ where $\mathfrak{M}^2 = m^2 exp(Q)$ and $Q = (\hbar^2/m^2c^2)(\Box\sqrt{\rho}/\sqrt{\rho}) \sim (\Box\sqrt{\rho}/m^2\sqrt{\rho})$ via Section 3.2 (for signature $(-,+,+,+)$ - recall here $g^{\mu\nu}\nabla_\mu S\nabla_\nu S = \mathfrak{M}^2c^2$). Thus for $exp(Q) \sim 1 + Q$ one has $m^2 - (R/6) \sim m^2(1+Q) \Rightarrow (R/6) \sim -Qm^2 \sim -(\Box\sqrt{\rho}/\sqrt{\rho})$. This agrees also with [**219**] where the whole matter is analyzed incisively (cf. also Remark 3.3.5). We recall also here from [**798**] (cf. Section 3.2.2) that in the conformal geometry the particles do not follow geodesics of the conformal metric alone. We will sketch an elaboration of this now from [**841**] (paper one). Thus summarizing [**840**] and the second paper in [**841**] one shows that traditional QM is equivalent (in some sense) to classical statistical mechanics in Weyl spaces. The following two points of view are taken to be equivalent

(1) **(A)** The spacetime is a Riemannian manifold and the statistical behavior of a spinless particle is described by the KG equation while probabilities combine according to Feynman quantum rules.
(2) **(B)** The spacetime is a generic affinely connected manifold whose actual geometric structure is determined by the matter content. The statistical behavior of a spinless particle is described by classical statistical mechanics and probabilities combine according to Laplace rules.
(3) In nonrelativistic applications the words spacetime, Riemannian, and KG are to be replacedby space, Euclidean, and SE.

We are skipping over the second paper in [**841**] here and going to the first paper which treates matters in a gauge invariant manner. The moral seems to be (loosely) that quantum mechanics in Riemannian spacetime is the same as classical statistical mechanics in a Weyl space. In particular one wants to establish that traditional QM, based on wave equations and ad hoc probability calculus (as in (1) above) is merely a convenient mathematical construction to overcome the complications arising from a nontrivial spacetime geometric structure. Here one works from first principles and includes gauge invariance (i.e. invariance with respect to an arbitrary choice of the spacetime calibration). The spacetime is supposed to be a generic 4-dimensional differential manifold with torsion free connections $\Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu}$ and a metric tensor $g_{\mu\nu}$ with signature $(+,-,-,-)$ (one takes $\hbar = c = 1$ - which I deem unfortunate since the role and effect of such quantities is not revealed). Here the (restrictive) hypothesis of assuming a Weyl geometry from the beginning is released, both the particle motion and the spacetime geometric structure are derived from a single average action principle. A result of this approach is that the spacetime connections are forced to be integrable Weyl connections by the extremization principle.

The particle is supposed to undergo a motion in spacetime with deterministic trajectories and random initial conditions taken on an arbitrary spacelike 3-dimensional hypersurface; thus the theory describes a relativistic Gibbs ensemble of particles (cf. Remark 3.3.3). Both the particle motion and the spacetime connections can be obtained from the average stationary action principle

$$\delta\left[E\left(\int_{\tau_1}^{\tau_2} L(x(\tau,\dot{x}(\tau))d\tau\right)\right]=0 \tag{3.47}$$

This action integral must be parameter invariant, coordinate invariant, and gauge invariant. All of these requirements are met if L is positively homogeneous of the first degree in $\dot{x}^\mu = dx^\mu/d\tau$ and transforms as a scalar of Weyl type $w(L)=0$. The underlying probability measure must also be gauge invariant. A suitable Lagrangian is then

$$L(x,dx)=(m^2-(R/6))^{1/2}ds+A_\mu dx^\mu \tag{3.48}$$

where $ds=(g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu)^{1/2}d\tau$ is the arc length and R is the space time scalar curvature; m is a parameterlike scalar field of Weyl type (or weight) $w(m)=-(1/2)$. The factor 6 is essentially arbitrary and has been chosen for future convenience. The vector field A_μ can be interpreted as a 4-potential due to an externally applied EM field and the curvature dependent factor in front of ds is an effective particle mass. This seems a bit ad hoc but some feeling for the nature of the Lagrangian can be obtained from Section 3.2 (cf. also [**63**]). The Lagrangian will be gauge invariant provided the A_μ have Weyl type $w(A_\mu)=0$. Now one can split A_μ into its gradient and divergence free parts $A_\mu=\bar{A}_\mu-\partial_\mu S$, with both S and $\bar{A}_\mu$ having Weyl type zero, and with $\bar{A}_\mu$ interpreted as and EM 4-potential in the Lorentz gauge. Due to the nature of the action principle regarding fixed endpoints in variation one notes that the average action principle is not invariant under EM gauge transformations $A_\mu\to A_\mu+\partial_\mu S$; but one knows that QM is also not invariant under EM gauge transformations (cf. [**17**]) so there is no incompatability with QM here.

Now the set of all spacetime trajectories accessible to the particle (the particle path space) may be obtained from (3.47) by performing the variation with respect to the particle trajectory with fixed metric tensor, connections, and an underlying probability measure. Thus (cf. Remark 3.3.3) the solution is given by the so-called Carathéodory complete figure (cf. [**826**]) associated with the Lagrangian

$$\bar{L}(x,dx)=(m^2-(R/6))^{1/2}ds+\bar{A}_\mu dx^\mu \tag{3.49}$$

(note this leads to the same equations as (3.48) since the Lagrangians differ by a total differential dS). The resulting complete figure is a geometric entity formed by a one parameter family of hypersurfaces $S(x)=const.$ where S satisfies the HJ equation

$$g^{\mu\nu}(\partial_\mu S-\bar{A}_\mu)(\partial_\nu S-\bar{A}_\nu)=m^2-\frac{R}{6} \tag{3.50}$$

and by a congruence of curves intersecting this family given by

$$\frac{dx^\mu}{ds}=\frac{g^{\mu\nu}(\partial_\nu S-\bar{A}_\nu)}{[g^{\rho\sigma}(\partial_\rho S-\bar{A}_\rho)(\partial_\sigma S-\bar{A}_\sigma)]^{1/2}} \tag{3.51}$$

The congruence yields the actual particle path space and the underlying probability measure on the path space may be defined on an arbitrary 3-dimensional hypersurface intersecting all of the members of the congruence without tangencies (cf. [**443**]). The measure will be completely identified by its probability current density j^μ (see [**841**] and Remark 3.3.3). Moreover, since the measure is independent of the arbitrary choice of the hypersurface, j^μ must be conserved, i.e. $\partial_\mu j^\mu = 0$ (see Remark 3.3.3). Since the trajectories are deterministically defined by (3.51), j^μ must be parallel to the particle 4-velocity (3.51), and hence

$$j^\mu = \rho\sqrt{-g}g^{\mu\nu}(\partial_\nu S - \bar{A}_\nu) \tag{3.52}$$

with some $\rho > 0$. Now gauge invariance of the underlying measure as well as of the complete figure requires that j^μ transforms as a vector density of Weyl type $w(j^\mu) = 0$ and S as a scalar of Weyl type $w(S) = 0$. From (3.52) one sees then that ρ transforms as a scalar of Weyl type $w(\rho) = -1$ and ρ is called the scalar probability density of the particle random motion.

The actual spacetime affine connections are obtained from (3.47) by performing the variation with respect to the fields $\Gamma^\lambda_{\mu\nu}$ for a fixed metric tensor, particle trajectory, and probability measure. It is expedient to tranform the average action principle to the form of a 4-volume integral

$$\delta\left[\int_\Omega d^4x[(m^2 - (R/6))(g_{\mu\nu}j^\mu j^\nu]^{1/2} + A_\mu j^\mu\right] = 0 \tag{3.53}$$

where Ω is the spacetime region occupied by the congruence (3.51) and j^μ is given by (3.52) (cf. [**841**] and Remark 3.3.3 for proofs). Since the connection fields $\Gamma^\lambda_{\mu\nu}$ are contained only in the curvature term R the variational problem (3.53) can be further reduced to

$$\delta\left[\int_\Omega \rho R\sqrt{-g}d^4x\right] = 0 \tag{3.54}$$

(here the HJ equation (3.50) has been used). This states that the average spacetime curvature must be stationary under a variation of the fields $\Gamma^\lambda_{\mu\nu}$ (principle of stationary average curvature). The extremal connections $\Gamma^\lambda_{\mu\nu}$ arising from (3.54) are derived in [**841**] using standard field theory techniques and the result is

$$\Gamma^\lambda_{\mu\nu} = \left\{ \begin{array}{c} \lambda \\ \mu\ \nu \end{array} \right\} + \frac{1}{2}(\phi_\mu\delta^\lambda_\nu + \phi_\nu\delta^\lambda_\mu - g_{\mu\nu}g^{\lambda\rho}\phi_\rho);\ \phi_\mu = \partial_\mu log(\rho) \tag{3.55}$$

This shows that the resulting connections are integrable Weyl connections with a gauge field ϕ_μ (cf. [**840**], Section 3, and Section 3.1). The HJ equation (3.50) and the continuity equation $\partial_\mu j^\mu = 0$ can be consolidated in a single complex equation for S, namely

$$e^{iS}g^{\mu\nu}(iD_\mu - \bar{A}_\mu)(iD_\nu - \bar{A}_\nu)e^{-iS} - (m^2 - (R/6)) = 0;\ D_\mu\rho = 0 \tag{3.56}$$

Here D_μ is (doubly covariant - i.e. gauge and coordinate invariant) Weyl derivative given by (cf. [**63**])

$$D_\mu T^\alpha_\beta = \partial_\mu T^\alpha_\beta + \Gamma^\alpha_{\mu\epsilon}T^\epsilon_\beta - \Gamma^\epsilon_{\mu\beta}T^\alpha_\epsilon + w(T)\phi_\mu T^\alpha_\beta \tag{3.57}$$

It is to be noted that the probability density (but not the rest mass) remains constant relative to D_μ. When written out (3.56) for a set of two coupled partial differential equations for ρ and S. To any solution corresponds a particular random motion of the particle.

Next one notes that (3.56) can be cast in the familiar KG form, i.e.

$$[(i/\sqrt{-g})\partial_\mu\sqrt{-g} - \bar{A}_\mu]g^{\mu\nu}(i\partial_\nu - \bar{A}_\nu)\psi - (m^2 - (\dot{R}/6))\psi = 0 \tag{3.58}$$

where $\psi = \sqrt{\rho}exp(-iS)$ and $\dot{R}$ is the Riemannian scalar curvature built out of $g_{\mu\nu}$ only. We have the (by now) familiar formula

$$R = \dot{R} - 3[(1/2)g^{\mu\nu}\phi_\mu\phi_\nu + (1/\sqrt{-g})\partial_\mu(\sqrt{-g}g^{\mu\nu}\phi_\nu)] \tag{3.59}$$

According to point of view **(A)** above in the KG equation (3.58) any explicit reference to the underlying spacetime Weyl structure has disappeared; thus the Weyl structure is hidden in the KG theory. However we note that no physical meaning is attributed to ψ or to the KG equation. Rather the dynamical and statistical behavior of the particle, regarded as a classical particle, is determined by (3.56), which, although completely equivalent to the KG equation, is expressed in terms of quantities having a more direct physical interpretation.

REMARK 3.3.3. We extract here from the Appendices to paper 1 of [**841**]. In Appendix A one shows that the Carathéodory complete figure formed by the congruence (3.51) solves the variational problem (3.47). One needs the notion of the Gibbs ensemble in relativistic mechanics (cf. [**443**]). Roughly a relativistic Gibbs ensemble of particles may be assimilated to an incoherent globule of matter moving in spacetime. More precisely a relativistic Gibbs ensemble is given by (i) A congruence of timelike curves in spacetime (the path space of the particles) and (ii) A probability measure defined on this congruence (note a congruence of spacelike curves could also be envisioned but causality is affected - a physical intepretation is unclear although it could be related to a statistical formulation of virtual phenomena). The construction here goes as follows. Let K be a 3-parameter congruence of time like curves in spacetime be given via (♦) $x^\mu = x^\mu(\tau, u^k)$ where $k = 1, 2, 3$ and τ is an arbitrary parameter along each curve of the congruence. For simplicity assume that the congruence covers a region Ω of spacetime simply (i.e. one and only one curve of K passes through each point of Ω). Then one can regard (♦) as a change of coordinates from x^μ to y^μ where $y^0 = t$, $y^k = u^k$ (assume the Jacobian is nonzero in Ω). Consider then the action integral $L = \int_{\tau_1}^{\tau_2} L(x(\tau, u^k), \dot{x}(\tau, u^k)d\tau$ with L homogeneous of the first degree in the derivatives $\dot{x}^\mu = \partial x^\mu/\partial\tau$. Given a 1-1 correspondence between the u^k and members of the congruence K one may introduce a formula for the probability that the particle follows a sample path having parameters u^k in some 3-dimensional region B as $prob(B) = \int_{B\subset\mathbf{R}} \mu(u^k)du^1du^2du^3$ where $\mu(u^k)$ is some probability density defined on $\mathbf{R}^3$. Hence the average action integral in (3.47) may be written as

$$I = E\left[\int_{\tau_1}^{\tau_2} Ld\tau\right] = \int_{\mathbf{R}^3}\int_{\tau_1}^{\tau_2} \mu(u^k)L(x^\mu(\tau, u^k), \dot{x}^\mu(\tau, u^k)d\tau \prod du^i \tag{3.60}$$

The last term is a 4-dimensional volume integral over the zone between the hyperplanes $y^0 = \tau_1$ and $y^0 = \tau_2$ in the y coordinate. In the x coordinates these hyperplanes are mapped on two 3-dimensional hypersurfaces $\tau(x^\mu) = \tau_1$ and $\tau(x^\mu) = \tau_2$ where $\tau(x^\mu)$ is obtained by solving (♦) with respect to τ; since they are merely a result of the parametrization of K they can be regarded as essentially arbitrary.

The integrand in (3.60) depends on the 4 unknown functions $x^\mu(y^\nu)$ and on their first derivatives $\partial x^\mu/\partial y^0$, and on the coordinates y^ν themselves. Therefore the variational problem $\delta I = 0$ is reduced to a standard variational problem whose solution will yield the functions $x^\mu(\tau, u^k)$, i.e. the actual congruence that renders the average action stationary.

Now the Lagrangian density in (3.60) is $\Lambda = \mu(u^k)L(x^\mu(\tau,u^k), x^\mu_{,\tau}(\tau,u^k)$ in which $x^\mu_{,\tau} = \dot{x}^\mu$ with τ and u^k are the independent variables. By standard methods the EL expressions are ($x^\mu_{,k} = \partial x^\mu/\partial u^k$)

$$E(\Lambda) = \frac{\partial}{\partial u^k}\left[\frac{\partial \Lambda}{\partial x^\mu_{,k}}\right] + \frac{\partial}{\partial \tau}\left[\frac{\partial \Lambda}{\partial x^\mu_{,\tau}}\right] - \frac{\partial \Lambda}{\partial x^\mu} \tag{3.61}$$

In this case howver $\partial \Lambda/\partial x^\mu_{,k} = 0$ and hence the fixed equations $E(\Lambda) = 0$ reduce to (note μ does not depend explicitly on τ)

$$\frac{\partial}{\partial \tau}\left[\mu \frac{\partial L}{\partial x^\mu_{,\tau}}\right] - \mu \frac{\partial L}{\partial x^\mu} = 0 \Rightarrow \frac{\partial}{\partial \tau}\left[\frac{\partial L}{\partial \dot{x}^\mu}\right] - \frac{\partial L}{\partial x^\mu} = 0 \tag{3.62}$$

and this coincides with the EL equations associated with the action integral above. This means that the actual congruence must be a congruence of extremals or equivalently that the particle obeys equations of motion (3.62) with probability one. Even if the congruence is extremal however we are left with nonvanishing surface terms in the variation of I, namely

$$\delta I = \int_{\mathbf{R}^3} \mu(u^k) \prod du^i \left[\frac{\partial L}{\partial \dot{x}^\mu}(\tau_2, u^k)\delta x^\mu(\tau_2, u^k) - \frac{\partial L}{\partial \dot{x}^\mu}(\tau_1, u^k)\delta x^\mu(\tau_1, u^k)\right] = 0 \tag{3.63}$$

In (3.63) the quantities δx^μ at $\tau = \tau_2$ and $\tau = \tau_1$ are displacements between points P and $P + \delta P$ where the curves x^μ and $x^\mu + \delta x^\mu$ intersect the hypesurfaces $\tau = \tau_2$ and $\tau = \tau_1$ so $\delta x^\mu(\tau_1, u^k)$ and $\delta x^\mu(\tau_1, u^k)$ are tangential to the hypersurfaces. Since the hypersurfaces $\tau(x^\mu) = const.$ are essentially arbitrary so must be the displacements δx^μ and $\delta I = 0$ implies then ($\bullet$) $\partial L/\partial \dot{x}^\mu(\tau, u^k) = 0$. Finally relating L with the Lagrangian (3.48) and comparing with $\bar{L}$ as defined in (3.49) one has $\partial L/\partial \dot{x}^\mu = \partial \bar{L}/\partial \dot{x}^\mu - \partial_\mu S$ so ($\bullet$) yields $\partial \bar{L}/\partial \dot{x}^\mu = \partial_\mu S$. Moreover L and $\bar{L}$, differing only by a total differential dS, lead to the same EL equations and hence one can replace L by $\bar{L}$ in (3.62). In conclusion the congruence that renders the average action stationary must be (i) A congruence of curves that are extremal with respect to Lagrangian $\bar{L}$ and (ii) A congruence satisfying the integrability conditions $\partial \bar{L}/\partial \dot{X}^\mu = \partial_\mu S$. However by standard HJ theory such a congruence is given by (3.51) provided $S(x^\mu)$ obeys the HJ equation associated with $\bar{L}$, namely (3.50).

In appendix B the current density j^μ is introduced and the equivalence between the average action (3.47) and the 4-volume integral (3.53) is proved. This provides a useful connection between ensemble averages and 4-volume integrals appearing in field theories. Here (3.60) is expressed in terms of the y coordinates (τ, u^k) and it can also be expressed in terms of the x coordinates. For this one introduces the current density j^μ associated with the relativistic Gibbs ensemble. The surface element normal to the hypersurface $\tau(u^k) = const.$ is given by $d\sigma_\mu = \pi_\mu du^1 du^2 du^3$ where π_μ are Jacobians

$$\pi_0 = \frac{\partial(x^1, x^2, x^3)}{\partial(u^1, u^2, u^3)};\ \pi_1 = \frac{\partial(x^0, x^2, x^3)}{\partial(u^1, u^2, u^3)}, \cdots \tag{3.64}$$

Then define the current density via $\mu = j^\mu \pi_\mu$ so that $prob(B)$ becomes

$$prob(B) = \int_{B\subset\mathbf{R}^3} \mu du^1 du^2 du^3 = \int_{B\subset\mathbf{R}^3} j^\mu d\sigma_\mu \tag{3.65}$$

The direction of j^μ is still not defined so one is free to choose the current direction parallel to the congruence K, i.e. $j^\mu = \lambda \dot{x}^\mu$. The independence of the underlying measure on the chosen hypersurface $\tau = const.$ is exprssed analytically by the fact that $\mu = \mu(u^1, u^2, u^3)$ does not depend on τ explicitly. Consequently $\partial_\mu j^\mu = 0$ since by the Gauss theorem

$$\int_{\tau(x^\mu)=\tau_2} j^\mu d\sigma_\mu - \int_{\tau(x^\mu)=\tau_1} j^\mu d\sigma_\mu = \int_\Omega \partial_\mu j^\mu d^4x = 0 \tag{3.66}$$

where Ω is the strip between the essentially arbitrary hypersurfaces $\tau = \tau_1$ and $\tau = \tau_2$. The same result could be obtained by differentiating $\mu = j^\mu \pi_\mu$ and using properties of Jacobians. Passing to x coordinates (3.60) becomes

$$I = \int_\Omega \mu L J^{-1} d^4x;\ J = \frac{\partial(x^0, x^1, x^2, x^3)}{\partial(\tau, u^1, u^2, u^3)} \tag{3.67}$$

Note that by definition $J = (\partial x^\mu/\partial\tau)\pi_\mu$ so

$$I = \int_\Omega \mu[L(x^\mu, \dot{x}^\mu)/(\dot{x}^\mu \pi_\mu)] d^4x \tag{3.68}$$

Since L is homogeneous of the first degree in the $\dot{x}^\mu$ the term in square brackets in (3.68) is homogeneous of degree zero in the $\dot{x}^\mu$. Hence we can replace $\dot{x}^\mu$ with the current $j^\mu = \lambda\dot{x}^\mu$ without affecting the integral to obtain $I = \int_\Omega L(x^\mu j^\mu) d^4x$ where $\mu = j^\mu \pi_\mu$ has been used. Thus the average action I may be converted to a four volume integral of $L(x^\mu, j^\mu)$. When this formal substitution is made in (3.48), (3.53) is obtained. This substitution does not alter the functional dependence of the average action integral I on the connection fields $\Gamma^\lambda_{\mu\nu}$ so the variational problems (3.47) and (3.53) are equivalent as long as the variation is performed with respect to these fields.

In Appendix C one derives (3.55); since similar calculations have already been used earlier (and will recur again) we omit this here.

REMARK 3.3.4. The formula (3.59) goes back to Weyl [**986**] and the connection of matter to geometry arises from (3.55). The time variable is treated in a special manner here related to a Gibbs ensemble and $\rho > 0$ is built into the theory. Thus problems of statistical transparancy as in Remark 2.3.3 will apparently not arise.

REMARK 3.3.5. As mentioned at the beginning of Section 3.2, in [**219**] the Santamato theory is analyzed in depth from several points of view and a number of directions for further study are indicated (in [**224**] the importance of a complex velocity is emphasized - see also Section 7.1.2). There is also a related development for the Dirac equation using an approach related to [**397, 463, 464**], where both relativistic and nonrelativistic spin 1/2 particles can be classically treated using anticommuting Grassmanian variables. However we prefer to treat the Dirac equation in a different manner later (cf. also [**89**] and Section 2.1.1).

4. SCALE RELATIVITY AND KG

In [**186**] and Section 1.2 we sketched a few developments in the theory of scale relativity. This is by no means the whole story and we want to give a taste of some further main ideas while deriving the KG equation in this context (cf. [**11, 232, 272, 273, 274, 715, 716, 717, 718, 719, 720, 721**]). A main idea here is that the Schrödinger, Klein-Gordon, and Dirac equations are all geodesic equations in the fractal framework. They have the form $D^2/ds^2 = 0$ where D/ds represents the appropriate covariant derivative. The complex nature of the SE and KG equation arises from a discrete time symmetry breaking based on nondifferentiability. For the Dirac equation further discrete symmetry breakings are needed on the spacetime variables in a biquaternionic context (cf. here [**232**]). First we go back to [**715, 716, 720**] and sketch some of the fundamentals of scale relativity. This is a very rich and beautiful theory extending in both spirit and generality the relativity theory of Einstein (cf. also [**225**] for variations involving Clifford theory). The basic idea here is that (following Einstein) the laws of nature apply whatever the state of the system and hence the relevant variables can only be defined relative to other states. Standard scale laws of power-law type correspond to Galilean scale laws and from them one actually recovers quantum mechanics (QM) in a nondifferentiable space. The quantum behavior is a manifestation of the fractal geometry of spacetime. In particular (as indicated in Section 1.2) the quantum potential is a manifestation of fractality in the same way as the Newton potential is a manifestation of spacetime curvature. In this spirit one can also conjecture (cf. [**720**]) that this quantum potential may explain various dynamical effects presently attributed to dark matter (cf. also [**16**] and Chapter 4). Now for basics one deals with a continuous but nondifferentiable physics. It is known for example that the length of a continuous nondifferentiable curve is dependent on the resolution ϵ. One approach now involves smoothing a nondifferentiable function f via $f(x,\epsilon) = \int_{-\infty}^{\infty}\phi(x,y,\epsilon)f(y)dy$ where ϕ is smooth and say "centered" at x (we refer also to Remark 1.2.8 and [**11, 272, 273, 274**] for a more refined treatment of such matters). There will now arise differential equations involving $\partial f/\partial log(\epsilon)$ and $\partial^2 f/\partial x \partial log(\epsilon)$ for example and the $log(\epsilon)$ term arises as follows. Consider an infinitesimal dilatation $\epsilon \to \epsilon' = \epsilon(1+d\rho)$ with a curve length

$$\ell(\epsilon) \to \ell(\epsilon') = \ell(\epsilon + \epsilon d\rho) = \ell(\epsilon) + \epsilon \ell_\epsilon d\rho = (1 + \tilde{D}d\rho)\ell(\epsilon) \tag{4.1}$$

Then $\tilde{D} = \epsilon\partial_\epsilon = \partial/\partial log(\epsilon)$ is a dilatation operator and in the spirit of renormalization (multiscale approach) one can assume $\partial \ell(x,\epsilon)/\partial log(\epsilon) = \beta(\ell)$ (where $\ell(x,\epsilon)$ refers to the curve defined by $f(x,\epsilon)$). Now for Galilean scale relativity consider

$\partial\ell(x,\epsilon)/\partial log(\epsilon) = a + b\ell$ which has a solution

$$\ell(x,\epsilon) = \ell_0(x)\left[1+\zeta(x)\left(\frac{\lambda}{\epsilon}\right)^{-b}\right] \tag{4.2}$$

where $\lambda^{-b}\zeta(x)$ is an integration constant and $\ell_0 = -a/b$. One can choose $\zeta(x)$ so that $<\zeta^2(x)>=1$ and for $a \neq 0$ there are two regimes (for $b<0$)

(1) $\epsilon << \lambda \Rightarrow \zeta(x)(\lambda/\epsilon)^{-b} >> 1$ and ℓ is given by a scale invariant fractal like power with dimension $D = 1-b$, namely $\ell(x,\epsilon) = \ell_0(\lambda/\epsilon)^{-b}$.
(2) $\epsilon >> \lambda \Rightarrow \zeta(x)(\lambda/\epsilon)^{-b} << 1$ and ℓ is independent of scale.

Here $\epsilon = \lambda$ constitutes a transition point between fractal and nonfractal behavior. Only the special case $a = 0$ yields unbroken scale invariance of $\ell = \ell_0(\lambda/\epsilon)^{\delta}$ ($\delta = -b$) and one has then $\tilde{D}\ell = b\ell$ so the scale dimension is an eigenvalue of $\tilde{D}$. Finally the case $b > 0$ corresponds to the cosmological domain.

Now one looks for scale covariant laws and checks this for power laws $\phi = \phi_0(\lambda/\epsilon)^{\delta}$. Thus a scale transformation for $\delta(\epsilon') = \delta(\epsilon)$ will have the form

$$log\frac{\phi(\epsilon')}{\phi_0} = log\frac{\phi(\epsilon)}{\phi_0} + V\delta(\epsilon);\ V = log\frac{\epsilon}{\epsilon'} \tag{4.3}$$

In the same way that only velocity differences have a physical meaning in Galilean relativity here only V differences or scale differences have a physical meaning. Thus V is a "state of scale" just as velocity is a state of motion. In this spirit laws of linear transformation of fields in a scale transformation $\epsilon \to \epsilon'$ amount to finding $A, B, C, D(V)$ such that

$$log\frac{\phi(\epsilon')}{\phi_0} = A(V)log\frac{\phi(\epsilon)}{\phi_0} + B(V)\delta(\epsilon);\ \delta(\epsilon') = C(V)log\frac{\phi(\epsilon)}{\phi_0} + D(V)\delta(\epsilon) \tag{4.4}$$

Here $A = 1$, $B = V$, $C = 0$, $D = 1$ corresponds to the Galileo group. Note also $\epsilon \to \epsilon' \to \epsilon'' \Rightarrow V'' = V + V'$. Now for the analogue of Lorentz transformations there is a need to preserve the Galilean dilatation law for scales larger than the quantum classical transition. Note $V = log(\epsilon/\epsilon') \sim \epsilon/\epsilon' = exp(-V)$ and set $\rho = \epsilon'/\epsilon$ with $\rho' = \epsilon''/\epsilon'$ and $\rho'' = \epsilon''/\epsilon$; then $log\rho'' = \log\rho + \log\rho'$ and one is thinking here of $\rho : \epsilon \to \epsilon'$, $\rho' : \epsilon' \to \epsilon''$ and $\rho'' : \epsilon \to \epsilon''$ with compositions (the notation is meant to somehow correspond to (4.1)). Now recall the Einstein-Lorentz law $w = (u+v)/[1+(uv/c^2)]$ but one now has several regimes to consider. Following [**716, 720**] small scale symmetry is broken by mass via the emergence of $\lambda_c = \hbar/mc$ (Compton length) and $\lambda_{dB} = \hbar/m\nu$ (deBroglie length), while for extended objects $\lambda_{th} = \hbar/m <\nu^2>^{1/2}$ (thermal deBroglie length) affects transitions. The transition scale in (4.2) is the Einstein-deBroglioe scale (in rest frame $\lambda \sim \tau = \hbar/mc^2$) and in the cosmological realm the scale symmetry is broken by the emergence of static structure of typical size $\lambda_g = (1/3)(GM/<\nu^2>)$. The scale space consists of three domains (quantum, classical - scale independent, and cosmological). Another small scale transition factor appears in the Planck length scale $\lambda_P = (\hbar G/c^3)^{1/2}$ and at large scales the cosmological constant Λ comes into play. With this background the composition of dilatations is taken to be

$$\text{(4.5)} \qquad log\frac{\epsilon'}{\lambda} = \frac{log\rho + \log\frac{\epsilon}{\lambda}}{1 + \frac{log\rho log\frac{\epsilon}{\lambda}}{log^2(L/\lambda)}} = \frac{log\rho + \log\frac{\epsilon}{\lambda}}{1 + \frac{log\rho log(\epsilon/\lambda)}{C^2}}$$

where $L \sim \lambda_P$ near small scales and $L \sim \Lambda$ near large scales (note $\epsilon = L \Rightarrow \epsilon' = L$ in (A.4)). Comparing with $w = (u+v)/(1+(uv/c^2))$ one thinks of $log(L/\lambda) = C \sim c$ (note here $log^2(a/b) = log^2(b/a)$ in comparing formulas in [**716, 720**]). Lengths now change via

$$\text{(4.6)} \qquad log\frac{\ell'}{\ell_0} = \frac{log(\ell/\ell_0) + \delta log\rho}{\sqrt{1 - \frac{\log^2\rho}{C^2}}}$$

and the scale variable δ (or djinn) is no longer constant but changes via

$$\text{(4.7)} \qquad \delta(\epsilon') = \frac{\delta(\epsilon) + \frac{log\rho log(\ell/\ell_0)}{C^2}}{\sqrt{1 - \frac{log^2\rho}{C^2}}}$$

where $\lambda \sim$ fractal-nonfractal transition scale.

We have derived the SE in Section 1.2 (cf. also [**186**]) and go now to the KG equation via scale relativity. The derivation in the first paper of [**232**] seems the most concise and we follow that at first (cf. also [**716**]). All of the elements of the approach for the SE remain valid in the motion relativistic case with the time replaced by the proper time s, as the curvilinear parameter along the geodesics. Consider a small increment dX^μ of a nondifferentiable four coordinate along one of the geodesics of the fractal spacetime. One can decompose this in terms of a large scale part $\overline{LS} < dX^\mu >= dx^\mu = v_\mu ds$ and a fluctuation $d\xi^\mu$ such that $\overline{LS} < d\xi^\mu >= 0$. One is led to write the displacement along a geodesic of fractal dimension $D = 2$ via

$$\text{(4.8)} \qquad dX^\mu_\pm = d_\pm x^\mu + d\xi^\mu_\pm = v^\mu_\pm ds + u^\mu_\pm \sqrt{2\mathcal{D}} ds^{1/2}$$

Here $u^\mu_\pm$ is a dimensionless fluctuation andd the length scale $2\mathcal{D}$ is introduced for dimensional purposes. The large scale forward and backward derivatives d/ds_+ and d/ds_- are defined via

$$\text{(4.9)} \qquad \frac{d}{ds_\pm} f(s) = lim_{s\to 0_\pm} \overline{LS} \left\langle \frac{f(s+\delta s) - f(s)}{\delta s} \right\rangle$$

Applied to x^μ one obtains the forward and backward large scale four velocities of the form

$$\text{(4.10)} \qquad (d/dx_+)x^\mu(s) = v^\mu_+;\ (d/ds_-)x^\mu = v^\mu_-$$

Combining yields

$$\text{(4.11)} \qquad \frac{d'}{ds} = \frac{1}{2}\left(\frac{d}{ds_+} + \frac{d}{ds_-}\right) - \frac{i}{2}\left(\frac{d}{ds_+} - \frac{d}{ds_-}\right);$$

$$\mathcal{V}^\mu = \frac{d'}{ds} x^\mu = V^\mu - iU^\mu = \frac{v^\mu_+ + v^\mu_-}{2} - i\frac{v^\mu_+ - v^\mu_-}{2}$$

For the fluctuations one has

$$\overline{LS} < d\xi_\pm^\mu d\xi_\pm^\nu >= \mp 2\mathcal{D}\eta^{\mu\nu} ds \tag{4.12}$$

One chooses here $(+,-,-,-)$ for the Minkowski signature for $\eta^{\mu\nu}$ and there is a mild problem because the diffusion (Wiener) process makes sense only for positive definite metrics. Various solutions were given in [**314, 859, 1013**] and they are all basically equivalent, amounting to the transformatin a Laplacian into a D'Alembertian. Thus the two forward and backward differentials of $f(x,s)$ should be written as

$$(df/ds_\pm) = (\partial_s + v_\pm^\mu \partial_\mu \mp \mathcal{D}\partial^\mu \partial_\mu) f \tag{4.13}$$

One considers now only stationary functions f, not depending explicitly on the proper time s, so that the complex covariant derivative operator reduces to

$$(d'/ds) = (\mathcal{V}^\mu + i\mathcal{D}\partial^\mu)\partial_\mu \tag{4.14}$$

Now assume that the large scale part of any mechanical system can be characterized by a complex action $\mathfrak{S}$ leading one to write

$$\delta\mathfrak{S} = -mc\delta \int_a^b ds = 0;\ ds = \overline{LS} < \sqrt{dX^\nu dX_\nu} > \tag{4.15}$$

This leads to $\delta\mathfrak{S} = -mc\int_a^b \mathcal{V}_\nu d(\delta x^\nu)$ with $\delta x^\nu = \overline{LS} < dX^\nu >$. Integrating by parts yields

$$\delta\mathfrak{S} = -[mc\delta x^\nu]_a^b + mc\int_a^b \delta x^\nu (d\mathcal{V}_\mu/ds) ds \tag{4.16}$$

To get the equations of motion one has to determine $\delta\mathfrak{S} = 0$ between the same two points, i.e. at the limits $(\delta x^\nu)_a = (\delta x^\nu)_b = 0$. From (4.16) one obtains then a differential geodesic equation $d\mathcal{V}/ds = 0$. One can also write the elementary variation of the action as a functional of the coordinates. So consider the point a as fixed so $(\delta x^\nu)_a = 0$ and consider b as variable. The only admissable solutions are those satisfying the equations of motion so the integral in (4.16) vanishes and writing $(\delta x^\nu)_b$ as δx^ν gives $\delta\mathfrak{S} = -mc\mathcal{V}_\nu \delta x^\nu$ (the minus sign comes from the choice of signature). The complex momentum is now

$$\mathcal{P}_\nu = mc\mathcal{V}_\nu = -\partial_\nu \mathfrak{S} \tag{4.17}$$

and the complex action completely characterizes the dynamical state of the particle. Hence introduce a wave function $\psi = exp(i\mathfrak{S}/\mathfrak{S}_0)$ and via (4.17) one gets

$$\mathcal{V}_\nu = (i\mathfrak{S}_0/mc)\partial_\nu log(\psi) \tag{4.18}$$

Now for the scale relativistic prescription replace the derivative in d/ds by its covariant expression d'/ds. Using (4.18) one transforms $d\mathcal{V}/ds = 0$ into

$$-\frac{\mathfrak{S}_0^2}{m^2c^2}\partial^\mu log(\psi)\partial_\mu\partial_\nu log(\psi) - \frac{\mathfrak{S}_0\mathcal{D}}{mc}\partial^\mu\partial_\mu\partial_\nu log(\psi) = 0 \tag{4.19}$$

The choice $\mathfrak{S}_0 = \hbar = 2mc\mathcal{D}$ allows a simplification of (4.19) when one uses the identity

$$\frac{1}{2}\left(\frac{\partial_\mu\partial^\mu\psi}{\psi}\right) = \left(\partial_\mu log(\psi) + \frac{1}{2}\partial_\mu\right)\partial^\mu\partial^\nu log(\psi) \tag{4.20}$$

Dividing by $\mathcal{D}^2$ one obtains the equation of motion for the free particle $\partial^\nu[\partial^\mu\partial_\mu\psi/\psi] = 0$. Therefore the KG equation (no electromagnetic field) is

$$\partial^\mu\partial_\mu\psi + (m^2c^2/\hbar^2)\psi = 0 \tag{4.21}$$

and this becomes an integral of motion of the free particle provided the integration constant is chosen in terms of a squared mass term $m^2c^2/\hbar^2$. Thus the quantum behavior described by this equation and the probabilistic interpretation given to ψ is reduced here to the description of a free fall in a fractal spacetime, in analogy with Einstein's general relativity. Moreover these equations are covariant since the relativistic quantum equation written in terms of d'/ds has the same form as the equation of a relativistic macroscopic and free particle using d/ds. One notes that the metric form of relativity, namely $V^\mu V_\mu = 1$ is not conserved in QM and it is shown in [**775**] that the free particle KG equation expressed in terms of $\mathcal{V}$ leads to a new equality

$$\mathcal{V}^\mu\mathcal{V}_\mu + 2i\mathcal{D}\partial^\mu\mathcal{V}_\mu = 1 \tag{4.22}$$

In the scale relativistic framework this expression defines the metric that is induced by the internal scale structures of the fractal spacetime. In the absence of an electromagnetic field $\mathcal{V}^\mu$ and $\mathfrak{S}$ are related by (4.17) which can be writen as $\mathcal{V}_\mu = -(1/mc)\partial_\mu\mathfrak{S}$ so (4.22) becomes

$$\partial^\mu\mathfrak{S}\partial_\mu\mathfrak{S} - 2imc\mathcal{D}\partial^\mu\partial_\mu\mathfrak{S} = m^2c^2 \tag{4.23}$$

which is the new form taken by the Hamilton-Jacobi equation.

REMARK 3.4.1. We go back to [**716, 775**] now and repeat some of their steps in a perhaps more primitive but revealing form. Thus one omits the $\overline{LS}$ notation and uses $\lambda \sim 2\mathcal{D}$; equations (4.8) - (4.14) and (4.11) are the same and one writes now $\mathfrak{d}/ds$ for d'/ds. Then $\mathfrak{d}/ds = \mathcal{V}^\mu\partial_\mu + (i\lambda/2)\partial^\mu\partial_\mu$ plays the role of a scale covariant derivative and one simply takes the equation of motion of a free relativistic quantum particle to be given as $(\mathfrak{d}/ds)\mathcal{V}^\nu = 0$, which can be interpreted as the equations of free motion in a fractal spacetime or as geodesic equations. In fact now $(\mathfrak{d}/ds)\mathcal{V}^\nu = 0$ leads directly to the KG equation upon writing $\psi = exp(i\mathfrak{S}/mc\lambda)$ and $\mathfrak{P}^\mu = -\partial^\mu\mathfrak{S} = mc\mathcal{V}^\mu$ so that $i\mathfrak{S} = mc\lambda log(\psi)$ and $\mathcal{V}^\mu = i\lambda\partial^\mu log(\psi)$. Then

$$\left(\mathcal{V}^\mu\partial_\mu + \frac{i\lambda}{2}\partial^\mu\partial_\mu\right)\partial^\nu log(\psi) = 0 = i\lambda\left(\frac{\partial^\mu\psi}{\psi}\partial_\mu + \frac{1}{2}\partial^\mu\partial_\mu\right)\partial^\nu \log(\psi) \tag{4.24}$$

Now some identities are given in [**775**] for aid in calculation here, namely

$$\frac{\partial^\mu\psi}{\psi}\partial_\mu\frac{\partial^\nu\psi}{\psi} = \frac{\partial^\mu\psi}{\psi}\partial^\nu\left(\frac{\partial_\mu\psi}{\psi}\right) = \tag{4.25}$$

$$= \frac{1}{2}\partial^\nu\left(\frac{\partial^\mu\psi}{\psi}\frac{\partial_\mu\psi}{\psi}\right);\ \partial_\mu\left(\frac{\partial^\mu\psi}{\psi}\right) + \frac{\partial^\mu\psi}{\psi}\frac{\partial_\mu\psi}{\psi} = \frac{\partial^\mu\partial_\mu\psi}{\psi}$$

The first term in the last equation of (4.24) is then $(1/2)[(\partial^\mu\psi/\psi)(\partial_\mu\psi/\psi)]$ and the second is

$$(4.26)\qquad (1/2)\partial^\mu\partial_\mu\partial^\nu log(\psi) = (1/2)\partial^\mu\partial^\nu\partial_\mu log(\psi) =$$

$$= (1/2)\partial^\nu\partial^\mu\partial_\mu log(\psi) = (1/2)\partial^\nu\left(\frac{\partial^\mu\partial_\mu\psi}{\psi} - \frac{\partial^\mu\psi\partial_\mu\psi}{\psi^2}\right)$$

Combining we get $(1/2)\partial^\nu(\partial^\mu\partial_\mu\psi/\psi) = 0$ which integrates then to a KG equation

$$(4.27)\qquad -(\hbar^2/m^2c^2)\partial^\mu\partial_\mu\psi = \psi$$

for suitable choice of integration constant (note $\hbar/mc$ is the Compton wave length).

Now in this context or above we refer back to Section 2.2 for example and write $Q = -(1/2m)(\Box R/R)$ (cf. Section 2.2 before Remark 2.2.1 and take $\hbar = c = 1$ for convenience here). Then recall $\psi = exp(i\mathfrak{S}/m\lambda)$ and $\mathfrak{P}_\mu = m\mathcal{V}_\mu = -\partial_\mu\mathfrak{S}$ with $i\mathfrak{S} = m\lambda log(\psi)$. Also $\mathcal{V}_\mu = -(1/m)\partial_\mu\mathfrak{S} = i\lambda\partial_\mu log(\psi)$ with $\psi = Rexp(iS/m\lambda)$ so $log(\psi) = i\mathfrak{S}/m\lambda = log(R) + iS/m\lambda$, leading to

$$(4.28)\qquad \mathcal{V}_\mu = i\lambda[\partial_\mu log(R) + (i/m\lambda)\partial_\mu S] = -\frac{1}{m}\partial_\mu S + i\lambda\partial_\mu log(R) = V_\mu + iU_\mu$$

Then $\Box = \partial^\mu\partial_\mu$ and $U_\mu = \lambda\partial_\mu log(R)$ leads to

$$(4.29)\qquad \partial^\mu U_\mu = \lambda\partial^\mu\partial_\mu log(R) = \lambda\Box log(R)$$

Further $\partial^\mu\partial_\nu log(R) = (\partial^\mu\partial_\nu R/R) - (R_\nu R_\mu/R^2)$ so

$$(4.30)\qquad \Box log(R) = \partial^\mu\partial_\mu log(R) = (\Box R/R) - (\sum R_\mu^2/R^2) =$$

$$= (\Box R/R) - \sum(\partial_\mu R/R)^2 = (\Box R/R) - |U|^2$$

for $|U|^2 = \sum U_\mu^2$. Hence via $\lambda = 1/2m$ for example one has

$$(4.31)\qquad Q = -(1/2m)(\Box R/R) = -\frac{1}{2m}\left[|U|^2 + \frac{1}{\lambda}\Box log(R)\right] =$$

$$= -\frac{1}{2m}\left[|U|^2 + \frac{1}{\lambda}\partial^\mu U_\mu\right] = -\frac{1}{2m}|U|^2 - \frac{1}{2}div(\vec{U})$$

(cf. Section 2.2).

REMARK 3.4.2. The words fractal spacetime as used in the scale relativity methods of Nottalle et al for producing geodesic equations (SE or KG equation) are somewhat misleading in that essentially one is only looking at continuous nondifferentiable paths for example. Scaling as such is of course considered extensively at other times. It would be nice to create a fractal derivative based on scaling properties and H-dimension alone for example which would permit the powerful techniques of calculus to be used in a fractal context. There has been of course some work in this direction already in e.g. [**187, 257, 411, 437, 466, 471, 562, 721, 748, 816**].

5. QUANTUM MEASUREMENT AND GEOMETRY

We consider here a paper [**989**], which is based in part on a famous paper of London [**611**] (reprinted in [**731**]). In [**611**] it was shown that the ratio of the Weyl scale factor to the Schrödinger wave function is constant if the proportionality constant between the Weyl potential and the EM potential is taken to be imaginary; this observation gave birth to modern gauge theories and the original Weyl theory was absorbed into QM with the original scale freedom becoming invariance under unitary gauge transformations (cf. also Section 3.5.1). Both the Weyl theory and the Schrödinger theory describe the evolution of a field in time and given the factor of i and the Kaluza-Klein framework used by London, those evolutions are the same. In the Weyl picture the field characterizes the length scales of fundamental matter, while in the Schrödinger picture it is the wave function corresponding to a fundamental particle. This analogy is pursued further in [**989**] with a main theme being the equivalence between Weyl measurement and quantum measurement; a complete theory of measurement in a Weyl geometry is said to contain the crucial elements of quantization and analogies of the following sort are indicated.

(5.1)

$Weyl - quantum\ correspondence$	$Quantum\ mechanics$
$Zero - Weyl - weight\ number$	$Real\ eigenvalue$
$Diffusion\ equation$	SE
$Weiner\ path\ integral$	$Feynman\ path\ integral$
$Weightful\ length\ field\ \psi_w$	$Complex\ state\ function\ \psi$
$Weyl\ conjugate\ \psi_{-w}$	ψ^*
$Probability\ \psi_w\psi_{-w}$	$Probability\ \|\psi\|^2$
$\psi_w \to e^{w\phi}\psi_w\ (conformal)$	$\psi \to e^{i\phi}\psi\ (unitary)$

We will try to make sense out of this following [**989**] (cf. also [**63, 64**]). Begin with a real 4-dimensional manifold $(M,[g])$ where $[g]$ is a conformal equivalence class of Lorentz metrics. In addition to local coordinate transformations one has Weyl (conformal) transformations given via $T(x)' = exp[w(T)\Lambda(x)]T(x)$ where T is a tensor field and $w(T)$ is the Weyl weight (a real number). One takes a coordinate basis $E_\alpha = \partial/\partial x^\alpha$ and $E^\alpha = dx^\alpha$ in the tangent and cotangent space satisfying $w(E_\alpha) = w(E^\alpha) = 0$.

DEFINITION 5.1. One defines a torsion free derivative D via

- Linearity: $D(aT_1 + bT_2) = aDT_1 + bDT_2$ for real a, b
- Leibniz: $D(T_1T_2) = (DT_1)T_2 + T_1(DT_2)$
- Weyl covariant: $D(fT) = [df + w(f)Wf]T + fDT$ where W is a real 1-form (Weyl potential)
- Zero weight: $w(DT) = w(T)$

Under a Weyl transformation $W \to W' = W - d\Lambda$ and one has

$$DT = D_\mu T^\alpha_{\ \beta} E^\mu \otimes E_\alpha \otimes E^\beta;\ D_\mu T^\alpha_{\ \beta} = \tag{5.2}$$

$$= \partial_\mu T^\alpha_{\ \beta} + T^\rho_{\ \beta}\Gamma^\alpha_{\ \rho\mu} - T^\alpha_{\ \rho}\Gamma^\rho_{\ \beta\mu} + w(T)W_\mu T^\alpha_{\ \beta}$$

There is no unique metric on the space; instead the metric is to be taken of the Weyl type $w(g) = 2$ so that under a Weyl transformation $g' = exp[2\Lambda(x)]g$.

The principle fields of the theory are related by the requirement $Dg = 0$, or in components

$$D_\mu g_{\alpha\beta} = 0 = \partial_\mu g_{\alpha\beta} - g_{\rho\beta}\Gamma^\rho{}_{\alpha\mu} - g_{\alpha\rho}\Gamma^\rho{}_{\beta\mu} + 2W_\mu g_{\alpha\beta} \tag{5.3}$$

This can be solved to give

$$\Gamma^\alpha{}_{\beta\mu} = \left\{ \begin{array}{c} \alpha \\ \beta\ \ \mu \end{array} \right\} + (\delta^\alpha{}_\beta W_\mu + \delta^\alpha{}_\mu W_\beta - g_{\beta\mu}W^\alpha) \tag{5.4}$$

Vanishing torsion has been assumed in (5.4) so that the bracket expression is the usual Christoffel connection. The curvature tensor is then

$$R^\alpha{}_{\beta\mu\nu} = \Gamma^\alpha{}_{\beta\nu,\mu} - \Gamma^\alpha{}_{\beta\mu,\nu} + \Gamma^\alpha{}_{\rho\mu}\Gamma^\rho{}_{\beta\nu} - \Gamma^\alpha{}_{\rho\nu}\Gamma^\rho{}_{\beta\mu} \tag{5.5}$$

Unlike the Riemannian curvature tensor the Weyl curvature has nonvanishing trace on the first pair of indices so that $(1/2)R^\alpha{}_{\alpha\mu\nu} = W_{\nu,\mu} - W_{\mu,\nu} = W_{\mu\nu}$ where $W_{\mu\mu}$ is the gauge invariant field strength of the Weyl potential. One says that two fields are Weyl conjugate if they have the same Lorenz transformation properties but opposite Weyl weights.

Now for a theory of measurement one first looks at zero weight fields. In this direction note that fields with nonvanishing Weyl weight will experience changes under parallel transport. For example the mass squared transported along a path with unit tangent vector $u^\mu = dx^\mu/d\tau$ satisfies

$$0 = u^\mu D_\mu(m^2) = u^\mu \partial_\mu(m^2) + w(m^2)u^\mu W_\mu m^2 \tag{5.6}$$

Integrating along the path of motion one finds a path dependence of the form $m^2 = m_0^2 exp[w(m^2)\int W_\mu u^\mu d\tau]$ where the line integral has been written in terms of the path parameter τ. Note this is analogous to $m^2 = m_0^2 exp(Q)$ in the Shojai theory of Section 3.2 suggesting some relation to a quantum potential $Q \sim w(m^2)\int W_\mu u^\mu d\tau$. However at this point there is no quantum matter posited and no density ρ so a Weyl vector $W_\mu \sim \partial_\mu log(\rho)$ as in Remark 3.3.1 is untenable and no comparison to (3.28) can be undertaken. However this does show a geometrical dependence of mass in general and in the flat space of Remark 3.3.1 it is replaced by a quantum potential. Indeed this (Schouten-Haantjes) conformal mass thus depends on the Weyl vector and if two particles of identical mass are allowed to propagate freely (by parallel transport) along different paths and brought together there will be a mass difference

$$\Delta m^2 = m_0^2 e^{w(m^2)\oint W_\mu u^\mu d\tau} \equiv m_0^2 e^{w(m^2)\int_S W_{\mu\nu} dS^{\mu\nu}} \tag{5.7}$$

where $dS^{\mu\nu}$ is an element of any 2-surface S bounded by the closed curve defined by the two particles. Hence unless the surface integral of the Weyl field strength vanishes there will be a path dependence for masses and of any other field of nonzero weight. One postulates now **(I)** that all quantities of vanishing Weyl weight should be physically meaningful (observables) and **(II)** that all fields occur in conjugate pairs satisfying conjugate equations of motion. Assume that $M_\pm$ evolves by parallel transport along a path as above via

$$0 = u^\mu D_\mu D_\pm = u^\mu \bar{D}_\mu M_\pm \pm w(M)M_\pm W_\mu u^\mu \tag{5.8}$$

where $\bar{D}$ is a derivation using the full connection (5.4) and one sets $w(M) = w(M_+) > 0$ for convenience. One has also

$$M_{\pm} = \mathfrak{M} exp[\mp w(M) \int W_\mu u^\mu d\tau] \tag{5.9}$$

where $\mathfrak{M}$ is weightless with $u^\mu \bar{D}_\mu \mathfrak{M} = 0$. Now suppose one wants to measure some characteristic of M (i.e. of M_+ or M_-). M can be scaled by an arbitrary gauge function and one transports M along a path so that its covariant derivative in the dirction of motion vanishes. Then the change in size is specified by (5.9) but it is not clear that we can tell what path a particle has taken. In a Riemannian space there are geodesics determining the paths of classical matter but that is not true in a Weyl space (in this regard we refer to [**188**], Section 3.2, and to [**133, 796, 797, 798, 799**]).

In order to study the motion of M one begins with the observation that a Weyl geometry provides a probability $P_{AB}(M)$ of finding a value M at a point B for a system which is known to have had a value M_0 at point A. Finding $P_{AB}(M)$ is tantamount to finding the fraction of paths which the system may follow leading to any given value of M. Since there may be no special paths in a Weyl geometry one has to settle for moments of the distribution. To find the average value of magnitude of M denoted by $< M >$ one integrates (5.9) over all paths via

$$< M >= \int \mathcal{D}[x] M_0 exp[w(M) \int_A^B W_\mu u^\mu d\tau] \tag{5.10}$$

where the usual path integral normalization is included implicitly in $\mathcal{D}[x]$ (see e.g. [**362, 457, 855**]) for path integrals). However this gives no information as to whether one should expect M to actualy reach B. In [**989**] there is then a long discussion (and a detailed Appendix) involving path averages, probability, Wiener integrals, etc. plus a postulate **(III)** that the probability a system will undergo a given infinitesimal displacement x^μ is inversely proportional to the change in length such a displacement produces in the system. Now $d\ell = w(M) W_\mu dx^\mu = w(M) W_\mu u^\mu d\tau$ and a plausible (rigorous) argument is given then to represent the probability of the system reaching any spacetime point x from x_0 as

$$G(x_0; x) = \int \mathcal{D}[x] exp[w(M) \int_{x_0}^x W_\mu u^\mu d\tau] \tag{5.11}$$

(which bears an obvious resemblence to (5.10)). Comparison of (5.10) and (5.11) involves noting first that (5.11) is gauge dependent but the gauge dependence comes out of the path integral since it depends only on the end points. Thus

$$G'(x_0; x) = \int \mathcal{D}[x] exp[w(M) \int_{x_0}^x (W_\mu - \partial_\mu \phi) u^\mu d\tau] = \tag{5.12}$$

$$= e^{-w(M)[\phi(x) - \phi(x_0)]} \int \mathcal{D}[x] exp[w(M) \int_{x_0}^x W_\mu u^\mu d\tau]$$

This means that one can eleiminate the gauge factor by multiplying by the Weyl conjugate expression

$$\bar{G}'(x_0;x) = e^{w(M)[\phi(x)-\phi(x_0)]}\int \mathcal{D}[x]exp[-w(M)\int_{x_0}^{x} W_\mu u^\mu d\tau] \tag{5.13}$$

to give a meaningful gauge invariant probability $P(x_0,x) = \bar{G}(x_0;x)G(x_0;x)$ which is the probability of detecting the dilating system at x given its presence at x_0. It may be thought of as the joint probability of finding both M and $\bar{M}$ at x. Here one is dealing with a real path integral, unlike QM, and the phase invariance of a wave function $\psi' = exp(i\phi)\psi$ is replaced by conformal invariance $M' = exp(\phi)M$ (this is the same factor of i introduced by London in 1927). Since that time gauge transformations have appeared as phases and the wave interpretation has been maintained; now one maintains a real gauge transformation and changes the interpretation of physical phenomena (see [**989**] for more discussion in this direction).

Now one shows the equivalence to QM of the nonrelativistic limit of (5.11) when the exponent in the path integral is identified with a multiple of the classical action, i.e. $\int W_\mu u^\mu d\tau = \lambda S = \lambda\int L d\tau$. The integrands here may also be equated except for the possible addition of the total derivative of a function of τ. But such a derivative is already known to be both a gauge freedom of W_μ and a transformation of L that leaves the equations of motion unaltered. So the possible equivalent versions of L may be understood as gauge changes of the underlying geometry. This identification fixes the physical interpretation of W_μ up to the gauge choice and since $u^\mu = \dot{x}^\mu$ equating the integrands gives

$$\lambda P_\mu = \lambda(\partial L/\partial u^\mu) = W_\mu \tag{5.14}$$

so that W_μ is proportional to the generalized momentum P_μ conjugate to x^μ. Now Weyl had originally identified W_μ with the derivative of an EM potential $\partial_\mu U \sim A_\mu$ and the present approach suggests $W_\mu = \lambda(p_\mu + A_\mu)$ so that all energy provides a surce of expansion rather than just EM energy. This still allows gauge transformations of W_μ to be identified with gauge transformations of A_μ. Next one goes to the nonrelativistic limit of the path integral to find a differential equation for the amplitudes $G(x_0;x)$. It is convenient to explicitly separate the kinetic term $p_\mu u^\mu$ from $W_\mu u^\mu$ which will enable one to identify the path integral in (5.11) with a Wiener integral. Thus with full generality one writes $W_\mu = \lambda(p_\mu + \tilde{W}_\mu)$ where any gauge transformation is understood to apply to $\tilde{W}_\mu$. Now consider the nonrelativistic limit where the integral $\int p_\mu u^\mu d\tau \sim mc^2\int d\tau$ so that $mc^2\int d\tau \sim \int[mc^2 - (m/2)\mathbf{v}^2]dt$. To this order the path integral becomes (suppressing limits of integration)

$$G(x_0;x) = \int \mathcal{D}[x]e^{\lambda w(M)\int[(1/2)m\mathbf{v}^2 + \tilde{W}\cdot\mathbf{v} - \tilde{W}^0 - mc^2)]dt} \tag{5.15}$$

This is of the form

$$P(x_0;x) = \int \mathcal{D}[x]exp[-(1/2)\int((\dot{\mathbf{q}}+\mathbf{w})^2 - \nabla\cdot\mathbf{w})dt] \tag{5.16}$$

where $P(x_0;x)$ is the propagator for the Fokker-Planck equation $\partial_t P = (1/2)\nabla^2 P + \nabla\cdot(\mathbf{w}P)$ provided one makes the identifications

$$\dot{\mathbf{q}} = \sqrt{-w(M)\lambda m}\mathbf{v};\ \nabla_x = \sqrt{-w(M)\lambda m}\nabla_q;\ \psi = Pe^{-2mc^2}; \tag{5.17}$$
$$\mathbf{w} = \sqrt{-w(M)\lambda/m}\tilde{\mathbf{W}};\ 2w(M)\lambda(mc^2+\tilde{\mathbf{W}}^0) = \mathbf{w}^2 - \nabla\cdot\mathbf{w}$$

(cf. [**457, 672, 674, 698, 856**]). Carrying out the substitutions and setting $\lambda\tilde{\mathbf{W}}_\mu = -U(\lambda\phi,\mathbf{A})$ one obtains $\psi(x) = \int\psi(x')G(x,x')dx'$ as a solution to

$$\frac{1}{w(M)\lambda}\partial_t\psi = -\frac{1}{2m[w(M)\lambda]^2}[\nabla + w(M)\lambda\mathbf{A}]^2 + (mc^2+U\phi)\psi \tag{5.18}$$

with initial condition $\psi = \psi(x')$ (this should be checked to clarify the roles of U and ϕ). If one sets $\lambda = \hbar^{-1}$ and the time is allowed to become imaginary the SE minimally coupled to EM arises. Thus choose $\lambda = \hbar^{-1}$ but leave time alone since it is not needed; then (5.18) can be interpreted as a stochastic form of QM. Evidently the Weyl weight serves the function of i, changing sign appropriately for the conjugate field. The emergence of the Fokker-Planck equation indicates diffusion and this is discussed at length in [**186, 672, 674, 698, 856**]. In addition the matter is discussed in [**989**] from various points of view. In particular one takes $(1/\hbar)S = \int W_\mu u^\mu d\tau$ and observes that a classical limit of the Weyl geometry will exist whenever there is an extremum to the action (as in the Feynman path integral). Thus a classical limit of (5.11) occurs whenever $\Psi = exp[w(M)\int_{x_0}^x W_\mu u^\mu d\tau]$ is extremal. However there is a difference here involving Ψ as a length factor. One shows that $\delta\Psi = 0$ corresponds to a special case of the Weyl field since $\int_A^B d\tau(W_{\mu,\nu} - W_{\nu,\mu})u^\mu\delta x^\nu = 0$ arises via variation which means $W_{\mu\nu}u^\nu = 0$. Some calculation then shows that $W_\alpha = \xi\partial_\alpha\chi$ (up to a gauge transformation) for any appropriately normalized functions ξ, χ satisfying

$$(D_\mu\chi)u^\nu = (D_\mu\chi)v^\mu = (D_\mu\xi)u^\nu = (D_\mu\xi)v^\mu = 0; \tag{5.19}$$
$$(1/2)\epsilon^{\mu\nu\alpha\beta}W_{\alpha\beta} = u^\mu v^\nu - u^\nu v^\mu$$

with ϵ the Levi-Civita tensor (cf. [**302, 989**]). Now $W_\alpha = \xi\partial_\alpha\chi$ is a rather remarkable relation; it represents a restricted form of W^α since it is easy to find a Weyl vector such that $W_{\mu\nu}u^\nu \sim W_{\mu 0} \neq 0$ for all nonspacelike u^ν. Since this formula arises for an arbitrary set of paths u^α it is clear that not all Weyl fields will have a classical limit. Thus as argued at the beginning the generic Weyl geometry lacks preferred paths and requires a path average. On the other hand if one chooses a gauge where $W_\alpha u^\alpha = 0$ (which is possible) then weightful bodies followed the preferred classical trajectories and experience no dilation. There is considerable discussion along these lines in [**989**] which is omitted here; there is also interesting material on relations to general relativity. In particular it is pointed out that size changes associated with nonvanishing Weyl field strength are not necessarily classically observable. However the Weyl field itself must be present and consequently must be detectable. Finding the physical field that it corresponds to simply requires substituting the appropriate conjugate momentum for W_μ in the classical equation of motion $W_{\mu\nu}u^\nu = 0$. Since the only long range forces are gravity and EM and gravity is still accounted for by the Riemannian curvature, W_μ must be electromagnetic. The most general classical conjugate momentum is therefore that of a point particle with charge q moving in an EM field. Then in an arbitrary gauge

$$W_\mu = (1/\hbar)(p_\mu + qA_\mu + \partial_\mu \Lambda) \tag{5.20}$$

where $p_\mu = mu_\mu$ and $u_\mu u^\mu = -1$. Then

$$0 = W_{\mu\nu}u^\nu = (1/\hbar)(p_{\mu,\nu} - p_{\nu,\mu} + qA_{\mu,\nu} - qA_{\nu\mu})u^\nu \tag{5.21}$$

or (using $(u_\mu u^\mu)_{,\nu} = 0$) $dp^\mu/d\tau = qu_\nu F^{\mu\nu}$ which is the Lorenz force law (note that Planck's constant drops out). For the interpretation of W_μ itself one can combine the curl of (5.20) with

$$W_{\alpha\beta} = D_\alpha\chi D_\beta\xi - D_\beta\chi D_\alpha\xi = \partial_\alpha\chi\partial_\beta\xi - \partial_\beta\chi\partial_\alpha\xi \tag{5.22}$$

(cf. (5.19) and the surrounding discussion); this leads to

$$\partial_\alpha\chi\partial_\beta\xi - \partial_\beta\chi\partial_\alpha\xi = (1/\hbar)(p_{\alpha,\beta} - p_{\beta,\alpha} + qA_{\alpha,\beta} - qA_{\beta,\alpha}) \tag{5.23}$$

the time component of which gives again the Lorenz law. The spatial components can be solved for the magnetic field to give

$$\mathbf{B} = (\hbar/q)(\nabla\chi \times \nabla\xi) - (m/q)(\nabla \times \mathbf{v}) \tag{5.24}$$

The two fields χ and ξ on the right side of $\mathbf{B}$ are sufficient to guarantee the existence of any type of physical magnetic field. Conversely one can use (5.24) to solve for the Weyl field in terms of $\mathbf{B}$ and $\mathbf{v}$ (which of course depend on $\hbar$). One notes that for vanishing Weyl field (5.24) reduces to the London equation for superconductivity. This means that matter fields which conspire to produce a Riemannian geometry become superconducting.

5.1. MEASUREMENT ON A BICONFORMAL SPACE. We continue the theme of Section 3.5 with a more general perspective from [**35**] based on biconformal geometry (cf. Appendix E for some background material and see also [**35, 36, 113, 497, 558, 987, 980, 981, 989, 990, 991, 992, 993, 994, 1010**]). We regard this approach via biconformal geometry as very interesting and will try to present it faithfully. The background material in Appendix E should be read first; results in [**994**] for example create a unified geometrical theory of gravity and electromagnetism based on biconformal geometry. One develops in [**35**] an interpretation for quantum behavior within the context of biconformal gauge theory based on the following postulates:

(1) A σ_C biconformal space provides the physical arena for quantum and classical physics.
(2) Quantities of vanishing conformal weight comprise the class of physically meaningful observables.
(3) The probability that a system will follow any given infinitesimal displacement is inversely proportional to the dilatation the displacement produces in the system.

From these assumptions follow the basic properties of classical and quantum mechanics. The symplectic structure of biconformal space is similar to classical phase space and also gives rise to Hamilton's equations, Hamilton's principal function, conjugate variables, fundamental Poisson brackets, and Liouville theory when postulate 3 is replaced by a postulate of extremal motion. We sketch this here (somewhat brutally) and refer to [**35**] for details, philosophy, and further references; the details for the biconformal geometry are spelled out in [**992, 994**]. Thus one wants a physical arena which contains 4-D spacetime in a straightforward manner but which is large enough and structured so as to contain both general relativity (GR) and quantum theory (QT) at the same time. One demands therefore invariance under global Lorentz transformations, translations, and scalings (see below) and the Lie group characterizing this is the conformal group $O(4,2)$ or its covering group $SU(2,2)$. In Appendix E the basic facts about Lorentz transformations $M^a_{\ b} = -M_{ba} = \eta_{ac}M^c_{\ b}$, translations P_a, special conformal transformations K^a, and dilatations D are exhibited in the context of conformal gauge theory $(a,b = 0,1,2,3)$. One has two involutive automorphisms of the conformal algebra, first

$$\sigma_1:\ (M^a_{\ b}, P_a, K^a, D) \to (M^a_{\ b}, -P_a, -K^a, D) \tag{5.25}$$

which identifies the residual local Lorentz and dilatation symmetry characteristic of biconformal gauging and this corresponds (resp. for the Poincaré Lie algebra or the Weyl algebra) to

$$\sigma_1:\ (M^a_{\ b}, P_a) \to (M^a_{\ b}, -P_a)\ \textit{or}\ \sigma_1:\ (M^a_{\ b}, P_a, D) \to (M^a_{\ b}, -P_a, D) \tag{5.26}$$

There is also a second involution for the conformal group, namely

$$\sigma_2:\ (M^a_{\ b}, P_a, K^a, D) \to (M^a_{\ b}, K_a, P^a, -D) \tag{5.27}$$

Some representations of the conformal algebra, namely $su(2,2)$, are necessarily complex and σ_2 can be realized as complex conjugation. Specifically one thinks of a representation in which P_a and K^a are complex conjugates while $M^a_{\ b}$ is real and D is purely imaginary and such representations will be called σ_C representations. Biconformal spaces for which the connection 1-forms (and hence curvatures) have this property are then called σ_C spaces (see Appendix E for examples). This leads to postulate 1 above, namely the physical arena for QT and classical physics is a σ_C biconformal space. Now biconformal gauging of the conformal group provides in particular a symplectic structure as follows. Gauging D introduces a single gauge 1-form ω (the Weyl vector) and the corresponding dilatational curvature 2-form is

$$\Omega = d\omega - 2\omega^a\omega_a \tag{5.28}$$

where ω^a, ω_a are 1-form gauge fields for the translation and special conformal transformations respectively, which span an 8-dimensional space as an orthonormal basis (note $\omega_a = \eta_{ab}\bar{\omega}^b$ for σ_C representations and products are wedge products). Now for all torsion free solutions to the biconformal field equations (i.e. $*d*d\omega^0_0 = J$, $\omega^0_a = T_a + \cdot$, etc. - cf. Appendix E) the dilatational curvature takes the form $(\bullet)$ $\Omega = \kappa\omega^a\omega_a$ with κ constant, so the structure equation becomes $(\bullet\bullet)$ $d\omega = (\kappa + 2)\omega^a\omega_a$. As a result $d\omega$ is closed and nondegenerate and hence symplectic (since ω^a, ω_a span the space). There is also a biconformal metric arising from the group invariant Killing metric $K_{\Sigma\Pi} = c^{\Lambda}_{\Delta\Sigma}c^{\Delta}_{\Lambda\Pi}$ where $c^{\Lambda}_{\Delta\Sigma}$ $(\Sigma,\Pi,\cdots = 1,2,\cdots,15)$ are the real structure constants from the Lie algebra. This metric has a nondegenerate projection to the 8-D subspace spanned by P_a, K^a and provides a natural pseudo-Riemannian metric on biconformal manifolds. The projection takes the form

$$K_{AB} = \begin{pmatrix} & \eta_{ab} \\ \eta_{ab} & \end{pmatrix} \ (A, B = 0, 1, \cdots, 7) \tag{5.29}$$

One defines now conformal weights w of a definite weight field F via (♦) $D_\phi : F \to [exp(w\phi)]F$ where D_ϕ is dilatation by $exp(\phi)$ (cf. [**989**] and Section 3.5). One assumes now postulate 2 and concludes that for a field with nontrivial Weyl weight to have physical meaning it must be possible to construct weightless scalars by combining it with other fields (easily done with conjugate fields); one notes that zero weight fields are self conjugate. The symplectic form $\Theta = \omega^a \omega_a$ defines a symplectic bracket via

$$\{f, g\} = \Theta^{MN} \frac{\partial f}{\partial u^M} \frac{\partial g}{\partial u^N} \tag{5.30}$$

where $u^M = (x^a, y^b)$. For real solutions f, g to the field equations f and g are conjugate if they satisfy $\{f, f\} = 1$, $\{f, f\} = \{g, g\} = 0$. However for σ_C representations ω is a pure imaginary 1-form since it is defined as the dual to the dilatation generator D which is pure imaginary. One sees then that

$$\overline{\omega^a \omega_a} = \bar{\omega}^a \bar{\omega}_a = \eta^{ab} \omega_b \eta_{ac} \omega^c = -\omega^a \omega_a \tag{5.31}$$

so the dilatational curvature and the symplectic form are imaginary (cf. also [**35, 529**]). Consequently, for use of a complex gauge vector with real gauge transformations, the fundamental brackets should take here the form

$$\{f, g\} = i; \ \{f, f\} = \{g, g\} = 0; \ w_f = -w_g \tag{5.32}$$

In an arbitrary biconformal space one sets either

$$\frac{1}{\hbar} S = \frac{1}{\hbar} \int L d\lambda = \int \omega = \int (W_a dx^a + \bar{W}_a dy^a) \ or \tag{5.33}$$

$$\frac{i}{\hbar} S = \frac{i}{\hbar} \int L d\lambda = \int \omega = \int (W_a dx^a + \bar{W}_a dy^a)$$

The second form holds in a σ_C representation for the conformal group. An arbitrary parameter λ is OK since the integral of the Weyl 1-form is independent of parametrization. This integral also governs measurable size change since under parallel transport the Minkowski length of a vector V^a changes by

$$\ell = \ell_0 exp \int \omega; \ \ell^2 = \eta_{ab} V^a V^b \tag{5.34}$$

(cf. Appendix E). This change occurs because $\eta_{ab} = (-1, 1, 1, 1)$ is not a natural structure for biconformal space. This is in contrast to the Killing metric K_{AB} where lengths are of zero conformal weight. In a σ_C representation the Weyl vector is imaginary so the measurable part of the change in ℓ is not a real dilatation - rather, it is a change of phase. Now for classical mechanics one uses a variation of postulate 3, namely: **The motion of a (classical) physical system is given by extrema of the integral of the Weyl vector**. Biconformal spaces are real symplectic manifolds so the Weyl vector can be chosen so that the symplectic form satisfies the Darboux theorem $\omega = W_a dx^a = -y_z dx^a$; for σ_C representations the Darboux equations still holds but now with

$$\omega = W_a dx^a = -iy_a dx^a \tag{5.35}$$

and the classical motion is independent of which form is chosen. Thus the symplectic form for the σ_C case is $\Theta = d\omega = -idy_a dx^a$ and one has (♦♦) $\{x^a, y_b\} = i\delta^a_b$. Thus from (♦♦) it follows that y_b is the conjugate variable to the position coordinate x^b and in mechanical units one may set $y_a = \alpha p_a$ with

$$i\alpha S = \int \omega = -i\alpha \int (p_0 dt + p_i dx^i) \tag{5.36}$$

(α can be any constant with appropriate dimensions). Now if one requires t as an invariant parameter (so $\delta t = 0$) one can vary the corresponding canonical bracket to find

$$0 = \delta\{t, p_0\} = \{\delta t, p_0\} + \{t, \delta p_0\} = \frac{\partial(\delta p_0)}{\partial p_0} \tag{5.37}$$

Thus δp_0 can depend only on the remaining coordinates so $\delta p_0 = -\delta H(y_i, x^j, t)$ and the existence of a Hamiltonian is a consequence of choosing time as a nonvaried parameter of the motion. Applying the postulate $\delta S = 0$ variation leads to

$$0 = i\alpha\delta S = -i\alpha \int (\delta p_0 dt + \delta p_i dx^i - dp_i \delta x^i) = \tag{5.38}$$
$$= -i\alpha \int \left(-\frac{\partial H}{\partial x^i}\delta x^i dt - \frac{\partial H}{\partial p_i}\delta p_i dt + \delta p_i dx^i - dp_i \delta x^i \right)$$

and this gives the standard Hamilton's equations

$$0 = -\frac{\partial H}{\partial p_i} dt + dx^i;\ 0 = -\frac{\partial H}{\partial x^i} dt - dp_i \tag{5.39}$$

(note i and α drop out of the equations).

In the presence of nonvanishing dilatational curvature one then considers a classical experiment to measure size (or phase) change along C_1, while a ruler measured by λ moves along C_2 (C_i are classical paths between two fixed points). Some argument (see [**35**]) leads to an unchanged ratio of lengths via

$$\frac{\ell}{\lambda} = \frac{\ell_0}{\lambda_0} exp \int_{C-1-C-2} \omega = \frac{\ell_0}{\lambda_0} exp \oint \omega = \frac{\ell_0}{\lambda_0} exp \int\int_S d\omega = \frac{\ell_0}{\lambda_0} \tag{5.40}$$

where S is any surface bounded by the closed curve $C_1 - C_2$ (cf. also Section 3.5). Thus no dilatations are observable along classical paths. This calculation also shows that the restriction of ω to classical paths is exact and proves the existence of Hamilton's principal function S with

$$\alpha S(x) = \int^x W_a dx^a = \int^x W_a \frac{dx^a}{dt} dt \tag{5.41}$$

There is further argument in [**35**] via gauge freedom to show that classical objects do not exhibit measurable length change (in the complex case the phase changes cannot be removed by gauge choice but they are unobservable). Relations between phase space and biconformal space are discussed and one arrives at QM.

From the above one knows that there is no measurable size change along classical pathes in a biconformal geometry but for systems evolving along other than extremal paths (where the Hamilton equations do not apply for example) there may be measurable dilatation. To deal with this one needs nonclassical motion and one goes to the basic postulate 3, namely that the probability a system will follow any given infinitesimal displacement is inversely proportional to the dilatation the displacement produces in the system. The properties of biconformal space determine the evolution of Minkowski lengths along arbitrary curves and the imaginary Weyl vector produces measurable phase changes in the same way as the wave function. Combining this with the classically probabilistic motion of postulate 3, together with the necessary use of a standard of length to comply with postulate 2, one concludes that the probability of a system at x_0^a arriving at the point x_1^a is given by

$$P(x_1^i) = \int \mathcal{D}[x_{C'}]exp\left(\int_{C'} \omega\right) \int \mathcal{D}[x_C]exp\left(-\int_C \omega\right) = \tag{5.42}$$

$$= \mathcal{P}(x_1^i)\mathcal{P}(-x_1^i) = \mathcal{P}(x_1^i)\bar{\mathcal{P}}(x_1^i)$$

where a path average over all paths connecting the two points is involved and $\bar{\mathcal{P}}(x)$ is simulaneously the probability amplitude of the conformally conjugate system reaching x_1^i. Here one considers ratios ℓ/λ as above and includes all possible ruler paths. These are standard Feynman path integrals which are known to lead to the Schrödinger equation (not Wiener integrals as in [**989**]) and it is the requirement of a length standard that forces the product structure in (5.42). Note that the phase invariance of a wave function $\psi' = exp(i\phi)\psi$ is created by the σ_C conformal invariance $M' = exp(\lambda w)M$. The i in the Weyl vector is the crucial i noted by London in [**611**] (cf. [**989**] and Section 3.5). Note also that the path integral in (5.42) and the biconformal paths depend generically on the spacetime and momentum variables so one can immediately generalize to the usual integrals of QM, namely

$$\mathcal{P}(x_1^i) = \int \mathcal{D}[x_C]\mathcal{D}[y_C]exp(\int C\omega) \tag{5.43}$$

Note also that the failure of the base space to break into space like and momentum like submanifolds indicates a fundamental coupling between position and momentum and suggests a connection to the Heisenberg uncertainty principle. The arguments in [**35**] have a somewhat heuristic flavor at times but are certainly plausible and do refine the techniques of [**989**] (sketched in Section 3.5) in many ways. Given the success of biconformal geometry in unifying GR and EM it would seem only natural and just that QM could be encompassed as well in the same framework and further developments are eagerly awaited.

REMARK 3.5.1 We note from [**993**] that when identifying biconformal coordinates (x^μ, y_ν) with phase space coordinates (x^μ, p_ν) one sets naturally $y_\nu = \beta p_\nu$. This β must account for a sign difference in $\eta^{\mu\nu}\beta p_\mu \beta p_\nu = -\eta^{\mu\nu}y_\mu y_\nu$ (cf. [**993**]) so β is pure imaginary. Further to account for the different units of y_ν ($length^{-1}$) and p_ν (momentum) one chooses $y_\nu = (i/\hbar)p_\nu$ and this relation between the geometric variables of conformal gauge theory and the physical momentum variables is the source of complex quantities in QM.

CHAPTER 4

GEOMETRY AND COSMOLOGY

This chapter and the next will cover a number of more or less related topics having to do with cosmology, the zero point field (ZPF), the aether and vacuum, quantum geometry, electromagnetic (EM) phenomena, and Dirac-Weyl geometry.

1. DIRAC-WEYL GEOMETRY

A sketch of Dirac Weyl geometry following [**302**] was given in [**188**] in connection with deBroglie-Bohm theory in the spirit of the Tehran school (cf. [**117, 118, 668, 669, 831, 832, 833, 834, 835, 836, 837, 838, 864, 865, 866, 867, 868, 869, 870, 871, 872, 873, 874, 875, 876, 877, 878, 879**]). We go now to [**498, 499, 500, 501, 502, 503, 817**] for generalizations of the Dirac Weyl theory involved in discussing magnetic monopoles, dark matter, quintessence, matter creation, etc. We skip [**500**] where some notational problems seem to arise in the Lagrangian and go to [**499**] where in particular an integrable Weyl-Dirac theory is developed (the book [**498**] is a lovely exposition but the work in [**499**] is somwhat newer). Note, as remarked in [**645**] (where twistors are used), the integrable Weyl-Dirac geometry is desirable in order that the natural frequency of an atom at a point should not depend on the whole world line of the atom. The first paper in [**499**] is designed to investigate the integrable Weyl-Dirac (Int-W-D) geometry and its ability to create massive matter. For example in this theory a spherically symmetric static geometric formation can be spatially confined and an exterior observer will recognize it as a massive entity. This may be either a fundamental particle or a cosmic black hole both confined by a Schwarzschild surface. We summarize again some basic features in order to establish notation, etc. Thus in the Weyl geometry one has a metric $g_{\mu\nu} = g_{\nu\mu}$ and a length connection vector w_μ along with an idea of Weyl gauge transformation (WGT)

$$g_{\mu\nu} \to \tilde{g}_{\mu\nu} = e^{2\lambda} g_{\mu\nu};\ g^{\mu\nu} \to \tilde{g}^{\mu\nu} = e^{-2\lambda} g^{\mu\nu} \tag{1.1}$$

where $\lambda(x^\mu)$ is an arbitrary differerentiable function (cf. also [**319, 320, 762, 853, 854**] for Weyl geometry). One is interested in covariant quantities satisfying $\psi \to \tilde{\psi} = exp(n\lambda)\psi$ where the Weyl power n is described via $\pi(\psi) = n,\ \pi(g_{\mu\nu}) = 2$, and $\pi(g^{\mu\nu}) = -2$. If $n = 0$ the quantity ψ is said to be gauge invariant (in-invariant). Under parallel displacement one has length changes and for a vector

$$(i)\ dB^\mu = -B^\sigma \Gamma^\mu_{\sigma\nu} dx^\nu;\ (ii)\ B = (B^\mu B^\nu g_{\mu\nu})^{1/2};\ (iii)\ dB = Bw_\nu dx^\nu \tag{1.2}$$

(note $\pi(B) = 1$). In order to have agreement between (i) and (iii) one requires

$$\Gamma^{\lambda}_{\mu\nu} = \left\{ \begin{array}{c} \lambda \\ \mu\ \nu \end{array} \right\} + g_{\mu\nu} w^{\lambda} - \delta^{\lambda}_{\nu} w_{\mu} - \delta^{\lambda}_{\mu} w_{\nu} \tag{1.3}$$

where $\left\{ \begin{array}{c} \lambda \\ \mu\ \nu \end{array} \right\}$ is the Christoffel symbol based on $g_{\mu\nu}$. In order for (iii) to hold in any gauge one must have the WGT $w_{\mu} \to \tilde{w}_{\mu} = w_{\mu} + \partial_{\mu}\lambda$ and if the vector B^{μ} is transported by parallel displacement around an infinitesimal closed parallelogram one finds

$$\Delta B^{\lambda} = B^{\sigma} K^{\lambda}_{\sigma\mu\nu} dx^{\mu} \delta x^{\nu};\ \Delta B = B W_{\mu\nu} dx^{\mu} \delta x^{\nu} \tag{1.4}$$

where

$$K^{\lambda}_{\sigma\mu\nu} = -\Gamma^{\lambda}{}_{\sigma\mu,\nu} + \Gamma^{\lambda}{}_{\sigma\nu,\mu} - \Gamma^{\alpha}{}_{\sigma\mu}\Gamma^{\lambda}_{\alpha\nu} + \Gamma^{\alpha}_{\sigma\nu}\Gamma^{\lambda}_{\alpha\mu} \tag{1.5}$$

is the curvature tensor formed from (1.3) and $W_{\mu\nu} = w_{\mu,\nu} - w_{\nu,\mu}$. Equations for the WGT $w_{\mu} \to \tilde{w}_{\mu}$ and the definition of $W_{\mu\nu}$ led Weyl to identify w_{μ} with the potential vector and $W_{\mu\nu}$ with the EM field strength; he used a variational principle $\delta I = 0$ with $I = \int L\sqrt{-g}d^4x$ with L built up from $K^{\lambda}_{\sigma\mu\nu}$ and $W_{\mu\nu}$. In order to have an action invariant under both coordinate transformations and WGT he was forced to use R^2 (R the Riemannian curvature scalar) and this led to the gravitational field.

Dirac revised this with a scalar field $\beta(x^{\nu})$ which under WGT changes via $\beta \to \tilde{\beta} = e^{-\lambda}\beta$ (i.e. $\pi(\beta) = -1$). His in-invariant action integral is then ($f_{,\mu} \equiv \partial_{\mu} f$)

$$I = \int [W^{\lambda\sigma} W_{\lambda\sigma} - \beta^2 R + \beta^2(k-6) w^{\sigma} w_{\sigma} + 2(k-6)\beta w^{\sigma}\beta_{,\sigma} + \tag{1.6}$$
$$+ k\beta_{,\underline{\sigma}}\beta_{,\sigma} + 2\Lambda\beta^4 + L_M]\sqrt{-g}d^4x$$

Here k is a parameter, Λ is the cosmological constant, L_M is the Lagrangian density of matter, and an underlined index is to be raised with $g^{\mu\nu}$. Now according to (1.4) this is a nonintegrable geometry but there may be situations when geometric vector fields are ruled out by physical constraints (e.g. the FRW universe). In this case one can preserve the WD character of the spacetime by assuming that w_{ν} is the gradient of a scalar function w so that $w_{\nu} = w_{,\nu} = \partial_{\nu} w$. One has then $W_{\mu\nu} = 0$ and from (1.4) results $\Delta B = 0$ yielding an integrable spacetime (Int-W-D spacetime). To develop this begin with (1.6) but with w_{ν} given by $w_{\nu} = \partial_{\nu} w$ so the first term in (1.6) vanishes. The parameter k is not fixed and the dynamical variables are $g_{\mu\nu}$, w, and β. Further it is assumed that L_M depends on ($g_{\mu\nu}$, w, β). For convenience write

$$b_{\mu} = (log(\beta))_{,\mu} = \beta_{,\mu}/\beta \tag{1.7}$$

and use a modified Weyl connection vector $W_{\mu} = w_{\mu} + b_{\mu}$ which is a gauge invariant gradient vector. Write also $k - 6 = 16\pi\kappa$ and varying w in (1.6) one gets a field equation

$$2(\kappa\beta^2 W^{\nu})_{;\nu} = S \tag{1.8}$$

where the semicolon denotes covariant differentiation with the Christoffel symbols and S is the Weylian scalar charge given by $16\pi S = \delta L_M/\delta w$. Varying $g_{\mu\nu}$ one gets also

$$G^{\nu}_{\mu} = -8\pi\frac{T^{\nu}_{\mu}}{\beta^2} + 16\pi\kappa\left(W^{\nu}W_{\mu} - \frac{1}{2}\delta^{\nu}_{\mu}W^{\sigma}W_{\sigma}\right) + \tag{1.9}$$

$$+2(\delta^{\nu}_{\mu}b^{\sigma}_{;\sigma} - b^{\nu}_{;\mu}) + 2b^{\nu}b_{\mu} + \delta^{\nu}_{\mu}b^{\sigma}_{\sigma} - \delta^{\nu}_{\mu}\beta^2\Lambda$$

where G^{ν}_{μ} represents the Einstein tensor and the EM density tensor of ordinary matter is

$$8\pi\sqrt{-g}T^{\mu\nu} = \delta(\sqrt{-g}L_M)/\delta g_{\mu\nu} \tag{1.10}$$

Finally the variation with respect to β gives an equation for the β field

$$R + k(b^{\sigma}_{;\sigma} + b^{\sigma}b_{\sigma}) = 16\pi\kappa(w^{\sigma}w_{\sigma} - w^{\sigma}_{;\sigma}) + 4\beta^2\Lambda + 8\pi\beta^{-1}B \tag{1.11}$$

Note in (1.11) R is the Riemannian curvature scalar and the Dirac charge B is a conjugate of the Dirac gauge function β, namely $16\pi B = \delta L_M/\delta\beta$.

By a simple procedure (cf. [**302**]) one can derive conservation laws; consider e.g. $I_M = \int L_M\sqrt{-g}d^4x$. This is an in-invariant so its variation due to coordinate transformation or WGT vanishes. Making use of $16\pi S = \delta L_M/\delta w$, (1.10), and $16\pi B = \delta L_M/\delta\beta$ one can write

$$\delta I_M = 8\pi\int(T^{\mu\nu}\delta g_{\mu\nu} + 2S\delta w + 2B\delta\beta)\sqrt{-g}d^4x \tag{1.12}$$

Via $x^{\mu} \to \tilde{x}^{\mu} = x^{\mu} + \eta^{\mu}$ for an arbitrary infinitesimal vector η^{μ} one can write

$$\delta g_{\mu\nu} = g_{\lambda\nu}\eta^{\lambda}_{;\mu} + g_{\mu\lambda}\eta^{\lambda}_{;\nu};\ \delta w = w_{,\nu}\eta^{\nu};\ \delta\beta = \beta_{,\nu}\eta^{\nu} \tag{1.13}$$

Taking into account $x^{\mu} \to \tilde{x}^{\mu}$ we have $\delta I_M = 0$ and making use of (1.13) one gets from (1.12) the energy momentum relations

$$T^{\lambda}_{\mu;\lambda} - Sw_{\mu} - \beta Bb_{\mu} = 0 \tag{1.14}$$

Further considering a WGT with infinitesimal $\lambda(x^{\mu})$ one has from (1.12) the equation $S+T-\beta B = 0$ with $T = T^{\sigma}_{\sigma}$. One can contract (1.9) and make use of (1.8) and $S + T = \beta B$ giving again (1.11), so that (1.11) is a corollary rather than an independent equation and one is free to choose the gauge function β in accordance with the gauge covariant nature of the theory. Going back to the energy-momentum relations one inserts $S + T = \beta B$ into (1.14) to get $T^{\lambda}_{\mu;\lambda} - Tb_{\mu} = SW_{\mu}$. Now go back to the field equation (1.9) and introduce the EM density tensor of the W_{μ} field

$$8\pi\Theta^{\mu\nu} = 16\pi\kappa\beta^2[(1/2)g^{\mu\nu}W^{\lambda}W_{\lambda} - W^{\mu}W^{\nu}] \tag{1.15}$$

Making use of (1.8) one can prove $\Theta^{\lambda}_{\mu;\nu} - \Theta b_{\mu} = -SW_{\mu}$ and using $T^{\lambda}_{\mu;\lambda} - TB_{\mu} = SW_{\mu}$ one has an equation for the joint energy momentum density

$$(T^{\lambda}_{\mu} + \Theta^{\lambda}_{\mu})_{;\lambda} - (T + \Theta)b_{\mu} = 0 \tag{1.16}$$

One can derive now the equation of motion of a test particle (following [**817**]). Consider matter consisting of identical particles with rest mass m and Weyl scalar charge q_s, being in the stage of a pressureless gas so that the EM density tensor can be written $T^{\mu\nu} = \rho U^\mu U^\nu$ where U^μ is the 4-velocity and the scalar mass density ρ is given by $\rho = m\rho_n$ with ρ_n the particle density. Taking into account the conservation of particle number one obtains from $T^\lambda_{\mu;\lambda} - Tb_\mu = SW_\mu$ the equation of motion

$$\frac{dU^\mu}{ds} + \left\{ \begin{array}{c} \mu \\ \lambda\ \sigma \end{array} \right\} U^\lambda U^\sigma = \left(b_\lambda + \frac{q_s}{m} W_\lambda\right)(g^{\mu\lambda} - U^\mu U^\lambda) \tag{1.17}$$

In the Einstein gauge ($\beta = 1$) we are then left with

$$\frac{dU^\mu}{ds} + \left\{ \begin{array}{c} \mu \\ \lambda\ \sigma \end{array} \right\} U^\lambda U^\sigma = \frac{q_s}{m} w_\lambda (g^{\mu\lambda} - U^\mu U^\lambda) \tag{1.18}$$

EXAMPLE 1.1. Following [**499**] one considers a static spherically symmetric situation with line element

$$ds^2 = e^\nu dt^2 - e^\lambda dr^2 - r^2(d\theta^2 + Sin^2(\theta)d\phi^2) \tag{1.19}$$

and all functions λ, ν, β, w, T^ν_μ, S, B depend only on r. One looks for local phenomena so $\Lambda = 0$. The field equations (1.9) can be written explicitly for $(\mu\nu) = (0,0)$, $(1,1)$, $(2,2)$, or $(3,3)$ to obtain

$$e^{-\lambda}\left(-\frac{\lambda'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} = -\frac{8\pi T^0_0}{\beta^2} + \tag{1.20}$$

$$+2e^{-\lambda}\left(-\frac{(b')^2}{2} - b'' + \frac{\lambda' b'}{2} - \frac{2b'}{r}\right) + 8\pi\kappa e^{-\lambda}(W')^2;$$

$$e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} = -\frac{8\pi T^1_1}{\beta^2} - 2e^{-\lambda}\left(\frac{\nu' b'}{2} + \frac{2b'}{r} + \frac{3(b')^2}{2}\right) - 8\pi\kappa e^{-\lambda}(W')^2;$$

$$\frac{1}{4}\left(\nu'' + \frac{(\nu')^2}{2} + \frac{\nu' - \lambda'}{r} - \frac{\nu'\lambda'}{2}\right) =$$

$$= -\frac{4e^\lambda \pi T^2_2}{\beta^2} - \left(b'' + \frac{(\nu' - \lambda')b'}{2} + \frac{b'}{r} + \frac{(b')^2}{2}\right) + 4\pi\kappa e^{-\lambda}(W')^2$$

From (1.8) one has the equation for the W field

$$2\kappa\left[W'' + \left(2b' + \frac{\nu' - \lambda'}{2} + \frac{2}{r}\right)W'\right] = -\frac{e^\lambda S}{\beta^2} \tag{1.21}$$

The most intriquing situation is when ordinary matter is absent, so $T^\mu_\nu = 0$, and then from $S + T = \beta B$ and $T^\lambda_{\mu;\lambda} - Tb_\mu = SW_\mu$ one has $S = 0$ and $B = 0$. Take first the simple case when $W' = 0$ or $\kappa = 0$ so (1.21) is satisfied identically and (1.20) takes the simple form

$$e^{-\lambda}\left(-\frac{\lambda'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} = 2e^{-\lambda}\left(-\frac{(b')^2}{2} - b'' + \frac{\lambda' b'}{2} - \frac{2b'}{r}\right); \tag{1.22}$$

$$e^{-\lambda}\left(\frac{\nu'}{r} + \frac{1}{r^2}\right) - \frac{1}{r^2} = -2e^{-\lambda}\left(\frac{\nu' b'}{2} + \frac{2b'}{r} + \frac{2(b')^2}{2}\right);$$

$$\frac{e^{-\lambda}}{2}\left(\nu'' + \frac{(\nu')^2}{2} + \frac{\nu' - \lambda'}{r} - \frac{\nu'\lambda'}{2}\right) = -2e^{-\lambda}\left(b'' + \frac{\nu' - \lambda'}{2} + \frac{b'}{r} + \frac{(b')^2}{2}\right)$$

Subtracting the first equation from the second one obtains $(1/r)(\lambda' + \nu') = 2b' - (\lambda' + \nu')b' - 2(b')^2$. The scalar $b = log(\beta)$ is still arbitrary so that one can impose a condition on it. Thus writing $b'' - (b')^2 = 0$ one can integrate to get $b(r) = log[1/(a - cr)]$ (curiously enough this is true) with a, c arbitrary constants which are taken to be positive. Using $b'' = (b')^2$ one obtains the equation $\lambda' + \nu' = 0$ and hence via $b = log[1/(a - cr)]$ there rsults from (1.22) a solution $exp(\nu) = exp(-\lambda) = (a - cr)^2/a^2$. Now go back to the Einstein equations $G^\nu_\mu = -8\pi T^\nu_\mu$; if one thinks of β as creating matter we can then calculate the matter density and pressure. From (1.22) the density is given by $8\pi\rho = -(3c^2/a^2) + (4c/ar)$ and the radial pressure is $P_r = -\rho$ (so there is tension rather than pressure). One notes that $P_r = -\rho$ has been used as the equation of state of prematter in cosmology (cf. [**502**]). Finally one can calculate the transverse pressure from (1.22) as $8\pi P_t = (3c^2/a^2) - (2c/ar)$ (which is anisotropic). Now suppose there is a spherically symmetric body filled with matter described by $8\pi\rho = -3c^2/a^2 + 4c/ar,\ P_r = -\rho$, and $8\pi P_t = (3c^2/a^2(-(2c/ar)$. Since the matter density can take only nonnegative values one has a limit on the size of the body $r_{boundary} \leq (4a/3c)$. Several models are possible; take e.g. a body with maximum radius $r_{bound} = 4a/3c$. One sees that on the boundary the density and radial pressure vanish so that this is an open model. Go back for a moment to the first equation in (1.3). It may be integrated, giving $exp(-\lambda) = exp(\nu) = 1 - (8\pi/r)\int_0^r \rho r^2 dr$. Assume that outside of the body the Einstein gauge holds, i.e. $\beta = 1$ $(b = 0)$ for $r > (4a/3c)$ so that one is left with the ordinary Riemannian geometry and with the exterior Schwarzschld solution $exp(-\lambda) = exp(\nu) = 1 - (2m/r)$. Comparing and using the equations at hand one obtains

$$m = 4\pi \int_0^{4a/3c} \rho r^2 dr = (16a/27c) = (4/9) r_{bound} \tag{1.23}$$

Note that in the body (at $r_s = a/c$) there is a singularity of β and of the metric; however the physical quantities ρ, P_r, P_t are regular there (cf. [**503**]). An external observer staying in the Riemannian spacetime will recognize the above entity, made of Weyl-Dirac geometry, as a body having mass (1.23) and radius $4a/3c$.

EXAMPLE 1.2. Another example of matter creation via geometry is also given in [**499**] with a homogeneous and isotropic FRW universe and line element

$$ds^2 = dt^2 - R^2(t)\left[\frac{dr^2}{1 - \tilde{k}r^2} + r^2 d\Omega^2\right] \tag{1.24}$$

Here $R(t)$ is the cosmic scale factor, $\tilde{k} = 0, \pm 1$ stands for the spatial curvature parameter, and $d\Omega^2 = d\theta^2 + Sin^2(\theta)d\phi^2$ is the line element on the unit sphere. The universe is filled with ordinary cosmic matter in the state of a perfect fluid at rest and with the cosmic scalar fields β and w. One considers, for $\kappa \geq 0$, $T^0_0 = \rho(t),\ T^1_1 = T^2_2 = T^3_3 = -P(t),\ T = \rho - 3P,\ \beta = \beta(t)$, and $= w(t)$. Use also the Einstein gauge $\beta = 1 \sim b = 0$ and the Einstein-Friedmann equations (cf. [**499**]). After much calculation one looks at the expansion of the universe (with no ordinary matter) and the model provides a high rate of matter creation from an initial empty egg (i.e. geometry brings matter into being). Another model along similar lines looks at interaction between geometric fields and matter during radiation and dust dominated periods with a number of interesting results. In particular matter creation takes place in the radiation dominated universe and also for open and flat models in a dust dominated era while in a closed dust universe there is matter creation for awhile after which matter annihilation arises stimulated by the w field.

EXAMPLE 1.3. We go now to the second paper in [**498**] which builds up a singularity free cosmological model that originates from pure geometry. The Planckian state (characterized by $\rho_P = c^3/\hbar G = 3.83 \cdot 10^{65}\ cm^{-2}$, $R_I = (3/8\pi\rho_P)^{1/2} = 5.58 \cdot 10^{-34}\ cm$, and $T_I = 2.65 \cdot 10^{-180}\ K$) is preceeded by a pre-Planckian period. This starts from a primary empty spacetime entity, described by an integrable WD geometry. During the pre-Planckian period geometry creates cosmic matter and a the end of this creation process one has the Planckian cosmic egg filled with prematter. The prematter model of [**499**] will be updated according to present observational data and also modified by the introduction of a nonzero cosmological constant. Thus, reviewing a bit, we have $g_{\mu\nu}$, β, and w_μ as above with $w_\mu = \partial{-}\mu w$ to provide an integrable WD theory. There is an action (1.6) and one uses (1.7) and $W_\mu = w_\mu + b_\mu$. Also $k-6 = 16\pi\kappa$ (here σ is used in place of κ and later σ becomes $-\kappa^2$ so we will think of $\kappa \to -\kappa^2$ later on. As before one has (1.8), $16\pi S = \delta L_M/\delta w$, (1.9), (1.10), (1.11), $16\pi B = \delta L_M/\delta B$, $I_M = \int L_M\sqrt{-g}d^4x$, (1.14), $S + T = \beta B$, and $T^\lambda_{\mu;\lambda} - Tb_\mu = SW_\mu$. Recall also (1.11) is a corollary so that β and the Dirac charge B can be chosen arbitrarily. Further (1.15), (1.16), etc. will still apply after which one has an Int-W-D theory. One considers a homogeneous isotropic spatially closed ($\tilde{k} = 1$) universe described by the FRW line element of (1.24). For a universe filled with cosmic matter in the state of a perfect fluid at rest its EM tensor has nonvanishing components T^i_i ($i = 0, 1, 2, 3$) and in addition to any matter field there are two cosmic scalar fields $\beta(t)$ and $W(t)$ stemming from the geometric framework. Taking into account (1.24), one obtains from (1.9) the cosmological equations

$$\frac{\dot{R}^2}{R^2} = \frac{8\pi T^0_0}{3\beta^2} - \frac{8\pi\sigma\dot{W}^2}{3} - \frac{2\dot{R}\dot{b}}{R} - \dot{b}^2 + \frac{\beta^2\Lambda}{3} - \frac{1}{R^2} \tag{1.25}$$

$$\frac{\ddot{R}}{R} = \frac{4\pi}{\beta^2}\left(T^1_1 - \frac{T^0_0}{3}\right) + \frac{16\pi\sigma\dot{W}^2}{3} - \ddot{b} - \frac{\dot{R}\dot{b}}{R} + \frac{\beta^2\Lambda}{3}$$

From (1.9) one also gets the trace equation

$$G^\sigma_\sigma = -\frac{8\pi T}{\beta^2} - 16\pi\sigma\dot{W}^2 + 6\ddot{b} + 18\frac{\dot{R}\dot{b}}{R} + 6\dot{b}^2 - 4\beta\Lambda \tag{1.26}$$

Comparing (1.25) with the usual Einstein equations for cosmology one concludes that the observable density and pressure of the cosmic matter are $\rho = T^0_0/\beta^2$ and $P = -T^1_1/\beta^2$. Now we let $\sigma \to -\kappa^2$ and regard $\beta(t)$ as a function of $R(t)$ so that

$$\beta = \beta(R),\ \dot{\beta} = \beta'\dot{R};\ \ddot{\beta} = \beta''\dot{R}^2 + \beta'\ddot{R} \tag{1.27}$$

Taking into account (1.27) one can rewrite (1.25) as

$$\frac{\dot{R}^2}{R^2}\left(1 + \frac{\beta' R}{\beta}\right)^2 = \frac{8\pi}{3}(\rho + \kappa^2\dot{W}^2) + \frac{\beta^2\Lambda}{3} - \frac{1}{R^2}; \tag{1.28}$$

$$\frac{\ddot{R}}{R}\left(1 + \frac{\beta' R}{\beta}\right) + \frac{\dot{R}^2}{R^2}\left(\frac{\beta'' R^2}{\beta} - \frac{(\beta')^2R^2}{\beta^2} + \frac{\beta' R}{\beta}\right) = -\frac{4\pi}{3}(3P + \rho + 4\kappa^2\dot{W}^2) + \frac{\beta^2\Lambda}{3}$$

Further the equation (1.8) of the W field takes the form

$$\ddot{W}+\left(\frac{2\dot{\beta}}{\beta}+\frac{3\dot{R}}{R}\right)=-\frac{S}{2\kappa^2\beta^2}\equiv\partial_t(\dot{W}\beta^2R^3)=-\frac{SR^3}{2\kappa^2} \tag{1.29}$$

With (1.24) (and the density and pressure terms above) the EM relation for matter is now

$$\dot{\rho}+3(\dot{R}/R)(\rho+P)+(\dot{\beta}/\beta)(\rho+3P)=(S\dot{W}/\beta^2) \tag{1.30}$$

and $S+T=\beta B$ can be written as

$$S+(\rho-3P)\beta^2-B\beta=0 \tag{1.31}$$

For the FRW line element (1.24) the β field equation (1.11) takes the form ($\sigma\sim-\kappa^2$)

$$R^\sigma_\sigma+\kappa\left[\frac{(\dot{b}R^3)_{,t}}{R^3}+\dot{b}^2\right]+16\pi\sigma\left[\frac{(\dot{w}R^3)_{,t}}{R^3}-\dot{w}^2\right]-4\beta^2\Lambda-8\pi\beta^{-1}B=0 \tag{1.32}$$

By (1.29) and (1.31) this turns out to be identical with the trace equation (1.26) and B may be cancelled from the equations (i.e. β and B are arbitrary - recall that (1.11) is a corollary, etc.). Now introduce the energy density and pressure of the W field via

$$\rho_w=\Theta^0_0;\ P_w=-\Theta^1_1=-\Theta^2_2=-\Theta^3_3 \tag{1.33}$$

Making use of (1.24) one obtains $\rho_w=P_w=\kappa^2\beta^2\dot{W}^2$ leading to

$$(\rho+\kappa^2\dot{W}^2)_{,t}+\frac{\dot{\beta}}{\beta}(\rho+3P+4\kappa^2\dot{W}^2)+\frac{3\dot{R}}{R}(\rho+P+2\kappa^2\dot{W}^2)=0 \tag{1.34}$$

Then introduce the reduced energy density and pressure $\bar{\rho}_w$ and $\bar{P}_w$ via

$$\bar{\rho}_w=\rho_w/\beta^2;\ \bar{P}_w=P_w/\beta^2;\ \bar{\rho}_w=\bar{P}_w=\kappa^2\dot{W}^2 \tag{1.35}$$

and write $\bar{\rho}=\rho+\bar{\rho}_w=\rho+\kappa^2\dot{W}^2$ and $\bar{P}=P+\bar{P}_w=P+\kappa^2\dot{W}^2$; then (1.28) becomes

$$\frac{\dot{R}^2}{R^2}\left(1+\frac{\beta' R}{\beta}\right)^2=\frac{8\pi\bar{\rho}}{3}+\frac{\beta^2\Lambda}{3}-\frac{1}{R^2}; \tag{1.36}$$

$$\frac{\ddot{R}}{R}\left(1+\frac{\beta' R}{\beta}\right)+\frac{\dot{R}^2}{R^2}\left(\frac{\beta'' R^2}{\beta}-\frac{(\beta')^2R^2}{\beta^2}+\frac{\beta' R}{\beta}\right)=-\frac{4\pi}{3}(3\bar{P}+\bar{\rho})+\frac{\beta^2\Lambda}{3}$$

and the energy momentum relation (1.34) is

$$\bar{\rho}+\frac{\dot{\beta}}{\beta}(\bar{\rho}+3\bar{P})+3\frac{\dot{R}}{R}(\bar{\rho}+\bar{P})=0 \tag{1.37}$$

Now for the pre-Planckian period one looks for matter production by geometry and returns to (1.29) and (1.30). From (1.30) one sees that the W field can act as a creator of matter even if at the beginning moment no matter was present (surely someone has thought of a Higgs role for the W field ?). According to (1.29) the W field depends on the source function S. On the whole one adopts the singularity free cosmological model of [**502**] with its initial prematter period but completed with a nonzero cosmological constant. Also some constants such as the Hubble constant, matter densities, etc. are updated. The initial Planckian egg is thus preceded by a pre-Planckian period originating from a primary geometric state (primary state) at $R_0 = 5.58 \cdot 10^{-36}$ cm and lasting up to the initial Planckian egg at $R_I = 5.58 \cdot 10^{-34}$ cm. This spherically symmetric homogeneous and isotropic universe is described by the Int-W-D geometry (no cosmic matter) with nothing but geometry, including the W and β fields, at $R_0 = 5.58 \cdot 10^{-36}$ cm, $\rho_0 = 0$, $P_0 = 0$, $\beta_0 \neq 0$, and $W_0 \neq 0$. Assume this was quasistatic with $\dot{R}_0 = 0$ and via (1.27) also $\dot{\beta}_0 = 0$. Thus one obtains from (1.28) at the beginning moment

$$(8\pi/3)\kappa^2\dot{W}_0^2 + (1/3)\beta_0^2\Lambda - (1/R_0^2) = 0 \tag{1.38}$$

Since B may be chosen arbitrarily one can take a suitable function for the Weylian scalar charge S and then calculate the Dirac charge B according to (1.31). An "appropriate" choice is $S = S_0(\beta^2\dot{\beta}/R^3$ with S_0 constant (explanation omitted). Inserting this into (1.29) and integrating one gets $\dot{W} = -(S_0/6\kappa^2)(\beta/R^3)$ so $\ddot{W} = 0$ and one can rewrite (1.38) as

$$(8\pi/3)(S_0^2/36\kappa^2)(\beta_0^2/R_0^6) + (1/3)\Lambda\beta_0^2 - (1/R_0^2) = 0 \tag{1.39}$$

One will use this below to calculate the value of β_0 but first the scenario of the very early universe must be completed by an equation of state of cosmic matter during the pre-Planckian period. According to (1.30) and (1.34) the matter is created by the W field which has an EM density tensor (1.14), etc. The components of this tensor are related by (1.33), etc., so that the pressure of this field is equal to its energy density. Thus one writes $P = \rho$ and then one can rewrite (1.30) as

$$\dot{\rho} + 6(\dot{R}/R)\rho + 4(\dot{\beta}/\beta)\rho = S\dot{W}/\beta^2 \equiv \rho = (1/\beta^4R^6)\int S\dot{W}\beta^2R^6dt \tag{1.40}$$

The density of matter created by geometry is given now by the expression

$$\rho = (S_0^2\beta_0^6/36\kappa^2\beta^4R^6)[1 - (\beta/\beta_0)^6] \tag{1.41}$$

Thus there was a zero matter density and pressure (with the assumptions above) at the beginning moment and for a rapidly decreasing $\beta(R)$ one can have $(\beta_I/\beta_0)^6 << 1$ where $\beta_I = \beta(R_I)$. According to this scenario at $R = R_I$ the matter density reaches its maximum $\rho_P = 3.83 \cdot 10^{65}$ cm^{-2} there so that from (1.41), etc., one gets $\rho_P = S_0^2\beta_0^6/36\kappa^2\beta_I^4R_I^6$. From this one can calculate $S_0^2/36\kappa^2$ for a given gauge function $\beta(R)$. Now for a moment go back to the energy equation (1.40); it can be rewritten as

$$\dot{\rho} + 6(\dot{R}/R)\rho + 4(\dot{\beta}/\beta)\rho = -(S_0^2/6\kappa^2)(\beta^2/R^6)(\dot{\beta}/\beta) \tag{1.42}$$

Then the term on the right side of (1.42), which describes matter creation by the W field can be compared with the third term on the left side which represents the existing amount of matter. Making use of (1.41) this gives

$$(1/4\rho)(S_0^2\beta^2/6\kappa^2R^6) = [3\beta^6/2(\beta_0^6 - \beta^6)] \tag{1.43}$$

Further, comparing the matter energy with that of the W field in the equations (1.28) one gets

$$(\kappa^2\dot{W}^2/\rho) = [\beta^6/(\beta_0^6 - \beta^6)] \tag{1.44}$$

Thus in the beginning when $\beta_0 - \beta$ is small $\dot{W}$ dominates the matter creation while for large R, when $\beta << \beta_0$, the matter creation term becomes negligible.

This is just a sampling of results in [**498, 499, 500, 501, 502, 503**]. Many other cosmological questions of great interest including dark matter, quintessence, etc. are also treated. There are in addition many fascinating papers speculating about the original universe from many points of view and we attempt no survey here. The approach here via Weyl-Dirac geometry seems however too lovely to ignore and it may provide further insight into questions of quantum fluctuations. The inroads into cosmology here are an inevitable consequence of the presence of Weyl-Dirac theory in dealing with quantum fluctuations and once wave functins and Bohmian ideas are introduced the quantum potential will automatically arise via β and w_μ.

2. REMARKS ON COSMOLOGY

We begin with some background information (cf. [**1, 186, 187, 188, 189, 741, 882, 883, 903**]. Thus recall that the deBroglie wave length is $\lambda = \hbar/p$ and the Compton wave length is $\Lambda = \hbar/mc$. The uncertainty principle states that $(\Delta x)(\Delta p) \geq \hbar$ and the diffusion coefficient for Brownian motion is proportional to $D = \hbar/m$. The fractal dimension of a quantum path is $d_f = 2$ at scales between λ and Λ but becomes $d_f = 1$ at scales smaller than Λ. Brownian motion characterizes the domain between Λ and λ (cf. [**1, 903**]). Heuristically from $\Delta p = m(\Delta x/\Delta t)$ and $\Delta x \Delta p \geq \hbar$ we have $m(\Delta x/\Delta t) = \Delta p \geq \hbar/\Delta x$ or $(\Delta x)^2 \geq (\hbar/m)\Delta t$ which can be rephrased as $< x^2 >\sim Dt$ for $D = \hbar/m$. Further note that from $E \sim p^2/2m$ one has $\Delta E \sim (1/2m)(\Delta p)^2$ (working around $p_0 = 0$ say). Then from $\Delta p \geq \hbar/\Delta x$ and $(\Delta x)^2 \sim D\Delta t$ (with $\Delta x \Delta p \geq \hbar$) we obtain $(1/2m)(\Delta p)^2 \geq (\hbar^2/2mD\Delta t)$ from which $\Delta E \Delta t \sim \hbar/2$.

Now from [**883**] one looks at Weyl geometry and refers to the Lagrangian of [**840**] of the form $L = L_C(q, \dot{q}, t) + \gamma(\hbar^2/m)R(q, t)$ where R is the Ricci scalar curvature in the Weyl geometry (cf. Section 3.3). Then it turns out that the quantum potential Q has the form $Q = -\gamma(\hbar^2/m)R$ and the Q can be related to quantum fluctuations via Fisher information. In [**883**] one replaces the Weyl vector ϕ (which measures length dilations) by a noncommutative (NC) geometry $ds^2 = (h_{\mu\nu} + \bar{h}_{\mu\nu})dx^\mu dx^\nu$ with a tensor density $\bar{h}_{\mu\nu}$ arising via the antisymmetric part. This corresponds then to $[dx^\mu, dx^\nu] \sim \ell^2 \neq 0$ and in a certain sense legitimizes the approach of [**840**]. Moreover the NC geometry produces a multiply connected space in which a closed circuit cannot be shrunk to a point so for a circle C of diameter λ in e.g. a doubly connected space one will have ($V = (\hbar/m)\vec{\nabla}S$ and v is some average velocity)

$$\Gamma = \int_C m\vec{V} \cdot d\vec{r} = \hbar \int_C \vec{\nabla}S \cdot d\vec{r} = \hbar \oint dS = mv\pi\lambda = \pi\hbar \tag{2.1}$$

Consequently $\lambda = \hbar/mv$ and this shows an emergence of the deBroglie wavelength following from the NC geometry. We note also from [**188, 189, 840**] that for $\psi = \sqrt{\rho}exp(iS/\hbar)$ the Weyl vector $\phi \sim -\nabla log(\rho)$ and ψ satisfies (for $A_k = 0$) an equation

$$i\hbar\psi_t = -\frac{\hbar^2}{2m}\left(\frac{1}{\sqrt{g}}\partial_i\sqrt{g}\right)g^{ik}\partial_k\psi + \left(V - \frac{\gamma\hbar^2}{m}\dot{R}\right)\psi \tag{2.2}$$

where $\dot{R}$ is the Riemannian curvature. Here $grad(f) \sim \partial_k f$ and Δf corresponds to taking the divergence of the associated contravariant vector $g^{ik}\partial_k f$, i.e. $\Delta f = div(grad(f)) = (1/\sqrt{g})\partial_i\sqrt{g}g^{ik}\partial_k f)$. Note also in forming a Lagrangian $\dot{x}^\alpha = 2g^{\alpha\beta}p_\beta$ or $p_\alpha = (1/2)g_{\alpha\beta}\dot{x}^\beta$ so that (cf. [**12**])

$$H = g^{\alpha\beta}p_\alpha p_\beta \mapsto L = \dot{x}^\alpha p_\alpha - g^{\alpha\beta}p_\alpha p_\beta = (1/4)g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta \tag{2.3}$$

Thus one has a complete geometrization of quantum mechanics (QM) via (2.2) (recall also that the Ricci-Weyl curvature has the form

$$R = \dot{R} + (1/2\gamma\sqrt{\rho})[(1/\sqrt{g})\partial_i(\sqrt{g}g^{ik}\partial_k\sqrt{\rho})] \tag{2.4}$$

where $\gamma = 1/12$ here.

REMARK 4.2.1. We recall now that in the Nottale derivation of the Schrödinger equation SE) one has a complex velocity $V - iU$ due to fractal quantum paths (cf. [**186, 187, 188, 272, 715**]). Here $U = D(d/dx)(log(\rho))$ can be "conveniently" taken to be a constant α (cf. [**272**]) which would imply $log(\rho) = (\alpha/D)x = \beta x$ and $Q = -(\hbar^2/8m)\beta^2$ as pointed out in [**882**] (recall $Q = -(\hbar^2/2m)(\partial^2\sqrt{\rho}/\sqrt{\rho})$ with $\sqrt{\rho} = exp(\beta x/2)$ here). Now one can apparently make a case for the Zitterbewegung or self interaction effects within a minimum cutoff Compton wavelength to generate inertial mass. If Q is inertial energy, say $Q = -\delta mc^2$, with $(\hbar^2/8m)\beta^2 = (\hbar^2/8m)(\alpha^2/D^2) = m\alpha^2/8 = \delta mc^2$ one arrives at $\alpha \sim (8\delta)c$ (omitting constants such as 8δ etc. in approximations involving large and small numbers). It is then argued that the stochastic-fractal formulation of Nottale leads to the emergence of spacetime coordinates (x, ict) and such matters are obviously intriguing.

There is a great deal of fascinating information available concerning various fundamental constants and large numbers in physics (we refer here to [**604, 715, 723, 961, 972**] for example and for more "adventurous" material to e.g. [**850, 837, 884, 885, 886, 887, 888**] and the numerous papers of M. El Naschie in the journal Chaos, Solitons, and Fractals). Dirac had previously spoken eloquently about the importance of large numbers and their relations and in this spirit one is compelled to look as such matters. One seems at times to be simply playing with numbers (sometimes called numerology) but there are too many remarkable coincidences to be ignored and e.g. El Naschie's program of tying matters together via Cantor sets and the golden mean ϕ is in my opinion worth serious consideration. Sidharth's arguments about Cantorian $\mathcal{E}^\infty$ also lend some structural meaning and should be pursued further. In any event we gather here first a collection of numbers and ideas in no particular order following [**882, 883, 884, 885**].

(1) The average distance ℓ covered in N steps in a random walk is $\ell = R/\sqrt{N}$ where R is the dimension of the system. Such a relation with $R \sim 10^{28}$ cm (radius of the universe) and $N \sim 10^{80}$ (number of particles in the universe) gives $\ell \sim 10^{-12} \sim 10^{-13} = \ell_\pi$ which is the Compton wavelength

of a typical elementary particle (the pion). The stipulation that $10^{-12} \sim 10^{-13}$ is of course reasonable but somewhat unsettling).

(2) The Planck scale is defined via $\ell_P = (\hbar G/c^3)^{1/2} \sim 10^{-33}$ cm and $t_P = (\hbar G/c^5)^{1/2} \sim 10^{-42}$ sec with $m_P = 10^{-5}$ gm the Planck mass. Here t_P is also the Compton time for the Planck mass ($\ell_P = ct_P$).

(3) $R \sim cT$ where T is the age of the universe so, from item 1, $T \sim \sqrt{N}\tau$ where $\tau = \ell_\pi/c$ is the Compton time for a pion. Further $M = Nm$ where M is the mass of the universe and m is the (typical) pion mass.

(4) The energy of fluctuations of the magnetic field $\vec{B}$ in a region of length ℓ is $B^2 \sim \hbar c/\ell^4$ so for $\ell \sim$ Compton wave length the resultant particle fluctuation energy in a volume $\sim \ell^3$ is $\ell^3 B^2 \sim \hbar c/\ell = mc^2$. Thus the entire energy of an elementary particle of mass m is generated by fluctuations alone.

(5) The fluctuation in particle number is of order $\sqrt{N}$ (cf. [**483**]) and a typical time interval of uncertainty is $\Delta t \sim \hbar/mc^2$ (via $\Delta E \Delta t \sim \hbar$). In the spirit of Prigogine Heisenberg uncertainty gives rise to production of energy over short intervals of time leading to a one way creation of particles. Thus $dN/dt \sim \sqrt{N}(mc^2/\hbar)$ leading to $T \sim (\hbar/2mc^2)\sqrt{N} = \sqrt{N}\tau$ as in item 3 ($\tau = \ell/c = (\hbar/mc)/c$ - a factor of 2 is included here).

(6) Now recall $R \sim GM/c^2$ where $Nm = M$ where m is the mass of a typical elementary particle. Then random walk considerations and fluctuations of order $\sqrt{N}$ from the ZPF give $R = \sqrt{N}\ell$. Going to [**888**] we note first for $H = Gm^3c/\hbar^2$ ($H \sim$ Hubble constant) and $R = GmN/c^2$ one has, for G constant, $\dot{R} = (Gm/c^2)\dot{N} \sim (Gm/c^2)(mc^2\sqrt{N}/\hbar) = (Gm^3c/\hbar^2)(\hbar/mc)\sqrt{N} = HR$ as it should. In fact H is often defined as $\dot{R}/R$. In particular one can conclude from $\dot{R} = RH$ and H constant that $\ddot{R} = RH^2$.

Consider now [**715, 716, 814**] where scale relations in micro and macro physics abound. Let us begin with $H = Gm^3c/\hbar^2$ or $m = (\hbar^2 H/cG)^{1/3}$ (m presumably refers to pion mass here). Then one notes that the cosmological constant Λ has dimension $1/L^2$ and this suggests that there should be a maximal scale length $L = 1/\sqrt{\Lambda}$. Next a version of Mach's principle is achieved by requiring that the gravitational energy of interaction of a body with the universe (described as a mass M at average distance R) should be equal to its self energy of inertial origin $E = mc^2$, namely

$$GmM/R = mc^2 \Rightarrow (GM/Rc^2) \sim 1 \tag{2.5}$$

Now $2GM/c^2$ corresponds to the classical radius of a Schwartzschild black hole so (2.5) says that the universe is like a black hole. Next imagine $M = 4\pi\rho R^3/3$ with $\dot{R} = c = RH$ so from $2GM/c^2R = 1$ there results $(2G/c^2R)(4\pi\rho R^3/3) = 1$ which implies $(8\pi G\rho/3H^2) = \Omega = 1$ (space flatness condition - with a cosmological constant the Schwartzschild relation is $(2GM/c^2R) + (\Lambda R^2/3) = 1 \sim (8\pi G\rho + \Lambda c^2)/3H^2 = 1$ - cf. [**716, 720**]). Another formula arises by introducing the Planck mass as a natural unit and writing Newton's law with $Gm_P^2 = \hbar c$ (following from $R = \hbar/mc$ and $(Gm^2/R) = mc^2$ which implies $Gm^2 = \hbar c$ for $m = m_P$) in the form $F = \hbar c[(m/m_P)(m'/m_P)]/R^2$ (since $\hbar cmm'/m_P^2R^2 = Gmm'/R^2$); such a formula appears also in [**176, 417**]. Regarding the cosmological constant one notes first that the Planck length $\Lambda_P = (\hbar G/c^3)^{1/2}$ is the only length that can be envisioned with the three fundamental constants $\hbar$, G, c. Here the maximum scale length L should have the form $L = \Lambda_P K = 1/\sqrt{\Lambda}$. This gives a number $K \sim 10^{61}$. Now if the universe is a black hole, looking at a resolution scale $1/L$ in the Einstein model the maximal separation between points is πL so the effective mass should be characterized by $2GM/c^2\pi L = 1$. This leads to one of the classical large number coincidences $m/m_P = (\pi/2)K$ which for $K \sim 10^{61}$ gives a characteristic mass $\sim 10^{23}$ solar masses (which corresponds to 10^{11} galaxies of 10^{12} solar masses). To get $m/m_P = (\pi/2)K$ one uses an argument comparing lengths $\ell = Gm/ < v^2 >$ and $\lambda = \hbar/mv$ which if equivalent yields Planck mass with $v = c$ and $Gm_P^2 = \hbar c$ or $m_P^2 = \hbar c/G$. Then from $2GM/c62\pi L = 1$ one has

$$\frac{\pi}{2}K = \frac{GM}{c^2 L}\frac{L}{\Lambda_P} = \frac{GM}{c^2\Lambda_P} = \frac{\hbar c M}{m_P^2 c^2 \Lambda_P} = \frac{\hbar M}{m_P^2 c \Lambda_P} = \frac{M}{m_P} \tag{2.6}$$

Finally consider a characteristic minimal energy $E_{min} = \hbar c/L$ and, for the electron of purely electromagnetic (EM) origin, a scale r_0 is defined where $e^2/r_0 = m_e c^2$ (i.e. $r_0 = \alpha\lambda_C$ is the classical radius of the electron - note $r_0 = e^2/mc^2 = \lambda(\hbar/mc) = \alpha\lambda_C$ where λ_C is the Compton wavelength and $\alpha\hbar = e^2/c$ or $\alpha = e^2/\hbar c$ is the fine structure constan - sometimes written as $\alpha = e^2/4\pi\hbar c$ in suitable units). Then assume the gravitational self energy of the electron at scale r_0 equals the minimal energy E_{min}; this implies $Gm^2(r_0)/r_0 = \hbar c/L$ (here $m(r_0) \sim \alpha^{-1}m_e$ modulo a small scale dependence of α) and leads to $\alpha(m_P/m_e) = K^{1/3}$. To see this write $G = \hbar c/m_P^2$ as before and recll $m = \alpha^{-1}m_e$; then

$$\frac{Gm^2}{r_0} = \frac{\hbar c}{L} = \frac{\hbar c}{\Lambda_P K} \Rightarrow \frac{\hbar c m^2}{r_0 m_P^2} = \frac{\hbar c}{\Lambda_P K} \Rightarrow K = \frac{r_0 m_P^2}{m^2 \Lambda_P} \tag{2.7}$$

But $\Lambda_P = \hbar/m_P c$ and $r_0 = \hbar/mc$ so $K = m_P^2\hbar/m^3 c\Lambda_P = m_P^4/m^3$ which means $K^{1/3} = m_P/m = \alpha(m_P/m_e)$. For completeness we note

$$\ell_P = \left(\frac{\hbar G}{c^3}\right)^{1/2} \sim 1.62\cdot 10^{-33}\ cm;\ \lambda_P = \frac{\ell_P}{c} \sim 5.4\cdot 10^{-44}\ sec; \tag{2.8}$$

$$m_P = \frac{\hbar}{\ell_P c} \sim 2.17\cdot 10^{-5}\ gram$$

We jump ahead now to the more recent articles [**716, 720, 717, 814**] and to the discussion in [**220, 221, 222**] (we also find it curious that Nottale's scale relativity has not been "blessed" with any establishment interest). We recall first a few basic facts following [**814**]. The simplest form for a scale differential equation describing the dependence of a fractal coordinate X in terms of resolution ϵ is given by a first order, linear, renormalization group like equation $(\partial X(t,\epsilon)/\partial log(\epsilon)) = a - \delta X$ with solution $X(t,\epsilon) = x(t)[1 + \zeta(t)(\lambda/\epsilon)^\delta]$. This involves a fractal asymptotic behavior at small scales with fractal dimension $D = 1 + \delta$, which is broken at large scale beyond the transition scale λ (cf. [**715, 716, 720**] for more detail). By differentiating one obtains $dX = dx + d\xi$ where $d\xi$ is a scale dependent fractal part and dx is a scale independent classical part such that $dx = vdt$ and $d\xi = \eta\sqrt{2\mathcal{D}}(dt^2)^{1/2D}$ where $< \eta >= 0$ and $< \eta^2 >= 1$ (one considers here only $D = 2$). Here each individual trajectory is assumed fractal and the test paricles can follow an infinity of possible trajectories. This leads one to a nondeterministic, fluid like description, in terms of $v = v(x(t), t)$. The reflection invariance $dt \to -dt$ is broken via nondifferentiablity leading to a two valued velocity vector (cf. [**186, 187, 272, 715**]) and one arrives at a complex time derivative $(d'/dt) = \partial_t + \mathcal{V} \cdot \nabla - i\mathcal{D}\Delta$. Setting $\psi = exp(i\mathfrak{S}/2m\mathcal{D})$ one obtains a geodesic equation in fractal space via $(d'/dt)\mathcal{V} = 0$ which becomes the free Schrödinger equation (SE) $\mathcal{D}^2\Delta\psi + i\mathcal{D}\partial_t\psi = 0$. Now in [**814**] one is interested in macrophysical applications and considers a free particle in a curved space time whose spatial part is also fractal beyond some time or space transition. The equation of motion can be written (to first order approximation) by a free motion geodesic equation combining the relativistic covariant derivative (describing curvature) and the scale relativistic covariant derivative d'/dt (describing fracticality). We refer to [**188, 189, 873**] for variations on this. Thus one considers in the Newtonian limit

$$(D/dt)\mathcal{V} = (d'/dt)\mathcal{V} + \nabla(\phi/m) = 0 \tag{2.9}$$

where ϕ is the Newton potential energy. In terms of ψ one obtains

$$\mathcal{D}^2\Delta\psi + i\mathcal{D}\partial_t\psi = (\phi/2m)\psi \tag{2.10}$$

Since the imaginary part of this equation is the equation of continuity (and thinking of the motion in terms of an infinite family of geodesics) $\rho = \psi\psi^\dagger$ can be interpreted as the probability density of the particle postions. For a Kepler potential (in the stationary case) one has then

$$2\mathcal{D}^2\Delta\psi + [(E/m) + (GM/r)]\psi = 0 \tag{2.11}$$

Via the equivalence principle (cf. [**13**]) this must be independent of the test particle mass while GM provides the natural length unit; hence $\mathcal{D} = (GM/2w)$ where w is a fundamental constant with the dimensions of velocity. The ratio $\alpha_g = w/c$ actually plays the role of a macroscopic gravitation coupling constant (cf. [**13**]). One shows in [**814**] that the solutions of this gravitational SE are characterized by a universal quantization of velocities in terms of the constant $w = 144.7 \pm 0.5$ km/s (or its multiples or submultiples); the precise law of quantization depends on the potential. Depending on the scale either the classical or the fractal part dominates. Various situations are examined and we only indicate a few here. The evolution equations are the Schrödinger - Newton equation and the classical Poisson equation (cf. [**632**])

$$\mathcal{D}^2\Delta\psi + i\mathcal{D}\frac{\partial\psi}{\partial t} - \frac{\phi}{2m}\psi = 0;\ \ \Delta\Phi = 4\pi G\rho\ \ (\phi = m\Phi) \tag{2.12}$$

Here Φ is the potential and $\phi = m\Phi$ the potential energy. Separating the real and imaginary parts one arives at

(2.13)
$$m\left(\frac{\partial}{\partial t} + V \cdot \nabla\right)V = -\nabla(\phi + Q);\ \ \frac{\partial P}{\partial t} + div(PV) = 0;\ \ Q = -2m\mathcal{D}^2\frac{\Delta\sqrt{P}}{\sqrt{P}}$$

In the situation where the particles are assumed to fill the "orbitals" the density of matter becomes proportional to the probability density, i.e. $\rho \propto P = \psi\psi^\dagger$ and the two equations combine to form a single Hartree equation for matter alone, namely

$$\Delta\left(\frac{\mathcal{D}^2\Delta\psi + i\mathcal{D}\partial_t\psi}{\psi}\right) - 2\pi G\rho_0|\psi|^2 = 0 \tag{2.14}$$

Another case arises when the number of bodies is small and they follow at random one among the possible trajectories so that $P = \psi\psi^\dagger$ is nothing else than a probability density while space remains essentially empty. It is suggested that this allows one to explain some effects that up to now have been attributed to dark matter (cf. [**290, 716, 720**] and below for more on this).

We mention in particular the case where $\Phi = -GM/r$ and the SE becomes $\mathcal{D}^2\Delta\psi + i\mathcal{D}\partial_t\psi + (GM/2r)\psi = 0$. One looks for solutions $\psi = \psi(r)exp(-iEt/2m\mathcal{D})$ and makes substitutions $\hbar/2m \sim \mathcal{D}$ and $e^2 \sim GMm$ where m is the test particle inertial mass. This yields an equation similar to the quantum hydrogen atom equation whose solution involves Laguerre polynomials $\psi(r) = \psi_{n\ell m}(r,\theta,\phi) = R_{n\ell}(r)Y_\ell^m(\theta,\phi)$ and the energy/mass ratio is quantized as

$$E_n/m = -(G^2M^2/8\mathcal{D}^2n^2) = -(1/2)(w_0^2/n^2) \tag{2.15}$$

while the natural length unit is the Bohr radius $a_0 = 4\mathcal{D}^2/GM = GM/w_0^2$. Consider now particles such as gas, dust, etc. in a highly chaotic and irreversible motion in a central Kepler potential; via the SE (2.11) there are solutions characterized by well defined and quantized values of conservative quantities such as energy etc. One therefore expects the particles to self-organize into "orbitals" and then to form planets etc. by accretion. Once so accreted one can recover classical elements such as eccentricity, semi-major axis, etc. We refer to [**716, 720, 814**] for more discussion and details of this and many other examples.

One refers later in the paper [**814**] to the dark matter problem and its connection to the formation of galaxies and large scale structure. Scale relativity allows one to suggest some solutions. Indeed the fractal geometry of a nondifferentiable space time solves the problem of formation on many scales and it also implies the appearance of a new scalar potential (as in (2.13) which manifests the fractality of space in the same way as Newton's potential manifests its curvature. It is suggested that this new potential (Q) may explain the anomalous dynamical effects without needing any missing mass. In this direction consider the case of the flat rotation curves of spiral galaxies. The formation of an isolated galaxy from a cosmological background of uniform density is obtaind in its first steps as the fundamental level solution $n = 0$ of the SE with an harmonic oscillator gravitational potential $(2\pi/3)G\rho r^2$ (for which some details are worked out in [**814**]). Once the galaxy is formed let r_0 be its outer radius beyond which the amount of visible matter becomes small. The potential energy at this point is given via $\phi_0 = -(GMm/r_0) = -mv_0^2$ where $M \sim$ mass of the galaxy. Observational data says that the velocity in the exterior region of the galaxy keeps the constant value v_0 and from the virial theorem the potential energy is proportional to the kinetic energy so that it also keeps the constant value $\phi_0 = -GMm/r_0$. Therefore r_0

is the distance at which the rotation curve begins to be flat and v_0 is the corresponding constant velocity. In the standard approach this flat rotation curve is in contradiction with the visible matter alone from which one would expect to observe a variable Keplerian potential energy $\phi = -GMm/r$. This means that one observes an additional potential energy $Q_{obs} = -(GMm/r_0)[1-(r_0/r)]$. Now the regions exterior to the galaxy are described in the scale relativity approach by a SE with a Kepler potential enery $\phi = -GMm/r$ where M is still the sole visible mass, since we assume here no dark matter. The radial solution for the fundamental level is $\sqrt{P} = 2exp(-r/r_B)$ where $r_B = GM/w_0^2$ is the macroscopic Bohr radius of the galaxy. Now one computes the theoretically predicted new potential Q from (2.13) (using $\mathcal{D} = GM/2w_0$) to get

$$Q_{pred} = -2m\mathcal{D}^2\frac{\Delta\sqrt{P}}{\sqrt{P}} = -\frac{GMm}{2r_B}\left(1-\frac{2r_B}{r}\right) = -\frac{1}{2}w_0^2\left(1-\frac{2r_B}{r}\right) \tag{2.16}$$

Thus one obtains, without any added hypotheses, the observed form Q_{obs} of the new potential. Moreover the visible radius and the Bohr radius are now related via $r_0 = 2r_B$. The constant velocity v_0 of the flat rotation curve is also linked to the fundamental gravitational constant w_0 via $w_0 = \sqrt{2}v_0$. Observational data supporting all this is also given.

3. WDW EQUATION

We go now to the famous Wheeler-deWitt equation (which might also be thought of as an Einstein-Schrödinger equation). Some background about this is given in Appendix C along with an introduction to the Ashtekar variables (following the beautiful exposition of [**69**]). The approach here follows [**870, 876**] which provides a Bohmian interpretation of quantum gravity and we cite also [**55, 56, 57, 58, 62, 70, 119, 303, 394, 476, 545, 551, 556, 630, 665, 819, 820, 896, 897, 929, 930, 931, 932**] for material on WDW and quantum gravity. Extracting liberally (and optimistically) now from [**876**] (first paper) one writes the Lagrangian density for general relativity (GR) in the form ($16\pi G = 1$)

$$\mathfrak{L} = \sqrt{-g}\mathfrak{R} = \sqrt{q}N({}^3\mathfrak{R} + Tr(K^2)) \tag{3.1}$$

where ${}^3\mathfrak{R}$ is the 3-dimensional Ricci scalar, K_{ij} is the extrinsic curvature, and q_{ij} is the induced spatial metric. The canonical momentum of the 3-metric is given by

$$p^{ij} = \frac{\partial\mathfrak{L}}{\partial\dot{q}_{ij}} = \sqrt{q}(K^{ij} - q^{ij}Tr(K)) \tag{3.2}$$

The classial Hamiltonian is

$$H = \int d^3x\mathfrak{H};\ \mathfrak{H} = \sqrt{q}(NC + N^iC_i) \tag{3.3}$$

where the lapse and shift functions, N and N_i, are given via (cf. [**69**])

$$C = {}^3\mathfrak{R} + \frac{1}{q}\left(Tr(p^2) - \frac{1}{2}(Tr(p))^2\right) = -2G_{\mu\nu}n^\mu n^\nu; \tag{3.4}$$

$$C_i = -2^3\nabla^j\left(\frac{p_{ij}}{\sqrt{q}}\right) = -2G_{\mu i}n^\mu$$

Here n^μ is the normal vector to the spatial hypersurfaces given by $n^\mu = (1/N, -\vec{N}/N)$. Now in the Bohmian approach one must add the quantum potential to the Hamiltonian to get the correct equations of motion so $H \to H + Q$ via $\mathfrak{H} \to \mathfrak{Q}$ where

(3.5)
$$Q = \int d^3x\mathfrak{Q};\ \mathfrak{Q} = \hbar^2 NqG_{ijk\ell}\frac{1}{|\psi|}\frac{\delta^2|\psi|}{\delta q_{ij}\delta_{k\ell}}$$

Here $G_{ijk\ell}$ is the superspace metric and ψ is the wavefunction satisfying the WDW equation. This means that we must modify the classical constraints via

(3.6)
$$C \to C + \frac{\mathfrak{Q}}{\sqrt{q}N};\ C_i \to C_i$$

Now for the constraint algebra one uses the integrated forms of the constraints defined as

(3.7)
$$C(N) = \int d^3x\sqrt{q}NC;\ \tilde{C}(\vec{N})\int d^3x\sqrt{q}N^iC_i$$

Then (cf. [**69, 870**] and Appendix C for notation)

(3.8)
$$\{\tilde{C}(\vec{N}), \tilde{C}(\vec{N}')\} = \tilde{C}(\vec{N}\cdot\nabla\vec{N}' - \vec{N}'\cdot\nabla\vec{N}) \equiv \tilde{C}(N^i\vec{\nabla}N_i' - N'^i\vec{\nabla}N_i);$$
$$\{\tilde{C}(\vec{N}), C(N)\} = C(\vec{N}\cdot\vec{\nabla}N);\ \{C(N), C(N')\} \sim 0$$

The first 3-diffeomorphism subalgebra does not change with respect to the classical situation and the second, representing the fact that the Hamiltonian constraint is a scalar under the 3-diffeomorphism, is also the same as in the classical case. In the third the quantum potential changes the Hamiltonian constraint algebra dramatically giving a result weakly equal to zero (i.e. zero when the equations of motion are satisfied). Following [**876**] we will give a number of calculations now regarding this Hamiltonian constraint. Thus first write the Poisson bracket explicitly as

(3.9)
$$\{C(N), C(N')\} = \int d^3z\sqrt{q(z)}\left(\frac{\delta C(N)}{\delta q_{ij}(z)}\frac{\delta C(N')}{\delta p^{ij}(z)} - \frac{\delta C(N)}{\delta p^{ij}(z)}\frac{\delta C(N')}{\delta q_{ij}(z)}\right) =$$
$$= \tilde{C}(N\vec{\nabla}N' - N'\vec{\nabla}N) + 2\int d^3zd^3x\sqrt{q(z)}G_{ijk\ell}(z)p^{k\ell}(z)\times$$
$$\times(-N(z)N'(x) + N(x)N'(z))\frac{\delta(Q/\sqrt{q}N))}{\delta q_{ij}(z)}$$

To simplify one differentiates the Bohmian HJ equation to get (cf. [**870**])

(3.10)
$$\frac{1}{N}\frac{\delta}{\delta q_{ij}}\frac{Q}{\sqrt{q}} = \frac{3}{4\sqrt{q}}q_{k\ell}p^{ij}p^{k\ell}\delta(x-z) - \frac{\sqrt{q}}{2}q^{ij}({}^3\mathfrak{R} - 2\Lambda)\delta(x-z) - \sqrt{q}\frac{\delta^3\mathfrak{R}}{\delta q_{ij}}$$

and use this in evaluation of the Poisson bracket giving the result indicated in (3.8). This calculation is given in the Appendix to [**876**] and we repeat it here for clarity.

REMARK 4.3.1. We follow here the Appendix to [**876**] and to evaluate the integral (3.9), in view of (3.10), one needs to consider $\int d^3zF(q,p,N,N')(\delta\,{}^3\mathfrak{R}/\delta q_{ij})$.

First look at the variation of the Ricci scalar with respect to the metric. Using the Palatini identity and dropping the superscript 3 one has

$$\delta\mathfrak{R}_{ij} = (1/2)q^{k\ell}(\nabla_j\nabla_i\delta q_{k\ell} - \nabla_k\nabla_j\delta q_{\ell i} - \nabla_k\nabla_i\delta q_{\ell j} + \nabla_k\nabla_\ell q_{ij}) \tag{3.11}$$

Consequently

$$\frac{\delta\mathfrak{R}_{ij}(x)}{\delta q_{ab}(z)} = \frac{1}{2}\sqrt{q}q^{k\ell}\left(\delta^a_k\delta^b_\ell\nabla_j\nabla_i\frac{\delta(x-z)}{\sqrt{q}} - \delta^a_\ell\delta^b_i\nabla_k\nabla_j\frac{\delta(x-z)}{\sqrt{q}} - \right.$$
$$\left. -\delta^a_\ell\delta^b_j\nabla_k\nabla_i\frac{\delta(x-z)}{\sqrt{q}} + \delta^a_i\delta^b_j\nabla_k\nabla_\ell\frac{\delta(x-z)}{\sqrt{q}}\right) \tag{3.12}$$

and therefore

$$\frac{\delta\mathfrak{R}(x)}{\delta q_{ab}} = \frac{\delta(q^{ij}\mathfrak{R}_{ij})(x)}{\delta q_{ab}} = -\mathfrak{R}^{ab}\delta(x-z) + \sqrt{q}(q^{ab}\nabla^2 - \nabla^a\nabla^b)\frac{\delta(x-z)}{\sqrt{q}} \tag{3.13}$$

Using this identity in the equation of motion (3.10) the only nonvanishing terms in (3.9) are
(3.14)

$$\{C(N), C(N')\} \sim \tilde{C}(N\vec{\nabla}N' - N'\vec{\nabla}N) - 2\int d^3z d^3x\sqrt{q(x)q(z)}G_{ijk\ell}(z)p^{k\ell}(z)$$
$$\times(N(x)N'(z) - N(z)N'(x))\left(q^{ij}(x)\nabla^2_x\frac{\delta(x-z)}{\sqrt{q}} - \nabla^i_x\nabla^j_x\frac{\delta(x-z)}{\sqrt{q}}\right)$$

where $\sim$ means the equality is weak (i.e. modulo the equation of motion). Integrating by parts gives

$$\{C(N), C(N')\} \sim 2\int d^3x(\nabla_j(N\nabla_i N') - \nabla_j(N'\nabla_i N))p^{ij} +$$
$$+\int d^3x\sqrt{q}G_{ijk\ell}p^{k\ell}(N'\nabla^i\nabla^j N - N\nabla^i\nabla^j N') -$$
$$-\int d^3x\sqrt{q}G_{ijk\ell}p^{k\ell}q^{ij}(N'\nabla^2 N - N\nabla^2 N') \tag{3.15}$$

Hence there results

$$\{C(N), C(N')\} \sim 2\int d^3x(N\nabla_j\nabla_i N' - N'\nabla_j\nabla_i N)p^{ij} +$$
$$+\int d^3x p^{k\ell}(N'\nabla_k\nabla_\ell N - N\nabla_k\nabla_\ell N' + N'\nabla_\ell\nabla_k N - N\nabla_\ell\nabla_k N' -$$
$$-q_{k\ell}(N'\nabla^2 N - N\nabla^2 N')) + \int d^3x q_{k\ell}p^{k\ell}(N'\nabla^2 N - N\nabla^2 N') = 0 \tag{3.16}$$

as desired.

One sees that the presence of the quantum potential means that the quantum algebra is the 3-diffeomorphism algebra times an Abelian subalgebra and the only difference with [**631**] is that this algebra is weakly closed. One sees that the algebra (3.8) is a clear projection of the general coordinate transformations to the spatial and temporal diffeomorphisms and in fact the equations of motion are invariant under such transformations (cf. also [**769**]). In particular although the form of the quantum potential will depend on regularization and ordering, in the quantum constraint algebra the form of the quantum potential is not important; the algebra holds independently of the form of the quantum potential. Further it appears that the inclusion of matter terms will not change anything.

One goes next to the quantum Einstein equations (QEI). For the dynamical part consider the Hamiltonian equations

$$\dot{q}^{ij} = \{H, q^{ij}\};\ \dot{p}_{ij} = \{H, p_{ij}\} \tag{3.17}$$

which produce the quantum equations (note the square bracket [ ] means that one is to antisymmetrize over all permutations of the enclosed indices, multiplying each term in the sum by the sign (±1) of the permutation)

$$\dot{q}_{ij} = \frac{2}{\sqrt{q}} N \left(p_{ij} - \frac{1}{2} p^k_k q_{ij} \right) + 2\, {}^3\nabla_{[i} N_{j]} \tag{3.18}$$

$$\dot{p}^{ij} = -N\sqrt{q}\left({}^3\mathfrak{R}^{ij} - \frac{1}{2}\,{}^3\mathfrak{R} q^{ij}\right) + \frac{N}{2\sqrt{q}} q^{ij}\left(p^{ab}p_{ab} - \frac{1}{2}(p^a_a)^2\right) - \\ -\frac{2N}{\sqrt{q}}\left(p^{ia}p^j_a - \frac{1}{2}p^a_a p^{ij}\right) + \sqrt{q}(\nabla^i\nabla^j N - q^{ij}\,{}^3\nabla^a\,{}^3\nabla_a N) + \\ +\sqrt{q}\,{}^3\nabla_a\left(\frac{N^a}{\sqrt{q}}p^{ij}\right) - 2p^{a[i}\,{}^3\nabla_a N^{j]} - \sqrt{q}\frac{\delta Q}{\delta q_{ij}} \tag{3.19}$$

Combining these two equations one obtains after some calculation

$$\mathfrak{G}^{ij} = -\frac{1}{N}\frac{\delta \mathfrak{Q}}{\delta q_{ij}} \tag{3.20}$$

which means that the quantum force modifies the dynamical part of the Einstein equations. For the nondynamical parts one uses the constraint relations (3.4) to get

$$\mathfrak{G}^{00} = \frac{\mathfrak{Q}}{2N^3\sqrt{q}};\ \mathfrak{G}^{0i} = -\frac{\mathfrak{Q}}{2N^3\sqrt{q}} N^i \tag{3.21}$$

These last two equations can be written via

$$\mathfrak{G}^{0\mu} = \frac{\mathfrak{Q}}{2\sqrt{-g}} g^{0\mu} \tag{3.22}$$

and the nondynamical parts are also modified by the quantum potential.

One addresses next the possibility that for a reparametrization invariant theory the equations obtained by the Hamiltonian may differ from those given by the phase of the wavefunction and the guidance formula (in a Bohmian spirit). However it is seen that there is no difference. Indeed write the Bohmian HJ equation (cf. [**123, 477, 557**]) by decomposing the phase part of the WDW equation; this gives

$$G_{ijk\ell}\frac{\delta S}{\delta q_{ij}}\frac{\delta S}{\delta q_{k\ell}} - \sqrt{q}\,({}^3\mathfrak{R} - \mathfrak{Q}) = 0 \tag{3.23}$$

where S is the phase of the WDW wave function. In order to get the equation of motion one differentiates the HJ equation with respect to q_{ab} and uses the guidance formula $p^{k\ell} \equiv \sqrt{q}(K^{k\ell} - q^{k\ell}K) = \delta S/\delta q_{k\ell}$. After considerable calculation one arrives again at (3.20). Thus the evolution generated by the Hamiltonian is compatible with the guidance formula, i.e. the Poisson brackets of the Hamiltonian and the guidance relation ($\chi^{k\ell} = p^{k\ell} - \delta S/\delta q_{k\ell}$) are zero. This can be evaluated explicitly and equals zero weakly so consistency prevails.

Next one shows explicitly that these modified Einstein equations (MEI) are covariant under spatial and temporal diffeomorphisms. Consider first $t \to t' = f(t)$ with $\vec{x}$ unchanged; one has

$$q'_{ij} = q_{ij};\ N'_i = (df/dt)N_i;\ N' = (df/dt)N \tag{3.24}$$

Putting these in the MEI one sees that the right side transforms as a second rank tensor under time reparametrization. Similarly consider $\vec{x} \to \vec{x}' = \vec{g}(\vec{x})$ with t unchanged; one has

$$q'_{ij} = \frac{\partial x^\ell}{\partial x'^i}\frac{\partial x^m}{\partial x'^j}q_{\ell m};\ N'_i = \frac{\partial x^\ell}{\partial x'^i}N_\ell;\ N' = N \tag{3.25}$$

Again the right side of MEI is a second rank tensor under a spatial 3-diffeomorphism. Inclusion of matter field is straightforward via

$$\mathfrak{G}^{ij} = -\kappa\mathfrak{T}^{ij} - \frac{1}{N}\frac{\delta(\mathfrak{Q}_G + \mathfrak{Q}_m)}{\delta g_{ij}};\ \mathfrak{G}^{0\mu} = -\kappa\mathfrak{T}^{0\mu} + \frac{\mathfrak{Q}_G + \mathfrak{Q}_m}{2\sqrt{-g}}g^{0\mu} \tag{3.26}$$

where

$$\mathfrak{Q}_m = \hbar^2\frac{N\sqrt{q}}{2}\frac{1}{|\psi|}\frac{\delta^2|\psi|}{\delta\phi^2} \tag{3.27}$$

where ϕ is the matter field and, as before,

$$\mathfrak{Q}_G = \hbar^2 NqG_{ijk\ell}\frac{1}{|\psi|}\frac{\delta^2|\psi|}{\delta q_{ij}\delta q_{k\ell}} \tag{3.28}$$

$$Q_G = \int d^3x\mathfrak{Q}_G;\ Q_m = \int d^3x\mathfrak{Q}_m \tag{3.29}$$

Equations (3.26) are the Bohm-Einstein equations which are in fact the quantum version of the Einstein equations; regularization only affects the quantum potential but the QEI are the same. They are invariant under temporal and spatial diffeomorphisms and can be written as

$$\mathfrak{G}^{\mu\nu} = -\kappa\mathfrak{T}^{\mu\nu} + \mathfrak{S}^{\mu\nu} \tag{3.30}$$

$$\mathfrak{S}^{0\mu} = \frac{\mathfrak{Q}_G + \mathfrak{Q}_m}{2\sqrt{-g}}g^{0\mu} = \frac{\mathfrak{Q}}{2\sqrt{-g}}g^{0\mu};\ \mathfrak{S}^{ij} = -\frac{1}{N}\frac{\delta(\mathfrak{Q}_G + \mathfrak{Q}_m)}{\delta g_{ij}} = -\frac{1}{N}\frac{\delta\mathfrak{Q}}{\delta g_{ij}} \tag{3.31}$$

($\mathfrak{S}^{\mu,\nu}$ is the quantum correction tensor - under the temporal $\otimes$ spatial diffeomorphism subgroup which is peculiar to the ADM decomposition). Note that the QEI were derived for a Robertson-Walker metric in [**961**] but without symmetry considerations or more general metrics. One concludes with the conservation law via taking the divergence of (3.30) to get

$$\nabla_\mu\mathfrak{T}^{\mu\nu} = \frac{1}{\kappa}\nabla_\mu\mathfrak{S}^{\mu\nu} \tag{3.32}$$

REMARK 4.3.2. We refer to [**724**] for a discussion of the Lichnerowicz-York equation as a solution to the constraint equations of GR.

3.1. CONSTRAINTS IN ASHTEKAR VARIABLES. We go now to the third paper in [**876**] where the new variables (or Ashtekar variables) are employed and it is shown that the Poisson bracket of the Hamiltonian with itself changes with respect to its classical counterpart but is still weakly equal to zero (as above in Section 4.3). Caution is advised however since ill defined terms have not been regularized; for this one needs a background metric and the result must be independent of such a metric. Thus the dynamical variables are the self dual connection A_a^i and athe canonical momenta are $\tilde{E}_i^a$ with constraints given via

$$G_i = \mathfrak{D}\tilde{E}_i^a;\ C_b = \tilde{E}_i^a F_{ab}^i;\ H = \epsilon_k^{ij}\tilde{E}_i^a\tilde{E}_j^b\mathfrak{F}_{ab}^k \tag{3.33}$$

where $\mathfrak{D}_a$ represents the self-dual covariant derivative and F_{ab}^k is the self-dual curvature (we refer to Appendix for the new variables). Canonical quantization of these constraints can be achieved via changing $\tilde{E}_i^a \to -\hbar(\delta/\delta A_a^i)$ to get

$$\hbar\mathfrak{D}_a\frac{\delta\psi(A)}{\delta A_a^i} = 0;\ \hbar F_{ab}^i\frac{\delta\psi(A)}{\delta A_a^i};\ \hbar^2\epsilon_k^{ij}F_{ab}^k\frac{\delta^2\psi(A)}{\delta A_a^i\delta A_b^j} = 0 \tag{3.34}$$

In order to get the causal interpretation one puts a definition $\psi = Rexp(iS/\hbar)$ into these relations to obtain

$$(\mathbf{A})\ \ \mathfrak{D}_a\frac{\delta R(A)}{\delta A_a^i} = 0;\ \ (\mathbf{B})\ \ \mathfrak{D}_a\frac{\delta S(A)}{\delta A_a^i} = 0; \tag{3.35}$$

$$(\mathbf{C})\ \ F_{ab}^i\frac{\delta R(A)}{\delta A_a^i} = 0;\ \ (\mathbf{D})\ \ F_{ab}^i\frac{\delta S(A)}{\delta A_a^i} = 0;$$

$$(\mathbf{E})\ \ \epsilon_k^{ij}F_{ab}^k\frac{\delta}{\delta A_a^i}\left(R^2\frac{\delta S(A)}{\delta A_b^j}\right) = 0;\ \ (\mathbf{F})\quad -\epsilon_k^{ij}F_{ab}^k\frac{\delta S(A)}{\delta A_a^i}\frac{\delta S(A)}{\delta A_b^j} + Q = 0$$

where the quantum potential is defined as

$$Q = -\hbar^2\epsilon_k^{ij}F_{ab}^k\frac{1}{R}\frac{\delta^2 R(A)}{\delta A_a^i\delta A_b^j} \tag{3.36}$$

Here **(E)** is the continuity equation while **(F)** is the quantum Einstein-Hamilton-Jacobi (EHJ) equation. The quantum trajectories would be achieved via the guidance relation

$$\tilde{E}_i^a = i\frac{\delta S(A)}{\delta A_a^i} \tag{3.37}$$

Now for the constraint algebra; in terms of smeared out Gauss, vector, and scalar constraints (the notation $\underset{\sim}{N}$ is not clearly defined here - cf. (3.3))

$$\mathfrak{G}(\Lambda_i) = -i\int d^3x\Lambda_i\mathfrak{D}_a\tilde{E}_i^a;\ \mathfrak{H}(\underset{\sim}{N}) = \frac{1}{2}\int d^3x\underset{\sim}{N}\epsilon_k^{ij}\tilde{E}_i^a\tilde{E}_j^b\mathfrak{F}_{ab}^k; \tag{3.38}$$

$$C(\vec{N}) = i\int d^3xN^b\tilde{E}_i^aF_{ab}^i - \mathfrak{G}(N^aA_a^i)$$

the classical algebra is

$$\{\mathfrak{G}(\Lambda_i),\mathfrak{G}(\Theta_j)\} = \mathfrak{G}(\epsilon_{ijk}^i\Lambda^j\Theta^k);\ \{C(\vec{N}),C(\vec{M})\} = C(\mathfrak{L}_{\vec{M}}\vec{N}); \tag{3.39}$$

$$\{C(\vec{N}), \mathfrak{G}(\Lambda_i)\} = \mathfrak{G}(\mathfrak{L}_{\vec{N}}\Lambda_i);\ \{C(\vec{N}), \mathfrak{H}(M)\} = \mathfrak{H}(\mathfrak{L}_{\vec{N}}M)$$
$$\{\mathfrak{G}(\Lambda_i), \mathfrak{H}(N)\} = 0;\ \{\mathfrak{H}(N), \mathfrak{H}(M)\} = C(\vec{K}) + \mathfrak{G}(K^a A_a^i)$$

where $K^a = \tilde{E}_i^a \tilde{E}^{bi}(N\partial_b M - M\partial_b N)$. The quantum trajectories can now be obtained from the quantum Hamiltonian given by $H_Q = H + Q$ and the smeared out gauge and diffeomorphism constraints will not change; the Hamiltonian constraint becomes

$$\mathfrak{H}_Q(N) = \frac{1}{2}\int d^3x N\epsilon_k^{ij}\tilde{E}_i^a \tilde{E}_j^b \mathfrak{F}_{ab}^k + \mathfrak{Q}(N) \tag{3.40}$$

where $\mathfrak{Q}(N) = \int d^3x NQ$ (the notation Q and $\mathfrak{Q}$ is switched here from previous use above). The first three constraint Poisson brackets in (3.39) will not change and the fourth is still valid because the quantum potential is a scalar density and one has

$$\{C(\vec{N}), \mathfrak{H}_Q(M)\} = \mathfrak{H}_Q(\mathfrak{L}_{\vec{N}}M) \tag{3.41}$$

This applies also to the fifth bracket because

$$\{\mathfrak{G}(\Lambda_i), \mathfrak{H}_Q(N)\} = 0 \tag{3.42}$$

but the last bracket changes via

$$\{\mathfrak{H}_Q(N), \mathfrak{H}_Q(M)\} = \{\mathfrak{H}(N), \mathfrak{H}(M)\} + \{\mathfrak{Q}(N), \mathfrak{H}(M)\} + \tag{3.43}$$
$$+\{\mathfrak{H}(N), \mathfrak{Q}(M)\} + \{\mathfrak{Q}(N), \mathfrak{Q}(M)\}$$

Here the last term is identically zero, since the quantum potential is a functional of the connection only. The sum of the second and third terms is

$$\{\mathfrak{Q}(N), \mathfrak{H}(M)\} + \{\mathfrak{H}(N), \mathfrak{Q}(M)\} \sim \tag{3.44}$$
$$\sim -\int d^3x \left(N\epsilon_k^{ij}F_{ab}^k \tilde{E}_j^b \mathfrak{D}_c(M\epsilon_i^{\ell m}\tilde{E}_\ell^a \tilde{E}_m^c) - M\epsilon_k^{ij}F_{ab}^k \tilde{E}_j^b \mathfrak{D}_c(N\epsilon_i^{\ell m}\tilde{E}_\ell^a \tilde{E}_m^c)\right)$$

A calculation then shows that the Poisson bracket of the quantum Hamiltonian with itself is given via

$$\{\mathfrak{H}_Q(N), \mathfrak{H}_Q(M)\} \sim 0 \tag{3.45}$$

which is similar to the situation with the old variables (cf. Remark 4.3.1).

Now in order to obtain the quantum equations of motion via the Hamilton equations one has

$$\dot{A}_a^i = -i\epsilon^{ijk}N\tilde{E}_j^b F_{abk} - N^b F_{ab}^i;\ \dot{\tilde{E}} = i\epsilon_i^{jk}\mathfrak{D}_b(N\tilde{E}_j^a \tilde{E}_k^b); \tag{3.46}$$
$$-2\mathfrak{D}_b(N^{[a}\tilde{E}_i^{b]}) + \frac{i}{2}\int d^3x \frac{\delta\mathfrak{Q}(N)}{\delta A_a^i(z)}$$

Further to recover the real quantum general relativity one must set the reality conditions, which are

$$\tilde{E}_i^a \tilde{E}^{bi}\ \textit{must be real}; \tag{3.47}$$
$$i\epsilon^{ijk}\tilde{E}_i^{(a}\mathfrak{D}_a(\tilde{E}_k^{b)}\tilde{E}_j^c) + \frac{i}{2}\int d^3x \frac{\delta\mathfrak{Q}}{\delta A_{(a}^i(z)}\tilde{E}^{b)}(x)\ \textit{must be real}$$

(note round brackets ( ) mean symmetrization with respect to the indices concerned). Thus formally one can construct a causal version of canonical quantum gravity using the Bohm-deBroglie interpretation of QM. All of the quantum behavior is encoded in the quantum potential. One has a well defined trajectory and no operators arise; the algebra action is in fact the Poisson bracket and only the Poisson bracket of the Hamiltonian with itself will change relative to the classical algebra by being weakly instead of strongly equal to zero. The result is similar to that obtained above with the old variables and one can give meaning to the idea of time generator for the Hamiltonian constraint. The equations of motion when finally written out should contain the quantum force. Regularization of ill defined terms remains and is promised in forthcoming papers of F. and A. Shojai.

The approach here in Sections 4.3 and 4.3.1 is so clean and beautiful that we suspend any attempt at criticism. Eventually one will have to reconcile this with results of [**770, 772**] for example (cf. Section 4.5). We remark also that in [**880**] one makes a preliminary study of Bohmian ideas in loop quantum gravity using Ashtekar variables.

4. REMARKS ON REGULARIZATION

In [**961**] (second paper) for example one considers the classical and WDW description of a gravity-minisuperspace model (cf. also [**573**]). Thus consider a homogeneous and isotropic metric defined via

$$ds^2 = -N(t)^2 dt^2 + a(t)^2 d\Omega_3^2 \tag{4.1}$$

where $d\Omega_3^2$ is a standard metric on 3-space. The lapse function N and the scale factor a depend on a time parameter t. A minisuperspace model represented by a single homogeneous mode ϕ is defined by the Lagrangian

$$L = -a^3\left[\frac{1}{2N}\left(\frac{\dot a}{a}\right)^2 + NV_G(a)\right] + a^3\left[\frac{\phi^2}{2N} - NV_M(\phi)\right] \tag{4.2}$$

One uses the Planck mass $m_P^2 = 3/4\pi G$ to scale all dimensional quantities so $a \equiv am_P$, $\phi \equiv \phi/m_P$, etc.; V_M is the potential for the scalar mode ϕ and the gravitational potential $V_G(a) = -(1/2)Ka^{-2} + (1/6)\Lambda$ may contain a cosmological constant Λ and a curvature constant $K = 1$, 0, or -1 for a spherical, planar, or hyperspherical 3-space. From the Lagragian one derives now the classical equations of motion by varying N, a, and ϕ respectively to obtain

$$\frac{1}{2}\left(\frac{\dot a}{a}\right)^2 - V_G(a) = \frac{1}{2}\dot\phi^2 + V_M(\phi); \tag{4.3}$$

$$\frac{1}{2}\left(\frac{\dot a}{a}\right)^2 + \frac{\ddot a}{a} - 3V_g(a) - a\partial_a V_G(a) + 3\left(\frac{1}{2}\dot\phi^2 - V_M(\phi)\right) = 0;$$

$$\ddot\phi + 3\left(\frac{\dot a}{a}\right)\dot\phi + \partial_\phi V_M(\phi) = 0$$

Here one has chosen the gauge $N = 1$ and t is identified with the classical time. However the time parameter is not directly observble, only the correlations $a(\phi)$ or $\phi(a)$ which follow from the solutions of (4.3). One could imagine an additional degree of freedom $\tau(t)$ to be used as a clock but this need not be done. One derives the WDW equation from (4.2) in the standard manner. After computing the classical Hamiltonian and replacing the canonical momenta $p_a = -a\dot{a}$ and $p_\phi = a^3\dot{\phi}$ by operators $p_\phi \to -i\partial_\phi$ and $p_a \to -i\partial_a$ the WDW Hamiltonian is

$$H_{WDW} = \left[\frac{1}{2}a^{-3}(a\partial_a)^2 + a^3V_G(a)\right] + \left[\frac{1}{2}a^{-3}\partial_\phi^2 + a^3V_M(\phi)\right] = H_G + H_M \tag{4.4}$$

The WDW equations is $H_{WDW}\psi = 0$ and there is an operator ordering ambiguity; the "Lagrangian" ordering has been chosen which makes H_{WDW} formaly selfadjoint in the inner product $(\psi,\psi) = \int a^2 dad\phi\psi^*(a,\phi)\phi(a,\phi)$. It is not clear that H_{WDW} can be used as a generator for time evolution since time has disappeared altogether in (4.4).

Now in the dBB treatment one has

$$\dot{S} + \frac{1}{2m}(\partial_x S)^2 + V + Q = 0;\ \dot{R}^2 + \partial_x(R^2\partial_x S) = 0 \tag{4.5}$$

where $Q = -\partial_x^2 R/2mR$. The trajectories are found by solving the autonomous system $\dot{x} = (1/m)\partial_x S$ and the measure R^2dx gives the probability for trajectories crossing the interval $(x, x+dx)$. Now treat the WDW equation as a SE which happens to be time independent and write it in the quantum potential form with $\psi = R(a,\phi)exp(iS(a,\phi))$ leading to

$$\frac{1}{2}a^{-3}[-(a\partial_a S)^2 + (\partial_\phi S)^2] + a^3[V_G + V_M + Q_{GM}] = 0; \tag{4.6}$$

$$-a\partial_a(R^2 a\partial_a S) + \partial_\phi(R^2\partial_\phi) = 0$$

These equations come from the real and imaginary part of $H_{WDW}\psi = 0$ and the quantum potential is

$$Q_{GM} = -\frac{1}{2}a^{-6}\left[-\frac{-(a\partial_a)^2 R}{R} + \frac{\partial_\phi^2 R}{R}\right] \tag{4.7}$$

Trajectories $(a(t),\phi(t))$ ae obtained from $S(a,\phi)$ by identifying $\partial_a S$ with the momentum p_a and $\partial_\phi S$ with p_ϕ; thus one uses the definition of the canonical momenta given before (4.4) to define trajectories parametrized by a time parameter t, via

$$\dot{a} = -a^{-1}\partial_a S(a,\phi);\ \dot{\phi} = a^{-3/2}\partial_\phi S(a,\phi) \tag{4.8}$$

One notes that the probability measure $R^2(a,\phi)a^2 dad\phi$ is conserved in time t if (a,ϕ) are solutions of (4.7). This is a consequence of using the measure $a^2 dad\phi$ together with the Lagrangian factor ordering making H_{WDW} formally self adjoint. In using t as in (4.8) one should eliminate t after solving in order to determine $a(\phi)$ or $\phi(a)$; for trajectories where e.g $a(t)$ is 1-1 the scale factor can be used as a clock. The analogues of equations (4.3) are obtained by differentiating the first equation in (4.6) with respect to a and ϕ and then eliminating S, using

$$-\partial_t(a\dot{a}) = \partial_a^2 S\dot{a} + \partial_\phi\partial_a S\dot{\phi};\ \partial_t(a^3\dot{\phi}) = \partial_a\partial_\phi\dot{a} + \partial_\phi^2 S\dot{\phi} \tag{4.9}$$

leading to

$$\frac{1}{2}a\dot{a}^2 - V_G(a) - a^3\left[\frac{1}{2}\dot{\phi}^2 + V_M(\phi)\right] = Q_{GM}(a,\phi); \tag{4.10}$$

$$\frac{1}{2}\dot{a}^2 + a\ddot{a} - \partial_a V_G(a) + a^2\left[\frac{1}{2}\dot{\phi}^2 - V_M(\phi)\right] = \partial_a Q_{GM}(a,\phi);$$

$$\ddot{\phi} + 3\frac{\dot{a}}{a} + \partial_\phi V_M(\phi) = -a^{-3/2}\partial_\phi Q_{GM}(a,\phi)$$

This evidently generalizes (4.3). There is more in this paper which we omit here, namely a semiclassical development is given as well as and some exact solutions of WDW for toy models.

In [**123**] one looks at an equation

$$H\psi(x) = \left(\frac{1}{2}g^{ij}(x)\nabla_i\nabla_j - V(x)\right)\psi(x) = \left(\frac{1}{2}\Box - V(x)\right)\psi(x) = 0 \tag{4.11}$$

Assume $\psi = R(x)exp(iS(x)/\hbar)$ with R, S real leading to

$$H[S(x)] = \frac{1}{2}g_{ij}\frac{\partial S}{\partial x_i}\frac{\partial S}{\partial x^j} + V(x) = \frac{\hbar^2}{2R}\Box R;\ R\Box S + 2g^{ij}\frac{\partial S}{\partial x^i}\frac{\partial S}{\partial x^j} = 0 \tag{4.12}$$

Introduce time via

$$\frac{dx^i}{dt} = g^{ij}\frac{\delta H[S(x)]}{\delta(\partial S/\partial x^j)} \tag{4.13}$$

This defines the trajectory $x^i(t)$ in terms of the phase of the wave function S. Put this now back into (1.2) to obtain

$$\frac{1}{2}g_{ij}\dot{x}^i\dot{x}^j + V(x) + Q = 0;\ Q = -\frac{\hbar^2}{2R}\Box R \tag{4.14}$$

Now define classical momenta

$$p_i = \frac{\delta H[S(x)]}{\delta(\partial S/\partial x^i)} = g_{ij}\dot{x}^j \tag{4.15}$$

and write (4.14) in the form

$$H = \frac{1}{2}g^{ij}p_ip_j + V(x) + Q = 0;\ \dot{p}_i = -\frac{\partial H}{\partial x^i};\ \dot{x}^i = \frac{\partial H}{\partial p_i} \tag{4.16}$$

This gives a method of identifying the time evolution corresponding to the wave function of a WDW type equation (e.g. the wave function of the universe). The following points are mentioned in [**123**]:

- (4.16) is equivalent to the classical equations of motion except for the quantum potential which therefore. in some sense, clarifies the effect of quantum matter on gravity.
- A semiclassical theory can be constructed from the above.

- The quantum potential provides a natural way to introduce time even when the Hamiltonian constraint (which doesn't involve time) is acting. In this connection note that the definition of time is not unique. One could use a more general expression

$$\dot{x}^j = N(x)\frac{\delta H[S(x)]}{\delta(\partial S/\partial x^j)} \tag{4.17}$$

where N could be identified with the lapse function for the ADM formulation.

In [**123**] one also studies situations of the form

$$ds^2 = -N^2 dt^2 + \sum_1^3 g_{ii}(\omega^i)^2 \tag{4.18}$$

where N is a lapse function and ω^i are appropriate 1-forms (some Bianchi types are classified) and we refer to [**123**] for details and [**84**] for criticism.

In [**124**] one starts with the classical constraints of Einstein's gravity, namely the diffeomorphism and Hamiltonian constraint in the form

$$\mathfrak{D}_a = \nabla_a \pi^{ab};\ \mathfrak{H} = \kappa^2 G_{abcd}\pi^{ab}\pi^{cd} - \frac{1}{\kappa^2}\sqrt{h}(R + 2\Lambda) \tag{4.19}$$

Here π^{ab} are momenta associated with the 3-metric h_{ab} where

$$G_{abcd} = \frac{1}{2\sqrt{h}}(h_{ac}h_{bd} + h_{ad}h_{bc} - h_{ab}h_{cd} \tag{4.20}$$

is the WDW metric where R is the 3-dimensional scalar curvature, κ is the gravitational constant, and Λ the cosmological constant. The constraints satisfy the algebra

$$[\mathfrak{D},\mathfrak{D}] \sim \mathfrak{D};\ [\mathfrak{D},\mathfrak{H}] \sim \mathfrak{H};\ [\mathfrak{H},\mathfrak{H}] \sim \mathfrak{D} \tag{4.21}$$

The rules of quantization given by the metric representation of the canonical commutation relations are

$$[\pi^{ab}(x), h_{cd}(y)] = -i\delta^a_c\delta^b_d\delta(x,y);\ \pi^{ab}(x) = -\frac{\delta}{\delta h_{ab}(x)} \tag{4.22}$$

There are problems here of all types (cf. [**124**]) which we will not discuss but in [**572**] a class of exact solutions of the WDW equation was found via heat kernel regularization of the Hamiltonian with a suitable ordering and the question addressed here is the level of arbitrariness in this construction. One bases now such constructions on the principle that the algebra of constraints should be anomaly free, i.e. the algebra should be weakly identical with the classical one. One chooses a starting space of states to consist of integrals over compact 3-space of scalar densities like $\mathfrak{V} = \int_M \sqrt{h}$, $\mathfrak{R} = \int_M \sqrt{h}R$, etc. so $\psi = \psi(\mathfrak{V},\mathfrak{R},\cdots)$. For the diffeomorphism constraint one takes the representation $\mathfrak{D}_a(x) = -i\nabla^x_b(\delta/\delta h_{ab}(x))$ where ∇^x_b means the covariant derivative acting at the point x. This constraint then annihilates all the states and the first commutator relation (1.21) is satisfied. Further the second relation in (4.21) reduces to the formal relation

$$\mathfrak{D}(\mathfrak{H}\psi) \sim \mathfrak{H}\psi \tag{4.23}$$

Now for the construction of the WDW operator one makes a point split in the kinetic term of the form

$$G_{abcd}(x)\pi^{ab}(x)\pi^{cd}(x) \rightsquigarrow \int dx' K_{abcd}(x,x',t)\frac{\delta}{\delta h_{ab}(x)}\frac{\delta}{\delta h_{cd}(x')} \tag{4.24}$$

where $lim_{t\to 0^+}K_{abcd}(x,x',t) = \delta(x,x')$ and in particular one takes

$$K_{abcd}(x,x',t) = G_{abcd}(x')\Delta(x,x',t)(1+K(x,t)); \tag{4.25}$$

$$\Delta = \frac{exp(-(1/4t)h_{ab}(x-x')^a(x-x')^b)}{4\pi t^{3/2}}$$

with $K(x,t)$ analytic in t. Next to resolve the ordering ambiguity in $\mathfrak{H}$ one adds a new term $L_{ab}(x)(\delta/\delta h_{ab}(x))$ where L_{ab} is a tensor to be derived along with $K(x,t)$. Thus the WDW operator will have the form

$$\mathfrak{H}(x) = \kappa^2 \int dx' K_{abcd}(x,x',t)\frac{\delta}{\delta h_{ab}(x)}\frac{\delta}{\delta h_{cd}(x')} + \tag{4.26}$$

$$+L_{ab}(x)\frac{\delta}{\delta h_{ab}(x)} + \frac{1}{\kappa^2}\sqrt{h}(R+2\Lambda)$$

Next one needs to define the action of operators on states which involves discussions of regularization and renormalization. Regularization is a trade off here of + and - powers of t for $\delta(0)$ type singularities and for renormalization one drops positive powers of t and replaces singular terms $t^{-k/2}$ by renormalization coefficients ρ^k (cf. [**124**] for more details and references). Then to interpret (4.23) for example one thinks of an operator acting on a state and the resulting state after renormalization can be acted upon by another operator. Thus (4.23) means

$$\mathfrak{D}(\mathfrak{H}\psi)_{ren} \sim (\mathfrak{H}\psi)_{ren} \tag{4.27}$$

Similarly for the Hamiltonian constraint

$$(\mathfrak{H}[N](\mathfrak{H}(M)\psi)_{ren})_{ren} - (\mathfrak{H}[M](\mathfrak{H}[N]\psi)_{ren})_{ren} = 0 \tag{4.28}$$

(since ψ is diffeomorphism invariant); here one is using the smeared form of the WDW operatr $\mathfrak{H}[M] = \int dx M(x)\mathfrak{H}(x)$. After some calculation one also concludes that for the states which are integrals of scalar densities there is no anomaly in the diffeomorphism - Hamiltonian commutator. As for the Hamiltonian - Hamiltonian commutator one claims that if $(\mathfrak{H}\psi)_{ren}$ contains terms which contain 4 or more derivatives of the metric like R^2, $R_{ab}R^{ab}$, etc. then (4.28) cannot be satisfied. This was checked for some low order terms but is otherwise open. The form of the wave function must also be examined and this leads to further conditions on the WDW operator, which is finally advanced in the form

$$\mathfrak{H}(x) = \kappa^2 \int dx' G_{agcd}(x')\Delta(x,x',t)\frac{\delta}{\delta h_{ab}(x)}\frac{\delta}{\delta h_{cd}(x')} + \tag{4.29}$$

$$+\left(\frac{1}{\kappa}\alpha h_{ab} + \kappa\gamma\left(\frac{1}{4}h_{ab}R + R_{ab}\right)\right)(x)\frac{\delta}{\delta h_{ab}(x)} + \frac{1}{\kappa^2}\sqrt{h}(R+2\Lambda)$$

where $\alpha,\ \gamma$ are independent constants.

When one now works with a Bohmian approach where $\psi = exp(\Gamma)exp(i\Sigma)$ and one considers only the real part of the resulting equation, namely

$$(4.30)\quad -\kappa^2 G_{abcd}(x)\frac{\delta\Sigma}{\delta h_{ab}(x)}\frac{\delta\Sigma}{\delta h_{cd}(x)} + \frac{1}{\kappa}\sqrt{h(x)}(R(x)+2\Lambda) + \Re(L)_{ab}(x)\frac{\delta\Gamma}{\delta h_{ab}(x)} -$$

$$-\Im(L)_{ab}(x)\frac{\delta\Sigma}{\delta h_{ab}(x)} + e^{-\Gamma}\kappa^2\left(\frac{\delta^2 e^{\Gamma}}{\delta h^2}\right)_{ren}(x) = 0$$

Identifying now

$$(4.31)\qquad p^{ab}(x) = \frac{\delta\Sigma}{\delta h_{ab}(x)}$$

we see that the first two terms in (4.30) are the same as the Hamiltonian constraint of classical relativity. The remaining terms are quantum corrections (and if $\hbar$ were inserted they would all be proportional to $\hbar^2$). The wave function is subject to the second set of equations, namely those enforcing the 3-dimensional diffeomorphism invariance, which read (for the imaginary part)

$$(4.32)\qquad \nabla^a \frac{\delta\Sigma}{\delta h_{ab}(x)} = \nabla^a p_{ab} = 0$$

Thus the theory is defined by two equations (1.30) (with the p^{ab} inserted) and (4.32). This leads then to the full set of ten equations governing the quantum gravity in the quantum potential approach, namely

$$(4.33)\qquad 0 = \mathfrak{H}^a = \nabla_a p^{ab};$$

$$0 = \mathfrak{H}_\perp = -\kappa^2 G_{abcd}(x)p^{ab}p^{cd} + \frac{1}{\kappa^2}\sqrt{h(x)}(R(x)+2\Lambda) +$$

$$+\Re(L)_{ab}(x)\frac{\delta\Gamma}{\delta h_{ab}(x)} - \Im(L)_{ab}(x)p^{ab} + \kappa^2 e^{-\Gamma}\left(\frac{\delta^2 e^{\Gamma}}{\delta h^2}\right)_{ren}(x);$$

$$\dot{h}_{ab}(x,t) = \{h_{ab}(x,t), \mathfrak{H}[N,\vec{N}]\};\ \dot{p}^{ab}(x,t) = \{p^{ab}(x,t), \mathfrak{H}[N,\vec{N}]\}$$

where (cf. Section 4.3)

$$(4.34)\qquad \mathfrak{H}[N,\vec{N}] = \int d^3x(N(x)\mathfrak{H}_\perp(x) + N^a(x)\mathfrak{H}_a(x))$$

This all shows in particular that when questions of regularization and renormalization are taken into account life becomes more complicated. Various examples are treated in [**123, 124, 571**] where the quantum potential approach works very well but others where time translation becomes a problem.

5. PILOT WAVE COSMOLOGY

We refer here to [**74, 75, 77, 84, 85, 86, 81, 769, 770, 771, 772, 881**] (other references to be given as we go along). First from [**881**] a set of nonrelativistic spinless particles are described via spatial coordinates $x = (x_1, \cdots, x_n)$ and the wave function ψ satisfies the SE

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2}\sum\frac{1}{m_n}\Delta_n\psi + V\psi \tag{5.1}$$

Putting in $\psi = Rexp(iS/\hbar)$ gives then

$$\frac{\partial S}{\partial t} + \sum_n\frac{1}{2m_n}(\nabla_n S)^2 + V + Q = 0;\ Q = -\sum_n\frac{\hbar^2}{2m_n}\frac{\Delta_n R}{R}; \tag{5.2}$$

$$\frac{\partial R^2}{\partial t} + \sum_n\frac{1}{m_n}\nabla_n(R^2\nabla_n S) = 0$$

In the pilot wave interpretation the evolution of coordinates is governed by the phase via $m_n\dot{x}_n = \nabla_n S$. In the relativistic theory of spin 1/2 one continues to describe particles by their spatial coordinates but the guidance conditions and the equations for the wave are now different, namely for the multispinor $\psi_{\alpha_1\cdots\alpha_n}(x,t)$ the Dirac equation is $(\alpha \sim (\alpha_1\cdots\alpha_n))$

$$i\dot{\psi}_\alpha = \sum_n(H_D)\psi)_\alpha;\ H_D = -i\gamma^0\gamma^i\nabla_i + m\gamma^0 \tag{5.3}$$

The guidance condition is

$$\frac{dx_n^\mu}{dt} = \frac{\psi^\dagger(\gamma^0\gamma^\mu)_n\psi}{\psi^\dagger\psi} \tag{5.4}$$

Here the label n enumerates the arguments of the multispinor ψ and the γ_n^μ act on the corresponding spinor index α_n; ψ is chosen to be antisymmetric with respect to interchange of any pair of its arguments. For integer spin the formulation in which the role of configuration variables is played by coordinates seems to be impossible; instead one has to consider the field spatial configurations as fundamental configuration variables guided by the corresponding wave functionals. For example the wave functional $\chi[\phi(\mathbf{x}), t]$ for a scalar field ϕ will obey the standard SE for the case of curved spacetime with guidance equation as indicated (notational gaps are filled in below)

$$\dot{\phi}(\mathbf{x},t) = \frac{\delta}{\delta\phi(\mathbf{x})}S[\phi(\mathbf{x}),t]\Big|_{\phi(\mathbf{x})=\phi(\mathbf{x},t)} \tag{5.5}$$

(here $S[\phi(\mathbf{x},t]$ is $\hbar$ times the phase of $\chi[\phi(\mathbf{x}),t]$). The classical limit in the dynamics here is achieved for those configuration variables for which the quantum potential becomes negligible and such variables evolve in accord with classical laws (cf. [**471**]). One can emphasize that the temporal dynamics of the particle coordinates and bosonic field configurations completely determine the state, be it microscopic or macroscopic. The role of the wave function in all physical situations is also the same, namely to provide the guidance laws for configuration variables. In order that the probabilities of of different measurements coincide with those calculated in the standard approach it is necessary to assume that the configuration variables of the system in a pure quantume ensemble are distributed in accord with the rule $p(x) = |\psi(x)|^2$ (quantum equilibrium condition - cf. [**954, 327**], Section 2.5, and references to Dürr, Goldstein, Holland, Valentini, Zanghi, et al for discussion).

Now for quantum gravity recall the ADM formulation for bosonic fields

$$I = \int_M d^3\mathbf{x}dt(\pi^{ab}\dot{g}_{ab} + \pi_\Phi\dot{\Phi} - N\mathfrak{H} - N^a\mathfrak{H}_a) \tag{5.6}$$

(here Φ is the set of bosonic fields). The restraints of GR are

$$\mathfrak{H} = \frac{1}{2\mu}\mathfrak{G}_{abcd}\pi^{ab}\pi^{cd} + \mu\sqrt{g}(2\Lambda - {}^3\mathfrak{R}) + \mathfrak{H}^\Phi \approx 0;\ \mathfrak{H}_a \equiv -2\nabla_b\pi^b_a + \mathfrak{H}^\Phi_a \approx 0 \tag{5.7}$$

(only the gravitational parts of the constraints are explicitly written out). Here $\mu = (16\pi G)^{-1}$ with G the Newton constant and

$$\mathfrak{G}_{abcd} = \frac{1}{\sqrt{g}}(g_{ac}g_{bd} + g_{ad}g_{bc} - g_{ab}g_{cd}) \tag{5.8}$$

while $\nabla_a \sim$ covariant derivative relative to g_{ab} and ${}^3\mathfrak{R}$ is the scalar curvature for the metric g. The classical equations of motion for g are

$$\dot{g}_{ab} = \frac{N}{\mu}\mathfrak{G}_{abcd}\pi^{cd} + \nabla_a N_b + \nabla_b N_a \tag{5.9}$$

Recall that in the Schrödinger representation the GR quantum system will be described by the wave functional $\Psi[g_{ab}(\mathbf{x}, \Phi(\mathbf{x}), t]$ over a manifold Σ with coordinates $\mathbf{x}$ and the quantum constraint equations are $\hat{\mathfrak{H}}_\mu\Psi = 0$ ($\hat{\mathfrak{H}}$ refers to all the components mentioned above (operator ordering and regularization are not treated in [**881**]). Putting in now $\Psi = Rexp(iS/\hbar)$ one arrives at

$$\frac{1}{2\mu}\delta S\circ\delta S + \mu\sqrt{g}(2\Lambda - {}^3\mathfrak{R}) - \frac{\hbar^2}{2\mu}\frac{\delta\circ\delta R}{R} + \frac{\Re(\Psi^\dagger\hat{\mathfrak{H}}^\Psi)}{R^2} = 0; \tag{5.10}$$
$$\delta\circ(R^2\delta S) - \frac{2\mu}{\hbar}\Im(\Psi^\dagger\hat{\mathfrak{H}}^\Phi\Psi) = 0$$

(here $\delta \sim \delta/\delta g_{ab}(\mathbf{x})$ and $\circ$ means contraction with respect to Wheeler's supermetric (5.8)). Note that for $\hbar \to 0$ the first equation in (5.10) reduces to the classical Einstein-HJ equation. Via the general guidance rules the quantum evolution of the g_{ab} is now given by (5.9) with the substitution

$$\pi^{ab}(\mathbf{x}) \to \left.\frac{\delta S}{\delta g_{ab}(\mathbf{x})}\right|_{g_{ab}(\mathbf{x})=g_{ab}(\mathbf{x},t)} \tag{5.11}$$

The Lagrange multipliers N and N^a in (5.9) remain undetermined and are to be specified arbitrarily. This is analogous to the classical situation where this arbitrariness reflects reparameterization freedom. Thus to get a solution g_{ab} and Φ depending on $(\mathbf{x}, t)$ one must first solve the constraint equation $\hat{\mathfrak{H}}_\mu\Psi = 0$, then specify the initial configuration (e.g. at $t = 0$) for g_{ab} and Φ, then specify arbitrarily $N(\mathbf{x}, t)$ and $N^a(\mathbf{x}, t)$, and then solve the guidance equations (5.9) and the analogous equations for Φ. The solution should then represent a 4-geometry foliated by spatial hypersurfaces $\Sigma(t)$ on which the 3-metric is $g_{ab}(\mathbf{x}, t)$, the lapse function is $N(\mathbf{x},t)$, the shift vector is $N^a(\mathbf{x},t)$ and the field configuration is $\Phi(\mathbf{x},t)$. This would be lovely but unfortunately there are complications as indicated below from [**770, 772**]; such matters are partially anticipated in [**881**] however and there is some discussion and calculation. The question of quantum randomness in pilot wave QM is picked up again in the first paper of [**881**] along with a continuation of time considerations. We only remark here that time in (5.9) is just a universal label of succession for spatial field configurations; it is not an observable.

5.1. EUCLIDEAN QUANTUM GRAVITY. The discussion here revolves around [**770**] and the first paper of [**772**] (cf. also [**263, 264, 266, 218**]). We go to [**770**] directly and refer to [**772**] for some background calculations and philosophy. [**770**] is a review paper of deBroglie-Bohm theory in quantum cosmology. Extracting liberally one can argue convincingly against the Copenhagen interpretation of quantum phenomena in cosmology, in particular because it imposes the existence of a classical domain (cf. [**729**]). Decoherence is discussed but this does not seem to be a complete answer to the measurement problem (cf. [**551, 412, 770**]) and one can also argue against the many-worlds theory (cf. [**320, 471, 770**]). Thence one goes to deBroglie-Bohm as in [**84, 881, 954, 961**] etc. and the quantum potential enters in a natural manner as we have already seen. Let us follow the notation of [**770**] here (with some repetition of other discussions) and write $H=\int d^3x(N\mathfrak{H}+N^j\mathfrak{H}_j)$ where (in standard notation) for GR with a scalar field ϕ

$$\mathfrak{H}_j=-2D_i\pi^i_j\pi_\phi\partial_j\phi;\ \mathfrak{H}=\kappa G_{ijk\ell}\pi^{ij}\pi^{k\ell}+\frac{1}{2}h^{-1/2}\pi^2_\phi+ \tag{5.12}$$
$$+h^{1/2}\left[-\kappa^{-1}(R^{(3)}-2\Lambda)+\frac{1}{2}h^{ij}\partial_i\phi\partial_j\phi+U(\phi)\right]$$

The canonical momentum is expressed via (we use π^{ij} instead of Π^{ij})

$$\pi^{ij}=-h^{1/2}(K^{ij}-h^{ij}K)=G^{ijk\ell}(\dot{h}_{k\ell}-D_kN_\ell-D_\ell N_k); \tag{5.13}$$
$$K_{ij}=-\frac{1}{2N}(\dot{h}_{ij}-D_iN_j-D_jN_i)$$

K is the extrinsic curvature of the 3-D hypersurface Σ in question with indices lowered and raised via the surface metric h_{ij} and its inverse). The canonical momentum of the scalar field is

$$\pi_\phi=\frac{h^{1/2}}{N}(\dot{\phi}-N^j\partial_j\phi) \tag{5.14}$$

As usual $R^{(3)}$ is the intrinsic curvature of the hypersurfaces and N, N_j are the standard Lagrange multipliers for the super-Hamiltonian constraint $\mathfrak{H}\approx 0$ and the super momentum constraint $\mathfrak{H}^i\approx 0$. These multipliers are present due to the invariance of GR under spacetime coordinate transformations. Recall also

$$G^{ijk\ell}=\frac{1}{2}h^{1/2}(h^{ik}h^{j\ell}+h^{i\ell}h^{jk}-2h^{ij}h^{k\ell}); \tag{5.15}$$
$$G_{ijk\ell}=\frac{1}{2}h^{-1/2}(h_{ik}h_{j\ell}+h_{i\ell}h_{jk}-h_{ij}h_{k\ell})$$

(called the deWitt metric). Here D_i is the i-component of the covariant derivative on the hypersurface and $\kappa = 16\pi G/c^4$. The classical 4-metric

$$ds^2 = -(N^2 - N^i N_i)dt^2 + 2N_i dx^i dt + h_{ij}dx^i dx^j \tag{5.16}$$

and the scalar field which are solutions of the Einstein equations can be obtained from the Hamilton equations

$$\dot{h}_{ij} = \{h_{ij}, H\};\ \dot{\pi}^{ij} = \{\pi^{ij}, H\};\ \dot{\phi} = \{\phi, H\};\ \dot{\pi}_\phi = \{\pi_\phi, H\} \tag{5.17}$$

for some choice of $N,\ N^j$ provided initial conditions are compatible with the constraints $\mathfrak{H} \approx 0$ and $\mathfrak{H}_j \approx 0$. It is a feature of the Hamiltonian structure that the 4-metrics (5.6) constructed in this way with the same initial conditions describe the same 4-geometry for any choice of N and N^j. The algebra of constraints closes in the following form (cf. [**470**])

$$\{\mathfrak{H}(x), \mathfrak{H}(x')\} = \mathfrak{H}^i(x)\partial_i\delta^3(x,x') - \mathfrak{H}^i(x')\partial_i\delta(x',x); \tag{5.18}$$
$$\{\mathfrak{H}_i(x), \mathfrak{H}(x')\} = \mathfrak{H}(x)\partial_i\delta(x,x');$$
$$\{\mathfrak{H}_i(x), \mathfrak{H}_j(x')\} = \mathfrak{H}_i(x)\partial_j\delta^3(x,x') + \mathfrak{H}_j(x')\partial_i\delta^3(x,x')$$

One quantizes following Dirac to get $\hat{\mathfrak{H}}_i|\psi >= 0$ and $\hat{\mathfrak{H}}|\psi >= 0$ and in the metric and field representation the first equation is

$$-2h_{\ell i}D_j\frac{\delta\psi(h_{ij},\phi)}{\delta h_{\ell j}} + \frac{\delta\psi(h_{ij}\phi)}{\delta\phi}\partial_i\phi = 0 \tag{5.19}$$

which implies that the wave functional ψ is invariant under space coordinate transformations. The second equation is the WDW equation which (in unregularized form) is

$$\left\{-\hbar^2\left[\kappa G_{ijk\ell}\frac{\delta}{\delta h_{ij}}\frac{\delta}{\delta h_{k\ell}} + \frac{1}{2}h^{-1/2}\frac{\delta^2}{\delta\phi^2}\right] + V\right\}\psi(h_{ij},\phi) = 0; \tag{5.20}$$
$$V = h^{1/2}\left[-\kappa^{-1/2}(R^{(3)} - 2\Lambda) + \frac{1}{2}h^{ij}\partial_i\phi\partial_j\phi + U(\phi)\right]$$

(V is the classical potential). This equation involves products of local operators at the same point and hence must be regulated (cf. also Section 4.4). After that one should find a factor ordering which makes the theory free of anomalies, in the sense that the commutator of the operator version of the constraints closes in the same way as their respective classical Poisson brackets (cf. [**476, 572, 616**]).

Consider now the dBB interpretation of (5.19)-(5.20) where $\psi = Aexp(iS/\hbar)$ with A, S functionals of h_{ij} and ϕ. One arrives at

$$-2h_{\ell i}D_j\frac{\delta S(h_{ij}\phi)}{\delta h_{\ell j}} + \frac{\delta S(h_{ij},\phi)}{\delta\phi}\partial_i\phi = 0;\ -2h_{\ell i}D_j\frac{\delta A(h_{ij},\phi)}{\delta h_{\ell j}} + \frac{\delta A(h_{ij},\phi)}{\delta\phi} = 0 \tag{5.21}$$

upon writing $\psi = Aexp(iS/\hbar)$. These equations will depend on the factor ordering; however in any case one of the equations will have the form

$$\kappa G_{ijk\ell}\frac{\delta S}{\delta h_{ij}}\frac{\delta S}{\delta h_{k\ell}} + \frac{1}{2}h^{-1/2}\left(\frac{\delta S}{\delta\phi}\right)^2 + V + Q = 0 \tag{5.22}$$

where V is the classical potential. Contrary to the other terms in (5.22), which are already well defined, the precise form of Q depends on the regularization and factor ordering which are prescribed for the WDW equation. In the unregulated form of (5.20)

$$Q = -\frac{\hbar^2}{A}\left(\kappa G_{ijk\ell}\frac{\delta^2 A}{\delta h_{ij}\delta h_{k\ell}} + \frac{h^{-1/2}}{2}\frac{\delta^2 A}{\delta\phi^2}\right) \tag{5.23}$$

The other equation (in addition to (5.22)) is

$$\kappa G_{ijk\ell}\frac{\delta}{\delta h_{ij}}\left(A^2\frac{\delta S}{\delta h_{k\ell}}\right) + \frac{1}{2}h^{-1/2}\frac{\delta}{\delta\phi}\left(A^2\frac{\delta S}{\delta\phi}\right) = 0 \tag{5.24}$$

Now consider the dBB interpretation. First (5.21) and (5.22), which are valid irrespective of any factor ordering, are like the HJ equations for GR supplemented by an extra term Q for (5.22) (which does depend on factor ordering etc.). One postulates that the 3-metric, the scalar field, and their canonical momenta always exist (independent of observation) and that the evolution of the 3-metric and scalar field can be obtained from the guidance relations

$$\pi^{ij} = \frac{\delta S(h_{ab},\phi)}{\delta h_{ij}};\ \pi_\phi = \frac{\delta S(h_{ij},\phi)}{\delta\phi} \tag{5.25}$$

(cf. (5.13)-(5.14)). The evolution of these fields will be different from the classical one due to the presence of Q in (5.22). The only difference between the cases of the nonrelativistic particle and QFT in flat spacetime is the fact that (5.24) for canonical QG cannot be interpreted as a continuity equation for a probability density A^2 because of the hyperbolic nature of the deWitt metric $G_{ijk\ell}$. However even without a notion of probability density one can extract a lot of information from (5.22), whatever Q may be. First note that whatever the form of Q it must be a scalar density of weight one (via the HJ equation (5.22)). From this equation one can express Q via

$$Q = -\kappa G_{ijk\ell}\frac{\delta S}{\delta h_{ij}}\frac{\delta S}{\delta h_{k\ell}} - \frac{1}{2}h^{-1/2}\left(\frac{\delta S}{\delta\phi}\right)^2 - V \tag{5.26}$$

Since S is an invariant (via (5.21)) it follows that $\delta S/\delta h_{ij}$ and $\delta S/\delta\phi$ must be a second rank tensor density and a scalar density respectively, both of weight one. When their products are contracted with $G_{ijk\ell}$ and multiplied by $h^{-1/2}$ they form a scalar density of weight one. As V is also a scalar density of weight one then so must be Q. Further Q must depend only on h_{ij} and ϕ because it comes from the wave functional which depends only on these variabes. Of course it can be nonlocal but it cannot depend on the momenta.

A minisuperspace is the set of spacelike geometries where all but a set of $h^{ij}_{(n)}(t)$ and the corresponding momenta $\pi^{(n)}_{ij}(t)$ are put identically equal to zero (this violates the uncertainty principle but one hopes to retain suitable qualitative features - cf. [**770**] for references). In the case of a minisuperspace of homogeneous models, one puts $\mathfrak{H}^i = 0$ and N_i in H can be set equal to zero without losing any of the Einstein equations. The Hamiltonian becomes $H_{GR} = N(t)\mathfrak{H}(p^\alpha(t), q_\alpha(t))$ where the q_α and p^α represent homogeneous degrees of freedom coming from $\pi^{ij}(x,t)$ and $h_{ij}(x,t)$. Equations (5.23)-(5.25) become then

$$\frac{1}{2}f_{\alpha\beta}(q_\mu)\frac{\partial S}{\partial q_\alpha}\frac{\partial S}{\partial q_\beta}+U(q_\mu)+Q(q_\mu)=0; \tag{5.27}$$

$$Q(q_\mu)=-\frac{1}{R}f_{\alpha\beta}\frac{\partial^2 R}{\partial q_\alpha \partial q_\beta};\ p^\alpha=\frac{\partial S}{\partial q_\alpha}=f^{\alpha\beta}\frac{1}{N}\frac{\partial q_\beta}{\partial t}=0$$

where $f_{\alpha\beta}(q_\mu)$ and $U(q_\mu)$ are the minisuperspace particularizations of $G_{ijk\ell}$ and $-h^{1/2}R^{(3)}(h_{ij})$. The last equation is invariant under time reparametrization and hence even at the quantum level different choices of $N(t)$ yield the same spacetime geometry for a given nonclassical solution $q_\alpha(x,t)$.

After some discussion and computations involving the avoidance of singularities and quantum isotropization of the universe one goes now in [**770**] to the general situation in full superspace. From the guidance equations (5.25) one obtains

$$\dot{h}_{ij}=2NG_{ijk\ell}\frac{\delta S}{\delta h_{k\ell}}+D_iN_j+D_jN_i;\ \dot{\phi}=Nh^{-1/2}\frac{\delta S}{\delta\phi}+N^i\partial_i\phi \tag{5.28}$$

The question here is, given some initial 3-metric and scalar field, what kind of structure do we obtain upon integration in t? In particuar does this structure form a 4-dimensional geometry with a scalar field for any choice of the lapse and shift functions? Classically all is well but with S a solution of the modified HJ equation containing Q there is no guarantee. One goes to the Hamiltonian picture because strong results have been obtained from that point of view historically (cf. [**470, 926**]). One constructs now a Hamiltonian formalism consistent with the guidance relations (5.25) which yields the Bohmian trajectories (5.28). Given this Hamiltonian one can then use results from the literature to obtain new results for the dBB picture of quantum geometrodynamics. Thus from (5.21)-(5.22) one can easily guess that the Hamiltonian which generates the Bohmian trajectories, once the guidance relations (5.25) are satisfied initially, should be

$$H_Q=\int d^3x\left[N(\mathfrak{H}+Q)+N^i\mathfrak{H}_i\right];\ \mathfrak{H}_Q=\mathfrak{H}+Q \tag{5.29}$$

Here $\mathfrak{H}$ and $\mathfrak{H}_i$ are the usual GR quantities from (5.12) and in fact the guidance relations (5.25) are consistent with the constraints $\mathfrak{H}_Q\approx 0$ and $\mathfrak{H}_i\approx 0$ because S satisfies (5.21)-(5.22). Furthermore they are conserved by the Hamiltonian evolution given by (5.29). Then one can show that indeed (5.28) can be obtained from H_Q with the guidance relations (5.25) viewed as additional constraints (cf. [**769, 772**] for details). Thus one has a Hamiltonian H_Q which generates the Bohmian trajectories once the guidance relations (5.25) are imposed initially. Now one asks about the evolution of the fields driven by H_Q forms a 4-geometry as in classical geometrodynamics. First recall a result from [**926**] which shows that if the 3-geometries and field configurations defined on hypersurfaces are evolved by some Hamiltonian with the form $\tilde{H}=\int d^3x(N\tilde{\mathfrak{H}}+N^i\tilde{\mathfrak{H}}_i)$ and if this motion can be viewed as the motion of a 3-dimensional cut in a 4-dimensional spacetime then the constraints $\tilde{\mathfrak{H}} \approx 0$ and $\tilde{\mathfrak{H}}_i \approx 0$ must satisfy the algebra

$$\{\tilde{\mathfrak{H}}(x), \tilde{\mathfrak{H}}(x')\} = -\epsilon[\tilde{\mathfrak{H}}^i(x)\partial_i\delta^3(x',x) - \tilde{\mathfrak{H}}(x')\partial_i\delta^3(x,x');$$
$$\{\tilde{\mathfrak{H}}_i(x), \tilde{\mathfrak{H}}(x')\} = \tilde{\mathfrak{H}}(x)\partial_i\delta^3(x,x');$$
$$\{\tilde{\mathfrak{H}}_i(x), \tilde{\mathfrak{H}}_j(x')\} = \tilde{\mathfrak{H}}_i(x)\partial_j\delta^3(x,x') - \tilde{\mathfrak{H}}_j(x')\partial_i\delta^3(x,x') \tag{5.30}$$

The constant ϵ can be ± 1 (if the 4-geometry is Euclidean $(+1)$ or hyperbolic (-1)); these are the conditions for the existence of spacetime. For $\epsilon = -1$ this algebra is the same as (5.18) for GR, but the Hamiltonian (5.29) is different via Q in $\mathfrak{H}_Q$. The Poisson bracket $\{\mathfrak{H}_i(x), \mathfrak{H}_j(x')\}$ satisfies (5.30) because the $\mathfrak{H}_i$ of H_Q defined in (5.29) is the same as in GR. Also $\{\mathfrak{H}_i(x), \mathfrak{H}_Q(x')\}$ satisfies (5.30) because $\mathfrak{H}_i$ is the generator of spatial coordinate transformations and since $\mathfrak{H}_Q$ is a scalar density of weight one (recall Q is a scalar density of weight one) it must satisfy this Poisson bracket relation with $\mathfrak{H}_i$. What remains to be verified is whether the Poisson bracket $\{\mathfrak{H}_Q(x), \mathfrak{H}_Q(x')\}$ closes as in (5.30). For this one recalls a result of [**470**] where there is a general super Hamiltonian $\tilde{\mathfrak{H}}$ which satisfies (5.30), is a scalar density of weight one (whose geometrical degrees of freedom are given only by h_{ij} and its canonical momentum), and which contains only even powers and no non-local term in the momenta. From [**470**] this means that $\tilde{\mathfrak{H}}$ must have the form

$$\tilde{\mathfrak{H}} = \kappa G_{ijk\ell}\pi^{ij}\pi^{k\ell} + \frac{1}{2}h^{-1/2}\pi_\phi^2 + V_G; \tag{5.31}$$
$$V_G = -\epsilon h^{1/2}\left[-\kappa^{-1}(R^{(3)} - 2\Lambda) + \frac{1}{2}h^{ij}\partial_i\phi\partial_j\phi + U(\phi)\right]$$

Note that $\mathfrak{H}_Q$ satisfies the hypotheses since it is quadraic in the momenta and the quantum potential does not contain any nonlocal term in the momenta. Consequently there are three possible scenarios for the dBB quantum geometrodynamics, depending on the form of the quantum potential. First assume that quantum geometrodynamics is consistent (independent of the choice of lapse and shift functions) and forms a nondegenerate 4-geometry. Then $\{\mathfrak{H}_Q, \mathfrak{H}_Q\}$ must satisfy the first equation in (5.30) and consequently, combining with (5.30) for $\tilde{\mathfrak{H}}$, Q must be such that $V + Q = V_G$ with V given by (5.20) yielding

$$Q = -h^{1/2}\left[(\epsilon+1)\left(-\kappa^{-1}R^{(3)} + \frac{1}{2}h^{ij}\partial_i\phi\partial_j\phi\right) + \frac{2}{\kappa}(\epsilon\tilde{\Lambda} + \Lambda) + \epsilon\tilde{U}(\phi) + U(\phi)\right] \tag{5.32}$$

For this situation there are two possibilities, namely

(1) The spacetime is hyperbolic with $\epsilon = -1$ and

$$Q = -h^{1/2}\left[\frac{2}{\kappa}(\Lambda - \tilde{\Lambda}) - \tilde{U}(\phi) + U(\phi)\right] \tag{5.33}$$

Hence Q is like a classical potential; its effect is to renormalize the cosmological constant and the classical scalar potential, nothing more. The quantum geometrodynamics is indistinguishable from the classical one. It is not necessary to require $Q = 0$ since $V_G = V + Q$ may already describe the classical universe in which we live.

(2) The spacetime is Euclidean with $\epsilon = 1$ in which case

$$Q = -h^{1/2}\left[2\left(-\kappa^{-1}R^{(3)} + \frac{1}{2}h^{ij}\partial_i\phi\partial_j\phi\right) + \frac{2}{\kappa}(\tilde{\Lambda}+\Lambda) + \tilde{U}(\phi) + U(\phi)\right] \tag{5.34}$$

Now Q not only renormalizes the cosmological constant and the classical scalar field but also changes the signature of spacetime. The total potential $V_G = V + Q$ may describe some era of the early universe when it had Euclidean signature but not the present era when it is hyperbolic. The transition between these two phases must happen in a hypersurface where $Q = 0$ which is the classical limit. This result points in the direction of [**410**].

There remains the possiblity that the evolution is consistent but does not form a nondegenerate 4-geometry. In this case the Poisson bracket $\{\mathfrak{H}_Q, \mathfrak{H}_Q\}$ does not satisfy (5.30) but is weakly zero in some other manner. Consider for example

(1) For real solutions of the WDW equation, which is a real equation, the phase S is zero and from (5.22) one can see that $Q = -V$. Hence the quantum super Hamiltonian (5.29) will contain only the kinetic term and $\{\mathfrak{H}_Q, \mathfrak{H}_Q\} = 0$ (strong equality). This case is connected with the strong gravity limit of GR (cf. [**770**] for references and further discussion).

(2) Any nonlocal quantum potential breaks spacetime and as an example consider $Q = \gamma V$ where γ is a function of the functional S (hence is nonlocal). Calculating one obtains (cf. [**769, 772**])

$$\{\mathfrak{H}_Q(x), \mathfrak{H}_Q(x')\} = (1+\gamma)[\mathfrak{H}^i(x)\partial_i\delta^3(x,x') - \mathfrak{H}^i(x')\partial_i\delta^3(x',x)] - \tag{5.35}$$

$$-\frac{d\gamma}{dS}V(x')[2\mathfrak{H}_Q(x) - 2\kappa G_{ijk\ell}(x)\pi^{ij}(x)\Phi^{k\ell}(x) - h^{-1/2}\pi_\Phi(x)\Phi_\phi(x) +$$

$$+\frac{d\gamma}{dS}V(x)[2\mathfrak{H}_Q(x') - 2\kappa G_{ijk\ell}(x')\pi^{ij}(x')\Phi^{k\ell}(x') - h^{-1/2}\pi_\phi(x')\Phi_\phi(x')] \approx 0$$

The last equation is weakly zero because it is a combination of the constraints and the guidance relations of Bohmian theory. This means that the Hamiltonian evolution with the quantum potential $Q = \gamma V$ is consistent only when restricted to the Bohmian trajectories. For other trajectories it is inconsistent.

In these examples one makes contact with the structure constants of the algebra characterizing the foam like pregeometry structure pointed out long ago by J.A. Wheeler. Another fact here of interest is that there are no inconsistent Bohmian trajectories (cf. [**772**]). We call attention also to [**77**] where in particular one considers noncommutative geometry and cosmology in connection with Bohmian theory; the results are very interesting.

6. BOHM AND NONCOMMUTATIVE GEOMETRY

We extract here from [**77, 78, 399**] with other references as we go along (cf. in particular [**230, 261, 318, 341, 552, 771, 772, 773, 897, 901**]). First from [**77**] one refers to [**78**] where a new interpretation of the canonical commutation

$$[\hat{X}^\mu, \hat{X}^\nu] = i\theta^{\mu\nu} \tag{6.1}$$

was proposed. The idea was that it is possible to interpret the commutation relation as a property of the particle coordinate observables rather than of the spacetime coordinates and this enforced a reinterpretation of the meaning of the wave function in noncommutative QM (NCQM). In [**77**] one develops a Bohmian interpretation for NCQM amd forms a deterministic theory of hidden variables that exhibit canonical noncommutativity (6.1) between the particle position observables. There are several motivations for reconsideration of hidden variable theory (see e.g. [**475**]) and we begin with a Moyal star product defined via

$$(6.2)\qquad (f*g)=\frac{1}{(2\pi)^n}\int d^mkd^npe^{i(k_\mu+p_\mu)x^\mu-(1/2)k_\mu\theta^{\mu\nu}p_\nu}f(k)g(p)=$$
$$=e^{(1/2)\theta^{\mu\nu}(\partial/\partial\xi^\mu)(\partial/\partial\eta^\nu)}f(x+\xi)g(x+\eta)|_{\xi=\eta=0}$$

(cf. [**192**] for an extensive treatment of star products and some noncommutative geometry). The commutative coordinates x^i are called the Weyl symbols (WS) of position operators $\hat{X}^i$ and one will consider them as spacetime coordinates following [**78**]. One assumes here that $\theta^{0i}=0$ and the Hilbert space of states of NCQM can be taken as in commutative QM with noncommutative SE now given by

$$(6.3)\qquad i\hbar\frac{\partial\psi(x^i,t)}{\partial t}=-\frac{\hbar^2}{2m}\nabla^2\psi(x^i,t)+V(x^i)*\psi(x^i,t)=$$
$$=\frac{\hbar^2}{2m}\nabla^2\psi(x^i,t)+V\left(x^j+i\frac{\theta^{jk}}{2}\partial_k\right)\psi(x^i,t)$$

The operators

$$(6.4)\qquad \hat{X}^j=x^j+\frac{i\theta^{jk}\partial_k}{2}$$

are the observables corresponding to the physical positions of the particles and x^i are the associated canonical coordinates. For intition one could think here in terms of a "half dipole" whose extent is proportional to the canonical momentum $\Delta x^i=\theta^{ij}p_j/2\hbar$; one of its endpoints carries the mass and is responsible for its interaction. The NCQM formulated with (6.3)-(6.4) can be considered as the uaual QM with a Hamiltonian not quadratic in momenta and "unusual" position operators. From this point of view the BNCQM below can be considered as an extension along the same lines. Any attempt to localize the particles must satisfy the uncertainty relations

$$(6.5)\qquad \Delta X^i\Delta X^j\geq|\theta^{ij}|/2$$

The expression for the definition of probability density $\rho(x^i,t)=|\psi(x^i,t)|^2$ has a meaning that differs from ordinary QM. Namely the quantity $\rho(x^i,t)d^3x$ must be interpreted as the probability that the system is found in a configuration such that the canonical coordinate of the particle is contained in a volume d^3x around the point x at time t. Given an arbitrary physical observable characterized by a Hermitian operator $\hat{A}(\hat{x}^i,\hat{p}^i)$ (which includes e.g. $\hat{A}(\hat{X}^i(\hat{x}^i,\hat{p}^i),\hat{p}^i)$) its expected value is

$$(6.6)\qquad <\hat{A}>_t=\int d^3x\psi^*(x^i,t)\hat{A}(x^j,-i\hbar\partial_j)\psi(x^i,t)$$

A HJ formalism for NCQM is found by writing $\psi = Rexp(iS/\hbar)$, putting it in (6.3), and splitting real and imaginary terms; the real part is

$$\frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2m} + V + V_{nc} + Q_K + Q_I = 0 \tag{6.7}$$

Here the three new potential terms are defined as

$$V_{nc} = V\left(x^i - \frac{\theta^{ij}}{2\hbar}\partial_j S\right) - V(x^i); \tag{6.8}$$

$$Q_K = \Re\left(-\frac{\hbar^2}{2m}\frac{\nabla^2\psi}{\psi}\right) - \left(\frac{\hbar^2}{2m}(\nabla S)^2\right) = -\frac{\hbar^2}{2m}\frac{\nabla^2 R}{R};$$

$$Q_I = \Re\left(\frac{V[x^j + (i\theta^{jk}/2)\partial_k]\psi}{\psi}\right) - V\left(x^i - \frac{\theta^{ij}}{2\hbar}\partial_j S\right)$$

V_{nc} is the potential that accounts for the NC classical interactions while Q_K and Q_I account for the quantum effects. The NC contributions contained in the latter two can be split out by defining

$$Q_{nc} = Q_K + Q_I - Q_c;\ Q_c = -\frac{\hbar^2}{2m}\frac{\nabla^2 R_c}{R_c};\ R_c = \sqrt{\psi_c^*\psi_c} \tag{6.9}$$

Here ψ_c is the wave function obtained from the commutative SE containing the usual potential $V(x^i)$, i.e. the equation obtained by setting $\theta^{ij} = 0$ in (6.3) before solving. The imaginary part of the SE which yields the probability conservation law, is

$$\frac{\partial R^2}{\partial t} + \nabla\cdot\left(\frac{R^2\nabla S}{m}\right) + \Sigma_\theta = 0;\ \Sigma_\theta = -\frac{2R}{\hbar}\Im\left[e^{-iS/\hbar}V * (Re^{iS/\hbar})\right] \tag{6.10}$$

By integrating the first equation we obtain

$$\frac{d}{dt}\int R^2 d^3x = 0;\ \int \Sigma_\theta d^3x = 0 \tag{6.11}$$

when R^2 vanishes at infinity.

Now for an ontological theory of motion one follows the traditional methods (cf. [**126, 471**]). Necessary conditions for the theory to be capable of reproducing the same statistical results as the standard interpretation of NCQM constrain the admissible form for the functions $X^i(t)$ which eliminates a certain arbitrariness in the constructions. First for the rules, with an arbitrary physical characterized by a Hermitian operator $\hat{A}(\hat{x}^i, \hat{p}^i)$ it is possible to associate a function $\mathfrak{A}(x^i, t)$, the local expectation value of $\hat{A}$ (cf. [**471**]), which when averated over the ensemble of density $\rho(x^i, t) = |\psi|^2$ gives the same expectation value obtained by the standard operator formalism. Thus it is natural to define the ensemble average via

$$< \hat{A} >_t = \int \rho(x^i, t)\mathfrak{A}(x^i, t) d^3x \tag{6.12}$$

For this to agree with (6.6) $\mathfrak{A}(x^i, t)$ must be defined as

$$\mathfrak{A}(x^i, t) = \frac{\Re\left[\psi^*(x^i, t)\hat{A}(x^j, -i\hbar\partial_j)\psi(x^i, t)\right]}{\psi^*(x^i, t)\psi(x^i, t)} = A(x^i, t) + \mathfrak{Q}_A(x^i, t) \tag{6.13}$$

where the real part is taken to account for the hemiticity of $\hat{A}(\hat{x}^i, \hat{p}^i)$ and $A(x^i, t) = A[x^i, p^i = \partial^i S(x^i, t)]$ while $\mathfrak{Q}_A$ is defined via

$$\mathfrak{Q}_A = \Re\left[\frac{\hat{A}(x^j, -i\hbar\partial_j)\psi(x^k, t)}{\psi(x^i, t)}\right] - A(x^i, t) \tag{6.14}$$

and this is the quantum potential that accompanies $A(x^i, t)$. From (6.13) one finds that the local expectation value of (6.4) is

$$X^i = x^i - \frac{\theta^{ij}}{2\hbar}\partial_j S(x^i, t) \tag{6.15}$$

Now to find the $X^i(t)$ the relevant information for particle motion can be extracted from the guiding wave $\psi(x^i, t)$ by first computing the associated canonical position tracks $x^i(t)$ and then evaluating (6.15) at $x^i = x^i(t)$. In order to find a good equation for the $x^i(t)$ it is interesting to consider the Heisenberg formulation and the equations of motion for the observables. Thus for the $\hat{x}^i$ one has

$$\frac{d\hat{x}^i_H}{dt} = \frac{1}{i\hbar}[\hat{x}^i_H, \hat{H}] = \frac{\hat{p}^i_H}{m} + \frac{\theta^{ij}}{2\hbar}\frac{\partial\hat{V}(\hat{X}^k_H)}{\partial\hat{X}^j_H} \tag{6.16}$$

By passing the right side of (6.16) to the Schrödinger picture one can define the velocity operators

$$\hat{v}^i = \frac{1}{\hbar}[\hat{x}^i, \hat{H}] = \frac{\hat{p}^i}{m} + \frac{\theta^{ij}}{2\hbar}\frac{\partial\hat{V}(\hat{X})}{\partial\hat{X}} \tag{6.17}$$

The differential equation for the canonical positions $x^i(t)$ is now found by identifying $dx^i(t)/dt$ with the local expectation value of $\hat{v}^i$, thus

$$\frac{dx^i(t)}{dt} = \left[\frac{\partial^i S(x^i, t)}{m} + \frac{\theta^{ij}}{2\hbar}\frac{\partial V(X^k)}{\partial X^j} + \frac{\mathfrak{Q}^i}{2}\right]\Bigg|_{x^i = x^i(t)} \tag{6.18}$$

where X^i is given in (6.15), $S(x^i, t0$ is the phase of ψ, and

$$\mathfrak{Q}^i = \Re\left(\frac{(\theta^{ij}/\hbar)[\partial\hat{V}(\hat{X}^k)/\partial\hat{X}^j]\psi(x^i, t)}{\psi(x^i, t)}\right) - \frac{\theta^{ij}}{\hbar}\frac{\partial V(X^k)}{\partial X^j} \tag{6.19}$$

The potentials $\mathfrak{Q}^i$ account for quantum effects coming from derivatives of order 2 and higher contained in $\partial\hat{V}(\hat{X}^i)/\partial\hat{X}^j$. Then once the $x^i(t)$ are known the particle trajectories are given via

$$X^i(t) = x^i(t) - \frac{\theta^{ij}}{2\hbar}\partial_j S(x^k(t), t) \tag{6.20}$$

One notes that the particles positions are not defined on nodal regions of ψ, where S is undefined, so the particles cannot run through such regions. Hence the vanishing of the wave function can be adopted as a boundary condition, implying that the particle does not run through such a region (see [**77**] for more discussion). The preceeding theory is now summarized in a formal structure as follows:

(1) The spacetime is commutative and has a pointwise manifold structure with canonical coordinates x^i. The observables correspondin to operators of position coordinates $\hat{X}^i$ of particles satisfy the commutation rules

$[\hat{X}^k, \hat{X}^j] = i\theta^{kj}$. The position observables can be represented in the coordinate space as $\hat{X}^j = x^j + i\theta^{jk}\partial_k/2$ and the x^j are canonical coordinates associated with the particle.

(2) A quantum system is composed of a point particle and a wave ψ. The particle moves in spacetime under the guidance of the wave which satisfies the SE $i\hbar\partial_t\psi(x^i,t) = -(\hbar^2/2m)\nabla^2\psi + V(\hat{X}^i)\psi$ ($\psi = \psi(x^i,t)$).

(3) The particle moves with trajectory $X^i(t) = x^i(t) - (\theta^{ij}/2\hbar)\partial_j S(x^i(t),t)$ independently of observation, where S is the phase of ψ and the $x^i(t)$ describe the canonical position trajectories found by solving

$$\frac{dx^i(t)}{dt} = \left[\frac{\partial^i S(\vec{x},t)}{m} + \frac{\theta^{ij}}{2\hbar}\frac{\partial V(X^k)}{\partial X^j} + \frac{\mathfrak{Q}^i}{2}\right]\Bigg|_{x^i=x^i(t)}$$

To find the path followed by a particle, one must specify its initial canonical position $x^i(0)$, solve the second equation, and then obtain the physical path from the first equation.

These three postulates constitute on their own a consisent theory of motion, and is intended to give a finer view of QM, namely a detailed description of the individual physical processes and to provide the same statistical predictions. In ordinary commutative Bohmian mechanics, in order to reproduce the statistics the additional requirement that $\rho(x^i,t_0) = |\psi(\vec{x},t_0)|^2$ is imposed for some initial time t_0. Then ρ is said to be equivariant if this property persists under evolution $\dot{x}^i(t) = f^i(x^j,t)$; in such a case

$$\frac{\partial\rho}{\partial t} + \frac{\partial(\rho\dot{x}^i)}{\partial x^i} = 0 \tag{6.21}$$

In ordinary commutative QM the equivariance property is satisfied via $\dot{x}^i(t) = J^i/\rho$ which is a consequenc of the identification between $\dot{x}^i$ and the local expectation value of the $\hat{v}^i$. In the BNCQM proposed here the same identification is valid but it is not sufficient to guarantee equivariance in all cases. This is clear after computing the canonical probability current

$$J^i(x^i,t) = \Re[\psi^*\hat{v}\psi] = |\psi|^2\left[\frac{\partial^i S(\vec{x},t)}{m} + \frac{\theta^{ij}}{2\hbar}\frac{\partial V(X^k)}{\partial X^j} + \frac{\mathfrak{Q}^i}{2}\right] = \rho\dot{x}^i \tag{6.22}$$

Then regrouping the terms in (6.10) so that the canonical probability flux (6.22) appears explicitly one obtains

$$\frac{\partial\rho}{\partial t} + \frac{\partial(\rho\dot{x}^i}{\partial x^i} - \frac{\partial}{\partial x^i}\left[\rho\left(\frac{\theta^{ij}}{2\hbar}\frac{\partial V(X^k)}{\partial X^j} + \frac{\mathfrak{Q}^i}{2}\right)\right] + \Sigma_\theta = 0 \tag{6.23}$$

For equivariance to occur an additional condition that the sum of the last two terms in the right side of (6.23) vanishes is required. When $V(X^i)$ is a linear or quadratic function, as in many applications, such a condition is trivially satisfied, and then $\rho(x^i,t) = |\psi(x^i,t)|^2$ as desired. The same may occur for other situations when other potentials are considered in #2 above and this is also discussed in [**77**].

In [**77**] (second paper) one looks at a Kantowski-Sachs (KS) universe (see e.g.

[**230, 261, 399, 901**]). Recall in the ADM formulation a line element is written in the form

$$ds^2 = (N_iN^i - N^2)dt^2 + 2N_i dx^i dt + h_{ij}dx^i dx^j \tag{6.24}$$

and the Hamiltonian of GR without matter is

$$H = \int d^3x(N\mathfrak{H} + N)i\mathfrak{H}^j);\ \mathfrak{H} = G_{ijk\ell}\pi^{ij}\pi^{k\ell} - h^{1/2}R^{(3)};\ \mathfrak{H}^j = 2D_i\pi^{ij} \tag{6.25}$$

Units are chosen so that $\hbar = c = 16\pi G = 1$. The momenta π_{ij} are canonically conjugate to h^{ij} and the deWitt metric $G_{ijk\ell}$ are given via

$$\pi_{ij} = -h^{1/2}(K_{ij} - h_{ij}K);\ G_{ijk\ell} = \frac{1}{2}h^{-1/2}(h_{ik}h_{j\ell} + h_{i\ell}h_{jk} - h_{ij}h_{k\ell}) \tag{6.26}$$

where $K_{ij} = -(\partial_t h_{ij} - D_iN_j - D_jN_i)/(2N)$ is the second fundamental form. The super Hamiltonian constraint $\mathfrak{H} \approx 0$ yields the WDW equation

$$\left(G^{ijk\ell}\frac{\delta}{\delta h^{ij}}\frac{\delta}{\delta h^{k\ell}} + h^{1/2}R^{(3)}\right)\psi[h^{ij}] = 0 \tag{6.27}$$

In the Bohmian approach now one has

$$\pi_{ij} = -h^{1/2}(K_{ij} - h_{ij}D) = \Re\left\{\frac{1}{\psi^*\psi}\left[\psi^*\left(-i\frac{\delta}{\delta h^{ij}}\right)\psi\right]\right\} = \frac{\delta S}{\delta h^{ij}} \tag{6.28}$$

If one puts $\psi = Aexp(iS)$ in (6.27) there results

$$G^{ijk\ell}\frac{\delta S}{\delta h^{ij}}\frac{\delta S}{\delta h^{k\ell}} - h^{1/2}R^{(3)} + Q = 0; \tag{6.29}$$

$$G^{ijk\ell}\frac{\delta S}{\delta h^{ij}}\left(A^2\frac{\delta S}{\delta h^{k\ell}}\right) = 0;\ Q = -\frac{1}{A}G^{ijk\ell}\frac{\delta^2 A}{\delta h^{ij}h^{k\ell}}$$

(one should really include $\hbar^2$ here in dealing with Q).

The Kantowski-Sachs universe is an important anisotropic model; the line element is

$$ds^2 = -Ndt^2 + X^2(t)dr^2 + Y^2(t)(d\theta^2 + Sin^2(\theta)d\phi^2) \tag{6.30}$$

In the Misner parametrization this becomes (cf. [**399**])

$$ds^2 = -N^2dt^2 + e^{2\sqrt{3}\beta}dr^2 + e^{-2\sqrt{3}\beta}e^{-2\sqrt{3}\Omega}(d\theta^2 + Sin^2(\theta)d\phi^2) \tag{6.31}$$

The Hamiltonian is then

$$H = N\mathfrak{H} = Nexp\left(\sqrt{3}\beta + 2\sqrt{3}\Omega\right)\left[-\frac{P_\Omega^2}{24} + \frac{P_\beta^2}{24} - 2exp(-2\sqrt{3}\Omega)\right] \tag{6.32}$$

One sets $\Theta = V^\alpha_{;\alpha}$ (volume expansion - $V^\alpha = \delta^\alpha_0/N$), and $\sigma^2 = \sigma^{\alpha\beta}\sigma_{\alpha\beta}/2$ (shear) where $\sigma_{\alpha\beta} = (h^\mu_\alpha h^\nu_\beta + h^\mu_\beta h^\nu_\alpha)V_{\mu;\nu}/2$. The semicolon denotes 4-dimensional covariant differentiation and $h^\mu_\alpha = \delta^\mu_\alpha + V^\mu V_\alpha$ is the projector orthogonal to the observer V^α (cf. [**458**]). A characteristic length scale is defined via $\Theta = 3\dot{\ell}/(\ell N)$ and in the gauge $N = 24exp(-\sqrt{3}\beta - 2\sqrt{3}\Omega)$ one has

$$\Theta(t) = \frac{1}{N}\left(\frac{\dot{X}}{X} + 2\frac{\dot{Y}}{Y}\right) = -\frac{\sqrt{3}}{24}(\dot{\beta} + 2\dot{\Omega})e^{\sqrt{3}\beta + 2\sqrt{3}\Omega}; \tag{6.33}$$

$$\sigma(t) = \frac{1}{N\sqrt{3}}\left(\frac{\dot{X}}{X} - \frac{\dot{Y}}{Y}\right) = \frac{1}{24}(2\dot{\beta} + \dot{\Omega})e^{\sqrt{3}\beta + 2\sqrt{3}\Omega};$$

$$\ell^3(t) = X(t)Y^2(t) = e^{-\sqrt{3}\beta(t) - 2\sqrt{3}\Omega(t)}$$

In order to distinguish the role of the quantum and noncommutative effects in a NC quantum universe one starts now with a KS geometry in the commutative classical version. The Poisson brackets for the classical phase space variables are

$$\{\Omega, P_\Omega\} = 1 = \{\beta, P_\beta\};\ \{P_\Omega, P_\beta\} = 0 = \{\Omega, \beta\} \tag{6.34}$$

For the metric (6.31) the constraint $\mathfrak{H} \approx 0$ is reduced to

$$\mathfrak{H} = \xi h \approx 0;\ \xi = \frac{1}{24}e^{\sqrt{3}\beta + 2\sqrt{3}\Omega};\ h = -P_\Omega^2 + P_\beta^2 - 48e^{-2\sqrt{3}\Omega} \approx 0 \tag{6.35}$$

The classical equations of motion for the phase space variables $\Omega,\ P_\Omega,\ \beta$, and P_β are then

$$\dot{\Omega} = N\{\Omega, \mathfrak{H}\} = -2P_\Omega;\ \dot{\beta} = N\{\beta, \mathfrak{H}\} = 2P_\beta; \tag{6.36}$$

$$\dot{P}_\Omega = N\{P_\Omega, \mathfrak{H}\} = -96\sqrt{3}e^{-2\sqrt{3}\Omega};\ \dot{P}_\beta = N\{P_\beta, \mathfrak{H}\} = 0$$

Explicit formulas are found and exhibited in [**77**].

Now for a NC classical model one considers

$$\{\Omega, P_\Omega\} = 1;\ \{\beta, P_\beta\} = 1;\ \{P_\Omega, P_\beta\} = 0;\ \{\Omega, \beta\} = \theta \tag{6.37}$$

The equations of motion can be written as

$$\dot{\Omega} = -2P_\Omega;\ \dot{P}_\Omega = -96\sqrt{3}e^{-2\sqrt{3}\Omega};\ \dot{\beta} = 2P_\beta - 96\sqrt{3}\theta e^{-2\sqrt{3}\Omega};\ \dot{P}_\beta = 0 \tag{6.38}$$

The solutions for Ω and β are then

$$\Omega(t) = \frac{\sqrt{3}}{6}log\left\{\frac{48}{P_{\beta_0}^2}Cosh^2\left[2\sqrt{3}P_{\beta_0}(t - t_0)\right]\right\}; \tag{6.39}$$

$$\beta(t) = 2P_{\beta_0}(t - t_0) + \beta_0 - \theta P_{\beta_0}Tanh[2\sqrt{3}P_{\beta_0}(t - t_0]$$

Further calculations appear in [**77**].

For the commutative quantum model one works with the minisuperspace construction of homogeneous models and freezing out an infinite number of degrees of freedom. First an Ansatz of the form (6.31) is introduced and the spatial dependence of the metric is integrated out. The WDW equation is then reduced to a KG equation which for the KS universe has the form

$$\left[-\hat{P}_\Omega^2 + \hat{P}_\beta^2 - 48e^{-2\sqrt{3}\Omega}\right]\psi(\Omega, \beta) = 0 \tag{6.40}$$

where $\hat{P}_\Omega = -i\partial/\partial\Omega$ and $\hat{P}_\beta = -i\partial/\partial\beta$. A solution to (6.40) is then (cf. [**399**])

$$\psi_\nu(\Omega, \beta) = e^{i\nu\sqrt{3}\beta}K_{i\nu}\left(4e^{-\sqrt{3}\Omega}\right) \tag{6.41}$$

where $K_{i\nu}$ is a modified Bessel function and ν is a real constant. Once a quantum state of the universe is given as, e.g. a superposition of states

$$\psi(\Omega,\beta) = \sum_{\nu} C_{\nu} e^{i\nu\sqrt{3}\beta} K_{i\nu}\left(4e^{-\sqrt{3}\Omega}\right) = Re^{iS} \tag{6.42}$$

the evolution can be determined by integrating the guiding equation (6.28). In the minisuperspace approach the analogue of that equation is

$$P_{\Omega} = -\frac{1}{2}\dot{\Omega} = \Re\left\{\frac{[\psi^*(-i\hbar\partial_{\Omega})\psi]}{\psi^*\psi}\right\} = \frac{\partial S}{\partial\Omega}; \tag{6.43}$$

$$P_{\beta} = \frac{1}{2}\dot{\beta} = \Re\left\{\frac{[\psi^*(-i\hbar\partial_{\beta})\psi]}{\psi^*\psi}\right\} = \frac{\partial S}{\partial\beta}$$

As before one has fixed the gauge $N = 24\ell^3 = 24exp(-\sqrt{3}\beta - 2\sqrt{3}\Omega)$. Usually different choices of time yield different quantum theories but when one uses the Bohmian interpretation in minisuperspace models the situation is identical to that of the classical case (but not beyond minisuperspace - cf. [**772**]), namely different choices yield the same theory (cf. [**84**]). Hence as long as $\ell^3(t)$ does not pass through zero (a singularity) the above choice for $N(t)$ is valid for the history of the universe. The minisuperspace analogue of the HJ equation in (6.29) is

$$-\frac{1}{24}\left(\frac{\partial S}{\partial\Omega}\right)^2 + \frac{1}{24}\left(\frac{\partial S}{\partial\beta}\right)^2 - 2e^{-2\sqrt{3}\Omega} + \frac{1}{24R}\left(\frac{\partial^2 R}{\partial\Omega^2} - \frac{\partial^2 R}{\partial\beta^2}\right) = 0 \tag{6.44}$$

Explicit calculations are then given in [**77**] with graphs and pictures.

Now for the NC quantum model one takes

$$[\hat{\Omega},\hat{\beta}] = i\theta \tag{6.45}$$

According to the Weyl quantization procedure (cf. [**318**]) the realization of (6.45) in terms of commutative functions is made by the Moyal star product defined via (cf. [**192**])

$$f(\Omega_c,\beta_c) * g(\Omega_c,\beta_c) = f(\Omega_c,\beta_c) e^{i(\theta/2)(\overleftarrow{\partial}_{\Omega_c}\overrightarrow{\partial}_{\beta_c} - \overleftarrow{\partial}_{\beta_c}\overrightarrow{\partial}_{gO_c})} g((\Omega_c,\beta_c) \tag{6.46}$$

The commutative coordinates Ω_c, β_c are called Weyl symbols of the operators $\hat{\Omega}$, $\hat{\beta}$. In order to compare evolutions with the same time parameter as above one again fixes the gauge $N = 24exp(-\sqrt{3}\beta - 2\sqrt{3}\Omega)$ and the WDW equation for the NC KS model is (cf. [**399**])

$$\left[-P_{\Omega_c}^2 + P_{\beta_c}^2 - 48e^{-2\sqrt{3}\Omega_c}\right] * \psi(\Omega_c,\beta_c) = 0 \tag{6.47}$$

which is the Moyal deformed version of (6.40). By using properties of the Moyal bracket (cf. [**192**]) one can write the potential term (denoted by V to include the general case) as

$$V(\Omega_c,\beta_c) * \psi(\Omega_c,\beta_c) = V\left(\Omega_c + i\frac{\theta}{2}\partial_{\beta_c}, \beta_c - i\frac{\theta}{2}\partial_{\Omega_c}\right)\psi(\Omega_c,\beta_c) = \tag{6.48}$$

$$= V(\hat{\Omega},\hat{\beta})\psi(\Omega_c,\beta_c);\ \hat{\Omega} = \hat{\Omega}_c - \frac{\theta}{2}\hat{P}_{\beta_c};\ \hat{\beta} = \hat{\beta}_c + \frac{\theta}{2}\hat{P}_{\Omega_c}$$

The WDW equation then reads as

$$\left[-\hat{P}^2_{\Omega_c} + \hat{P}^2_{\beta_c} - 48e^{-2\sqrt{3}\hat{\Omega}_c + \sqrt{3}\theta\hat{P}_{\beta_c}}\right]\psi(\Omega_c, \beta_c) \tag{6.49}$$

Two consistent interpretations for the cosmology emerging from these equations are possible. One consists in considering the Weyl symbols Ω_c and β_c as the constituents of the physical metric, which makes things essentially commutative with a modified interaction. The second, as adopted in [**78, 79**], involves the Weyl symbols being considered as auxiliary coordinates, and thereby one studies the evolution of a NC quantum universe.

Next for the Bohmian formulation of the NC minisuperspace one looks at

$$[\hat{x}^i, \hat{p}^j] = i\hbar\delta^{ij} \tag{6.50}$$

Note the operator formalism of QM is not a primary concept in Bohmian mechanics. Thus it is reasonable to expect that in Bohmian NC quantum cosmology it should be possible to describe the metric variables as well defined entities, although the operators $\hat{\Omega}$ and $\hat{\beta}$ satisfy (6.45). This is exactly what is proposed here. One wants to give an objective meaning to the wavefunction and the metric variables Ω and β and the wave function is obtained by solving (6.47). What is missing is the evolution law for Ω and β. To find this one associates a function $\mathfrak{A}(\Omega_c, \beta_c)$ to $\hat{A}(\hat{\Omega}_c, \hat{\beta}_c, \hat{P}_{\Omega_c}, \hat{P}_{\beta_c})$ according to the rule

$$\mathfrak{B}[\hat{A}] = \frac{\Re[\psi^*(\Omega_c, \beta_c)\hat{A}(\Omega_c, \beta_c, -i\hbar\partial_{\Omega_c}, -i\hbar\partial_{\beta_c})\psi(\Omega_c, \beta_c)]}{\psi^*(\Omega_c, \beta_c)\psi(\Omega_c, \beta_c)} = \mathfrak{A}(\Omega_c, \beta_c) \tag{6.51}$$

where the real part takes into account the hermiticity of $\hat{A}$ (the $\mathfrak{B}$ here refers to the idea of "beable"). Applying this to the operators $\hat{\Omega}$ and $\hat{\beta}$ one arrives at

$$\Omega(\Omega_c, \beta_c) = \mathfrak{B}[\hat{\Omega}] = \frac{\Re[\psi^*(\Omega_c, \beta_c)\hat{\Omega}(\Omega_c, -i\hbar\partial_{\beta_c})\psi(\Omega_c, \beta_c)]}{\psi^*(\Omega_c, \beta_c)\psi(\Omega_c, \beta_c)} = \Omega_c - \frac{\theta}{2}\partial_{\beta_c}S; \tag{6.52}$$

$$\beta(\Omega_c, \beta_c) = \mathfrak{B}[\hat{\beta}] = \frac{\Re[\psi^*(\Omega_c, \beta_c)\hat{\beta}(\beta_c, -i\hbar\partial_{\Omega_c})\psi(\Omega_c, \beta_c)]}{\psi^*\psi} = \beta_c + \frac{\theta}{2}\partial_{\Omega_c}S$$

Thus the relevant information for universe evolution can be extracted from the guiding wave $\psi(\Omega_c, \beta_c)$ by first computing the associated canonical position tracks $\Omega_c(t)$ and $\beta_c(t)$. Then one obtains $\Omega(t)$ and $\beta(t)$ by evaluating (6.52) at $\Omega_c = \Omega_c(t)$ and $\beta_c = \beta_c(t)$ (similar procedures are worked out in [**77**] for the NC classical situation). Differential equations for the canonical positions $\Omega_c(t)$ and $\beta_c(t)$ can be found by identifying $\dot{\Omega}_c(t)$ and $\dot{\beta}_c(t)$ with the beables associated with their time evolution and formulas are worked out in [**77**].

The combination of NC geometry and Bohmian type quantum physics is somewhat like a fusion of two apparently opposite ways of thinking, one fuzzy and the other refering ontologically to point particles. Every Hermitian operator can be associated with an ontological element and by averaging the beable $\mathfrak{B}[\hat{A}]$ over an ensemble of particles with probability density $\rho = |\psi|^2$ one gets the same result as computing the expectation value of the observable $\hat{A}$ in standard operational formalism. In the KS universe, where WDW is of KG type (with no notion of probability) the beable mapping is well defined, even in the NC case. In the commutative context this formulation leads to the Bohmian quantum gravity proposed by Holland in [**471**] in the minisuperspace approximation. The work here shows that noncommutativity can modify appreciably the universe evolution in the quantum context (qualitatively as well as quantitively).

We go next to [**78**] and consider NC in the evolution of Friedman-Robertson-Walker (FRW) universes with a conformally coupled scalar field. First take the commutative situation and restrict attention to the case of constant positive curvature of the spatial sections. The action is then

$$S=\int d^4x\sqrt{-g}\left[-\frac{1}{2}\phi_{;\mu}\phi^{;\mu}+\frac{R}{16\pi G}-\frac{R\phi^2}{12}\right] \tag{6.53}$$

Units are chosen so that $\hbar = c = 1$ and $8\pi G = 3\ell_P^2$ where ℓ_P is the Planck length. For the FRW model with a homogeneous scalar field the following Ansatz of minisuperspace can be adopted

$$ds^2=-N^2(t)dt^2+a^2(t)\left[\frac{dr^2}{1-r^2}+r^2(d\theta^2+Sin^2(\theta)d\phi^2)\right];\ \phi=\phi(t) \tag{6.54}$$

Rescaling the scalar field via $\chi=\phi a\ell_P/\sqrt{2}$ one obtains the minisuperspace action, Hamiltonian, and momenta

$$S=\int dt\left(Na-\frac{a\dot{a}^2}{N}+\frac{a\dot{\chi}^2}{N}-\frac{N\chi^2}{a}\right);\ P_a=-\frac{2a\dot{a}}{N}; \tag{6.55}$$
$$H=N\left[-\frac{P_a^2}{4a}+\frac{P_\chi^2}{4a}-a+\frac{\chi^2}{a}\right]=N\mathfrak{H};\ P_\chi=\frac{2a\dot{\chi}}{N}$$

For the classical phase space variables one knows $\{a,\chi\}=0=\{P_a,P_\chi\}$ and $\{a,P_a\}=1=\{\chi,P_\chi\}$ and the equations for the metric and matter field variables following from this and (6.54) are

$$\dot{a}=\{a,H\}=-\frac{NP_a}{2a};\ \dot{P}_a=\{P_a,H\}=2N; \tag{6.56}$$
$$\dot{\chi}=\{\chi,H\}=\frac{NP_\chi}{2a};\ \dot{P}_\chi=\{P_\chi,H\}=-\frac{2N\chi}{a}$$

Now one adopts the conformal time gauge $N=\dot{a}$ and the general solution of (6.56) in this gauge is

$$a(t)=(A+C)Cos(t)+(B+D)Sin(t); \tag{6.57}$$
$$\chi(t)=(A-C)Cos(t)+(B-D)Sin(t)$$

where the constraint $\mathfrak{H}\approx 0$ imposes the relation $AC+BD=0$.

Next look at a NC deformation by keeping the Hamiltonian with the same functional form as in (6.55) but now with NC variables

$$H=N\left[-\frac{P_{a_{nc}}^2}{4a_{nc}}+\frac{P_{\chi_{nc}}^2}{4a_{nc}}-a_{nc}+\frac{\chi_{nc}^2}{a_{nc}}\right] \tag{6.58}$$

where

$$\{a_{nc}, \chi_{nc}\} = \theta;\ \{a_{nc}, P_{a_{nc}}\} = 1 = \{\chi_{nc}, P_{\chi_{nc}}\};\ \{P_{a_{nc}}, P_{\chi_{nc}}\} = 0 \tag{6.59}$$

Now make the substitution

$$a_{nc} = a_c - \frac{\theta}{2}P_{\chi_c};\ \chi_{nc} = \chi_c + \frac{\theta}{2}P_{a_c};\ P_{a_c} = P_{a_{nc}},\ P_{\chi_c} = P_{\chi_{nc}} \tag{6.60}$$

Then the theory defined by (6.58)-(6.59) can be mapped to a theory where the metric and matter variables satisfy

$$\{a_c, \chi_c\} = 0 = \{P_{a_c}, P_{\chi_c}\};\ \{a_c, P_{a_c}\} = \{\chi_c, P_{\chi_c}\} = 1 \tag{6.61}$$

In the case where a_c, χ_c are taken as the preferred variables one has a commutative theory referred to as a theory realized in the C-frame (cf. [**399**]). When a_{nc} and χ_{nc} are used as constituents of the physical metric and matter field one refers to the NC frame (cf. [**77, 78, 79**]). Some work assumes the difference between the C and NC variables is negligible (cf. [**234**]) but as shown in [**79**] even in simple models the difference in behavior between these two types of variables can be appreciable; here one shows that the assumption of C or NC frame leads to dramatic differences in the analysis of universe history. Calculations are made for the classical situation in both sets of variables which we omit here.

Next comes the quantum version of the commutative universe model for the FRW universe with conformally coupled scalar field; this has been investigated on the basis of the WDW equation in e.g. [**108**] using Bohmian trajectories but there was there a restriction to the regime of small scale parameters and the wavefunctions there were different from those used here. One writes then $P_a = -\partial/\partial a$ and $P_\chi = -i\partial/\partial\chi$ to get from (6.55) the WDW equation

$$\left[-\frac{\partial^2}{\partial a^2} + \frac{\partial^2}{\partial\chi^2} + 4(a^2 - \chi^2)\right]\psi(a,\chi) = 0 \tag{6.62}$$

One can separate variables as in [**108, 552**] but here one chooses a different route more suitable for application in the NC situation. Thus write $a = \xi Cosh(\eta)$ and $\chi = \xi Sinh(\eta)$ to get

$$\left[\left(\frac{\partial^2}{\partial\xi^2} + \frac{1}{\xi}\frac{\partial}{\partial\xi} - \frac{1}{\xi^2}\frac{\partial^2}{\partial\eta^2}\right) - 4\xi^2\right]\psi(\xi,\eta) = 0 \tag{6.63}$$

Putting in $\psi = R(\xi)exp(i\alpha\eta)$ one obtains

$$\frac{\partial^2 R}{\partial\xi^2} + \frac{1}{\xi}\frac{\partial R}{\partial\xi} + \left(\frac{\alpha^2}{\xi^2} - 4\xi^2\right)R = 0 \tag{6.64}$$

A solution is $R(\xi) = AK_{i\alpha/2}(\xi^2) + BI_{i\alpha/2}(\xi^2)$ where K_ν and I_ν are Bessel functions of the second kind, A, B are constants, and α is a real number. The solution of WDW is then

$$\psi(\xi,\eta) = AK_{i\alpha/2}(\xi^2)e^{i\alpha\eta} + BI_{i\alpha/2}(\xi^2)e^{i\alpha\eta} \tag{6.65}$$

Such wavefunctions appear in the study of quantum wormholes (cf. [**230**]) and in quantum cosmology for the KS universe (cf. [**399, 892**]). One often discards the

I_ν solution (not always wisely) and uses

$$\psi(\xi,\eta)=\sum_\alpha A_\alpha K_{i\alpha/2}(\xi^2)e^{i\alpha\eta} \tag{6.66}$$

as a solution.

Now for the Bohmian approach one recalls from quantum information theory that the wavefunction has a nonphysical character (cf. [**758**]) and Bohmian theory should be in accord with this in some way. In the present picture the object of attention is the primordial quantum universe, characterized in the minisuperspace formalism by the configuration variables a and χ. Having fixed the ontological objects one must determine how they evolve in time and this is done with the aid of the wave function. Thus an evolution law is ascribed to point particles via

$$\dot{x}^i=\Re\left\{\frac{1}{m}\frac{[\psi^*(-i\hbar\partial_i)\psi]}{\psi^*\psi}\right\}=\frac{\nabla S}{m};\ i\hbar\partial_\psi=-\frac{\hbar^2}{2m}\nabla^2\psi+V\psi \tag{6.67}$$

where ψ is the wavefunction of the universe and S comes from $\psi=Aexp(iS)$. All phenomena governed by nonrelativistic QM follow from the analysis of this dynamical system. The expectation value of a physical quantity associated with a Hermitian operator $\hat{A}(\hat{x}^i,\hat{p}^i)$ can be computed in the Bohmian formalism via

$$\mathfrak{B}(\hat{A})=\Re\left\{\frac{[\psi^*\hat{A}(\hat{x}^i,-i\hbar\partial_i)\psi]}{\psi^*\psi}\right\}=A(x^i,t) \tag{6.68}$$

which represents the same quantity when seen from the Bohmian perspective (cf. [**471**]). In the context of nonrelativistic QM it can be shown from first principles that for an ensemble of particles obeying the evolution law (6.67) (first equation) the associated probability density is $\rho=|\psi|^2$ (cf. [**327**]). This is why computing the ensemble average of $A(x^i,t)$ via

$$\int d^3x\rho A(x^i,t)=\int d^3x\psi^*\hat{A}(\hat{x}^i,-i\hbar\partial_i)\psi=<\hat{A}>_t \tag{6.69}$$

leads to the same result as the standard operator formalism. Note that the law of motion in (6.67) can itself be obtained from (6.68) by associating $\dot{x}^i$ with the beable corresponding to the velocity operator, namely

$$\dot{x}^i=\mathfrak{B}(i[\hat{H},\hat{x}^i])=\nabla S/m \tag{6.70}$$

Again Bohmian mechanics does not give to probability a privileged role, but, as discussed in [**327**], probability is a derived concept arising from the laws of motion of point particles. In this sense the Bohmian approach is suited to an isolated system (such as the universe) but on the other hand there might be an ensemble of universes.

Now for quantum cosmology on the present context of a FRW universe with a conformally coupled scalar field. In the commutative case the resulting Bohmian minisuperspace formalism matches with the minisuperspace version of the Bohmian quantum gravity proposed in [**471**] and employed in [**108**]. From (6.70) one has in the gauge $N = a$

$$\dot{a} = \Re\left\{\frac{[\psi^*(-i\partial_a/2)\psi]}{\psi^*\psi}\right\} = -\frac{1}{2}\frac{\partial S}{\partial a};\ \dot{\chi} = \Re\left\{\frac{[\psi^*(-i\partial_\chi/2)\psi]}{\psi^*\psi}\right\} = \frac{1}{2}\frac{\partial S}{\partial \chi} \tag{6.71}$$

Changing into the (ξ, η) coordinates we obtain

$$\frac{d\xi}{dt} = -\frac{1}{2}\frac{\partial S(\xi,\eta)}{\partial \xi};\ \frac{d\eta}{dt} = \frac{1}{2\xi^2}\frac{\partial S(\xi,\eta)}{\partial \eta} \tag{6.72}$$

For a single Bessel function in (6.66) one has $\psi(\xi, \eta) = AK_{i\alpha/2}(\xi^2)exp(i\alpha\eta)$ where A is a constant. From $S = \alpha\eta$ the equations of motion in (6.72) reduce to

$$\frac{d\xi}{dt} = 0;\ \frac{d\eta}{dt} = \frac{\alpha}{2\xi^2} \Rightarrow \xi = \xi_0;\ \eta = \frac{\alpha}{2\xi_0^2}t + \eta_0 \tag{6.73}$$

leading to

$$a(t) = \xi_0 Cosh\left(\frac{\alpha}{2\xi_0^2} + \eta_0\right);\ \chi(t) = \xi_0 Sinh\left(\frac{\alpha}{2\xi_0^2} + \eta_0\right) \tag{6.74}$$

Quantum effects can therefore remove the cosmological singularity giving rise to bouncing universes. The case of a superposition of two Bessel functions of type $K_{i\nu}$ is also written out in part but the calculations become difficult.

For the NC quantum model one takes the quantum version of (6.59) as

$$[\hat{a}, \hat{\chi}] = i\theta;\ [\hat{a}, \hat{P}_a] = i;\ [\hat{\chi}, \hat{P}_\chi] = i;\ [\hat{P}_a, \hat{P}_\chi] = 0 \tag{6.75}$$

These commutation relations can be realized in terms of commutative functions by making use of star products as in (6.46). The commutative coordinates a_c, χ_c are called Weyl symbols as before and a WDW equation is

$$\left[\hat{P}^2_{a_c} - \hat{P}^2_{\chi_c}\right]\psi(a_c, \chi_c) + 4(a_c^2 - \chi_c^2) * \psi(a_c, \chi_c) = 0 \tag{6.76}$$

(obtained by Moyal deforming (6.62)). The resulting equations are simply operator versions of (6.60).

For the NC Bohmian version one departs from the C-frame and uses the beable mapping (6.68) to ascribe an evolution law to the canonical variables. In time gauge $N = a_{nc}$ the Hamiltonian (6.58) reduces to

$$h = \left[-\frac{P^2_{a_{nc}}}{4} + \frac{P^2_{\chi_{nc}}}{4} - a^2_{nc} + \chi^2_{nc}\right] \tag{6.77}$$

One can therefore use h to generate time dependence and obtain the Bohmian equations as

$$\frac{da_c}{dt} = \mathfrak{B}(i[\hat{h}, \hat{a}_c]) = -\frac{1}{2}(1-\theta^2)\frac{\partial S}{\partial a_c} + \theta\chi_c; \tag{6.78}$$

$$\frac{d\chi_c}{dt} = \mathfrak{B}(i[\hat{g}, \hat{\chi}_c]) = \frac{1}{2}(1-\theta^2)\frac{\partial S}{\partial \chi_c} + \theta a_c$$

The connection between the C and NC frame variables is established b applying the "beable" mapping to the operator equations based on (6.60), namely $\hat{a} = \hat{a}_c - (\theta/2)\hat{P}_{\chi_c}$, etc.; that is by defining $a \equiv \mathfrak{B}(\hat{a})$ and $\chi = \mathfrak{B}(\hat{\chi})$. Once the trajectories are determined in the C frame one can find their counterparts in the NC frame by evaluating the variables a and χ along the C-frame trajectories

$$a(t) = \mathfrak{B}(\hat{a})|_{\substack{a_c=a_c(t)\\ \chi_c=\chi_c(t)}} = a_c(t) - \frac{\theta}{2}\partial_{\chi_c} S[a_c(t), \chi_c(t)]; \tag{6.79}$$

$$\chi(t) = \mathfrak{B}(\hat{\chi})|_{\substack{a_c=a_c(t)\\ \chi_c=\chi_c(t)}} = \chi_c(t) + \frac{\theta}{2}\partial_{a_c} S[a_c(t), \chi_c(t)]$$

Now for an application to NC quantum cosmology use $P_{a_c} = -i\partial_{a_c}$ and $P_{\chi_c} = -i\partial_{\chi_c}$ with NC WDW from (6.76) written as

$$\left[\beta\left(-\frac{\partial^2}{\partial a_c^2} + \frac{\partial^2}{\partial \chi_c^2}\right) + 4(a_c^2 - \chi_c^2) + 4i\theta\left(\chi_c\frac{\partial}{\partial a_c} + a_c\frac{\partial}{\partial \chi_c}\right)\right]\psi(a_c, \chi_c) = 0 \tag{6.80}$$

where $\beta = 1 - \theta^2$. Separation of variables works, after writing $a_c = \xi Cosh(\eta)$ and $\chi_c = \xi Sinh(\eta)$, allowing (6.80) to be rewritten as

$$\left[\beta\left(\frac{\partial^2}{\partial \xi^2} + \frac{1}{\xi}\frac{\partial}{\partial \xi} - \frac{1}{\xi^2}\frac{\partial^2}{\partial \eta^2}\right) - 4i\theta\frac{\partial}{\partial \eta} - r\xi^2\right]\psi(\xi, \eta) = 0 \tag{6.81}$$

Using (6.78) one can write the Bohmian equations of motion as

$$\frac{d\xi}{dt} = -\frac{1}{2}(1-\theta^2)\frac{\partial S(\xi,\eta)}{\partial \xi}; \ \frac{d\eta}{dt} = \frac{1}{2\xi^2}(1-\theta^2)\frac{\partial S(\xi,\eta)}{\partial \eta} + \theta \tag{6.82}$$

(6.79) can be written in the new set of coordinates as

$$a_{nc}(t) = a_c(t) + \frac{\theta}{2}Sinh(\eta)\partial_\xi S[\xi(t), \eta(t)] - \frac{\theta}{2}\xi^{-1}Cosh(\eta)\partial_\eta S[\xi(t), \eta(t)]; \tag{6.83}$$

$$\chi_{nc}(t) = \chi_c(t) + \frac{\theta}{2}Cosh(\eta)\partial_\xi S[\xi(t), \eta(t)] - \frac{\theta}{2}\xi^{-1}Sinh(\eta)\partial_\eta S[\xi(t), \eta(t)]$$

The paper continues with extensive computations for a variety of situations and we hope to have captured the spirit of investigtion.

7. EXACT UNCERTAINTY AND GRAVITY

We go here to [**444**] (cf. Remarks 1.1.4 and 1.1.5 along with Section 3.1) and [**186, 187, 189, 203**] for background (for the original sources see e.g.[**446, 447, 448, 449, 450, 805, 806, 807**]). The theme here is that the exact uncertainty approach may be promoted to *the* fundamental element distinguishing quantum and classical mechanics. Nonclassical fluctuations are addded to the usual deterministic connection between the configuration and momentum properties of a physical system. Assuming that the uncertainty introduced to the momentum (i.e. the fluctuation strength) is fully determined by the uncertainty in the configuration via the configuration probability density one arrives at QM from CM. We remark that the quantum potential arises from variation of the Fisher metric with respect to the probability P and this is another significant feature relating the quantum potential to fluctuations and indirectly to the Bohmian formulation of QM (cf. [**186, 187, 189, 203**]). For a quick review of the particle situation let $H = (p^2/2m) + V(x)$ be the Hamiltonian for a spinless particle with SE $i\hbar\partial_t\psi = H(x, -i\hbar\nabla, t)\psi = -(\hbar^2/2m)\nabla^2\psi + V\psi$ and recall that the probability density P is specified as $|\psi|^2$. Thus in this canonical approach there is a lot of black magic while in the exact uncertainty approach one uses statistical concepts from the beginning and the wavefunction and SE are derived rather than postulated. Thus assume an ensemble picture from the beginning (due to uncertainty in the position) and assume that a fundamental position probability density P follows from an action principle involving

$$A = \int dt \left[\tilde{H} + \int dxP\frac{\partial S}{\partial t}\right];\ \frac{\partial P}{\partial t} = \frac{\delta\tilde{H}}{\delta S};\ \frac{\partial S}{\partial t} = -\frac{\delta\tilde{H}}{\delta P} \tag{7.1}$$

(no ψ is assumed here). One shows that conservation of probability requires $\tilde{H}$ is invariant under $S \to S + c$ and if $\tilde{H}$ has no explicit time dependence then its value is a conserved quantity corresponding to energy. As an example here consider the classical ensemble Hamiltonian

$$\tilde{H}_c[P,S] = \int dxP\left[\frac{|\nabla S|^2}{2m} + V\right] \tag{7.2}$$

Then as above

$$\frac{\partial P}{\partial t} + \nabla\cdot\left[P\frac{\nabla S}{m}\right] = 0;\ \frac{\partial S}{\partial t} + \frac{|\nabla S|^2}{2m} + V = 0 \tag{7.3}$$

This formalism based on an action principle for the position probability density successfully describes the motion of ensembles of classical particles; moreover it is considerably more general (see e.g. [**450**]). In particular the essential difference between classical and quantum ensembles becomes a matter of form, being characterized by a simple difference in the forms of the emsemble Hamiltonians $\tilde{H}_c$ and $\tilde{H}_q$.

Thus assume the physical momentum is given via (♣) $p = \nabla S + f$ where the fluctuation field f vanishes almost everywhere on average. This is not dissimilar from e.g. Nelson's mechanics where a Brownian motion is attached, or the approach of scale relativity (cf. [**186, 187, 188, 189, 203**] for an extensive treatment of such matters). One will see that such fluctuations introduce indeterminism at the level of individual particles but *not* at the ensemble level. Write now an overline to denote averaging over the fluctuations at a given position so $\overline{f} = 0$ and $\overline{p} = \nabla S$ by assumption and the classical ensemble energy is replaced by

$$<E> = \int dxP[(2m)^{-1}\overline{|\nabla S + f|^2} + V] = \tag{7.4}$$

$$= \int dxP[(2m)^{-1}(|\nabla S|^2 + 2\overline{f}\cdot\nabla S + \overline{f\cdot f}) + V] = \tilde{H}_c + \int dxP\frac{\overline{f\cdot f}}{2m}$$

Now one asks whether this modified classical ensemble can be subsumed within the general formalism above and this is OK proved that $\overline{f\cdot f}$ is determined by some function of P, S and their derivatives, i.e.

$$\overline{f\cdot f} = \alpha(x, P, S, \nabla P, \nabla S, \cdots) \tag{7.5}$$

In this case one can define a modified ensemble Hamiltonian

$$\tilde{H}_q = \tilde{H}_c + \int dxP\frac{\alpha(x,p,S,\nabla P,\nabla S,\cdots)}{2m} \tag{7.6}$$

The aim of the exact uncertainty approach is to fix the form of α uniquely and this is done by requiring first three generally desirable principles to be satisfied (causality, invariance, and independence), plus an exact uncertainty principle, and given this the resulting equations of motion are equivalent to the SE for a quantum ensemble of particles. This is covered at length in [**186, 205, 203**] and in [**447**] for example and slightly different versions are given here, following [**449**] for bosonic fields, in order to make a connection with qauntum gravity. The requirements are: (i) The modified ensemble Hamiltonian $\tilde{H}_q$ leads to causal equations of motion (so α cannot depend on second and higher derivatives of P and S) (ii) The respective fluctuation strengths for noninteracting uncorrelated ensembles are independent (thus $\overline{f_1\cdot f_1}$ and $\overline{f_2\cdot f_2}$ are independent of P_1 and P_2 respectively when $P(x) = P_1(x_2)P_2(x_2)$) (iii) The fluctuations transform correctly under linear canonical transformations (thus $f\to L^Tf$ for any invertible linear coordinate transformation $x\to L^{-1}x$). The fourth assumption is: (iv) The strength of the momentum fluctuations $\alpha=\overline{f\cdot f}$ is determined solely by the uncertainty in position - hence α can only depend on x, P and derivatives. It is shown in the references above that these four principles lead to the unique form

$$\tilde{H}_q[P,S] = \tilde{H}_c[P,S] + C\int dx\frac{\nabla P\cdot\nabla P}{2mP} \tag{7.7}$$

where C is a positive universal constant (i.e. having the same value for all ensembles). Moreover if one sets $\hbar = 2\sqrt{C}$ and $\psi=\sqrt{P}exp(iS/\hbar)$ the SE results as above.

Now for gravitational situations we have an ADM metric

$$ds^2 = -(N^2 - h^{ij}N_iN_j)dt^2 + 2N_idx^idt + h_{ij}dx^idx^j \tag{7.8}$$

where N and N_j refer to lapse and shifts with h_{ij} the spatial metric. Consider now the possibility that the configuration of the field is an inherently imprecise notion, hence requiring a probability functional $P[h_{ij}]$ for its description. Assume that the dynamics of the corresponding statistical ensemble are generated by an action principle $\delta A = 0$ where $A=\int dt[\tilde{H}+\int DhP(\partial S/\partial t)]$ analogous to (7.1). Here $\int Dh$ denotes a functional integral over configuration space and $\tilde{H}$ depends on $P[h_{ij}]$ and its conjugate functional $S[h_{ij}]$. The equations of motion are then

$$\frac{\partial P}{\partial t} = \frac{\Delta\tilde{H}}{\Delta S};\ \frac{\partial S}{\partial t} = -\frac{\Delta\tilde{H}}{\Delta P} \tag{7.9}$$

where $\Delta/\Delta F$ denotes the variational derivative with respect to the functional F (cf. [**449**] for details and Remark 5.1). A suitable "classical" ensemble Hamiltonian may be constructed from knowledge of the classical equations of motion for an individual field via (cf. [**449**])

$$\tilde{H}_c[P,S] = \int DhPH_0[h_{ij},\delta S/\delta h_{ij}] \tag{7.10}$$

where

$$H_0[h_{ij},\pi^{ij}] = \int dx \left[N\left(\frac{1}{2}G_{ijk\ell}\pi^{ij}\pi^{k\ell} + V(h_{ij})\right) - 2N_i\pi^{ij}_{\ |j}\right] \tag{7.11}$$

where V is the negative of twice the product of the 3-curvature scalar with $[det(h)]^{1/2}$ and $|j$ denotes the covariant 3-derivative (this is the single field Hamiltonian). For the ensemble Hamiltonian $\tilde{H}_c$ in (7.10) one has now

$$\frac{\partial P}{\partial t} + \int dx \frac{\delta}{\delta h_{ij}}(P\dot{h}_{ij}) = 0;\ \frac{\partial S}{\partial t} + H_0[h_{ij}, \delta S/\delta h_{ij}] = 0; \tag{7.12}$$

$$\dot{h}_{ij} = NG_{ijk\ell}\frac{\delta S}{\delta h_{k\ell}} - N_{i|j} - N_{j|i}$$

These equations of motion correspond to the conservation of probality with probability flow $\dot{h}_{ij}$ and the HJ equation for an individual gravitational field with configuration h_{ij} As is well known the lack of conjugate momenta for the lapse and shift components N and N_i places constraints on the classical equations of motion. In the ensemble formalism these constraints take the form (cf. [**449**])

$$\frac{\delta P}{\delta N} = \frac{\delta P}{\delta N_i} = \frac{\partial P}{\partial t} = 0;\ \left(\frac{\delta P}{\delta h_{ij}}\right)\Big|_{|j} = 0; \tag{7.13}$$

$$\frac{\delta S}{\delta N} = \frac{\delta S}{\delta N_i} = \frac{\partial S}{\partial t} = 0;\ \left(\frac{\delta S}{\delta h_{ij}}\right)\Big|_{|j} = 0$$

and this corresponds to invariance of the dynamics with respect to the choice of lapse and shift functions and initial time - and to the invariance of P and S under arbitrary spatial coordinate transformations. Applying these constraints to the above classical equations of motion yields for the Gaussian choice $N = 1,\ N_i = 0$ the reduced classical equations

$$\frac{\delta}{\delta h_{ij}}\left(PG_{ijk\ell}\frac{\delta S}{\delta h_{k\ell}}\right) = 0;\ \frac{1}{2}G_{ijk\ell}\frac{\delta S}{\delta h_{ij}}\frac{\delta S}{\delta h_{k\ell}} + V = 0 \tag{7.14}$$

Now the exact uncertainty approach can be adapted in a straightforward way to obtain a modified ensemble Hamiltonian that generates the quantum equations of motion. It is assumed first that the classical deterministic relation between the field configuration h_{ij} and its conjugate momentum density π^{ij} is relaxed to

$$\pi^{ij} = \frac{\delta S}{\delta h_{ij}} + f^{ij} \tag{7.15}$$

analogous to (♣) where here f^{ij} vanishes on average for all configurations. This adds a kinetic term to the average ensemble energy analogous to (7.4) with

$$\tilde{H}_q = <E> = \tilde{H}_x + \frac{1}{2}\int DhP\int dxNG_{ijk\ell}\overline{f^{ij}f^{k\ell}} \tag{7.16}$$

Note here that the term in (7.11) linear in the derivative of π^{ij} can be integrated by parts to give a term directly proportional to π^{ij}, which remains unchanged when the fluctuations are added and averaging is performed. Next one fixes the form of $\tilde{H}_q$ using the same principles of causality, independence, invariance and exact uncertainty used before (cf. [**449**] for details) leading to

$$\tilde{H}_q[P,S] = \tilde{H}_c[P,S] + \frac{C}{2}\int Dh \int dx N G_{ijk\ell}\frac{1}{P}\frac{\delta P}{\delta h_{ij}}\frac{\delta P}{\delta h_{k\ell}} \tag{7.17}$$

analogous to (7.7) where C is a positive constant with the same value for all fields. The corresponding modified equations of motion may be calculated via (7.9) and the constraints in (7.13) applied to obtain reduced equations analogous to (7.14). If one now *defines* $\hbar = 2\sqrt{C}$ and $\Psi[h_{ij}] = \sqrt{P}exp(iS/\hbar)$ these reduced equations can be rewritten in the form (cf. [**449**])

$$\left[-\frac{\hbar^2}{2}\frac{\delta}{\delta h_{ij}}G_{ijk\ell}\frac{\delta}{\delta h_{k\ell}} + V\right]\Psi = 0 \tag{7.18}$$

This is of course the WDW equation for quantum gravity. Note also that the constraints in (7.13) may be rewritten in terms of the wavefunctional Ψ as

$$\frac{\delta\Psi}{\delta N} = \frac{\delta\Psi}{\delta N_i} = \frac{\partial\Psi}{\partial t} = 0;\ \left.\left(\frac{\delta\Psi}{\delta h_{ij}}\right)\right|_{|j} = 0 \tag{7.19}$$

An interesting aspect of the WDW equation in (7.19) is that it has been obtained with a particular operator ordering, namely the supermetric $G_{ijk\ell}$ is sandwiched between the two functional derivatives. This constrasts with the canonical approach which is unable to specify a unique ordering (cf. [**444**] for references). It should be noted that different orderings can lead to different physical predictions and hence the exact uncertainty approach is able to remove ambiguity in this respect. An analogous removal of ambiguity is obtained for quantum particles having a position dependent mass (cf. [**449**]) with the exact uncertainty approach specifying, via (7.7), the unique sandwich ordering

$$i\hbar\frac{\partial\psi}{\partial t} = -\frac{\hbar^2}{2}\nabla\cdot\frac{1}{m}\nabla\psi + V\psi \tag{7.20}$$

for the SE.

Summarizing now it follows that physical ensembles are described by a probability density on configuration space (P), a corresponding conjugate quantity (S), and an ensemble Hamiltonian $\tilde{H}[P,S]$. The transition from classical ensembles to quantum ensembles then follows as a consequence of the addition of nonclassical fluctuations, under the assumption that the fluctuation uncertainty is fully determined by the configuration uncertainty. In contrast to the canonical approach the SE and WDW equations are derived, rather than postulated, and the probability connection $P = |\psi|^2$ is a simple consequence of the definition of ψ in terms of P and S. Planck's constant appears as a consequence of a derived universal scale for the nonclassical momentum fluctuations instead of being an unexplained constant in the canonical approach. A (nonserious) limitation appear in that the momentum of a classical ensemble must contribute quadratically to the ensemble energy for the exact uncertainty approach to go through whereas the canonical approach is indifferent to this. We refer to [**444, 449**] for more details and discussion.

REMARK 4.7.1. We extract here from the appendix to [**449**] for some notation and constructions involving functional derivatives. One considers functionals $F[f]$ with

$$\delta F = F[f+\delta f] - F[f] = \int dx \frac{\delta F}{\delta f_x}\delta f_x \tag{7.21}$$

Here $f \sim f(x)$ refers to fields with real or complex values. Thus the functional derivative is a field density $\delta F/\delta f$ having the value $\delta F/\delta f_x$ at position x. For curved spaces one would need a more elaborate notation, and volume element, etc. The choice $F[f] = f_{x'}$ in (7.21) yields $\delta f_{x'}/\delta f_x = \delta(x-x')$ and if the field depends on a parameter t then writing $\delta f_x = f_x(t+\delta t) - f_x(t)$ one arrives at (♠) $dF/dt = \partial_t F + \int dx(\delta F/\delta f_x)\partial_t f_x$ for the rate of change of F with respect t. Functional integrals correspond to integration of functionals over the vector space of physical fields (or equivalence classes thereof) and the only property one requires for the present discussion is the existence of a measure Df on this vector space which is translation invariant (i.e. $\int Df \equiv \int Df'$ for $f' = f + h$). This property implies e.g.

$$\int Df \frac{\delta F}{\delta f} = 0 \;\; if \;\; \int Df F[f] < \infty \tag{7.22}$$

This follows immediately via

$$0 = \int Df(F[f+\delta f] - F[f]) = \int dx \delta f_x \left(\int Df \frac{\delta F}{\delta f_x}\right) \tag{7.23}$$

In particular if $F[f]$ has a finite expectation value with respect to some probability density functional $P[f]$ then (7.22) gives an integration by parts formula

$$\int Df P(\delta F/\delta f) = -\int Df(\delta P/\delta f)F \tag{7.24}$$

Moreover again via (7.22) the total probability $\int Df P$ is conserved for any probability flow satisfying a continuity condition

$$\frac{\partial P}{\partial t} + \int dx \frac{\delta}{\delta f_x}[PV_x] = 0 \tag{7.25}$$

provided that the average flow rate $< V_x >$ is finite. Next consider a functional integral of the form $I[F] = \int Df\xi(F, \delta F/\delta f)$; then variation of $I[F]$ with respect to F gives to first order

$$\Delta I = I[F+\Delta F] - I[F] = \tag{7.26}$$

$$= \int Df \left\{(\partial\xi/\partial F)\Delta F + \int dx[\partial\xi/\partial(\delta F/\delta f_x)][\delta(\Delta F)/\delta f_x]\right\} =$$

$$= \int Df \left\{(\partial\xi/\partial F) - \int dx \frac{\delta}{\delta f_x}[\partial\xi/\partial(\delta F/\delta f_x)]\right\}\Delta F +$$

$$+ \int dx \int Df \frac{\delta}{\delta f_x}\{[\partial\xi/\partial(\delta F/\delta f_x)]\Delta F\}$$

One assumes here that Df is translation invariant and hence if the functional integral of the expression in curly brackets in the last term of (7.26) is finite the term will vanish, yielding the result $\Delta I = \int Df(\Delta I/\Delta F)\Delta F$ where $\Delta I/\Delta F =$

$\partial_F\xi - \int dx(\delta/\delta f_x)[\partial\xi/\partial(\delta F/\delta f_x)]$.

REMARK 4.7.2. Regarding the general HJ formulation of classical field theory we go to Appendix B of [**449**]. Two classical fields f, g are canonically conjugate if there is a Hamiltonian functional $H[f,g,t]$ such that

$$\frac{\partial f}{\partial t} = \frac{\delta H}{\delta g};\ \frac{\partial g}{\partial t} = -\frac{\delta H}{\delta f} \tag{7.27}$$

These equations follow from the action principle $\delta A = 0$ with $A = \int dt[-H + \int dx g_x(\partial f_x/\partial t))]$. The rate of change of an arbitrary functional $G[f,g,t]$ follows from (7.27) and (♠) as

$$\frac{dG}{dt} = \frac{\partial G}{\partial t} + \int dx\left(\frac{\delta G}{\delta f_x}\frac{\delta H}{\delta g_x} - \frac{\delta G}{\delta g_x}\frac{\delta H}{\delta f_x}\right) = \frac{\partial G}{\partial t} + \{G,H\} \tag{7.28}$$

A canonical transformation maps f, g, H to f', g', H' such that the equations of motion for the latter retain the canonical form of (7.27). Equating the variations of the corresponding actions A and A' t zero it follows that physical trajectories must satisfy

$$-H + \int dx g_x(\partial f_x/\partial t) = -H' + \int dx g'_x(\partial f'_x/\partial t) + (dF/dt) \tag{7.29}$$

for some generating functional F. Now any two of the fields f, g, f', g' determine the remaining two fields for a given canonical transformation; choosing f, g' as independent and defining the new generating functional $G[f,g',t] = F + \int dx f'_x g'_x$ gives then via (♠)

$$H' = H + \frac{\partial G}{\partial t} + \int dx\left[\frac{\partial f_x}{\partial t}\left(\frac{\delta G}{\delta f_x} - g_x\right) + \frac{\partial g'_x}{\partial t}\left(\frac{\delta G}{\delta g'_x} - f'_x\right)\right] \tag{7.30}$$

The terms in round brackets therefore vanish identically yielding the generating realtions

$$H' = H + \partial G/\partial t;\ g = \delta G/\delta f;\ f' = \delta G/\delta g' \tag{7.31}$$

A canonical transformation is thus completely specified by the associated generating function. To obtain the HJ formulation of the equations of motion consider a canonical transformation to fields f', g' which are time independent. From (7.27) one may choose the corresponding Hamiltonian $H' = 0$ without loss of generality and hence from (7.31) the momentum density and the associated generating functional S are specified by

$$g = \frac{\delta S}{\delta f};\ \frac{\partial S}{\partial t} + H[f, \delta S/\delta f, t] = 0 \tag{7.32}$$

the latter being the desired HJ equation. Solving this equation for S is equivalent to solving (7.27) for f and g.

Note that along a physical trajectory one has $g' = \textit{constant}$ and hence from (♠) and (7.32)

$$\frac{dS}{dt} = \frac{\partial S}{\partial t} + \int dx \frac{\delta S}{\delta f_x}\frac{\partial f_x}{\partial t} = -H + \int dx g_x \frac{\partial f_x}{\partial t} = \frac{dA}{dt} \tag{7.33}$$

Thus the HJ functional S is equal to the action functional A up to an additive constant. This relation underlies the connection between the derivation of the HJ equaiton from a particular type of canonical transformation as above and the derivation from a particular type of variation of the action. The HJ formulation has the feature that once S is specified the momentum density is determined by the relation $g = \delta S/\delta f$, i.e. it is a functional of f. Thus unlike the Hamiltonian formulation of (7.27) an ensemble of fields is specified by a probability density functional $P[f]$, not by a phase space density functional $\rho[f, g]$. In either case the equation of motion for the probability density corresponds to the conservation of probability, i.e. to a continuity equation as in (7.25). For example in the Hamiltonian formulation the associated continuity equation for $\rho[f, g]$ is

$$\frac{\partial \rho}{\partial t} + \int dx\{(\delta/\delta f_x)[\rho(\partial f_x/\partial t)] + (\delta/\delta g_x)[\rho(\partial g_x/\partial t)]\} = 0 \tag{7.34}$$

which reduces to the Liuville equation $\partial_t \rho = \{H, \rho\}$ via (7.27). Similarly in the HJ formulation the rate of change of f follows from (7.27) and (7.32) as

$$V_x[f] = \frac{\partial f_x}{\partial t} = \left(\frac{\delta H}{\delta g_x}\right)\Big|_{g=\delta S/\delta f} \tag{7.35}$$

and hence the associated continuity equation for an ensemble of fields described by $P[f]$ follows as in (7.25) to be

$$\frac{\partial P}{\partial t} + \int dx \frac{\delta}{\delta f_x}\left[P\frac{\delta H}{\delta g_x}\Big|_{g=\delta S/\delta f}\right] \tag{7.36}$$

Everything generalizes naturally for multicomponent fields.

Given the background in Remarks 4.7.1 and 4.7.2 the development in [**449**] is worth sketching in connection with general bosonic field calculations. Thus one looks at the HJ formalism which provides a straightforward mechanism for adding momentum fluctuations to an ensemble of fields. First the equation of motion for an individual classical field is given by the HJ equation (•) $\partial_t S + H[f, \delta S/\delta f, t] = 0$ where $S[f]$ denotes the HJ functional. The momentum density associated with the field f is $g = \delta S/\delta f$ and hence S is called a momentum potential. Next the description of an ensemble of such fields further requires a probability density functional $P[f]$ whose equation of motion corresponds to conservation of probability, i.e. to the continuity equation (cf. (7.36))

$$\frac{\partial P}{\partial t} + \sum_a \int dx \frac{\delta}{\delta f_x^a}\left(P\frac{\delta H}{\delta g_x^a}\Big|_{g=\delta S/\delta f}\right) = 0 \tag{7.37}$$

Equations (7.37) and (•) describe the motion of the ensemble completely via P and S and this can be written in the Hamiltonian form

$$\frac{\partial P}{\partial t} = \frac{\Delta \tilde{H}}{\Delta S};\ \frac{\partial S}{\partial t} = -\frac{\Delta \tilde{H}}{\Delta P};\ \tilde{H}[P, S, t] = < H > = \int Df P H[f, (\delta S/\delta f), t] \tag{7.38}$$

(cf. Remark 4.7.1). The functional integral $\tilde{H}$ in (7.38) will be referred to then as the ensemble Hamiltonian and in analogy to (7.27) P and S may be regarded as canonically conjugate functionals. Note from (7.38) that $\tilde{H}$ typically corresponds to the mean energy of the ensemble; moreover (7.38) follows from the action principle $\Delta\tilde{A} = 0$ with action $\tilde{A} = \int dt[-\tilde{H} + \int DfS(\partial P/\partial t)]$. In the following one specializes to ensembles for which the associated Hamiltonian is quadratic in the momentum field density, i.e. of the form

$$H[f,g,t] = \sum_{a,b}\int dxK_x^{ab}[f]g_x^a g_x^b + V[f] \tag{7.39}$$

Here $K_x^{ab} = K_x^{ba}$ is a kinetic factor coupling components of the momentum density and $V[f]$ is a potential energy functional. The corresponding ensemble Hamiltonian is given via (7.38) and one notes that cross terms of the form $g_x^a g_{x'}^b$ with $x \neq x'$ are not permitted in local field theories and hence are not considered here.

The approach of [**449**] to obtain a quantum ensemble of fields is now simply to add nonclassical fluctuations to the momentum density with the magnitude of the fluctuations derermined by the uncertainty in the field. This leads to equations of motion equivalent to those of a bosonic field with the advantage of a unique operator ordering for the associated SE. Note here also the analogy with adding a quantum potential to the HJ equation in the Bohmian theory. Thus suppose now that $\delta S/\delta f$ is in fact an average momentum density associated with the field f in the sense that the true momentum density is given by

$$g = \frac{\delta S}{\delta f} + N \tag{7.40}$$

where N is a fluctuation field that vanishes on the average for any given f. The meaning of S becomes then that of being an average momentum potential. No specific underlying model for N is assumed or necessary; one may in fact interpret the source of the fluctuations as the field uncertainty itself. The main effect of the fluctuation field is to remove any deterministic connection between f and g. Since the momentum fluctuations my conceivably depend on the field f the average over such fluctuations for a given quantity $A[f,N]$ will be denoted by $\overline{A}[f]$ and the average over fluctuations and the field by $< A >$. Thus $\overline{N} = 0$ by assumption and in general $< A >= \int DfP[f]\overline{A}[f]$. Assuming a quadratic dependence on momentum as in (7.39) it follows that when the fluctuations are significant the classical ensemble Hamiltonian $\tilde{H} =< H >$ in (7.38) should be replaced by

$$\tilde{H}' =< H[f,(\delta S/\delta f) + N,t] >= \tilde{H} + \sum_{a,b}\int Df\int dxPK_x^{ab}\overline{N_x^a N_x^b} = \tag{7.41}$$

$$= \sum_{a,b}\int Df\int dxPK_x^{ab}\overline{[(\delta S/\delta f_x^a) + N_x^a][(\delta S/\delta f_x^b) + N_x^b]} + < V >$$

Thus the momentum fluctuations lead to an additional nonclassical term in the ensemble Hamiltonian specified via the covariance matrix

$$[Cov_x(N)]^{ab} = \overline{N_x^a N_x^b} \tag{7.42}$$

This covariance matrix is uniquely determined (up to a multiplicative constant) by the following four assumptions:

(1) Causality: $\tilde{H}'$ is an ensemble Hamiltonian for the canonical conjugate functionals P and S which yield causal equations of motion. Thus no higher than first order functional derivatives can appear in the additional term in (7.41) which implies

$$Cov_x(N) = \alpha\left(P, \frac{\delta P}{\delta f_x}, S, \frac{\delta S}{\delta f_x}, f_x, t\right)$$

for some symmetric matrix function α. Note the fourth assumption below removes the possibility of dependence here on auxillary fields and functionals.

(2) Independence: If the ensemble has two independent noninteracting subensembles 1 and 2 with a factorisable probability density functional $P[f^1, f^2] = P_1[f^1]P_2[f^2]$ then any dependence of the corresponding N^1, N^2 on P only enters via P_1, P_2 in the form

$$Cov_x(N^1)|_{P_1P_2} = Cov_x(N^1)|_{P_1};\ Cov_x(N^2)|_{P_1P_2} = Cov_x(N^2)|_{P_2}$$

This implies that the ensemble Hamiltonian $\tilde{H}'$ in (7.41) is additive for independent noninteracting ensembles (as is the corresponding action $\tilde{A}'$).

(3) Invariance: The covariance matrix transforms correctly under linear canonical transormations of the field components. Thus $f \to \Lambda^{-1}f,\ g \to \Lambda^T g$ is a canonical transformation for any invertible matrix Λ preserving the quadratic form of H in (7.39) and leaving the momentum potential S invariant (since $\delta/\delta f \to \Lambda^T\delta/\delta f$) and thus from (7.40) $N \to \Lambda^T N$ so $Cov_x(N) \to \Lambda^T cov_x(N)\Lambda$ for $f \to \Lambda^{-1}f$ is required. For single component fields this reduces to a scaling relation for the variance of the fluctuations at each point x.

(4) Exact uncertainty: The uncertanty of the momentum density fluctuations, as characterized by the covariance matrix, is specified by the field uncertainty and hence by the probability density functional P; hence $Cov_x(N)$ cannot depend on S or explicitly on t.

One proves then in [**449**]

THEOREM 7.1. Under the above four assumptions one has

$$\overline{N_x^a N_x^b} = \frac{C}{P^2}\frac{\delta P}{\delta f_x^a}\frac{\delta P}{\delta f_x^b} \tag{7.43}$$

where C is a positive universal constant.

COROLLARY 7.1. The equations of motion corresponding to the ensemble Hamiltonian $\tilde{H}'$ can be expressed in the form

$$i\hbar\frac{\partial\Psi}{\partial t} = H\left[f, -i\hbar\frac{\delta}{\delta f}, t\right]\Psi = -\hbar^2\left(\sum_{a,b}\int dx\frac{\delta}{\delta f_x^a}K_x^{ab}[f]\frac{\delta}{\delta f_x^b}\right)\Psi + V[f]\Psi$$

where $\hbar = 2\sqrt{C}$ and $\Psi = \sqrt{P}exp(iS/\hbar)$.

One notes that the corollary specifies a unique operator ordering for the functional derivative operators. The proofs are given below following [**449**] and this is substantially different from (and stronger than) proofs of analogous theorems for quantum particles in [**447**].

PROOF OF THEOREM AND COROLLARY:

From the causality and exact uncertainty assumptions one has $Cov_x(N) = \alpha(P, (\delta P/\delta f_x), f_x)$. Next to avoid issues of regularisation it is convenient to consider a position dependent canonical transformation $f_x \to \Lambda_x^{-1} f_x$ such that $A[\Lambda] = exp[\int dxlog(|det(\Lambda_x)|)] < \infty$. Then the probability density functional P and the measure Df transform as $P \to AP$ and $Df \to A^{-1}Df$ respectively (for conservation of probability) and the invariance assumption requires

$$\alpha(AP, A\Lambda_x^T u, \Lambda_x^{-1} w) \equiv \Lambda_x^T \alpha(P, u, w)\Lambda_x \tag{7.44}$$

where u^a, w^a denote respectively $\delta P/\delta f_x^a$, f_x^a for a given value of x. This must hold for A and Λ_x independently so choosing Λ_x to be the identity matrix at some point x, one has $\alpha(AP, Au, w) = \alpha(P, u, w)$ for all A, which implies that α can involve P only via the combination $v = u/P$. Consequently

$$\alpha(\Lambda^T v, \Lambda^{-1} w) = \Lambda^T \alpha(v, w)\Lambda \tag{7.45}$$

This equation is linear and invariant under multiplication of α by any function of the scalar $J = v^T w$. Moreover one checks that if σ and τ are solutions then so are $\sigma\tau^{-1}\sigma$ and $\tau\sigma^{-1}\tau$. Choosing the two independent solutions $\sigma = vv^T$ and $\tau = (ww^T)^{-1}$ it follows that the general solution has the form

$$\alpha(v, w) = \beta(J)vv^T + \gamma(J)(ww^T)^{-1} \tag{7.46}$$

for arbitrary functions β and γ. Now for $P = P_1P_2$ one has $v = (v_1, v_2)$ and $w = (w_1, w_2)$ so the independence assumption reduces to the requirements

$$\beta(J_1 + J_2)v_1v_1^T + \gamma(J_1 + J_2)(w_1w_1^T)^{-1} = \beta_1(J_1)v_1v_1^T + \gamma_1(J_1)(w_1w_1^T)^{-1}; \tag{7.47}$$

$$\beta(J_1 + J_2)v_2v_2^T + \gamma(J_1 + J_2)(w_2w_2^T)^{-1} = \beta_2(J_2)v_2v_2^T + \gamma_2(J_2)(w_2w_2^T)^{-1}$$

Thus $\beta = \beta_1 = \beta_2 = C$ and $\gamma = \gamma_1 = \gamma_2 = D$ for universal C and D yielding the general form

$$[cov_x(N)]^{ab} = C\frac{1}{P^2}\frac{\delta P}{\delta f_x^a}\frac{\delta P}{\delta f_x^b} + DW_x^{ab}[f] \tag{7.48}$$

where $W_x[f]$ denotes the inverse of the matrix with ab-coefficient $f_x^a f_x^b$. Since $W_x[f]$ is purely a functional of f it merely contributes a classical additive potential term to the ensemble Hamiltonian of (7.41); thus it has no nonclassical role and can be absorbed directly into the classical potential $< V >$. In fact for fields with more than one component this term is ill-defined and can discarded on physical grounds; consequently one can take $D = 0$ without loss of generality. Positivity of C follows from positivity of the covariance matrix and the theorem is proved.

For the corollary one notes first that the equations of motion corresponding to the ensemble Hamiltonian $\tilde{H}'$ follow via the theorem and (7.38) as first: The continuity equation (7.37) as before (since the additional term does not depend on S) and following (7.39) this takes the form

$$\frac{\partial P}{\partial t} + 2\sum_{a,b}\int dx \frac{\delta}{\delta f_x^a}\left(PK_x^{ab}\frac{\delta S}{\delta f_x^b}\right) = 0 \tag{7.49}$$

and second: The modified HJ equation

$$\frac{\partial S}{\partial t} = -\frac{\Delta \tilde{H}'}{\Delta P} = -H[f,(\delta S/\delta f),t] - \frac{\Delta(\tilde{H}' - \tilde{H})}{\Delta P} \tag{7.50}$$

Calculating the last term via (7.43) and Remark 5.1 this simplifies to

$$\frac{\partial S}{\partial t} + H[f,(\delta S/\delta f),t] - 4CP^{-1/2}\sum_{a,b}\int dx\left(K_x^{ab}\frac{\delta^2 P^{1/2}}{\delta f_x^a \delta f_x^b} + \frac{\delta K_x^{ab}}{\delta f_x^a}\frac{\delta P^{1/2}}{\delta f_x^b}\right) = 0 \tag{7.51}$$

Now writing $\Psi = P^{1/2}exp(iS/\hbar)$, multiplying each side of the equation in the corollary by Ψ^{-1}, and expanding, one obtains a complex equation for P and S. The imaginary part is just the continuity equation (7.49) and the real part is the modified HJ equation (7.50) provided $C = \hbar^2/4$.

EXAMPLE 7.1. This formulation is applied to gravitational fields for example to obtain a version of the WDW equation as in (7.18). Thus write (with some repetition from before)

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu = -(N^2 - \mathbf{N}\cdot\mathbf{N})dt^2 + 2N_i dx^i dt + h_{ij}dx^i dx^j \tag{7.52}$$

as before and recall that the Einstein field equations follow from the Hamiltonian functional

$$H[h,\pi,N,\mathbf{N}] = \int dx N\mathfrak{H}_G[h,\pi] - 2\int dx N_i \pi^{ij}{}_{|j} \tag{7.53}$$

where $\pi = (\pi^{ij})$ is the momentum density conjugate to h and $|j$ denotes the covariant 3-derivative. Further

$$\mathfrak{H}_G = (1/2)G_{ijk\ell}[h]\pi^{ij}\pi^{k\ell} - 2\,{}^3R[h](det(h))^{1/2} \tag{7.54}$$

Here ${}^3R$ is the scalar curvature corresponding to h and

$$G_{ijk\ell}[h] = (h_{ik}h_{j\ell} + h_{i\ell}h_{jk} - h_{ij}h_{k\ell})(det(h))^{-1/2} \tag{7.55}$$

The Hamiltonian functional corresponds to the standard Lagrangian given via $L = \int dx(-det(g))^{1/2}R[g]$ where the momenta π^0 and π^i conjugate to N and N_i vanish identically. However the lack of dependence of H on π^0 and π^i is consistently maintained only if the rates of change of these momenta also vanish, i.e. (noting (7.27), only if the constraints

$$\frac{\delta H}{\delta N} = \mathfrak{H}_G = 0;\ \frac{\delta H}{\delta N_i} = -2\pi^{ij}{}_{|j} = 0 \tag{7.56}$$

are satisfied. Thus the dynamics of the field are independent of N and N_i so that these functions may be fixed arbitrarily. Moreover these constraints immediately yield $H = 0$ in (7.53) and hence the system is static with no explicit time dependence. It follows that in the HJ formulation of the equations of motion the momentum potential S is independent of N, $\mathbf{N}$, and t. Noting that $\pi = \delta S/\delta h$ in this formulation (7.56) yields the corresponding constraints

$$\frac{\delta S}{\delta N} = \frac{\delta S}{\delta N_i} = \frac{\partial S}{\partial t} = 0; \ \left(\frac{\delta S}{\delta h_{ij}}\right)\bigg|_{|j} = 0 \tag{7.57}$$

As shown in [**759**] a given functional $F[h]$ is invariant under spatial coordinae transformations if and only if $(\delta F/\delta h_{ij})|_{|j} = 0$ and hence the fourth constraint in (7.57) is equivalent to the invariance of S under such transformations. This constraint implies moreover that the second term in (7.53) may be dropped from the Hamiltonian yielding the reduced Hamiltonian

$$H_G[h, \pi, N] = \int dx N \mathfrak{H}_G[h, \pi] \tag{7.58}$$

For an ensemble of classical gravitational fields the independence of the dynamics with respect to (N, $\mathbf{N}$, t) implies that members of the ensemble are distinguishable only by their corresponding 3-metric h. Moreover it is natural to impose the additional geometric requirement that the ensemble is invariant under spatial coordinate transformations. Consequently one has constraints

$$\frac{\delta P}{\delta N} = \frac{\delta P}{\delta N_i} = \frac{\partial P}{\partial t} = 0; \ \left(\frac{\delta P}{\delta h_{ij}}\right)\bigg|_{|j} = 0 \tag{7.59}$$

The first two constraints imply that ensemble averages only involve integration over h_{ij}.

Now in view of (7.54) the Hamiltonian H_G in (7.58) has the quadratic form of (7.39) so the exact uncertainty approach is applicable and leads immediately to the SE

$$i\hbar\frac{\partial \Psi}{\partial t} = \int dx N \mathfrak{H}_G[h, -i\hbar(\delta/\delta h)]\Psi \tag{7.60}$$

for a quantum ensemble of gravitational fields. One follows the guiding principle used before that all constraints imposed on the classical ensemble should be carried over to the corresponding constraints on the quantum ensemble. Thus from (7.57) and (7.59) one requires that P and S and hence Ψ in the equation of the Corollary above are independent of N, $\mathbf{N}$, and t as well as invariant under spatial transformations. Thus

$$\frac{\delta \Psi}{\delta N} = \frac{\delta \Psi}{\delta N_i} = \frac{\partial \Psi}{\partial t} = 0; \ \left(\frac{\delta \Psi}{\delta h_{ij}}\right)\bigg|_{|j} = 0 \tag{7.61}$$

Applying the first and third of these constraints to (7.60) gives then, via (7.54), the reduced SE

(7.62)

$$H_G[h, -i\hbar(\delta/\delta h)]\Psi = (-\hbar^2/2)\frac{\delta}{\delta h_{ij}}G_{ijk\ell}[h]\frac{\delta}{\delta h_{k\ell}}\Psi - 2\,{}^3R[h](det(h))^{1/2}\Psi = 0$$

which is again the WDW equation.

CHAPTER 5

FLUCTUATIONS AND GEOMETRY

1. THE ZERO POINT FIELD

The zero point field (ZPF) arising from the quantum vacuum is still a contentious idea and we only make a few remarks following [**18, 144, 145, 148, 183, 251, 252, 253, 293, 310, 321, 438, 439, 440, 441, 442, 487, 488, 490, 606, 650, 651, 753, 754, 791, 792, 793, 804, 813, 823, 825, 824, 955, 968, 969**] (see also [**136, 377, 378, 379, 1006**] for stochastic spacetime and gravity fluctuations). It is to be noted that a certain amount of research on ZPF is motivated (and funded) by the desire to extract energy from the vacuum for "space travel" and in this direction one is referred to publications of the CIPA (see http://www.calphysics.org/sci.html) where a number of papers discussed here originate or are referenced. In a sense the quest here seems also to be an effort to really understand the equation $E = mc^2$. There is of course some firm physical evidence for forces induced by quantum fluctuations via the Casimir effect for example and it is very stimulating to see so much speculation now in the literature about questions of mass, inertia, Zitterbewegung, and the vacuum. We extract here first from [**438**] (cf. also [**441**]). No attempt is made here to be complete; the quantum vacuum is a relatively hot topic and there are many unsolved matters (for survey material see e.g. [**655, 651, 753, 968, 969**]). There are apparently at least two main views on the origin of the EM ZPF as embodied in QED and Stochastic electrodynamics (SED). QED is "standard" physics and the arguments go as follows. The Heisenberg uncertainty principle sets a fundamental limit on the precision wieh thich conjugate quantities are allowed to be determined. Thus $\Delta x\Delta p \geq \hbar/2$ and $\Delta E\Delta t \geq \hbar/2$. It is standard to work via harmonic oscillators (see below) and there are two non-classical results for a quantized harmonic oscillator. First the energy levels are discrete; one can add energy but only in units of $\hbar\nu$ where ν is a frequency. The second stems from the fact that if an oscillator were able to come completely to rest Δx would be zero and this would violate $\Delta x\Delta p \geq \hbar/2$. Hence there is a minimum energy of $\hbar\nu/2$ and the oscillator can only take on values $E = (n + 1/2)\hbar\nu$ which can never be zero. The argument is then made that the EM field is analogous to a mechanical harmonic oscillator since the fields $\vec{E}$ and $\vec{B}$ are modes of oscillating plane waves with minimum energy $\hbar\nu/2$. The density of modes between ν and $\nu + d\nu$ is given by the density of states function $N_\nu d\nu = (8\pi\nu^2/c^3)d\nu$. Each state has a minimum $\hbar\nu/2$ of energy and thus the ZPF spectral density function is

$$\rho(\nu)d\nu = (8\pi\nu^2/c^3)(\hbar\nu/2)d\nu \quad (1.1)$$

It is instructive to compare this with the blackbody radiation

$$\rho(\nu,T)d\nu = \frac{8\pi\nu^2}{c^3}\left(\frac{\hbar\nu}{e^{\hbar\nu/kT}-1}+\frac{\hbar\nu}{2}\right)d\nu \tag{1.2}$$

If one takes away all the thermal energy ($T \to 0$) what remains is the ZPF term. It is traditionally assumed in QM that the ZPF can be ignored or subtracted away. In SED is is assumed that the ZPF is as real as any other EM field and just came with the universe. In this spirit the Heisenberg uncertainty relations for example are not a result of quantum laws but a consequence of ZPF. Philosophically one could ask whether or not the universe is classical with ZPF (as envisioned at times by Planck, Einstein, and Nerst and later by Nelson et al) or has two sets of laws, quantum and classical.

It was shown in the 1970's that a Planck like component of the ZPF will arise in a uniformly accelerated coordinate system having constant proper acceleration a with what amounts to an effective temperature $T_a = \hbar a/2\pi ck$ (cf. [**293**]). More precisely one says that an observer who accelerates in the conventional quantum vacuum of Minkowski space will perceive a bath of radiation, while an inertial observer of course perceives nothing. This is a quantum phenomenon and the temperature is negligible for most accelerations, becoming significant only in extremely large gravitational fields. Thus for the case of no true external thermal radiation ($T = 0$), but including this acceleration effect T_a, equation (1.1) becomes

$$\rho(\nu,T_a)d\nu = \frac{8\pi\nu^2}{c^3}\left[1+\left(\frac{a}{2\pi c\nu}\right)^2\right]\left[\frac{\hbar\nu}{2}+\frac{\hbar\nu}{e^{\hbar\nu/kT_a}-1}\right]d\nu \tag{1.3}$$

(the pseudo-Planckian component at the end is generally very small).

There have been (at least) two approaches demonstrating how a reaction force proportional to acceleration ($\vec{f}_r = -m_{ZP}\vec{a}$) arises out of properties of the ZPF. One, called HRP and based on the first paper in [**438**], identified the Lorentz force arising from the stochastically averaged magnetic component of the ZPF, namely $<\vec{B}^{ZP}>$, as the basis of $\vec{f}_r$. The second, called RH after [**823**], considers only the relativistic transformations of the ZPF itself to an accelerated frame, leading to a nonzero stochastically averaged Poynting vector $(c/4\pi) < \vec{E}^{ZP} \times \vec{B}^{ZP} >$ which leads immediately to a nonzero EM ZPF momentum flux as viewed by an accelerating object. If the quarks and electrons in such an accelerating object scatter this asymmetric radiation an acceleration dependent reaction force $\vec{f}_r$ arises. In this context $\vec{f}_r$ is the space part of a relativistic four vector so that the resulting equation of motion is not simply the classical $\vec{f} = m\vec{a}$ but rather the properly relativistic $\mathfrak{F} = d\mathfrak{P}/d\tau$ (which becomes exactly to $\vec{f} = m\vec{a}$ for subrelativistic velocities). The expression for inertial mass in HRP for an individual particle is $m_{ZP} = \Gamma_z\hbar\omega_c^2/2\pi c^2$ where Γ_z represents a damping constant for Zitterbewegung oscillations (a free parameter) and ω_c represents an assumed cutoff frequency for the ZPF spectrum (another free parameter). The expression for inertial mass in RH for an object with volume V_0 is

$$m_i = m_{ZP} = \left(\frac{V_0}{c^2}\int \eta(\omega)\frac{\hbar\omega^3}{2\pi^2c^3}d\omega\right) = \frac{V_0}{c^2}\int \eta(\omega)\rho_{ZP}d\omega \tag{1.4}$$

(note from (1.1) $\omega \sim 2\pi\nu$ and m_i here refers to inertial mass). Also from [**440**] recall that 4-momentum is defined as $\vec{P} = (E/c, \vec{p}) = (\gamma m_0 c, \gamma m_0 \vec{v})$ where $|\vec{P}| = m_0 c$ and $E = \gamma m_0 c^2$ (γ is a Lorentz contraction factor). Recall that the Compton frequency is given via $\hbar\nu_C = m_0 c^2$ for a particle of rest mass m_0 (note $\lambda_C = \hbar/m_0 c$ so $c = (\hbar/m_0 c)\nu_C = \lambda_C\nu_C \Rightarrow \nu_C = c/\lambda_C$ has dimension T^{-1}). Further $\lambda_B = \hbar/p$ for a particle of momentum p and here $p \sim m_0\gamma v$ so one expects

$$\lambda_B = \hbar/m_0\gamma v = m_0 c\lambda_C/m_0\gamma v = (c/\gamma v)\lambda_C \tag{1.5}$$

(cf. also [**577**]). In any event in [**824**] one argues that what appears as inertial mass in a local frame corresonds to gravitational mass m_g and identifies m_i in (1.4) with m_g.

REMARK 5.1.1. We mention here a thoughtful analysis about the origin of inertial mass etc. in [**488**] (second paper), which refers to the material developed above in [**438, 439, 823, 824**]. It is worthwhile extracting in some detail as follows (cf. also [**441, 487, 650, 753**]). The purpose of the paper is stated to be that of suggesting qualifications in some claims that the classical equilibrium spectrum of charged matter is that of the classically conceived ZFF (cf. here [**487**] where one introduces an alternative classical ZPF with a different stochastic character which reproduces the statistics of QED). It is pointed out that a classical massless charge cannot acquire mass from nothing as a result of immersion in any EM field and therefore that the ZPF alone cannot provide a full explanation of intertial mass. Thus as background one mentions several works where classical representations of the ZPF have been used to derive a variety of quantum results, e.g. the van der Waals binding (cf. [**145**]), the Casimir effect (cf. [**650**]), the Davies-Unruh effect (cf. [**293**]), the ground state behavior of the QM harmonc oscillator (cf. [**487**]), and the blackbody spectrum (cf. [**145, 252**]). In its role as the originator of inertial mass the ZPF has been evisined as an external energizing influence for a classical particle whose mass is to be explained. In SED a free charged particle is deemed to obey the (relativistic version of the) Braffort-Marshall equation

$$m_0 a^\mu - m_0\tau_0\left[\frac{da^\mu}{d\tau} + \frac{a^\lambda a_\lambda}{c^2}u^\mu\right] = eF^{\mu\lambda}u_\lambda \tag{1.6}$$

where $\tau_0 = e^2/6\pi\epsilon_0 m_0 c^3$ and f is the field tensor of the ZPF interpreted classically (cf. [**487**]) for the correspondence between this and the vacuum state of the EM field - indicated also later in this paper). If F is the ZPF field tensor operator then (1.6) is a relativistic generalization of the Heisenberg equation of motion for the QM position operator of a free charged particle, properly taking into account the vacuum sate of the quantized EM field (cf. [**650**]). From the standpoint of QED, once coupling to the EM field is switched on and radiation reaction admitted, the action of the vacuum field is not an optional extra, but a necessary component of the fluctuation dissipation relation between atom and field. In the ZPF inertial mass studies the electrodynamics of the charge in its pre-mass condition has not received attention, presumably on the grounds that the ZPF energization will quickly render the particle massive so that the intermediate state of masslessness is on no import. However letting $m_0 \to 0$ one sees that $F^{\mu\lambda}u_\lambda \to 0$ which demands that $\mathbf{E}\cdot\mathbf{v} \to 0$ (the massless particle moves orthogonal to E - cf. [**488**], first paper). It is concluded that if a charge is initially massless, there is no means by which it can acquire inertial mass energy from an EM field, including the ZPF, and one must discount the possibility that the given ZPF can alone explain inertial mass of such a particle (this does not take into consideration however the possible effects of Dirac-Weyl geomety on mass creation - cf. Section 4.1). This is not to deny that mass may yet emerge from a process involving the ZPF or some other EM field, only that it cannot be the whole story. Note here that in earlier works [**441, 823**] an internal structure is implied by the use of a mass-specific frequency dependent coupling constant between the charged particles and the ZPF so there is no contradiction to the arguments above, namely to the statement that the classical structureless charge particle cannot acquire mass solely as a result of immersion in the ZPF. One must also be concerned with a distinction between inertia as a reaction force and inertial mass as energy. For example one could ask whether the ZPF could be the cause of resistance to acceleration without it having to be the cause of mass-energy. To address this consider the geometric form of the mass action

$$I = -m_0c^2\int \gamma^{-1}dt \tag{1.7}$$

which simultaneously gives both the mass acceleration $f^\mu = m_0a^\mu$ and the Noether conserved quantity under time translations $E = \gamma m_0c^2 = m(v)c^2$ (i.e. the traditional mechanical energy (γ is the Lorenz factor). Thus the distinction between the two qualities of mass appears to be one of epistemology. In some work (cf. [**438**]) the ZPF has been envisioned as an external energizing influence for an explicitly declared local interal degree of freedom, intrinsic to the charged particle whose mass is to be energized. Upon immersion in the ZPF this (Planck) oscillator is energized and the energy so acquired is some of all of its observed inertial mass. Such a particle is not a structureless point in the usual classical sense and so does not suffer from an inability to acquire mass from the ZPF, provided the proposed components (sub-electron charges) already carry some inertia. There is much more interesting discussion which should be read. Note that one is excluding here such matters as the (mythical) Higgs field as well as renormalization.

One of the first objections typically raised against the existence of a real ZPF is that the mass equivalent of the energy embodied in (1.1) would generate an enormous spacetime curvature that would shrink the universe to microscopic size (apparently refuted however via the principle of equivalence - cf. [**438**]). One notes that all matter at the level of quarks and electrons is driven to oscillate (Zitterbewegung) by the ZPF and every oscillating charge will generate its own minute EM fields. Thus any particle wil experience the ZPF as modified ever so slightly by the fields of adjacent particles - but that might be gravitation as a kind of long range van der Waals force. A ZPF based theory of gravitation is however only exploratory at this point and there are disputes (see e.g. [**183, 253, 310, 792**]

- discussed in part later). In any event following [**440, 441, 824**] the preceeding analysis leads to $\mathfrak{F} = d\mathfrak{P}/d\tau = (d/d\tau)(\gamma m_i c, p)$ for the relativistic force where $m_i \sim m_{ZP} \sim m_g$. More generally via [**823**] (which improves [**438**]) plausible arguments are given to show that the EM quantum vacuum makes a contribution to the inertial mass m_i in the sense that at least part of the inertial force of opposition to acceleration, or inertia reaction force, springs from the EM quantum vacuum. Specifically, the properties of the EM vacuum as experienced in a Rindler constant acceleration frame were investigated, and the existence of an EM flux was discovered, called for convenience the Rindler flux (RF). The RF, and its relative, Unruh-Davies radiation, both stem from event horizon effects in accelerating reference frames. The force of radiation pressure produced by the RF proves to be proportional to the acceleration of the reference fame, which leads to the hypothesis that at least part of the inertia of an object should be due to the individual and collective interaction of its quarks and electrons with the RF. This is called the quantum vacuum inertia hypothesis (QVIH). This is consistent with general relativity (GR) and it answers a fundamental question left open within GR, namely, whether there is a physical mechanism that generates the reaction force known as weight when a specific nongeodesic motion is imposed on an object. Put another way, while geometrodynamics dictates the spacetime metric and thus specifies geodesics, whether there is an identifiable mechanism for enforcing the motion of freely falling bodies along geodesic trajectories. The QVIH provides such a mechanism since by assuming local Lorentz invariance (LLI) one can immediately show that the same RF arises due to curved spacetime geometry as for acceleration in flat spacetime. Thus the previously derived expression for the inertial mass contribution from the EM quantum vacuum field is exactly equal to the corresonding contribution to the gravitational mass m_g and the Newtonian weak equivalence principle $m_i = m_g$ ensues. One also adopts the assumption of space and time uniformity (uniformity assumption (UA) stating that the laws of physics are the same at any time or place within the universe. With such hypotheses one also derives in [**824**] the Newtonian gravitational law $\mathbf{f} = -(GMm/r^2)\dot{\mathbf{r}}$ while disclaiming the success of such attempts in earlier work (cf. [**183, 253, 439, 792**]). We mention also some quite appropriate comments about the mystical Higgs field in [**441**] (cf. also [**442**]), the main point being perhaps that even if the Higgs exists it does not necessaarily explain inertial mass.

ZPF also plays the role of a Lorentz invariant EM component of a Dirac ether (cf. [**439**]). Thus Newton's equation of motion in special relativity can be written as $m(d^2x^\mu/d\tau^2) = F^\mu$ where four vectors are involved and F^μ represents a nongravitational force. In general relativity this becomes

$$m\left(\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\nu\rho}\frac{dx^\nu}{d\tau}\frac{dx^\rho}{d\tau}\right) = F^\mu \tag{1.8}$$

The velocity $dx^\mu/d\tau$ is a time-like 4-vector and F^μ is a space-like 4-vector orthogonal to the velocity. If $F^\mu \neq 0$ in any coordinate frame it will be nonzero in all coordinate frames. The $\Gamma^\mu_{\nu\rho}$ represent the gravitational force and can be set equal to zero by a coordinate transformation but F^μ cannot be transformed away.

Turning the arguments around the absolute nature of nongravitational acceleration is demonstrated by the manifestation of a force that cannot be transformed away. Thus there is need for a special reference frame that is not perceptible on account of uniform motion but that is perceptible on account of non gravitational acceleration and one proposes that the Lorentz invariant ZPF plays this role. We refer to [**144, 825**] for background here.

REMARK 5.1.2. We mention for heuristic purposes some developments following [**442, 700**] (cf. also [**494**] and Section 5.2 for possible comparison). One addresses the quantum theoretic prediction that the Zitterbewegung of particles occurs at the speed of light, that particles exihbit spin, and that pair creation can occur. Given motion at the speed of light the particles involved in Zitterbewegung would seem to be massless (and in this respect we refer to [**494**]). A suitable equation for motion of a massless charge can be derived as a massless limit of the Lorentz force equation $ma^{\mu} = (q/c)F^{\mu\nu}u_{\nu}$ where q is charge and m will be go to zero; $F^{\mu\nu}$ is the EM field tensor of impressed fields, including the ZPF. The acceleration a and velocity u are four vectors and in terms of the usual three space quanties $\mathbf{a}$ and $\mathbf{v}$ one can write

$$m\left[\gamma^2 a^j + \gamma^4 \mathbf{v}\cdot\mathbf{a}\frac{v^j}{c}\right] = \frac{q}{c}\gamma\left[\sum_1^3 F^{jk}v_k - F^{j0}c\right]; \tag{1.9}$$

$$m\gamma^4\frac{\mathbf{v}\cdot\mathbf{a}}{c} = \frac{q}{c}\gamma\left[\sum_1^3 F^{0k}v_k - F^{00}c\right]$$

where $\gamma = 1/\sqrt{1-(v^2/c^2)}$. Combining leads to

$$m\gamma a^j = \frac{q}{c}\left[\sum_1^3\left(F^{jk} - F^{0k}\frac{v^j}{c}\right)v_k - \left(F^{j0} - F^{00}\frac{v^j}{c}\right)c\right] \tag{1.10}$$

Now let $m \to 0$ and $\gamma \to \infty$ with $m\gamma$ remaining finite; thus one will have $lim_{m\to 0, v\to c} m\gamma = m_*$ and m_* has the dimensions of mass but is not mass. We need further $\mathbf{v} = c\mathbf{n}$ where $\mathbf{n}$ is a unit vector in the direction of the particle motion. Acceleration can therefore only be due to changes in the direction of the form $\mathbf{a} = c(d\mathbf{n}/dt)$ so that

$$m_* c\frac{dn^j}{dt} = \frac{q}{c}\left[\sum_1^3\left(F^{jk} - F^{0k}\frac{v^j}{c}\right)v_k - \left(F^{j0} - F^{00}\frac{v^j}{c}\right)c\right] \tag{1.11}$$

In terms of EM fields this is $(d\mathbf{n}/dt) = (q/m_*c)[\mathbf{n}\times\mathbf{B} - (\mathbf{n}\cdot\mathbf{E})\mathbf{n} + \mathbf{E}]$. Since the particle is moving at the speed of light it should see a universe Lorentz contracted to two transverse dimensions and can only be accelerated by forces from the side. When the impressed fields include the ZPF this motion can be regarded as Schrödinger's Zitterbewegung. When a field above the vacuum is applied the charge will be observed to drift in a preferred direction in its Zitterbewegung wandering. When viewed in a zoom picture one sees spin line orbital motion driven by the ZPF. Now equation (1.11) does not exhibit inertia which seems in violation of relativity. However the ZPF fields have the vacuum energy density spectrum of the form $\rho(\omega)d\omega = (\hbar\omega^3/2\pi^2c^3)d\omega$ where $\omega = 2\pi\nu$ (cf. (1.1)). The cubic frequency dependence endows the spectrum with Lorentz invariance and all inertial frames see an isotropic ZPF. A Lorentz transformation will cause a Doppler shift of each frequency component but an equal amount of energy is shifted into and out of each frequency bin. When there are no fields above the vacuum in an inertial frame an observer in that frame should expect to see a zero-mean random walk due to the isotropic ZPF. Thus in our example an observer in a frame comoving with the average motion of the charge just before the driving field is switched off should expect to see continued zero-mean Zitterbewegung in his frame whereas (1.11) produces zero-mean motion in whatever frame the calculation is performed. To have a consistent theory Lorentz covariance must be restored. Thus assume (1.11) holds in the spacetime of the particle and describes a null geodesic. The curvature is defined by the EM fields in the particles history and since the particle is massless and moving at the speed of light one cannot use proper time as the affine parameter of the geodesic (since proper time intervals vanish for null geodesics). However normal time serves as well for time parameter and (1.11) is replaced with

$$\frac{dp^\mu}{dt} + \frac{1}{m_*}\Gamma^\mu_{\nu\rho}p^\nu p^\rho = 0;\; p^\mu = m_* c n^\mu;\; n^\mu = (n^0, \mathbf{n}) \tag{1.12}$$

One equates the connection terms with the Lorentz force terms of $ma^\mu = (q/c)F^{\mu\nu}u_\nu$, i.e.

$$\Gamma^\mu_{\nu\rho}p^\nu p^\rho = -(q/c)F^\mu_\nu p^\nu \tag{1.13}$$

and these equations can be solved for the metric of the particle's spacetime (although not uniquely - there is apparently a class of metrics). Further constraints are needed to select a particular solution and in particular the geodesic should be a null curve as expected for a massless object. Using (1.13) the geodesic equation in (1.12) is claimed to be

$$\frac{dn^j}{dt} = \frac{q}{m_* c}\left[F^j_\nu n^\nu - F^o_\nu n^\nu \frac{n^j}{n^0}\right] + \frac{n^j}{n^0}\frac{dn^0}{dt};\; \frac{dm_*}{dt} = \frac{q}{c}F^0_\nu \frac{n^\nu}{n^0} - \frac{m_*}{n^0}\frac{dn^0}{dt} \tag{1.14}$$

In general n^0 does not retain a value of 1 but should change in a way that preserves the null curve property $g_{\mu\nu}p^\mu p^\nu = 0$. We note that the second equation in (1.14) is an equation for the parameter m_*, which is therefore not a constant but rather varies in response to applied forces. The effect is to introduce time dilation (or Doppler shifting) in the EM 4-vector analogous to the gravitational redshift of GR. Also note that inertia is not assumed here by the requirement that the particle travel on a null geodesic; the particle is not restricted but its motion is used to define the spacetime and metric that the particle sees. It is only after a transformation to Minkowski spacetime that inertial behavior appears. To find solutions of (1.13) consider an infinitesimal region around the charge and require $g_{\mu\nu} = \eta_{\mu\nu} + g_{\mu\nu,\rho}dx^\rho$ where $\eta_{\mu\nu}$ is the flat spacetime metric (signature $(-1,1,1,1)$). The Christoffel symbols are then calculated in terms of the derivatives $g_{\mu\nu,\rho}$ and substituted into (1.13). A simple local solution is written down with some interesting properties for which we refer to [**442, 700**].

1.1. REMARKS ON THE AETHER AND VACUUM. Regarding massless particles and Maxwell's equations we refer here to [**9, 19, 29, 152, 233, 331, 405, 442, 543, 666, 700, 863, 864, 920**] and consider first [**405**]. Thus the Dirac equation is derived from the relativistic condition $(E^2-c^2\mathbf{p}^2-m^2c^4)I^4\psi=0$ where I^4 is a 4×4 unit matrix and ψ is a fourcomponent (bispinor) wave function. This can be decomposed via

(1.15)
$$\left[EI^4+\begin{pmatrix} mc^2I^2 & c\mathbf{p}\cdot\sigma \\ c\mathbf{p}\cdot\sigma & -mc^2I^2\end{pmatrix}\right]\times\left[EI^4-\begin{pmatrix} mc^2I^2 & c\mathbf{p}\cdot\sigma \\ c\mathbf{p}\cdot\sigma & -mc^2I^2\end{pmatrix}\right]\psi=0;$$
$$\sigma_x=\begin{pmatrix} 0 & 1\\ 1 & 0\end{pmatrix};\ \sigma_y=\begin{pmatrix} 0 & -i\\ i & 0\end{pmatrix};\ \sigma_z=\begin{pmatrix} 1 & 0\\ 0 & -1\end{pmatrix}$$

and I^2 is a 2×2 unit matrix. The two component neutrino equation can be derived from the decomposition

$$(E^2-c^2\mathbf{p}^2)I^2\psi=[EI^2-c\mathbf{p}\cdot\sigma][EI^2+c\mathbf{p}\cdot\sigma[\psi=0 \tag{1.16}$$

where ψ is a two component spinor wavefunction. The photon equation can be derived from the decomposition

(1.17)
$$\left(\frac{E^2}{c^2}-\mathbf{p}^2\right)I^3=\left(\frac{E}{c}I^3-\mathbf{p}\cdot\mathbf{S}\right)\left(\frac{E}{c}I^3+\mathbf{p}\cdot\mathbf{S}\right)-\begin{pmatrix} p_x^2 & p_xp_y & p_xp_z\\ p_yp_x & p_y^2 & p_yp_x\\ p_xp_z & p_xp_y & p_z^2\end{pmatrix}=0;$$
$$S_x=\begin{pmatrix} 0 & 0 & 0\\ 0 & 0 & -i\\ 0 & i & 0\end{pmatrix};\ S_y=\begin{pmatrix} 0 & 0 & i\\ 0 & 0 & 0\\ -i & 0 & 0\end{pmatrix};\ S_z=\begin{pmatrix} 0 & -i & 0\\ i & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$$

and I^3 is a 3 unit matrix. Further

$$[S_x,S_y]=iS_z,\ [S_z,S_x]=iS_y,\ [S_y,S_z]=iS_x,\ \mathbf{S}^2=2I^3 \tag{1.18}$$

One notes that the last matrix in (1.17) (first line) can be written as $(p_x\,p_y\,p_z)^T\cdot(p_x\,p_y\,p_z)$. This leads to the photon equation in the form

$$\left(\frac{E^2}{c^2}-\mathbf{p}^2\right)\psi=\left(\frac{E}{c}I^3-\mathbf{p}\cdot\mathbf{S}\right)\left(\frac{E}{c}I^3+\mathbf{p}\cdot\mathbf{S}\right)\psi-\begin{pmatrix} p_x\\ p_y\\ p_z\end{pmatrix}(\mathbf{p}\cdot\psi)=0 \tag{1.19}$$

where ψ is a 3-component column wave function. (1.19) will be satisfied if

$$\left(\frac{E}{c}I^3+\mathbf{p}\cdot\mathbf{S}\right)\psi=0;\ \mathbf{p}\cdot\psi=0 \tag{1.20}$$

For real energies and momenta conjugation leads to

$$\left(\frac{E}{c}I^3-\mathbf{p}\cdot\mathbf{S}\right)\psi^*=0;\ \mathbf{p}\cdot\psi^*=0 \tag{1.21}$$

The only difference here is that (1.20) is the negative helicity equation while (1.21) represents positive helicity. Now in (1.20) make the substitutions $E \sim i\hbar\partial_t$ and $\mathbf{p} \sim -i\hbar\nabla$ with $\psi\mathbf{E} - i\mathbf{B}$. Then $(\mathbf{p}\cdot\mathbf{S})\psi = \hbar\nabla\times\psi$ and this leads to

$$\frac{i\hbar}{c}\partial_t\psi = -\hbar\nabla\times\psi;\ -i\hbar\nabla\cdot\psi = 0 \tag{1.22}$$

Cancelling $\hbar$ (!) one obtains then

$$\nabla\times(\mathbf{E}-i\mathbf{B}) = -\frac{i}{c}\partial_t(\mathbf{E}-i\mathbf{B});\ \nabla\cdot(\mathbf{E}-i\mathbf{B}) = 0 \tag{1.23}$$

If the electric and magnetic fields are real one obtains the Maxwell equations

$$\nabla\times\mathbf{E} = -\frac{1}{c}\partial_t\mathbf{B};\ \nabla\times\mathbf{B} = \frac{1}{c}\partial_t\mathbf{E};\ \nabla\cdot\mathbf{E} = \nabla\cdot\mathbf{B} = 0 \tag{1.24}$$

The Planck constant $\hbar$ does not appear since it cancelled out! In any event this shows the QM nature of the Maxwell equations and hence of EM fields.

REMARK 5.1.3. One notes here a similar formula arising in [**9**] going back to work of Majorana, Weinberg, et al (see [**9**] for references). The idea there involves the matrices S of (1.17) written as s^1, s^2, s^3 with

$$i\frac{\partial\mathbf{E}}{\partial t} = \frac{1}{i}(\mathbf{s}\cdot\nabla)i\mathbf{B};\ i\frac{\partial(i\mathbf{B})}{\partial t} = \frac{1}{i}(\mathbf{s}\cdot\nabla)\mathbf{E};\ \mathbf{E} = \begin{pmatrix} E_1\\ E_2\\ E_3\end{pmatrix};\ \mathbf{B} = \begin{pmatrix} B_1\\ B_2\\ B_3\end{pmatrix} \tag{1.25}$$

corresponding to the first two equations in (1.24). Then a fermion-like formulation is created via 6×6 matrices

$$\Gamma_0 = \begin{pmatrix} I & 0\\ 0 & -I\end{pmatrix};\ \vec{\Gamma} = \begin{pmatrix} 0 & \mathbf{s}\\ -\mathbf{s} & 0\end{pmatrix};\ \Gamma_5 = \begin{pmatrix} 0 & I\\ I & 0\end{pmatrix} \tag{1.26}$$

For $S = \begin{pmatrix} 0 & \mathbf{s}\\ \mathbf{s} & 0\end{pmatrix}$ (1.25) becomes

$$i\partial_t\psi = (1/i)(S\cdot\nabla)\psi \Rightarrow (\partial_t + S\cdot\nabla)\psi = 0 \tag{1.27}$$

Equation (1.27) resembles the massless Dirac equation and one can consider ψ as a (quantum) wave function for the photon of type $\psi = (\mathbf{E}\ i\mathbf{B})^T$ with $\bar{\psi} = (\mathbf{E}^\dagger\ i\mathbf{B}^\dagger)$ where $\bar{\psi} = \psi^\dagger\Gamma_0$ is an analogue of Hermitian conjugate. Note $\psi^\dagger\psi = \mathbf{E}^2 + \mathbf{B}^2$ (where $\mathbf{E}^\dagger = \mathbf{E}$ and $\mathbf{B}^\dagger = \mathbf{B}$) corresponds to the local mean number of photons. The construction of ψ thus mimics a Dirac spinor and one can write

$$i\Gamma_0\partial - t\psi = \frac{1}{i}(\Gamma_0 S\cdot\nabla)\psi = \frac{1}{i}(\vec{\Gamma}\cdot\nabla)\psi \Rightarrow (\Gamma_0\partial_t + \vec{\Gamma}\cdot\nabla)\psi = 0 \tag{1.28}$$

or more compactly $\Gamma^\mu\partial_\mu\psi = 0$ (although this is not manifestly covariant). One goes on to construct a Lagrangian with a duality transformation and we refer to [**9**] for more details.

REMARK 5.1.4. Going to [**331**] one finds some generalizations of [**405**] in the form

$$\nabla\times\mathbf{E} = -\frac{1}{c}\partial_t\mathbf{B} + \nabla\Im(\chi);\ \nabla\times\mathbf{B} = \frac{1}{c}\partial_t\mathbf{E} + \nabla\Re(\chi); \tag{1.29}$$

$$\nabla \cdot \mathbf{E} = -\frac{1}{c}\partial_t \Re(\chi);\ \nabla \cdot \mathbf{B} = \frac{1}{c}\partial_t \Im(\chi)$$

and some further analysis of spin situations.. If one assumes no monopoles it may be suggested that $\chi(x)$is a real field and its derivatives play the role of charge and current densities; there is also some flexibility here in interpretation.

1.2. A VERSION OF THE DIRAC AETHER. We go here to [**215, 216**] (cf. also [**302, 499, 500, 727**]). Dirac adopted the idea of using spurious degrees of freedom associated to the gauge potential to describe the electron via a gauge condition $A^2 = k^2$. In fact in [**302**] this was introduced as a gauge fixing term in the Lagrangian $L = -(1/4)F^2 + (\lambda/2)(A^2 - k^2)$ leading to $\partial_\nu F^{\mu\nu} = J^\mu \equiv \lambda A^\mu$. Here the gauge condition doesn't intend to eliminate spurious degrees of freedom but rather acquires a physical meaning as the condition allowing the right description of the physics without having to introduce extra fields. From this Dirac argued that it would be possible to consider an aether provided that one interprets its four velocity v as a quantity subjected to uncertainty conditions. Admitting the aether velocity as defining a point in a hyperboloid with equation $v_0^2 - \vec{v}^2 = 1$ with $v_0 > 0$ it could be related to the gauge potential (satisfying $A^2 = k^2$) via $(1/k)A_\mu = v_\mu$. Then v (the aether velocity) would be the velocity with which an electric charge would flow if placed in the aether. The model in [**215**] is a continuation of [**727**] and it is implicit in the present formulation that there is an inertial frame (the aether frame) in which the aether is at rest; one writes $(1, 0, 0, 0)$ $(\sim v^\mu_{aether})$ for the aether frame. Then an action $S = \int dx(-(1/4)F^2 + \sigma v_\alpha F^{\alpha\mu} A_\mu)$ is proposed with v being the aether's velocity relative to a generic observer (inertial or not). For inertial observers $v^\mu = \Lambda^\mu_\nu v^\nu_{aether} = \Lambda^\mu_0$ is still constant and one obtains from the equations of motion a current $J^\mu = -\sigma v^\mu \partial \cdot A + \sigma v_\nu \partial^\mu A^\nu$ that is understood as being induced in the aether by the presence of the EM field. Therefore it defines a polarization tensor $M_{\alpha\beta}$ from which one obtains the vectors of polarization and magnetization of the medium. In the aether reference frame this allows one to define the electric displacement vector as $\mathbf{D} = \mathbf{E} + \sigma\mathbf{A}$ while $\mathbf{H} = \mathbf{B}$; the resulting equations are similar in form with the macroscopic Maxwell equations in a medium, in agreement with [**537, 959**] but with the difference that $\mathbf{D} \neq \epsilon\mathbf{E}$. Hence the aether cannot be thought of as an isotropic medium. Moreover in a generic refernce frame moving relative to the aether $\mathbf{D}$ and $\mathbf{H}$ will depend on v. There is more summary material in [**215**] but we proceed directly here to the equations.

For flat spacetime one takes for $\eta_{\mu\nu} = diag(+, -, -, -)$

$$x^\mu = (x^0, x^i) = (t, \mathbf{x});\ x_\mu = (x_0, x_i) = (t, -\mathbf{x});\ A_\mu = (A_0, A_i) = (\phi, -\mathbf{A}); \tag{1.30}$$

$$\partial^\mu = (\partial^0, \partial^i) = (\partial_t, -\nabla);\ \partial_\mu = (\partial_0, \partial_i) = (\partial_t, \nabla);\ A^\mu = (A^0, A^i) = (\phi, \mathbf{A});$$

$$F_{0i} = \mathbf{E}_i;\ F_{ij} = -\epsilon_{ijk}\mathbf{B}_k;\ F^{0i} = -F_{0i} = -\mathbf{E}_i;\ F^{ij} = F_{ij} = -\epsilon_{ijk}\mathbf{B}_k$$

Take now

$$S = \int dx(-(1/4)F^2 + \tilde{\mathbf{J}} \cdot \mathbf{A}); \tag{1.31}$$

$$\tilde{J}^\mu = \sigma v_\alpha F^{\alpha\mu};\ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

The constant σ is associated to the aether conductivity and $\tilde{J}$ will be conserved (we drop the bold face here); $\tilde{J} \cdot A$ defines an interaction of the gauge field with itself. Then the equation of motion for A_μ is

$$\partial_\nu F^{\nu\mu} + \sigma v^\mu \partial \cdot A - \sigma v_\nu \partial^\mu A^\nu = 0 \tag{1.32}$$

and this assumes the form of the Maxwell equations in the presence of a source $\partial_\nu F^{\nu\mu} \equiv J^\mu$ provided that one identifies $J^\mu = -\sigma v^\mu \partial \cdot A + \sigma v_\nu \partial^\mu A^\nu$ with a conserved 4-current. This is the same as Dirac in which the term $j^\mu = \lambda A^\mu$ is interpreted as a 4-current but here this arises from the interaction term $\tilde{J} \cdot A$ instead of via a gauge fixing term $(1/2)\lambda(A^2 - k^2)$ in the action. Now taking the divergence of $\partial_\nu F^{\nu\mu} = J^\mu$ one gets

$$0 = \partial_\mu J^\mu = \sigma v_\mu (\Box A^\mu - \partial^\mu \partial \cdot A) = \sigma v_\mu \partial_\nu F^{\nu\mu} = \sigma v_\mu J^\mu = \sigma^2(-v^2 \partial \cdot A + \\ + v_\alpha v_\beta \partial^\alpha A^\beta); \;\; \partial \cdot A = (v_\alpha v_\beta / v^2) \partial^\alpha A^\beta \tag{1.33}$$

This constraint is a new feature of this model; its origin is independent of any local symmetry of the action.

For global gauge invariance one considers an invariance defined via the equation $A_\mu \to A'_\mu = A_\mu + \lambda v_\mu$ (where $\partial_\mu \lambda = 0$); this is associated to the Noether current

$$\Theta^\mu = F^{\mu\nu} v_\nu - \sigma v^\mu A \cdot v + \sigma v^2 A^\mu \equiv \Theta^{\mu\nu} v_\nu \tag{1.34}$$

where $\Theta^{\mu\nu} = F^{\mu\nu} - \sigma v^\mu A^\nu + \sigma v^\nu A^\mu$. Later this $\Theta^{\mu\nu}$ will be interpreted as $H^{\mu\nu}$ which in the aether frame becomes $H^{\mu\nu} = (\mathbf{D}, \mathbf{H})$. In a system at rest relative to the aether one has $\Theta^\mu = (0, \mathbf{E} + \sigma \mathbf{A})$ and (1.33) implies $\nabla \cdot \mathbf{A} = 0$. Then the conservation equation for Θ^μ gives $\nabla \cdot E = 0$. In this model there is another conserved current

$$\hat{J}^\mu = -\sigma v^\mu \partial \cdot A + \sigma v \cdot \partial A^\mu \tag{1.35}$$

where $\tilde{J} = \hat{J} - J$ so that conservation of $\tilde{J}$ will follow immediately. Equivalently one can think of $\tilde{J}^\nu$ as originating from the divergence of $\Theta^{\mu\nu}$, i.e. $\partial_\mu \Theta^{\mu\nu} = -\tilde{J}^\nu$. In the classical formulation of electrodynamics in conducting media the nonhomogeneous Maxwell equations are written covariantly as $\partial_\mu H^{\mu\nu} = -j^\nu_{ext}$ with $H^{\mu\nu}$ having $\mathbf{D}$ and $\mathbf{H}$ as its components. Here $\Theta^{\mu\nu}$ above generalizes $H^{\mu\nu}$ and $\partial_\mu \Theta^{\mu\nu} = -\tilde{J}^\nu$ corresponds to $\partial_\mu H = -j^\nu_{ext}$. This allows one to interpret $\tilde{J}^\mu$ as the corresponding 4-current in much the same way as Dirac interpreted $j_\mu = \lambda A_\mu$ as a 4-current in [**302**]. The interaction term $\tilde{J} \cdot A$ then parallels the same term in the usual electrodynamics. The global symmetry above is a new feature with no counterpart in the Maxwell formulation. It is also possible to add a mass term to S above that preserves this global symmetry; in fact the action

$$S = \int dx(-(1/4)F^2 + \tilde{J} \cdot A - (1/2)\sigma^2 A_\mu (v^2 g^{\mu\nu} - v^\mu v^\nu) A_\nu) \tag{1.36}$$

is invariant. From this one obtains the equation

$$\partial_\nu F^{\nu\mu} \equiv \bar{J}^\mu = -\sigma v^\mu \partial \cdot A + \sigma v_\nu \partial^\mu A^\nu + \sigma^2 (v^2 g^{\mu\nu} - v^\mu v^\nu) A_\nu \sim \\ \sim [g^{\mu\nu}(\Box - \sigma^2 v^2) = \partial^\mu \partial^\nu + \sigma(v^\mu \partial^\nu - v^\nu \partial^\mu) + \sigma^2 v^\mu v^\nu] A_\nu = 0 \tag{1.37}$$

Here the conservation of the current $\bar{J}$ that follows from (1.37) doesn't produce any constraint on A; further the conserved current in (1.34) is associated to the global symmetry. Finally the local gauge invariance depends on a parameter $\theta(x)$ and has the usual form $A_\mu \to A'_\mu = A_\mu + \partial_\mu\theta$ and (1.33) adds some new features to the analysis. In fact let A' and A be two fields related by $A'_\mu = A_\mu + \partial_\mu\theta$. Since both field configurations should obey (1.33) one must have

$$\partial \cdot A' = (1/v^2)(v \cdot \partial)(v \cdot A') \iff \Box\theta = (v_\alpha v_\beta/v^2)\partial^\alpha\partial^\beta\theta \tag{1.38}$$

Now let $\partial \cdot A \neq 0$ and choose θ so that it ensures $\partial \cdot A' = 0$; one should then have θ satisfying $\Box\theta = -\partial \cdot A$. This last condition together with the constraints (1.33) and (1.38) give then $\partial^\alpha\partial^\beta\theta = -(1/2)(\partial^\alpha A^\beta + \partial^\beta A^\alpha)$ which represents a stronger restriction than that shown in $\Box\theta = -\partial \cdot A$. Equivalently one can obtain this stronger restriction directly from $0 = \partial \cdot A' = (v_\alpha v_\beta/v^2)\partial^\alpha(A^\beta + \partial^\beta\theta)$ using (1.33).

Now one considers this model in the aether reference frame where $v = (1, 0, 0, 0)$ and it is supposed that the aether is a medium without any given density of charge or current. From $\partial_\nu F^{\nu\mu} = J^\mu$ one has

$$\nabla \cdot \mathbf{E} = -\sigma\nabla \cdot \mathbf{A};\ \nabla \times \mathbf{B} = \frac{\partial \mathbf{E}}{\partial t} + \sigma\frac{\partial \mathbf{A}}{\partial t} + \sigma\mathbf{E} \tag{1.39}$$

to which is added the homogeneous equations $\nabla \cdot \mathbf{B} = 0$ and $\nabla \times \mathbf{E} = -\partial_t\mathbf{B}$. Since $\nabla \cdot \mathbf{E} \neq 0$ we see that (1.39) introduces a new feature for the physical vacuum (cf. also [**597**]). Essentially a divergenceless equation for $\mathbf{E}$ signals that the vacuum is not merely an empty space but it is also capable of becoming electrically polarized. The presence of additional terms depending on the potential vector in (1.39) indicates the response of the medium to the presence of the fields $(\mathbf{E}, \mathbf{B})$, which resembles the phenomena of polarization and magnetization of a medium. Therefore one rewrites (1.39) as $\nabla \cdot \mathbf{D} = 0$ with $\mathbf{D} = \mathbf{E} + \sigma\mathbf{A} + \nabla \times \mathbf{K}$. At this point $\mathbf{K}$ is an arbitrary vector that can be thought of as playing the role of a gauge parameter. Now rewrite (1.39) with an assignment for $\mathbf{K}$ as

$$\nabla \times \mathbf{B} = \partial_t\mathbf{D} + \sigma\mathbf{E} - \partial_t\nabla \times \mathbf{K}\ \ \textit{with}\ \ \partial_t\nabla \times \mathbf{K} = \sigma\mathbf{E} \tag{1.40}$$

Then using $\nabla \cdot \mathbf{B} = 0$ and $\nabla \times \mathbf{E} = -\partial_t\mathbf{B}$ one obtains $\partial_t\nabla \times (\sigma\mathbf{A} + \nabla \times \mathbf{K}) = 0$. The vector $\mathbf{K}$ can be still further restricted so that $\sigma\mathbf{A} + \nabla \times \mathbf{K} = 0$ and this gives $\mathbf{E} = \mathbf{D}$ and $\mathbf{B} = \mathbf{H}$ with the Maxwell equations in free space, namely

$$\nabla \cdot \mathbf{E} = 0;\ \nabla \times \mathbf{B} = \partial_t\mathbf{E};\ \nabla \cdot \mathbf{B} = 0;\ \nabla \times \mathbf{E} = -\partial_t\mathbf{B} \tag{1.41}$$

Conditions (1.40) with $\sigma\mathbf{A} + \nabla \times \mathbf{K} = 0$ can be interpreted as originating from the imposition of the temporal gauge and together they imply $\nabla\mathbf{A}_0 = 0$ which is naturally satisfied by putting $\mathbf{A}_0 = 0$. It is possible to give another description for the present electrodynamics without using $\mathbf{K}$. Thus from (1.39) one can simply identify $\mathbf{D} = \mathbf{E} + \sigma\mathbf{A}$ and $\mathbf{B} = \mathbf{H}$, leading to

$$\nabla \cdot \mathbf{D} = 0;\ \nabla \times \mathbf{B} = \partial_t\mathbf{D} + \sigma\mathbf{E};\ \nabla \cdot \mathbf{B} = 0;\ \nabla \times \mathbf{E} = -\partial_t\mathbf{B} \tag{1.42}$$

and this coincides with the aether equations of [**537, 959**]. In the identification $\mathbf{D} = \mathbf{E} + \sigma\mathbf{A}$ and $\mathbf{B} = \mathbf{H}$ the aether behaves like a medium that responds to the presence of the electric field by creating a polarization $\mathbf{P} = \sigma\mathbf{A}$. One also has a curreent $\mathbf{J} = \sigma\mathbf{E}$ which is in agreement with the supposition of the aether being a medium with conductivity σ. According to Schwinger's idea of a structureless vacuum an EM field disturbs the vacuum affecting its properties of homogeneity and isotropy. This is exactly the situation obtained in the present model where the presence of an EM field in a vacuum with conductivity σ produces a response of the medium ($\mathbf{D} \neq \epsilon\mathbf{E}$) that signals its nonisotropy.

Thus for a flat spacetime and a reference frame at rest relative to the aether one has $\mathbf{D} = \mathbf{E} + \sigma\mathbf{A}$ and $\mathbf{B} = \mathbf{H}$. In the case of a curved spacetime and a noninertial reference frame moving relative to the aether one will have a more complicated relation between H and F. Indeed in a medium that is at rest in any reference fame with a metric $g_{\alpha\beta}$ one knows from [**967**] that the relation between H and F has the form

$$\sqrt{-g}H^{\alpha\beta} = \sqrt{-g}g^{\alpha\gamma}g^{\beta\kappa}S^{\mu}_{\gamma}S^{\nu}_{\kappa}F_{\mu\nu} \tag{1.43}$$

where S^{α}_{β} characterizes the EM properties of the medium. It is convenient to rewrite (1.43) as

$$\sqrt{-g}H^{\alpha\beta} = \sqrt{-g}g^{\alpha\gamma}g^{\beta\kappa}S^{\mu}_{\gamma\kappa}A_{\mu}; \tag{1.44}$$
$$S^{\mu}_{\gamma\kappa} = (S^{\nu}_{\gamma}S^{\mu}_{\kappa} - S^{\mu}_{\gamma}S^{\nu}_{\kappa})\partial_{\nu}$$

As an application of (1.43) it was shown in [**967**] that for the vacuum (considered from an inertial reference frame) the tensor S^{α}_{β} has the form $S^{\alpha}_{\beta} = \delta^{\alpha}_{\beta}$ and the material equations become $\sqrt{-g}H^{\alpha\beta} = \sqrt{-g}g^{\alpha\mu}g^{\beta\nu}F_{\mu\nu}$ which reproduces the usual equations of free electrodynamics in a curved background (cf. [**793**]). Also in the case of a linear isotropic medium that is at rest in an inertial reference frame the tensor S^{α}_{β} is given by $S^0_0 = \epsilon\sqrt{\mu},\ S^1_1 = S^2_2 = S^3_3 = 1/\sqrt{\mu}$ and one obtains the usual relations $\mathbf{D} = \epsilon\mathbf{E}$ and $\mathbf{H} = (1/\mu)\mathbf{B}$. In the present model in order to define $H^{\alpha\beta}$ for a generic reference frame in a curved background and t find a suitable material relation of the type (1.44) one should first follow the preceeding approach which allows one to define $H^{\mu\nu}$ directly from the equation of motion for A_{μ}. Explicitly take the equation of motion in a curved background to be

$$\partial_{\nu}(\sqrt{-g}F^{\mu\nu} - \sigma\sqrt{-g}v^{\nu}A^{\mu} + \sigma\sqrt{-g}v^{\mu}A^{\nu}) = J^{\mu} = -\sigma\sqrt{-g}v_{\nu}F^{\nu\mu} \tag{1.45}$$

Then one defines

$$\sqrt{-g}H^{\alpha\beta} = \sqrt{-g}(F^{\alpha\beta} - \sigma v^{\alpha}A^{\beta} + \sigma v^{\beta}A^{\alpha}) \tag{1.46}$$

This is equivaent to the introduction of an antisymmetric polarization tensor $M^{\alpha\beta}$ (cf. [**967**]) of the form

$$\sqrt{-g}H^{\alpha\beta} = \sqrt{-g}(F^{\alpha\beta} + M^{\alpha\beta}) \iff \sqrt{-g}H^{\alpha\beta} = \sqrt{-g}g^{\alpha\nu}(F_{\nu\mu} + M_{\nu\mu}) \tag{1.47}$$

provided we identify

$$M^{\alpha\beta} = -\sigma v^{\alpha}A^{\beta} + \sigma v^{\beta}A^{\alpha} \iff M_{\alpha\beta} = -\sigma v_{\alpha}A_{\beta} + \sigma v_{\beta}A_{\alpha}; \tag{1.48}$$
$$F^{\mu\nu} = g^{\mu\alpha}g^{\nu\beta}F_{\alpha\beta},\ F_{\mu\nu} = \partial_{\mu}A_{\nu} - \partial_{\nu}A_{\mu};\ M^{\mu\nu} = g^{\mu\alpha}g^{\nu\beta}M_{\alpha\beta}$$

Finally in order to obtain the material equations one extends (1.44) by allowing $S^{\mu}_{\alpha\beta}$ to be a generic operator not restricted as in (1.44) but given by $S^{\mu}_{\alpha\beta} = \delta^{\mu}_{\beta}(\partial_{\alpha} - \sigma v_{\alpha}) - \delta^{\mu}_{\alpha}(\partial_{\beta} - \sigma v_{\beta})$. Here the tensor $S^{\mu}_{\alpha\beta}$ may contain not only EM properties of the medium (as in the case of (1.44)) but also information about the reference frame (implicit in the 4-velocity v_α). Adopting the convention $D_i = \sqrt{-g}H^{i0}$ and $H_i = -(1/2)\epsilon_{ijk}\sqrt{-g}H^{jk}$ (cf. [**967**]) and considering a flat spacetime one obtains (in the aether frame) the relations $\mathbf{D} = \mathbf{E} + \sigma\mathbf{A}$ and $\mathbf{B} = \mathbf{H}$.

1.3. MASSLESS PARTICLES. We review again some classical and quantum features of EM with some repetition of earlier material. First, following [**650**], recall the Maxwell equations

$$\nabla \cdot E = 0;\ \nabla \cdot B = 0;\ \nabla \times E = -\frac{1}{c}B_t;\ \nabla \times B = \frac{1}{c}E_t \tag{1.49}$$

(bold face is omitted). Recall now the vector potential A eners via $B = \nabla \times A$ and from the third equation in (1.49) $E = -(1/c)A_t - \nabla\phi$ where ϕ is the scalar potential. Hence $\nabla^2 A - (1/c^2)A_{tt} = 0$ in the Coulomb gauge defined via $\nabla \cdot A = 0$ and in the absence of sources $\phi = 0$. Separation of variables gives monochromatic solutions

$$A(x,t) = \alpha(t)A_0(x) + \alpha^*(t)A_0^*(x) = \alpha(0)e^{-i\omega t}A_0(x) + \alpha^*(0)e^{i\omega t}A_0^*(x) \tag{1.50}$$

where (**F2**) $\nabla^2 A_0(x) + k^2 A_0(x) = 0$ ($k = \omega/c$) and $\ddot{\alpha} = -\omega^2\alpha$. Consequently

$$E(x,t) = -\frac{1}{c}[\dot{\alpha}(t)A_0(r) + \dot{\alpha}^*(t)A_0^*(x); \tag{1.51}$$

$$B(x,t) = \alpha(t)\nabla \times A_0(x) + \alpha^*(t)\nabla \times A_0^*(x)$$

and a calculation gives the EM energy as

$$H_F = (1/8\pi)\int d^3x(E^2 + B^2) = (k^2/2\pi)|\alpha(t)|^2 \tag{1.52}$$

where A_0 is normalized via $\int d^3x|A_0(x)|^2 = 1$ Now defining

$$q(t) = \frac{i}{c\sqrt{4\pi}}[\alpha(t) - \alpha^*(t)];\ p(t) = \frac{k}{\sqrt{4\pi}}[\alpha(t) + \alpha^*(t)] \tag{1.53}$$

gives $H_F = (1/2)(p^2 + \omega^2 q^2)$ so the field mode of frequency ω is mathematically equivalent to a harmonic oscillator of frequency ω. Note q and p are canonically conjugate since $\dot{q} = p$ and $\dot{p} = -\omega^2 q$ corresponds to Hamiltonian equations with Hamiltonian H_F.

REMARK 5.1.5. We assume familiarity with harmonic oscillator calculations. For $H = (p^2/2m) + (1/2)m\omega^2 q^2$ one has $\dot{q} = (i\hbar)^{-1}[q,H] = p/m$ and $\dot{p} = (i\hbar)^{-1}[p,H] = -m\omega^2 q$. Then defining

$$a = \frac{1}{\sqrt{2m\hbar\omega}}(p - iq);\ a^\dagger = \frac{1}{\sqrt{2m\hbar\omega}}(p + im\omega q) \equiv \tag{1.54}$$

$$\equiv q = i\sqrt{\frac{\hbar}{2m\omega}}(a - a^\dagger);\ p = \sqrt{\frac{m\hbar\omega}{2}}(a + a^\dagger)$$

which yields $[q,p] = i\hbar$ and $[a, a^\dagger] = 1$ with $H = (1/2)\hbar\omega(aa^\dagger + a^\dagger a) = \hbar\omega(a^\dagger a + (1/2))$. For $N = a^\dagger a$ one has eigenkets $N|n >= n|n >$ and $a|n >= \sqrt{n}|n-1>$ with $a^\dagger|n >= \sqrt{n+1}|n+1>$; further $E_n = [n + (1/2)]\hbar\omega$.

One can now replace (1.50) by

$$A(x,t) = \left(\frac{2\pi\hbar c^2}{\omega}\right)^{1/2}[a(t)A_0(x) + a^\dagger(t)A_0^*(x)]; \tag{1.55}$$
$$E(x,t) = i(2\pi\hbar\omega)^{1/2}[a(t)A_0(x) - a^\dagger(t)A_0^*(x)];$$
$$B(x,t) = \left(\frac{2\pi\hbar c^2}{\omega}\right)^{1/2}[a(t)\nabla\times A_0(x) + a^\dagger(t)A_0^*(x)]$$

and H_F becomes $H_F = \hbar\omega[a^\dagger a + (1/2)]$. Now the vacuum state $|0>$ has no photons but has an energy $(1/2)\hbar\omega$ and QM thus predicts the ZPF. In all stationary states $|n>$ one has $< E(x,t) >=< B(x,t) >= 0$ since $< n|a|n >= 0$. This means that E and B fluctuate with zero mean in the state $|n>$ even though the state has a definite nonfluctuating energy $[n + (1/2)]\hbar\omega$. A computation also gives

$$< E^2(x,t)[n + (1/2)]4\pi\hbar\omega|A_0(x)|^2 = 4\pi\hbar\omega|A_0(x)|^2 n + < E^2(x) >_0 \tag{1.56}$$

The factor n is the number of photons and $|A_0(x)|^2$ gives the same spatial intensity as in the classical theory. Normal ordering is used of course in QM ($a^\dagger$ to the left of a) and this eliminates contributions of ZPF to various calculations. However one does not eliminate the ZPF by dropping its energy from the Hamiltonian. In the vacuum state E and B do not have definite values (they fluctuate about a zero mean value) and one arrives at energy densities etc. as before. An atom is often considered to be "dressed" by emmision and reabsorption of virutal photons from the vacuum which itself has an infinite energy. Some thermal aspects were mentioned before (cf. (1.2) for example) and this is enormously important (in this direction there is much information in [**650, 950**]).

REMARK 5.1.6. We refer next to the first paper in [**488**] on massless classical electrodynamics for some interesting calculations (some details are omitted here). One considers a bare charge, free of self action, compensating forces, and radiation reaction (cf. [**488**] for a long discussion on all this). Use the convention $u^a v_a = u_0 v_0 - \mathbf{u}\cdot\mathbf{v}$ and Heavyside-Lorentz units with $c = 1$ in general. With the fields given the Euler equation for the (massless) lone particle degree of freedom is simply that the Lorentz force on the particle in question must vanish, i.e. $F^{\nu\mu}u_{\mu\ell} = 0$ where F is the EM field stregth tensor and the fields E, B are to be evaluated along the trajectory. In $3+1$ form, omitting particle labels, this is

$$(dt(\lambda)/d\lambda)E(x(\lambda),t(\lambda)) + (dx(\lambda)/d\lambda)\times B(x(\lambda),t(\lambda)) = 0 \tag{1.57}$$

In order for $F^{\nu\mu}u_{\mu\ell} = 0$ to have a solution the determinant of F must vanish which gives

$$S(x(\lambda)) \equiv E(x(\lambda))\cdot B(x(\lambda)) = 0 \tag{1.58}$$

This imposes a constraint on the fields along the trajectory which can be interpreted as the condition that the Lorentz force on a particle must vanish (recognized as the constraint on the fields such that there exist a frame in which the electric field is zero). Calculation leads to

$$v(x,t) = \frac{dx/d\lambda}{dt/d\lambda} = \frac{E\times\nabla S - BS_t}{B\cdot\nabla S} \tag{1.59}$$

as the ordinary velocity of the trajectory passing through $(t(\lambda), x(\lambda))$. The right side is an arbitrary function of (x, t) decided by the fields and in general (1.59) will not admit a solution of the form $x = f(t)$ since the solution trajectory may be nonmonotonic in time. Various situations are discussed including the particles advanced and retarded fields. In particular the particle does not respond to force in the traditional sense of Newton's second law. Its motion is precisely that which causes it to feel no force. Yet its motion is uniquely prescribed by E and B, which decide the particle trajectory (given some initial condition) just as the Lorentz force determines the motion of a massive particle. The important difference is that traditionally the fields determine acceleration whereas here they determine the velocity. The massless particle discussed cannot be a relative of the neutrino and it does not seem to be a traditional classical object in need of quantization.

1.4. EINSTEIN AETHER WAVES. We go now to [**510**] where the violation of Lorentz invariance by quantum gravity effects is examined (cf. also [**28, 51, 337, 338, 380, 459, 518, 642**]). In a nongravitational setting it suffices to specify fixed background fields violating Lorentz symmetry in order to formulate the Lorentz violating (LV) matter dynamics. However this would break general covariance, which is not an option, so one promotes the LV background fields to dynamical fields, governed by a generally covariant action. Virtually any configuration of matter fields breaks Lorentz invariance but here the LV fields contemplated are constrained dynamically or kinematically not to vanish, so that every relevant field configuration violates local Lorentz symmetry everywhere, even in the "vacuum". If the Lorentz violation preserves a 3-dimensional rotation subgroup then the background field must be a timelike vector and one considers here the case where the LV field is a unit timelike vector u^a which can be viewed as the minimal structure required to determine a local preferred rest frame. One opts to call this field the "aether" as it is ubiquitous and determines a local preferred rest frame. Kinetic terms in the action couple the ether directly to the spacetime metric in addition to any couplings that might be present between the aether and the matter fields. This system of the metric coupled to the aether will be referred to as Einstein-aether theory. In [**401**] an essentially equivalent theory appears based on a tetrad formalism (cf. also [**510, 642**] for various special cases).

In the spirit of effective field theory consider a derivative expansion of the action for the metric g_{ab} and aether u^a. The most general action that is diffeomorphism invariant and quadratic in derivatives is

(1.60)
$$S = \frac{1}{16\pi G}\int d^4x\sqrt{-g}(-R + \mathfrak{L}_u - \lambda(j^a u_a - 1));\ \mathfrak{L}_u = -K^{ab}_{\ \ mn}\nabla_a u^m \nabla_b u^n;$$

$$K^{ab}_{\ \ mn} = c_1 g^{ab} g_{mn} + c_2 \delta^a_m \delta^b_n + c_3 \delta^a_n \delta^b_m + c_4 u^a u^b g_{mn}$$

Here R is the Ricci scalar and λ is a Lagrange multiplier enforcing the unit constraint. The signature is $(+, -, -, -)$ and $c = 1$ (cf. [**972**] for other notation). The possible term $R_{ab}u^a u^b$ is proportional to the difference of the c_2 and c_3 terms via integration by parts and hence has been omitted. Also any matter coupling is omitted since one wants to concentrate on the metric-aether sector in vacuum. Varying the action with respect t u^a, g^{ab}, and λ yields the field equations

$$\nabla_a J^a_{\ m} = c_4 \dot{u}_a \nabla_m u^a = \lambda u_m;\ G_{ab} = T_{ab};\ g_{ab}u^a u^b = 1 \tag{1.61}$$

Here one has $J^a_{\ m} = K^{ab}_{\ \ mn}\nabla_b u^n$ and $\dot{u}^m = u^a \nabla_a u^m$ and the aether stress tensor is

$$\begin{aligned} T_{ab} = \nabla_m (J_{(a}^{\ \ m} u_{b)} - J^m_{\ (a} u_{b)} - J_{(ab)} u^m) + \\ + c_1[(\nabla_m u_a)(\nabla^m u_b) - (\nabla_a u_m)(\nabla_b u^m)] + c_4 \dot{u}_a \dot{u}_b + \\ + [u_n(\nabla_m J^{mn}) - c_4 \dot{u}^2] u_a u_b - \frac{1}{2} g_{ab} \mathfrak{L}_u \end{aligned} \tag{1.62}$$

Here the constraint has been used to eliminate the term that arises from varying $\sqrt{-g}$ in the constraint term in (1.60) and in the last line λ has been eliminated using the aether field equations.

Now the first step in finding the wave modes is to linearize the field equaitons about the flat Minkowski background η_{ab} and the constant unit vector $\underline{u}^a$. The expanded fields are then

$$g_{ab} = \eta_{ab} + \gamma_{ab};\ u^a = \underline{u}^a + v^a \tag{1.63}$$

The Lagrange multiplier λ vanishes in the background so we use the same notation for its linearized version. Indices will be raised and lowered now with η_{ab} and one adopts Minkowski coordinates (x^0, x^i) aligned with $\underline{u}^a$; i.e. for which $\eta_{ab} = diag(1,-1,-1,-1)$ and $\underline{u}^a = (1,0,0,0)$. Keeping only the first order terms in v^a and γ_{ab} the field equations become

$$\partial_a J^{(1)a}_{\ \ m} = \lambda \underline{u}_m;\ G^{(1)}_{ab} = T^{(1)}_{ab};\ v^0 + \frac{1}{2}\gamma_{00} = 0 \tag{1.64}$$

where the superscript (1) denotes the first order part. The linearized Einstein tensor is

$$G^{(1)}_{ab} = -\frac{1}{2}\Box\gamma_{ab} - \frac{1}{2}\gamma_{,ab} + \gamma_{m(a,b)}^{\ \ m} + \frac{1}{2}\eta_{ab}(\Box\gamma - \gamma_{mn,}^{\ \ mn}) \tag{1.65}$$

where $\gamma_m^{\ m}$ is the trace, while the linearized aether stress tensor is

$$T^{(1)}_{ab} = \partial_m [J^{(1)m}_{(a} \underline{u}_{b)} - J^{(1)m}_{(a} \underline{u}_{b)} - J^{(1)}_{(ab)} \underline{u}^m] + [\underline{u}_n(\partial_m J^{(1)mn})] \underline{u}_a \underline{u}_b \tag{1.66}$$

In one imposes the linearized aether field equation (1.64) then the second and last terms of this expression for $T^{(1)}_{ab}$ cancel, yielding

$$T^{(1)}_{ab} = -\partial_0 J^{(1)}_{(ab)} + \partial_m J^{(1)m}_{(a} \underline{u}_{b)}; \tag{1.67}$$

$$J^{(1)}_{ab} = c_1 \nabla_a u_b + c_2 \eta_{ab} \nabla_m u^m + c_3 \nabla_b u_a + c_4 \underline{u}_a \nabla_0 u_b$$

where the covariant derivatives of u^a are expanded to linear order, i.e. replaced by

$$(\nabla_a u_b)^{(1)} = (v_b + (1/2)\gamma_{0b})_{,a} + (1/2)\gamma_{ab,0} - (1/2)\gamma_{a0,b} \tag{1.68}$$

Were it not for the aether background the linearized aether stress tensor (1.66) would vanish and the metric would drop out of the aether field equation, leaving all modes uncoupled.

Now diffeomorphism invariance of the action (1.60) implies that the field equations are tensorial and hence covariant under diffeomorphisms. The linearized equations inherit the linearized version of this symmetry and to find the independent physical wave modes one must fix the corresponding gauge symmetry. Thus an infinitesimal diffeomorphism generated by a vector field ξ^a transforms g_{ab} and u^a by

$$\delta g_{ab} = \mathfrak{L}_\xi g_{ab} = \nabla_a\xi_b + \nabla_b\xi_a;\ \delta u^a = \mathfrak{L}_\xi u^a = \xi^m\nabla_m u^a - u^m\nabla_m\xi^a \tag{1.69}$$

ξ^a is itself first order in the perturbations so the linearized gauge transformations take the form

$$\gamma'_{ab} = \gamma_{ab} + \partial_a\xi_b + \partial_b\xi_a;\ v^{'a} = v^a - \partial_0\xi^a \tag{1.70}$$

The usual choice of gauge in vacuum GR is the Lorentz gauge $\partial^a\bar{\gamma}_{ab} = 0$ where $\bar{\gamma}_{ab} = \gamma_{ab} - (1/2)\gamma\eta_{ab}$ but for various reasons this is inappropriate here (cf. [**510**]). Instead one imposes directly the four gauge conditions

$$\gamma_{0i} = 0;\ v_{i,i} = 0 \tag{1.71}$$

To see that this is accessible note that the gauge variations of γ_{0i} and $v_{i,i}$ are, according to (1.70)

$$\delta\gamma_{0i} = \xi_{i,0} + \xi_{0,i};\ \delta v_{i,i} = -\xi_{i,i0} \tag{1.72}$$

Thus to achieve (1.71) one must choose ξ_0 and ξ_i to satisfy equations of the form

$$(\mathbf{A})\ \xi_{i,0} + \xi_{0,i} = X_i \quad (\mathbf{B})\ \xi_{i,i0} = Y \tag{1.73}$$

Subtracting the second equation from the divergence of the first gives $\xi_{0,ii} = X_{i,i} - Y$ which determines ξ_0 up to constants of integration by solving a Poisson equation. Then ξ_i can be determined up to a time independent field by integrating **(A)** in (1.73) with respect to time. From these choices of ξ_0 and ξ_i **(A)** in (1.73) holds and the divergence of this gives **(B)** in (1.73). In the gauge (1.71) the tensors in the aether and spatial metric equations in (1.64) take the forms

$$J_{ai,}^{\ \ a} = c_{14}(v_{i,00} - (1/2)\gamma_{00,i0}) - c_1 v_{i,kk} - (1/2)c_{13}\gamma_{ik,k0} - (1/2)c_2\gamma_{kk,0i}; \tag{1.74}$$

$$G_{ij}^{(1)} = -(1/2)\Box\gamma_{ij} - (1/2)\gamma_{,ij} - \gamma_{k(i,j)k} - (1/2)\delta_{ij}(\Box\gamma - \gamma_{00,00} - \gamma_{k\ell,k\ell});$$

$$T_{ij}^{(1)} = -c_{13}(v_{(i,j)0} + (1/2)\gamma_{ij,00} - (1/2)c_2\delta_{ij}\gamma_{kk,00}$$

where e.g. $c_{14} = c_1 + c_4$ etc. Various wave modes are calculated and in particular there are a total of 5 modes, 2 with an unexcited aether which correspond to the usual GR modes, two "transverse" aether-metric modes, and a fifth trace aether-metric mode. We refer to [**510**] for details and discussion.

REMARK 5.2.1. The Lorentz violation theme and the idea of a preferred reference frame is presently of a certain general interest in connection with proposals of quantum gravity and we cite as before [**28, 51, 76, 337, 338, 380, 518, 642**]. We rephrase matters here as in [**380**]. Thus in an effective field theory description the Lorentz symmetry breaking can be realized by a vector field that defines the preferred frame. In the flat spacetime of the standard model this field can be treated as non-dynamical background structure but in the context of GR diffeomorphism invariance (a symmetry distinct from local Lorentz invariance) can be preserved by elevating this field to a dynamical quantity. This leads to investigation of vector-tensor theories of gravity and one such model couples gravity to a vector field that is constrained to be everywhere timelike and of unit norm. The unit norm condition embodies the notion that the theory assigns no physical importance to the norm of the vector and this corresponds to the Einstein aether (AE) theory as in [**337**]. In [**380**] one demonstrates the effect of a field redefinition on the conventional second order AE theory action. Thus take $g_{ab} \to g'_{ab} = A(g_{ab} - (1-B)u_a u_b)$ with $u^a \to (u')^a = (1/\sqrt{AB})u^a$ where g_{ab} is a Lorentzian metric and u^a is the aether field. The action is taken as the most general form which is generally covariant, second order in derivatives, and is consistent with the unit norm constraint. The redefinition preserves this form and the net effect is a rescaling of the action and a transformation of the coupling constants (generalizing the work of [**76**]). Thus start with $S = -(1/16\pi G)\int \sqrt{|g|}\mathfrak{L}$ where

$$\mathfrak{L} = R + c_1(\nabla_a u_b)(\nabla^a u^b) + c_2(\nabla_a u^a)(\nabla_b u^b) + \tag{1.75}$$
$$+ c_3(\nabla_a u^b)(\nabla_b u^a) + c_4(u^a\nabla_a u^c)(u^b\nabla_b u_c)$$

where R is the scalar curvature of g_{ab} with signature $(+,-,-,-)$ and the c_i are dimensionless constants. After the substitution indicated above one has

$$(g')^{ab} = \frac{1}{A}\left(g^{ab} - \left(1 - \frac{1}{B}\right)u^a u^b\right);\ u'_a = \sqrt{AB}u_a \tag{1.76}$$

(note $(u')^a u'_a = 1$ is preserved). The net effect of A is a rescaling of the action by a factor of A (with the Lagrangian scaling as $1/A$ while the volume scales as A^2). There are many calculations (omitted here) and the constructions simplify the problem of characterizing solutions for a specific set of c_i by transforming that set into one in which one or more of the c_i vanish. If non-aether matter is included a metric redefinition not only changes the c_i but also modifies the matter action suggesting perhaps a universal metric to which the matter couples (cf. [**213, 518**]).

2. STOCHASTIC ELECTRODYNAMICS

From topics in Chapters 1,2,3, and 4 we are familiar with some stochastic aspects of QM. Further the ZPF has been seen to be related to quantum phenomena. The idea of stochastic electrodynamcs (SED) is essentially an attempt to establish SED as the foundation for QM. There has been some partial success in this direction but the methods break down when trying to deal with nonlinearities. The paper [**754**] is a recent version of nonperturbative linear SED (LSED) which provides a speculative mechanism leading to the quantum behavior of field and matter based on 3 fundamental principles; it purports to explain for example why all systems described by it (and hence by QM ?) behave as if they consisted of a set of harmonic oscillators. We review here first some basic background issues in SED arising from [**753**] (cf. also [**250, 509, 650**]) and then will sketch a few matters from [**754**].

Thus start with the homogeneous Maxwell equations

$$\nabla\cdot\mathbf{D}=0;\ -\frac{1}{c}\frac{\partial\mathbf{D}}{\partial t}+\nabla\times\mathbf{H}=0;\ \nabla\cdot\mathbf{B}=0;\ \frac{1}{c}\frac{\partial\mathbf{B}}{\partial t}+\nabla\times\mathbf{E}=0 \tag{3.1}$$

One obtains then (we refer to [**753**] for details and discussion)

$$\mathbf{B}=\nabla\times\mathbf{A};\ \mathbf{E}=-\nabla\Phi-\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t};\ \nabla\times\left(\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t}+\mathbf{E}\right)=0 \tag{3.2}$$

Then the third and fourth Maxwell equations are satisfied identically and the first two in combination with $\mathbf{D}=\epsilon\mathbf{E}$ and $\mathbf{B}=\mu\mathbf{H}$ determine the evolution of the potentials $\mathbf{A}$ and Φ. In particular one has

$$\nabla^2\Phi+\frac{1}{c}\partial_t\nabla\cdot\mathbf{A}=0;\ \nabla^2\underline{A}-\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2}-\nabla\left(\nabla\cdot\mathbf{A}+\frac{1}{c}\frac{\partial\Phi}{\partial t}\right)=0 \tag{3.3}$$

There is then room for gauge transformations

$$\mathbf{A}\to\mathbf{A}'=\mathbf{A}+\nabla\Lambda;\ \Phi\to\Phi'=\Phi-\frac{1}{c}\frac{\partial\Lambda}{\partial t} \tag{3.4}$$

Now choose the potentials to uncouple (3.3) via

$$\nabla\cdot\mathbf{A}+\frac{1}{c}\frac{\partial\Phi}{\partial t};\ \nabla^2\Phi-\frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2}=0;\ \nabla^2\mathbf{A}-\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2}=0 \tag{3.5}$$

This set is equivalent in all respects to the Maxwell equations in vacuum (Lorentz gauge) and this arrangement can always be achieved via

$$\nabla^2\Lambda-\frac{1}{c^2}\frac{\partial^2\Lambda}{\partial t^2}=0 \tag{3.6}$$

Another gauge selection is the Coulomb gauge defined via

$$\nabla\cdot\mathbf{A}=0;\ \nabla^2\Phi=0;\ \nabla^2\mathbf{A}-\frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2}=\frac{1}{c}\nabla\partial_t\Phi \tag{3.7}$$

One can take $\Phi=0$ and the last equation reduces to the last equation of (3.5). In any case for any gauge one has

$$\nabla^2\mathbf{E}-\frac{1}{c^2}\frac{\partial^2\mathbf{E}}{\partial t^2}=0;\ \Box^2\mathbf{A}=\Box^2\mathbf{B}=\Box^2\mathbf{E}=0 \tag{3.8}$$

where $\Box^2=(1/c^2)\partial_t^2-\nabla^2$. One now goes into mode expansions with Fourier series and integrals and we simply list formulas of the form ($a_\alpha=a_\alpha(t)=a_{n\lambda}exp(-i\omega_n t)=a_{n\lambda}(t)$)

$$q_\alpha=i\sqrt{\frac{E_\alpha}{2\omega_\alpha^2}}(a_\alpha-a_\alpha^*);\ p_\alpha=\sqrt{\frac{E_\alpha}{2}}(a_\alpha+a_\alpha^*); \tag{3.9}$$

$$H=\sum H_\alpha=\sum E_\alpha a_\alpha^* a_\alpha=\sum\frac{1}{2}(p_\alpha^2+\omega_\alpha^2 q_\alpha^2)$$

$$\mathbf{A}=\sum\sqrt{\frac{4\pi c^2}{\omega_n^2 V}}e_n^\lambda[p_{n\lambda}Cos(k\cdot x)+\omega_n q_{n\lambda}Sin(k\cdot x)]; \tag{3.10}$$

$$\mathbf{E}=\sum\sqrt{\frac{4\pi}{V}}e_n^\lambda[-p_{n\lambda}Sin(k\cdot x)+\omega_n q_{n\lambda}Cos(k\cdot x)];$$

$$\mathbf{B} = \sum \sqrt{\frac{4\pi c^2}{\omega_n^2 V}}(k \times e_n^\lambda)[-p_{n\lambda} Sin(k\cdot x) + \omega_n q_{n\lambda} Cos(k\cdot x)]$$

(we have written k for $\mathbf{k} \sim \mathbf{k_n}$, x for $\mathbf{x}$, e_n^λ for $\mathbf{e}_n^\lambda$, n for $\mathbf{n} \sim (n_1, n_2, n_3)$, etc. and one is thinking of a reference volume $V = L_1L_2L_3$ with sides L_i and $\omega_n = c|\mathbf{k_n}|$. Further $\mathbf{k_n}$ has components $k_i = (2\pi/L_i)$ and, setting $\hat{\mathbf{k}}_\mathbf{n} = \mathbf{k_n}/|\mathbf{k_n}|$, one has

(3.11) $\mathbf{k_n}\cdot e_n^\lambda = 0;\ e_n^\lambda \cdot e_n^{\lambda'} = \delta_{\lambda\lambda'};\ \hat{\mathbf{k}}_\mathbf{n} = e_n^1 \times e_n^2;\ e_n^1 = -\hat{\mathbf{k}}_n \times e_n^2;\ e_n^2 = \hat{\mathbf{k}}_\mathbf{n} \times e_n^1$

Now the nonrelativistic Hamiltonian describing a charged particle in interaction with the radiation field is

(3.12) $$H = \frac{1}{2m}\left(\mathbf{p} - \frac{e}{c}\mathbf{A}\right)^2 + e\Phi + \frac{1}{8\pi}\int d^3x(\mathbf{E}^{\perp 2} + \mathbf{B}^2)$$

Here $\mathbf{E} = \mathbf{E}^\perp + \mathbf{E}^{||}$ and the contribution from the longitudinal part has been written as the Coulomb potential $e\Phi$; in the Coulomb gauge $\mathbf{E}^{||} = -\nabla\Phi$ so

(3.13) $$\int d^3\mathbf{E}^{||2} = \int d^3x(\nabla\Phi)^2 = \int d^3x\nabla\cdot(\Phi\nabla\Phi) - \int d^3x\Phi\nabla^2\Phi = 4\pi\int d^3x\rho(x)\Phi$$

since $\nabla^2\Phi = -\nabla\cdot\mathbf{E}^{||} = -4\pi\rho(x)$ (Gauss law). From this one obtains the equations of motion for the particle ($x \sim \mathbf{x}$)

(3.14) $$m\dot{x} = \mathbf{p} - \frac{e}{c}\mathbf{A};\ \dot{\mathbf{p}} = \frac{e}{c}[\dot{x}\times\mathbf{B} + (\dot{x}\cdot\nabla)\mathbf{A}] - e\nabla\Phi$$

and also the following Maxwell equations in the presence of the charge and its current

(3.15) $$\nabla\cdot\mathbf{D} = 4\pi\rho;\ \nabla\times\mathbf{H} = \frac{4\pi}{c}\mathbf{J} + \frac{1}{c}\frac{\partial\mathbf{D}}{\partial t};\ \rho = e\delta^3(x - x_p(t));\ \mathbf{J} = e\dot{x}_p\delta^3(x - x_p(t))$$

where $x_p(t)$ is the actual position of the particle. Combining these equations one obtains the Newton second law with the Lorentz force

(3.16) $$m\ddot{x} = e\mathbf{E} + \frac{e}{c}\dot{x}\times\mathbf{B}$$

Some simplification arises if one expresses matters via field modes $(q_\alpha(t), p_\alpha(t))$ and $(x(t), p(t))$ in the form ($p \sim \mathbf{p},\ x \sim \mathbf{x}$, etc. with $e' = e\sqrt{4\pi/V}$)

(3.17) $$m\dot{x} = p - \frac{e}{c}\mathbf{A};\ \dot{p} = F + e'\sum(\dot{x}\cdot e_\alpha)k\left(q_\alpha cos(k\cdot x) - \frac{p_\alpha}{\omega_\alpha}Sin(k\cdot x)\right);$$

$$\dot{q}_\alpha = p_\alpha - e'(\dot{x}\cdot e_\alpha)\frac{1}{\omega_\alpha}Cos(k\cdot x);\ \dot{p}_\alpha = -\omega_\alpha^2 q_\alpha + e'(\dot{x}\cdot e_\alpha)Sin(k\cdot x)$$

where $F = -\nabla(e\Phi)$ (without the Coulomb self-interaction). Write now

(3.18) $$c_\alpha(t) = \frac{1}{\sqrt{2E_\alpha}}(p_\alpha - i\omega_\alpha q_\alpha);\ c_\alpha^* = \frac{1}{\sqrt{2E_\alpha}}(p_\alpha + i\omega_\alpha q_\alpha)$$

Some calculation gives ($x' = x(t')$)

(3.19) $$c_\alpha(t) = a_\alpha(t) - ie\sqrt{\frac{2\pi}{E_\alpha V}}e^{-i\omega_\alpha t}\int_0^t(\dot{x}'\cdot e_\alpha)e^{-ik\cdot x' + i\omega_\alpha t'}dt'$$

Combining (3.17) and (3.19) one gets the equation of motion $m\ddot{x} = F + F_{Lor}^{free} + F_{self}$ where

$$F_{Lor}^{free} = ie\sum\sqrt{\frac{2\pi E_\alpha}{V}}\left(e_\alpha + \frac{\dot{x}}{c}\times(\hat{k}\times e_\alpha)\right)\times(a_\alpha e^{-ik\cdot x} - a_\alpha^* e^{ik\cdot x}); \tag{3.20}$$

$$F_{self} = -\frac{4\pi e^2}{V}\sum\left(e_\alpha + \frac{\dot{x}}{c}\times(\hat{k}\times e_\alpha)\right)\times\int dt'(\dot{x}'\cdot e_\alpha)Cos(\omega_\alpha(t'-t) - k(x'-x))$$

The self force is too complicated to handle here and some approximations are made (cf. [**753**] for details) leading to an expression

$$F_{self} = -\frac{4e^2}{3c^2}\left(\frac{1}{2}\dddot{x}(t) - \frac{\ddot{x}(t)}{\pi}\int_0^\infty d\omega\right) = m\tau\dddot{x}(t) - \delta m\ddot{x}(t) \tag{3.21}$$

where $\tau = (2e^2/3mc^3)$ and $\delta m = (4e^2/3\pi c^2)\int_0^\infty d\omega$. The self radiation has two effects now, within the present approximation (cf. also [**650**]). First there is a reaction force on the particle proportional to the time derivative of the acceleration (radiation reaction) and secondly there is a contribution to the term $m\ddot{x}(t)$ involving a total or dressed mass of the particle $m_T = m + \delta m$. This contribution is infinite for the point particle since $\int_0^\infty d\omega$ is divergent but a cure is to take a cutoff ω_c so that $\delta m = (2/\pi)m\tau\omega_c$. Even for huge ω_c the quantity $\delta m/m$ is smaller than 1 since τ is very small and one conjectures that in a more precise calculation the mass correction would be at most of order $\alpha = e^2/\hbar c = 1/137$ (fine structure constant). In any case one is led to the Abraham-Lorentz equation

$$m_T\ddot{x} \equiv (m + \delta m)\ddot{x} = F + F_{Lor}^{free} + m_T\tau\dddot{x} \tag{3.22}$$

Note that the self field terms have this simple form only in free space. In any event such an equation is beset with problems; as an example one looks at a homogeneous time dependent force $F(t)$ and the equation $m\ddot{x} = F(t) + m\tau\dddot{x}$ or $a - \tau a = F(t)/m$ with solution

$$a(t) = e^{t/\tau}\left(a(0) - \frac{1}{m\tau}\int_0^t e^{t'/\tau}F(t')dt'\right) \tag{3.23}$$

Problems with acausality arise and one comes to the conclusion that there seems to be no (classical or relativistic) equation of motion for a radiating particle in interaction with the radiation field that is free of conceptual difficulties.

A variation on the Abraham-Lorentz equation in the form

$$m\ddot{x} = -m\omega_0^2 x + m\tau\dddot{x} + eE_x(x,t) + e\left(\frac{\dot{x}}{c}\times\mathbf{B}\right) \tag{3.24}$$

is the Braffort-Marshall equation; it is the analogue of the Langevin equation in Brownian motion (cf. here one refers back to the form $m\ddot{x} = F + F_{Lor}^{free} + F_{self}$ with $m \sim m_T$ and $\tau = 2e^2/3mc^3$). Upon approximating $\dddot{x}$ by$-\omega_0^2\dot{x}$ and linearizing one has something tractable but we do not discuss this here. There is also considerable discussion of harmonic oscillators and Fokker-Planck equations which we omit here. A Braffort-Marshall equation arises again in linear SED in the form

$$m\ddot{x} = m\tau\dddot{x} + F(x) + eE(t) \tag{3.25}$$

(1-D suffices here and one observes on p.303 of [**753**] that SED and QM are incompatible theories). This is subsequently modified to

$$m\ddot{x} = m\tau\dddot{x} + F(x) + e\sum \tilde{E}_k a_k^0 e^{-i\omega t} + c.c. \tag{3.26}$$

Here one has started from a standard Fourier representation of the ZPF in the form

$$\mathbf{E} = \sum \tilde{E}_k a_k e^{-i\omega_n t} + c.c.;\ a_k^0 \sim e^{i\phi_k};\ a_k \to a_k^0 \sim c \to \infty \tag{3.27}$$

with random phases ϕ_k uniformly distributed over $(0, 2\pi)$ (x here describes the response of the particle to the effective field). A lot of partial averaging has gone into this and we refer to [**753**] for details and discussion. Even for simple examples the calculations are a kind of horror story!

Let us try to summarize now some of [**754**]. One recalls that the central premise of SED is that the quantum behavior of the particle is a result of its interaction with the vacuum radiation field or ZPF. This field is assumed to pervade the space and is considered here to be in a stationary state with well defined stochastic properties. Its action on the particle is to impress upon it at every point a stochastic motion with an intensity characterized by Planck's constant which is a measure of the magnitude of fluctuations of the vacuum field. One begins with the Braffort-Marshall equation in the form

$$m\ddot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + m\tau\dddot{\mathbf{x}} + e\mathbf{E}(t) \tag{3.28}$$

where $\tau = 2e^2/3mc^3 \sim 10^{-23}$ for the electron. The term $e\mathbf{E}(t)$ stands for the electric force exerted by the ZPF on the particle (the magnetic term is omitted since one deals here with the nonrelativistic case). LSED is now to be based on three principles:

(1) Principle 1. **The system under study reaches an equilibrium state at which the rate of energy radiated by the particle equals the average rate of energy absorbed by it from the field.** To make this quantitative multiply (3.28) by $\dot{\mathbf{x}}$ to obtain

$$\left\langle \frac{dH}{dt} \right\rangle = -m\tau\left\langle \ddot{\mathbf{x}} \right\rangle + e\left\langle \dot{\mathbf{x}} \cdot \mathbf{E} \right\rangle \tag{3.29}$$

where H is the particle Hamiltonian including the Schott energy

$$H = \frac{1}{2}m\dot{\mathbf{x}}^2 + V(\mathbf{x}) - m\tau\dot{\mathbf{x}} \cdot \ddot{\mathbf{x}} \tag{3.30}$$

and V is the potential associated to the external force $\mathbf{f}$. The average is over the realizations of the background ZPF and when the system has reached the state of energetic equilibrium we have

$$\left\langle \frac{dH}{dt} \right\rangle = 0 \Rightarrow m\tau\left\langle \ddot{\mathbf{x}}^2 \right\rangle = e\left\langle \dot{\mathbf{x}} \cdot \mathbf{E} \right\rangle \tag{3.31}$$

When this equilibrium is reached (or nearly so) one says the system has reached the quantum regime. In LSED it is claimed that detailed energy balance will hold (i.e. for every frequency). One sees that at equilibrium the term $< \ddot{\mathbf{x}} >$ is determined by the ZPF and hence the acceleration itself should be determined by the field.

(2) Principle 2. **Once the quantum regime has been attained the vacuum field has gained control over the motion of the material part of the system.** To apply this principle consider the free particle with $m\ddot{x} = m\tau\dddot{x} + eE(t)$ and express the field via

$$E(t) = \sum \tilde{E}_\beta a_\beta e^{i\omega_\beta t} = \sum_{\omega_\beta > 0} (\tilde{E}^+_\beta a_\beta e^{i\omega_\beta t} + \tilde{E}^-_\beta a^*_\beta e^{-i\omega_\beta t}) \tag{3.32}$$

The $a_\beta = a_\beta(\omega_\beta)$ are stochastic variables and the present approach leaves them momentarily unspecified. The amplitudes $\tilde{E}_\beta$ will be selected to assign to each mode of the field the mean energy $E_\beta = (1/2)\hbar\omega_\beta$ and this is the unique door through which Planck's constant enters the theory. Write now

$$E_\beta = \frac{1}{2} < p^2_\beta + \omega^2_\beta q^2_\beta >; \; p_\beta = \sqrt{\frac{E_\beta}{2}}(a_\omega + a^*_\omega); \; i\omega_\beta q_\beta = \sqrt{\frac{E_\beta}{2}}(a_\omega - a^*_\omega) \tag{3.33}$$

The solution to $m\ddot{x} = m\tau \overset{\dots}{[\,x]} + eE(t)$ is then

$$x(t) = \sum \tilde{x}_\beta a_\beta e^{i\omega_\beta t} = -\sum \frac{e\tilde{E}_\beta a_\beta}{m\omega^2_\beta + im\tau\omega^3_\beta} e^{i\omega_\beta t} \tag{3.34}$$

Here all quantities except the a_β are "sure" numbers but upon introduction of an external force $f(x)$ these parameters become in principle stochastic variables. Indeed from (3.28)

$$\sum \left(-m\omega^2_\beta \tilde{x}_\beta - im\tau\omega^2_\beta \tilde{x}_\beta + \frac{\tilde{f}_\beta}{a_\beta} \right) = e \sum \tilde{E}_\beta a_\beta e^{i\omega_\beta t} \tag{3.35}$$

For a generic force the Fourier coefficients $\tilde{f}_\beta$ will be a complicated function of $(\tilde{x}_\beta)$ and (a_β). Writing now

$$\tilde{x}_\beta = -\frac{e\tilde{E}_\beta}{m\omega^2_\beta + im\tau\omega^3_\beta + (\tilde{f}_\beta/\tilde{x}_\beta a_\beta)} \tag{3.36}$$

and putting this into (3.34) one gets

$$x(t) = -\sum \frac{e\tilde{E}_\beta a_\beta}{m\omega^2_\beta + im\tau\omega^3_\beta + (\tilde{f}_\beta/\tilde{x}_\beta a_\beta)} \tag{3.37}$$

The problem of determining $x(t)$ in general seems impossible. One tries now to simplify matters by looking for stable "orbits" and this leads to

(3) Principle 3. **There exist states of matter (quantum states) that are unspecific to (or basically independent of) the particular realization of the ZPF.** The ensuing calculations in [**754**] seem to be somewhat mysterious and we refer to this paper for further discussion (cf. also [**440, 784, 858**]).

REMARK 5.3.1. We refer here to [**177, 363, 364, 365, 366**]) where, following Feynman, the idea is to introduce QM via the relativistic theory of free photons. The arguments are very physical and historically based which would make a welcome complement to the mainly mathematical features of the rest of this book and there is a lovely interplay of physical ideas (cf. also [**1021**]).

3. PHOTONS AND EM

We go here to [**940, 941**] and will concentrate on the third paper in [**940**]. In [**941**] one sketches an heuristic approach to develop QM as a field theory with quantum paricles (or rather clouds) an emergent phenomenon. This is continued in [**940**] for EM fields and the photon but one comes to the conclusion that it is wrong to think of the photon as the particle like duality partner of the wave associated to the EM field. This leads to the idea that it is photons (not EM fields) that are the basic ontology and EM fields are an emergent collective property of an ensemble of photons. Quantization of the fields is thus not necessary. We sketch this now following the second and third papers in [**940**]. First one notes that quantization of the Lagrangian field theory for EM presents several difficulties and may not be the way to go (cf. [**940, 982**]). Second in order to maintain relativistic covariance every Lorentz transformation must be accompanied by a gauge transformation (cf. [**1007**]). This seems to indicate a clear preference for thinking of the photons as the entities responsible for the EM fields. The elements of physical reality for the classical relativistic photon can be encoded in a photon tensor $f^{\mu\nu}$ described in terms of three vectors **e** and **b**. To arrive at this one uses $g^{\mu\nu} = g_{\mu\nu} = diag(1, -1, -1, -1)$ and the completely antisymmetric tensors $\epsilon_{jk\ell}$, $\epsilon_{k\mu\sigma\rho}$. One postulates the existence of a massless physical system, called photons, transporting energy E, momentum **P**, and intrinsic angular momentum or spin **S**. The relativistic kinemmatics of a massless particle requires that its speed must be c and the energy, momentum, and spin are related via

$$E = c|\mathbf{P}|;\ \mathbf{S} \times \mathbf{P} = 0 \tag{4.1}$$

The second relation, called the transversality constraint, follows from the fact that the intrinsic angular momentum of an extended object moving at the speed c can not have any component perpendicular to the direction of propagation (easy argument). Further one thinks of particles as having spatial extension since massive particles can not be localized beyond their Compton wavelength and a photon moving in a well defined direction must be extended in a direction perpendicular to the direction of movement via the uncertainty principle. The massless character of the photon suggests that its energy must be proportional to some frequency; i.e. something in the photon must be changing periodically in time. Thus for motion with $p^{\mu} = (E, E, 0, 0)$ for example (and $c = 1$) a Lorentz transformation with speed β and Lorentz factor $\gamma = (1 - \beta^2)^{-1/2}$ in the direction of the x^1 axis leads to an energy decrease

$$E' = E\gamma(1 - \beta) = E\sqrt{\frac{1-\beta}{1+\beta}} \tag{4.2}$$

in the primed reference frame. However this is precisely the transformation property of a frequency (Doppler shift) and one can therefore expect that the energy of a photon depends on some frequency (note that the argument is classical and one knows from QM that $E = \hbar\nu$). An argument is then made in [**940**] that the natural object representing the photon will be a tensor (cf. [**1021**])

$$f^{\mu\nu} = \begin{pmatrix} 0 & e_1 & e_2 & e_3 \\ -e_1 & 0 & b_3 & -b_2 \\ -e_2 & -b_3 & 0 & b_1 \\ -e_3 & b_2 & -b_1 & 0 \end{pmatrix} \tag{4.3}$$

Note $\mathbf{e}$ and $\mathbf{b}$ are not the space components of a 4-vector but they transform as 3-vectors under the subgroup of Lorentz transformations corresponding to space rotations and translations.The energy, momentum, and spin are related by (4.1), and in any reference frame one visualizes a positive or negative helicity photon of energy E and spin $\hbar$, propagating with speed c in a direction $\mathbf{k}$, as a unit vector $\hat{\mathbf{e}}$ rotating clockwise of counterclockwise in a plane orthogonal to $\mathbf{k}$ with frequency $\omega = E/\hbar$. In the same plane there is another unit vector $\hat{\mathbf{b}} = \mathbf{k} \times \hat{\mathbf{e}}$ and with the vectors $\mathbf{e} = \omega\hat{\mathbf{e}}$ and $\mathbf{b} = \omega\hat{\mathbf{b}}$ one can build the photon tensor $f^{\mu\nu}$ whose Lorentz transformations provide the description of the photon in other reference frames. The rotating vectors $\mathbf{e}_s$, corresponding to a photon of helicity $s = \pm 1$, can be given more conveniently via circular polarization vectors ϵ_s defined by

$$\epsilon_+ = \frac{1}{\sqrt{2}}(\hat{\mathbf{e}} + i\hat{\mathbf{b}});\ \epsilon_- = \frac{1}{\sqrt{2}}(i\hat{\mathbf{e}} + \hat{\mathbf{b}}) \tag{4.4}$$

resulting in

$$\mathbf{e}_+(t) = \omega(\hat{\mathbf{e}}Cos(\omega t) + \hat{\mathbf{b}}Sin(\omega t)) = \left(\frac{\omega}{\sqrt{2}}\epsilon_+ e^{-i\omega t} + c.c\right); \tag{4.5}$$

$$\mathbf{e}_- = \omega(\hat{\mathbf{b}}Cos(\omega t) + \hat{\mathbf{e}}Sin(\omega t)) = \left(\frac{\omega}{\sqrt{2}}\epsilon_- e^{-i\omega t} + c.c.\right)$$

(c.c. means complex conjugation of the previous terms). Another initial position of the vector can be achieved via multiplication of the circular polarization complex vectors by a phase $exp(-i\theta)\epsilon_s$ to get

$$\mathbf{e}_s(t) = \left(\frac{\omega}{\sqrt{2}}\epsilon_s e^{-i\omega t} + c.c\right) \tag{4.6}$$

The other vector $\mathbf{b}_s$ needed to build the photon tensor is simply $\mathbf{b}_s(t) = \mathbf{k} \times \mathbf{e}_s(t)$ (note one should write $\epsilon_s(\mathbf{k})$ but this will be omitted). The usual orthogonality equations are

$$\epsilon_s^* \cdot \mathbf{k} = 0;\ \epsilon_s^* \cdot \epsilon_{s'} = \delta_{s,s'};\ \epsilon_s^* \times \epsilon_{s'} = is\mathbf{k}\delta_{s,s'};\ \mathbf{k} \times \epsilon_s = s\epsilon_{-s}^* \tag{4.7}$$

A few other relations are

$$\epsilon_- = i\epsilon_+^*;\ \epsilon_s \cdot \epsilon_{s'} = i(1 - \delta_{s,s'}) = i\delta_{s,-s'};\ \epsilon_s \times \epsilon_{s'} = s\mathbf{k}(1 - \delta_{s,s'}) = s\mathbf{k}\delta_{s,-s'} \tag{4.8}$$

$$\sum_{s=\pm}(\epsilon_s^*)_i(\epsilon_s)_j = \delta_{ij} - (\mathbf{k})_i(\mathbf{k})_j \tag{4.9}$$

where the components refer to an arbitrary set of orthogonal unit vectors $(\hat{\mathbf{x}}_1, \hat{\mathbf{x}}_2, \hat{\mathbf{x}}_3)$.

The QM description of a photon in a Hilbert space $\mathfrak{H} = \mathfrak{H}^S \otimes \mathfrak{H}^K$ (spin and kinematic part) is done in terms of eigenstates of fixed helicity $s = \pm 1$ and momentum $\mathbf{p}$ (in the direction $\mathbf{k}$), denoted by $\phi_{s,p} = \chi_s^k \otimes \phi_p$. One takes spin operators

(4.10)

$$S_x = \hbar \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}; \ S_y = \hbar \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ -i & 0 & 0 \end{pmatrix}; \ S_z = \hbar \begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Thus $(S_j)_{k\ell} = -i\hbar\epsilon_{jk\ell}$ and the helicity states are

(4.11)

$$\chi_\pm = \frac{1}{2\sqrt{1-k_xk_y-k_yk_z-k_zk_x}} \begin{pmatrix} 1-k_x(k_x+k_y+k_z) \pm i(k_y-k_z) \\ 1-k_y(k_x+k_y+k_z) \pm i(k_z-k_x) \\ 1-k_z(k_x+k_y+k_z) \pm i(k_x-k_y) \end{pmatrix}$$

For the kinematic part use (rigged) square integrable functions (cf. [**202**]) of the form

$$\phi_p(\mathbf{r}) = \frac{1}{(\sqrt{2\pi}\hbar)^3} exp\left(\frac{i}{\hbar}\mathbf{p}\cdot\mathbf{r}\right) \tag{4.12}$$

One will show now how the EM fields are built and emerge as an observable property of an ensemble of photons.

For many photons one builds up a Fock space with Hilbert spaces for $1, 2, \cdots$ photons with annihilation and creation operators via (we write p for $\mathbf{p}$ at times)

$$a_s^\dagger(p)\phi_{s_1p_1,\cdots,s_np_n} = \sqrt{n+1}\phi_{sp,s_1p_1,\cdots,s_np_n}; \tag{4.13}$$

$$a_s(p)\phi_{s_1p_1,\cdots,s_np_n} = \frac{1}{\sqrt{n}}\sum_1^n \delta_{s,s_i}\delta(p-p_i)\phi_{s_1p_1,\cdots,\widehat{s_ip_i},\cdots,s_np_n}$$

where $\widehat{s_ip_i}$ means that these indices are eliminated. The vacuum state ϕ_0 with zero photons is such that $a_s(p)\phi_0 = 0$ and an n-photon state can be built up via

$$\phi_{s_1p_1,\cdots,s_np_n} = \frac{1}{\sqrt{n!}}a_{s_1}^\dagger(p_1)\cdots a_{s_n}^\dagger(p_n)\phi_0 \tag{4.14}$$

The symmetry requirements for identical boson states impose the commutation relations

$$[a_s(p), a_{s'}(p')] = \delta_{s,s'}\delta^3(p-p'); \ [a_s^\dagger(p), a_{s'}(p')] = [a_s(p), a_{s'}(p')] = 0 \tag{4.15}$$

Finally the number operator is $N_s(p) = a_s^\dagger(p)a_s(p)$ (number of photons with helicity s and momentum in d^3p centered at p) and the operator for the total number of photons in the system is $N = \sum_s \int d^3p N_s(p)$.

If we accept that the photons have objective existence and that each of them

carries momentum p, energy $E = c|\mathbf{p}| = \hbar\omega$ and spin $\hbar$ in the direction of propagation $\mathbf{k}$ then for a system of many noninteracting particles the total energy, momentum, and spin are ($\pm$ means $s = \pm 1$)
(4.16)

$$H = \sum_s \int d^3p\hbar\omega N_s(p);\ P = \sum_s \int d^3ppN_s(p);\ S = \int d^3p\hbar\mathbf{k}(N_+(p) - N_-(p))$$

The noninteraction is a reasonably good approximation since the leading interaction photon-photon contribution is of fourth order in perturbation theory. Now a nonhermitian operator like $a_s^\dagger(p)$ or $a_s(p)$ is related to two hermitian operators, the number operator (corresponding to a modulus squared) and another observable of the form

$$\sum_s \int d^3p(f(s,p,E,\mathbf{r},t)a_s(p) \pm c.c) \tag{4.17}$$

and one can guess its form. This leads to the two hermitian operators

$$\mathbf{E}(\mathbf{r},t) = \frac{1}{2\pi\hbar}\sum_s \int d^3p\sqrt{\omega}(ia_s(p)\epsilon_s e^{(i/\hbar)(\mathbf{p}\cdot\mathbf{r}-Et)} + h.c.); \tag{4.18}$$

$$\mathbf{B}(\mathbf{r},t) = \frac{1}{2\pi\hbar}\sum_s \int d^3p\sqrt{\omega}(ia_s(p)(\mathbf{k}\times\epsilon_s)e^{(i/\hbar)(\mathbf{p}\cdot\mathbf{r}-Et)} + h.c.)$$

One can show that these operators are indeed the EM fields and one has

$$\mathbf{A}(\mathbf{r},t) = \frac{c}{2\pi\hbar}\sum_s \int d^3p\frac{1}{\sqrt{\omega}}(a_s(p)\epsilon_s e^{(i/\hbar)(\mathbf{p}\cdot\mathbf{r}-Et)} + h.c.); \tag{4.19}$$

$$\mathbf{E}(\mathbf{r},t) = -\frac{1}{c}\partial_t\mathbf{k}\mathbf{A}(\mathbf{r},t);\ \mathbf{B}(\mathbf{r},t) = \nabla\times\mathbf{A}(\mathbf{r},t)$$

$$H = \frac{1}{8\pi}\int d^3\mathbf{r}(\mathbf{E}^2 + \mathbf{B}^2);\ P = \frac{1}{8\pi c}\int d^3\mathbf{r}(\mathbf{E}\times\mathbf{B} - \mathbf{B}\times\mathbf{E}; \tag{4.20}$$

$$S = \frac{1}{8\pi c}\int d^3\mathbf{r}(\mathbf{E}\times\mathbf{A} - \mathbf{A}\times\mathbf{E})$$

$$-\nabla\times\mathbf{E} = \frac{1}{c}\partial_t\mathbf{B};\ \nabla\times\mathbf{B} = \frac{1}{c}\partial_t\mathbf{E};\ \nabla\cdot\mathbf{E} = 0;\ \nabla\cdot\mathbf{B} = 0 \tag{4.21}$$

Further, defining

$$D(\rho,\tau) = \frac{-1}{(2\pi\hbar)^3}\int d^3pe^{(i/\hbar)\mathbf{p}\cdot\rho}\frac{Sin)\omega\tau)}{\omega} = \frac{-1}{8\pi^2c\rho}[\delta)\rho - c\tau) - \delta(\rho + c\tau)] \tag{4.22}$$

one finds commutator formulas for the components of $\mathbf{E}$ and $\mathbf{B}$ in terms of $D(\rho,\tau)$. The singular character of such formulas casts some question on the meaningfulness of the EM fields as QM observables but in any event one can also calculate commutator relations between $\mathbf{E}$, $\mathbf{B}$, $\mathbf{A}$ and the number operator N. There are a number of interesting calculations in [**940**] which we omit here but we do mention the divergent vacuum expectation value

$$< \phi_0, \mathbf{E}^2(\mathbf{r},t)\phi_0 >= \frac{1}{(2\pi\hbar)^2}\int d^3p\omega \tag{4.23}$$

which indicates the presence of fluctuations of $\mathbf{E}$ in the vacuum; this is infinite but if averaged over a small region the value would be finite. One shows also that the EM field of an indefinite number of photons, all with the same helicity and momentum, is a plane wave with circular polarization; in fact for $\psi \sum_n C_n \phi_{n(sp)}$ one has

$$(4.24) \qquad <\psi, \mathbf{E}(\mathbf{r},t)\psi>= \frac{\sqrt{\omega}}{2\pi\hbar}\left(i\sum_n C_n^* C_{n+1}\epsilon_s e^{(i/\hbar)\mathbf{p}\cdot\mathbf{r}_E t)} + c.c.\right)$$

In any event it is wrong to think that the photon is the particle like duality partner of the wave like EM field. Another confusion analyzed in [**940**] is the erroneous identification of Maxwell's equations for the EM fields with the SE for the photon. The standard derivations of Maxwell's equation from the SE is shown to be incorrect.

REMARK 5.4.1. We refer to [**724**] for further approaches to wave-particle duality and a correspondence between linearized spacetime metrics of GR and wave equations of QM.

4. QUANTUM GEOMETRY

First we sketch the relevant symbolism for geometrical QM from [**54**] without much philosophy; the philosophy is eloquently phrased there and in [**153, 244, 550, 654**] for example. Thus let H be the Hilbert space of QM and write it as a real Hilbert space with a complex structure J. The Hermitian inner product is then $<\phi,\psi>= (1/2\hbar)G(\phi,\psi) + (i/2\hbar)\Omega(\phi,\psi)$ (note $G(\phi,\psi) = 2\hbar\Re(\phi,\psi)$ is the natural Fubini-Study (FS) metric - cf. [**244**]). Here G is a positive definite real inner product and Ω is a symplectic form (both strongly nondegenerate). Moreover $<\phi, J\psi>= i<\phi,\psi>$ and $G(\phi,\psi) = \Omega(\phi, J\psi)$. Thus the triple (J,G,Ω) equips H with the structure of a Kähler space. Now, from [**998**], on a real vector space V with complex structure J a Hermitian form satisfies $h(JX,JY) = h(X,Y)$. Then V becomes a complex vector space via $(a+ib)X = aX + bJX$. A Riemannian metric g on a manifold M is Hermitian if $g(X,Y) = g(JX,JY)$ for X,Y vector fields on M. Let ∇_X be he Levi-Civita connection for g (i.e. parallel transport preserves inner products and the torsion is zero. A manifold M with J as above is called almost complex. A complex manifold is a paracompact Hausdorff space with complex analytic patch transformation functions. An almost complex M with Kähler metric (i.e. $\nabla_X J = 0$) is called an almost Kähler manifold and if in addition the Nijenhuis tensor vanishes it is a Kähler manifold (cf. (4.1) below). Here the defining equations for the Levi-Civita connection and the Nijenhuis tensor are

$$(4.1) \qquad \Gamma^k_{ij} = \frac{1}{2}g^{hk}[\partial_i g_{jk} + \partial_j g_{ik} - \partial_k g_{ji}];$$

$$N(X,Y) = [JX,JY] - [X,Y] - J[X,JY] - J[JX,Y]$$

Further discussion can be found in [**998**]. Material on the Fubini-Study metric will be provided later. Next by use of the canonical identification of the tangent space (at any point of H) with H itself, Ω is naturally extended to a strongly nondegenerate, closed, differential 2-form on H, denoted also by Ω. The inverse of Ω may be used to define Poisson brackets and Hamiltonian vector fields. Now in

QM the observables may be viewed as vector fields, since linear operators associate a vector to each element of the Hilbert space. Moreover the Schrödinger equation, written here as $\dot{\psi} = -(1/\hbar)J\hat{H}\psi$, motivates one to associate to each quantum observable $\hat{F}$ the vector field $Y_{\hat{F}}(\psi) = -(1/\hbar)J\hat{F}\psi$. The Schrödinger vector field is defined so that the time evolution of the system corresponds to the flow along the Schrödinger vector field and one can show that the vector field $Y_{\hat{F}}$, being the generator of a one parameter family of unitary mappings on H, preserves both the metric G and the symplectic form Ω. Hence is is locally, and indeed globally, Hamiltonian. In fact the function which generates this Hamiltonian vector ield is simply the expectation value of $\hat{F}$. To see this write $F : H \to \mathbf{R}$ via $F(\psi) =< \psi, \hat{F}\psi >=< \hat{F} >= (1/2\hbar)G(\psi, \hat{F}\psi)$. Then if η is any tangent vector at ψ

$$(dF)(\eta) = \frac{d}{dt} < \psi + t\eta, \hat{F}(\psi + t\eta) > |_{t=0} =< \psi, \hat{F}\eta > + < \eta, \hat{F}\psi >= \tag{4.2}$$

$$= \frac{1}{\hbar}G(\hat{F}\psi, \eta) = \Omega(Y_{\hat{F}}, \eta) = (i_{Y_{\hat{F}}}\Omega)(\eta)$$

where one uses the selfadjointness of $\hat{F}$ and the definition of $Y_{\hat{F}}$ (recall the Hamiltonian vector field X_f generated by f satisfies the equation $i_{X_f}\Omega = df$ and the Poisson bracket is defined via $\{f, g\} = \Omega(X_f, X_g)$). Thus the time evolution of any quantum mechanical system may be written in terms of Hamilton's equation of classical mechanics; the Hamiltonian function is simply the expectation value of the Hamiltonian operator. Consequently Schrödinger's equation is simply Hamilton's equation in disguise. For Poisson brackets we have

$$\{F, K\}_{\Omega} = \Omega(X_F, X_K) = \left\langle \frac{1}{i\hbar}[\hat{F}, \hat{K}] \right\rangle \tag{4.3}$$

where the right side involves the quantum Lie bracket. Note this is not Dirac's correspondence principle since the Poisson bracket here is the quantum one determined by the imaginary part of the Hermitian inner product.

Now look at the role played by G. It enables one to define a real inner product $G(X_F, X_K)$ between any two Hamiltonian vector fields and one expects that this inner product is related to the Jordan product. Indeed

$$\{F, K\}_{+} = \frac{\hbar}{2}G(X_F, X_K) = \left\langle \frac{1}{2}[\hat{F}, \hat{K}]_{+} \right\rangle \tag{4.4}$$

Since the classical phase space is generally not equipped with a Riemannian metric the Riemann product does not have a classical analogue; however it does have a physical interpretation. One notes that the uncertainty of the observable $\hat{F}$ at a state with unit norm is $(\Delta\hat{F})^2 =< \hat{F}^2 > - < \hat{F}^2 >= \{F, F\}_{+} - F^2$. Hence the uncertainty involves the Riemann bracket in a simple manner. In fact Heisenberg's uncertainty relation has a nice form as seen via

$$(\Delta\hat{F})^2(\Delta\hat{K})^2 \geq \left\langle \frac{1}{2i}[\hat{F}, \hat{K}] \right\rangle^2 + \left\langle \frac{1}{2}[\hat{F}_{\perp}, \hat{K}_{\perp}]_{+} \right\rangle^2 \tag{4.5}$$

where $\hat{F}_\perp$ is the nonlinear operator defined by $\hat{F}_\perp(\psi) = \hat{F}(\psi) - F(\psi)$. Thus $\hat{F}_\perp(\psi)$ is orthogonal to ψ if $\|\psi\| = 1$. Using this one can write (4.5) in the form

$$(\Delta\hat{F})^2(\Delta\hat{K})^2 \geq \left(\frac{\hbar}{2}\{F,K\}_\Omega\right)^2 + (\{F,K\}_+ - FK)^2 \tag{4.6}$$

The last expression in (4.6) can be interpreted as the quantum covariance of $\hat{F}$ and $\hat{K}$.

The discussion in [**54**] continues in this spirit and is eminently worth reading; however we digress here for a more "hands on" approach following [**189, 244, 245, 246, 247, 248**]. Assume H is separable with a complete orthonormal system $\{u_n\}$ and for any $\psi \in H$ denote by $[\psi]$ the ray generated by ψ while $\eta_n = (u_n|\psi)$. Define for $k \in \mathbf{N}$

$$U_k = \{[\psi] \in P(H);\ \eta_k \neq 0\};\ \phi_k : U_k \to \ell^2(\mathbf{C}) : \tag{4.7}$$
$$\phi_k([\psi]) = \left(\frac{\eta_1}{\eta_k}, \cdots, \frac{\eta_{k-1}}{\eta_k}, \frac{\eta_{k+1}}{\eta_k}, \cdots\right)$$

where $\ell^2(\mathbf{C})$ denotes square summable functions. Evidently $P(H) = \cup_k U_k$ and $\phi_k \circ \phi_j^{-1}$ is biholomorphic. It is easily shown that the structure is independent of the choice of complete orthonormal system. The coordinates for $[\psi]$ relative to the chart (U_k, ϕ_k) are $\{z_n^k\}$ given via $z_n^k = (\eta_n/\eta_k)$ for $n < k$ and $z_n^k = (\eta_{n+1}/\eta_k)$ for $n \geq k$. To convert this to a real manifold one can use $z_n^k = (1/\sqrt{2})(x_n^k + iy_n^k)$ with

$$\frac{\partial}{\partial z_n^k} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x_n^k} + i\frac{\partial}{\partial y_n^k}\right);\ \frac{\partial}{\partial \bar{z}_n^k} = \frac{1}{\sqrt{2}}\left(\frac{\partial}{\partial x_n^k} - i\frac{\partial}{\partial y_n^k}\right) \tag{4.8}$$

etc. Instead of nondegeneracy as a criterion for a symplectic form inducing a bundle isomorphism between TM and T^*M one assumes here that a symplectic form on M is a closed 2-form which induces at each point $p \in M$ a toplinear isomorphism between the tangent and cotangent spaces at p. For $P(H)$ one can do more than simply exhibit such a natural symplectic form; in fact one shows that $P(H)$ is a Kähler manifold (meaning that the fundamental 2-form is closed). Thus one can choose a Hermitian metric $\mathfrak{G} = \sum g_{mn}^k dz_m^k \otimes d\bar{z}_n^k$ with

$$g_{mn}^k = (1 + \sum_i z_i^k \bar{z}_i^k)^{-1}\delta_{mn} - (1 + \sum_1 z_i^k \bar{z}_i^k)^{-2}\bar{z}_m^k z_n^k \tag{4.9}$$

relative to the chart $U_k, \phi_k)$. The fundamental 2-form of the metric $\mathfrak{G}$ is $\omega = i\sum_{m,n} g_{mn}^k dz_m^k \wedge d\bar{z}_n^k$ and to show that this is closed note that $\omega = i\partial\bar{\partial}f$ where locally $f = log(1 + \sum z_i^k \bar{z}_i^k)$ (the local Kähler function). Note here that $\partial + \bar{\partial} = d$ and $d^2 = 0$ implies $\partial^2 = \bar{\partial}^2 = 0$ so $d\omega = 0$ and thus $P(H)$ is a K manifold.

Now on $P(H)$ the observables will be represented via a class of real smooth functions on $P(H)$ (projective Hilbert space) called Kählerian functions. Consider a real smooth Banach manifold M with tangent space TM, and cotangent space T^*M. We remark that the extension of standard differential geometry to the infinite dimensional situation of Banach manifolds etc. is essentially routine modulo some functional analysis; there are a few surprises and some interesting technical machinery but we omit all this here. One should also use bundle terminology at various places but we will not be pedantic about this. One hopes here to simply give a clear picture of what is happening. Thus e.g. $L(T_x^*M, T_xM)$ denotes bounded linear operators $T_x^*M \to T_xM$ and $L_n(T_xM, \mathbf{R})$ denotes bounded n-linear forms on T_xM. An almost complex structure is provided by a smooth section J of $L(TM)$ = vector bundle of bounded linear operators with fibres $L(T_xM)$ such that $J^2 = -1$. Such a J is called integrable if its torsion is zero, i.e. $N(X,Y) = 0$ with N as in (4.1). An almost Kähler (K) manifold is a triple (M, J, g) where M is a real smooth Hilbert manifold, J is an almost complex structure, and g is a K metric, i.e. a Riemannian metric such that

- g is invariant; i.e. $g_x(J_xX_x, J_xY_x) = g_x(X_x, Y_x)$.
- The fundamental two form of the metric is closed; i.e.

$$\omega_x(X_x, Y_x) = g_x(J_xX_x, Y_x) \tag{4.10}$$

is closed (which means $d\omega = 0$).

Note that an almost K manifold is canonically symplectic and if J is integrable one says that M is a K manifold. Now fix an almost K manifold (M, J, g). The form ω and the K metric g induce two top-linear isomorphisms I_x and G_x between T_x^*M and T_xM via $\omega_x(I_xa_x, X_x) =< a_x, X_x >$ and $g_x(G_xa_z, X_x) =< a_x, X_x >$. Denoting the smooth sections by I, G one checks that $G = J \circ I$.

Definition 5.4.1. For $f, h \in C^\infty(M, \mathbf{R})$ the Poisson and Riemann brackets are defined via $\{f, h\} =< df, Idh >$ and $((f, h)) =< df, Gdh >$. From the above one can reformulate this as

(4.11)

$$\{f, h\} = \omega(Idf, Idh) = \omega(Gdf, Gdh);\ \ ((f, h)) = g(Gdf, Gdh) = g(Idf, Idh)$$

Definition 5.4.2. For $f, h \in C^\infty(M, \mathbf{C})$ the K bracket is $< f, h >= ((f, h)) + i\{f, h\}$ and one defines products $f \circ_\nu h = (1/2)\nu((f, h)) + fh$ (ν will be determined to be $\hbar$) and $f *_\nu h = (1/2)\nu < f, h > +fh$. One observes also that

$$f *_\nu h = f \circ_\nu h + (i/2)\nu\{f, h\};\ \ f \circ_\nu h = (1/2)(f *_\nu h + h *_\nu f); \tag{4.12}$$

$$\{f, h\} = (1/i\nu)(f *_\nu h - h *_\nu f)$$

Definition 5.4.3. For $f \in C^\infty(M, \mathbf{R})$ let $X = Idf$; then f is called Kählerian (K) if $L_X g = 0$ where L_X is the Lie derivative along X (recall $L_X f = Xf,\ L_X Y = [X, Y],\ L_X(\omega(Y)) = (L_X\omega)(Y) + \omega(L_X(Y)), \cdots$). More generally if $f \in C^\infty(M, \mathbf{C})$ one says that f is K if $\Re f$ and $\Im f$ are K; the set of K functions is denoted by $K(M, \mathbf{R})$ or $K(M, \mathbf{C})$.

Remark 5.4.1. In the language of symplectic manifolds $X = df$ is the Hamiltonian vector field corresponding to f and the condition $L_X g = 0$ means that the integral flow of X, or the Hamiltonian flow of f, preserves the metric g. From this follows also $L_X J = 0$ (since J is uniquely determined by ω and g via (2.26)). Therefore if f is K the Hamiltonian flow of f preserves the whole K structure. Note also that $K(M, \mathbf{R})$ (resp. $K(M, \mathbf{C})$) is a Lie subalgebra of $C^\infty(M, \mathbf{R})$ (resp. $C^\infty(M, \mathbf{C})$).

Now $P(H)$ is the set of one dimensional subspaces or rays of H; for every $x \in H/\{0\}$, $[x]$ is the ray through x. If H is the Hilbert space of a Schrödinger quantum system then H represents the pure states of the system and $P(H)$ can be regarded as the state manifold (when provided with the differentiable structure below). One defines the K structure as follows. On $P(H)$ one has an atlas $\{(V_h, b_h, C_h)\}$ where $h \in H$ with $\|h\| = 1$. Here (V_h, b_h, C_h) is the chart with domain V_h and local model the complex Hilbert space C_h where

$$V_h = \{[x] \in P(H); (h|x) \neq 0\}; \; C_h = [h]^{\perp}; \tag{4.13}$$

$$b_h : V_h \to C_h; \; [x] \to b_h([x]) = \frac{x}{(h|x)} - h$$

This produces a analytic manifold structure on $P(H)$. As a real manifold one uses an atlas $\{(V_h, R \circ b_h, RC_h)\}$ where e.g. RC_h is the realification of C_h (the real Hilbert space with $\mathbf{R}$ instead of $\mathbf{C}$ as scalar field) and $R: C_h \to RC_h; v \to Rv$ is the canonical bijection (note $Rv \neq \Re v$). Now consider the form of the K metric relative to a chart $(V_h, R \circ b_h, RC_h)$ where the metric g is a smooth section of $L_2(TP(H), \mathbf{R})$ with local expression $g^h : RC_h \to L_2(RC_h, \mathbf{R}); Rz \mapsto g^h_{Rz}$ where

$$g^h_{Rz}(Rv, Rw) = 2\nu \Re \left(\frac{(v|w)}{1 + \|z\|^2} - \frac{(v|z)(z|w)}{(1 + \|z\|^2)^2} \right) \tag{4.14}$$

The fundamental form ω is a section of $L_2(TP(H), \mathbf{R})$, i.e. one can write ω^h : $RC_h \to L_2(RC_h, \mathbf{R}); \; Rz \to \omega^h_{Rz}$, given via

$$\omega^h_{Rz}(Rv, Rw) = 2\nu \Im \left(\frac{(v|w)}{1 + \|z\|^2} - \frac{(v|z)(z|w)}{(1 + \|z\|^2)^2} \right) \tag{4.15}$$

Then using e.g. (4.14) for the FS metric in $P(H)$ consider a Schrödinger Hilbert space with dynamics determined via $\mathbf{R} \times P(H) \to P(H) : (t, [x]) \mapsto [exp(-(i/\hbar)tH)x]$ where H is a (typically unbounded) self adjoint operator in H. One thinks then of Kähler isomorphisms of $P(H)$ (i.e. smooth diffeomorphisms $\Phi : P(H) \to P(H)$ with the properties $\Phi^* J = J$ and $\Phi^* g = g$). If U is any unitary operator on H the map $[x] \mapsto [Ux]$ is a K isomorphism of $P(H)$. Conversely (cf. [**246**]) any K isomorphism of $P(H)$ is induced by a unitary operator U (unique up to phase factor). Further for every self adjoint operator A in H (possibly unbounded) the family of maps $(\Phi_t)_{t \in \mathbf{R}}$ given via $\Phi_t : [x] \to [exp(-itA)x]$ is a continuous one parameter group of K isomorphisms of $P(H)$ and vice versa (every K isomorphism of $P(H)$ is induced by a self adjoint operator where boundedness of A corresponds to smoothness of the Φ_t). Thus in the present framework the dynamics of QM is described by a continuous one parameter group of K isomorphisms, which automatically are symplectic isomorphisms (for the structure defined by the fundamental form) and one has a Hamiltonian system. Next ideally one can suppose that every self adjoint operator represents an observable and these will be shown to be in $1-1$ correspondence with the real K functions.

Definition 5.4.4. Let A be a bounded linear operator on H and denote by $< A >$ the mean value function of A defined via $< A >: P(H) \to \mathbf{C}, \; [x] \mapsto < A >_{[x]} = (x|Ax)/\|x\|^2$. The square dispersion is defined via $\Delta^2 A : P(H) \to \mathbf{C}, \; [x] \mapsto \Delta^2_{[x]} A = < (A - < A >_{[x]})^2 >_{[x]}$.

These maps in Definition 2.4 are smooth and if A is self adjoint $< A >$ is real, $\Delta^2 A$ is nonnegative, and one can define $\Delta A = \sqrt{\Delta^2 A}$. To obtain local expressions one writes $< A >^h$: $C_h \to \mathbf{R}$ and $(d < A >)^h : C_h \to (C_h)^*$ via $< A >^h (R) = (z+h)|A(z+h))/(1+\|z\|^2)$ and

(4.16)

$$< (d < A >)^h_{Rz}|Rv >= 2\Re\left(\frac{A(z+h)}{1+\|z\|^2} - \frac{(h|A(z+h))}{1+\|z\|^2}h - \frac{(A(z+h)|z+h)}{(1+\|z\|^2)^2}z\middle| Rv\right)$$

Further the local expressions $X^h : RC_h \to RC_h$ and $Y^h : RC_h \to RC_h$ of the vector fields $X = Id < A >$ and $Y = Gd < A >$ are

$$X^h(Rz) = (1/\nu)R(i(h|A(z+h))(z+h) - iA(z+h)); \tag{4.17}$$

$$Y^h(Rz) = (1/\nu)R(-(h|A(z+h))(z+h) + A(z+h))$$

One proves then (cf. [**246, 446**]) that the flow of the vector field $X = Id < A >$ is complete and is given via $\Phi_t([x]) = [exp(-i(t/\nu)A)x]$. This leads to the statement that if f is a complex valued function on $P(H)$ then f is Kählerian if and only if there is a bounded operator A such that $f =< A >$. From the above it is clear that one should take $\nu = \hbar$ for QM if we want to have $< \mathfrak{H} >$ represent Hamiltonian flow ($\mathfrak{H} \sim$ a Hamiltonian operator) and this gives a geometrical interpretation of Planck's constant. The following formulas are obtained for the Poisson and Riemann brackets

$$\{< A >, < B >\}^h(Rz) = \tag{4.18}$$

$$= \frac{(z+h|(1/i\nu)(AB-BA)(z+h))}{1+\|z\|^2};\ ((< A >, < B >))^h(Rz) =$$

$$= \frac{1}{\nu}\frac{(z+h|(AB+BA)(z+h))}{1+\|z\|^2} - \frac{2}{\nu}\frac{(z+h|A(z+h))}{1+\|z\|^2}\frac{(z+h|B(z+h))}{1+\|z\|^2}$$

This leads to the results

(1) $\{< A >, < B >\} =< (1/i\nu)[A,B] >$
(2) $((< A >, < B >)) = (1/\nu) < AB + BA > -(2/\nu) < A >< B >$; $((< A >< A >)) = (2/\nu)\Delta^2 A$
(3) $<< A >, < B >>= (2/\nu)(< AB > - < A >< B >)$
(4) $< A > \circ_\nu < B >= (1/2) < AB + BA >$
(5) $< A > *_\nu < B >=< AB >$

Remark 5.4.2. One notes that (setting $\nu = \hbar$) item 1 gives the relation between Poisson brackets and commutators in QM. Further the Riemann bracket is the operation needed to compute the dispersion of observables. In particular putting $\nu = \hbar$ in item 2 one sees that for every observable $f \in K(P(H), \mathbf{R})$ and every state $[x] \in P(H)$ the results of a large number of measurements of f in the state $[x]$ are distributed with standard deviation $\sqrt{(\hbar/2)((f,f))([x])}$ around the mean value $f([x])$. This explains the role of the Riemann structure in QM, namely it is the structure needed for the probabilistic description of QM. Moreover the $\circ_\nu$ product corresponds to the Jordan product between operators (cf. item 5) and item 4 tells us that the $*_\nu$ product corresponds to the operator product. This allows one to formulate a functional representation for the algebra $L(H)$. Thus put $\|f\|_\nu = \sqrt{sup_{[x]}(\bar{f} *_\nu f)([x])}$. Equipped with this norm $K(P(H), \mathbf{C})$ becomes a W^* algebra and the map of W^* algebras between $K(P(H), \mathbf{C}$ and $L(H)$ is an isomorphism. This makes it possible to develop a general functional representation theory for C^* algebras generalizing the classical spectral representation for commutative C^* algebras. The K manifold $P(H)$is replaced by a topological fibre bundle in which every fibre is a K manifold isomorphic to a projective space. In particular a nonzero vector $x \in H$ is an eigenvector of A if and only if $d_{[x]} < A >= 0$ or equivalently if and only if $[x]$ is a fixed point for the vector field $Id < A >$ (in which case the corresponding eigenvalue is $< A >_{[x]}$).

4.1. PROBABILITY ASPECTS. We go here to [**33, 54, 55, 151, 153, 189, 244, 247, 257, 268, 389, 408, 409, 446, 447, 612, 621, 661, 742, 765, 766, 805, 937, 1005**] and refer also to Section 3.1 and Remark 3.3.2. First from [**151, 1005**] one defines a (Riemann) metric (statistical distance) on the space of probability distributions $\mathcal{P}$ of the form

(4.19) $$ds^2_{PD} = \sum (dp_j^2/p_j) = \sum p_j (dlog(p_j))^2$$

Here one thinks of the central limit theorem and a distance between probability distributions distinguished via a Gaussian $exp[-(N/2)(\tilde{p}_j - p_j)^2/p_j]$ for two nearby distributions (involving N samples with probabilities p_j, $\tilde{p}_j$). This can be generalized to quantum mechanical pure states via (note $\psi \sim \sqrt{p} exp(i\phi)$ in a generic manner)

(4.20)
$$|\psi >= \sum \sqrt{p_j} e^{i\phi_j} |j >; \ |\tilde{\psi} >= |\psi > + |d\psi >= \sum \sqrt{p_j + dp_j} e^{i(\phi_j + d\phi_j)} |j >$$

Normalization requires $\Re(< \psi|d\psi >) = -1/2 < d\psi|d\psi >$ and measurements described by the one dimensional projectors $|j >< j|$ can distinguish $|\psi >$ and $|\tilde{\psi} >$ according to the metric (4.19). The maximum (for optimal disatinguishability) is given by the Hilbert space angle $cos^{-1}(| < \tilde{\psi}|\psi > |)$ and the corresponding line element ($PS \sim$ pure state)

(4.21) $$\frac{1}{4} ds^2_{PS} = [cos^{-1}(| < \tilde{\psi}|\psi > |)]^2 \sim 1 - | < \tilde{\psi}|\psi > |^2 =< d\psi_\perp|d\psi_\perp >\sim$$
$$\sim \frac{1}{4} \sum \frac{dp_j^2}{p_j} + \left[\sum p_j d\phi_j^2 - (\sum p_j d\phi_j)^2\right]$$

(called the Fubini-Study (FS) metric) is the natural metric on the manifold of Hilbert space rays. Here

(4.22) $$|d\psi_\perp >= |d\psi > - |\psi >< \psi|d\psi >$$

is the projection of $|d\psi >$ orthogonal to $|\psi >$. Note that if $cos^{-1}(| < \tilde{\psi}|\psi > | = \theta$ then $cos(\theta) = | < \tilde{\psi}|\psi > |$ and $cos^2(\theta) = | < \tilde{\psi}|\psi > |^2 = 1 - Sin^2(\theta) \sim 1 - \theta^2$ for small θ. Hence $\theta^2 \sim 1 - cos^2(\theta) = 1 - | < \tilde{\psi}|\psi > |^2$. The term in square brackets (the variance of phase changes) is nonnegative and an appropriate choice of basis makes it zero. In [**151**] one then goes on to discuss distance formulas in terms of density operators and Fisher information but we omit this here. Generally as in [**1005**] one observes that the angle in Hilbert space is the only Riemannian metric on the set of rays which is invariant uder unitary transformations. In any event $ds^2 = \sum(dp_i^2/p_i)$, $\sum p_i = 1$ is referred to as the Fisher metric (cf. [**661**]). Note in terms of $dp_i = \tilde{p}_i - p_i$ one can write $d\sqrt{p} = (1/2)dp/\sqrt{p}$ with $(d\sqrt{p})^2 = (1/4)(dp^2/p)$ and think of $\sum(d\sqrt{p_i})$ as a metric. Alternatively from $cos^{-1}(| < \tilde{\psi}|\psi > |$ one obtains $ds_{12} = cos^{-1}(\sum \sqrt{p_{1i}}\sqrt{p_{2i}})$ as a distance in $\mathcal{P}$. Note from (4.21) that $ds_{12}^2 = 4cos^{-1}| < \psi_1|\psi_2 > | \sim 4(1 - |(\psi_1|\psi_2)|^2 \equiv 4(< d\psi|d\psi > - < d\psi|\psi >< \psi|d\psi >)$ begins to look like a FS metric before passing to projective coordinates. In this direction we observe from [**661**] that the FS metric can be expressed also via

$$\partial\bar{\partial}log(|z|^2) = \phi = \frac{1}{|z|^2}\sum dz_i \wedge d\bar{z}_i - \frac{1}{|z|^4}\left(\sum \bar{z}_i dz_i\right) \wedge \left(\sum z_i d\bar{z}_i\right) \tag{4.23}$$

so for $v \sim \sum v_i\partial_i + \bar{v}_i\bar{\partial}_i$ and $w \sim \sum w_i\partial_i + \bar{w}_i\bar{\partial}_i$ and $|z|^2 = 1$ one has $\phi(v,w) = (v|w) - (v|z)(z|w)$ (cf. (3.4)).

REMARK 5.4.3. We refer now to Section 3.1 and Remark 3.3.2 for connections between quantum geometry and the quantum potential via Fisher information and probability.

CHAPTER 6

INFORMATION AND ENTROPY

Information and entropy have been discussed in Sections 1.3.2, 3.3.1, 4.7, etc. and we continue with a further elaboration (see in particular [**10, 23, 72, 146, 173, 174, 175, 240, 343, 388, 396, 400, 431, 446, 452, 481, 512, 634, 637, 639, 694, 740, 749, 755, 765, 766, 856, 914, 916, 915, 906, 976**]). As before we will again encounter relations to the quantum potential which serves as a persistent theme of development. There is an enormous literature on entropy and we try to select aspects which fit in with ideas of quantum diffusion and information theory.

1. THE DYNAMICS OF UNCERTAINTY

We begin with some topics from [**396**] to which we refer for certain tutorial aspects. Given events A_j $(1 \leq j \leq N)$ with probabilities μ_j of occurance in some game of chance with N possible outcomes one calls $log(\mu_j)$ an uncertainty function for A_j. We write the natural logarithm as log and recall that e.g. $log_2(b) = log(b)/ln(2)$ (the information theoretic base is taken as 2 in some contexts). The quantity (Shannon entropy)

$$\mathfrak{S}(\mu) = -\sum_1^N \mu_j log_2(\mu_j) \tag{1.1}$$

stands for the measure of the mean uncertainty of the possible outcomes of the game and at the same time quantifies the mean information which is accessible from an experiment (i.e. actually playing the game). Thus if one identifies the A_i as labels for discrete states of a system (1.1) can be interpreted as a measure of uncertainty of the state before this state is chosen and the Shannon entropy is a measure of the degree of ignorance concerning which possibility (event A_j) may hold true in the set of all $A_i^{\prime s}$ with a given a priori probability distribution (μ_i). Note also that $0 \leq \mathfrak{S}(\mu) \leq log_2(N)$ (since certainty means one entry with probability 1 and maximum uncertainty occurs when all events are equally probable with $\mu_j = 1/N$). There is some discussion of the Boltzman law $\mathfrak{S} = k_B log(W) = -k_B log(P)$ $(P = 1/W)$ and its relation to Shannon entropy, coarse graining, and differential entropy defined as

$$\mathfrak{S}(\rho) = -\int \rho(x) log(\rho(x)) dx \tag{1.2}$$

(cf. Sections 1.1.6 and 1.1.8). One recalls also the vonNeumann entropy

$$\mathfrak{S}(\hat{\rho}) = -k_B Tr(\hat{\rho} log(\hat{\rho})) \tag{1.3}$$

where $\hat{\rho}$ is the density operator for a quantum state ($\hat{\rho}log(\rho)$ is defined via functional calculus for selfadjoint operators (cf. [**916**]). For diagonal density operators with eigenvalues p_i this will coincide with the Shannon entropy $\sum p_i log(p_i)$. We go now directly to an extension of the discussion in Sections 1.1.6-1.1.8. It is known from [**862**] that among all one dimensional distributions $\rho(x)$ with a finite mean, subject to the condition that the standard deviation is fixed at σ, it is the Gaussian with half width σ which sets a maximum of the differential entropy. Thus for the Gaussian with $\rho(x) = (1/\sigma\sqrt{2\pi})exp[-(x-x_0)^2/2\sigma^2]$ one has

$$\mathfrak{S}(\rho) \leq \frac{1}{2}log(2\pi e\sigma^2) \Rightarrow \frac{1}{\sqrt{2\pi e}}exp[\mathfrak{S}(\rho)] \leq \sigma \tag{1.4}$$

A result of this is that the major role of the differential entropy is to be a measure of localization in the configuration space (note that even for relatively large mean deviations $\sigma < 1/\sqrt{2\pi e} \simeq .26$ the differential entropy $\mathfrak{S}(\rho)$ is negative. Consider now a one parameter family of probability densities $\rho_\alpha(x)$ on $\mathbf{R}$ whose first (mean) and second moments (variance) are finite. Write $\int x\rho_\alpha(x)dx = f(\alpha)$ with $\int x^2\rho_\alpha dx < \infty$. Under suitable hypotheses (implying that $\partial\rho_\alpha/\partial\alpha$ is bounded by a function $G(x)$ which together with $xG(x)$ is integrable on $\mathbf{R}$) one obtains

$$\int (x-\alpha)^2\rho_\alpha(x)dx \cdot \int \left(\frac{\partial log(\rho_\alpha)}{\partial\alpha}\right)^2 \rho_\alpha dx \geq \left(\frac{df(\alpha)}{d\alpha}\right)^2 \tag{1.5}$$

which results from

$$\frac{df}{d\alpha} = \int [(x-\alpha)\rho_\alpha^{1/2}] \left[\frac{\partial(log(\rho_\alpha))}{\partial\alpha}\rho_\alpha^{1/2}\right] dx \tag{1.6}$$

and the Schwartz inequality. Assume now that the mean value of ρ_α actually is α and fix at σ^2 the value of the variance $<(x-\alpha)^2> = <x^2> - \alpha^2$. Then (1.5) takes the familiar form

$$\mathfrak{F}_\alpha = \int \frac{1}{\rho_\alpha}\left(\frac{\partial\rho_\alpha}{\partial\alpha}\right)^2 dx \geq \frac{1}{\sigma^2} \tag{1.7}$$

where the left side is the Fisher information for ρ_α. This says that the Fisher information is a more sensitive indicator of the wave packet localization than the entropy power in (1.4). Consider now $\rho_\alpha = \rho(x-\alpha)$ so $\mathfrak{F}_\alpha = \mathfrak{F}$ is no longer dependent on α and one can transform this to the QM form (up to a factor of D^2 where $D = \hbar/2m$ which we acknowledge here via the symbol $\sim$)

$$\frac{1}{2}\mathfrak{F} = \frac{1}{2}\int \frac{1}{\rho}\left(\frac{\partial\rho}{\partial x}\right)^2 dx \sim \int \rho \cdot \frac{u^2}{2}dx \sim -<\tilde{Q}> \tag{1.8}$$

where $u = \nabla log(\rho)$ is the osmotic velocity field and the average $<\tilde{Q}> = \int \rho \cdot \tilde{Q}dx$ involves the quantum potential $\tilde{Q} = 2(\Delta\sqrt{\rho}/\sqrt{\rho})$ (cf. equations (6.1.13) - (6.1.16) where $Q = -(\hbar^2/2m)(\Delta\sqrt{\rho}/\sqrt{\rho})$, $\tilde{Q} = -(1/m)Q$, $D = \hbar/2m$, and $u = D\nabla log(\rho)$). Consequently $-<\tilde{Q}> \geq (1/2\sigma^2)$ for all relevant probability densities with any finite mean (with variance fixed at σ^2). We continue in this section with the notation $\tilde{Q} = 2(\Delta\sqrt{\rho}/\sqrt{\rho})$ and note that $D^2\tilde{Q}$ is in fact the correct $\tilde{Q} = -(1/m)Q$ (which occasionally arises here as well, in a diffusion context).

Next one defines the Kullback entropy $K(\theta,\theta')$ for a one parameter family of probability densities ρ_θ so that the distance between any two densities can be directly evaluated. Let $p_{\theta'}$ be the reference density and one writes

$$K(\theta,\theta') = K(\rho_\theta|\rho_{\theta'}) = \int \rho_\theta(x) log\frac{\rho_\theta(x)}{\rho_{\theta'}(x)}dx \tag{1.9}$$

(note this is positive and sometimes one refers to $\mathfrak{H}_c = -K$ as a conditional entropy). If one takes $\theta' = \theta + \Delta\theta$ with $\Delta\theta << 1$ then under a number of standard assumptions

$$K(\theta,\theta+\Delta\theta) \simeq \frac{1}{2}\mathfrak{F}_\theta \cdot (\Delta\theta)^2 \tag{1.10}$$

where $\mathfrak{F}_\theta$ denotes the Fisher information measure as in (1.7). More generally for a two parameter family $\theta \sim (\theta_1,\theta_2)$ of densities one has

$$K(\theta,\ \theta+\Delta\theta) \simeq \frac{1}{2}\sum \mathfrak{F}_{ij}\Delta\theta_i\Delta\theta_j;\ \mathfrak{F}_{ij} = \int \rho_\theta \frac{\partial log(\rho_\theta)}{\partial\theta_i}\frac{\partial log(\rho_\theta)}{\partial\theta_j}dx \tag{1.11}$$

For Gaussian densities at fixed σ with $\theta = \alpha$ one has then $K(\alpha,\alpha+\Delta\alpha) \simeq (\Delta\alpha)^2/2\sigma^2$. Various related formulas are derived and in particular one relates the Shannon entropy for a coarse grained density ρ_B to the differential entropy of the density ρ leading to a formula $\mathfrak{S}(\rho_B) - \mathfrak{S}(\rho'_B) \simeq \mathfrak{S}(\rho) - \mathfrak{S}(\rho')$. One considers also spatial Markov diffusion processes in $\mathbf{R}$ with a diffusion coefficient D which drive space-time inhomogeneous probability density densities $\rho(x,t)$. For example a free Brownian motion characterized by $v = -u = -D\nabla log(\rho(x,t))$ and diffusion current $j = v\cdot\rho$ obeys the continuity equation $\partial_t\rho = -\nabla j$ which is equivalent to the heat equation. As in Sections 1.1.6-1.1.8 and 6.1 we have the important relations

$$\tilde{Q} = 2D^2\frac{\Delta\rho^{1/2}}{\rho^{1/2}} = \frac{1}{2}u^2 + D\nabla\cdot u;\ \partial_t v + (v\cdot\nabla)v = -\nabla\tilde{Q} \tag{1.12}$$

A straightforward generalization refers to a diffusive dynamics of a mass m in a conservative force field $F = -\nabla V$. The associated Smoluchowski diffusion with a forward drift $b(x) = F/m\beta$ is analyzed in terms of a Fokker-Planck (FP) equation $\partial_t\rho = D\Delta\rho - \nabla(b\cdot\rho)$ with initial data $\rho_0(x) = \rho(x,0)$. For standard Brownian motion in an external force field one has $D = k_BT/m\beta$ where $\beta \sim$ friction, T is temperature and k_B is the Boltzman constant. With suitable hypotheses one has the following compatibility equations in the form of hydrodynamical conservation laws

$$\partial_t\rho + \nabla(v\rho) = 0;\ (\partial_t + v\cdot\nabla)v = \nabla(\Omega - \tilde{Q}) \tag{1.13}$$

where $\Omega(x)$ is the volume potential for the process, namely

$$\Omega = \frac{1}{2}\left(\frac{F}{m\beta}\right)^2 + D\nabla\cdot\left(\frac{F}{m\beta}\right) \tag{1.14}$$

Here$v = b - u = (F/m\beta) - D(\nabla\rho/\rho)$ defines the current velocity of Brownian partices in an external force field. With a solution ρ of the FP equation one associates a differential entropy $\mathfrak{S}(t) = -\int \rho log(\rho)dx$ which is typically not conserved.

With boundary conditions on ρ, $v\rho$, and $b\rho$ involving vanishing at boundaries or at infinity one obtains

$$\frac{d\mathfrak{S}}{dt} = \int \left[\rho(\nabla \cdot b) + D\frac{(\nabla\rho)^2}{\rho}\right] dx \tag{1.15}$$

One emphasizes that it is not obvious whether the differential entropy grows, decreases, or whatever. One can rewrite (1.15) in the forms

$$D\dot{\mathfrak{S}} = D < \nabla \cdot b > + < u^2 >= D < \nabla \cdot v >; \tag{1.16}$$

$$D\dot{\mathfrak{S}} =< v^2 > - < b \cdot v >= - < v \cdot u >$$

where $< \ >$ denotes the mean value relative to ρ. For $b = F/m\beta$ and $j = v\rho$ this leads to a characteristic "power release" expression

$$\frac{d\tilde{Q}}{dt} = \frac{1}{D}\int \frac{1}{m\beta} F \cdot jcx = \frac{1}{D} < b \cdot v > \tag{1.17}$$

Again $\dot{Q}$ can be positive (power removal) or negative (power absorption). In thermodynamic terms one deals here with the time rate at which the mechanical work per unit of mass is dissipated (removed from the reservoir) in the form of heat in the course of the Smoluchowski diffusion process - i.e. $k_BT\dot{\tilde{Q}} = \int F \cdot jdx$ where T is the temperature of the bath. For $b = 0$ (no external forces) one has $D\dot{\mathfrak{S}} = D^2\int[(\nabla\rho)^2/\rho]dx = D^2\mathfrak{F} = -D^2 < \tilde{Q} >$ and one can also write

$$\frac{d\mathfrak{S}}{dt} = \left(\frac{d\mathfrak{S}}{dt}\right)_{in} - \frac{d\tilde{Q}}{dt} \tag{1.18}$$

from (1.15) and (1.16) (here $(\dot{\mathfrak{S}})_{in} = (1/D) < v^2 >$.

One goes now to mean energy and the dynamics of Fisher information and considers $-\rho$ and s where $v = \nabla s$ as canonically conjugate fields; then one can use variational calculus to derive the continuity and FP equations together with the HJ type equations whose gradient gives the hydrodynamical conservation law

$$\partial_t s + (1/2)(\nabla s)^2 - (\Omega - \tilde{Q}) = 0 \tag{1.19}$$

Here the mean Lagrangian is

$$\mathfrak{L} = -\int \rho\left[\partial_t s + \frac{1}{2}(\nabla s)^2 - \left(\frac{u^2}{2} + \Omega\right)\right] dx \tag{1.20}$$

The related Hamiltonian (mean energy of the diffusion process per unit of mass) is

$$\mathfrak{H} = \int \rho\left[\frac{1}{2}(\nabla s)^2 - \left(\frac{u^2}{2} + \Omega\right)\right] dx = \frac{1}{2}(< v^2 > - < u^2 >) - < \Omega > \tag{1.21}$$

(note here $v = \nabla s$ satisfies $v = b - u$ with $u = D\nabla log(\rho)$ and we refer to Section 1.1 for clarification). One defines a thermodynamic force $F_{th} = v/D$ associated with the Smoluchowski diffusion with a corresponding potential $-\nabla\Psi = k_BTF_{th} = F - k_BT\nabla log(\rho)$ so in the absence of external forces $F_{th} = -\nabla log(\rho) = -(1/D)u$. The mean value of the thermodynamic force associates with the diffusion process an analogue of the Helmholz free energy $<\Psi>=<V>-T\mathfrak{S}_G$ where the dimensional version $\mathfrak{S}_G = k_B\mathfrak{S}$ of information entropy has been introduced (it is a configuration space analogue of the Gibbs entropy). Here the term $<V>$ plays the role of (mean) internal energy and assuming ρv vanishes at boundaries (or infinity) one obtains the time rate of change of Helmholz free energy at a constant temperature, namely

(1.22)
$$\frac{d}{dt}<\Psi>=-k_BT\dot{\tilde{Q}}-T\dot{\mathfrak{S}}_G \Rightarrow \frac{d}{dt}<\Psi>=-(k_BT)\left(\frac{d\mathfrak{S}}{dt}\right)_{in}=-(m\beta)<v^2>$$

Now one can evaluate an expectation value of (1.19) which implies an identity $\mathfrak{H}=-<\partial_t s>$. Then using $\Psi=V+k_BTlog(\rho)$ (with time independent V) one arrives at $\dot{\Psi}=(k_BT/\rho)\nabla(v\rho)$ and since $v\rho=0$ at integration boundaries we get $<\dot{\Psi}>=0$. Since $v=-(1/m\beta)\nabla\Psi$ define then $s(x,t)=(1/m\beta)\Psi(x,t)$ so that $<\partial_t s>=0$ and hence $\mathfrak{H}=0$ identically. This gives an interplay between the mean energy and the information entropy production rate in the form

$$\frac{D}{2}\left(\frac{d\mathfrak{S}}{dt}\right)_{in}=\frac{1}{2}<v^2>=\int\rho\left(\frac{u^2}{2}+\Omega\right)dx\geq 0 \tag{1.23}$$

Next recalling (1.7)-(1.8) and setting $\mathfrak{F}=D^2\mathfrak{F}_\alpha$ one obtains

$$\mathfrak{F}=<v^2>-2<\Omega>\geq 0 \tag{1.24}$$

where $(1/2)\mathfrak{F}=-<\tilde{Q}>$ holds for probability densities with finite mean and variance. One also derives the following formulas (under suitable hypotheses)

$$\partial_t(\rho v^2)=-\nabla\cdot[(\rho v^3)]-2\rho v\cdot\nabla(\tilde{Q}-\Omega); \tag{1.25}$$

$$\frac{d}{dt}<\Omega>=<v\cdot\nabla\Omega>;\ \frac{d}{dt}\mathfrak{F}=\frac{d}{dt}[<v^2>-2<\Omega>]=-2<v\cdot\nabla\tilde{Q}>$$

Then since $\nabla\tilde{Q}=\nabla P/\rho$ where $P=D^2\rho\Delta log(\rho)$ (this is the real $\tilde{Q}$) the previous equation takes the form $\dot{\mathfrak{F}}=-\int\rho v\nabla\tilde{Q}dx=-\int v\nabla Pdx$ which is an analogue of the familiar expression for the power release $(dE/dt)=F\cdot v$ with $F=-\nabla V$ in classical mechanics.

Next in [**396**] there is a discussion of differential entropy dynamics in quantum theory. Assume one has an arbitrary continuous function $\mathcal{V}(x,t)$ with dimensions of energy and consider the SE in the form $i\partial_t\psi=-D\Delta\psi+(\mathcal{V}/2mD)\psi$. Using $\psi=\rho^{1/2}exp(is)$ with $v=\nabla s$ one arrives at the standard equations $\partial_t\rho=-\nabla(v\rho)$ and $\partial_t s+(1/2)(\nabla s)^2+(\Omega-\tilde{Q})=0$ where $\Omega=\mathcal{V}/m$ and $\tilde{Q}$ has the same form as in (1.12) (note a sign change of the $\Omega-\tilde{Q}$ term in comparison with (1.19)). These two equations generate a Markovian diffusion type process the probability density of which is propagated by a FP dynamics as before with drift $b=v-u$ (instead of $v=b-u$) where $u=D\nabla log(\rho)$ is an osmotic velocity field. Repeating the variational calculations one looks at (cf. (1.21))

$$\mathfrak{H}=\int\rho\left[\frac{1}{2}(\nabla s)^2+\left(\frac{u^2}{2}+\Omega\right)\right]dx \tag{1.26}$$

Then

$$\mathfrak{H} = (1/2)[< v^2 > + < u^2 >] + < \Omega > = - < \partial_t s > \tag{1.27}$$

For time independent $\mathcal{V}$ one has $\mathfrak{H} = - < \partial_t s > = \mathcal{E} = const.$ and the FP equation propagates a probability density $|\psi|^2 = \rho$ whose differential entropy $\mathfrak{S}$ may nontrivially evolve in time. Maintaining the previous derivations involving $(\dot{\mathfrak{S}})_{in}$ one arrives at

$$(\dot{\mathfrak{S}})_{in} = \frac{2}{D}\left[\mathcal{E} - \left(\frac{1}{2}\mathfrak{F} + < \Omega >\right)\right] \geq 0 \tag{1.28}$$

One recalls $(1/2)\mathfrak{F} = - < \tilde{Q} > > 0$ so $\mathcal{E} - < \Omega > \geq (1/2)\mathfrak{F} > 0$. Hence the localization measure $\mathfrak{F}$ has a definite upper bound and the pertinent wave packet cannot be localized too sharply. Note also that $\mathfrak{F} = 2(\mathcal{E} - < \Omega >) - < v^2 >$ in general evolves in time (here $\mathcal{E}$ is a constant and $\dot{\Omega} = 0$). Using the hydrodynamical conservation laws one sees that the dynamics of Fisher information follows the rules

$$\frac{d\mathfrak{F}}{dt} = 2 < v\nabla\tilde{Q} >;\ \frac{1}{2}\dot{\mathfrak{F}} = -\frac{d}{dt}\left[\frac{1}{2} < v^2 > + < \omega >\right] \tag{1.29}$$

However $\dot{\mathfrak{F}} = \int v\nabla P dx$ where $P = D^2\rho\Delta log(\rho)$ and one interprets $\dot{\mathfrak{F}}$ as the measure of power transfer - keeping intact an overall mean energy $\mathfrak{H} = \mathcal{E}$. We refer to [**396**] for much more discussion and examples. We have concentrated on topics where the quantum potential appears in some form.

1.1. INFORMATION DYNAMICS. We go here to [**173, 174, 175**] and consider the idea of introducing some kind of dynamics in a reasoning process (Fisher information can apparently be linked to semantics - cf. [**907, 970**]). In [**173, 174**] one looks at the Fisher metric defined by

$$g_{\mu\nu} = \int_X d^4x p_\theta(x)\left(\frac{1}{p_\theta(x)}\frac{\partial p_\theta(x)}{\partial\theta^\mu}\right)\left(\frac{1}{p_\theta(x)}\right)\left(\frac{\partial p_\theta(x)}{\partial\theta^\nu}\right) \tag{1.30}$$

and constructs a Riemannian geometry via

$$\Gamma^\sigma_{\lambda\nu} = \frac{1}{2}g^{\nu\sigma}\left(\frac{\partial g_{\mu\nu}}{\partial\theta^\lambda} + \frac{\partial g_{\lambda\nu}}{\partial\theta^\mu} - \frac{\partial g_{\mu\lambda}}{\partial\theta^\nu}\right); \tag{1.31}$$

$$R^\lambda_{\mu\nu\kappa} = \frac{\partial\Gamma^\lambda_{\mu\nu}}{\partial\theta^\kappa} - \frac{\partial\Gamma^\lambda_{\mu\kappa}}{\partial\theta^\nu} + \Gamma^\eta_{\mu\nu}\Gamma^\lambda_{\kappa\eta} - \Gamma^\eta_{\mu\kappa}\Gamma^\lambda_{\nu\eta}$$

Then the Ricci tensor is $R_{\mu\kappa} = R^\lambda_{\mu\lambda\kappa}$ and the curvature scalar is $R = g^{\mu\kappa}R_{\mu\kappa}$. The dynamics associated with this metric can then be described via functionals

$$J[g_{\mu\nu}] = -\frac{1}{16\pi}\int\sqrt{g(\theta)}R(\theta)d^4\theta \tag{1.32}$$

leading upon variation in $g_{\mu\nu}$ to equations

$$R^{\mu\nu}(\theta) - \frac{1}{2}g^{\mu\nu}(\theta)R(\theta) = 0 \tag{1.33}$$

Contracting with $g_{\mu\nu}$ gives then the Einstein equations $R^{\mu\nu}(\theta) = 0$ (since $R = 0$). J is also invariant under $\theta \to \theta + \epsilon(\theta)$ and variation here plus contraction leads to a contracted Bianchi identity. Constraints can be built in by adding terms $(1/2)\int\sqrt{g}T^{\mu\nu}g_{\mu\nu}d^4\theta$ to $J[g_{\mu\nu}]$. If one is fixed on a given probability distribution

$p(x)$ with variable θ^μ attached to give $p_\theta(x)$ then this could conceivably describe some gravitational metric based on quantum fluctuations for example. As examples a Euclidean metric is produced in 3-space via Gaussian $p(x)$ and complex Gaussians will give a Lorentz metric in 4-space. However it seems to be very restrictive to have a fixed $p(x)$ as the basis; it would be nice if one could vary the probability distribution in some more general manner and study the corresponding Fisher metrics (and this seems eminently doable with a Fisher metric over a space of probability distributions).

1.2. INFORMATION MEASURES FOR QM. We follow here [**749**] and derive the SE within an information theoretic framework somewhat different from the exact uncertainty principle of Hall and Reginatto (cf. Sections 1.1, 3.1, and 4.7). Begin with a SE for N particles in $d+1$ dimensions of the form $i\hbar\psi_t = [-(\hbar^2/2m)g_{ij}\partial_i\partial_j + V]\psi$ with $g_{ij} = \delta_{ij}/m_{[i]}$ where $i, j = 1, \cdots, dN$ and $[i]$ is the smallest integer $\geq i/d$. Use the Madelung transformation $\psi = \sqrt{\rho}exp(iS/\hbar)$ (cf. [**614**]) to get
(1.34)

$$\partial_t S + \frac{g_{ij}}{2}\partial_i S\partial_j S + V - \frac{\hbar^2}{8}g_{ij}\left(\frac{2\partial_i\partial_j\rho}{\rho} - \frac{\partial_i\rho\partial_j\rho}{\rho^2}\right) = 0;\ \partial_t\rho + g_{ij}\partial_i(\rho\partial_j S) = 0$$

These equations can be obtained from a variational principle, minimizing the action

$$\Phi = \int \rho\left[\partial_t S + \frac{g_{ij}}{2}\partial_i S\partial_j S + V\right]dx^{Nd}dt + \frac{\hbar^2}{8}I_F; \tag{1.35}$$

$$I_F = \int dx^{Nd}dtg_{ij}\rho(\partial_i log(\rho))(\partial_j log(\rho))$$

Here I_F resembles the Fisher information of [**369**] whose inverse sets a lower bound on the variance of the probabiliy distribution ρ via the Cramer-Rao inequaliity (see Section 1.1). (1.35) was used to derive the SE through a procedure analogous to the principle of maximum entropy in [**807, 806**] (cf. also Section 1.1). However the method of [**806**] does not explain a priori the form of information measure that should be used; i.e. why must the Fisher information be minimized rather than something else. The aim of [**749**] is to construct permissible information measures I. Thus the relevant action is

$$\mathfrak{A} = \int \rho\left[\partial_t S + \frac{g_{ij}}{2}\partial_i S\partial_j kS + V\right]dx^{Nd}dt + \lambda I \tag{1.36}$$

with λ a Lagrange multiplier. Varying this action will lead in general to a nonlinear SE

$$i\hbar\partial_t\psi = \left[-\frac{\hbar^2}{2}g_{ij}\partial_i\partial_j + V\right]\psi + F(\psi,\psi^\dagger)\psi \tag{1.37}$$

In order to have deformations of the linear theory that permit maximal preservation of the usual interpretation of the wave function one considers the following conditions:

(1) I should be real valued and positive definite for all $\rho = \psi^\dagger\psi$ and should be independent of V.

(2) I should be of the form $I = \int dx^{Nd} dt\rho H(\rho)$ where H is a function of $\rho(x,t)$ and its spatial derivatives. This will insure the weak superposition principle in the equations of motion.
(3) H should be invariant under scaling, i.e. $H(\lambda\rho) = H(\rho)$ which allows solutions of (1.37) to be renormalized, etc.
(4) H should be separable for the case of two independent subsystems for which the wave function factorizes, i.e. $H(\rho_1\rho_2) = H(\rho_1) + H(\rho_2)$.
(5) H should be Galilean invariant.
(6) The action should not contain derivatives beyond second order (Absence of higher order derivatives or AHD condition). This will insure that the multiplier λ, and hence Planck's constant, will be the only new parameter that is required in making the transition from classical to quantum mechanics.

The conditions 2-6 are already satisfied by the classical part of the action so it is quite minimalist to require them also of I. The homogeneity requirement 3 cannot be satisfied if H depends only on ρ; it must contain derivatives and the AHD and rotational invariance conditions imply then that $H = g_{ij}(U_1\partial_i U_2\partial_j U_3 + V_1\partial_i\partial_j V_2)$ where the U_i, V_i are functions of ρ. One can write then

$$H = g_{ij}\left(\frac{\partial_i\rho\partial_j\rho}{\rho^2}[U_1U_2'U_3'\rho^2 + V_1V_2''\rho^2] + \frac{\partial_i\partial_j\rho}{\rho}[V_1V_2'\rho]\right) \tag{1.38}$$

where the prime denotes a derivative with respect to ρ. Scaling conditions plus positivity and universality then lead to

$$I = \int dtdx^{Nd}\rho g_{ij}\frac{\partial_i\rho\partial_j\rho}{\rho^2} \tag{1.39}$$

Consequently the unique solution of the conditions 1-6 is the Fisher information measure and one arrives at the linear SE since the Lagrange multiplier must then have the dimension of $action^2$ thereby introducing the Planck constant. Note condition 4 was not used but it will be useful below. Further one notes that the AHD condition ensures that within the information theoretic approach the SE is the unique single parameter extension of the classical HJ equations. One also argues that a different choice of metric in the information term would in fact lead back to the original g_{ij} after a nonlinear gauge transformation; this suggests that a nonlinear SE is not automatically pathological. Further argument also shows that I should not depend on S. The main difference between this and the Hall-Reginatto method is to replace the exact uncertainly principle by condition 3.

1.3. PHASE TRANSITIONS. Referring here to [**512**] the introduction of a metric onto the space of parameters in models in statistical mechanics gives an alternative perspective on their phase structure. In fact the scalar curvature $\mathcal{R}$ plays a central role where for a flat geometry $\mathcal{R} = 0$ (noninteracting system) while $\mathcal{R}$ diverges at the critical point of an interacting one. Thus models are characterized by certain sets of parameters and given a probability distribution $p(x|\theta)$ and a sample x_i the object is to estimate the parameter θ. This can be done by maximizing the so-called likelihood function $L(\theta) = \prod_1^n p(x_i|\theta)$ or its

logarithm. Thus one writes
(1.40)

$$log(L(\theta)) = \sum_1^n log(p(x_i|\theta));\ U(\theta) = \frac{d\,log(L(\theta))}{d\theta};\ Var[U(\theta)] = -\left[\frac{-d^2log(L(\theta))}{d\theta^2}\right]$$

The last term $Var[U(\theta))]$ is called the expected or Fisher information and we note that it is the same as (1.30) (see below) and in multidimensional form is expressed via

$$G_{ij}(\theta) = -E\left[\frac{\partial^2 log(p(x|\theta))}{\partial\theta_i\partial\theta_j}\right] = -\int p(x|\theta)\frac{\partial^2 log(p(x|\theta))}{\partial\theta_i\partial\theta_j}dx \tag{1.41}$$

In generic statistical-physics models one often has two parameters β (inverse temperature) and h (external field); in this case the Fisher-Rao metric is given by $G_{ij} = \partial_i\partial_j f$ where f is the reduced free energy per site and this leads to a scalar curvature

$$\mathcal{R} = -\frac{1}{2G^2}\begin{vmatrix} \partial_\beta^2 f & \partial_\beta\partial_h f & \partial_h^2 f \\ \partial_\beta^3 f & \partial_\beta^2\partial_h f & \partial_\beta\partial_h^2 f \\ \partial_\beta^2\partial_h f & \partial_\beta\partial_h^2 f & \partial_h^3 f \end{vmatrix} \tag{1.42}$$

where $G = det(G_{ij})$. In some sense $\mathcal{R}$ measures the complexity of the system since for $\mathcal{R} = 0$ the system is not interacting and (in all known systems) the curvature diverges at, and only at, a phase transition point. As an example under standard scaling assumptions one can anticipate the behavior of $\mathcal{R}$ near a second order critical point. Set $t = 1 - (\beta/\beta_c)$ and consider

$$f(\beta, h) = \lambda^{-1} f(t\lambda^{a_t}, h\lambda^{a_h}) = t^{1/a_t}\psi(ht^{-a_h/a_t});\ a_t = \frac{1}{\nu d};\ a_h = \frac{\beta\delta}{\nu d} \tag{1.43}$$

$a_t,\ a_h$ are the scaling dimensions for the energy and spin operators and d is the space dimension. For the scalar curvature there results

$$\mathcal{R} = -\frac{1}{2G^2}\begin{vmatrix} t^{(1/a_t)-2} & 0 & t^{(1/a_t)-2(a_h/a_t)} \\ t^{(1/a_t)-3} & 0 & t^{(1/a_t)-2(a_h/a_t)-1} \\ 0 & t^{(1/a_t)-2(a_h/a_t)-1} & t^{(1/a_t)-3(a_h/a_t)} \end{vmatrix}; \tag{1.44}$$

$$G \sim t^{(2/a_t)+2(a_h/a_t)-2} \Rightarrow \mathcal{R} \sim \xi^d \sim |\beta - \beta_c|^{\alpha-2}$$

where hyperscaling ($\nu d = 2 - \alpha$) is assumed and ξ is the correlation length. We refer to [**512**] for more details, examples, and references.

1.4. FISHER INFORMATION AND HAMILTON'S EQUATIONS. Going to [**755**] one shows that the mathematical form of the Fisher information I for a Gibb's canonical probability distribution incorporates important features of the intrinsic structure of classical mechanics and has a universal form in terms of forces and accelerations (i.e. one that is valid for all Hamiltonians of the form $T + V$). First one has shown that the Fisher information measure provides a powerful variational principle, that of extreme information, which yields most of the canonical Lagrangians of theoretical physics. In addition I provides an interesting characterization of the "arrow of time", alternative to the one associated with the Boltzman entropy (cf. [**776, 777**]). Following [**381, 384**] one considers a (θ, z) "scenario" in which we deal with a system specified by a physical parameter θ while z is a stochastic variable ($z \in \mathbf{R}^M$) and $f_\theta(z)$ is a probability density for z. One makes a measurement of z and has to infer θ, calling the resulting estimate $\tilde{\theta} = \tilde{\theta}(z)$. Estimation theory states that the best possible estimator $\tilde{\theta}(z)$, after a large number of samples, suffers a mean-square error e^2 from θ that obeys a relationship involving Fisher's I, namely $Ie^2 = 1$, where $I(\theta) = \int dz f_\theta(z)[\partial log(f_\theta(z))/\partial\theta]^2$ (only unbiased estimators with $<\tilde{\theta}>= \theta$ are in competition). The result here is that $Ie^2 \geq 1$ (Cramer-Rao bound). A case of great importance here concerns shift invariant distribution functions where the form does not change under θ displacements and one can write

$$I = \int dz f(z) \left(\frac{\partial log(f(z))}{\partial z}\right)^2 \tag{1.45}$$

If one is dealing with phase space where z is a M=2N dimensional vector with coordinates r and p then $I(z) = I(r) + I(p)$ (cf. [**755**]). Now assume that one wishes to describe a classical system of N identical particles of mass m with Hamiltonian

$$\mathfrak{H} = \mathfrak{T} + \mathfrak{V} = \sum_1^N \frac{p_i^2}{2m} + \sum_1^N V(r_i) \tag{1.46}$$

This is a simple situation but the analysis is not limited to such systems. Assume also that the system is in equilibrium at temperature T so that in the canonical ensemble the probability density is

$$\rho(r,p) = \frac{e^{-\beta\mathfrak{H}(r,p)}}{Z};\ Z = \int \frac{d^{3N}r d^{3N}p}{N!h^{3N}} e^{-\beta\mathfrak{H}(r,p)} \tag{1.47}$$

(here for h an elementary cell in phase space one writes $d\tau = d^{3N}rd^{3N}p/(N!h^{3N}$, $\beta = 1/kT$ with k the Boltzman constant, and Z is the partition function). Then from Hamilton's equations $\partial_p\mathfrak{H} = \dot{r}$ and $\partial_r\mathfrak{H} = -\dot{p}$ there results

$$-kT\frac{\partial log(\rho(r,p))}{\partial p} = \dot{r};\ -kT\frac{\partial log(\rho(r,p))}{\partial r} = -\dot{p} \tag{1.48}$$

One can now write the Fisher information measure in the form

$$I_\tau = \int \frac{d^{3N}rd^{3N}p}{N!h^{3N}}\rho(r,p)\mathfrak{A}(r,p);\ \mathfrak{A} = a\left(\frac{\partial log(\rho(r,p))}{\partial p}\right)^2 + b\left(\frac{\partial log(\rho(r,p))}{\partial r}\right)^2 \tag{1.49}$$

One needs two coefficients for dimensional balance (cf. [**755**]). One notes that

$$\frac{\partial log(\rho(r,p))}{\partial p} = -\beta\frac{\partial\mathfrak{H}}{\partial p};\ \frac{\partial log(\rho(r,p))}{\partial r} = -\beta\frac{\partial\mathfrak{H}}{\partial r} \tag{1.50}$$

leading to the Fisher information in the form

$$(kT)^2 I_\tau = a\left\langle\left(\frac{\partial\mathfrak{H}}{\partial p}\right)^2\right\rangle + b\left\langle\left(\frac{\partial\mathfrak{H}}{\partial r}\right)^2\right\rangle \Rightarrow I_\tau = \beta^2[a<\dot{r}^2> + b<\dot{p}^2>] \tag{1.51}$$

This gives the universal Fisher form for any Hamiltonian of the form (1.46) and we refer to [**607**] for connections to kinetic theory. Many other interesting results on Fisher can be found in [**381, 382, 755**].

1.5. UNCERTAINTY AND FLUCTUATIONS. We go first to [**38**] and recall the idea of a phase space distribution in the form (♣) $\mu(p,q) =< z|\rho|z >$ where ρ is the density matrix and $|z>$ denotes coherent states (cf. [**191, 757**] for coherent states). The chosen measure of uncertainty here is the Shannon information

$$I = -\int \frac{dpdq}{2\pi\hbar}\mu(p,q)log(\mu(p,q)) \tag{1.52}$$

The uncertainty principle manifests itself via the inequality (♠) $I \geq 1$ with equality if and only if ρ is a coherent state (cf. [**609, 987**]). In [**38**] one wants to generalize this to include the effects of thermal fluctuations in nonequilibrium systems and we sketch some of the ideas at least for equilibrium systems. There are in general three contributions to the uncertainty:

(1) The quantum mechanical uncertainty (quantum fluctuations) which is not dependent on the dynamics.
(2) The uncertainty due to spreading or reassembly of the wave packet. This is a dynamical effect and it may increase or decrease the uncertainty.
(3) The uncertainty due to the coupling to a thermal environment (diffusion and dissipation).

The time evolution I_t of I is studied for nonequilibrium systems and it is shown to generally settle down to monotone increase. I_t^{min} is a measure of the amount of quantum and thermal noise the system must suffer after a nonunitary evolution for time t (we do not deal with this here but refer to [**38**] for the nonequilibrium situation where the system decomposes into a distinguished system $\mathcal{S}$ plus the rest, referred to as the environment; the resulting time evolution of ρ is then nonunitary). In any event the lower bound I_t^{min} includes the effects of 1 and 3 but avoids 2.

One recalls the Shannon information (discussed earlier)

$$I(S) = -\sum_1^N p_i log(p_i);\ 0 \leq I(S) \leq log(N) \tag{1.53}$$

This is often referred to as entropy but here the word entropy is reserved for the vonNeumann entropy. In a similar manner, for continuous distributions (X a random variable with probability density $p(x)$ and $\int p(x)dx = 1$), the information of X is defined as

$$I(X) = -\int dxp(x)log(p(x)) \tag{1.54}$$

One emphasize that $p(x)$ here is a density (so it may be greater that 1 and $I(X)$ may be negative). However it retains its utility as a measure of uncertainty and e.g. for a Gaussian

$$p(x) = \frac{1}{[2\pi(\Delta x)^2]^{1/2}}exp\left(-\frac{(x-x_0)^2}{2(\Delta x)^2}\right);\ I(X) = log\left(2\pi e(\Delta x)^2\right)^{1/2} \tag{1.55}$$

Thus $I(X)$ is unbounded from below and goes to $-\infty$ as $\Delta x \to 0$ and $p(x)$ goes to a delta function. $I(X)$ is also unbounded from above but if the variance is fixed then $I(X)$ is maximized by the Gaussian distribution (1.55). Hence one has

$$I(X) \leq log\left(2\pi e(\Delta x)^2\right)^{1/2} \tag{1.56}$$

The generalization to more than one variable is straightforward, e.g.

$$I(X,Y) = -\int dxdyp(x,y)log(p(x,y)) \Rightarrow I(X,Y) \leq I(X) + I(Y) \tag{1.57}$$

where e.g. $I(X) = \int dyp(x,y)$. It is useful to introduce QM phase space distributions of the form

(1.58)

$$\mu(p,q) = < z|\rho|z >;\ < x|z > = < x|p,q > = \left(\frac{1}{2\pi\sigma_q^2}\right)^{1/4} exp\left(-\frac{(x-q)^2}{4\sigma_q^2} + ipx\right)$$

Here $< x|z >$ is a coherent state with $\sigma_q\sigma_p = (1/2)\hbar$ and there is a normalization $\int (dpdq/2\pi\hbar)\mu(p,q) = 1$. One can also show that

$$\mu(p,q) = 2\int dp'dq'exp\left(-\frac{(p-p')^2}{2\sigma_p^2} - \frac{(q-q')^2}{2\sigma_q^2}\right) W_\rho(p',q'); \tag{1.59}$$

$$W_\rho(p,q) = \frac{1}{2\pi\hbar}\int d\xi e^{-(i/\hbar)p\xi}\rho(q+(1/2)\xi, q-(1/2)\xi)$$

(Wigner function - cf. [**191, 192**]). One is interested in the extent to which $\mu(p,q)$ is peaked about some region in phase space and the Shannon information is a natural measure of the extent to which a probability distribution is peaked. Thus one takes as a measure of uncertainty the information

$$I(P,Q) = -\int \frac{dpdq}{2\pi\hbar}\mu(p,q)log(\mu(p,q)) \tag{1.60}$$

One expects there to be a lower bound for I and it should be achieved on a coherent state and this was in fact proved (cf. [**609, 987**]) in the form $I(P,Q) \geq 1$ with equality if and only if ρ is the density matrix of a coherent state $|z' >< z'|$. Further

$$log\left(\frac{e}{\hbar}\Delta_\mu q\Delta_\mu p\right) \geq I(Q) + I(P) \geq I(P,Q) \tag{1.61}$$

The variances here have the form

$$(\Delta_\mu q)^2 = (\Delta_\rho q)^2 + \sigma_q^2;\ (\Delta_\mu p)^2 = (\Delta_\rho p)^2 + \sigma_p^2 \tag{1.62}$$

where Δ_ρ denotes the QM variance and hence

$$\left((\Delta_\rho q)^2 + \sigma_q^2\right)\left((\Delta_\rho p)^2 + \sigma_p^2\right) \geq \hbar^2 \tag{1.63}$$

Minimizing (1.63) over σ_q (and recalling that $\sigma_q\sigma_p = (1/2)\hbar$) one obtains the standard uncertainty relation $\Delta x\Delta p \geq (\hbar/2)$. Now suppose one has a genuinely mixed state so that

$$\rho = \sum_n p_n|n >< n|;\ p_n < 1;\ \mu(p,q) = \sum p_n| < z|n > |^2 \tag{1.64}$$

The information of (1.64) will always satisfy $I(P,Q) \geq 1$ but this is a very low lower bound; indeed from the inequality

$$-\left(\int dx f(x)g(x)\right) log\left(\int dy f(y)g(y)\right) \geq -\int dx g(x)g(x)log(x) \quad (1.65)$$

we have

(1.66)
$$I \geq -\int \frac{dpdq}{2\pi\hbar}\sum_n | < z|n > |^2 p_n log(p_n) = -\sum p_n log(p_n) = -Tr(\rho log(\rho)) \equiv S[\rho]$$

Thus I is bounded from below by the vonNeumann entropy $S[\rho]$ and this is a virtue of the chosen measure of uncertainty. One sees that I is a useful measure of both quantum and thermal fluctuations. It has a lower bound expressing the effect of quantum fluctuations which is connected to entropy and this in turn is a measure of thermal fluctuations.

Consider now the situation of thermal equilibrium. Let the density matrix be thermal, $\rho = Z^{-1}exp(-\beta H)$ where $Z = Tr(e^{-\beta H})$ is the partition function and $\beta = 1/kT$. Then

$$< z|\rho|z >= \frac{1}{Z}\sum e^{-\beta E_n} | < z|n > |^2 \quad (1.67)$$

where $|n >$ are energy eigenstates with eigenvalue E_n. For simplicity look at a harmonic oscillator for which

$$H = \frac{1}{2}\left(\frac{p^2}{M} + M\omega^2 q^2\right);\ | < z|n > |^2 = \frac{|z|^{2n}}{n!}e^{-|z|^2};\ E_n = \hbar\omega(n + (1/2)) \quad (1.68)$$

Here $z = (1/2)[(q/\sigma_q) + i(p/\sigma_p)]$ where $\sigma_q\sigma_q = (1/2)\hbar$ and $\sigma_q = (\hbar/2M\omega)^{1/2}$ (cf. [**191, 559, 757**] for coherent states). There results

$$\mu(q,p) =< z|\rho|z >= \left(1 - e^{-\beta\hbar\omega}\right) exp\left(-(1 - e^{-\beta\hbar\omega})|z|^2\right) \quad (1.69)$$

The information (1.60) is then (•) $I = 1 - log(1 - e^{-\beta\hbar\omega})$ which is exactly what one expects; as $T \to 0$ one has $\beta \to \infty$ and the uncertainty reduces to the Lieb-Wehrl result $I(P,Q) \geq 1$ expressing purely quantum fluctuations. For nonzero temperature however the uncertainty is larger tending to the value $-log(\beta\hbar\omega)$ as $T \to \infty$ which expresses purely thermal fluctuations. It is interesting to compare (•) with the entropy $S = -Tr(\rho log(\rho))$. Here the partition function is $Z = [2Sinh((1/2)\beta\hbar\omega)]^{-1}$ and the entropy is then $S = -\beta(\partial_\beta(log(Z)) + log(Z)$ or

$$S = -log[2Sinh((1/2)\beta\hbar\omega)] + (1/2)\beta\hbar\omega Coth[(1/2)\beta\hbar\omega] \quad (1.70)$$

For large T one has then $S \simeq -log(\beta\hbar\omega)$ coinciding with I but $S \to 0$ as $T \to 0$ while I goes to a nontrivial lower bound. Hence one sees that I is a useful measure of uncertainty in both the quantum and thermal regimes. We refer also to [**4**] where an information theoretic uncertainty relation including the effects of thermal fluctuations at thermal equilibrium has been derived using thermofield dynamics (cf. [**950**]); their information theoretic measure is however different than that in [**38**]. On goes next to non-equilibrium systems and proves for linear systems that, for each t, I has a lower bound I_t^{min} over all possible initial states. It coincides with the Lieb-Wehrl bound in the absence of an environment and is related to the vonNeumann entropy in the long time limit. We refer to [**38**] for details.

2. A TOUCH OF CHAOS

For quantum chaos we refer to [**33, 96, 97, 185, 218, 282, 359, 435, 484, 491, 517, 518, 584, 659, 747, 787, 936**] and begin here with [**747**]. Chaos is quantitatively measured by the Lyapunov spectrum of characteristic exponents which represent the principal rates of orbit divergence in phase space, or alternatively by the Kolmogorov-Sinai (KS) invariant, which quantifies the rate of information production by the dynamical system. Chaos is conspicuously absent in finite quantum systems but the chaotic nature of a given classical Hamiltonian produces certain characteristic features in the dynamical behavior of its quantized version; these features are referred to as quantum chaos (cf. [**218, 435**]). They include short term instabilities and diffusive behavior versus dynamical localization and other effects. One is concerned here with an approach to the information dynamics of the quantum-classical transition based on the HJ formalism with the KS invariant playing a central role. The extension to the quantum domain is accomplished via the orbits introduced by Madelung and Bohm (cf. [**129, 614**]); these are natural extensions of the classical phase space flow to QM and provide the required bridge across the transition. One striking result is that the quantum KS invariant for a given Madelung-Bohm (MB) orbit is equal to the mean decay rate of the probability density along the orbit. Further one shows that the quantum KS invariant averaged over the ensemble of MB orbits equals the mean growth rate of configuration space information and a general and rigorous argument is given for the conjecture that the standard quantum-classical correspondence (or the classical limit) breaks down for classically chaotic Hamiltonians.

We give only a sketch of results here. Thus consider a classical system of N degrees of freedom described by canonical variables (q_i, p_i) with $1 \leq i \leq N$ and denote the Hamiltonian as $H(\mathbf{q}, \mathbf{p}, t)$ with Hamilton principal function $S(\mathbf{q}, t, \mathbf{p}_0)$ where $\mathbf{p}_0$ being the initial momenta. In matrix form Hamilton's equations are $\dot{\xi} = \mathcal{J}\nabla_\xi H(\xi, t)$ where ξ stands for the 2N dimensional phase space vector $(\mathbf{q}, \mathbf{p})$. Here $\mathcal{J}$ is a real antisymmetric matrix of order 2N with a $2 \otimes N$ block form $(0_N, I_N, -I_N, 0_N)$ which is a listing of blocks in the order $(11, 12, 21, 22)$. The tangent dynamics of the system is described by the $2N \times 2N$ nonsingular matrix $\mathcal{T}_{\mu\nu}(t, \xi_0) = \partial\xi_\mu(t, \xi_0)/\partial\xi_{0\nu}$ (the sensitivity matrix) where $\xi(t, \xi_0)$ is the trajectory starting from ξ_0 at time t_0. One can in fact write ($\tilde{S} \equiv S^T$ - matrix transpose)

$$\mathcal{T} = (S^{-1}_{\mathbf{p}_0\mathbf{q}}, -S_{\mathbf{p}_0\mathbf{q}}S_{\mathbf{p}_0\mathbf{p}_0}, S_{\mathbf{q}\mathbf{q}}S^{-1}_{\mathbf{p}_0\mathbf{q}}, \tilde{S}_{\mathbf{p}_0\mathbf{q}} - S_{\mathbf{q}\mathbf{q}}S^{-1}_{\mathbf{p}_0\mathbf{q}}S_{\mathbf{p}_0\mathbf{p}_0}) \tag{2.1}$$

where $(S_{\mathbf{p}_0\mathbf{q}})_{ij} = \partial^2 S/\partial_{q_j}\partial_{p_0 i}$. It is shown that one can write $\mathcal{T}$ in an upper triangular block form $\Gamma = \Omega(\Theta)\mathcal{T}$ where $\Omega(\Theta) = (cos(\Theta), -Sin(\Theta), Sin(\Theta), Cos(\Theta))$ and $-\sigma = Tan(\Theta) = -S_{\mathbf{q}\mathbf{q}}$. Here Θ is a real symmetric matrix of order N while Ω is orthogonal and symplectic (symplectic phase matrix). The upper triangular form $(\Gamma_{11}, \Gamma_{12}, 0_N, \Gamma_{22})$ of Γ satisfies $\Gamma_{11}^{-1} = \tilde{\Gamma}_{22}$ and the upper half of the Lyapunov spectrum is obtained from the singular values of Γ_{11} (see [**747**]). In particular the

Kolmogorov-Sinai (KS) entropy is given via

$$k = lim_{t\to\infty} log[det(\Gamma_{11})]/t \tag{2.2}$$

For illustration consider the standard form $H = \mathbf{p}^2/2 + V(\mathbf{q}, t)$ with N-dimensional vectors $\mathbf{q}$, $\mathbf{p}$. Then

$$k =< Tr(\sigma) >_{p.v.};\ < f >= lim_{t\to\infty} \frac{1}{t}\int_0^t f(t')dt' \tag{2.3}$$

where p.v. stipulates a principal value evaluation (σ will have simple pole behavior near singularities and the principal value contribution vanishes). Since $Tr(\sigma) = \nabla_q^2 S$ along the orbit (2.3) simply states that the KS invariant equals the time average of the Laplacian of the action along the orbit. Now the MB formalism associates a phase space flow with a quantum system via

$$\psi = exp[iS(\mathbf{x}, t)/\hbar + R(\mathbf{x}, t)];\ \dot{\mathbf{q}}(t, \mathbf{q}_0, \mathbf{p}_0) = \mathbf{p} = \nabla S[\mathbf{q}, t] \tag{2.4}$$

It can be verified that the expectation value of any observable in the state ψ is given by its average over the ensemble of orbits thus defined (e.g. Ehrenfest's equations arise in this manner). The correspondence thus allows us to define the quantum KS invariant for a given orbit as

$$\mathbf{k} =< \nabla^2 S >_{p.v.} \tag{2.5}$$

(the averaging process is with respect to the time along the MB orbit to which S is restricted). Now intuitively one would expect that orbits neighboring a hypothetical chaotic orbit in the ensemble diverge from it on the average thus causing the orbit density along the chaotic orbit to decrease with a mean rate related to $\mathbf{k}$. This is fully realized here as one sees by considering the equation of motion for $R(\mathbf{x}, t)$ as inherited from the SE, namely $\partial_t R + \nabla R\cdot\nabla S = -(1/2)\nabla^2 S$. The characteristic curves for this equation are the MB orbits so that it takes the following form along these orbits;

$$\frac{dR}{dt} = -\frac{1}{2}\nabla^2 S \Rightarrow \mathbf{k} = -2\left\langle \frac{dR}{dt}\right\rangle = -\left\langle \frac{d\,log(|\psi|^2)}{dt}\right\rangle_{p.v.} \tag{2.6}$$

This says that the quantum KS invariant for a given orbit is the mean decay rate of the probability density along the orbit. Comparing this to a classical system where $k \neq 0$ while $\mathbf{k} = 0$ for the quantum version one sees that the classical limit cannot hold for chaotic Hamiltonians and since chaotic classical Hamiltonians are certainly more common than regular ones the idea of classical limit is not a reliable test for quantum systems. Finally let $\bar{\mathbf{k}}$ be the MB ensemble average, which is the same as the QM expectation value, leading to

$$\bar{\mathbf{k}} = lim_{t\to\infty}\frac{1}{t}\int dq|\psi|^2 log(|\psi|^2) \tag{2.7}$$

which is an information entropy measure. The discussion here is very incomplete but should motivate further investigation and we refer to [**747**] for more detail (cf. also [**976, 977**] involving chaos, fractals, and entropy.

2.1. CHAOS AND THE QUANTUM POTENTIAL. The paper [**745**] offers an interesting perspective on the quantum potential. Thus consider a system of n particles with the SE

$$i\hbar\partial_t\psi = \left[\sum_1^n\left(\frac{-\hbar^2}{2m_i}\right)\nabla_i^2 + V\right]\psi;\ \nabla_i = \left(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}, \frac{\partial}{\partial z_i}\right) \tag{2.8}$$

(here $\mathbf{x}_i = (x_i, y_i, z_i)$). Set $\psi = Rexp[i(S/\hbar)]$ and there results as usual

$$\partial_t S + \sum_1^n (\nabla_i S)^2(2m_i)^{-1} + Q + V = 0;\ \partial_t R^2 + \sum_1^n \nabla_i\cdot\left(\frac{R^2\nabla_i S}{m_i}\right) = 0 \tag{2.9}$$

where $Q = -\sum_1^n(\hbar^2/2m_iR)\nabla_i^2 R$. Now just as the causal form of the HJ equation contains the additional term Q so the causal form of Newton's second law contains Q as follows

$$\dot{P}_i = -\nabla_i V - \nabla_i Q;\ P = \sum_1^n P_i;\ \dot{P} = \frac{dP}{dt} = -\sum_1^n \nabla_i V - \sum_1^n \nabla_i Q \tag{2.10}$$

The author cites a number of curious and conflicting statements in the literature concerning the effect of the quantum potential on Bohmian trajectories, for clarification of which he observes that for an isolated system one has

$$-\sum\nabla_i V = 0;\ \dot{P} = 0;\ -\sum\nabla_i Q = -\sum F_i = 0 \tag{2.11}$$

Thus the sum of all the quantum forces is zero so $F_i = \sum_{j\neq i}^n(-F_j)$. Thus the net quantum force on a given particle is the result of all the other particles exerting force on this particle via the intermediary of the quantum potential. This then is his explanation for the guidance role of the wave function.

Next it is noted that removing Q from the HJ equation is equivalent to adding the term

$$\left(\frac{\hbar^2}{2m}\right)exp(iS/\hbar)\nabla^2 R = \left(\frac{\hbar^2}{2m}\right)|\psi|^{-1}\psi\nabla^2|\psi| = -Q\psi \tag{2.12}$$

to the SE so that the effective Hamiltonian becomes

$$H_{eff} = -\left(\frac{\hbar^2}{2m}\right)\nabla^2 + V + \left(\frac{\hbar^2}{2m}\right)|\psi|^{-1}\nabla^2|\psi| \tag{2.13}$$

Since $H_{eff} = H_{eff}(\psi)$ depends on ψ the superposition principle no longer applies. When $\phi \neq \psi$ we have

$$\int(\phi^* H_{eff}\psi - \psi H_{eff}\phi^*)d\tau = \left(\frac{\hbar^2}{2m}\right)\int\phi^*\psi[|\psi|^{-1}\nabla^2|\psi| - |\phi|^{-1}\nabla^2|\phi|] \neq 0 \tag{2.14}$$

so H_{eff} is not Hermitian. Hence the time development operator $exp[(i/\hbar)H_{eff}t]$ is not unitary and the time dependent SE is a nonunitary flow. Then since $i\hbar\partial_t(\psi^*\psi) = \psi^* H_{eff}\psi - \psi H_{eff}\psi^*$ one has

$$\partial_t\int|\psi - \phi|^2 d\tau = \left(\frac{\hbar^2}{2m}\right)\int i[\psi^*\phi - \phi^*\psi][|\psi|^{-1}\nabla^2|\psi| - |\phi|^{-1}|\phi|]d\tau \neq 0 \tag{2.15}$$

Consider then the case where two initial conditions for the time dependent SE differ only infinitesimally. As time progresses the two corresponding wave functions can become quite different, indicating the possibity of deterministic chaos, and this is a consequence of H_{eff} being a functional of the state upon which it is acting. If the term $(\hbar^2/2m)|\psi|^{-1}\nabla^2|\psi|$ is removed from (2.13) one is left with a Hermitian Hamiltonian and the normalization of $(\psi - \phi)$ is time independent, so there can be no deterministic chaos. Thus in particular Q acts as a constraining force preventing deterministic chaos (cf. also [**623**]).

REMARK 6.2.1. There are many different aspects of quantum chaos and the perspective of [**?**] just mentioned does not deal with everything covered in the references already cited (cf. also [**633, 983, 1000, 1001**] for additional referencers). We are not expert enough to attempt any kind of in depth coverage but extract here briefly from a few papers. First from [**1000**] one notes that the dBB theory of quantum motion provides motion in deterministic orbits under the influence of the quantum potential. This quantum potential can be very intricate because it generates wave interferences and further numerical work has shown the presence of chaos and complex behavior of quantum trajectories in various systems (cf. [**746**]). In [**1000**] one indicates that movement of the zeros of the wave function (called vortices) implies chaos in the dynamics of quantum trajectories. These vortices result from wave function interferences and have no classical explanation. In systems without magnetic fields the bulk vorticity $\nabla \times \mathbf{v}$ in the probability fluid is determined by points where the phase S is singular (which can occur when the wave function vanishes). Due to singlevaluedness of the wave function the circulation $\Gamma = \int_C \dot{\mathbf{r}} d\mathbf{r} = (2\pi n/m)$ around a closed contour C encircling a vortex is quantized with n an integer and the velocity must diverge as one approaches a vortex. This leads to a universal mechanism producing chaotic behavior of quantum trajectories (cf. also [**746, 1001**]).

Next in [**983**] one speaks of the edge of quantum chaos (the border between chaotic and non-chaotic regions) where the Lyapunov exponent goes to zero; it is then replaced by a generalized Lyapunov coefficient describing power-law rather than exponential divergence of classical trajectories. In [**983**] one characterizes quantum chaos by comparing the evolution of an initially chosen state under the chaotic dynamics with the same state evolved under a perturbed dynamics (cf. [**761**]). When the initial state is in a regular region of a mixed system (one with regular and chaotic regions) the overlap remains close to one; however when the initial state is in a chaotic zone the overlap decay is exponential. It is shown that at the edge of quantum chaos there is a region of polynomial overlap decay. Here the overlap is defined as $O(t) = | < \psi_u(t)|\psi_p(t) > |$ where $\psi_u$is the state evolved under the unperturbed system operator and ψ_p is the state evolved under the perturbed operator.

In various papers (e.g. [**185, 484, 485, 517, 518, 584**]) one characterizes quantum chaos via the quantum action. This is defined via

$$\tilde{S}[x] = \int dt \frac{\tilde{m}}{2}\dot{x}^2 - \tilde{V}(x) \tag{2.16}$$

for a given classical action

$$S[x] = \int dt \frac{m}{2}\dot{x}^2 - V(x) \tag{2.17}$$

so that the QM transition amplitude is

$$G(x_f, t_f; x_i, t_i) = \tilde{Z} exp\left[\frac{i}{\hbar}\, \tilde{\Sigma}\Big| + x_i, t_i{}^{x_f, t_f}\right]; \tag{2.18}$$

$$\tilde{\Sigma}\Big|_{x_i,t_i}^{x_f,t_f} = \tilde{S}[\tilde{x}_{cl}]\Big|_{x_i,t_i}^{x_f,t_f} = \int_{t_i}^{t_f} dt \frac{\tilde{m}}{2}\dot{\tilde{x}}_{cl}^2 - \tilde{V})\tilde{x}_{cl})\Big| + x_i{}^{x_f}$$

where $\tilde{x}_{cl}$ is the classical path corresponding to the action $\tilde{S}$. One requires here 2-point boundary conditions $\tilde{x}_{cl}(t = t_i) = x_i$ and $\tilde{x}_{cl}(t = t_f) = x_f$ and $\tilde{Z}$ stands for a dimensionful normalisation factor. The parameters of the quantum action (i.e. mass and potential) are independent of the boundary points but depend on the transition time $T = t_f - t_i$. A general existence proof is lacking but such quantum actions exist in many interesting cases. Then quantum chaos is defined as follows. Given a classical system with action S the corresponding quantum system displays quantum chaos if the corresponding quantum action $\tilde{S}$ in the asymptotic regime $T \to \infty$ generates a chaotic phase space.

3. GENERALIZED THERMOSTATISTICS

We refer to [**38, 262, 382, 384, 694, 696, 755, 778, 779**] for discussion of various entropies based on deformed exponential functions (generalizations of the Boltzman-Gibbs formalism for equilibrium statistical physics), the entropies of Beck-Cohen, Kaniadakis, Renyi, Tsallis, etc., maximum entropy ideas, escort density operators, and a host of other matters in generalized theormstatistics. We sketch here first a few ideas following the third paper in [**694**]. Thus a model of thermostatistics is described by a density of states $\rho(E)$ and a probability distribution $p(E)$ and for a system in thermal equilibrium at temperature T one has

$$p(E) = \frac{1}{Z(T)} e^{-E/T};\ Z(T) = \int dE \rho(E) e^{-E/T} \tag{3.1}$$

(Boltzman's constant is set equal to one here). Thermal averages are defined via $< f >= \int dE\rho(E)p(E)f(E)$ (this is a simplified treatment with T not made explicit - i.e. $p(E) \sim p(E, T)$). A microscopic model of thermostatistics is specified via an energy functional $H(\gamma)$ over phase space Γ which is the set of all possible microstates. Using $\rho(E)dE = d\gamma$ one can write

$$< f >= \int_\Gamma d\gamma p(\gamma) f(\gamma);\ p(\gamma) = \frac{e^{-H(\gamma)/T)}}{Z(T)};\ Z(T) = \int_\Gamma d\gamma e^{-H(\gamma)/T)} \tag{3.2}$$

In the quantum case the integration is replaced by a trace to obtain

$$< f >= \frac{1}{Z(T)} Tr\, exp(-H/T) f;\ Z(T) = Tr\, exp(-H/T) \tag{3.3}$$

In relevant examples of thermostatistics the density of states $\rho(E)$ increases as a power law $\rho(E) \sim E^{\alpha N}$ with N the number of particles and $\alpha > 0$. There is an energy - entropy balance where the increase of density of states $\rho(E)$ compensates for the exponential decrease of probability density $p(E)$ with a maximum of $\rho(E)p(E)$ reached at some macroscopic energy far above the ground state energy. One can write $\rho(E)p(E) = (1/Z)exp(log(\rho(E)) - E/T)$ with the argument of the exponential maximal when E satisfies

$$\frac{1}{\rho(E)}\rho'(E) = \frac{1}{T} \tag{3.4}$$

where $\rho'(E)$ is the derivative $d\rho/dE$. If $\rho(E) \sim E^{\alpha N}$ then $E \sim \alpha NT$ follows which is the equipartition theorem. The form of the theory here indicaters that the actual form of the probability distribution is not very essential; alternative expressions for $p(E)$ are acceptable provided they satisfy the equipartition theorem and reproduce thermodynamics. One begins here by generalizing the equipartition result (3.4) and postulates the existence of an increasing positive function $\phi(x)$ defined for $x \geq 0$ such that ($\bullet$) $(1/T) = -[p'(E)/\phi(p(E))]$ holds for all E and T. Then the equation for the maximum of $\rho(E)p(E)$ becomes

$$0 = \frac{d}{dE}[\rho(E)p(E)] = \rho'(E)p(E) - \frac{1}{T}\rho(E)\phi(p(E)) \equiv \frac{\rho'(E)}{\rho(E)} = \frac{1}{T}\frac{\phi(p(E))}{p(E)} \tag{3.5}$$

The Boltzman-Gibbs case is recovered when $\phi(x) = x$. Now ($\bullet$) fixes the form of the probability distribution $p(E)$; to see this introduce a function $log_\phi(x)$ via

$$log_\phi(x) = \int_1^x \frac{1}{\phi(y)}dy \tag{3.6}$$

The inverse is $exp_\phi(x)$ and from the identity $1 = exp'_\phi(log_\phi(x))log'_\phi(x)$ there results ($\blacklozenge$) $\phi(x) = exp'_\phi(log_\phi(x))$. Hence ($\bullet$) can be written as

$$p'(E) = -\frac{1}{T}exp'_\phi[log_\phi(p(E))] \Rightarrow p(E) = exp_\phi(G_\phi(T) - (E/T)) \tag{3.7}$$

The function $G_\phi(T)$ is the integration constant and it must be chosen so that $1 = \int dE\rho(E)p(E)$ is satisfied. The formula (3.7) resembles the Boltzman-Gibbs distribution but the normalization constant appears inside the function $exp_\phi(x)$; for $\phi(x) = x$ one has then $G_\phi(T) = -log(Z(T))$.

In general it is difficult to determine $G_\phi(T)$ but an expression for its temperature derivative can be obtained via escort probabilities (cf. [**146, 943**]). The general definition is

$$P(E) = \frac{1}{Z(T)}\phi(p(E));\ Z(T) = \int dE\rho(E)\phi(p(E)) \tag{3.8}$$

Then expectation values for $P(E)$ are denoted by

$$< f >_* = \int dE\rho(E)P(E)f(E) \tag{3.9}$$

Note $P(E) = p(E)$ in the Boltzman-Gibbs case $\phi(x) = x$. Now calculate using (♦) and (3.8) to get

$$\frac{d}{dT}p(E) = exp'_\phi(G_\phi(T) - (E/T))\left(\frac{d}{dT}G_\phi(T) + \frac{E}{T^2}\right) = \tag{3.10}$$

$$= Z(T)P(E)\left(\frac{d}{dT}G_\phi(T) + \frac{E}{T^2}\right)$$

from which follows (recall $\int dE\rho(E)p(E) = 1$)

$$0 = \int dE\rho(E)\frac{d}{dT}p(E) = Z(T)\frac{d}{dT}G_\phi(T) + \frac{1}{T^2}Z(T) < E >_* \Rightarrow \tag{3.11}$$

$$\Rightarrow \frac{d}{dT}G_\phi(T) = -\frac{1}{T^2} < E >_*$$

Note also that combining (3.10) and (3.11) one obtains

$$\frac{d}{dT}p(E) = \frac{1}{T^2}Z(T)P(E)(E- < E >_*) \tag{3.12}$$

One wants now to show that generalized thermodynamics is compatible with thermodynamics begins by establishing thermal stability. Internal energy $U(T)$ is defined via $U(T) =< E >$ with $p(E)$ given by (3.7), so using (3.12) one obtains (3.13)

$$\frac{d}{dT}U(T) = \int dE\rho(E)E\frac{d}{dT}p(E) = \int dE\rho(E)\frac{E}{T^2}Z(T)P(E)(E- < E >_*) =$$

$$= \frac{1}{T^2}Z(T)(< E^2 >_* - < E >_*^2) \geq 0$$

Hence average energy is an increasing function of T but thermal stability requires more so define ϕ entropy (relative to $\rho(E)dE$ via

$$S_\phi(p) = \int dE\rho(E)[(1 - p(E))F_\phi(0) - F_\phi(p(E))];\ F_\phi(x) = \int_1^x dylog_\phi(y) \tag{3.14}$$

One postulates that thermodynamic entropy $S(T)$ equals the value of the above entropy $S_\phi(p)$ with p given by (3.7). Then

$$\frac{d}{dT}S(T) = \int dE\rho(E)(-log_\phi(p(E)) - F_\phi(0))\frac{d}{dT}p(E) = \tag{3.15}$$

$$= \int dE\rho(E)\left(-G_\phi(T) + \frac{E}{T} - F_\phi(0)\right)\frac{d}{dT}p(E) = \frac{1}{T}\frac{d}{dT}U(T)$$

(recall that $p(E)$ is normalized to 1). This shows that temperature T satisfies the thermodynamic relation $(1/T) = dS/dU$ and since E is an increasing function of T one concludes that S is a concave function of U; this is called thermal stability. One can also introduce the Helmholz free energy $F(T)$ via the well known $F(T) = U(T) - TS(T)$ so from (3.15) it follows that

$$\frac{d}{d\beta}\beta F(T) = U(T)\ \ (\beta = 1/T) \tag{3.16}$$

Going back to (3.11) which is similar to (3.16) with $F(T)$ replaced by $TG_\phi(T)$ and with $U(T) =< E >$ replaced by $< E >_*$ the comparison shows that $TG_\phi(T)$

is the free energy associated with the escort probability distribution $P(E)$ up to a constant independent of T.

The most obvious generalization now involves $\phi(x) = x^q$ with $q > 0$ and this essentially produces the Tsallis entropy where one has

$$(3.17) \quad log_q(x) = \int_1^x dyy^{-q} = \frac{1}{1-q}(x^{1-q} - 1);\ exp_q(x) = [1 + (1-q)x]_+^{1/(1-q)}$$

(cf. [**?**]). The probability distribution (3.17) becomes
(3.18)

$$p(E) = [1 + (1-q)(G_q(T) - (E/T))]_+^{1/(1-q)} = \frac{1}{z_q(T)}[1 - (1-q)\beta_q^*(T)E]_+^{1/(1-q)}$$

$$z_q(T) = (1 + (1-q)G_q(T))^{1/(1-q)};\ \beta_q^*(T) = z_q(T)^{1-q}/T$$

A nice feature of the Tsallis theory is that the correspondence between $p(E)$ and the escort $P(E)$ leads to a dual structure $q \leftrightarrow 1/q$; indeed

$$(3.19) \qquad P(E) = \frac{1}{Z_q(T)} p(E)^q \Rightarrow p(E) = \frac{1}{Z_{1/q}(T)} P(E)^{1/q}$$

Moreover there is also a $q - 2 \leftrightarrow q$ duality; given $log_\phi(x)$ a new deformed $log_\psi(x)$ is obtained via

$$(3.20) \quad log_\psi(x) = (x-1)F_\phi(0) - xF_\phi(1/x);\ \frac{1}{\psi(x)} = F_\phi(0) - F_\phi(1/x) + \frac{1}{x} log_\phi(1/x)$$

and for $\phi = x^q$ one has $\psi = (2-q)x^{2-q}$. One notes also that the definition (3.14) of entropy $S_\phi(p)$ can be written as

$$(3.21) \qquad S_\phi(p) = \int dE\rho(E)p(E)log_\psi(1/p(E))$$

and with $\psi(x) = x^q$ we get the Tsallis entropy(cf. also [**1025, 1027**])

$$(3.22) \qquad S_q(p) = \int dE\rho(E)\frac{1}{1-q}(p(E)^q - p(E))$$

3.1. NONEXTENSIVE STATISTICAL THERMODYNAMICS. We go here to [**944**] for an lovely introduction and extract liberally. The Boltzman-Gibbs entropy is given via

$$(3.23) \qquad S_{BG} = -k\sum_1^W p_i log(p_i);\ \sum_1^W p_i = 1$$

Here p_i is the probability for the system to be in the i^{th} microstate and k is the Boltzman constant k_B (taken now to be 1). If every microstate has the same probability $p_i = 1/W$ then $S_{BG} = klog(W)$. The entropy (3.23) can be shown to be nonnegative, concave, extensive, and stable (or experimentally robust). By extensive one means that if A and B are two independent systems (i.e. $p_{ij}^{A+B} = p_i^A p_j^B$) then

$$(3.24) \qquad S_{BG}(A+B) = S_{BG}(A) + S_{BG}(B)$$

One can still not derive this form of entropy (3.23) from first principles. There is also good reason to conclude that physical entropies different from (3.23) would be more appropriate for anomalous systems. In this spirit the Tsallis entropy was proposed in [**945**] and the property thereby generalized is extensivity. One discusses motivations etc. in [**841**] and in particular observes that the function

$$y = \frac{x^{1-q} - 1}{1-q} = log_q(x) \tag{3.25}$$

satisfies

$$log_q(x_A x_B) = log_q(x_A) + log_q(X_B) + (1-q)(log_q(X_A))(log_q(X_B)) \tag{3.26}$$

Now rewrite (3.23) in the form ($k = 1$)

$$S_{BG} = -\sum_1^W p_i log(p_i) = \sum_1^W p_i log(1/p_i) = \left\langle log\frac{1}{p_i}\right\rangle \tag{3.27}$$

The quantity $log(1/p_i)$ is called surprise or unexpectedness and one thinks of a q-surprise $log_q(1/p_i)$ in defining

$$S_q = \left\langle log_q\frac{1}{p_i}\right\rangle = \sum_1^W p_i log_q(1/p_i) = \frac{1 - \sum_1^W p_i^q}{q-1} \tag{3.28}$$

In the limit $q \to 1$ one gets $S_1 = S_{BG}$ and assuming equiprobability $p_i = 1/W$ one gets

$$S_q = \frac{W^{1-q} - 1}{1-q} = log_q(W) \tag{3.29}$$

Consequently S_q is a genuine generalization of the BG entropy and the pseudo-additivity of the q-logarithm implies (restoring momentarily k)

$$\frac{S_q(A+B)}{k} = \frac{S_q(A)}{k} + \frac{S_q(B)}{k} + (1-q)\frac{S_q(A)}{k}\frac{S_q(B)}{k} \tag{3.30}$$

if A and B are two independent systems (i.e. $p_{ij}^{A+B} = p_i^A p_j^B$). Thus $q = 1,\ q < 1$, and $q > 1$ respectively correspond to the extensive, superextensive, and subextensive cases and the q-generalization of statistical mechanics is referred to as nonextensive statistical mechanics. (3.30) is true for independent A and B but if A and B are correlated in some way one can ask if extensivity would hold for some q. For example a system whose elements are correlated at at scales might correspond to $W(N) \sim N^\rho\ \rho > 0$ with entropy

$$S_q(N) = log_q W(N) \sim \frac{N^{\rho(1-q)} - 1}{1-q} \tag{3.31}$$

and extensivity is obtained if and only if $q = 1 - (1/\rho) < 1$ or $S_q(N) \propto N$. Shannon and Khinchin gave early similar sets of axioms for the form of the entropy functional, both leading to (3.23). These were generalized in [**5, 781, 780, 843**] leading to the entropy

$$S(p_1, \cdots, p_W) = k\frac{1 - \sum_1^W p_i^q}{q-1} \tag{3.32}$$

and it was shown that S_q is the only possible entropy extending the Boltzman-Gibbs entropy maintaining all the basic properties except extensivity for $q \neq 1$.

Some other properties are also discussed, e.g. bias, concavity, and stability. First note

$$S_{BG} = -\left[\frac{d}{dx}\sum_1^W p_i^x\right]_{x=1} \tag{3.33}$$

(x here is referred to as a bias). Similarly

$$S_q = -\left[D_q\sum_1^W p_i^x\right]_{x=1};\ D_q h(x) = \frac{h(qx)-h(x)}{qx-1} \tag{3.34}$$

(Jackson derivative) and this may open the door to quantum groups (see e.g. [**192**]). As for concavity consider for $p_i'' = \mu p_i + (1-\mu)p_i'$ $(0 < \mu < 1)$ concavity defined via

$$S(\{p_i''\}) \geq \mu S(\{p_i\}) + (1-\mu)S(\{p_i'\}) \tag{3.35}$$

It can be shown that S_q is concave for every $\{p_i\}$ and $q > 0$. This implies theormdynamic stability in the framework of statistical mechanics (i.e. stability of the system with regard to energetic perturbations). This means that the entropy functional is defined such that the stationary state (thermodynamic equilibrium) makes it extreme.

There are also other generalizations of the BG entropy and we mention the Renyi entropy

$$S_q^R = \frac{log\sum_1^W p_i^q}{1-q} = \frac{log[1+(1-q)S_q]}{1-q} \tag{3.36}$$

and an entropy due to Landsberg, Vedral, Rajagopal, Abe defined via

$$S_q^N = S_q^{LVRA} = \frac{1-\frac{1}{\sum_1^W p_i^q}}{1-q} = \frac{S_q}{1+(1-q)S_q} \tag{3.37}$$

These are however not concave nor experimentally robust and seem unsuited for thermodynamical purposes; on the other hand Renyi entropy seems useful for geometrically characterizing multifractals.

Various connections of S_q to thermodynamics are indicated in [**944**] and we mention here first the Legendre structure. Thus for all values of q

$$\frac{1}{T} = \frac{\partial S_q}{\partial U_q};\ T = \frac{1}{k\beta};\ U_q = -\frac{\partial}{\partial\beta}log_q Z_q; \tag{3.38}$$

$$log_q Z_q = \frac{Z_q^{1-q}-1}{1-q} = \frac{\bar{Z}^{1-q}-1}{1-q} - \beta U_q;\ F_q = U_q - TS_q = -\frac{1}{\beta}log_q Z_q$$

Here $U_q \sim$ internal energy and $F_q \sim$ free energy and the specific heat is

$$C_q = T\frac{\partial S_q}{\partial T} = \frac{\partial U_q}{\partial T} = -T\frac{\partial^2 F_q}{\partial T^2} \tag{3.39}$$

Finally a list of other properties follows supporting the thesis that S_q is a correct road for generalizing the BG theory (see [**944**] for details and references); we mention a few here via

(1) Boltzmann H-theorem (macroscopic time irreversibility) $q(dS_q/dt) \geq 0$ $(\forall q)$
(2) Ehrenfest theorem: For an observable $\hat{O}$ and a Hamiltonian $\hat{H}$ one has $d < \hat{O} >_q /dt = (i/\hbar) < [\hat{H}, \hat{O} >_q \;\; (\forall q)$
(3) Pesin theorem (connection between sensitivity to initial conditions and the entropy production per unit time). Define the q-generalized Kolmogorov-Sinai entropy as

$$K_q = lim_{t\to\infty} lim_{W\to\infty} lim_{N\to\infty} \frac{< S_q > (t)}{t} \tag{3.40}$$

where N is the number of initial conditions, W is the number of windows in the partition (fine graining), and t is discrete time (cf. also [**593**]). The q-generalized Lyapunov coefficient λ_q can be defined via sensitivity to initial conditions

$$\xi = lim_{\Delta x(0)\to 0} \frac{\Delta x(t)}{\Delta x(0)} = e_q^{\lambda_q t} \tag{3.41}$$

(focusing on a 1-D system, basically $x(t+1) = g(x(t))$ with g nonlinear). It was proved in [**73**] that for unimodal maps $K_q = \lambda_q$ if $\lambda_q > 0$ and $K_q = 0$ otherwise. More explicitly $K_1 = \lambda_1$ if $\lambda_1 \geq 0$ (and $K_1 = 0$ if $\lambda_1 < 0$). But if $\lambda_1 = 0$ then there is a special value of q such that $K_q = \lambda_q$ if $\lambda_q \geq 0$ (and $K_q = 0$ if $\lambda_q < 0$).

We refer also to [**30, 31, 785, 636**] for other results and approaches to thermodynamics, temperature, fluctuations, etc. in generalized thermostatistics and to [**595**] for relativistic nonextensive thermodynamics.

4. FISHER PHYSICS

The book [**381**] purports (with notable success) to unify several subdisciplines of physics via Fisher information and this theme appears also in many papers, e.g. [**217, 262, 239, 382, 383, 384, 385, 660, 755, 776, 777, 778, 779, 780, 781, 949**]. We sketch some of this here and note in passing an interesting classical-quantum trajectory in [**386**] which differs from a Bohmian trajectory (cf. also [**93, 269, 376, 579, 629, 812, 999**]). First let us sketch some summary items from [**381**] and then provide some details. Thus in Chapter 12 of [**381**] Frieden lists (among other things) the following items:

(1) Writing $p = q^2$ in the standard formulas one can express the Fisher information as $I = 4\int dx(dq/dx)^2$ with q a real probability amplitude for fluctuations in measurement. Under suitable conditions (see below) the information I obeys an I-theorem $dI/dt \leq 0$. In the same spirit by which a positive increment in therodynamic time corresponds to an increase in Boltzman entropy there is a positive increment in Fisher time defined by a decrease in information I (the two times do not always agree). Let θ be

the measured phenomenon and define the Fisher temperature T_θ via

$$\frac{1}{T_\theta} = -k_\theta \frac{\partial I}{\partial \theta} \ \ (k_\theta = const.) \tag{4.1}$$

When θ is taken to be the sysem energy E then the Fisher temperature has analogous properties to the ordinary Boltzman temperature, in particular there is a perfect gas law $\bar{p}V = k_E T_E I$ where $\bar{p}$ is the pressure. The I theorem can be extended to a multiparameter, multicomponent scenario with

$$I = 4\int dx \sum_n \nabla q_n \cdot \nabla q_n \tag{4.2}$$

(2) Any measurement of physical parameters initiates a transformation of Fisher information $J \to I$ connecting the phenomenon with the "intrinsic data". The phenomenological or "bound" information is denoted by J and the acquired information is I; J is ultimately identified by an invariance principle that characterizes the measured phenomenon. In any exchange of information one must $\delta J = \delta I$ (conservation law) and for $K = I - J$ one arrives at a variational principle (extreme physical information or EPI) $K = I - J = extremum$. Since $J \geq I$ always the EPI zero principle involves $I - \kappa J = 0$ $(0 \leq \kappa \leq 1)$. These equations follow (independently of the axiomatic approach taken and of the I-theorem) if there is e.g. a unitary transformation connecting the measurement space with a physically meaningful conjugate space. In this manner one arrives at the Lagrangian approach to physics, often using the Fourier transform to connect I and J. This seems a little mystical at first but many convincing examples are given involving the SE, wave equations, KG equation, Dirac equation, Maxwell equations, Einstein equations, WDW equation, etc.

There is much more summary material in [**381**] which we omit here. A certain amount of metaphysical thinking seems necessary and Frieden remarks that John Wheeler (cf. [**988**]) anticipated a lot of this in his remarks that "All things physical are information-theoretic in origin and this is a participatory universe....Observer participancy gives rise to information and information gives rise to physics." Going now to [**381**] recall $I = \int dx[(p')^2/p] = 4\int dx(q')^2$ for $p = q^2$ and one derives the inequality $e^2 I \geq 1$ as follows. Look at estimators $\hat{\theta}$ satisfying

$$< \hat{\theta}(y) - \theta >= \int dy[\hat{\theta}(y) - \theta]p(y|\theta) = 0 \tag{4.3}$$

where $p(y|\theta)$ describes fluctuations in data values y. Hence

$$\int dy(\hat{\theta} - \theta)\frac{\partial p}{\partial \theta} - \int dyp = 0 \tag{4.4}$$

Use now $\partial_\theta p = p(\partial log(p)/\partial\theta)$ and normalization to get $\int dy(\hat{\theta}-\theta)(\partial log(p)/\partial\theta)p = 1$ which becomes

(4.5)
$$\int dy\left[\frac{\partial log(p)}{\partial\theta}\sqrt{p}\right][(\hat{\theta}-\theta)\sqrt{p}] = 1 \Rightarrow \left[\int dy\left(\frac{\partial log(p)}{\partial\theta}\right)^2 p\right]\left[\int dy(\hat{\theta}-\theta)^2 p\right] \geq 1$$

For $e^2 = \int dy(\hat{\theta}-\theta)^2 p$ this gives immediately $e^2 I \geq 1$. One notes that if $p(y|\theta) = p(y-\theta)$ then I is simply $I = \int dx(\partial log(p(x))/\partial x)^2 p(x)$ where $x \sim y - \theta$. We recall also the Shannon entropy as $H = -\int dxp(x)log(p(x))$ and the Kullback-Leibler entropy is defined as

$$G = -\int dxp(x)log\frac{p(x)}{r(x)} \tag{4.6}$$

where $r(x)$ is a reference probability distribution function (PDF). Consider now a discrete form of Fisher information

(4.7)
$$I = (\Delta x)^{-1}\sum_n \frac{[p(x_{n+1}) - p(x_n)]^2}{p(x_n)} = (\Delta x)^{-1}\sum_n p(x_n)\left[\frac{p(x_n+\Delta x)}{p(x_n)} - 1\right]^2$$

Here $p(x_n+\Delta x)/p(x_n)$is close to 1 for Δx small and one writes $[p(x_n+\Delta x)/p(x_n)] - 1 = \nu$. Then $log(1+\nu) \sim \nu - (\nu^2/2)$ or $\nu^2 = 2[\nu - log(1+\nu)]$. Hence I becomes

$$I = -2(\Delta x)^{-1}\sum_n p(x_n)log\frac{p(x_n+\Delta x)}{p(x_n)} + \tag{4.8}$$
$$2(\Delta x)^{-1}\sum_n p(x_n+\Delta x) - 2(\Delta x)^{-1}\sum_n p(x_n)$$

But each of the last two terms is $(\Delta x)^{-1}$ by normalization so they cancel leaving

$$I = -\frac{2}{\Delta x}\sum p(x_n)log\frac{p(x_n+\Delta x)}{p(x_n)} \to -\frac{2}{\Delta x}G[p(x), p(x+\Delta x)] \tag{4.9}$$

One notes (cf. [**381**]) that I results as a cross information between $p(x)$ and $p(x+\Delta x)$ for many different types of information measure, e.g. Renyi and Wooters information and in this sense serves as a kind of "mother" information. Next the I-theorem says that $dI/dt \leq 0$ and this can be seen as follows. Start with (4.9) in the form

$$I(t) = -2lim_{\Delta x\to 0}(\Delta x)^{-2}\int dx\, plog\frac{p_{\Delta_x}}{p};\ p_{\Delta_x} = p(x+\Delta x|t);\ p = p(x|t) \tag{4.10}$$

Under certain conditions (cf. [**381**]) p obeys a FK equation

$$\frac{\partial p}{\partial t} = -\frac{d}{dx}[D_1(x,t)p] + \frac{d^2}{dx^2}[D_2(x,t)p] \tag{4.11}$$

where D_1 is a drift function and D_2 a diffusion function. Then it is shown (cf. [**776, 811**]) that two PDF such as p and p_{Δ_x} that obey the FP equation have a cross entropy satisfying an H-theorem

$$G(t) = -\int dx\, plog\frac{p}{p_{\Delta_x}};\ \frac{dG(t)}{dt} \geq 0 \tag{4.12}$$

Hence I obeys an I theorem $dI/dt \leq 0$. We refer to [**381**] for more on temperature, pressure, and gas laws.

For multivariable situations one writes $I = 4\int dx \sum \nabla q_n \cdot \nabla q_n$ with $p_n = q_n^2$. An interesting notation here is

$$(4.13)\quad \psi_n = \frac{1}{\sqrt{N}}(q_{2n-1} + iq_{2n})\ \ (n = 1, \cdots, N/2);\ \sum_1^{N/2} \psi_n^* \psi_n = \frac{1}{N}\sum q_n^2 = p(x)$$

In such situations one finds for $I_n = 4\int dx \nabla q_n \cdot \nabla q_n$ and $I = \sum I_n$ (cf. [**381**])

$$(4.14)\qquad I_n = -\frac{2}{(\Delta x)^2} G_n[p_n(x|t), p_n(x + \Delta x|t)];\ \frac{\partial I_n}{\partial t} \leq 0; I(t) \to\ min.$$

Now one looks at minimization problems for I where $\delta I[\mathbf{q}(\mathbf{x}|t)] = 0$ and for anything meaningful to happen the physics has to be introduced via constraints and covariance (we refer to [**381**] for a more thorough discussion of these matters). Thus one is considering $K = I - J$ and the physics is introduced via J. One can write e.g. $I = \int dx \sum i_n(x)$ and $J = \int dx \sum j_n(x)$ where $i_n = 4\nabla q_n \cdot \nabla q_n$. In general now the functional form of J follows from a statement about invariance for the system. Examples of invariance are (i) unitary transformations such as that between the space and momentum space in QM (ii) gauge invariance as in EM or gravitational theory (iii) a continuity equation for the flow, usually involving sources. The answer $\mathbf{q}$ for EPI is completely dependent on the particular $J(\mathbf{q})$ for that problem and that in turn depends completely on the invariance principle that is used. If the invariance principle is not sufficiently strong in defining the system then one can expect the EPI output $\mathbf{q}$ to be only approximately correct. One has $I \leq J$ generally but $I = J$ for an optimally strong invariance principle. Note $\kappa = I/J$ measures the efficiency of the EPI in transferring Fisher information from the phenomenon (specfied by J) to the output (specified by I). Thus $\kappa < 1$ indicates that the answer $\mathbf{q}$ is only approximate. When the invariance principle is the statement of a unitary transformation between the measurement space and a conjugate coordinate space then the solution to the requirement $I - \kappa J = 0$ will simply be the reexpression of I in the conjugate space; when this holds then one can show that in fact $I = J$ (i.e. $\kappa = 1$). In this situation the out put $\mathbf{q}$ will be "correct", i.e. not explicitly incorrect due to ignored quantum effects for example. There are in fact nonquantum and nonunitary theories for which $\kappa = 1$ (or in fact any real number) and the nature of κ is not yet fully understood.

Let us call attention also to the information demon of Frieden and Soffer (cf. [**381, 384**]). For real Fisher coordinates x the EPI process amounts to carrying through a zero sum game betweem an observer (who wants to acquire maximal information) and an information demon (who wants to minimize the information transfer) with a limited resource of intrinsic information. The demon represents nature (and always wins or breaks even of course) and $K = I - J \leq 0$. Further since $\Delta I = K$ one has $\Delta I \leq 0$ while $\Delta t \geq 0$; hence the I-theorem follows.

We run through the EPI procedure here for the KG equation which illustrates many points. Define $x_1 = ix$, $x_2 = iy$, $x_3 = iz$, $x_4 = ct$ with $r = (x, y, z)$ and

$\mathbf{x} = (x_1, x_2, x_3, x_4)$ and use the ψ_n notation of (4.13). From $I = 4\int dx \sum \nabla q_n \cdot \nabla q_n$ we get

$$I = 4Nc\sum_1^{N/2}\int\int drdt\left[-(\nabla\psi_n)^*\cdot\nabla\psi_n + \left(\frac{1}{c^2}\right)^2(\partial_t\psi_n)^*(\partial_t\psi_n)\right] \tag{4.15}$$

The invariance principle here involves a unitary Fourier transformation from x to μ in the form

(4.16)
$$(ir, ct) \to (i\mu/\hbar, E/c\hbar);\ \psi_n(r,t) = \frac{1}{(2\pi\hbar)^2}\int\int d\mu dE\phi_n(\mu, E)e^{-i(\mu\cdot r - Et)/\hbar}$$

One recalls

$$\int\int drdt\psi_m^*\psi_n = \int\int d\mu dE\phi_m^*\phi_n \tag{4.17}$$

Differentiating in (4.16) one has $(\nabla\psi_n, \partial_t\psi_n) \to (-i\mu\phi_n/\hbar, iE\phi_n/\hbar)$ and via $\nabla\psi_n \sim -i\mu\phi_n/\hbar$ one gets

$$\int\int drdt(\nabla\psi_n)^*\cdot\nabla\psi_n = \frac{1}{\hbar^2}\int\int d\mu dE|\phi_n(\mu,E)|^2\mu^2; \tag{4.18}$$
$$\int\int drdt(\partial_t\psi_n)^*\partial_t\psi_n = \frac{1}{\hbar^2}\int\int d\mu dE|\phi_n(\mu,E)|^2E^2$$

Putting this in (4.15) gives

$$I = \left(\frac{4Nc}{\hbar^2}\right)\sum_1^{N/2}\int\int d\mu dE|\phi_n(\mu,E)|^2\left(-\mu^2 + \frac{E^2}{c^2}\right) = J \tag{4.19}$$

This is the invariance principle for the given scenario. The same value of I can be expressed in the new space (μ, E) where it is called J and J is then the bound (physical) information. Now one has from (4.17)

$$c\int\int drdt|\psi_n|^2 = c\int\int d\mu dE|\phi_n|^2\ \ (n = 1, \cdots, N/2) \tag{4.20}$$

Summing over n and using $p = \sum_1^{N/2}\psi_n^*\psi = (1/N)\sum q_n^2$ with normalization gives

$$1 = \int d\mu deP(\mu,E);\ P(\mu,E) = c\sum_1^{N/2}|\phi_n(\mu,E)|^2 \tag{4.21}$$

so P is a PDF in the (μ, E) space. One obtains then

(4.22)
$$I = J = \frac{4N}{\hbar^2}\int\int d\mu dEP(\mu,E)\left(-\mu^2 + \frac{E^2}{c^2}\right);\ J = \left(\frac{4N}{\hbar^2}\right)\left\langle -\mu^2 + \frac{E^2}{c^2}\right\rangle$$

One must have J a universal constant here so $-\mu^2 + (E^2/c^2) = const. = A^2(m,c)$ where A is some function of the rest mass m and c (which are the only other parameters ($\hbar$ must also be a constant). By dimensional analysis $A = mc$ so $E^2 = c^2\mu^2 + m^2c^4$ which links mass, momentum, and energy. This defines coordinates μ and E as momentum and energy values. One has then $I = 4N(mc/\hbar)^2 = J$ and the intrinsic information I in the 4-position of a particle is proportional to the square of its intrinsic energy mc^2. Since J is a universal constant (see comments below), c is fixed, and given that $\hbar$ has been fixed, one concludes that the rest mass m is a universal constant. Since I measures the capacity of the observed phenomenon to provide information about (in this case) 4-length it follows that I should translate into a figure for the ultimate fluctuation (resolution) length that is intrinsic to QM. Here the information is $I = (4N/L^2)$ with $L = \hbar/mc$ the reduced Compton wavelength. If all N estimates have the same accuracy some argument then leads to $e_{min} = L$ and e_{min} corresponds to a minimal resolution length (i.e. ability to know). Finally putting things together one gets

$$J = \frac{4Nm^2c^3}{\hbar^2} \int\int d\mu dE \sum_1^{N/2} \phi_n^* \phi_n = \frac{4Nm^2c^3}{\hbar^2} \int\int drdt \sum_1^{N/2} \psi_n^* \psi \tag{4.23}$$

$$K = I - J = \tag{4.24}$$

$$= 4Nc \sum_1^{N/2} \int\int drdt \left[-(\nabla\psi_n)^* \cdot \nabla\psi_n + \left(\frac{1}{c^2}\right) \partial_t \psi_n^* \partial_t \psi_n - \frac{m^2c^2}{\hbar^2} \psi_n^* \psi_n \right]$$

There is much more material in [**381**] to enhance and refine the above ideas. There are certain subtle features as well. In 4-dimensions the Fourier transform is unitary and covariance is achieved in all variables (treating t separately as in $q_n(x|t)$ is not a covariant formalism). EPI treats all phenomena as being statistical in origin and every Euler-Lagrange (EL) equation determines a kind of QM for the particular phenomenon (think here of the q_n as fields). This includes classical electromagnetism for example where the vector potential A is considered as a kind of probability "amplitude" for photons. In 4-D the Lorentz transformation satisfies the requirement that Fisher information I is invariant under a change of reference frame and this property is transmitted to J and K. Thus invariance of accuracy (or of error estimation) under a change of reference frame leads to the Lorentz transformation and to the requirement of covariance. Historically the classical Lagrangian has often been a contrivance for getting the correct answers and a main idea in [**381**] is to present a systematic approach to deriving Lagrangians. The Lagrangian represents the physical information $k(\mathbf{x}) = \sum k_n(\mathbf{x})$, $k_n(\mathbf{x}) = i_n(\mathbf{x}) - j_n(\mathbf{x})$, and $\int k(\mathbf{x})$ is the total physical information K for the system. The solution to the variational problem for the Lagrangian can represent then (for real coordinates) the payoff in a mathematical information game (e.g. the KG equation is a payoff expression). We exhibit now a derivation of the SE from [**381**] to show robustness of the EPI scheme. The position of a particle of mass m is measured as a value $y = x + \theta$ where x is a random excursion whose probability amplitude law $q(x)$ is sought. Since the time t is being ignored here one is in effect looking for a stationary solution to the problem. Note the issue of covariance does not arise here and the time dependent SE is not treated since in particular it is not covariant; it can however be obtained from the KG equation as a nonrelativistic limit. Assume that the particle is moving in a conservative field of scalar potential $V(x)$ with total energy W conserved. One defines complex wave functions as before and can write

$$I = 4N \sum_{1}^{N/2} \int dx \left| \frac{d\psi_n(x)}{dx} \right|^2 \tag{4.25}$$

A Fourier transform space is defined via $\psi_n(x) = (1/\sqrt{2\pi\hbar} \int d\mu \phi_n(\mu) exp(-i\mu x/\hbar)$ where $\mu \sim$ momentum. The unitary nature of this transformation guarantees the validity of the EPI variational procedure. One uses the Parseval theorem to get

$$I = \frac{4N}{\hbar^2} \int d\mu \mu^2 \sum_n |\phi_n(\mu)|^2 = J \tag{4.26}$$

This corresponds to (4.19) and is the invariance principle for the given measurement problem. The x-coordinate expressions analogous to (4.20) and (4.21) show that the sum in (4.26) is actually an expectation $J = (4N/\hbar^2) < \mu^2 >$. Now use the specifically nonrelativistic approximation that the kinetic energy E_{kin} of the particle is $\mu^2/2m$ and then

$$J = \frac{8Nm}{\mu^2} < E_{kin} >= \frac{8Nm}{\hbar^2} < [W - V(x)] >= \tag{4.27}$$

$$= \frac{8Nm}{\hbar^2} \int dx [W - V(x)] \sum |\psi_n(x)|^2$$

where the last expression is the PDF $p(x)$. This J is the bound information functional $J[q] = J(\psi)$ and $\kappa = 1$ here. This leads to a variational problem

$$K = N \sum_{1}^{N/2} \int dx \left[4 \left| \frac{d\psi_n(x)}{dx} \right|^2 - \frac{8m}{\hbar^2} [W - V(x)] |\psi_n(x)|^2 \right] = extremum \tag{4.28}$$

The Euler-Lagrange equations are then ($\psi^*_{nx} = \partial\psi^*_n/\partial x$)

$$\frac{d}{dx}\left(\frac{\partial \mathfrak{L}}{\partial \psi^*_{nx}}\right) = \frac{\partial \mathfrak{L}}{\partial \psi^*_n}; \ \psi''_n(x) + \frac{2m}{\hbar^2}[W - V(x)]\psi_n(x) = 0 \tag{4.29}$$

which is the stationary SE. Since the form of equation (4.29) is the same for each index value n the scenario admits $N = 2$ degrees of freedom $q_n(x)$ or one complex degree of freedom $\psi(x)$; hence the SE defines a single complex wave function. Since this derivation works with a real coordinate x the information transfer game is being played here and the payoff is the Schrödinger wave function.

REMARK 6.4.1. There are generalizations of EPI to nonextensive information measures in [**217, 262, 239**] (cf. also [**755, 756, 776, 778**]).

4.1. LEGENDRE THERMODYNAMICS. We go to the last paper in [**382**] which provides a discussion of Fisher thermodynamics and the Legendre transformation. It is shown that the Legendre transform structure of classical thermodynamics can be replicated without change if one replaces the entropy S by the Fisher information I. This produces a thermodynamics capable of treating equilibrium and nonequilibrium situations in a traditional manner. We recall the Shannon information measure $S = -\sum P(i)log[P(i)]$; it is known that if one chooses the Boltzmann constant as the informational unit and identifies Shannon's entropy with the thermodynamic entropy then the whole of statistical mechanics can be elegantly reformulated without any reference to the idea of ensemble. The success of thermodynamics and statistical physics depends crucially on the Legendre structure and one shows now that such relationships all hold if one replaces S by the Fisher information measure. We recall that for $\int g(x,\theta)dx = 1$ one writes $I = \int dx g(x,\theta)[\partial_\theta g/g]^2$ and for shift invariant g one has $I = \int dx[(g')^2/g]$. There are two approaches to using Fisher information, EPI and minimum Fisher information (MFI), and both lead to the same results here. We write (shifting to a probability function f)

$$\int dx f(x,\theta) = 1;\ I[f] = \int dx F_{Fisher}(f);\ F_{Fisher}(f) = f(x)[f'/f]^2 \tag{4.30}$$

Assume that for M functions $A_i(x)$ the mean values $< A_i >$ are known where

$$< A_i >= \int dx A_i(x) f(x) \tag{4.31}$$

This represents information at some appropriate (fixed) time t. The analysis will use MFI (or EPI) to find the probability distribution $f_I = f_{MFI}$ that extremizes I subject to prior conditions $< A_i >$ and the result will be given via solutions of a stationary Schrödinger like equation. The Fisher based extremization problem has the form $(F(f) = F_{Fisher}(f))$

$$\delta_f \left[I(f) - \alpha < 1 > - \sum_1^M \lambda_i < A_i > \right] = 0 \equiv \tag{4.32}$$

$$\delta_f \left[\int dx \left(F(f) - \alpha f - \sum_1^M \lambda_i A_i f \right) \right] = 0$$

Variation leads to $((\alpha, \lambda_1, \cdots, \lambda_M)$ are Lagrange multipliers)

$$\int dx \delta f \left[(f)^{-2} \left(\frac{\partial f}{\partial x} \right)^2 + \frac{\partial}{\partial x} \left(\frac{2}{f} \frac{\partial f}{\partial x} \right) + \alpha + \sum_1^M \lambda_i A_i \right] = 0 \tag{4.33}$$

and on account of the arbitrariness of δf this yields

$$(f)^{-2}(f')^2 + \frac{\partial}{\partial x}(2/f)f') + \alpha + \sum_1^M \lambda_i A_i = 0 \tag{4.34}$$

The normalization condition on f makes α a function of the λ_i and we assume $f_I(x,\lambda)$ to be a solution of (4.34) where $\lambda \sim (\lambda_i)$. The extreme Fisher information is then

$$I = \int dx f_I^{-1} (\partial_x f_I)^2 \tag{4.35}$$

Now to find a general solution of (4.34) define $G(x) = \alpha + \sum_1^M \lambda_i A_i(x)$ and write (4.34) in the form

$$\left[\frac{\partial log(f_I)}{\partial x} \right]^2 + 2 \frac{\partial^2 log(f_I)}{\partial x^2} + G(x) = 0 \tag{4.36}$$

Make the identification $f_I = (\psi)^2$ now the introduce a new variable $v(x) = \partial log(\psi(x))/\partial x$. Then (4.36) becomes

$$v'(x) = -\left[\frac{G(x)}{4} + v^2(x)\right] \tag{4.37}$$

which is a Riccati equation. This leads to

$$u(x) = exp\left[\int^x dx[v(x)]\right] = exp\left[\int^x dx\frac{dlog(\psi)}{dx}\right] = \psi; \tag{4.38}$$

$$-\frac{1}{2}\psi''(x) - \frac{1}{8}\sum_1^M \lambda_i A_i(x)\psi(x) = \frac{\alpha}{8}\psi(x)$$

where the Lagrange multiplier $\alpha/8$ plays the role of an energy eigenvalue and the sum of the $\lambda_i A_i(x)$ is an effective potential function $U(x) = (1/8)\sum_1^M \lambda_i A_i(x)$. We note (in keeping with the Lagrangian spirit of EPI) that the Fisher information measure corresponds to the expectation value of the kinetic energy of the SE. Note also that (4.38) has multiple solutions and it is reasonable to suppose that the solution leading to the lowest I is the equilibrium one. Now standard thermodynamics uses derivatives of the entropy S with respect to λ_i and $< A_i >$ and we start from (4.35) and write after an integration by parts

$$\frac{\partial I}{\partial \lambda_i} = \int dx \frac{\partial f_I}{\partial \lambda_i}\left[-f_I^{-2}(f_I')^2 - \frac{\partial}{\partial x}\left(\frac{2}{f_I}f_I'\right)\right] \tag{4.39}$$

Comparing this to (4.34) one arrives at

$$\frac{\partial I}{\partial \lambda_i} = \int dx \frac{\partial f_I}{\partial \lambda_i}\left[\alpha + \sum_1^M \lambda_j A_j\right] \tag{4.40}$$

which on account of normalization yields

$$\frac{\partial I}{\partial \lambda_i} = \sum_1^M \lambda_j \frac{\partial}{\partial \lambda_i}\int dx f_I A_j(x) \equiv \frac{\partial I}{\partial \lambda_i} = \sum_1^N \lambda_j \frac{\partial}{\partial \lambda_i} < A_j > \tag{4.41}$$

This is a generalized Fisher-Euler theorem whose thermodynamic counterpart is the derivative of the entropy with respect to the mean values. One computes easily

$$\sum_i \frac{\partial I}{\partial \lambda_i}\frac{\partial \lambda_i}{\partial < A_j >} = \sum_i\sum_k \lambda_k \frac{\partial < A_k >}{\partial \lambda_i}\frac{\partial \lambda_i}{\partial < A_j >} \Rightarrow \frac{\partial I}{\partial < A_j >} = \lambda_j \tag{4.42}$$

as expected. The Lagrange multipliers and mean values are seen to be conjugate variables and one can also say that $f_I = f_I(\lambda_1, \cdots, \lambda_M)$.

Now as the density f_I formally depends on $M + 1$ Lagrange multipliers, normalization $\int dx f_I(x) = 1$ makes α a function of the λ_i and we write $\alpha = \alpha(\lambda_1, \cdots, \lambda_M)$. One can assume that the input information refers to the λ_i and not to the $< A_i >$. Introduce then a generalized thermodynamic potential (Legendre transform of I) as

$$\lambda_J(\lambda_1, \cdots, \lambda_M) = I(< A_1 >, \cdots, < A_M >) - \sum_1^M \lambda_i < A_i > \tag{4.43}$$

Then

$$\frac{\partial\lambda_J}{\partial\lambda_i} = \sum_1^M \frac{\partial I}{\partial < A_j >}\frac{\partial < A_j >}{\partial\lambda_i} - \sum_1^M \lambda_j \frac{\partial < A_j >}{\partial\lambda_i} - < A_i) = - < A_i > \tag{4.44}$$

where (4.42) has been used. Thus the Legendre structure can be summed up in

$$\lambda_J = I - \sum_1^M \lambda_i < A_i >;\ \frac{\partial\lambda_J}{\partial\lambda_i} = - < A_i >;\ \frac{\partial I}{\partial < A_i >} = \lambda_i; \tag{4.45}$$

$$\frac{\partial\lambda_i}{\partial < A_j >} = \frac{\partial\lambda_j}{\partial < A_i >} = \frac{\partial^2 I}{\partial < A_i > \partial < A_j >};$$

$$\frac{\partial < A_j >}{\partial\lambda_i} = \frac{\partial < A_i >}{\partial\lambda_j} = -\frac{\partial^2\lambda_J}{\partial\lambda_i\partial\lambda_j}$$

As a consequence one can recast (4.41) in the form

$$\frac{\partial I}{\partial\lambda_i} = \sum_1^M \lambda_j \frac{\partial}{\partial\lambda_j} < A_i > \tag{4.46}$$

Thus the Legendre transform structure of thermodynamics is entirely translated into the Fisher context.

4.2. FIRST AND SECOND LAWS. We go here to [**779**] where one shows the coimplication of the first and second laws of thermodynamics. Thus macroscopically in classical phenomenological thermodynamics the first and second laws can be regarded as independent statements. In statistical mechanics an underlying microscopic substratum is added that is able to explain thermodynamics itself. Of this substratum a microscopic probability distribution (PD) that controls the population of microstates is a basic ingredient. Changes that affect exclusively microstate population give rise to heat and how these changes are related to energy changes provides the essential content of the first law (cf. [**809**]). In [**779**] one shows that the PD establishes a link between the first and second laws according to the following scheme.

- Given: An entropic form (or an information measure) S, a mean energy U and a temperature T, and for any system described by a microscopic PD p_i a heat transfer process via $p_i \to p_i + dp_i$ then
- If the PD p_i maximizes S this entails $dU = TdS$ and alternatively
- If $dU = TdS$ then this predetermines a unique PD that maximizes S.

For the second law one wants to maximize entropy S with M appropriate constraints A_k which take values $A_k(i)$ at the microstate i; the constrains have the form

$$< A_k >= \sum_i p_i A_k(i)\ \ (k = 1, \cdots M) \tag{4.47}$$

The Boltzman constant is k_B and assume that $k = 1$ in (4.47) corresponds to the energy E with $A_1(i) = \epsilon_i$ so that the above expression specializes to

$$U =< A_1 >= \sum p_i\epsilon_i \tag{4.48}$$

One should now maximize the "Lagrangian" Φ given by

$$\Phi = \frac{S}{k_B} - \alpha \sum_i p_i - \beta \sum_i p_i \epsilon_i - \sum_2^M \lambda_k \sum_i p_i A_k(p_i) \tag{4.49}$$

in order to obtain the actual distribution p_i from the equation $\delta_{p_i}\Phi = 0$. Since here one is interested just in the "heat" part the last term on the right of (4.49) will not be considered. It is argued that if p_i changes to $p_i + dp_i$ because of $\delta_{p_i}\Phi = 0$ one will have

$$0 = \frac{dS}{k_B} - \beta dU \tag{4.50}$$

(note $\sum_i \delta p_i = 0$ via normalization). Since $\beta = 1/k_B T$ we get $dU = TdS$ so MaxEnt implies the first law.

The central goal here is to go the other way so assume one has a rather general information measure of the form

$$S = k \sum_i p_i f(p_i) \tag{4.51}$$

where $k \sim k_B$. The sum runs over a set of quantum numbers denoted by i (characterizing levels of energy ϵ_i) that specify an appropriate basis in Hilbert space, $\mathcal{P} = \{p_i\}$ is an (as yet unknown) probability distribution with $\sum p_i = constant$, and f is an arbitrary smooth function of the p_i. Assume further that mean values of quantities A that take the value A_i with probability p_i are evaluated via

$$< A >= \sum_i A_i g(p_i) \tag{4.52}$$

In particular the mean energy U is given by $U = \sum_i \epsilon_i g(p_i)$. Assume now that the set $\mathcal{P}$ changes in the fashion

$$p_i \to p_i + dp_i; \ \sum dp_i = 0 \tag{4.53}$$

(the last via $\sum p_i = constant$. This in turn generates corresponding changes dS and dU and one is thinking here of level population changes, i.e. heat. To insure the first law one assumes ($\bullet$) $dU - TdS = 0$ and as a consequence of ($\bullet$) a little algebra gives (up to first order in the dp_i the condition

$$\epsilon_i g'(p_i) - kT[f(p_i) + p_i f'(p_i)] = 0 \tag{4.54}$$

This equation is now examined for several situations. First look at Shannon entropy with

$$f(p_i) = -log(p_i); \ g(p_i) = p_i \tag{4.55}$$

In this situation (4.54) becomes

$$-\epsilon_i = kT[log(p_i) + 1] \Rightarrow p_i = \frac{1}{e} exp(-\epsilon_i/kT) \tag{4.56}$$

After normalization this is the canonical Boltzmann distribution and this is the only distribution that guarantees obedience to the first law for Shannon's information measure. A posteriori this distribution maximizes entropy as well with U as a constraint which establishes a link with the second law. Several other measures are considered, in particular the Tsallis measure, and we refer to [**779**] for details. In summary if one assumes entropy is maximum one immediately derives the first law and if you assume the first law and an information measure this predetermines a probability distribution that maximizes entropy.

REMARK 6.4.2. There is currently a great interest in acoustic wave phenomena, sound and vortices, acoustic spacetime, acoustic black holes, etc. A prime source of material involves superfluid physics à la Volovik [**968, 969**] and Bose-Einstein condensates (see e.g. [**39, 40, 86, 81, 101, 115, 369, 370, 912, 913, 963**]). We had originally written out material from [**912, 913**] in preparation for sketching some material from [**968**]. However we realized that there is simply too much to include in this book; at least 2-3 more chapters would be needed to even get off the ground.

CHAPTER 7

ON THE QUANTUM POTENTIAL

1. RESUMÉ

We have seen already how the quantum potential arises in many contexts as a fundamental ingredient connected with quantum matter, and how it provides linkage between e.g. statistics and uncertainty, Fisher information and entropy, Weyl geometry, and quantum Kähler geometry. We expand further on certain aspects of the quantum potential in this Chapter aafter the resumé from Chapters 1-6 Moreover we have seen how the trajectory representation à la deBroglie-Bohm (dBB) can be used to develop meaningful insight and results in quantum field theory (QFT) and cosmology. This is achieved mainly without the elaborate machinery of Fock spaces, Feynman diagrams, operator algebras, etc. in a straightforward manner. The conclusion seems inevitable that dBB theory is essentially all pervasive and represents perhaps the most powerful tool available for understanding not only QM but the universe itself. There are of course many papers and opinions concerning such conclusions (some already discussed) and we will make further comments along these lines in this Chapter. We remark that there is some hesitation in postulating an ensemble interpretation when using dBB theory in cosmology with a wave function of the universe for example but we see no obstacle here, once an ensemble of universes is admitted. This is surely as reasonable as dealing with many string theories as is now fashionable. In any event we begin with a resumé of highlights from Chapters 1-6 and gather some material here (see also [**1026**] for information on the map $SE \to Q$).

1.1. THE SCHRÖDINGER EQUATION. We list some examples primarily concerned with QM.

(1) The SE in 1-dimension with $\psi = Rexp(iS/\hbar)$ and associated HJ and continuity equations are

$$-\frac{\hbar^2}{2m}\psi_{xx} + V\psi = i\hbar\psi_t;\ S_t + \frac{S_x^2}{2m} + V + Q = 0; \tag{1.1}$$

$$Q = -\frac{\hbar^2 R''}{2mR};\ \partial_t(R^2) + \frac{1}{m}(R^2 S_x)_x = 0$$

Here $P = R^2 = |\psi|^2$ is a probability density, $p = m\dot{x} = S_x$ (with S not constant!) is the momentum, $\rho = mP$ is a mass density, and Q is the quantum potential of Bohm. Classical mechanics involves a HJ equation with $Q = 0$ and can be derived as follows (cf. [**304**]). Consider a Lagrangian $L = p\dot{x} - H$ with Hamiltonian $H = (p^2/2m) + V$ and action

$S = \int_{t_1}^{t_2} dtL(x, \dot{x}, t)$. One computes

(1.2)
$$\delta S = \int_{t_1}^{t_2} dt \left[p\frac{d}{dt}\delta x + \dot{x}\delta p - \delta H - H\frac{d}{dt}\delta t\right]$$

where $\delta H = H_x\delta x + H_p\delta p + H_t\delta t = V_x\delta x + (p/m)\delta p + H_t\delta t$. Then

(1.3)
$$\delta S = \int_{t_1}^{t_2} dt \left[p\frac{d}{dt}\delta x + \dot{x}\delta p - V_x\delta x - \frac{p}{m}\delta p - H_t\delta t - H\frac{d}{dt}\delta t\right] =$$
$$= \int_{t_1}^{t_2} dt \left\{\frac{d}{dt}[p\delta x - H\delta t] + \delta p\left(\dot{x} - \frac{p}{m}\right) - \delta x\left(\frac{dp}{dt} + V_x\right) + \delta t[\dot{H} - H_t]\right\}$$

Consequently, writing $\delta S = (S_x\delta x + S_t\delta t)|_{t_1}^{t_2}$ one arrives at

(1.4)
$$\dot{x} = \frac{p}{m};\ \dot{p} = -V_x;\ \dot{H} = H_t;\ p = S_x;\ S_t + H = 0$$

Note here the "surface" term (from integration) is $G = p\delta x - H\delta t$ and $\delta S = G_2 - G_1$ which should equal $\delta S = (\partial S/\partial x_1)\delta x_1 + (\partial S/\partial x_2)\delta x_2 + (\partial S/\partial t_1)\delta t_1 + (\partial S/\partial t_2)\delta t_2$ where $x_1 = x(t_1)$ and $x_2 = x(t_2)$ One sees then directly how the addition of Q to a classical HJ equation produces a quantum situation.

(2) Another classical connection comes via hydrodynamics (cf. Section 1.1) where (1.1) can be put in the form

(1.5)
$$\partial_t(\rho v) + \partial(\rho v^2) + \frac{\rho}{m}(\partial V + \partial Q) = 0$$

which is like an Euler equation in fluid mechanics moduo a pressure term $-\rho^{-1}\partial\mathfrak{P}$ on the right. If we identify $(\rho/m)\partial Q = \rho^{-1}\partial\mathfrak{P} \equiv \mathfrak{P} = \partial^{-1}(\rho^2/m)\partial Q$ (with some definition of ∂^{-1} - cf. [**205**]) then the quantum term could be thought of as providing a pressure with Q corresponding e.g. to a stress tensor of a quantum fluid. We refer also to Remark 1.1.2 and work of Kaniadakis et al where the quantum state coressponds to a subquantum statistical ensemble whose time evolution is governed by classical kinetics in phase space.

(3) The Fisher information connection à la Remarks 1.1.4 - 1.1.5 involves a classical ensemble with particle mass m moving under a potential V

(1.6)
$$S_t + \frac{1}{2m}(S')^2 + V = 0;\ P_t + \frac{1}{m}\partial(PS')' = 0$$

where S is a momentum potential; note that no quantum potential is present but this will be added on in the form of a term $(1/2m)\int dt(\Delta N)^2$ in the Lagrangian which measures the strength of fluctuations. This can then be specified in terms of the probability density P as indicated in Remark 1.1.4 leading to a SE where by Theorem 1.1.1 $(\Delta N)^2 \sim c\int[(P')^2/P]dx$. A "neater" approach is given in Remark 1.1.5 leading in 1-D to

(1.7)
$$S_t + \frac{1}{2m}(S')^2 + V + \frac{\lambda}{m}\left(\frac{(P')^2}{P^2} - \frac{2P''}{P}\right) = 0$$

Note that $Q = -(\hbar^2/2m)(R''/R)$ becomes for $R = P^{1/2}$ an equation $Q = -(\hbar^2/8m)[(2P''/P) - (P'/P)^2]$. Thus the addition of the Fisher information serves to quantize the classical system via a quantum potential and this gives a direct connection of the quantum potential with fluctuations.

(4) The Nagasawa theory (based in part on Nelson's work) is very revealing and fascinating. The essense of Theorem 1.1.2 is that $\psi = exp(R+iS)$ satisfies the SE $i\psi_t + (1/2)\psi'' + ia\psi' - V\psi = 0$ if and only if

$$V = -S_t + \frac{1}{2}R'' + \frac{1}{2}(R')^2 - \frac{1}{2}(S')^2 - aS';\ 0 = R_t + \frac{1}{2}S'' + S'R' + aR' \tag{1.8}$$

Changing variables ($X = (\hbar/\sqrt{m})x$ and $T = \hbar t$) one arrives at $i\hbar\psi_T = -(\hbar^2/2m)\psi_{XX} - iA\psi_X + V\psi$ where $A = a\hbar/\sqrt{m}$ and

$$i\hbar R_T + (\hbar^2/m^2)R_X S_X + (\hbar^2/2m^2)S_{XX} + AR_X = 0; \tag{1.9}$$

$$V = -i\hbar S_T + (\hbar^2/2m)R_{XX} + (\hbar^2/2m^2)R_X^2 - (\hbar^2/2m^2)S_X^2 - AS_X$$

The diffusion equations then take the form

$$\hbar\phi_T + \frac{\hbar^2}{2m}\phi_{XX} + A\phi_X + \tilde{c}\phi = 0;\ -\hbar\hat{\phi}_T + \frac{\hbar^2}{2m}\hat{\phi}_{XX} - A\hat{\phi}_X + \tilde{c}\hat{\phi} = 0; \tag{1.10}$$

$$\tilde{c} = -\tilde{V}(X,T) - 2\hbar S_T - \frac{\hbar^2}{m}S_X^2 - 2AS_X$$

It is now possible to introduce a role for the quantum potential in this theory. Thus from $\psi = exp(R+iS)$ (with $\hbar = m = 1$ say) we have $\psi = \rho^{1/2}exp(iS)$ with $\rho^{1/2} = exp(R)$ or $R = (1/2)log(\rho)$. Hence $(1/2)(\rho'/\rho) = R'$ and $R'' = (1/2)[(\rho''/\rho) - (\rho'/\rho)^2]$ while the quantum potential is $Q = (1/2)(\partial^2\rho^{1/2}/\rho^{1/2}) = -(1/8)[(2\rho''/\rho) - (\rho'/\rho)^2]$. Equation (1.8) becomes then

$$V = -S_t + \frac{1}{8}\left(\frac{2\rho''}{\rho} - \frac{(\rho')^2}{\rho^2}\right) - \frac{1}{2}(S')^2 - aS' \equiv \tag{1.11}$$

$$\equiv S_t + \frac{1}{2}(S')^2 + V + Q + aS' = 0;\ \rho_t + \rho S'' + S'\rho' + a\rho' = 0 \equiv \rho_t + (\rho S')' + a\rho' = 0$$

Thus $-2S_t - (S')^2 = 2V + 2Q + 2aS'$ and one has

PROPOSITION 1.1. The creation-annihilation term c in the diffusion equations (cf. Theorem 1.1.2) becomes

$$c = -V - 2S_t - (S')^2 - 2aS' = V + 2Q \tag{1.12}$$

where Q is the quantum potential.

(5) Going to Remarks 1.1.6-1.1.8 we set

$$\tilde{Q} = \frac{\hbar^2}{2m^2}\frac{\partial^2\sqrt{\rho}}{\sqrt{\rho}} = -\frac{1}{m}Q;\ D = \frac{\hbar}{2m};\ u = D\partial(log(\rho)) = \frac{h}{2m}\frac{\rho'}{\rho} \tag{1.13}$$

Then u is called an osmotic velocity field and Brownian motion involves $v = -u$ for the diffusion current. In particular

$$\tilde{Q} = \frac{1}{2}u^2 + D\partial u \tag{1.14}$$

One defines an entropy term $\mathfrak{S} = -\int \rho log(\rho)dx$ leading, for suitable regions of integration and behavior of ρ at infinity, and using $\rho_t = -\partial(v\rho)$ from (1.1), to

$$\frac{\partial\mathfrak{S}}{\partial t} = -\int \rho_t(1+log(\rho)) = \int (1+log(\rho))\partial(v\rho) = \tag{1.15}$$

$$= -\int v\rho' = \int u\rho' = D\int \frac{(\rho')^2}{\rho}$$

Note also

$$\tilde{Q} = \frac{D^2}{2}\left(\frac{2\rho''}{\rho} - \left(\frac{\rho'}{\rho}\right)^2\right) \Rightarrow \int \rho\tilde{Q} = -\frac{D^2}{2}\int \frac{(\rho')^2}{\rho} \tag{1.16}$$

Thus generally $\partial\mathfrak{S}/\partial t \geq 0$ and $\mathfrak{F} = -(2/D^2)\int \rho\tilde{Q}$ is a functional form of Fisher information with $\mathfrak{S}_t = D\mathfrak{F}$.

(6) The development of the SE by Nottale, Cresson, et al in Section 1.2 is basically QM and is peripheral to scale relativity as such. The idea is roughly to to imagine e.g. continuous nondifferentiable quantum paths and to describe the velocity in terms of an average $V = (1/2)(b_+ + b_-)$ and a discrepancy $U = (1/2)(b_+ - b_-)$ where $b_\pm$ are given by (2.1). Nottale's derivation is heuristic but revealing and working with a complex velocity he captures the complex nature of QM. In particular the quantum potential can be written as $Q = -(m/2)U^2 - (\hbar/2)\partial U$ corresponding to (1.14) via $\tilde{Q} = -(1/m)Q$. As indicated in Proposition 1.2.1 this reveals the quantum potential as a manifestation of the "fractal" nature of quantum paths - smooth paths correspond to $Q = 0$ which seems to preclude smooth trajectories for quantum particles. In such a case the standard formula $\dot{x} = (\hbar/2m)\Im[\psi^*\partial\psi/|\psi|^2]$ requires a discontinuous $\dot{x}$ wich places some constraints on ψ and the whole guidance idea. This whole matter should be addressed further along with considerations of osmotic velocity, etc. Another approach to quantum fractals is given in Section 1.5.

(7) In section 2.3.1 we sketched some of the Bertoldi-Faraggi-Matone (BFM) version of Bohmian mechanics and in particular for the stationary quantum HJ equation (QHJE) $(1/2m)S_x^2 + W = E$ $(W = V + Q)$, arising from $-(\hbar^2/2m)\psi'' + V\psi = E\psi$, one can extract from [**347**] the formulas for trajectories (using Floydian time $t \sim \partial S/\partial E$). Thus $(\partial_E W = \partial_E Q)$

$$t \sim \partial_E \int S_x dx = \partial_E \int [E-W]^{1/2}dx = \left(\frac{m}{2}\right)^{1/2}\int \frac{(1-\partial_E Q}{\sqrt{E-W}}dx \tag{1.17}$$

Hence

$$\frac{dt}{dx} = \left(\frac{m}{2}\right)^{1/2}\frac{1-\partial_E Q}{\sqrt{E-W}} \Rightarrow \dot{x} = \frac{S_x}{m}\frac{1}{1-\partial_E Q} \tag{1.18}$$

Thus $m(1-\partial_E Q)\dot{x} = m_Q\dot{x} = S_x$ and this is defined as p with m_Q representing a quantum mass. Note $\dot{x} \neq p/m$ and we refer to [**194, 191, 347, 373, 374**] for discussion of all this. Further via $p' = m_Q'\dot{x} +$

$m_Q(\ddot{x}/\dot{x})$, etc., one can rewrite the QSHJ as a third order trajectory (or microstate) equation (see also Remark 7.4)

$$\frac{m_Q^2}{2m}\dot{x}^2 + V - E + \frac{\hbar^2}{4m}\left(\frac{m_Q''}{m_Q} - \frac{3}{2}\left(\frac{m_Q'}{m_Q}\right)^2 - \frac{m_Q'}{m_Q}\frac{\ddot{x}}{\dot{x}^2} + \frac{\dddot{x}}{\dot{x}^3} - \frac{5}{2}\frac{\ddot{x}^2}{\dot{x}^4}\right) = 0 \quad (1.19)$$

In Remark 2.2.2 with Theorem 2.1 we observed how the uncertainty principle of QM can be envisioned as due to incomplete information about microstates when working in the Hilbert space formulation of QM based on the SE. It was shown how $\Delta q \Delta x = O(\hbar)$ arises automatically from a BFM perspective. Thus the canonical QM in Hilbert space cannot see a single trajectory and hence is obliged to operate in terms of ensembles and probability. We have also seen how a probabilistic ensemble picture with quantum fluctuations comes about with the fluctuations corresponding to the quantum potential (see Item 3 above). This suggests that a background motivation for the Hilbert space may really exist (beyond its pragmatic black magic) since these fluctuations represent a form of information (and uncertainty). The hydrodynamic and diffusion models are also directly connected to this and produce as in Item 5 above a connection to entropy.

1.2. DEBROGLIE-BOHM. There are many approaches to dBB theory and in fact much of the book is concerned with this. David Bohm wrote extensively about the subject but we have omitted much of the philosophy (implicate order, etc.). The book by Holland [**471**] is excellent and a modern theory is being constructed by Dürr, Goldstein, Zanghi, et al (cf. also the work of Bertoldi, Farragi, Matone, and Floyd). Some new directions in QFT, Weyl geometry, and cosmology are also covered in the book, due to Barbosa, Pinto-Neto, Nikolić, A. and F. Shojai, et al, and we will try to summarize some of that here.

(1) The BFM theory is quite novel (and profound) in that it is based entirely on an equivalence principle (EP) stating that all physical systems can be connected by a coordinate transformation to the free situation with vanishing energy. One bases the stationary situation of energy E in the nonrelativistic case with the SE as in Item 7 above. In the relativistic case (Remark 2.2.3) one can work in the same spirit directly with a Minkowski metric to obtain the Klein-Gordon (KG) equation with a relativistic quantum potential

$$Q_{rel} = -\frac{\hbar^2}{2m}\frac{\Box R}{R} \quad (1.20)$$

Note that the probability aspects concerning R appear to be absent in the relativistic theory. It is interesting to note (cf. Remark 2.2.1) that the EP implies that all mass can be generated by a coordinate transformation and since mass can be expressed in terms of the quantum potential Q this provides yet another role for the quantum potential.

(2) Some quantum field theory (QFT) aspects of the Bohm theory are developed in [**471**] and sketched here in Example 2.1.1. One arrives at a

formula, namely

$$\Box\psi = -\left.\frac{\delta Q[\psi(x),t]}{\delta\psi(x)}\right|_{\psi(x)=\psi(x,t)};\ Q[\psi,t] = -\frac{1}{2R}\int d^3x\frac{\delta^2 R}{\delta\psi^2} \tag{1.21}$$

More recently there have been some impressive papers by Nikolić involving Bohmian theory and QFT. First there are papers on bosonic and fermionic Bohmian QFT sketched in Sections 3.2 and 3.3. These are lovely but even more attractive are two newer papers [**708, 713**] by Nikolić which we displayed in Sections 2.4 and 2.6. In [**708**] one utilizes the deDonder-Weyl formulation of QFT (reviewed in Appendix A) and a Bohmian formulation is not postulated but derived from the technical requirements of covariance and consistency with standard QM. One introduces a preferred foliation of spacetime with R^μ normal to the leaf Σ and writes $\mathfrak{R}([\phi],\Sigma) = \int_\Sigma d\Sigma_\mu R^\mu$ with $\mathfrak{S}([\phi],x) = \int_\Sigma d\Sigma_\mu S^\mu$. This produces a covariant version of Bohmian mechanics with $\Psi = \mathfrak{R}exp(i\mathfrak{S}/\hbar)$ via

$$\frac{1}{2}\frac{dS_\mu}{d\phi}\frac{dS^\mu}{d\phi} + V + Q + \partial_\mu S^\mu = 0;\ \frac{dR^\mu}{d\phi}\frac{dS^\mu}{d\phi} + J + \partial_\mu R^\mu = 0 \tag{1.22}$$

$$Q = -\frac{\hbar^2}{2\mathfrak{R}}\frac{\delta^2\mathfrak{R}}{\delta_\Sigma\phi^2(x)};\ J = \frac{\mathfrak{R}}{2}\frac{\delta^2\mathfrak{S}}{\delta_\Sigma\phi^2(x)} \tag{1.23}$$

In [**713**] one uses the many fingered time (MF) Tomonaga-Schwinger (TS) equation where a Cauchy hypersurface Σ is defined via $x^0 = T(\mathbf{x})$ with $\mathbf{x}$ corresponding to coordinates on Σ. The TS equation is

$$i\frac{\delta\Psi[\phi,T]}{\delta T(\mathbf{x})} = \hat{\mathfrak{H}}\Psi[\phi,T] \tag{1.24}$$

Take a free scalar field for convenience with

$$\hat{\mathfrak{H}}(\mathbf{x}) = -\frac{1}{2}\frac{\delta^2}{\delta\phi^2(\mathbf{x})} + \frac{1}{2}[(\nabla\phi(\mathbf{x}))^2 + m^2\phi^2(\mathbf{x})] \tag{1.25}$$

Then for a manifestly covariant theory one introduces parameters $\mathbf{s} = (s^1,s^2,s^3)$ to serve as coordinates on a 3-dimensional manifold Σ in spacetime with $x^\mu = X^\mu(\mathbf{s})$ the embedding coordinates. The induced metric on Σ is

$$q_{ij}(\mathbf{s}) = g_{\mu\nu}(X(\mathbf{s}))\frac{\partial X^\mu(\mathbf{s})}{\partial s^i}\frac{\partial X^\nu(\mathbf{s})}{\partial s^j} \tag{1.26}$$

Similarly a normal (resp. unit normal - transforming as as a spacetime vector) to the surface are

$$\tilde{n}(\mathbf{s}) = \epsilon_{\mu\alpha\beta\gamma}\frac{\partial X^\alpha}{\partial s^1}\frac{\partial X^\beta}{\partial s^2}\frac{\partial X^\gamma}{\partial s^3};\ n^\mu(\mathbf{s}) = \frac{g^{\mu\nu}\tilde{n}_\nu}{\sqrt{|g^{\alpha\beta}\tilde{n}_\alpha\tilde{n}_\beta|}} \tag{1.27}$$

Then for $\mathbf{x}\to\mathbf{s}$ and $\frac{\delta}{\delta T(\mathbf{x})}\to n^\mu(\mathbf{s})\frac{\delta}{\delta X^\mu(\mathbf{s})}$ the TS equation becomes

$$\hat{\mathfrak{H}}(\mathbf{s})\Psi[\phi,X] = in^\mu(\mathbf{s})\frac{\delta\Psi[\phi,X]}{\delta X^\mu(\mathbf{s})} \tag{1.28}$$

and the Bohmian equations of motion are ($\Psi = Rexp(iS)$)

(1.29) $$\frac{\partial\Phi(\mathbf{s},T]}{\partial\tau(\mathbf{s})} = \frac{1}{|q(\mathbf{s})|^{1/2}}\left.\frac{\delta S}{\delta\phi(\mathbf{s})}\right|_{\phi=\Phi};\ \frac{\partial}{\partial\tau(\mathbf{s})} \equiv lim_{\sigma_x\to 0}\int_{\sigma_x} d^3sn^{\mu}(\mathbf{s})\frac{\delta}{\delta X^{\mu}(\mathbf{s})}$$

In the same spirit the quantum MFT KG equation is

(1.30) $$\left[\left(\frac{\partial}{\partial\tau(\mathbf{s})}\right)^2 + \nabla^i\nabla_i + m^2\right]\Phi(\mathbf{s},X] = -\frac{1}{|q(\mathbf{s})|^{1/2}}\left.\frac{\partial\mathfrak{Q}(\mathbf{s},\phi,X]}{\partial\phi\mathbf{s})}\right|_{\phi=\Phi}$$

where ∇_i is the covariant derivative with respect to s^i and

(1.31) $$\mathfrak{Q}(\mathbf{s},\phi,X] = -\frac{1}{|q(\mathbf{s})|^{1/2}}\frac{1}{2R}\frac{\delta^2 R}{\delta\phi^2(\mathbf{s})}$$

(3) The QFT model in Section 2.5 involving stochastic jumps is quite technical and should be read in conjunction with Nagasawa's book [**674**]. The idea (cf. [**326**]) is that for the Hamiltonian of a QFT there is associated a $|\psi|^2$ distributed Markov process, typically a jump process (to accound for creation and annihilation processes) on the configuration space of a variable number of particles. One treats this via functional analysis, operator thery, and probability, which leads to mountains of detail, only a small portion of which is sketched in this book.

(4) In Section 3.2 we give a sketch of dBB in Weyl geometry following A. and F. Shojai [**873**]. This is a lovely approach and using Dirac-Weyl methods one is led comfortably into general relativity (GR), cosmology, and quantum gravity, in a Bohmian context. Such theories dominate Chapters 3 and 4. First one looks at the relativistic energy equation $\eta_{\mu\nu}p^{\mu}p^{\nu} = m^2c^2$ generalized to

(1.32) $$\eta_{\mu\nu}P^{\mu}P^{\nu} = m^2c^2(1+\mathcal{Q}) = \mathcal{M}^2c^2;\ \mathcal{Q} = (\hbar^2/m^2c^2)(\Box|\Psi|/|\Psi|)$$

(1.33) $$\mathcal{M}^2 = m^2\left(1+\alpha\frac{\Box|\Psi|}{|\Psi|}\right);\ \alpha = \frac{\hbar^2}{m^2c^2}$$

(obtained e.g. by setting $\psi = \sqrt{\rho}exp(iS/\hbar)$ in the KG equation). Here $\mathcal{M}^2$ is not positive definite and in fact (1.32) is the wrong equation! Some interesting arguments involving Lorentz invariance lead to better equations and for a particle in a curved background the natural quantum HJ equation is most comfortably phrased as

(1.34) $$\nabla_\mu(\rho\nabla^\mu S) = 0;\ g^{\mu\nu}\nabla_\mu S\nabla_\nu S = \mathfrak{M}^2c^2;\ \mathfrak{M}^2 = m^2e^{\mathfrak{Q}};\ \mathfrak{Q} = \frac{\hbar^2}{m^2c^2}\frac{\Box_g|\Psi|}{|\Psi|}$$

This is equivalent to

(1.35) $$\left(\frac{m^2}{\mathcal{M}^2}\right)g^{\mu\nu}\nabla_\mu S\nabla_\nu S = m^2c^2$$

showing that the quantum effects correspond to a change in spacetime metric $g_{\mu\mu} \to \tilde{g}_{\mu\nu} = (\mathfrak{M}^2/m^2)g_{\mu\nu}$. This is a conformal transformation and leads to Weyl geometry where (1.35) takes the form $\tilde{g}^{\mu\nu}\tilde{\nabla}_\mu S\tilde{\nabla}_\nu S =$

m^2c^2 with $\tilde{\nabla}_\mu$ the covariant derivative in the metric $\tilde{g}_{\mu\nu}$. The particle motion is then

$$\mathfrak{M}\frac{d^2x^\mu}{d\tau^2} + \mathfrak{M}\Gamma^\mu_{\nu\kappa}u^\nu u^\kappa = (c^2g^{\mu\nu} - u^\mu u^\nu)\nabla_\nu\mathfrak{M} \tag{1.36}$$

and the introduction of a quantum potential is equivalent to introducing a conformal factor $\Omega^2 = \mathcal{M}^2/m^2$ in the metric (i.e. QM corresponds to Weyl geometry). One considers then a general relativistic system containing gravity and matter (no quantum effects) and links it to quantum matter by the conformal factor Ω^2 (using an approximation $1 + Q \sim exp(Q)$ for simplicity); then the appropriate Einstein equations are written out. Here the conformal factor and the quantum potential are made into dynamical fields to create a scalar-tensor theory with two scalar fields. Examples are developed and we refer to Section 3.2 and [**873**] for more details. Back reaction effects of the quantum factor on the background metric are indicated in the modified Einstein equations. Thus the conformal factor is a function of the quantum potential and the mass of a relativistic particle is a field produced by quantum corrections to the classical mass. In general frames both the spacetime metric and the mass field have quantum properties.

(5) The Dirac-Weyl theory is developed also in [**873**] via the action

$$\mathfrak{A} = \int d^4x\sqrt{-g}(F_{\mu\nu}F^{\mu\nu} - \beta^2\ {}^W\mathcal{R} + (\sigma+6)\beta_{;\mu}\beta^{;\mu} + \mathfrak{L}_{matter} \tag{1.37}$$

The gravitational field $g_{\mu\nu}$ and Weyl field ϕ_μ plus β determine the spacetime geometry and one finds a Bohmian theory with $\beta \sim \mathcal{M}$ (Bohmian quantum mass field). We will say much more about Dirac-Weyl theory below.

(6) There is an interesting approach by Santamato in [**840, 841**] dealing with the SE and KG equation in Weyl geometry (cf. Section 3.3 and [**189, 203**]). In the first paper on the SE one assumes particle motion given by a random process $q^i(t,\omega)$ with probability density $\rho(q,t)$, $\dot{q}^i(t,\omega) = v^i(q(t,\omega),t)$, and random initial conditions $q_0^i(\omega)$ $(i = 1,\cdots,n)$. One begins with a stochastic construction of (averaged) classical type Lagrange equations in generalized coordinates for a differentiable manifold M in which a notion of scalar curvature R is meaningful (this is where statistics enters the geometry). It is then shown that a theory equivalent to QM (via a SE) can be constructed where the "quantum force" (arising from a quantum potential Q) can be related to (or described by) geometric properties of space. To do this one assumes that a (quantum) Lagrangian can be constructed in the form $L(q,\dot{q},t) = L_C(q,\dot{q},t) + \gamma(\hbar^2/m)R(q,t)$ where $\gamma = (1/6)(n-2)/(n-1)$ with $n = dim(M)$ and R is a curvature scalar. Now for a Riemannian geometry $ds^2 = g_{ik}(q)dq^idq^k$ it is standard that in a transplantation $q^i \to q^i + \delta q^i$ one has $\delta A^i = \Gamma^i_{k\ell}A^\ell dq^k$, and here it is assumed that for $\ell = (g_{ik}A^iA^k)^{1/2}$ one has $\delta\ell = \ell\phi_k dq^k$ where the ϕ_k are covariant components of an arbitrary vector of M (Weyl geometry). Thus the actual affine connections $\Gamma^i_{k\ell}$ can be found by comparing this

with $\delta\ell^2 = \delta(g_{ik}A^iA^k)$ and one finds

$$\Gamma^i_{k\ell} = -\left\{ \begin{matrix} i \\ k\ \ell \end{matrix} \right\} + g^{im}(g_{mk}\phi_\ell + g_{m\ell}\phi_k - g_{k\ell}\phi_m) \tag{1.38}$$

Thus we may prescribe the metric tensor g_{ik} and ϕ_i and determine via (1.38) the connection coefficients. Covariant derivatives are defined via commas and the curvature tensor $R^i_{k\ell m}$ in Weyl geometry is introduced via $A^i_{,k,\ell} - A^i_{,\ell,k} = F^i_{mk\ell}A^m$ from which arises the standard formula of Riemannian geometry $R^i_{mk\ell} = -\partial_\ell\Gamma^i_{mk} + \partial_k\Gamma^i_{m\ell} + \Gamma^i_{n\ell}\Gamma^n_{mk} - \Gamma^i_{nk}\Gamma^n_{m\ell}$ where (1.38) must be used in place of the Riemannian Christoffel symbols. The Ricci symmetric tensor R_{ik} and the scalar curvature R are defined via $R_{ik} = R^\ell_{i\ell k}$ and $R = g^{ik}R_{ik}$, while

$$R = \dot{R} + (n-1)[(n-2)\phi_i\phi^i - 2(1/\sqrt{g})\partial_i(\sqrt{g}\phi^i)] \tag{1.39}$$

where $\dot{R}$ is the Riemannian curvature built by the Christoffel symbols. Now the geometry is to be derived from physical principles so the ϕ_i cannot be arbitrary but are obtained by the same (averaged) least action principle giving the motion of the particle (statistical determination of geometry) and when $n \geq 3$ the minimization involves only (1.39). One shows that $\hat{\rho}(q,t) = \rho(q,t)/\sqrt{g}$ transforms as a scalar in a coordinate change and this will be called the scalar probability density of the random motion of the particle. Starting from $\partial_t\rho + \partial_i(\rho v^i) = 0$ a manifestly covariant equation for $\hat{\rho}$ is found to be

$$\partial_t\hat{\rho} + (1/\sqrt{g})\partial_i(\sqrt{g}v^i\hat{\rho}) = 0 \tag{1.40}$$

Some calculation then yields a minimum over R when

$$\phi_i(q,t) = -[1/(n-2)]\partial_i[log(\hat{\rho})(q,t)] \tag{1.41}$$

This shows that the geometric properties of space are indeed affected by the presence of the particle and in turn the alteration of geometry acts on the particle through the quantum force $f_i = \gamma(\hbar^2/m)\partial_i R$ which according to (1.39) depends on the gauge vector and its derivatives. It is this peculiar feedback between the geometry of space and the motion of the particle which produces quantum effects. In this spirit one goes next to a geometrical derivation of the SE. Thus inserting (1.41) into (1.39) one gets

$$R = \dot{R} + (1/2\gamma\sqrt{\hat{\rho}})[1/\sqrt{g})\partial_i(\sqrt{g}g^{ik}\partial_k\sqrt{\hat{\rho}})] \tag{1.42}$$

where the value $\gamma = (1/6)[(n-2)/(n-1)]$ has been used. On the other hand the HJ equation can be written as

$$\partial_t S + H_C(q, \nabla S, t) - \gamma(\hbar^2/m)R = 0 \tag{1.43}$$

When (1.42) is introduced into (1.43) the HJ equation and the continuity equation (1.40), with velocity field given by $v^i = (\partial H/\partial p_i)(q, \nabla S, t)$, form a set of two nonlinear PDE which are coupled by the curvature of space. Therefore self consistent random motions of the "particle" are obtained by solving (1.40) and (1.43) simultaneously. For every pair of solutions

$S(q,t,\hat{\rho}(q,t))$ one gets a possible random motion for the particle whose invariant probability density is $\hat{\rho}$. The present approach is so different from traditional QM that a proof of equivalence is needed and this is only done for Hamiltonians of the form $H_C(q,p,t) = (1/2m)g^{ik}(p_i - A_i)(p_k - A_k) + V$ (which is not very restrictive) leading to

$$\partial_t S + \frac{1}{2m}g^{ik}(\partial_i S - A_i)(\partial_k S - A_k) + V - \gamma\frac{\hbar^2}{m}R = 0 \tag{1.44}$$

(R in (1.43)). The continuity equation (1.40) is

$$\partial_t\hat{\rho} + (1/m\sqrt{g})\partial_i[\hat{\rho}\sqrt{g}g^{ik}(\partial_k S - A_k)] = 0 \tag{1.45}$$

Owing to (1.42), (1.44) and (1.45) form a set of two nonlinear PDE which must be solved for the unknown functions S and $\hat{\rho}$. Then a straightforward calculation shows that, setting $\psi(q,t) = \sqrt{\hat{\rho}(q,t)}exp](i/\hbar)S(q,t)]$ the quantity ψ obeys a linear SE

$$i\hbar\partial_t\psi = \frac{1}{2m}\left\{\left[\frac{i\hbar\partial_i\sqrt{g}}{\sqrt{g}} + A_i\right]g^{ik}(i\hbar\partial_k + A_k)\right\}\psi + \left[V - \gamma\frac{\hbar^2}{m}\dot{R}\right]\psi \tag{1.46}$$

where only the Riemannian curvature $\dot{R}$ is present (any explicit reference to the gauge vector ϕ_i having disappeared).

We recall that in the nonrelativistic context the quantum potential has the form $Q = -(\hbar^2/2m)(\partial^2\sqrt{\rho}/\sqrt{\rho})$ ($\rho \sim \hat{\rho}$ here) and in more dimensions this corresponds to $Q = -(\hbar^2/2m)(\Delta\sqrt{\rho}/\sqrt{\rho})$. The continuity equation in (1.45) corresponds to $\partial_t\rho + (1/m\sqrt{g})\partial_i[\rho\sqrt{g}g^{ik}(\partial_k S)] = 0$ ($\rho \sim \hat{\rho}$ here). For $A_k = 0$ (1.44) becomes $\partial_t S + (1/2m)g^{ik}\partial_i S\partial_k S + V - \gamma(\hbar^2/m)R = 0$. This leads to an identification $Q \sim -\gamma(\hbar^2/m)R$ where R is the Ricci scalar in the Weyl geometry (related to the Riemannian curvature built on standard Christoffel symbols via (1.39). Here $\gamma = (1/6)[(n-2)/(n-1)]$ which for $n = 3$ becomes $\gamma = 1/12$; further by (1.41) the Weyl field is $\phi_i = -\partial_i log(\rho)$. Consequently for the SE (1.46) in Weyl space the quantum potential is $Q = -(\hbar^2/12m)R$ where R is the Weyl-Ricci scalar curvature. For Riemannian flat space $\dot{R} = 0$ this becomes via (1.42)

$$R = \frac{1}{2\gamma\sqrt{\rho}}\partial_i g^{ik}\partial_k\sqrt{\rho} \sim \frac{1}{2\gamma}\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}} \Rightarrow Q = -\frac{\hbar^2}{2m}\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}} \tag{1.47}$$

as desired; the SE (1.46) reduces to the standard SE $i\hbar\partial_t\psi = -(\hbar^2/2m)\Delta\psi + V\psi$ ($A_k = 0$). Moreover (1.39) provides an interaction between gravity (involving $\dot{R}$ and g) and QM (which generates ϕ_i via ρ and R via Q).

(7) In [**841**] the KG equation is also derived via an average action principle with the restriction of a priori Weyl geometry removed. The spacetime geometry is then obtained from the average action principle to be of Weyl type with a gauge field $\phi_\mu = \partial_\mu log(\rho)$. One has a kind of "moral" equivalence between QM in Riemannian spaces and classical statistical mechanics in a Weyl space. Traditional QM based on wave equations and ad hoc probability calculus is merely a convenient tool to overcome the complications arisin from a nontrivial spacetime geometrical structure.

In the KG situation there is a relation $m^2 - (R/6) \sim \mathcal{M}^2 \sim m^2(1+Q)$ (approximating $m^2 exp(Q)$) and $Q \sim \Box\sqrt{\rho}/m^2\sqrt{\rho}$ which implies $R/6 \sim -\Box\sqrt{\rho}/\sqrt{\rho}$.

(8) Referring to Item 3 in Section 6.1.1 we note that for $\phi_\mu \sim A_\mu = \partial_\mu log(P)$ ($P \sim \rho$) one can envision a complex velocity $p_\mu + i\lambda A_\mu$ leading to

$$|p_\mu + i\sqrt{\lambda}A_\mu|^2 = p_\mu^2 + \lambda A_\mu^2 \sim g^{\mu\nu}\left(\frac{\partial S}{\partial x^\mu}\frac{\partial S}{\partial x^\nu} + \frac{\lambda}{P^2}\frac{\partial P}{\partial x^\mu}\frac{\partial P}{\partial x^\nu}\right) \tag{1.48}$$

This is exactly the term arising in a Fisher information Lagrangian

$$L_{QM} = L_{CL} + \lambda I = \int P\left\{\frac{\partial S}{\partial t} + \frac{1}{2}g^{\mu\nu}\left[\frac{\partial S}{\partial x^\mu}\frac{\partial S}{\partial x^\nu} + \frac{\lambda}{P^2}\frac{\partial P}{\partial x^\mu}\frac{\partial P}{\partial x^\nu}\right] + V\right\}dtd^nx \tag{1.49}$$

where I is the information term (see Section 3.1)

$$I = g^{ik}I_{ik} = \frac{g^{ik}}{2}\int\frac{1}{P}\frac{\partial P}{\partial y^i}\frac{\partial P}{\partial y^k}d^ny \tag{1.50}$$

known from ϕ_μ. Hence we have a direct connection between Fisher information and the Weyl field ϕ_μ along with a motivation for a complex velocity (cf. [**223**]). Further we note, via [**189**] and quantum geometry in the form $ds^2 \sim \sum dp_j^2/p_j$ on a space of probability distributions, that (1.50) can be defined as a Fisher information metric (positive definite via its connection to $(\Delta N)^2$) and

$$Q \sim -2\hbar^2 g^{\mu\nu}\left[\frac{1}{P^2}\frac{\partial P}{\partial x^\mu}\frac{\partial P}{\partial x^\nu} - \frac{2}{P}\frac{\partial^2 P}{\partial x^\mu \partial x^\nu}\right] \tag{1.51}$$

(corresponding to $-(\hbar^2/2m)(\partial^2\sqrt{\rho}/\sqrt{\rho}) = -(\hbar^2/8m)[(2\rho''/\rho)-(\rho'/\rho)^2]$). Further from $\mathbf{u} = -D\vec{\phi}$ with $Q = D^2((1/2)|\mathbf{u}|^2 - \nabla\cdot\vec{\phi})$, one expresses Q directly in terms of the Weyl vector. This enforces the idea that QM is built into Weyl geometry and moreover that fluctuations generate Weyl geometry.

(9) In Section 3.3 the WDW equation is treated following [**876, 870**] from a Bohmian point of view. One builds up a Lagrangian and Hamiltonian in terms of lapse and shift functions with a quantum potential

$$Q = \int d^3x\mathfrak{Q};\ \mathfrak{Q} = \hbar^2 NqG_{ijk\ell}\frac{1}{|\psi|}\frac{\delta^2|\psi|}{\delta q_{ij}\delta_{k\ell}} \tag{1.52}$$

The quantum potential changes the Hamiltonian constraint algebra to require weak closure (i.e. closure modulo the equations of motion); regularization and ordering are not considered here but will not affect the constraint algebra. The quantum Einstein equations are derived in the form

$$\mathfrak{G}^{ij} = -\frac{1}{N}\frac{\delta\mathfrak{Q}}{\delta q_{ij}};\ \mathfrak{G}^{0\mu} = \frac{\mathfrak{Q}}{2\sqrt{-g}}g^{0\mu} \tag{1.53}$$

The Bohmian HJ equation is

$$G_{ijk\ell}\frac{\delta S}{\delta q_{ij}}\frac{\delta S}{\delta q_{k\ell}} - \sqrt{q}\,({}^3\mathfrak{R} - \mathfrak{Q}) = 0 \tag{1.54}$$

where S is the phase of the WDW wave function and this leads to the same equations of motion (1.53). The modified Einstein equations are given in Bohmian form via

$$\mathfrak{G}^{ij} = -\kappa\mathfrak{T}^{ij} - \frac{1}{N}\frac{\delta(\mathfrak{Q}_G + \mathfrak{Q}_m)}{\delta g_{ij}};\ \mathfrak{G}^{0\mu} = -\kappa\mathfrak{T}^{0\mu} + \frac{\mathfrak{Q}_G + \mathfrak{Q}_m}{2\sqrt{-g}}g^{0\mu} \tag{1.55}$$

$$\mathfrak{Q}_m = \hbar^2\frac{N\sqrt{q}}{2}\frac{1}{|\psi|}\frac{\delta^2|\psi|}{\delta\phi^2};\ \mathfrak{Q}_G = \hbar^2 NqG_{ijk\ell}\frac{1}{|\psi|}\frac{\delta^2|\psi|}{\delta q_{ij};\delta q_{k\ell}} \tag{1.56}$$

$$Q_G = \int d^3x\mathfrak{Q}_G;\ Q_m = \int d^3x\mathfrak{Q}_m$$

In the third paper of [**876**] the Ashtekar variables are employed and it is shown that the Poisson bracket of the Hamiltonian with itself changes with respect to its classical counterpart but is still weakly equal to zero (modulo regularization, etc.); we refer to [**876**] and Section 4.3.1 for details.

1.3. GEOMETRY, GRAVITY, AND QM. We have already indicated some interaction of QM and geometry via Bohmian mechanics and remark here upon other aspects.

(1) It is known that one can develop a quantum geometry via Kähler geometry on a preHilbert space $P(H)$ (see e.g. [**54, 153, 188, 189, 203, 244, 245, 246, 247, 248**]). Thus $P(H)$ is a Kähler manifold with a Fubini-Study metric based on $(|d\psi_\perp>=|d\psi> - |\psi><\psi|d\psi>)$

$$\frac{1}{4}ds^2_{PS} = [cos^{-1}(|<\tilde{\psi}|\psi>|)]^2 \sim 1 - |<\tilde{\psi}|\psi>|^2 =<d\psi_\perp|d\psi_\perp> \tag{1.57}$$

where $ds^2_{PS} = \sum dp_j^2/p_j = \sum p_j(d\,log(p_j))^2$ gives the connection to probability distributions. We have already seen in Item 8 of Section 6.1.2 how this probability metric is related to Fisher information, fluctuations, and the quantum potential.

(2) There is a fascinating series of papers by Arias, Bonal, Cardenas, Gonzalez, Leyva, Martin, and Quiros dealing with general relativity (GR) and conformal variations (cf. Section 3.2.2). We omit details here but simply remark that conformal GR with $\hat{g}_{ab} = \Omega^2 g_{ab}$ is shown to be the only consistent formulation of gravity. Here consistent refers to invariance under the group of transformations of units of length, time, and mass.

(3) In Section 4.5.1 one goes into the Bohmian interpretation of quantum cosmology à la [**770, 772, 774, 961**] for example (cf. also [**123, 124, 571, 572, 573**]). Thus write $H = \int d^3x(N\mathfrak{H} + N^j\mathfrak{H}_j)$ where for GR with a scalar field

$$\mathfrak{H}_j = -2D_i\pi^i_j\pi_\phi\partial_j\phi;\ \mathfrak{H} = \kappa G_{ijk\ell}\pi^{ij}\pi^{k\ell} + \frac{1}{2}h^{-1/2}\pi^2_\phi + \tag{1.58}$$
$$+h^{1/2}\left[-\kappa^{-1}(R^{(3)} - 2\Lambda) + \frac{1}{2}h^{ij}\partial_i\phi\partial_j\phi + U(\phi)\right]$$

The canonical momentum is

$$\pi^{ij} = -h^{1/2}(K^{ij} - h^{ij}K) = G^{ijk\ell}(\dot{h}_{k\ell} - D_kN_\ell - D_\ell N_k); \tag{1.59}$$

$$K_{ij} = -\frac{1}{2N}(\dot{h}_{ij} - D_i N_j - D_j N_i)$$

K is the extrinsic curvature of the 3-D hypersurface Σ in question with indices lowered and raised via the surface metric h_{ij} and its inverse) and $\pi_\phi = (h^{1/2}/N)(\dot{\phi} - N^j \partial_j \phi)$ is the momentum of the scalar field (D_i is the covariant derivative on Σ). Recall also the deWitt metric

$$G_{ijk\ell} = \frac{1}{2}h^{-1/2}(h_{ik}h_{j\ell} + h_{i\ell}h_{jk} - h_{ij}h_{k\ell}) \quad (1.60)$$

The classical 4-metric and scalar field which satisfy the Einstein equations can be obtained from the Hamiltonian equations

$$ds^2 = -(N^2 - N^i N_i)dt^2 + 2N_i dx^i dt + h_{ij}dx^i dx^j; \quad (1.61)$$

$$\dot{h}_{ij} = \{h_{ij}, H\};\ \dot{\pi}^{ij} = \{\pi^{ij}, H\};\ \dot{\phi} = \{\phi, H\};\ \dot{\pi}_\phi = \{\pi_\phi, H\}$$

One has the standard constraint equations which when put in Bohmian form with $\psi = Aexp(iS/\hbar)$ become

(1.62)

$$-2h_{\ell i}D_j \frac{\delta S(h_{ij}\phi)}{\delta h_{\ell j}} + \frac{\delta S(h_{ij},\phi)}{\delta \phi}\partial_i \phi = 0;\ -2h_{\ell i}D_j \frac{\delta A(h_{ij},\phi)}{\delta h_{\ell j}} + \frac{\delta A(h_{ij},\phi)}{\delta \phi} = 0$$

These depend on the factor ordering but in any case will have the form

$$\kappa G_{ijk\ell}\frac{\delta S}{\delta h_{ij}}\frac{\delta S}{\delta h_{k\ell}} + \frac{1}{2}h^{-1/2}\left(\frac{\delta S}{\delta \phi}\right)^2 + V + Q = 0 \quad (1.63)$$

$$Q = -\frac{\hbar^2}{A}\left(\kappa G_{ijk\ell}\frac{\delta^2 A}{\delta h_{ij}\delta h_{k\ell}} + \frac{h^{-1/2}}{2}\frac{\delta^2 A}{\delta \phi^2}\right)$$

$$\kappa G_{ijk\ell}\frac{\delta}{\delta h_{ij}}\left(A^2 \frac{\delta S}{\delta h_{k\ell}}\right) + \frac{1}{2}h^{-1/2}\frac{\delta}{\delta \phi}\left(A^2 \frac{\delta S}{\delta \phi}\right) = 0 \quad (1.64)$$

Now in the dBB interpretation one has guidance relations

$$\pi^{ij} = \frac{\delta S(h_{ab},\phi)}{\delta h_{ij}};\ \pi_\phi = \frac{\delta S(h_{ij},\phi)}{\delta \phi} \quad (1.65)$$

One then develops the Bohmian theory and from (1.65) results

$$\dot{h}_{ij} = 2NG_{ijk\ell}\frac{\delta S}{\delta h_{k\ell}} + D_i N_j + D_j N_i;\ \dot{\phi} = Nh^{-1/2}\frac{\delta S}{\delta \phi} + N^i \partial_i \phi \quad (1.66)$$

The question posed now is to find what kind of structure arises from (1.66). The Hamiltonian is evidently $H_Q = \int d^3x \left[N(\mathfrak{H} + Q) + N^i \mathfrak{H}_i\right]$; $\mathfrak{H}_Q = \mathfrak{H} + Q$ and the first question is whether the evolution of the fields driven by H_Q forms a 4-geometry as in classical gravitational dynamics. Various situations are examined and (for Q of a specific form) sometimes the quantum geometry is consistent (i.e. independent of the choice of lapse and shift functions) and forms a nondegenerate 4-geometry (of Euclidean type). However it can also be consistent and not form a nondegenerate 4-geometry. In general, and always when the quantum potential is nonlocal, spacetime is broken and the evolving 3-geometries do not stick together to form a nondegenerate 4-geometry. These are very interesting results and mandate further study.

(4) Next (cf. Section 4.6) one goes to noncommutative (NC) theories folowing [**77, 78, 79, 772**]. First from [**77, 78**] one considers canonical comutation relations $[\hat{X}^\mu, \hat{X}^\nu] = i\theta^{\mu\nu}$ and develops a Bohmian theory for a noncommutative QM (NCQM) via a Moyal product

$$(f * g) = \frac{1}{(2\pi)^n}\int d^mkd^npe^{i(k_\mu+p_\mu)x^\mu-(1/2)k_\mu\theta^{\mu\nu}p_\nu}f(k)g(p) = \tag{1.67}$$

For $\theta^{0i} = 0$ one has a Hilbert space as in commutative QM with a NC SE

$$i\hbar\frac{\partial\psi(x^i,t)}{\partial t} = -\frac{\hbar^2}{2m}\nabla^2\psi(x^i,t) + V(x^i) * \psi(x^i,t) = \tag{1.68}$$

$$= \frac{\hbar^2}{2m}\nabla^2\psi(x^i,t) + V\left(x^j + i\frac{\theta^{jk}}{2}\partial_k\right)\psi(x^i,t)$$

The operators $\hat{X}^j = x^j + \frac{i\theta^{jk}\partial_k}{2}$ are the observables with canonical coordinates x^i and $\rho d^3x = |\psi|^2 d^3x$ is intepreted as the porbability that the system is in a region of volume d^3x around x at time t. One writes $\psi = Rexp(iS/\hbar)$ and there results

$$\frac{\partial S}{\partial t} + \frac{(\nabla S)^2}{2m} + V + V_{nc} + Q_K + Q_I = 0;\; V_{nc} = V\left(x^i - \frac{\theta^{ij}}{2\hbar}\partial_j S\right) - V(x^i); \tag{1.69}$$

$$Q_K = \Re\left(-\frac{\hbar^2}{2m}\frac{\nabla^2\psi}{\psi}\right) - \left(\frac{\hbar^2}{2m}(\nabla S)^2\right) = -\frac{\hbar^2}{2m}\frac{\nabla^2 R}{R};$$

$$Q_I = \Re\left(\frac{V[x^j + (i\theta^{jk}/2)\partial_k]\psi}{\psi}\right) - V\left(x^i - \frac{\theta^{ij}}{2\hbar}\partial_j S\right)$$

One arrives at a formal structure involving

$$\hat{X}^j = x^i + i\theta^{jk}\partial_k/2;\; X^i(t) = x^i(t) - (\theta^{ij}/2\hbar)\partial_j S(x^i(t),t); \tag{1.70}$$

$$\frac{dx^i(t)}{dt} = \left[\frac{\partial^i S(\vec{x},t)}{m} + \frac{\theta^{ij}}{2\hbar}\frac{\partial V(X^k)}{\partial X^j} + \frac{\mathfrak{Q}^i}{2}\right]\Bigg|_{x^i = x^i(t)}$$

One finds then

$$\frac{\partial\rho}{\partial t} + \frac{\partial(\rho\dot{x}^i}{\partial x^i} - \frac{\partial}{\partial x^i}\left[\rho\left(\frac{\theta^{ij}}{2\hbar}\frac{\partial V(X^k)}{\partial X^j} + \frac{\mathfrak{Q}^i}{2}\right)\right] + \Sigma_\theta = 0 \tag{1.71}$$

so for equivariance ($\rho = |\psi|^2$) it is necessary that the sum of the last two terms in (1.71) vanish and when $V(X^i)$ is linear or quadratic this holds. In [**77, 78**] one also looks at NC theory in Kantowski-Sachs (KS) universes and at Friedman-Robertson-Walker (FRW) universes with a conformally coupled scalar field. Many specific situations are examined, especially in a minisuperspace context.

(5) Next (cf. Item 3 in Section 6.1.1 and Item 8 in Section 6.1.2) we consider [**449, 444**]. One gives a new derivation of the SE via the exact uncertainty principle and a formula

$$\tilde{H}_q[P,S] = \tilde{H}_c[P,S] + C\int dx \frac{\nabla P\cdot\nabla P}{2mP} \tag{1.72}$$

for the quantum situation. Consider then the gravitational framework

$$ds^2 = -(N^2 - h^{ij}N_iN_j)dt^2 + 2N_i dx^i dt + h_{ij}dx^i dx^j \tag{1.73}$$

One introduces fluctuations via $\pi^{ij} = (\delta S/\delta h_{ij}) + f^{ij}$ and arrives at a WDW equation

$$\left[-\frac{\hbar^2}{2}\frac{\delta}{\delta h_{ij}}G_{ijk\ell}\frac{\delta}{\delta h_{k\ell}} + V\right]\Psi = 0 \tag{1.74}$$

Note that an operator ordering is implicit and thus ordering ambiguities do not arise (similarly for quantum particle motion). The work here in [**449, 444**] is significant and very interesting; it is developed in some detail in Section 4.7.

(6) In Section 4.1 we followed work of M. Israelit and N. Rosen on Dirac-Weyl geometry (see in particular [**498, 499, 817**]). Recall that in Weyl geometry $g_{\mu\nu} \to \tilde{g}_{\mu\nu} = exp(2\lambda)g_{\mu\nu}$ is a gauge transformation and for a vector $\vec{B}$ of length B one has $dB = Bw_\nu dx^\nu$ where $w_\nu \sim \phi_\nu$ is the Weyl vector. The Weyl connection coefficients are

$$\Delta B^\lambda = B^\sigma K^\lambda_{\sigma\mu\nu}dx^\mu\delta x^\nu;\ \Delta B = BW_{\mu\nu}dx^\mu\delta x^\nu \tag{1.75}$$

and under a gauge transformation $w_\mu \to \bar{w}_\mu = w_\mu + \partial_\mu\lambda$. One writes $W_{\mu\nu} = w_{\mu,\nu} - w_{\nu,\mu}$ (where commas denote partial derivatives). The Dirac-Weyl action here is given via a field β ($\beta \to \bar{\beta} = exp(-\lambda)\beta$ under a gauge transformation) in the form

$$I = \int [W^{\lambda\sigma}W_{\lambda\sigma} - \beta^2 R + \beta^2(k-6)w^\sigma w_\sigma + 2(k-6)\beta w^\sigma\beta_{,\sigma} + \tag{1.76}$$

$$+k\beta_{,\underline{\sigma}}\beta_{,\sigma} + 2\Lambda\beta^4 + L_M]\sqrt{-g}d^4x$$

Note here the difference in appearance from (1.37) or the Dirac form in Appendix D; these are all equivalent after suitable adjustment (cf. Remark 6.1). Under parallel transport $\Delta B = BW_{\mu\nu}dx^\mu\delta x^\nu$ so one takes $W_{\mu\nu} = 0$ via $w_\nu = \partial_\nu w$ and we have what is called an integrable Weyl geometry with generating elements $(g_{\mu\nu}, w, \beta)$. Further set $b_\mu = \partial_\mu(log(\beta)) = \beta_\mu/\beta$ and use a modified Weyl connection vector $W_\mu = w_\mu + b_\mu$. Then varying (1.76) in w and $g_{\mu\nu}$ gives

$$2(\kappa\beta^2 W^\nu)_{;\nu} = S;\ G^\nu_\mu = -8\pi\frac{T^\nu_\mu}{\beta^2} + \tag{1.77}$$

$$+16\pi\kappa\left(W^\nu W_\mu - \frac{1}{2}\delta^\nu_\mu W^\sigma W_\sigma\right) + 2(\delta^\nu_\mu b^\sigma_{;\sigma} - b^\nu_{;\mu}) + 2b^\nu b_\mu + \delta^\nu_\mu b^\sigma_\sigma - \delta^\nu_\mu\beta^2\Lambda$$

where S is the Weyl scalar charge $16\pi S = \delta L_M/\delta w$, G^ν_μ is the Einstein tensor, and the energy momentum tensor of ordinary matter is

$$8\pi\sqrt{-g}T^{\mu\nu} = \delta(\sqrt{-g}L_M)/\delta g_{\mu\nu} \tag{1.78}$$

Finally variation in β gives an equation for he β field

$$R + k(b^\sigma_{;\sigma} + b^\sigma b_\sigma) = 16\pi\kappa(w^\sigma w_\sigma - w^\sigma_{;\sigma}) + 4\beta^2\Lambda + 8\pi\beta^{-1}B \tag{1.79}$$

(here $16\pi B = \delta L_M/\delta\beta$ is the Dirac charge conjugate to β). Note

$$\delta I_M = 8\pi\int(T^{\mu\nu}\delta g_{\mu\nu} + 2S\delta w + 2B\delta\beta)\sqrt{-g}d^4x \tag{1.80}$$

yielding the energy momentum relation $T^\lambda_{\mu;\lambda} - Sw_\mu - \beta Bb_\mu = 0$. Actually via (1.77) with $S + T = \beta B$ one obtains again (1.79) which is seen therefore as a corollary and not an independent equation. One derives now conservation laws etc. and following [**817**] produces an equation of motion for a test particle. Thus consider matter made up on identical particles of rest mass m and Weyl scalar charge q_s, being in the stage of a pressureless gas so $T^{\mu\nu} = \rho U^\mu U^\nu$ where U^ν is the 4-velocity and note also $T^\lambda_{\mu;\lambda} - Tb_\mu = SW_\mu$. Then one arrives at

$$\frac{dU^\mu}{ds} + \left\{ \begin{array}{c} \mu \\ \lambda\ \sigma \end{array} \right\} U^\lambda U^\sigma = \left(b_\lambda + \frac{q_s}{m}W_\lambda\right)(g^{\mu\lambda} - U^\mu U^\lambda) \tag{1.81}$$

Further a number of illustrations are worked out involving the creation of mass like objects from Weyl-Dirac geometry, in a FRW universe for example (i.e. an external observer sitting in Riemannian spacetime would recognize the object as massive). Cosmological models are also constructed with the Weyl field serving to create matter. The treatment is extensive and profound.

REMARK 7.1.1. We have encountered Dirac-Weyl-Bohm (DWB) in Section 2.1 (Section 7.1.2, Item 5) and Dirac-Weyl geometry in Section 4.1 and Appendix D (Section 7.1.3, Item 6). The formulations are somewhat different and we try now to compare certain features. In Section 7.1.2 one has

$$\mathfrak{A} = \int d^4x\sqrt{-g}(F_{\mu\nu}F^{\mu\nu} - \beta^2\ {}^W\mathcal{R} + (\sigma + 6)\beta_{;\mu}\beta^{;\mu} + \mathfrak{L}_{matter} \tag{1.82}$$

for the action; in Section 7.1.3 the action is

$$I = \int[W^{\lambda\sigma}W_{\lambda\sigma} - \beta^2 R + \beta^2(k-6)w^\sigma w_\sigma + 2(k-6)\beta w^\sigma\beta_{,\sigma} + \tag{1.83}$$

$$+k\beta_{,\underline{\sigma}}\beta_{,\sigma} + 2\Lambda\beta^4 + L_M]\sqrt{-g}d^4x$$

and in Appendix D we have (for the simplest vacuum equations and $\mathfrak{g} = \sqrt{-g}$)

$$I = \int[(1/4)F_{\mu\nu}F^{\mu\nu} - \beta^2 R + 6\beta^\mu\beta_\mu + c\beta^4]\mathfrak{g}d^4x \tag{1.84}$$

We recall the idea of co-covariant derivative from Appendix D where for a scalar S of Weyl power n one has $S_{*\mu} = S_{;\mu} = nw_\mu S \equiv S_\mu - nw_\mu S$ $(S_\mu = \partial_\mu S)$ and I in

(1.84) is originally

$$I = \int [(1/4)F_{\mu\nu}F^{\mu\nu} - \beta^{2*}R + k\beta^{*\mu}\beta_{*\mu} + c\beta^4]\mathfrak{g}d^4x \tag{1.85}$$

However β is a co-scalar of weight -1 and $\beta^{*\mu}\beta_{*\mu} = (\beta^\mu + \beta w^\mu)(\beta_\mu + \beta w_\mu)$ so using

$$-\beta^{2*}R + k\beta^{*\mu}\beta_{*\mu} = -\beta^2 R + k\beta^\mu\beta_\mu + (k-6)\beta^2\kappa^\mu\kappa_\mu + \tag{1.86}$$

$$+6(\beta^2\kappa^\mu)_{:\mu} + (2k-12)\beta\kappa^\mu\beta_\mu$$

one obtains (1.84) for $k = 6$. Further recall that $W_\mu = w_\mu + \partial_\mu log(\beta)$ so $W_{\mu\nu} = F_{\mu\nu}$ and c in (1.84) corresponds to Λ in (1.83). The notation ${}^W\mathcal{R}$ in (1.82) is the same as R and $\beta_{;\mu} \sim \beta_\mu = \partial_\mu\beta$ so $(\sigma+6) = 6$ provides a complete identification (modulo matter terms to be added in (1.84)); note $\sigma = k-6$ from [**817**]. Now in [**872, 873**] one takes a Dirac-Weyl action of the form (1.82) and relates it to a Bohmian theory as in Section 3.2.1. The same arguments hold also for $\sigma = 0$ here with

$$\beta \sim \mathfrak{M};\ \frac{8\pi\mathfrak{T}}{R} \sim m^2;\ \alpha = \frac{\hbar^2}{m^2c^2} \sim -\frac{6}{R};\ \nabla_\nu\mathfrak{T}^{\mu\nu} - \mathfrak{T}\frac{\nabla^\mu\beta}{\beta} = 0 \tag{1.87}$$

Note also $16\pi\mathfrak{T} = \beta\psi$ where $\psi = \delta L_M/\delta\beta \sim 16\pi B$ so $B \sim \mathfrak{T}/\beta$. One assumes here that $16\pi J_\mu = \delta L_M/\delta w_\mu = 0$ where $w_\mu \sim \phi_\mu$.

REMARK 7.1.2. We recall from Section 3.2 that $\beta \sim \mathfrak{M}$ and $\mathfrak{M}^2/m^2 = exp(\mathfrak{Q})$ is a conformal factor. Further for $\beta_0 \to \beta_0 exp(-\Xi(x))$ one has $w_\mu \to w_\mu + \partial_\mu\Xi$ where $-\Xi = log(\beta/\beta_0)$ showing an interplay between mass and geometry. Recall also the relation $\nabla_\mu(\beta w^\mu + \beta\nabla^\mu\beta) = 0$. This indicates a number of connections between the quantum potential, geometry, and mass. Hence virtually any results in Dirac-Weyl theory models will involve the quantum potential. This is made explicit in Section 3.2 from [**873**] and could be developed for the examples and theory from [**499**] once wave functions and Bohmian ideas are inserted.

1.4. GEOMETRIC PHASES. We go now to [**283**] for some remarks on geometric phase and the quantum potential. One refers back here to geometric phases of Berry [**108**] and Levy-Leblond [**603**] for example where the latter shows that when a quanton propagates through a tube, within which it is confined by impenetrable walls, it acquires a phase when it comes out of the tube. Thus consider a tube with square section of side a and length L. Before entering the tube the quanton's wave function is $\phi = exp(ipx/\hbar)$ where p is the initial momentum. In the tube the wave function has the form

$$\psi = Sin\left(n_x\pi\frac{x}{a}\right)Sin\left(n_y\pi\frac{y}{a}\right)exp(ip'x/\hbar) \tag{1.88}$$

with appropriate transverse boundary conditions. After entering the tube the energy E of the quanton is unchanged but satisfies

$$E = \frac{(p')^2}{2m}(n_x^2 + n_y^2)\frac{\pi^2}{2ma} \tag{1.89}$$

For the simplest case $n_x = n_y = 1$ it was found that after the quanton left the tube there was an additional phase

$$\Delta\Phi = \frac{\pi^2\hbar^2}{pa^2}L \tag{1.90}$$

Subsequently Kastner [**539**] related this to the quantum potential that arises in the tube. Thus let the wave function in the tube be $Rexp[(iS/\hbar)+(ipx/\hbar)]$ in polar form. The eventual changes in the phase of the wave function, due to the tube, are now concentrated in S. In order to single out the influence of the tube on the wave function write $\psi_1 = \psi exp(ipx/\hbar)$ and the quantum potential corresponding to ψ is then

$$Q = -\frac{\hbar^2}{2m}\frac{\Delta R}{R} = \frac{\pi^2\hbar^2}{ma^2} \tag{1.91}$$

Now turn to the laws of parallel transport where for the Berry phase the law of parallel transport for the wave function is (cf. [**892**])

$$\Im < \psi|\dot{\psi} >= 0 \tag{1.92}$$

For the Levy-Leblond phase the law of parallel transport is given by

$$\Im < \psi|\dot{\psi} >= -\frac{1}{\hbar}Q|\psi|^2 \tag{1.93}$$

In this approach the wave function acquires an additional phase after the quanton has left the tube in the form

$$\psi(t+\Delta t) = exp(-iQ\Delta t/\hbar)\psi(t) \tag{1.94}$$

which after expansion in Δt leads to the law of parallel transport in (1.93). Indeed

$$Q\Delta t = \frac{\pi^2\hbar^2}{ma^2}\Delta t = \frac{\pi^2\hbar^2}{ma^2}\frac{mL}{p} = \frac{\pi^2\hbar^2}{pa^2}L = \Delta\Phi \tag{1.95}$$

If we use the polar form for the wave function (1.93) gives $(\partial S/\partial t) = -Q$ and this means that this new law of parallel transport eliminates the quantum potential from the quantum HJ equation. The whole quantum information is now carried by the phase of the wave function. One can see that the nature of this phase is quite different from Berry's phase; it is related to the presence or not of constraints in the system (in this case the tube).

Now consider a quite different type of constrained system where again a new geometric phase will arise. Look at a quantum particle constrained to move on a circle. The wave function has the form $\psi \sim sin(ns/\rho_0)$ where ρ_0 is the radius of the circle and s is the arc length with origin at a tangent point. Then the wave function will have a node at this tangent point. For a circle the value $n = 1/2$ is also allowed (cf. [**467, 570**]) and the corresponding quantum potential for $n = 1/2$ is

$$Q = \frac{h^2}{8m\rho_0^2} \tag{1.96}$$

which is exactly equal to the constant E_0 appearing in the Hamiltonian for a particle on a circle with radius ρ_0 following the Dirac quantization procedure for constrained systems (cf. [**846**]). The phase which a quanton would acquire traveling along the circle is then

$$Q\frac{2\pi\rho_0 m}{p} = \frac{\pi\hbar^2}{4\rho_0 p} \tag{1.97}$$

Note that if the circle becomes very small then the geometrical phase can not get bigger than $\sim (\hbar/m)$. This limit is imposed by the Heisenberg uncertainty relation $\rho_0 p \sim \hbar$. This is not the case for the Levy-Leblond phase which can get very large provided $L >> a$.

1.5. ENTROPY AND CHAOS. Connections of the quantum potential to Fisher information have already been recalled in e.g. Section 7.1.1 and we recall here from Chapter 6 a few matters.

(1) An extensive discussion relating Fisher information (as a "mother" information) to various forms of entropy is developed in Chapter 6 and this gives implicitly at least many relations between entropy, kinetic theory, uncertainty, and the quantum potential.
(2) A particular result of interest in Section 6.2.1 shows how the quantum potential acts as a constraining force to prevent deterministic chaos.

2. HYDRODYNAMICS AND GEOMETRY

We mentioned briefly some hydrodynamical aspects of the SE in Sections 1.1 and 1.3.2 and return to that now following [**294**]. Here one wants to limit the role of statistics and measurement to unveil some geometric features of the so called Madelung approach. Thus, with some repetition from Section 1.1, consider a SE $(\hbar/i)\psi_t + H(x, (\hbar/i)\nabla)\psi = 0$ with $\psi = Rexp(iS/\hbar)$ to arrive at

$$\frac{\partial S}{\partial t} + H(x, \nabla S) - \frac{\hbar^2}{2m}\frac{\Delta R}{R} = 0;\ \frac{\partial P}{\partial t} + \frac{\partial}{\partial x^i}(P\dot{x}^i) = 0;\ \dot{x}^i = \left[\frac{\partial H}{\partial p_i}\right]_{p=\nabla S} \tag{2.1}$$

(where $P = R^2$), and Madelung equations of the form (cf. (1.5))

$$\frac{\partial S}{\partial t} + H(x, \nabla S) - \frac{\hbar^2}{2m}\frac{\Delta\sqrt{\rho}}{\sqrt{\rho}} = 0;\ \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x^i}(\rho\dot{x}^i) = 0 \tag{2.2}$$

where, in a continuum picture, $\rho = mP$ is the mass density of an extended particle whose shape is dictated by P. Setting $v^i = \dot{x}^i$ one has then an Euler equation of the form

$$\frac{\partial}{\partial t}(\rho v^i) + \frac{\partial}{\partial x^k}(\rho v^i v^k) = -\frac{\rho}{m}\frac{\partial V}{\partial x^i} + \frac{\partial}{\partial x^k}\tau^{ik} \tag{2.3}$$

Following [**924**] one has expressed the quantum force term here as the divergence of a symmetric "quantum stress" tensor

$$\tau_{ij} = \left(\frac{\hbar}{2m}\right)^2 \rho\frac{\partial^2(log(\rho))}{\partial x^i \partial x^j};\ p_i = \frac{\hbar^2}{2m^2}\rho\frac{\partial^2\sigma}{\partial(x^i)^2} \tag{2.4}$$

where p_i denotes diagonal elements or principal stresses expressed in normal coordinates (with $\rho = exp(2\sigma)$). The stress p_i is tension like (resp. pressure like) if

$p_i > 0$ (resp. $p_i < 0$) and the mean pressure is

$$\bar{p} = -\frac{1}{3}Tr(\tau_{ij}) = -\frac{\hbar^2}{6m^2}\rho\Delta\sigma \tag{2.5}$$

In classical hydrodynamics negative pressures are often associated with cavitation which involves the formation of topological defects in the form of bubbles. For an ideal fluid one would need $\tau_{ij} = -\bar{p}\delta_{oj}$ and this occurs if and only if the mass density is Gaussian $\sigma \propto -x^i x_i$ in which case $\bar{p} \propto (\hbar^2/2m)\rho$. Generally the stress tensor will not be isotropic, and not an ideal fluid; moreover if one had a viscous fluid one would expect τ_{ij} to be coupled to the rate of deformation tensor (derived from $D\mathbf{v}$). Since this does not occur one does not call this form of matter a fluid but rather a Madelung continuum, corresponding to something like an inviscid fluid which also supports shear stresses, whereas the Gaussian wave packet of QM corresponds to an ideal compressible irrotational fluid medium.

If now one adds time as the zeroth coordinate and extends the velocity vector by $v^0 = 1$ then, defining the energy momentum tensor as

$$\mathfrak{T}^{\mu\nu} = \rho\left[v^\mu v^\nu - \left(\frac{\hbar}{2m}\right)^2 \frac{\partial^2(log(\rho))}{\partial x^\mu \partial x^\nu}\right] \tag{2.6}$$

then the Euler and continuity equations can be combined in the form $\partial\mathfrak{T}^{\mu\nu}/\partial x^\mu = -(\rho/m)\partial^\nu V$; this is somewhat misleading since it is based on a nonrelativistic approach but it leads now to the relativistic theory. First start with the KG equation

$$\left[-\hbar^2\eta^{\mu\nu}\frac{\partial^2}{\partial x^\mu \partial x^\nu} + m_0^2c^2\right]\psi = 0 \tag{2.7}$$

For $\psi = Rexp(iS/\hbar)$ one gets now

$$\eta^{\mu\nu}\frac{\partial S}{\partial x^\mu}\frac{\partial S}{\partial x^\nu} + m_0^2c^2 - \hbar^2\frac{\Box R}{R} = 0;\ \eta^{\mu\nu}\frac{\partial}{\partial x^\mu}\left(P\frac{\partial S}{\partial x^\nu}\right) = 0 \tag{2.8}$$

Define now the 4-velocity, rest mass energy, energy momentum, and stress tensor via

$$u_\mu = \frac{1}{m_0}\frac{\partial S}{\partial x^\mu};\ \rho = m_0 P;\ p_\mu = \rho u_\mu; \tag{2.9}$$

$$\tau_{\mu\nu} = \left(\frac{\hbar}{2m_0}\right)^2\frac{\partial^2(log(\rho))}{\partial x^\mu \partial x^\nu};\ \mathfrak{T}_{\mu\nu} = \rho[u_\mu u_\nu + \tau_{\mu\nu}]$$

to arrive at relativistic equations for the medium described by ρ and u_μ in the form (♣) $\partial^\mu\mathfrak{T}_{\mu\nu} = 0$ and $\partial^\mu p_\mu = 0$ (the second equation is an incompressibility equation and this does not contradict the nonrelativistic compressibility of the medium since in relativity incompressibility in fluid media is equivalant to an infinite speed of light corresponding to rigidity in solid media).

One then erects an elegant mathematical framework involving spacetime foliations related to the complex character of ψ (see also the second paper in [**294**] on foliated cobordism, etc.). This is lovely but rather too abstract for the style of this book so we will not try to reproduce it here; we can however skip to some calculations involving the geometric origin of the quantum potential. Thus consider the consequences of choosing a scale of unit norm via the function $\sqrt{\rho}$. One takes a conformally related metric $\bar{g} = \Omega^2 g$ on a manifold M (where $\Omega^2 > 0$) and, writing $\Omega = exp(\sigma)$, one obtains the following formulas for the Levi-Civita connection, Ricci curvature, and scalar curvature

$$\bar{\Gamma}^i_{jk} = \Gamma^i_{jk} = \delta^i_j \partial_k \sigma + \delta^i_k \partial_j \sigma - g_{jk} g^{i\ell} \partial_\ell \sigma; \tag{2.10}$$

$$\bar{R}_{ij} = R_{ij} - (n-2)\sigma_{ij} - [\Delta\sigma + (n-2)(\partial^k \sigma \partial_k \sigma)] g_{ij}$$

$$\bar{R} = e^{-2\sigma}[R - 2(n-1)\Delta\sigma - (n-1)(n-2)(\partial^i \sigma \partial_i \sigma)]$$

where $\sigma_{ij} = \partial_i \partial_j \sigma - \partial_j \sigma \partial_i \sigma$. Now if the constant m is replaced by the function ρ then one must contend with the derivatives $\partial_i \rho$ and for $\rho = exp(2\sigma)$ the Minkowski metric will be deformed from $g = \eta,\ \Gamma^i_{jk} = 0,\ R_{ij} = R = 0$ to

$$\bar{\Gamma}^i_{jk} = \delta^i_j \partial_k \sigma + \delta^i_k \partial_j \sigma - \eta_{jk} \eta^{i\ell} \partial_\ell \sigma;\ \bar{R}_{ij} = -2\sigma_{ij} - [\Box\sigma + 2(\partial^k \sigma \partial_k \sigma)]\eta_{ij} \tag{2.11}$$

$$\bar{R} = -6e^{-2\sigma}[\Box\sigma + (\partial^i \sigma \partial_i \sigma]$$

Putting Ω back into the equation for scalar curvature one obtains

$$\bar{R} = -\frac{6}{\Omega^2}\left(\frac{\Box\Omega}{\Omega}\right);\ \frac{\Box\sqrt{\rho}}{\sqrt{\rho}} = \frac{\Box\Omega}{\Omega} = -\frac{1}{6}\bar{R}\Omega^2 = -\frac{1}{6}\rho\bar{R} \tag{2.12}$$

This identifies the quantum potential as a mass density times a scalar curvature and resembles some results obtained earlier from [**840, 841**] for example (cf. Section 3.3 and 3.3.2). One has also

$$\partial^i \bar{R}_{ij} = \partial^i \left(\frac{1}{2} g_{ij} \bar{R}\right) = \frac{1}{2}\partial^j \bar{R} \Rightarrow \partial^i \bar{R}_{ij} = -3\partial^i \tau_{ij} \tag{2.13}$$

so the Takabayashi stress tensor differs from the Ricci curvature only by a term of vanishing divergence. Hence there is no loss of generality in using the Ricci curvature of $g = \rho\eta$ as the stress tensor since both define the same force field; this means in particular that one is dealing with principal curvatures instead of principal stresses. To extend all this to a more general Lorentz manifold one notes that under a conformal change of spacetime metric to the energy metric the Einstein tensor becomes

$$\bar{G}_{\mu\nu} = \bar{R}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{R} = G_{\mu\nu} - 2\sigma_{\mu\nu} + [2\Box\sigma + \partial^\lambda \sigma \partial_\lambda \sigma] g_{\mu\nu} \tag{2.14}$$

Thus it seems to assume that this implies a quantum correction to the Einstein equation

$$G_{\mu\nu} + \left(\Box\sigma + \frac{1}{2}\partial^\lambda \sigma \partial_\lambda \sigma\right) g_{\mu\nu} = 8\pi G T_{\mu\nu} + 2\sigma_{\mu\nu} \tag{2.15}$$

2.1. PARTICLE AND WAVE PICTURES. An interesting discussion of hydrodynamic features of QM, electrodynamics, and Bohmian mechanics appears in [**474**] and we will sketch more or less thoroughly a few ideas here. Thus a hydrodynamic model of QM provides an interpretation of two pictures, wave mechanical (Eulerian) and particle (Lagrangian), and the two versions of QM have associated Hamiltonian formulations that are connected by a canonical transformation. This gives a new and precise meaning to the notion of wave-particle duality. However it is necessary to distinguish the dBB corpuscle from a fluid particle. Consider a fluid as a continuum of particles with history encoded in the position variables $q(a,t)$ where each particle is distinguished by a continuous vector label a. The motion is continuous in that the mapping from a-space to q-space is single valued and differentiable with inverse $a(q,t)$. Let $\rho_0(a)$ be the initial quantum probability density with $\int \rho_0(a)d^3a = 1$. Introduce a mass parameter m so that the mass of an elementary volume d^3a attached to the point a is given by $m\rho_0(a)d^3a$. Note $\int m\rho_0(a)d^3a = m$ so this is a total mass of the system. The conservation of mass of a fluid element in the course of its motion is

$$m\rho(q(a,t))d^3q(a,t) = m\rho_0(a)d^3a;\ \ \rho(a,t) = J^{-1}(a,t)\rho_0(a); \tag{2.16}$$

$$J = \frac{1}{3!}\epsilon_{ijk}\epsilon_{\ell mn}\frac{\partial q_i}{\partial a_\ell}\frac{\partial q_j}{\partial a_m}\frac{\partial q_k}{\partial a_n}$$

J is the Jacobian of the transformation between the two sets of coordinates and ϵ_{ijk} is the completely antisymmetric tensor with $\epsilon_{123} = 1$. Let V be the potential of an external classical body force and U the internal potential energy of the fluid due to interparticle interactions. Here assume U depends on $\rho(q)$ and its first derivatives and hence via (2.16) on the second order derivatives of q with respect to a. The Lagrangian is then (with integrand $\ell = [\cdots]$)

$$L[q,q_t,t] = \int\left[\frac{1}{2}m\rho_0(a)\left(\frac{\partial q(a,t)}{\partial t}\right)^2 - \rho_0(a)U(\rho) - \rho_0(a)V(q(a))\right]d^3a \tag{2.17}$$

(one substitutes for ρ from (2.16)). It is the action of the conservative force derived from U on the trajectories that represents the quantum effects here. They are characterized by the following choice for U, motivated by the known Eulerian expression for internal energy,

$$U = \frac{\hbar^2}{8m}\frac{1}{\rho^2}\frac{\partial\rho}{\partial q_i}\frac{\partial\rho}{\partial q_i} = \frac{\hbar^2}{8m}\frac{1}{\rho_0^2}J_{ij}J_{ik}\frac{\partial}{\partial a_j}\left(\frac{\rho_0}{J}\right)\frac{\partial}{\partial a_k}\left(\frac{\rho_0}{J}\right); \tag{2.18}$$

$$\frac{\partial}{\partial q_i} = J^{-1}J_{ij}\frac{\partial}{\partial a_j};\ \ J_{i\ell} = \frac{\partial J}{\partial(\partial q_i/\partial a_\ell)} = \frac{1}{2}\epsilon_{ijk}\epsilon_{\ell mn}\frac{\partial q_j}{\partial a_m}\frac{\partial q_k}{\partial a_n};\ \ \frac{\partial q_k}{\partial a_j}J_{ki} = J\delta_{ij}$$

Thus $J_{i\ell}$ is the cofactor of $\partial q_i/\partial a_\ell$. The interaction in the quantum case is not conceptually different from classical fluid dynamics but differs in that the order of derivative coupling of the particles is higher than in a classical equation of state. The Euler-Lagrange equations for the coordinates are

$$\frac{\partial}{\partial t}\frac{\partial L}{\partial(\partial q_i(a,t)/\partial t)} - \frac{\delta L}{\delta q_i(a)} = 0; \tag{2.19}$$

$$\frac{\delta L}{\delta q_i}=\frac{\partial \ell}{\partial q_i}-\frac{\partial}{\partial a_j}\frac{\partial \ell}{\partial(\partial q_i/\partial a_j)}+\frac{\partial^2}{\partial a_j\partial a_k}\frac{\partial \ell}{\partial(\partial^2 q_i/\partial a_j\partial a_k)}$$

which yield

$$m\rho_0(a)\frac{\partial^2 q_i(a)}{\partial t^2}=-\rho_0(a)\frac{\partial V}{\partial q_i}-\frac{\partial W_{ij}}{\partial a_j}; \tag{2.20}$$

$$W_{ij}=-\rho_0(a)\frac{\partial U}{\partial(\partial q_i/\partial a_j)}+\frac{\partial}{\partial a_k}\left(\rho_0(a)\frac{\partial U}{\partial(\partial^2 q_i/\partial a_j\partial a_k)}\right)$$

This has the form of Newton's second law and instead of giving an explicit form for W_{ij} one uses a more useful tensor σ_{ij} defined via $W_{ik}=J_{jk}\sigma_{ij}$ where σ_{ij} is the analogue of the classical pressure tensor $p\delta_{ij}$. Using (2.18) one can invert to obtain

(2.21)

$$\sigma_{ij}=J^{-1}W_{ik}\frac{\partial q_j}{\partial a_k}=\frac{\hbar^2}{4mJ^3}J_{ik}\left[\rho_0^{-1}J_{j\ell}\frac{\partial\rho_0}{\partial a_k}\frac{\partial\rho_0}{\partial a_\ell}+(J^{-1}J_{j\ell}J_{mn}-J_{jm}J_{\ell n})\times\right.$$

$$\times\frac{\partial\rho_0}{\partial a_\ell}\frac{\partial^2 q_m}{\partial a_k\partial a_n}-J_{j\ell}\frac{\partial^2\rho_0}{\partial a_k\partial a_\ell}+\rho_0(J^{-1}J_{mn}J_{jr\ell s}+J^{-1}J_{j\ell}J_{mrns}-2J^{-2}J_{j\ell}J_{mn}J_{rs})\times$$

$$\left.\times\frac{\partial^2 q_r}{\partial a_k\partial a_s}\frac{\partial^2 q_m}{\partial a_\ell\partial a_n}+\rho_0J^{-1}J_{j\ell}J_{mn}\frac{\partial^3 q_m}{\partial a_k\partial a_\ell\partial a_n}\right];$$

$$J_{jm\ell n}=\frac{\partial J_{j\ell}}{\partial(\partial q_m/\partial a_n)}=\epsilon_{jmk}\epsilon_{\ell nr}\frac{\partial q_k}{\partial a_r}$$

One checks that σ_{ij} is symmetric and the equation of motion of the a^{th} particle moving in the field of the other particles and the external force is then

$$m\rho_0(a)\frac{\partial^2 q_i(a)}{\partial t^2}=-\rho_0(a)\frac{\partial V}{\partial q_i}-J_{kj}\frac{\partial\sigma_{ik}}{\partial a_j};\ \ \frac{\partial J_{ij}}{\partial a_j}=0 \tag{2.22}$$

(the latter equation is an identity used in the calculation). The result in (2.22) is the principal equation for the quantum Lagrangian method; its solutions, subject to specification of $\partial q_{i0}/\partial t$, lead to solutions of the SE. Multiplying by $\partial q_i/\partial a_k$ one obtains the Lagrangian form

$$m\rho_0(a)\frac{\partial^2 q_i(a)}{\partial t^2}\frac{\partial q_i}{\partial a_k}=-\rho_0(a)\frac{\partial V}{\partial a_k}-\frac{\partial q_i}{\partial a_k}J_{kj}\frac{\partial\sigma_{ik}}{\partial a_j}; \tag{2.23}$$

$$p_i(a)=\frac{\partial L}{\partial(\partial q_i(a)/\partial t)}=m\rho_0(a)\frac{\partial q_i(a)}{\partial t}$$

A Hamiltonian form can also be obtained via the canonical field momenta $p_i(a)=\partial L/\partial(\partial q_i(a)/\partial t)=m\rho_0(a)(\partial q_i(a)/\partial t)$ with

(2.24)

$$H=\int p_i(a)\frac{\partial q_i(a)}{\partial t}d^3a-L=\int\left[\frac{p(a)^3}{2m\rho_0(a)}+\rho_0(a)U(J^{-1}\rho_0)+\rho_0(a)V(q(a))\right]d^3a$$

Hamilton's equations via Poisson brackets $\{q_i(a),q_j(a')\}=\{p_i(a),p_j(a')\}=0$ and $\{q_i(a),p_j(a')\}=\delta_{ij}(a-a')$ are $\partial_t q_i(a)=\delta H/\delta p_i(a)$ and $\partial_t p_i(a)=-\delta H/\delta q_i(a)$ which when combined reproduce (2.22). Now to obtain a flow that is representative of QM one restricts the initial conditions for (2.22) to something corresponding to quasi-potential flow which means (♦) $\partial q_{i0}/\partial t = (1/m)(\partial S_0(a)/\partial a_i)$. However the flow is not irrotational everywhere because the potential $S_0(a)$ obeys the quantization condition

$$\oint_C \frac{\partial q_{i0}(a)}{\partial t} da_i = \oint_C \frac{1}{m}\frac{\partial S_0(a)}{\partial a_i} da_i = \frac{n\hbar}{m} \ \ (n \in \mathbf{Z}) \tag{2.25}$$

where C is a closed curve composed of material paricles. If it exists vorticity occurs in nodal regions (where the density vanishes) and it is assumed that C passes through a region of good fluid where $\rho_0 \neq 0$. To show that these assumptions imply motion characteristic of QM one demonstrates that they are preserved by the dynamical system. One first puts (2.23) into a more convenient form. Thus, using (2.16), the stress tensor (2.21) takes a simpler form

$$\sigma_{ij} = \frac{\hbar^2}{4m}\left(\frac{1}{\rho}\frac{\partial\rho}{\partial q_i}\frac{\partial\rho}{\partial q_j} - \frac{\partial^2\rho}{\partial q_i\partial q_j}\right) \tag{2.26}$$

Using

$$\frac{1}{\rho}\frac{\partial\sigma_{ij}}{\partial q_j} = \frac{\partial V_Q}{\partial q_i};\ V_Q = \frac{\hbar^2}{4m\rho}\left(\frac{1}{2\rho}\frac{\partial\rho}{\partial q_i}\frac{\partial\rho}{\partial q_i} - \frac{\partial^2\rho}{\partial q_i\partial q_i}\right) \tag{2.27}$$

(note V_Q is the dBB quantum potential) one sees that (2.23) can also be simplified as

$$m\frac{\partial^2 q_i}{\partial t^2}\frac{\partial q_i}{\partial a_k} = \frac{\partial}{\partial a_k}(V + V_Q) \tag{2.28}$$

Now integrate this equation between limits $(0,t)$ to get

$$m\frac{\partial q_i}{\partial t}\frac{\partial q_i}{\partial a_k} = m\frac{\partial q_{i0}}{\partial t} + \frac{\partial\chi(a,t)}{\partial a_k};\ \chi = \int_0^t\left(\frac{1}{2}m\left(\frac{\partial q}{\partial t}\right)^2 - V - V_Q\right)dt \tag{2.29}$$

Then using (♦) one has

$$\frac{\partial q_i}{\partial t}\frac{\partial q_i}{\partial a_k} = \frac{1}{m}\frac{\partial S}{\partial a_k};\ S(a,t) = S_0(a) + \chi(a,t) \tag{2.30}$$

with initial conditions $q = a,\ \chi_0 = 0$. To obtain the q-components multiply by $J^{-1}J_{ik}$ and use (2.18) to get (♠) $\partial q_i/\partial t = (1/m)(\partial S/\partial q_i)$ where $S = S(a(q,t),t)$. Thus the velocity is a gradient for all time and (♠) is a form of the law of motion. Correspondingly one can write (2.28) as

$$m\frac{\partial^2 q_i}{\partial t^2} = -\frac{\partial}{\partial q_i}(V + V_Q) \tag{2.31}$$

This puts the fluid dynamical law of motion (2.22) in a form of Newton's law for a particle of mass m. Note that the motion is quasi potential since the value (2.25) is preserved, i.e. (•) $\partial_t \oint_{C(t)}(\partial q_i/\partial t)dq_i = 0$ where $C(t)$ is the evolute of the material particles conposing C (cf. [**113**]). To obtain the equation governing S use the chain rule $F_t|_a = \partial_t F|_q + (\partial_t q_i)(\partial F/\partial q_i)$ and since $\chi = S - S_0$ from (2.30) one has

$$\left.\frac{\partial\chi}{\partial t}\right|_a = \left.\frac{\partial S}{\partial t}\right|_q + \frac{\partial q_i}{\partial t}\frac{\partial S}{\partial q_i} - \left(\left.\frac{\partial S_0}{\partial t}\right|_q + \frac{\partial q_i}{\partial t}\frac{\partial S_0}{\partial q_i}\right) \tag{2.32}$$

The two terms in the bracket sum to $\partial S_0(a)/\partial t = 0$ and using (♠) one obtains $(\partial\chi/\partial t)|_a = (\partial S/\partial t)|_q + m(\partial q/\partial t)^2$. Hence from (2.29) and (♠) one has

$$\left.\frac{\partial\chi}{\partial t}\right|_a = \frac{1}{2}m\left(\frac{\partial q}{\partial t}\right)^2 - V - V_Q \Rightarrow \frac{\partial S}{\partial t} + \frac{1}{2m}\left(\frac{\partial S}{\partial q}\right)^2 + V + V_Q = 0 \tag{2.33}$$

This is the quantum HJ equation and one has shown that the equations (2.16), (2.22), and (♦) are equivalent to the 5 equations (2.16), (2.30), and (2.33); they determine the functions (q_i, ρ, S). Note that although the particle velocity is orthogonal to a moving surface $S = c$ the surface does not keep step with the particles that initially compose it and hence is not a material surface. There is also some interesting discussion about vortex lines for which we refer to [**474**].

The fundamental link between the particle (Lagrangian) and wave mechanical (Eulerian) pictures is defined by the following expression for the Eulerian density

$$\rho(x,t) = \int \delta(x - q(a,t))\rho_0(a)d^3a \tag{2.34}$$

The corresponding formula for the Eulerian velocity is contained in the expression for the current

$$\rho(x,t)v_i(x,t) = \int \frac{\partial q_i(a,t)}{\partial t}\delta(x - q(a,t))\rho_0(a)d^3a \tag{2.35}$$

These relations play an analogous role in the approach to the Huygen's formula

$$\psi(x,t) = \int G(x,t;a,0)\psi_0(a)d^3a \tag{2.36}$$

in the Feynman theory; thus one refers to $\delta(x-q9a,t))$ as a propagator. Unlike the many to one mapping embodied in (2.36) the quantum evolution here is described by a local point to point development. Using the result

$$\delta(x - q(a,t)) = \left.J^{-1}\right|_{a(x,t)}\delta(a - a_0(x,t));\ x - q(a_0,t) = 0 \tag{2.37}$$

and evaluating the integrals (2.34) and (2.22) are equivalent to

$$\rho(x,t) = \left.J^{-1}\right|_{a(x,t)}\rho_0(a(x,t));\ \rho(x(a,t),t) = J^{-1}(a,t)\rho_0(a) \tag{2.38}$$

$$v_i(x,t) = \left.\frac{\partial q_i(a,t)}{\partial t}\right|_{a(x,t)};\ v_i(x,t)|_{a(x,t)} = \frac{\partial q_i(a,t)}{\partial t}$$

These restate the conservation equation (2.16) and give the relations between the velocities in the two pictures; J^{-1} could be called a local propagator. Now from (2.38) one can relate the accelerations in the two pictures via

$$\frac{\partial v_i}{\partial t} + v_j\frac{\partial v_i}{\partial x_j} = \left.\frac{\partial^2 q_i(a,t)}{\partial t^2}\right|_{a(x,t)};\ \left.\left(\frac{\partial v_i}{\partial t} + v_j\frac{\partial v_i}{\partial x_j}\right)\right|_{a(x,t)} = \frac{\partial^2 q_i(a,t)}{\partial t^2} \tag{2.39}$$

One can now translate the Lagrangian flow equations into Lagrangian language. Differentiating (2.34) in t and using (2.35) one finds the continuity equation

$$\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x_i}(\rho v_i) = 0 \tag{2.40}$$

Next differentiating (2.35) and using (2.31) and (2.40) one obtains the quantum analogue of Euler's equation

$$\frac{\partial v_i}{\partial t} + v_j \frac{\partial v_i}{\partial x_j} = -\frac{1}{m}\frac{\partial}{\partial x_i}(V + V_Q) \tag{2.41}$$

Finally the quasi potential condition (♠) becomes (★) $v_i = (1/m)(\partial S(x,t)/\partial x_i)$. (2.38) gives the general solutions of the continuity equation (2.40) and Euler's equation (2.41) in terms of the paths and initial density. To establish the connection between the Eulerian equations and the SE note that (2.41) and (★) can be written

$$\frac{\partial}{\partial x_i}\left(\frac{\partial S}{\partial t} + \frac{1}{2m}\frac{\partial S}{\partial x_i}\frac{\partial S}{\partial x_i} + V + V_Q\right) = 0 \tag{2.42}$$

The quantity in brackets is thus a function of time and since this does not affect the velocity field one may absorb it in S (i.e. set it equal to zero) leading to

$$\frac{\partial S}{\partial t} + \frac{1}{2m}\frac{\partial S}{\partial x_i}\frac{\partial S}{\partial x_i} + V + V_Q = 0 \tag{2.43}$$

Combining all this the function $\psi = \sqrt{\rho} exp(iS/\hbar)$ satisfies the SE

$$i\hbar\frac{\partial \psi}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 \psi}{\partial x_i \partial x_i} + V\psi \tag{2.44}$$

This has all been deduced from (2.22) subject to the quasi potential requirement. The quantization condition (•) becomes (••) $\oint_{C_0}(\partial S(x,t)/\partial x_i)dx_i = n\hbar$ ($n \in \mathbf{Z}$ where C_0 is a closed curve fixed in space that does not pass through nodes. Given the initial wave function $\psi_0(a)$ one can now compute ψ for all (x,t) as follows. first solve (2.22) subject to initial conditions $q_0(a) = a$, $\partial q_{i0}(a)/\partial t = m^{-1}\partial S_0(a)/\partial a$ to get the set of trajectories for all (x,t). Then substitute $q(a,t)$ and q_t in (2.36) to find ρ and $\partial S/\partial x$ which gives S up to an additive function of time $f(t)$. To fix this function up t a constant use (2.43) and one gets finally

$$\psi(x,t) = \sqrt{(J^{-1}\rho_0|_{a(x,t)}} exp\left[\frac{i}{\hbar}\left(\int m(\partial q_i(a,t)/\partial t)|_{a(x,t)}dx_i + f(t)\right)\right] \tag{2.45}$$

The Eulerian equations (2.40) and (2.41) form a closed system of four coupled PDE to determine the four independent fields ($\rho(x)$, $v_i(x)$) and do not refer to the material paths. One notes that the Lagrangian theory from which the Eulerian system was derived comprises seven independent fields (ρ, $q(a)$, $p(a)$). In the case of quasi potential flow there are respectively 2 or 5 independent fields. This may be regarded as an incompleteness in the Eulerian description or a redundancy in the Lagrangian description; it could also be viewed in terms of refinement. One notes also that the law of motion (2.29) for the fluid elements coincides with that of the dBB interpretation of QM and one must be careful to discriminate between the two points of view.

2.2. ELECTROMAGNETISM AND THE SE. We go next to the second paper in [**474**] which connects the electromagnetic (EM) fields to hydrodynamics and relates this to the quantum potential. Thus the source free Maxwell equations in free space are

$$\epsilon_{ijk}\partial_j E_k = -\frac{\partial B_i}{\partial t};\ \epsilon_{ijk}\partial_j B_k = \frac{1}{c^2}\frac{\partial E_i}{\partial t};\ \partial_i E_i = \partial_i B_i = 0 \tag{2.46}$$

One regards the last two equations as constraints rather than dynamical equations. First one goes to a representation of these equations in Schrödinger form and begins with the Riemann-Silberstein 3-vector $F_i = \sqrt{\epsilon_0/2}(E_i + icB_i)$ and 3×3 angular momentum matrices s_i so that

$$(s_i)_{jk} = -i\hbar\epsilon_{ijk};\ [s_i, s_j] = i\hbar\epsilon_{ijk}s_k \tag{2.47}$$

so that (2.46) is equivalent to

$$i\hbar\frac{\partial F_i}{\partial t} = -ic(s_j)_{ik}\partial_j F_k;\ \partial_i F_i = 0 \tag{2.48}$$

To formulate groundwork for continuous representation of the spin freedoms one transforms to a representation of the s_i where the z-component is diagonal via the unitary matrix

$$U_{ai} = \frac{1}{\sqrt{2}}\begin{pmatrix} -1 & i & 0\\ 0 & 0 & \sqrt{2}\\ 1 & i & 0\end{pmatrix} \tag{2.49}$$

and Maxwell's equations become

$$i\hbar\frac{\partial G_a}{\partial t} = -ic(J_j)_{ab}\partial_j G_b;\ G_a = U_{ai}F_i;\ J_i = Us_iU^{-1};\ (a,b = 1,0,-1) \tag{2.50}$$

Here one has

$$\begin{pmatrix} G_1\\ G_0\\ G_{-1}\end{pmatrix} = \frac{1}{\sqrt{2}}\begin{pmatrix} -F_1+iF_2\\ \sqrt{2}F_3\\ F_1+iF_2\end{pmatrix};\ J_1 = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0\\ 1 & 0 & 1\\ 0 & 1 & 0\end{pmatrix}; \tag{2.51}$$

$$J_2 = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0\\ i & 0 & -i\\ 0 & i & 0\end{pmatrix};\ J_3 = \hbar\begin{pmatrix} 1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & -1\end{pmatrix}$$

Next one passes to an angular coordinate representation using the Euler angles $(\alpha_r) = (\alpha,\beta,\gamma)$ and conventions of [**471**] so that

$$\hat{M}_1 = i\hbar(Cos(\beta)\partial_\alpha - Sin(\beta)Ctn(\alpha)\partial_\beta + Sin(\beta)Csc(\alpha)\partial_\gamma); \tag{2.52}$$

$$\hat{M}_2 = i\hbar(-Sin(\beta)\partial_\alpha - Cos(\beta)Ctn(\alpha)\partial_\beta + Cos(\beta)Csc(\alpha)\partial_\gamma);\ \hat{M}_3 = i\hbar\partial_\beta$$

The SE (2.50) becomes then

$$i\hbar\frac{\partial\psi(x,\alpha)}{\partial t} = -ic\hat{M}_i\partial_i\psi(x,\alpha) \equiv i\hbar\frac{\partial\psi}{\partial t} = -c\hbar\hat{\lambda}_i\partial_i\psi\ \left(\hat{\lambda}_i = \frac{\hat{M}_i}{-i\hbar}\right) \tag{2.53}$$

where ψ is a function on the 6-dimensional manifold $M = \mathbf{R}^3 \otimes SO(3)$ whose points are labeled by (x, α). The wave function may be expanded in terms of an orthornormal set of spin 1 basis functions $u_a(\alpha)$ (cf. [**471**]) in the form

$$\psi(x,\alpha,t) = G_a(x,t)u_a(\alpha)\ (a = 1, 0, -1);\ u_1(\alpha) = \frac{\sqrt{3}}{4\pi}Sin(\alpha)e^{-i\beta}; \tag{2.54}$$

$$u_0(\alpha) = \frac{ii\sqrt{3}}{2\sqrt{2}\pi}Cos(\alpha);\ u_{-1}(\alpha) = \frac{\sqrt{3}}{4\pi}Sin(\alpha)e^{i\beta};$$

where $\int u_a^*(\alpha)u_b(\alpha)d\Omega = \delta_{ab}$ with $d\Omega = Sin(\alpha)d\alpha d\beta d\gamma$ and $\alpha \in [0, \pi]$, $\beta \in [0, 2\pi]$, $\gamma \in [0, 2\pi]$. One can show that $\int u_a^* \hat{M}_i u_b(\alpha)d\Omega = (J_i)_{ab}$ and multiplying (2.53) (with use of (2.54)) one recovers the Maxwell equations in the form (2.50). In this formulation the field equations (2.53) come out as second order PDE and summation over i or a is replaced by integration in α_r. For example the energy density and Poynting vector have the alternate expressions

$$\frac{\epsilon_0}{2}(\mathbf{E}^2 + c^2\mathbf{B}^2) = F_i^*F_i = G_a^*G_a = \int |\psi(x,\alpha)|^2 d\Omega; \tag{2.55}$$

$$\epsilon_0 c^2(\mathbf{E}\times\mathbf{B})_i = \frac{c}{\hbar}F_j^*(s_i)_{jk}F_k = \frac{c}{\hbar}G_a^*(J_i)_{ab}G_b = \frac{c}{\hbar}\int \psi^*(x,\alpha)\hat{M}_i\psi(x,\alpha)d\Omega$$

For the hydrodynamic model one writes $\psi = \sqrt{\rho}exp(iS/\hbar)$ and splitting (2.53) into real and imaginary parts we get

$$\frac{\partial S}{\partial t} + \frac{c}{\hbar}\hat{\lambda}_i S \partial_i S + Q = 0;\ \frac{\partial \rho}{\partial t} + \frac{c}{\hbar}\partial_i(\rho\hat{\lambda}_i S) + \frac{c}{\hbar}\hat{\lambda}(\rho\partial_i S) = 0;\ Q = -c\hbar\frac{\hat{\lambda}_i\partial_i\sqrt{\rho}}{\sqrt{\rho}} \tag{2.56}$$

These equations are equivalent to the Maxwell equations provided ρ and S obey certain conditions; in particular single valuedness of the wave function requires

$$\oint_{C_0} \partial_i S dx_i + \partial_r S d\alpha_r = n\hbar\ (n \in \mathbf{Z}) \tag{2.57}$$

where C_0 is a closed curve in M. In the hydrodynamic model n is interpreted as the net strength of the vortices contained in C_0 (these occur in nodal regions ($\psi = 0$) where S is singular). Comparing (2.56) with the Eulerian continuity equation corresponding to a fluid of density ρ with translational and rotational freedom one expects

$$\frac{\partial \rho}{\partial t} + \partial_i(\rho v_i) + \hat{\lambda}_i(\rho\omega_i) = 0;\ v_i \sim (c/\hbar)\hat{\lambda}S;\ \omega_i \sim (c/\hbar)\partial_i S \tag{2.58}$$

Thus one obtains a kind of potential flow (strictly quasi-potential in view of (2.57)) with potential $(c/\hbar)S$. The quantity Q in (2.56) is of course the analogue for the Maxwell equations of the quantum potential and will have the classical form $-\nabla^2\sqrt{\rho}/\sqrt{\rho}$ when the appropriate metric on M is identified. From the Bernoulli-like (or HJ-like) equation in (2.56) we obtain the analogue of Euler's force law for the EM field. Thus applying first ∂_i and using (2.58) one gets

$$\left(\frac{\partial}{\partial t} + v_j\partial_j + \omega_j\hat{\lambda}_j\right)\omega_i = -\frac{c}{\hbar}\partial_i Q \tag{2.59}$$

Acting on this with $\hat{\lambda}_i$ and using $[\hat{\lambda}_i, \hat{\lambda}_j] = -\epsilon_{ijk}\hat{\lambda}_k$ yields

$$\left(\frac{\partial}{\partial t} + v_j\partial_j + \omega_i\hat{\lambda}_j\right) v_i = \epsilon_{ijk}\omega_j v_k - \frac{c}{\hbar}\hat{\lambda}_i Q \tag{2.60}$$

which contains a Coriolis type force in addition to the quantum contribution. The paths $x = x(x_0, \alpha_0, t)$ and $\alpha = \alpha(x_0, \alpha_0, t)$ of the fluid particles in M are obtained by solving the differential equations

$$v_i(x,\alpha,t) = \frac{\partial x_i}{\partial t};\ v_r(x,\alpha,t) = \frac{\partial \alpha_r}{\partial t} \tag{2.61}$$

These paths are an analogue in the full wave theory of a ray.

Now one generalizes this to coordinates x^μ in an N-dimensional Riemannian manifold M with (static) metric $g_{\mu\nu}(x)$. The history of the fluid is encoded in the positions $\xi(\xi_0, t)$ of distinct fluid elements and one assumes a single valued and differentiable map between coordinates (cf. Section 7.2.1). Let $P_0(\xi_0)$ be the initial density of some continuously distributed quantity in M (mass in ordinary hydrodynamics, energy here) and set $g = det(g_{\mu\nu})$. Then the quantity in an elementary volume $d^N\xi_0$ attached to the point ξ_0 is $P_0(\xi_0)\sqrt{-g(\xi_0)}d^N\xi_0$ and conservation of this quantity is expressed via

$$P(\xi(\xi_0,t))\sqrt{-g(\xi(\xi_0,t))}d^N\xi(\xi_0,t) = P_0(\xi_0)\sqrt{-g(\xi_0}d^N\xi_0 \equiv \tag{2.62}$$

$$\equiv P(\xi_0,t) = D^{-1}(\xi_0,t)P_0(\xi_0);\ D(\xi_0,t) = \sqrt{g(\xi)/g(\xi_0)}J(\xi_0,t)$$

where J is the Jacobian

$$J = \frac{1}{N!}\epsilon_{\mu_1\cdots\mu_n}\epsilon^{\nu_1\cdots\nu_n}\frac{\partial\xi^{\mu_1}}{\partial\xi_0^{\nu_1}}\cdots\frac{\partial\xi^{\mu_N}}{\partial\xi_0^{\nu_N}} \tag{2.63}$$

One assumes the Lagrangian for the set of fluid particles has a kinetic term and an internal potential representing a certain kind of particle interaciton

$$L = \int P_0(\xi_0)\left(\frac{1}{2}g_{\mu\nu}(\xi)\frac{\partial\xi^\mu}{\partial t}\frac{\partial\xi^\nu}{\partial t} - g^{\mu\nu}\frac{c^2\ell^2}{8}\frac{1}{P^2}\frac{\partial P}{\partial\xi^\mu}\frac{\partial P}{\partial\xi^\nu}\right)\sqrt{-g(\xi_0)}d^N\xi_0 \tag{2.64}$$

Here ℓ is a constant with the dimension of length (introduced for dimensional reasons) and $\xi = \xi(\xi_0, t)$; one substitutes now for P from (2.62) and writes

$$\frac{\partial}{\partial\xi^\mu} = J^{-1}J^\nu_\mu\frac{\partial}{\partial\xi_0^\nu};\ J^\nu_\mu = \frac{\partial J}{\partial(\partial\xi^\mu/\partial\xi_0^\nu)};\ \frac{\partial\xi^\mu}{\partial\xi_0^\nu}J^\sigma_\mu = J\delta^\sigma_\nu \tag{2.65}$$

One assumes suitable behavior at infinity so that surface terms in the variational calculations vanish and the Euler-Lagrange equations are then
(2.66)

$$\frac{\partial^2\xi^\mu}{\partial t^2} + \left\{\begin{matrix}\mu\\ \nu\ \sigma\end{matrix}\right\}\frac{\partial\xi^\nu}{\partial t}\frac{\partial\xi^\sigma}{\partial t} = -\frac{c\ell}{\hbar}g^{\mu\nu}\frac{\partial Q}{\partial\xi^\nu};\ Q = \frac{-\hbar c\ell}{2\sqrt{-gP}}\frac{\partial}{\partial\xi^\mu}\left(\sqrt{-g}g^{\mu\nu}\frac{\partial\sqrt{P}}{\partial\xi^\nu}\right)$$

$$\left\{\begin{matrix}\mu\\ \nu\ \sigma\end{matrix}\right\} = \frac{1}{2}g^{\mu\rho}\left(\frac{\partial g_{\sigma\rho}}{\partial xi^\nu} + \frac{\partial g_{\nu\rho}}{\partial\xi^\sigma} - \frac{\partial g_{\nu\sigma}}{\partial\xi^\rho}\right)$$

Now one restricts to quasi-potential flows with conditions

$$g_{\mu\nu}(\xi_0)\frac{\partial\xi_0^\mu}{\partial t}=\frac{c\ell}{\hbar}\frac{\partial S_0(\xi_0)}{\partial\xi_0^\nu};\ \oint_C\frac{\partial S_0(\xi_0)}{\partial\xi_0^\mu}d\xi_0^\mu=h\hbar\ (n\in\mathbf{Z}) \tag{2.67}$$

One follows the same procedures as in Section 7.2.1 so multiplying (2.66) by $g_{\sigma\mu}(\partial\xi^\sigma/\partial\xi_0^\rho)$ and integrating gives
(2.68)

$$g_{\sigma\mu}(\xi(\xi_0,t))\frac{\partial\xi^\sigma}{\partial\xi_0^\rho}\frac{\partial\xi^\mu}{\partial t}=g_{\rho\mu}(\xi_0)\frac{\partial\xi_0^\mu}{\partial t}+\frac{\partial}{\partial\xi_0^\rho}\int_0^t\left(\frac{1}{2}g_{\mu\nu}(\xi(\xi_0,t))\frac{\partial\xi^\mu}{\partial t}\frac{\partial\xi^\nu}{\partial t}-\frac{c\ell}{\hbar}Q\right)dt$$

Then substituting (2.67) one has

$$g_{\sigma\mu}\frac{\partial\xi^\sigma}{\partial\xi_0^\rho}\frac{\partial\xi^\mu}{\partial t}=\frac{c\ell}{\hbar}\frac{\partial S}{\partial\xi_0^\rho};\ S=S_0+\int_0^t\left(\frac{\hbar}{2c\ell}g_{\mu\nu}\frac{\partial\xi^\mu}{\partial t}\frac{\partial\xi^\nu}{\partial t}-Q\right)dt \tag{2.69}$$

The left side gives the velocity at time t relative to ξ_0 and this is a gradient. To obtain the ξ components multiply by $J^{-1}J^\rho_\nu$ and use (2.65) to get $g_{\mu\nu}(\partial\xi^\nu/\partial t)=(c\ell/\hbar)(\partial S/\partial\xi^\mu)$ where $S=S(\xi_0(\xi,t),t)$. Thus for all time the covariant velocity of each particle is the gradient of a potential with respect to the current position. Finally to see that the motion is quasi-potential since (2.66) holds and the value in (2.67) of the circulation is preserved following the flow, i.e.

$$\frac{\partial}{\partial t}\oint_C(t)g_{\mu\nu}\frac{\partial\xi^\nu}{\partial t}d\xi^\mu=0 \tag{2.70}$$

Finally for the SE one defines a fundamental link between the particle (Lagrangian) and wave-mechanical (Eulerian) pictures via

$$P(x,t)\sqrt{-g(x)}=\int\delta(x-\xi(\xi_0,g))P_0(\xi_0)\sqrt{-g(\xi_0)}d^N\xi_0 \tag{2.71}$$

The corresponding formla for the Eulerian velocity is contained in the current expression

$$P(x,t)\sqrt{-g(x)}v^\mu(x,t)=\int\frac{\partial\xi^\mu}{\partial t}\delta(x-\xi(\xi_0,t))P_0(\xi_0)\sqrt{-g(\xi_0)}d^N\xi_0 \tag{2.72}$$

These are equivalent to the following local expressions ($\xi_0\sim\xi_0(x,t)$)
(2.73)

$$P(x,t)\sqrt{-g(x)}=J^{-1}\big|_{\xi_0}P_0(\xi_0(x,t))\sqrt{-g(\xi_0(x,t))};\ v^\mu(x,t)=\left.\frac{\partial\xi^\mu(\xi_0,t)}{\partial t}\right|_{\xi_0}$$

We can now translate the Lagrangian flow equations into Eulerian language. First differentiate (2.71) in t and use (2.72) to get

$$\frac{\partial P}{\partial t}+\frac{1}{\sqrt{-g}}\frac{\partial}{\partial x^\mu}(P\sqrt{-g}v^\mu)=0 \tag{2.74}$$

Then, differentiating (2.72) and using (2.66) and (2.74) one obtains the analogue of the classical Euler equation

$$\frac{\partial v^\mu}{\partial t}+v^\nu\frac{\partial v^\mu}{\partial x^\nu}+\left\{\begin{array}{c}\mu\\ \nu\ \sigma\end{array}\right\}v^\nu v^\sigma=-\frac{c\ell}{\hbar}g^{\mu\nu}\frac{\partial Q}{\partial x^\nu} \tag{2.75}$$

where Q is given by (2.66) with ξ replaced by x. Finally the quasi-potential condition becomes

$$v^{\mu} = \frac{c\ell}{\hbar} g^{\mu\nu} \frac{\partial S(x,t)}{\partial x^{\nu}} \tag{2.76}$$

Formula (2.73) give the general solution of the coupled continuity and Euler equations (2.74) and (2.75) in terms of the paths and initial density. To establish the connection between the Eulerian equations and the SE note that (2.75) and (2.76) can be written

$$\frac{\partial}{\partial x^{\mu}} \left(\frac{\partial S}{\partial t} + \frac{c\ell}{2\hbar} g^{\nu\sigma} \frac{\partial S}{\partial x^{\nu}} \frac{\partial S}{\partial x^{\sigma}} + Q \right) = 0 \tag{2.77}$$

Again the quantity in brackets is a function of time which is incorporated into S if necessary and one arrives at

$$\frac{\partial S}{\partial t} + \frac{c\ell}{2\hbar} g^{\nu\sigma} \frac{\partial S}{\partial x^{\nu}} \frac{\partial S}{\partial x^{\sigma}} + Q = 0 \tag{2.78}$$

Combining (2.78) with (2.74) and using (2.76) one finds for $\psi = \sqrt{P} exp(i/\hbar)$ the equation

$$i\hbar \frac{\partial \psi}{\partial t} = -\frac{-\hbar c\ell}{2\sqrt{-g}} \frac{\partial}{\partial x^{\mu}} \left(\sqrt{-g} g^{\mu\nu} \frac{\partial \psi}{\partial x^{\nu}} \right) \tag{2.79}$$

(for a system of mass $\hbar/c\ell$). The quantization condition (2.70) becomes

$$\oint_{C_0} \frac{\partial S(x,t)}{\partial x^{\mu}} dx^{\mu} = n\hbar \ \ (n \in \mathbf{Z}) \tag{2.80}$$

Now an alternate representation of the internal angular motion can be given in terms of the velocity fields $v_r(x,\alpha,t)$ conjugate to the Euler angles. One has

$$\omega_i = (A^{-1})_{ir} v_r; \ v_r = A_{ri}\omega_i; \tag{2.81}$$

$$(A^{-1})_{ir} = \begin{pmatrix} -Cos(\beta) & 0 & -Sin(\alpha)Sin(\beta) \\ Sin(\beta) & 0 & -Sin(\alpha)Cos(\beta) \\ 0 & -1 & -Cos(\alpha) \end{pmatrix};$$

The relations (2.52) may be written as $\hat{\lambda}_i = A_{ir}\partial_r$ and hence $\omega_j \hat{\lambda}_j = v_r \partial_r$. In terms of the conjugate velocities Euler's equations (2.59) and (2.60) become

$$\left(\frac{\partial}{\partial t} + v_j\partial_j + v_r\partial_r \right) v_s + A_{si}\partial_r (A^{-1})_{iq} v_q v_r = -\frac{c}{\hbar} A_{si}\partial_i Q; \tag{2.82}$$

$$\left(\frac{\partial}{\partial t} + v_j\partial_j + v_r\partial_r \right) v_i + \epsilon_{ijk}(A^{-1})_{kr} v_j v_r = -\frac{c}{\hbar} \hat{\lambda}_i Q$$

Now one specializes the general treatment to the manifold $M = \mathbf{R}^3 \times SO(3)$ with coordinates $x^{\mu} = (x_i, \alpha_r)$ and metric

$$g^{\mu\nu} = \begin{pmatrix} 0 & \ell^{-1}A_{ir} \\ \ell^{-1}A_{ri} & 0 \end{pmatrix}; \ g_{\mu\nu} = \begin{pmatrix} 0 & \ell(A^{-1})_{ir} \\ \ell(A^{-1})_{ri} & 0 \end{pmatrix} \tag{2.83}$$

and density $P = \rho/\ell^3$. From $\partial_r(\sqrt{-g} g^{ir}) = 0$ and $\ell g^{ir}\partial_r = \hat{\lambda}_i$ one gets via $[\hat{\lambda}_i, \hat{\lambda}_j] = -\epsilon_{ijk}\hat{\lambda}_k$ the relation $g^{ir}(\partial_s g_{rj} - \partial_r g_{sj}) = \ell^{-1}\epsilon_{ijk} g_{sk}$. Then the relation (2.76) becomes (2.58), (2.74) becomes (2.56), (2.75) becomes (2.82), (2.66) (with ξ

replaced by x) becomes (2.56), (2.78) becomes (2.56), (2.79) becomes (2.53), and (2.80) becomes (2.57). Writing $\xi^\mu = (q_i, \theta_r)$ for the Lagrangian coordinates (2.64) becomes

$$(2.84)\quad L = \int \ell\rho_0(q_0,\theta_0)\left((A^{-1})_{ir}\frac{\partial q_i}{\partial t}\frac{\partial \theta_r}{\partial t} - A_{ir}\frac{c^2}{4\rho^2}\frac{\partial\rho}{\partial q_i}\frac{\partial \rho}{\partial\theta_r}\right)Sin(\theta_0)d^3\theta_0 d^3q_0$$

Newton's law (2.66) reduces to the coupled relations

$$(2.85)\qquad \frac{\partial^2 q_i}{\partial t^2} + \epsilon_{ijk}(A^{-1})_{ir}\frac{\partial q_j}{\partial t}\frac{\partial\theta_r}{\partial t} = -\frac{c}{\hbar}A_{ir}\frac{\partial Q}{\partial\theta_r};$$

$$\frac{\partial^2\theta_s}{\partial g^2} + A_{si}\frac{\partial}{\partial\theta_r}(A^{-1})_{iq}\frac{\partial\theta_q}{\partial t}\frac{\partial\theta_r}{\partial t} = -\frac{c}{\hbar}A_{si}\frac{\partial Q}{\partial q_i}$$

where A_{ir} is given via (2.81) with α_r replaced by $\theta_r(q_0,\theta_0,t)$ and one substitutes $\rho(q_0,\theta_0,t) = D^{-1}(q_0,\theta_0,t)\rho_0(q_0,\theta_0)$ along with

$$(2.86)\qquad Q = -c\hbar A_{ir}\frac{1}{\sqrt{\rho}}\frac{\partial^2\sqrt{\rho}}{\partial\theta_r\partial q_i};\ \ \frac{\partial}{\partial q_i} = J^{-1}\left(J_{ij}\frac{\partial}{\partial q_{0j}} + J_{is}\frac{\partial}{\partial\theta_{0s}}\right);$$

$$\frac{\partial}{\partial\theta_r} = J^{-1}\left(J_{rj}\frac{\partial}{\partial q_{0j}} + J_{rs}\frac{\partial}{\partial\theta_{0s}}\right)$$

Given the initial wavefunction $\psi_0(x,\alpha) = G_{0a}(x)u_a(\alpha) = \sqrt{\rho_0}exp(iS_0/\hbar)$ one can solve (2.85) subject to initial conditions $\partial q_{0i}/\partial t = (c/\hbar)A_{ir}(\partial S_0/\partial\theta_{0r})$ and $\partial\theta_{0r}/\partial t = (c/\hbar)A_{ri}(\theta_0)(\partial S_0/\partial q_{0i})$ to get the set of trajectories for all (q_0,θ_0,t). Then invert these functions and substitute $q_0(x,\alpha,t)$ and $\theta_0(x,\alpha,t)$ in the right side of (2.85) to find $\rho(x,\alpha,t)$ and in the right sides of the equations

$$(2.87)\qquad \partial_r S = \frac{\hbar}{c}(A^{-1})_{ir}\frac{\partial q_i}{\partial t};\ \ \partial_i S = \frac{\hbar}{c}(A^{-1})_{ri}\frac{\partial\theta_r}{\partial t}$$

to get S up to an additive function of time $\hbar f(t)$, which is adjusted as before (cf. (2.56)). There results

(2.88)

$$\psi = \sqrt{D^{-1}\rho_0)|_{q_0,\theta_0}}e^{[(i/c)\int(A^{-1})_{ir}(\partial q_i/\partial t)|_{q_0,\theta_0}d\alpha_r + (A^{-1})_{ri}(\partial\theta_r/\partial t)|_{q_0,\theta_0}dx_i + if(t)]}$$

Finally the components of the time dependent EM field may be read off from (2.50) where

$$(2.89)\qquad G_a = \int\psi(x,\alpha)u_a^*(\alpha)d\Omega$$

EXAMPLE 2.1. One computes the time dependence of the EM field whose initial form is $E_{0i} = (ECos(kz),0,0)$ with $B_{0i} = (0,(1/c)ECos(kz),0)$. The initial wavefunction is $\psi_0 = G_{01}u_1$ or

$$(2.90)\qquad \psi_0(q_0,\theta_0) = -\frac{\sqrt{3}}{2\sqrt{2\pi}}ECos(kq_{03})Sin(\theta_{01})e^{-i\theta_{02}}$$

One looks for solutions to (2.85) that generate a time dependent wavefunction whose spatial dependence is on z alone. The Hamiltonian in the SE (2.53) then reduces to $-ic\hat{M}_3\partial_3\psi(x,\alpha)$ alone which preserves the spin dependence of ψ_0. Hence, since ρ is independent of θ_3, the quantum potential Q in (2.86) vanishes. Some calculation leads to

(2.91) $$\psi(x,\alpha,t) = -\frac{\sqrt{3}}{2\sqrt{2}\pi} ECos(k(z-ct))Sin(\alpha)e^{-i\beta};$$

$$E_i = (Ecos(k(ct-z)),0,0);\ \ B_i = (0,(1/c)ECos(k(ct-z)),0)$$

Note that one obtains oscillatory behavior of the Eulerian variables from a model in which the individual fluid elements do not oscillate! This circumvents one of the classical problems where it was considered necessary for the elements of a continuum to vibrate in order to support a wave motion. Another interesting feature is that the speed of each element $|\mathbf{v}| = |cCosc(\theta_{01})|$ obeys $c \le |\mathbf{v}| < \infty$. One might regard the occurence of superluminal speeds as evidence that the Lagrangian model is only a mathematical tool. Indeed performing a weighed sum of the velocity over the angles to get the Poynting vector $\epsilon_0 c^2(\mathbf{E}\times\mathbf{B})_i = \int \rho v_i d\Omega$ the collective x and y motions cancel to give the conventional geometrical optics rays propagating at speed c in the z-direction.

3. SOME SPECULATIONS ON THE AETHER

We give first some themes and subsequently some details and speculations.

(1) In a hauntingly appealing paper [**494**] P. Isaev makes conjectures, with supporting arguments, which arrive at a definition of the aether as a Bose-Einstein condensate of neutrino-antineutrino pairs of Cooper type (Bose-Einstein condensates of various types have been considered by others in this context - cf. [**262, 263, 338, 398, 482, 510, 606, 893, 960**] and Remark 5.3.1). The equation for the ψ-aether is then a solution of the massless Klein-Gordon (KG) equation (photon equation)

(3.1) $$\left(\hbar^2\Delta - \frac{\hbar^2}{c^2}\partial_t^2\right)\psi = 0$$

(cf. also [**911**]). This ψ field heuristically acts as a carrier of waves (playground for waves) and one might say that special relativity (SR) is a way of including the influence of the aether on physical processes and consequently SR does not see the aether (cf. here also the idea of a Dirac aether in [**215, 216, 302, 537, 727**] and Einstein-aether theories as in [**338, 510**] - some of this is developed below). In the electromagnetic (EM) theory one looks at $\vec{\psi} = (\phi, \vec{A})$ with $\Box\psi_i = 0$ as the defining equation for a real ψ-aether, in terms of the potentials ϕ and $\vec{A}$ which therefore define the ψ-aether. EM waves are then considered as oscillations of the ψ aether and wave processes in the aether accompanying a moving particle determine wave properties of the particle. The ensemble point of view can be considered artificial, in accord wth a conclusion we made in [**194, 197, 203**], based on Theme 3 below, that uncertainty and the ensemble "cloud" are based on the lack of deterimation of particle trajectories when using the SE instead of a third order equation. Thus it is the SE context which automatically (and gratuitously) introduces probability; nevertheless, given the limitations of measurement, it produces an amazingly accurate theory.

(2) Next there is the classical deBroglie-Bohm (dBB) theory (cf. [**191, 186, 187, 188, 189, 205, 203, 471**] - and references in these papers) where, working from a Schrödinger equation (SE)

$$-\frac{\hbar^2}{2m}\Delta\psi + V\psi = i\hbar\psi_t;\ \psi = Re^{iS/\hbar} \tag{3.2}$$

one arrives at a quantum potential $Q = -(\hbar^2/2m)(\Delta\sqrt{\rho}/\sqrt{\rho})$ $(R = \sqrt{\rho})$ associated to a quantum Hamilton-Jacobi equation (QHJE)

$$S_t + \frac{(\nabla S)^2}{2m} + V + Q = 0 \tag{3.3}$$

The ensuing particle theory exhibits trajectory motion choreographed by ψ via $Q = -(\hbar^2/2m)(\Delta|\psi|/|\psi|)$ or directly via the guidance equation

$$\dot{\vec{x}} = \vec{v} = \hbar\Im\frac{\psi^*\nabla\psi}{\psi^*\psi} \tag{3.4}$$

(cf. [**186, 187, 188, 203**] for extensive references). Relativistic and geometrical aspects are also provided below.

(3) In [**346**] Faraggi and Matone develop a theory of $x-\psi$ duality, related to Seiberg-Witten theory in the string arena, which was expanded in various ways in [**2, 41, 110, 198, 194, 191, 249, 640, 751, 958**]. Here one works from a stationary SE $[-(\hbar^2/2m)\Delta + V(x)]\psi = E\psi$, and, assuming for convenience one space dimension, the space variable x is determined by the wave function ψ from a prepotential $\mathfrak{F}$ via Legendre transformations. The theory suggests that x plays the role of a macroscopic variable for a statistical system with a scaling term $\hbar$. Thus define a prepotential $\mathfrak{F}_E(\psi) = \mathfrak{F}(\psi)$ such that the dual variable $\psi^D = \partial\mathfrak{F}/\partial\psi$ is a (linearly independent) solution of the same SE. Take V and E real so that $\bar{\psi} = \psi^D$ qualifies and write $\partial_x\mathfrak{F} = \psi^D\partial_x\psi = (1/2)[\partial_x(\psi\psi^D) + W)]$ where W is the Wronskian. This leads to $(\psi^D = \bar{\psi})$ the relation $\mathfrak{F} = (1/2)\psi\bar{\psi} + (W/2)x$ (setting the integration constant to zero). Consequently, scaling W to $-2i\sqrt{2m}/\hbar$ one obtains

$$\frac{i\sqrt{2m}}{\hbar}x = \frac{1}{2}\psi\frac{\partial\mathfrak{F}}{\partial\psi} - \mathfrak{F} \equiv \frac{i\sqrt{2m}}{\hbar}x = \psi^2\frac{\partial\mathfrak{F}}{\partial\psi^2} - \mathfrak{F} \tag{3.5}$$

which exhibits x as a Legendre transform of $\mathfrak{F}$ with respect to ψ^2. Duality of the Legendre transform then gives also

$$\mathfrak{F} = \phi\partial_\phi\left(\frac{i\sqrt{2m}\,x}{\hbar}\right) - \left(\frac{i\sqrt{2m}\,x}{\hbar}\right);\ \phi = \partial_{\psi^2}\mathfrak{F} = \frac{\bar{\psi}}{2\psi} \tag{3.6}$$

so that $\mathfrak{F}$ and $(i\sqrt{2m}\,x/\hbar)$ form a Legendre pair. In particular one has $\rho = |\psi|^2 = \frac{2i\sqrt{2m}}{\hbar}x + 2\mathfrak{F}$ which also relates x and the probability density (but indirectly since the x term really only cancels the imaginary part of $2\mathfrak{F}$). In any event one sees that the wave function ψ specifically determines the exact location of the "particle" whose quantum evolution is described by

ψ. We mention here also that the (stationary) SE can be replaced by a third order equation

(3.7) $$4\mathfrak{F}''' + (V(x) - E)(\mathfrak{F}' - \psi\mathfrak{F}'')^3 = 0;\ \mathfrak{F}' \sim \frac{\partial\mathfrak{F}}{\partial\psi}$$

and a dual stationary SE has the form

(3.8) $$\frac{\hbar^2}{2m}\frac{\partial^2 x}{\partial\psi^2} = \psi[E - V]\left(\frac{\partial x}{\partial\psi}\right)^3$$

A noncommutative version of this is developed in the second paper of [**958**].

(4) We also note for comparison and analogy some relations between Legendre duals in mechanics, thermodynamics, and (x,ψ) duality. Thus (cf. [**202, 596**]) one has in mechanics $p\dot{x} - L = H$ via $L = (/2)m\dot{x}^2 - V$ and $H = (p^2/2m) + V$ with $p = \partial L/\partial\dot{x}$ and $\dot{x} = \partial H/\partial p$. In thermodynamics one has a Helmholtz free energy F with $F = U - TS$ for energy U, entropy S, and temperature T. Set $\mathcal{F} = -F$ to obtain $\mathcal{F} = T(\partial\mathcal{F}/\partial T) - U$ and $U = S\partial_S U - \mathcal{F}$ (where $\partial_T\mathcal{F} = S$ and $\partial_S U = T$). Now put this in a table where we write the (x,ψ) duality in the form $\chi = \psi^2(\partial\mathfrak{F}/\partial\psi^2) - \mathfrak{F}$ with $\mathfrak{F} = \phi(\partial\chi/\partial\phi) - \chi$ (for $\chi = (i\sqrt{2m}/\hbar)x$ and $\phi = (\partial\mathfrak{F}/\partial\psi^2)$). This leads to a table

(3.9)

Mechanics	*Thermodynamics*	(x,ψ) *duality*
$\dot{x},\ p,\ L,\ H$	$T,\ S,\ \mathcal{F},\ U$	$\psi^2,\ \phi,\ \mathfrak{F},\ \chi$
$p\dot{x} - H = L$	$TS - U = \mathcal{F}$	$\psi^2\phi - \mathfrak{F} = \chi$
$L = \dot{x}\frac{\partial H}{\partial p} - H$	$\mathcal{F} = S\frac{\partial U}{\partial S} - U$	$\mathfrak{F} = \phi\frac{\partial\chi}{\partial\phi} - \chi$
$H = p\frac{\partial L}{\partial\dot{x}} - L$	$U = T\frac{\partial\mathcal{F}}{\partial T} - \mathcal{F}$	$\chi = \psi^2\frac{\partial\mathfrak{F}}{\partial\psi^2} - \mathfrak{F}$

One says that e.g. $(\mathfrak{F},\chi)$ or $(\mathcal{F},U)$ or (L,H) form a Legendre dual pair and in the first situation one refers to (x,ψ) duality. One sees in particular that $\mathfrak{F} = \psi^2\phi - \chi$ where $\psi^2\phi \sim \dot{x}p$ in mechanics. Note that $\phi = \partial\mathfrak{F}/\partial\psi^2 = (1/2\psi)(\partial\mathfrak{F}/\partial\psi) = \bar{\psi}/2\psi$ with $\psi^2\phi = (1/2)\psi\bar{\psi} = (1/2)|\psi|^2$. In any event $\chi = -i(\sqrt{2m}/\hbar)x$ and we will see below how the physics can be expressed via ψ, $\partial/\partial\psi$, $d\psi$ etc. without mentioning x. This allows one to think of the coordinate x as an emergent entity and we like to think of $x - \psi$ duality in this spirit.

3.1. DISCUSSION OF A PUTATIVE PSI AETHER. We mention [**650, 753, 935**] for some material on the aether and the vacuum and refer to the bibliography for other references. We sketch first some material from [**2, 41, 110, 249, 751, 958**] which extends theme 3 to the Klein-Gordon (KG) equation. Following [**958**] take a spacetime manifold M with a metric field g and a scalar field ϕ satisfying the KG equation. Locally one has cartesian coordinates x^α ($\alpha = 0, 1.\cdots, n-1$) in which the metric is diagonal with $g_{\alpha\beta}(x) = \eta_{\alpha\beta}(x)$ and the KG equation has the form $(\Box_x + m^2)\phi(x) = 0$ ($\Box_x \stackrel{?}{\sim} (\hbar^2/c^2)[(\partial_t^2/c^2) - \nabla^2]$ - cf. (2.26)). Defining prepotentials such that $\tilde{\phi}^{(\alpha)} = \partial\mathfrak{F}^{(\alpha)}[\phi^{(\alpha)}]/\partial\phi^{(\alpha)}$ where $\phi^{(\alpha)}$ and $\tilde{\phi}^{(\alpha)}$ are two linearly independent solutions of the KG equation depending on a variable x^α (where the x^β for $\beta \neq \alpha$ enter ϕ^α and $\tilde{\phi}^\alpha$ as parameters) one has as above (with a different scaling factor)

$$\frac{\sqrt{2m}}{\hbar}x^{\alpha} = \frac{1}{2}\phi^{(\alpha)}\frac{\partial\mathfrak{F}^{(\alpha)}[\phi^{\alpha)}]}{\partial\phi^{(\alpha)}} - \mathfrak{F}^{(\alpha)};\ (\partial^{\alpha}\partial_{\alpha} - V^{\alpha})\,\phi^{\alpha} = 0 \tag{3.10}$$

This is suggested in [**346**] and used in [**958**]; the factor $\sqrt{2m}/\hbar$ is simply a scaling factor and it may be more appropriate to scale $x^0 \sim ct$ differently or in fact to scale all variables as indicated below with factors $\beta^i(x^j, t)$. Locally now $\mathfrak{F}^{(\alpha)}$ satisfies the third order equation

$$4\mathfrak{F}^{(\alpha)'''} + [V^{(\alpha)}(x^{\alpha}) + m^2](\phi^{(\alpha)}\mathfrak{F}^{(\alpha)''} - \mathfrak{F}^{(\alpha)'})^3 = 0 \tag{3.11}$$

where $' \sim \partial/\partial\phi^{(\alpha)}$ and the "effective" potential V has the form

$$V^{(\alpha)}(x^{\alpha}) = \left[\frac{1}{\phi(x)}\sum_{\beta=0,\,\beta\neq\alpha}^{n-1}\partial^{\beta}\partial_{\beta}\phi(x)\right]\Bigg|_{x^{\beta\neq\alpha}\ fixed} \tag{3.12}$$

REMARK 7.3.1. Strictly speaking V^{α} does not have the form $\Box R/R$ of a quantum potential; however since it is created by the wave function ϕ we could well think of it as a form of quantum potential. We will refer to it as the effective potential as in [**346**] and note from Section 3.2 that with $\eta^{\mu\nu} = (-1, 1, 1, 1)$ and $\Box = -(1/c^2)\partial_t^2 + \Delta$, one has for $\phi = Rexp(iS/\hbar)$

$$\frac{1}{2m}(\partial S)^2 = \frac{\hbar^2}{2m}\frac{\Box\phi}{\phi} + \frac{\hbar^2}{2m}\frac{\Box R}{R}; \tag{3.13}$$

$$\partial(R^2\partial S) = 0;\ Q_{rel} = -\frac{\hbar^2}{2m}\frac{\Box R}{R}$$

The discussion below indicates that much further development of these themes should be possible.

As indicated in [**346**], once x^{α} is replaced with its functional dependence on $\mathfrak{F}^{\alpha}$ given in (3.10), (3.11) becomes an ordinary differential equation for $\mathfrak{F}^{\alpha}(\phi^{\alpha})$; further the functional structure of $\mathfrak{F}^{\alpha}$ does not depend on the parameters x^{β} for $\beta \neq \alpha$ (which enter ϕ^{α} as parameters). Now as a consequence of (3.10) one has

$$\frac{\partial}{\partial x^{\alpha}} = \frac{(8m)^{1/2}}{\hbar}\frac{1}{E^{(\alpha)}}\frac{\partial}{\partial\phi^{(\alpha)}};\ dx^{\alpha} = \frac{\hbar}{(8m)^{1/2}}E^{(\alpha)}d\phi^{(\alpha)} \tag{3.14}$$

where $E^{(\alpha)} = \phi^{(\alpha)}\mathfrak{F}^{(\alpha)''} - \mathfrak{F}^{(\alpha)'}$. Here (3.14) represents an induced parametrization on the spaces $T_P(U)$ and $T_P^*(U)$ ($P \in U$ - local tangent and cotangent spaces). Note there is no summation over α in (3.14). Now using the linearity of the metric tensor field (cf. [**322**]) one sees that the components of the metric in the $\{(\phi^{(\alpha)}, \mathfrak{F}^{(\alpha)}\}$ are

$$G_{\alpha\beta}(\phi) = \frac{\hbar^2}{8m}E^{(\alpha)}E^{(\beta)}\eta_{\alpha\beta}(x) \tag{3.15}$$

Now let z^{μ} ($\mu = 0, 1, \cdots, n-1$) be a general coordinate system in U and write the coordinate transformation matrices via

$$A^{\alpha}_{\mu} = \frac{\partial x^{\alpha}}{\partial z^{\mu}};\ (A^{-1})^{\mu}_{\alpha} = \frac{\partial z^{\mu}}{\partial x^{\alpha}} \tag{3.16}$$

The metric then takes the form

$$g_{\mu\nu}(z) = \frac{8m}{\hbar^2}\frac{1}{E^{(\alpha)}E^{(\beta)}}A^\alpha_\mu A^\beta_\nu G_{\alpha\beta}(\phi) \tag{3.17}$$

The components of the metric connection can be computed via

$$\Gamma^\rho_{\mu\nu} = \frac{1}{2}g^{\rho\sigma}(z)\sum_{\mathcal{P}}\epsilon_{\mathcal{P}}\mathcal{P}\left[\partial g_{\sigma\nu}(z)/\partial z^\mu\right] \tag{3.18}$$

where $\mathcal{P}$ is a cyclic permutation of the ordered set of indices $\{\sigma\nu\mu\}$ and $\epsilon_{\mathcal{P}}$ is the signature of $\mathcal{P}$. Via the coordinate transformation (3.16) the function $\phi^{(\alpha)}$ depends on all the z^μ. The metric connection (3.18) can be expressed in the $\{\phi^{(\alpha)}, \mathfrak{F}^{(\alpha)}\}$ parametrization via

$$\Gamma^\rho_{\mu\nu} = \left(\frac{2m}{\hbar}\right)^{1/2}\frac{E^{(\rho)}E^{(\sigma)}}{E^{(\gamma)}}(A^{-1})^\rho_\tau(A^{-1})^\sigma_\chi G^{\tau\chi}\times$$
$$\times\sum_{\mathcal{P}}\epsilon_{\mathcal{P}}\mathcal{P}\left[A^\gamma_\mu\frac{\partial}{\partial\phi^{(\gamma)}}\left(\frac{1}{E^{(\alpha)}E^{(\beta)}}A^\alpha_\sigma A^\beta_\nu G_{\alpha\beta}\right)\right] \tag{3.19}$$

In [**958**] one computes, in the $(\phi^{(\alpha)}, \mathfrak{F}^{(\alpha)})$ parametrization, the components of the curvature tensor, the Ricci tensor, and the scalar curvature and gives an expression for the Einstein equations (we omit the details here). These matters are taken up again in [**41**] for a general curved spacetime and some sufficient constraints are isolated which make the theory work. Also in both papers a quantized version of the KG equation is also treated and the relevant $x-\psi$ duality is spelled out in operator form. We omit this also in remarking that the main feature here for our purposes is the fact that one can describe spacetime geometry (at least locally) in terms of (field) solutions of a KG equation and prepotentials (which are themselves functions of the fields). In other words the coordinates are programmed by fields and if the motion of some particle of mass m is involved then its coordinates are choreographed by the fields with a quantum potential entering the picture via (3.12). In [**2**] a similar duality is worked out for the Dirac field and cartesian coordinates and to connect this with the aether idea one should examine the above formulas for $m \to 0$.

Let us do some rescaling now and recall the origin of equations such as (3.10). Thus (cf. [**191, 198, 206, 346**]) one writes (in 1-D)

(3.20)

$$\partial_\psi\mathfrak{F} = \psi^D \sim \bar{\psi};\ \partial_x\mathfrak{F} = \partial_\psi\mathfrak{F}\partial_x\psi = \psi^D\psi_x = \frac{1}{2}[\partial_x(\psi^D\psi) + W];\ W = \psi^D\psi_x - \psi^D_x\psi$$

and $W = constant$ (this is the scaling factor). For example with $x \sim ct$ we write

$$\mathfrak{F} = \frac{1}{2}\psi^E\psi + Wct = \frac{1}{2}\psi^D\psi + \gamma ct \tag{3.21}$$

to find $(\chi^0 \sim \gamma ct)$

$$\gamma ct = \frac{1}{2}\phi^0\frac{\partial\mathfrak{F}^0}{\partial\phi^0} - \mathfrak{F}^0;\ E^0 = \phi^0\frac{\partial^2\mathfrak{F}^0}{\partial(\phi^0)^2} - \frac{\partial\mathfrak{F}^0}{\partial\phi^0} \tag{3.22}$$

$$dt = \frac{E^0 d\phi^0}{2\gamma c};\ \partial_t = \frac{2\gamma c}{E^0}\frac{\partial}{\partial \phi^0} \tag{3.23}$$

For the other variables x^1, x^2, x^3 we write $\gamma c = \beta$ and

$$dx^i = \frac{E^i d\phi^i}{2\beta};\ \frac{\partial}{\partial x^i} = \frac{2\beta}{E^i}\frac{\partial}{\partial \phi^i} \tag{3.24}$$

Again (3.11) holds along with (3.12). Now simply replace $\sqrt{2m}/\hbar$ by γc when $ct \sim t = x^0$ and by β^i for the x^i ($1 \leq i \leq 3$) to obtain for example (here $\beta^i = \beta^i(x^j, t)$ and $\beta^0 = \gamma c$)

$$G_{\alpha\sigma}(\phi) = \frac{4E^\alpha E^\sigma}{\beta^\alpha \beta^\sigma}\eta_{\alpha\sigma} \tag{3.25}$$

in place of (2.15). Consequently we obtain an heuristic result

THEOREM 3.1. One can then continue this process to find analogues of (3.16) - (3.19) and we think now of the KG equation as $[(\partial_t^2/c^2) - \nabla_x^2 + (c^2m^2/\hbar^2)]\phi = 0$ so letting $m \to 0$ we obtain the photon (or aether) equation (2.10) of Isaev (assuming the scaling factors β^i can be taken independently of m). Now however we have in addition a geometry for this putative aether via (3.22) - (3.25) and their continuations (see [**1021**]).

Let us sketch next some arguments from [**494**] where we omit the historical and philosophical introduction describing some opinions and ideas of famous people, e.g. Dirac, Einstein, Faraday, Lorentz, Maxwell, Planck, Poincaré, Schwinger, et al. One begins with a KG equation

$$\left(\hbar^2\nabla^2 - \frac{\hbar^2}{c^2}\frac{\partial^2}{\partial t^2} - m^2c^2\right)\psi(s,t) = 0 \equiv (\hbar^2\Box - m^2c^2)\psi = 0 \tag{3.26}$$

One asserts that any relativistic equation for a free particle with mass m whould be understood not as an equation in vacuum but as an equation for a particle with mass m in the aether; thence setting the mass equal to zero one arrives at (3.1) for the equation of the aether itself. This is called the ψ-aether in contrast to the (impossible!) Lorentz-Maxwell aether. Now consider the case of an EM field with $\mathbf{H} = curl(\mathbf{A})$ and $\mathbf{E} = -(1/c)\partial_t\mathbf{A} - \nabla\phi$ and use the Lorentz condition $div\mathbf{A} + (1/c)\partial_t\phi = 0$. Then the potentials $\mathbf{A}$ and ϕ satisfy

$$\Box\mathbf{A} = \nabla^2\mathbf{A} - \frac{1}{c^2}\frac{\partial^2\mathbf{A}}{\partial t^2} = 0;\ \Box\phi = \nabla^2\phi - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t} = 0 \tag{3.27}$$

Using the Lorentz gauge one can take $\phi = 0$ so the charge independent part of the potentials is determined via

$$\Box\mathbf{A} = 0;\ div\mathbf{A} = 0;\ \phi = 0;\ \mathbf{E} = -\frac{1}{c}\frac{\partial\mathbf{A}}{\partial t};\ \mathbf{H} = curl\mathbf{A} \tag{3.28}$$

This system (3.28) is completely equivalent to the Maxwell-Lorentz equations and the general solution is given by a superposition of transverse waves. For a more symmetric representation one can write $\vec{\psi} = (\phi, \mathbf{A})$ and (3.27) becomes $\Box\psi_i(x,t) = 0$; these are called the equations for the real ψ-aether.

The classical unphysicality of ϕ, $\mathbf{A}$ now is removed in attaching them to the physically observable reality of the ψ-aether. Indeed the KG equation can be written as a product of two commuting matrix operators

$$I_{\alpha\beta}(\Box - m^2) = \sum_{\delta}\left(i\gamma^n \frac{\partial}{\partial x_n} + m\right)_{\alpha\delta}\left(i\gamma^k \frac{\partial}{\partial x_k} - m\right)_{\delta\beta} \tag{3.29}$$

and in order that the field function satisfy the KG equation one could require that it satisfy also one of the first order equations

$$\left(i\gamma^n \frac{\partial}{\partial x_n} + m\right)\psi = 0 \ \ or \ \ \left(i\gamma^n \frac{\partial}{\partial x_n} - m\right)\psi = 0 \tag{3.30}$$

Putting $m = 0$ in (3.30) one has possible equations for the neutrino-anti-neutrino field (there may be some question about $m = 0$ here). Recall that particle solutions of the KG equation corresponding to single valued representations of the Lorentz group have integer spins while particles with half-integer spin are described by a spinor representation. One also knows that the neutrino has spin $\hbar/2$. In any event (cf. (3.27)-(3.28)) the potentials ϕ and $\mathbf{A}$ are not merely auxilliary functions but are connected to physical reality in the form of the ψ-aether by neutrino-anti-neutrino pairs (cf. [**494**] for further arguments along these lines).

For an interesting connection of the ψ-aether with QM consider a hydrogen atom with spherically symetric and time independent potential $V(\mathbf{r}) = V(r)$ where $r = |\mathbf{r}|$. The solution to the SE $-i\hbar\partial_t\psi = -(\hbar^2/2m)\nabla^2\psi + V(r)\psi$ is obtained by separation of variables $\psi = u(\mathbf{r})f(t)$ with $u(\mathbf{r}) = R(r)Y(\theta,\phi)$. This is a problem of two body interaction (a proton and an electron) and for stationary states with energy E one looks at $\psi(x,t) = Cexp(-iEt/\hbar)$ satisfying

$$\frac{1}{sin(\theta)}\frac{\partial}{\partial\theta}\left(sin(\theta)\frac{\partial Y}{\partial\theta} + \frac{1}{sin^2(\theta)}\frac{\partial^2 Y}{\partial\phi^2}\right) + \lambda Y = 0; \tag{3.31}$$
$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \left\{\frac{2\mu}{\hbar^2}[E - V(r)] - \frac{\lambda}{r^2}\right\}R = 0$$

Here μ is the reduced mass of the system (proton + electron), E is the energy level for the bound state $p+e$ ($E < 0$), and $V(r) = e^2/r$ is the potential energy. (3.31) is solved by further separation of variables $Y = \Theta(\theta)\Phi(\phi)$ leading to

$$\frac{\partial^2\Phi}{\partial\phi^2} + \nu\Phi = 0;\ \frac{1}{sin(\theta)}\frac{d}{d\theta}\left(sin(\theta)\frac{\partial\Theta}{\partial\theta}\right) + \left(\lambda - \frac{\nu}{sin^2\theta}\right)\Theta = 0 \tag{3.32}$$

The solution for Φ is $\Phi_m(\phi) = (1/2\pi)exp(im\phi)$ with $\nu = m^2$ and physically admissible solutions for Θ (associated Legendre polynomials) require $\lambda = \ell(\ell+1)$ with $|m| \leq \ell$. For R one has

$$\frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{2\mu}{\hbar^2}\frac{e^2}{r}R(r) + \frac{2\mu}{\hbar^2}ER(r) - \frac{\ell(\ell+1)}{r^2}R = 0 \tag{3.33}$$

Now here $V(r) = (2\mu/\hbar^2)(e^2/r) = [\ell(\ell+1)/r^2]$ and the term involving e^2/r is responsible for the Coulomb interaction of a proton with an electron; however the second term $\ell(\ell+1)/r^2$ does not depend on any physical interaction (even though in [**847**] it is said to be connected with angular momentum). Now putting the Coulomb interaction to zero the $\ell(\ell+1)/r^2$ term does not disappear and it makes no sense to attribute it to angular momentum. It is now claimed that in fact this term arises because of the ψ-aether and an argument based on standing waves in a spherical resonator is given. Thus following [**155**] one considers an associated Borgnis function $U(r,\theta,\phi)$, having definite connections to **E** and **H**, and when it satisfies

$$\frac{\partial^2 U}{\partial r^2}+\frac{1}{r^2 sin(\theta)}\left[\frac{\partial}{\partial\theta}sin(\theta)\frac{\partial U}{\partial\theta}+\frac{\partial}{\partial\phi}\frac{1}{sin(\theta)}\frac{\partial U}{\partial\phi}\right]+k^2U=0 \tag{3.34}$$

the Maxwell equations are also valid. Further U is connected by definite relations with **A** and ϕ, i.e. with the ψ-aether (presumably all this is spelled out in [**155**]). To solve (3.34) one writes $U=F_1(r)F_2(\theta,\phi)$ (following the notation of [**155**]) and there results

$$(\mathbf{A})\ \ \frac{1}{sin(\theta)}\frac{\partial}{\partial\theta}sin(\theta)\frac{\partial F_2}{\partial\theta}+\frac{1}{sin^2\theta}\frac{\partial^2F_2}{\partial\phi^2}+\gamma F_2=0 \tag{3.35}$$

$$(\mathbf{B})\ \ r^2\frac{\partial^2F_1}{\partial r^2}+k^2r^2F_1-\gamma F_1=0$$

One considers here EM waves harmonic in time and characterized either by the frequency $\nu=kc/2\pi$ or by the wave vector $k=2\pi\nu/c$ with $[k]=1/cm$. Now **(A)** in (3.35) is the same as (3.31) with spherical function solutions and regular solutions of **(B)** in (3.35) exist when $\gamma=n(n+1)$. Setting $F_1(r)=rf(r)$ one obtains then

$$\frac{d^2f}{dr^2}+\frac{2}{r}\frac{df}{dr}+\left[k^2-\frac{n(n+1)}{r^2}\right]f(r)=0 \tag{3.36}$$

A little calculation puts (3.33) into the form

$$\frac{d^2R}{dr^2}+\frac{2}{r}\frac{dR}{dr}+\left(\frac{2\mu E}{\hbar^2}+\frac{2\mu e^2}{\hbar^2 r}-\frac{\ell(\ell+1)}{r^2}\right)R=0 \tag{3.37}$$

Setting $2\mu e^2/\hbar^2 r=0$ and replacing E by $E=p^2/2\mu$ in (3.37) one obtains

$$\frac{d^2R}{dr^2}+\frac{2}{r}\frac{dR}{dr}+\left(k^2-\frac{\ell(\ell+1)}{r^2}\right)R=0 \tag{3.38}$$

where $2\mu^2p^2/2\mu\hbar^2=k^2\hbar^2/\hbar^2=k^2$ with k the wave vector). Now (3.36) and (3.38) are identical and are solved under the same boundary conditions (i.e. $f(r)$ should be finite as $r\to 0$ and when $r\to\infty$ one wants $f(r)\to 0$ on the boundary of a sphere). The corresponding solutions to (3.36) represent standing waves inside the sphere at values $n=0,1,\cdots$ with $m\leq n$. Since EM waves are nothing but oscillations of the ψ-aether the term $n(n+1)/r^2$ in (3.36) is responsible for standing waves of the ψ-aether in a sphere resonator. Thus (mathematically at least) one can say that the problem of finding the energy levels in a hydrogen atom via the SE is equivalent to the problem of finding natural EM oscillations in a spherical resonator. One recalls that one of the basic postulates of QM (quantization of orbits in a hydrogen atom à la Bohr with $mvr=n\hbar/2\pi$) is equivalent to determination of conditions for existence of standing waves of the ψ-aether in a spherical resonator. This suggests that QM may be equivalent to "mechanics" of the ψ-aether. One remarks that until now only a small part of the alleged ψ-aether properties have been observed, namely in superfluidity and superconductivity (see e.g. [**968**]). It is suggested that one might well rethink a lot of physics in terms of the aether, rather than, for example, the standard model. In any event there is much further discussion in [**494**], related to real physical situations, and well worth reading.

4. REMARKS ON TRAJECTORIES

There have been a number of papers written involving microstates and Bohmian mechanics (cf. [**138, 140, 139, 191, 194, 197, 203, 305, 306, 307, 308, 309, 347, 373, 374, 375, 376, 348, 349, 520**]) and we sketch here some features of the Bouda-Djama method following [**309**]. There are some disagreements regarding quantum trajectories, discussed in [**138, 375**], which we will not deal with here. Generally we have followed [**347**] in our previous discussion and microstates were not explicitly considered (beyond mentioning the third order equation and the comments in Remark 2.2.2). Thus, referring to [**309**] for philosophy, one begins with the SE $-(\hbar^2/2m)\Delta\psi + V\psi = i\hbar\psi_t$ where $\psi = Rexp(iS/\hbar)$ in 3-D and arrives at the standard

$$\frac{1}{2m}(\nabla S)^2 - \frac{\hbar^2}{2m}\frac{\Delta R}{R} + V = -S_t;\ \nabla\cdot\left(R^2\frac{\nabla S}{m}\right) + V = -\partial(R^2) \tag{4.1}$$

witeh $Q = -(\hbar^2/2m)(\Delta R/R)$. Then one sets

$$\mathbf{j} = \frac{\hbar}{2mi}(\psi^*\nabla\psi - \psi\nabla\psi^*) = R^2\frac{\nabla S}{m} \Rightarrow \nabla\cdot\mathbf{j} + \partial_t R^2 = 0 \tag{4.2}$$

and $\rho = |\psi|^2 = R^2$ as usual. The velocity $\mathbf{v}$ is taken as $\mathbf{v} = \mathbf{j}/\rho = \nabla S/m$ here in the spirit of Bohm (and Dürr, Goldstein, Zanghi, et al). Working in 1-D with $S = S_0(x,E) - Et$ one recovers the stationary HJ equation of Section 2.2 for example and there is some discussion about the situation $S_0 = constant$ referring to Floyd and Farragi-Matone. Explicit calculations for microstates are considered and comparisons are indicated. The EP of Faraggi-Matone is then discussed as in Section 2.2 and the quantum mass field $m_Q = m(1 - \partial_E Q)$ is introduced. This leads to the third order differential equation for $\dot{x}$ (where $P = \partial_x S_0 = m_Q\dot{x}$)

$$\frac{m_Q^2}{2m} + V(x) - E + \frac{|hbar^2}{4m}\left(\frac{m_Q''}{m_Q} - \frac{3}{2}\frac{(m_Q')^2}{m_Q^2} - \frac{m_Q'}{m_Q}\frac{\ddot{x}}{\dot{x}^2} + \frac{\dddot{x}}{\dot{x}^3} - \frac{5}{2}\frac{\ddot{x}^2}{\dot{x}^4}\right) = 0 \tag{4.3}$$

It is observed correctly that (4.3) is a difficult equation to manipulate, requiring a priori a solution of the QSHJE.

Now one proposes a Lagrangian which depends on x, $\dot{x}$ and the set of hidden variables Γ which is connected to constants of integration from an equation like (4.3). This approach was developed in order to avoid dealing with the Jacobi type formula $t - t_0 = \partial S_0/\partial E$ which the authors felt should be restricted to HJ equations of first order. Then one looks for a quantum Lagrangian L_q such that $(d/dt)(\partial L_q/\partial\dot{x}) - \partial_x L_q = 0$ and writes

$$L_q(x,\dot{x},\Gamma) = \frac{m}{2}\dot{x}^2 f(x,\Gamma) - V(x);\ \frac{\partial L_q}{\partial\dot{x}} = m\dot{x}f(x,\Gamma);\ \frac{\partial L_q}{\partial x} = \frac{m}{2}\dot{x}^2 f_x - V_x \tag{4.4}$$

This leads to

$$mf(x,\Gamma)\ddot{x} + \frac{m\dot{x}^2}{2}f_x + V_x = 0 \tag{4.5}$$

Then set $H_q = (\partial_x L_q)\dot{x} - L_q$ and $P = \partial L_q/\partial\dot{x} = m\dot{x}f$ so

$$H_q = \frac{m\dot{x}^2}{2}f(x,\Gamma) + V(x) = \frac{P^2}{2mf} + V(x) \tag{4.6}$$

Working with the stationary situation $S = S_0(x,\Gamma) - Et$ some calculation gives then

$$\frac{1}{2mf}S_x^2 + V = -S_t \Rightarrow \frac{1}{2mf}(\partial_x S_0)^2 + V(x) - E = 0 \tag{4.7}$$

Now referring to the general equation (2.18) in Chapter 2 (extracted from [**347**]) one writes here $w = \tilde{\theta}/\tilde{\phi} \sim \psi^D/\psi \in \mathbf{R}$ with $(\alpha \sim \omega)$ so that (cf. [**?, ?**])

$$e^{2iS_0/\hbar} = e^{i\omega}\frac{(\tilde{\theta}/\tilde{\phi}) + i\bar{\ell}}{(\tilde{gt}/\tilde{\phi}) - i\ell} \rightsquigarrow S_0 = \hbar Tan^{-1}\frac{\theta + \mu\phi}{\nu\theta + \phi} \tag{4.8}$$

(cf. [**139**] for details). For the QSHJE the basic equation is (2.2.17) which we repeat as

(4.9)
$$\frac{1}{2m}(S_0')^2 + \mathfrak{W} + Q = 0;\ \mathfrak{W} = -\frac{\hbar^2}{4m}\left\{e^{2iS_0/\hbar}, x\right\} \sim V - E;\ Q = \frac{\hbar^2}{4m}\{S_0, x\}$$

There is a "quantum" transformation $x \to \hat{x}$ described in [**347, 348**] with the QSHJE arising then from a conformal modification of the CSHJE. Thus note (•) $\{x, S_0\} = -(S_0')^{-2}\{S_0, x\}$ and define $\mathfrak{U}(S_0) = \{x, S_0\}/2 = -(1/2)(S_0')^{-2}\{S_0, x\}$. This gives a conformal rescaling $\frac{1}{2m}(S_0')^2[1 - \hbar^2\mathfrak{U}] + V - E = 0$ since

(4.10)
$$\frac{1}{2m}(S_0')^2[1 - \hbar^2\mathfrak{U}] = \frac{1}{2m}(S_0')^2[1 - \frac{\hbar^2}{2}\{x, S_0\}] = \frac{1}{2m}(S_0')^2[1 + \frac{\hbar^2}{2}(S_0')^{-2}\{S_0, x\}] =$$

$$= \frac{1}{2m}(S_0')^2 + \frac{\hbar^2}{4m}\{S_0, x\} = \frac{1}{2m}(S_0')^2 + Q \Rightarrow Q = -\frac{\hbar^2}{2m}(S_0')^2\mathfrak{U}$$

which agrees with $Q = (\hbar^2/4m)\{S_0, x\}$ using (•). Then from (4.10)

$$\left(\frac{\partial x}{\partial\hat{x}}\right)^2 = 1 = \hbar^2\mathfrak{U}(S_0) = 1 + 2m(S_0')^2 Q \Rightarrow \hat{x} = \int^x \frac{dx}{\sqrt{1 + 2m(\partial S_0)^{-2}Q}} \tag{4.11}$$

Similarly using the QSHJE (2.2.17) we have

$$\left(\frac{\partial x}{\partial\hat{x}}\right)^2 = (S_0')^{-2}[(S_0')^2 + 2mQ] = [(S_0')^2 - 2m\mathfrak{W}](S_0')^{-2} \Rightarrow \tag{4.12}$$

$$\Rightarrow \hat{x} = \int^x \frac{S_0' dx}{\sqrt{2m(E - V)}}$$

This all follows from [**347, 348**] and is used in [**309**]. Now from (4.7), (4.9), (4.10), and the QSHJE one can write (correcting a sign in [**309**])

$$f(x,\Gamma)=\left[1+\frac{\hbar^2}{2}(S_0')^{-2}\{S_0,x\}\right]^{-1}\Rightarrow f=\frac{(S_0')^2}{2m(E-V)} \tag{4.13}$$

and via (4.8) $\Gamma=\Gamma(E,\mu,\nu)$ with $f=f(x,E,\mu,\nu)$. Putting this in (4.7) gives then

$$E=\frac{m\dot{x}^2}{2}\frac{(S_0')^2}{2m(E-V)}+V\Rightarrow \dot{x}S_0'=2(E-V) \tag{4.14}$$

Note that this equation also follows from (4.5), namely $m\dot{x}f=\partial L_q/\partial\dot{x}$, and integration (cf. [**309**]). Now for the appropriate third order trajectory equation in this framework, one finds from (4.14) and the QSHJE

$$(E-V)^4-\frac{m\dot{x}^2}{2}(E-V)^3+\frac{\hbar^2}{8}\left[\frac{3}{2}\left(\frac{\ddot{x}}{\dot{x}}\right)^2-\frac{\dddot{x}}{\dot{x}}\right](E-V)^2- \tag{4.15}$$
$$-\frac{\hbar^2}{8}\left[\dot{x}^2\frac{d^2V}{dx^2}+\ddot{x}\frac{dV}{dx}\right](E-V)-\frac{3\hbar^2}{16}\left[\dot{x}\frac{dV}{dx}\right]^2=0$$

(cf. [**138, 309**]). This is somewhat simpler to solve that (4.3) since it is independent of the SE and the QSHJE. We refer now to [**138, 140, 139, 305, 306, 307, 308, 309**] for more in this direction.

CHAPTER 8

REMARKS ON QFT AND TAU FUNCTIONS

1. INTRODUCTION AND BACKGROUND

The idea here is to connect various aspects of quantum groups, quantum field theory (QFT), and tau functions by various algebraic techniques. Actually the principal connections are already in place but we want to make matters explicit and direct with some attempt at understanding. For references we have in mind [**191, 205**] for Wick's theorem and tau functions, [**157, 158, 159, 160, 161, 162, 168**] for quantum groups and QFT, and [**147, 192, 193, 207, 212, 656, 662, 664**] for q-tau functions and Hirota formulas.

We go first to the beautiful set of ideas in [**157, 158, 159, 161**] (and references there) and will sketch some of the results (sometimes without proof). The main theme is to create an algebraic framework which "tames" the intricate combinatorics of QFT. Thus the main concepts used in the practical calculation of observables are covered, namely, normal and time ordering and renormalization. One begins with a finite dimensional vector space V (e_i a basis) and then the symmetric algebra $S(V)$ is defined as $S(V) = \oplus_0^\infty S^n(V)$ where $S^0(V) = \mathbf{C}$, $S^1(V) = V$, and $S^n(V)$ is spanned by elements $e_{i_1} \vee \cdots \vee e_{i_n}$ with $i_1 \leq i_2 \leq \cdots \leq i_n$. The symbol $\vee$ denotes the associative and commutative product $\vee : S^m(V) \otimes S^n(V) \to S^{m+n}(V)$ defined on elements of the basis via $(e_{i_1} \vee \cdots \vee e_{i_m}) \vee (e_{i_{m+1}} \vee \cdots \vee e_{i_{m+n}}) = e_{i_{\sigma(1)}} \vee \cdots \vee e_{i_{\sigma(m+n)}}$ where σ is the permutation on $m+n$ elements such that $i_{\sigma(1)} \leq \cdots \leq i_{\sigma(m+n)}$ (extended by linearity and associativity to all elements of $S(V)$. The unit of $S(V)$ is $1 \in \mathbf{C}$ (i.e. for any $u \in S(V)$ one has $1 \vee u = u \vee 1 = u$). The algebra is graded via $|u| = n$ for $u \in S^n(V)$ and $\vee$ is a graded map (i.e. if $|u| = n$ and $|v| = m$ then $|u \vee v| = m + n$). In fact $S(V)$ is the algebra of polynomials in e_i with elements of $S^n(V)$ being homogeneous polynomials of degree n. Let now u, v, w be elements of $S(V)$, $a, a_i, b, c \in V$ and e_i basis elements of V as above. Define a coproduct over $S(V)$ via

$$\Delta 1 = 1 \otimes 1;\ \Delta a = a \otimes 1 + 1 \otimes a;\ \Delta(u \vee v) = \sum (u_1 \vee v_1) \otimes (u_2 \vee v_2) \quad (1.1)$$

(here e.g. $\Delta u = \sum u_1 \otimes u_2$ etc.). This coproduct is coassociative and cocommutative and equivalent to a coproduct from [**768**]. Note in particular

$$\Delta(a \vee b) = \sum (a_1 \vee b_1) \otimes (a_2 \vee b_2) \quad (1.2)$$

and since $\Delta a = a \otimes 1 + 1 \otimes a$ with $\Delta b = b \otimes 1 + 1 \otimes b$ one obtains

$$\Delta(a \vee b) = (a \vee b) \otimes 1 + 1 + a \otimes b + b \otimes a + 1 \otimes (a \vee b) \tag{1.3}$$

At the next order $\Delta(a \vee b \vee c) = 1 \otimes a \vee b \vee c + a \otimes b \vee c + b \otimes a \vee c + c \otimes a \vee b + a \vee b \otimes c + a \vee c \otimes b + b \vee c \otimes a + a \vee b \vee c \otimes 1$. Generally if $u = a_1 \vee a_2 \vee \cdots \vee a_n$ one has following [**610**]

$$\Delta u = u \otimes 1 + 1 \otimes u + \sum_{p=1}^{n-1} \sum_{\sigma} a_{\sigma(1)} \vee \cdots \vee a_{\sigma(p)} \otimes a_{\sigma(p+1)} \vee \cdots \vee a_{\sigma(n)} \tag{1.4}$$

where σ runs over the $(p, n-p)$ shuffles. Such a shuffle is a pemutation σ of $1, \cdots, n$ such that $\sigma(1) < \sigma(2) < \cdots < \sigma(p)$ and $\sigma(p+1) < \cdots < \sigma(n)$. The counit is defined via $\epsilon(1) = 1$ and $\epsilon(u) = 0$ if $u \in S^n(V)$ for $n > 0$. The antipode is defined via $s(u) = (-1)^n u$ for $u \in S^n(V)$ and in particular $s(1) = 1$. Since the symmetric product is commutative the antipode is an algebra morphism($s(u \vee v) = s(u) \vee s(v)$). This algebra is called the symmetric Hopf algebra.

One goes next to the Laplace pairing on $S(V)$ defined as a bilinear form $V \times V \to \mathbf{C}$ denoted $(a|b)$ and extended to $S(V)$ via

$$(u \vee v|w) = \sum (u|w_1)(v|w_2);\ (u|v \vee w) = \sum (u_1|v)(u_2|w) \tag{1.5}$$

A priori the bilinear form is neither symmetric nor antisymmetric (cf. [**433**] where the form is still more general). The term Laplace pairing comes from the Laplace identities in determinant theory where the determinant is expressed in terms of minors. In [**433**] it is proved that if $u = a_1 \vee a_2 \vee \cdots a_k$ and $v = b_1 \vee b_2 \vee \cdots \vee b_n$ then $(u|v) = 0$ if $n \neq k$ and $(u|v) = perm(a_i|b_j)$ if $n = k$ where

$$perm(a_i|b_j) = \sum_{\sigma} (a_1|b_{\sigma(1)}) \cdots (a_k|b_{\sigma(k)}) \tag{1.6}$$

over all permutations σ of $(1, \cdots, k)$. The permanent is a kind of determinant where all signs are positive. For example $(a \vee b|c \vee d) = (a|c)(b|d) + (a|d)(b|c)$. A Laplace-Hopf algebra now means the symmetric Hopf algebra with a Laplace pairing.

Now comes the circle product which treats in one stroke the operator product and the time ordered product according to the definition of the bilinear form $(a|b)$. Following [**804**] the circle product is defined via $v \circ v = \sum u_1 \vee v_1 (u_2|v_2)$ which by cocommutativity of the coproduct on $S(V)$ is equivalent to

$$u \circ v = \sum u_1 \vee v_2 (u_2|v_1) = \sum u_2 \vee v_1 (u_1|v_2) = \sum (u_1|v_1) u_2 \vee v_2 \tag{1.7}$$

A few examples are instructive, thus

$$a \circ b = a \vee b + (a|b);\ (a \vee b) \circ c = a \vee b \vee c + (a|c)b + (b|c)a;\ a \circ (b \vee c) = \tag{1.8}$$

$$= a \vee b \vee c + (a|c)b + (a|b)c;\ a \circ b \circ c = a \vee b \vee c + (a|b)c + (a|c)b + (b|c)a$$

Further some useful properties are

$$u \circ 1 = 1 \circ u = u;\ u \circ (v + w) = u \circ v + u \circ w;\ u \circ (\lambda v) = (\lambda u) \circ v = \lambda(u \circ v) \tag{1.9}$$

A detailed proof of the associativity of $\circ$ is given in [**158**] which is based on two lemmas which we record as $(u|v\circ w)=(u\circ v|w)$ and

$$\Delta(u\circ v)=\sum(u_1\vee v_1)\otimes(u_2\circ v_2)=\sum(u_1\circ v_1)\otimes(u_2\vee v_2) \tag{1.10}$$

The proofs are straightforward and associativity of $\circ$ follows directly. One has also a useful formula (proved in [**158**])

$$u\vee v=\sum(s(u_1)|v_1)u_2\circ v_2=\sum(u_1|s(v_1))u_2\circ v_2 \tag{1.11}$$

Next in [**158**] one proves a few additional formulas, namely

(1.12)

$$(u|v)=\sum s(u_1\vee v_1)\vee(u_2\circ v_2);\ \ u\circ(v\vee w)=\sum(u_{11}\circ v)\vee(u_{12}\circ w)\vee s(u_2)$$

1.1. WICK'S THEOREM AND RENORMALIZATION. One shows next that the circle product satisfies a generalized version of Wick's theorem. There are two such theorems for normal or time ordered products but they have an identical structure (cf. [**935**]). In verbal form one has e.g. that the time or normal ordered product of a given number of elements of V is equal to the sum over all possible pairs of contractions (resp. pairings). A contraction $a^{\bullet}b^{\bullet}$ is the difference between the time ordered and the normal product. Thus $a^{\bullet}b^{\bullet}=T(ab)-:ab:$. A pairing $a^{\diamond}b^{\diamond}$ is the difference between the operator product and the normal product, namely $a^{\diamond}b^{\diamond}=ab-:ab:$. Both the contraction and the pairing are scalars and one can identify the normal product $:ab:$ with $a\vee b$ because the normal product has all the properties required for a symmetric product (see remarks later). Now on the one hand the time ordered product is symmetric and $a^{\bullet}b^{\bullet}$ is a symmetric bilinear form that we can identify with $(a|b)$. Then the time ordered product of two operators is equal to the circle product obtained from the symmetric bilinear form. On the other hand the pairing obtained from the operator product is an antisymmetric bilinear form $a^{\diamond}b^{\diamond}=(ab-ba)/2$ that we can also identify with (a different) $(a|b)$. The operator product of two elements of V is now the circle product obtained with this antisymmetric bilinear form.

REMARK 8.1.1. To put this in a more visible form recall $a\circ b=a\vee b+(a|b)$ so $a^{\diamond}b^{\diamond}=ab-:ab:\sim ab=a\circ b=a\vee b+(a|b)$ with $(a|b)=a^{\diamond}b^{\diamond}$. Similarly for $a^{\bullet}b^{\bullet}=(a|b)=T(ab)-a\vee b\sim T(ab)=(a|b)+a\vee b$ with $T(ab)=a\circ b$.

Now the main ingredient in proving Wick's theorem as in [**995**] is to go from a product of n elements of V to a product of $n+1$ elements via

$$:a_1\cdots a_n:b=:a_1\cdots a_nb:+\sum_{j=1}^n a_j^{\bullet}b^{\bullet}:a_1\cdots a_{j-1}a_{j+1}\cdots a_n: \tag{1.13}$$

To prove this we use the definition of the circle product and (1.1) to get

$$u\circ b=u\vee b+\sum(u_1|b)u_2;\ \ u=u_1\vee\cdots\vee a_n \tag{1.14}$$

(note $\Delta b=b\otimes 1+1\otimes b$ and $u\circ b=\sum(u_1|b_1)u_2\vee b_2=\sum(u_1|b)u_2\vee 1+\sum(u_1|1)u_2\vee b$; but $w\vee 1=w$ and $u\circ 1=u=\sum(u_1|1)u_2$ since $\Delta 1=1\otimes 1$). The Laplace pairing $(u_1|b)$ is zero if the grading of u_1 is different from 1, so u_1 must be an element of V.

According to the general definition (1.4) for Δu this happens only for the $(1, n-1)$ shuffles. By definition then a $(1, n-1)$ shuffle is a permutation σ of $(1, \cdots, n)$ such that $\sigma(2) < \cdots < \sigma(n)$ and the corresponding terms of the coproduct in Δu are $\sum_1^n a_j \otimes a_1 \vee \cdots \vee a_{j-1} \vee a_{j+1} \vee \cdots \vee a_n$. This proves the required identity so the circle product satisfies Wick's theorem. One repeats that if the bilinear form is 1/2 of the commutator then one has the Wick theorem for operator products and if the bilinear form is the Feynman propagator then one has Wick's theorem for time ordered products.

Now one introduces a renormalised circle product $\bar{\circ}$. In a finite dimensional vector space the purpose of renormalisation is no longer to remove infinities, since everything is finite, but to provide a deformation of the circle product involving an infinite number of parameters. The physical meaning of the parameters goes as follows. Time ordering of operators is clear when two operators are defined at different times; however the meaning is ambiguous when the operators are defined at the same time. Renormalisation is present to parametrise the ambiguity. The parameters are defined as a linear map $\zeta : S(V) \to \mathbf{C}$ such that $\zeta(1) = 1$ and $\zeta(a) = 0$ for $a \in V$. The parameters form a renormalisation group with product $*$ defined via

$$(1.15) \qquad (\zeta * \zeta')(u) = \sum \zeta(u_1)\zeta'(u_2)$$

This is called the convolution of the Hopf algebra. The coassociativity and cocommutativity of the Hopf algebra implies that $*$ is associative and commutative. The unit of the group is the counit ϵ of the Hopf algebra and the inverse ζ^{-1} is defined via $(\zeta * \zeta^{-1})(u) = \epsilon(u)$ or recursively via

$$(1.16) \qquad \zeta^{-1}(1) = 1;\ \zeta^{-1}(u) = -\zeta(u) - \sum{}' \zeta(u_1)\zeta^{-1}(u_2)$$

where $\sum' u_1 \otimes u_2 = \Delta u - 1 \otimes u - u \otimes 1$ for $u \in S^n(V)$ with $n > 0$. As examples note

$$(1.17) \qquad \zeta^{-1}(a) = 0;\ \zeta^{-1}(a \vee b) = -\zeta(a \vee b);\ \zeta^{-1}(a \vee b \vee c) = -\zeta(a \vee b \vee c);$$

$$\zeta^{-1}(a\vee b\vee c\vee d) = -\zeta(a\vee b\vee c\vee d) + 2\zeta(a\vee b)\zeta(c\vee d) + 2\zeta(a\vee c)\zeta(b\vee d) + 2\zeta(a\vee d)\zeta(b\vee c)$$

Now from the renormalization parameters one defines a Z-pairing

$$(1.18) \qquad Z(u, v) = \sum \zeta^{-1}(u_1)\zeta^{-1}(v_1)\zeta(u_2 \vee v_2)$$

A few examples are

(1.19)

$$Z(u, v) = Z(v, u);\ Z(1, u) = \epsilon(u);\ Z(a, b) = \zeta(a \vee b);\ Z(a, b \vee c) = \zeta(a \vee b \vee c);$$

$$Z(a \vee b, c \vee d) = \zeta(a \vee b \vee c \vee d) - \zeta(a \vee b)\zeta(c \vee d)$$

The main property of the Z-pairing is the coupling identity

$$(1.20) \qquad \sum Z(u_1 \vee v_1, w)Z(u_2, v_2) = \sum Z(u, v_1 \vee w_1)Z(v_2, w_2)$$

One can check e.g. when $u = a$, $v = b$, $w = c \vee d$ that

$$(1.21) \qquad Z(a \vee b, c \vee d) = Z(a, b \vee c \vee d) + Z(a, c)Z(b, d) + Z(b, c)Z(a, d)$$

To show (1.20) one uses the lemma $\sum \zeta^{-1}(u_1 \vee v_1)Z(u_2, v_2) = \zeta^{-1}(u)\zeta^{-1}(v)$ (cf. [**158**] for proof). To prove the coupling identity one expands Z via

$$\sum Z(u_1 \vee v_1, w)Z(u_1, v_2) = \sum \zeta^{-1}(u_{11} \vee v_{11})\zeta^{-1}(w_1)\zeta(u_{12} \vee v_{12} \vee w_2)Z(u_2, v_2) \tag{1.22}$$

Some calculation (cf. [**158**]) yields

$$\sum Z(u_1 \vee v_1, w)Z(u_2, v_2) = \zeta^{-1}(u_1)\zeta^{-1}(v_1)\zeta^{-1}(w_1)\zeta(u_2 \vee v_2 \vee w_2) \tag{1.23}$$

The right side is fully symmetric in u, v, w so all permutations of u, v, w in the left side give the same result. In particular the permutation $(u, v, w) \to (v, w, u)$ and the symmetry of Z transform the left side of (1.20) into its right side, and this proves (1.20). One can check also that the Laplace pairing satisfies the coupling identity

$$\sum (u_1 \vee v_1|w)(u_2|v_2) = \sum (u|v_1 \vee w_1)(v_2|w_2) \tag{1.24}$$

It would be interesting to know the most general solution of the coupling identity.

One defines next a modified Laplace pairing via (♣) $\overline{(u|v)} = \sum kZ(u_1, v_1)(u_2|v_2)$; thus e.g.

$$\overline{(u|1)} = \overline{(1|u)} = \epsilon(u);\ \overline{(a|b)} = \zeta * a \vee b) + (a|b);\ \overline{(a|b \vee c)} = \zeta(a \vee b \vee c) \tag{1.25}$$

The modified Laplace pairing also satisfies the coupling identity

$$\sum \overline{(u_1 \vee v_1|w)(u_2|v_2)} = \sum \overline{(u|v_1 \vee w_1)(v_2|w_2)} \tag{1.26}$$

(cf. [**158**]). Finally one can define a renormalized circle product via (♠) $u\bar{\circ}v = \sum \overline{(u_1|v_1)}u_2 \vee v_2$. As examples one has

$$1\bar{\circ}u = u\bar{\circ}1 = u;\ \overline{(u|v)} = \epsilon(u\bar{\circ}v);\ a\bar{\circ}b = a \vee b + \zeta(a \vee b) + (a|b) \tag{1.27}$$

The renormalised circle product is associative (cf. [**158**]) and that is considered a main result. One notes that the renormalisation group acts on the circle product (not on the elements of the algebra) even thought the action is expressed via the algebra.

1.2. PRODUCTS AND RELATIONS TO PHYSICS. In QFT the boson operators commute inside a t-product and we consider now the circle product to be commutative. Then in particular $a \circ b = b \circ a$ so $(a|b) = (b|a)$. Conversely if $(a|b) = (b|a)$ for all $a, b \in V$ then $u \circ v = v \circ u$ for all $u, v \in S(V)$. To see this note that if $(a|b) = (b|a)$ then $(u|v) = (v|u)$ for all $u, v \in S(V)$ (cf. (1.6)). Then because of the commutativity of the symmetric product one obtains

$$u \circ v = \sum u_1 \vee v_1(u_2|v_2) = \sum v_1 \vee u_1(v_2|u_2) = v \circ u \tag{1.28}$$

Now one defines a T map from $S(V)$ to $S(V)$ via $T(1) = 1,\ T(a) = a$ for $a \in V$ and $T(u \vee v) = T(u) \circ T(v)$. More explicitly $T(a_1 \vee \cdots a_n) = a_1 \circ \cdots a_n$. The circle product is associative and here commutative so T is well defined. The main property we need is

$$\Delta T(u) = \sum u_1 \otimes T(u_2) = \sum T(u_1) \otimes u_2 \tag{1.29}$$

(cf. [**158**] for proof). Next one shows that the T map can be written as the exponential of an operator Σ. First define a derivation δ_k attached to a basis e_k of V by requiring

$$\delta_k 1 = 0,\ \delta_k e_j = \delta_{kj};\ \delta_k(u \vee v) = (\delta_k u) \vee v + u \vee (\delta_k v) \tag{1.30}$$

This gives a Leibnitz relation yielding $\delta_1(e_1 \vee e_2) = e_2$ and $\delta_2(e_1 \vee e_2) = e_1$. From this follows $\delta_i\delta_j = \delta_j\delta_i$; further one notes $\delta_i(u|v) = 0$. Now define the infinitesimal T map as $\Sigma = (1/2)\sum_{i,j}(e_i|e_j)\delta_i\delta_j$. One shows that $T = exp(\Sigma)$. First one needs $[\Sigma, a] = \sum(a|e_i)\delta_i$ and to see this apply Σ to an element $e_k \vee u$ to get

$$\Sigma(e_k \vee u) = \frac{1}{2}\sum_{i,j}(e_i|e_j)\delta_i\delta_j(e_k \vee u) = \frac{1}{2}\sum(e_i|e_j)\delta_i(\delta_{jk}u + e_k \vee \delta_j u) = \tag{1.31}$$

$$= \frac{1}{2}\sum_i(e_i|e_j)\delta_i u + \frac{1}{2}\sum_j(e_k|e_j)\delta_j u + \frac{1}{2}\sum(e_i|e_j)e_k \vee \delta_i\delta_j u = \sum_j(e_k|e_j)\delta_j u + e_k \vee \Sigma u$$

Extending this by linearity to V one obtains $\Sigma(a \vee u) = a \vee \Sigma u + \sum(a|e_j)\delta_j u$ and more generally

$$\Sigma(u \vee v) = (\Sigma u) \vee v + u \vee (\Sigma v) + \sum_{i,j}(e_i|e_j)(\delta_i u) \vee (\delta_j v) \tag{1.32}$$

From the commutation of the derivations we obtain $[\Sigma, \delta_k] = 0$ and $[\Sigma, [\Sigma, a]] = 0$; therefore the classical formula (cf. [**507**]) yields $exp(\Sigma)a exp(-\Sigma) = a + [\Sigma, a]$ so that

$$[exp(\Sigma), a] = \sum(a|e_i)\delta_i exp(\Sigma) = exp(\Sigma)\sum(a|e_i)\delta_i \equiv \tag{1.33}$$

$$\equiv exp(\Sigma)(a \vee u) = a \vee (exp(\Sigma)u) + [\Sigma, a](exp(\Sigma)u)$$

Note also $a \circ u = a \vee u + [\Sigma, a]u$. From this one preceeds inductively. One has $T(1) = exp(\Sigma)1 = 1$ and $T(a) = a = exp(\Sigma)a$ since $\Sigma a = 0$. Assuming the property $T = exp(\Sigma)$ is true up to grading k take $u \in S^k(V)$ and calculate

$$T(a \vee u) = a \circ T(u) = a \vee T(u) + [\Sigma, a]T(u) = a \vee e^{\sigma}u + [\Sigma, a]e^{\Sigma}u = e^{\Sigma}(a \vee u) \tag{1.34}$$

Thus the property is true for $a \vee u$ of grading $k + 1$.

Now one can write the T map as a sum of scalars multiplied by elements of $S(V)$ in the form $T(u) = \sum t(u_1)u_2$ where t is a linear map $S(V) \to \mathbf{C}$ defined recursively by $t(1) = 1,\ t(a) = 0$ for $a \in V$ and

$$t(u \vee v) = \sum t(u_1)t(v_1)(u_2|v_2) \tag{1.35}$$

This is called the t map and the details are in [**158**]. The t map is well defined because $t(u) = \epsilon(T(u))$ which can be proved by recursion. It is true for $u = 1$ and $u = a$ and if true up to grading k take $w = u \vee v$ and one has

$$\epsilon(T(w)) = \sum t(w_1)\epsilon(w_2) = t(\sum w_1\epsilon(w_2)) = t(w) \tag{1.36}$$

A few examples are

$$t(a \vee b) = (a|b);\ t(a \vee b \vee c \vee d) = (a|b)(c|d) + (a|c)(b|d) + (a|d)(b|c) \tag{1.37}$$

The general formula for $t(a_1 \vee \cdots \vee a_{2n})$ has $(2n-1)!!$ terms which can be written

$$t(a_1 \vee \cdots \vee a_{2n}) = \sum_{\sigma} \prod_{j=1}^{n} (a_{\sigma(j)}|a_{\sigma(j+n)}) \tag{1.38}$$

where the sum is over permutations σ of $(1, \cdots, 2n)$ such that $\sigma(1) < \cdots < \sigma(n)$ and $\sigma(j) < \sigma(j+n)$ for $j = 1, \cdots, n$. Alternatively as a sum over all permutations of $(1, \cdots, 2n)$ (since $(a|b) = (b|a)$ here) one has

$$t(a_1 \vee \cdots \vee a_{2n}) = \frac{1}{2^n n!} \sum_{\sigma} (a_{\sigma(1)}|a_{\sigma(2)}) \cdots (a_{\sigma(2n-1)}|a_{\sigma(2n)}) \tag{1.39}$$

Next one defines renormalised T maps and shows that they coincide with the renormalised t products of QFT. A renormalised T map $\bar{T}$ is a linear map $S(V) \to S(V)$ such that $\bar{T}(1) = 1$, $\bar{T}(a) = a$ for $a \in V$, and $\bar{T}(u \vee v)\bar{T}(u)\bar{\circ}\bar{T}(v)$. Since the circle product is assumed commutative here the renormalised circle product is also commutative and the map $\bar{T}$ is well defined. Now following the proof of (1.29) one can show that $\Delta\bar{T}(u) = \sum u_1 \otimes \bar{T}(u_2)$ and one derives two renormalisation identities which we state without proof, namely

$$T(u)\bar{\circ}T(v) = \sum Z(u_1, v_1)T(u_2) \circ T(v_2);\ \bar{T}(u) = \sum \zeta(u_1)T(u_2) \tag{1.40}$$

The second is a second main result of [**158**] and corresponds to Pinter's identity in [**770**]. Its importance stems from the fact that it is valid also for field theories that are only renormalisable with an infinite number of renormalisation parameters. A proof of the formula for $T(u)$ goes as follows. It is true for $u = 1$ or $u = a$ and $v = 1$ or $v = b$. Suppose it holds for u and v; then by definition and using the recursion hypothesis

$$\bar{T}(u \vee v) = \bar{T}(u)\bar{\circ}\bar{T}(v) = \sum \zeta(u_1)\zeta(v_1)T(u_2)\bar{\circ}T(v_2) \tag{1.41}$$

But $a \circ u = a \vee u + [\Sigma, a]u$ so

$$\bar{T}(u \vee v) = \sum \zeta(u_1)\zeta(v_1)Z(u_{21}, v_{21})T(u_{22}) \circ T(v_{22}) \tag{1.42}$$

From the coassociativity of the coproduct and the definition (1.18) of the Z pairing one obtains

$$\bar{T}(u \vee v) = \sum \zeta(u_1 \vee v_1)T(u_2) \circ T(v_2) = \sum \zeta((u \vee v)_1)T((u \vee v)_2) \tag{1.43}$$

which is the required identity for $u \vee v$.

Now the reasoning leading to the scalar t map can be followed exactly to define a scalar renormalised t map as a linear map $\bar{t} : S(V) \to \mathbf{C}$ such that $\bar{t}(1) = 1$, $\bar{t}(a) = 0$ for $a \in V$ and $\bar{t}(u \vee v) = \sum \bar{t}(u_1)\bar{t}(v_1)\overline{(u_2|v_2)}$. Consequently $\bar{t}(u) = \epsilon(\bar{T}(u))$ and $\bar{T}(u) = \sum \bar{t}(u_1)u_2$. Further $\bar{T}(u) = \sum \zeta(u_1) \otimes \bar{T}(u_2)$ enables one to show that $\bar{t}(u) = \sum \zeta(u_1)t(u_2)$. As examples consider

$$\bar{t}(a \vee b) = (a|b) + \zeta(a \vee b);\ \bar{t}(a \vee b \vee c) = \zeta(a \vee b \vee c) \tag{1.44}$$

The scalar renormalised t produce corresponds to the numerical distributions of the causal approach (cf. [**768, 848**]).

To define a normal product one starts from creation and annihilation operators a_k^+ and a_k^- taken as basis vectors of V^+ and V^-. These two bases are in involution via $(a_k^+)^* = a_k^-$ and $(a_k^-)^* = a_k^+$. The creation operators commute and there is no other relation between them (similarly for the annihilation operators). Thus the Hopf algebra of the creation operators is the symmetric algebra $S(V^+)$ and that of the annihilitation operators is $S(V^-)$. One defines $V = V^+ \oplus V^-$ $(a = a^+ + a^-)$ and let P (resp. M) be the projector $V \to V^+$ (resp. $V \to V^-$). Thus $P(a) = a^+$ and $M(a) = a^-$. There is an isomorphism ϕ between $S(V)$ and the tensor product $S(V^+) \otimes S(V^-)$ via $\phi(u) = \sum P(u_1) \otimes M(u_2)$. Now $S(V)$ is the vector space of normal products and one recovers the fact that a normal product puts all annihilation operators on the right of all creation operators. The isomorphism can be defined recursively via

$$P(1) = 1;\ M(1) = 1;\ P(a) = a^+;\ M(a) = a^-;\ \phi(1) = 1 \otimes 1;$$
$$\phi(a) = P(a) \otimes 1 + 1 \otimes M(a);$$
$$\phi(u \vee v) = \sum P(u_1)P(v_1) \otimes M(u_2)M(v_2);$$
$$P(u \vee v) = P(u)P(v);\ M(v \vee v) = M(u)M(v) \tag{1.45}$$

The algebra $S(V)$ is graded by the number of creation and annihilation operators. Note here a subtelty. It appears that one has forgotten the operator product of elements of V^+ with elements of V^-. One has replaced $a_k^+ a_\ell^-$ by $a_k^+ \otimes a_\ell^-$ and lost all information concerning the commutation of creation and annihilation operators. But an operator with creation operators on the left and annihilation operators on the right would not be well defined (cf. [**935**]). For instance $a_k^- a_k^+$ and $a_k^+ a_k^- + 1$ are equal as operators but their normal products $a_k^+ a_k^-$ and $a_k^+ a_k^- + 1$ are different!. Thus it is not consistent to consider a normal product as obtained from the transformation of an operator product and one naturally loses all information about commutation relations in QFT. However one recalls that an antisymmetric Laplace pairing enables one to build a circle product in $S(V)$ which is an operator product. In that sense normal products are a basic concept and operator products are a derived concept of QFT. Thus in the present picture QFT starts with the space $S(V)$ of normal products and the operator product and time ordered product are deformations of the symmetric product.

Note in QFT for scalar particles the basis a_k is not indexed by k but by a continuous variable x and (unfortunately) creation operators are denoted by $\phi^-(x)$ and annihilation operators by $\phi^+(x)$. This leads (cf. [**507**]) to a bilinear form for the definition of operator products of the type $(\phi(x)|\phi(y)) = iD(x-y)$ and a bilinear form for the time ordered products as $(\phi(x)|\phi(y)) = iG_F(x-y)$ where for massless bosons

$$D(x) = -\frac{1}{2\pi} sign(x^0)\delta(x^2);\ G_F(x-y) = \frac{1}{4\pi^2}\frac{1}{x^2 + i0} \tag{1.46}$$

are distributions.

There is a striking identity in $S(V)$, namely $\epsilon(u) =< 0|u|0 >$ which can be shown as follows. First $< 0|u|0 >$ is a linear map $S(V) \to \mathbf{C}$. Second for elements $u \in S^n(V)$ with $n > 0$ one has $\epsilon(u) = 0$ and $< 0|u|0 >= 0$ because

u is a normal product. Finally the symmetric Hopf algebra $S(V)$ is connected so all elements of $S^0(V)$ are multiples of the unit (i.e. $u = \lambda(u)1$). But then $\epsilon(u) = \lambda(u)\epsilon(1) = \lambda(u)$ so $u = \epsilon(u)1$. But the vacuum is assumed to be normalized so $< 0|u|0 >= \epsilon(u) < 0|1|0 >= \epsilon(u)$ showing that the counit and the vacuum expectation value are the same. We refer to [**158**] (and subsequent sections) for further comments on the physics.

2. QUANTUM FIELDS AND QUANTUM GROUPS

We follow now [**157, 160**]. First from [**160**] we recall that a quantum group (QG) is a quasitriangular Hopf algebra (cf. [**192**]). Thus we want a Hopf algebra H with an R matrix $R \in H \otimes H$. However since Hopf algebras have a self dual nature we can also describe a QG as a Hopf algebra with a coquasitriangular structure. This means there is an invertible linear map $\mathfrak{R} : H \times H \to \mathbf{C}$ such that

$$\mathfrak{R}(a \cdot b, c) = \sum \mathfrak{R}(a, c_1)\mathfrak{R}(b, c_2);\ \mathfrak{R}(a, b \cdot c) = \sum \mathfrak{R}(a_1, c)\mathfrak{R}(a_2, b) \tag{2.1}$$

For a commutative and cocommutative Hopf algebra no other condition is required for $\mathfrak{R}$ and one restricts consideration to this situation. One uses $\mathfrak{R}$ to define a twisted product

$$a \circ b = \sum \mathfrak{R}(a_1, b_1)a_2 \cdot b_2 \tag{2.2}$$

This is a special case of Sweedler's crossed product (see [**923**]) and given (2.1) with a cocommutative coproduct the twisted product is associative. Now consider a real scalar particle with field operator

$$\phi(x) = \int \frac{d\mathbf{k}}{(2\pi)^3\sqrt{2\omega_k}} \left(e^{-ip\cdot x}a(\mathbf{k}) + e^{ip\cdot x}a^\dagger(\mathbf{k})\right) \tag{2.3}$$

where $\omega_k = \sqrt{m^2 + |\mathbf{k}|^2}$, $p = (\omega_k, \mathbf{k})$, and $a^\dagger(\mathbf{k})$, $a(\mathbf{k})$ are the creation and annihilation operators acting in a symmetric Fock space of scalar particles F_s. One considers the vector space of smoothed fields $V = \{\phi(f),\ f \in \mathfrak{D}(\mathbf{R}^4)\}$ where $\phi(f) = \int dx f(x)\phi(x)$ ($\mathfrak{D}$ is the standard Schwartz space). Now from V one builds the symmetric Hopf algebra $S(V) = \oplus_0^\infty S^n(V)$ as in Section 8.1 with $S^0(V) = \mathbf{C} \cdot 1$, $S^1(V) = V$, and $S^n(V)$ generated by the symmetric product of n elements of V. Here one is thinking of $\phi(f) \vee \phi(g) \sim: \phi(f)\phi(g) :$ and 1 is the unit operator in F_s. The coproduct is defined via $\Delta\phi(f) = \phi(f) \otimes 1 + 1 \otimes \phi(f)$ and extended to $S(V)$ as before; thus e.g.

$$\Delta(\phi(f) \vee \phi(g)) = (\phi(f) \vee \phi(g)) \otimes 1 + \phi(f) \otimes \phi(g) + \tag{2.4}$$

$$+\phi(g) \otimes \phi(f) + 1 \otimes (\phi(f) \vee \phi(g))$$

(cf. (1.3)). This coproduct is cocommutative with counit $\epsilon(a) = 0$ for $a \in S^n(V)$ with $n > 0$ and $\epsilon(1) = 1$; further $\epsilon(a) =< 0|a|0 >$ as indicated in Section 8.1. Thus $S(V)$ is a commutative, cocommutative, star Hopf algebra with the involution $\phi(f)^* = \phi(f^*)$ where $f^* = \bar{f}$. To make $S(V)$ a quantum group now we can add a coquasitriangular structure determined entirely by its value on V. Thus if

$a = u_1 \vee \cdots \vee v_m \in S^m(V)$ and $b = v_1 \vee \cdots \vee v_n \in S^n(V)$ then (2.1) implies that $\mathfrak{R}(a,b) = 0$ if $m \neq n$ and $\mathfrak{R}(a,b) = perm(u_i, v_j)$ if $m = n$. Here

$$perm(u_i, v_j) = \sum \mathfrak{R}(u_1, v_{\sigma(1)}) \cdots \mathfrak{R}(u_n, v_{\sigma(n)}) \tag{2.5}$$

where the sum is over all permutations σ of $(1, \cdots, n)$. The most general Poincaré invariant coquasitriangular structure on $S(V)$ is determined by values on V which can be written

$$\mathfrak{R}(\phi(f), \phi(g)) = \int dxdy f(x) G(x-y) g(y) \tag{2.6}$$

where $G(x)$ is a Lorentz invariant distribution. Now $\mathfrak{R}$ gives rise to a twisted product on $S(V)$ as in (2.3) and by choosing the proper $G(x)$ this twisted product can become the operator product or the time ordered product. Explicitly if

$$G(x) = G_+(x) = \int \frac{d\mathbf{k} e^{-ip\cdot x}}{(2\pi)^3 2\omega_k} \tag{2.7}$$

the twisted product is the usual operator product. If $G(x)$ is the Feynman propagator

$$G_F(x) = i \int \frac{d^4 p e^{-ip\cdot x}}{(2\pi)^4 (p^2 - m^2 + i\epsilon)} \tag{2.8}$$

then the twisted product (2.3) is the time ordered product. The proof is sketched in [**160**] with the observation that (2.3) is of course Wick's theorem as discussed in Section 8.1.

Next we look at the second paper in [**160**]. The first sections are mainly a repetition from [**158**] which we have covered in Section 8.2. We pick up the story in Section 4 (with some repetition from [**158, 160**]). One takes a vector space V of operators from which the space of normal products $S(V)$ will be constructed. The elements of V are defined as operators acting in a Fock space. For scalar bosons one takes $M = \mathbf{R}^N$ ($N = 3$ for the usual space time) and we set $H = L^2(M)$. To define the symmetric Fock space set $H^{\otimes n} = H \otimes \cdots \otimes H$ and S_n acts on $H^{\otimes n}$ via

$$S_n(f_1(\mathbf{k}_1) \otimes \cdots f_n(\mathbf{k}_n)) = \frac{1}{n!} \sum_{\sigma \in S_n} f_{\sigma(1)}(\mathbf{k}_1) \otimes \cdots \otimes f_{\sigma(n)}(\mathbf{k}_n) \tag{2.9}$$

where S_n is the group of permutations of n elements. The symmetric Fock space is

$$\mathfrak{F}_s(H) = \oplus_{n=0}^{\infty} S_n H^{\otimes n};\ S_0 H^{\otimes 0} = \mathbf{C};\ S_1 H^{\otimes 1} = H; \tag{2.10}$$

$$|\psi> = (\psi_0, \psi_1(\mathbf{k}_1), \cdots, \psi_n(\mathbf{k}_1, \mathbf{k}_2, \cdots \mathbf{k}_n), \cdots)$$

where $\psi_0 \in \mathbf{C}$ and each $\psi_n(\mathbf{k}_1, \cdots, \mathbf{k}_n) \in L^2(M^n)$ is left invariant under any permutation of the variables. The components of $|\psi>$ satisfy

$$\| |\psi> \|^2 = <\psi|\psi> = |\psi_0|^2 + \sum_1^{\infty} |\psi_n(\mathbf{k}_1, \cdots, \mathbf{k}_n)|^2 d\mathbf{k}_1 \cdots d\mathbf{k}_n < \infty \tag{2.11}$$

An element of $\mathfrak{F}_s(H)$ for which $\psi_n = 0$ for all but finitely many n is called a finite particle vector and the set of such vectors is denoted by F_0. Define now the annihilation operator $a(\mathbf{k})$ by its action in $\mathfrak{F}_s(H)$. The action of $a(\mathbf{k})$ on $|\psi>$ is given by the coordinates $(a(\mathbf{k}|\psi>$ in the form

$$(a(\mathbf{k})|\psi>_n(\mathbf{k}_1,\cdots,\mathbf{k}_n)=\sqrt{n+1}\psi_{n+1}(\mathbf{k},\mathbf{k}_1,\cdots,\mathbf{k}_n) \tag{2.12}$$

The domain of $a(\mathbf{k})$ is $D_S=\{|\psi>\in F_0,\ \psi_n\in\mathfrak{S}(M^n)$ for $n>0\}$ where $\mathfrak{S}$ is the Schwartz space of functions of rapid decrease. The adjoint of $a(\mathbf{k})$ is not densely defined but $a^\dagger(\mathbf{k})$ is well defined as a quadratic form on $D_S\times D_S$ as follows. If $|\phi>$ and $|\psi>$ belong to D_S then (♦) $<\phi|a^\dagger(\mathbf{k})\psi>=<a(\mathbf{k})\phi|\psi>$. If now $|\psi>\in D_S$ then it can be checked that $a(\mathbf{k})|\psi>\in D_S$ so we can calculate the operator product $a(\mathbf{k}_1)\cdot a(\mathbf{k}_2)|\psi>\in D_S$ (the operator product is denoted by $\cdot$). By definitions and symmetry one knows that $a(\mathbf{k}_1)\cdot a(\mathbf{k}_2)=a(\mathbf{k}_2)\cdot a(\mathbf{k}_1)$ which implies $a^\dagger(\mathbf{k}_1)\cdot a^\dagger(\mathbf{k}_2)=a^\dagger(\mathbf{k}_2)\cdot a^\dagger(\mathbf{k}_1)$ but nothing is known about $a(\mathbf{k}_1)\cdot a^\dagger(\mathbf{k}_2)$ for example in view of the definition (♦) which demands that all the $a^\dagger(\mathbf{k})$ must be on the left of the $a(\mathbf{k})$. The vectors $a^\dagger(\mathbf{k}_1)\cdots a^\dagger(\mathbf{k}_m)\cdot a(\mathbf{q}_1)\cdots a(\mathbf{q}_n)$ for $m,n\geq 0$ generate a vector space W. The element where $m=n=0$ is called 1 (unit operator). The space W is equipped with a product (normal product) denoted by $\vee$. If $u=a^\dagger(\mathbf{k}_1)\cdots a^\dagger(\mathbf{k}_m)\cdot a(\mathbf{q}_1)\cdots a(\mathbf{q}_n)$ and $v\in W$ then

$$u\vee v=v\vee u=a^\dagger(\mathbf{k}_1)\cdots a^\dagger(\mathbf{k}_m)\cdot v\cdot a(\mathbf{q}_1)\cdots a(\mathbf{q}_n) \tag{2.13}$$

W is a commutative and associative star algebra with unit generated by the $a(\mathbf{k})$ and $a^\dagger(\mathbf{k})$ with star structure $a(\mathbf{k})^*=a^\dagger(\mathbf{k})$. For V one takes $V=\{\phi(f),\ f\in\mathfrak{D}(\mathbf{R}^{N+1})$ where (cf. (2.3))

$$\phi(f)=\int dx\int\frac{d\mathbf{k}}{(2\pi)^3\sqrt{2\omega_k}}\left(f(x)e^{-ip\cdot x}a(\mathbf{k})+f(x)e^{ip\cdot x}a^\dagger(\mathbf{k})\right) \tag{2.14}$$

(again $\omega_k=\sqrt{m^2+|\mathbf{k}|^2}$ and $p=(\omega_k,\mathbf{k})$. Note that the smoothed fields can also be written as $\phi(f)=\int dxf(x)\phi(x)$ with $\phi(x)$ given via (2.3) (note that $\phi(x)\notin V$). Now to define $S(V)$ in terms of operators we know $S^0(V)=\mathbf{C}1$ where 1 is the unit operator (i.e. for any $|\psi>\in F_s(H)$, $1|\psi>=|\psi>$). Also we know $S^1(V)=V$. To identify $S^n(V)$ we need only determine the symmetric product from the normal product of creation and annihilation operators. Thus e.g.

$$\begin{aligned}\phi(f)\vee\phi(g)=&\int dxdx'\int\frac{d\mathbf{k}}{(2\pi)^3\sqrt{2\omega_k}}\frac{d\mathbf{k}'}{(2\pi)^3\sqrt{2\omega_{k'}}}f(x)g(x')\times\\ &\times\Big(e^{-i(p\cdot x+p'\cdot x')}a(\mathbf{k})\cdot a(\mathbf{k}')+e^{-i(p\cdot x-p'\cdot x')}a^\dagger(\mathbf{k}')\cdot a(\mathbf{k})+\\ &+e^{i(p\cdot x-p'\cdot x')}a^\dagger(\mathbf{k})\cdot a(\mathbf{k}')+e^{i(p\cdot x+p'\cdot x')}a^\dagger(\mathbf{k})\cdot a^\dagger(\mathbf{k}')\Big)\end{aligned} \tag{2.15}$$

These involve Fourier transforms and elementary arguments show that everything is well defined and the product is an operator in D_S. The same reasoning shows that the symmetric product $\vee$ is well defined in $S(V)$ where the coproduct is generated by $\Delta\phi(f)=\phi(f)\otimes 1+1\otimes\phi(f)$ and the antipode is defined by $s(\phi(f_1)\vee\cdots\vee\phi(f_n))=(-1)^n\phi(f_1)\vee\cdots\vee\phi(f_n)$. Again the counit gives the vacuum expectation value as follows. Define the vacuum in D_S by $|0>=(1,0,\cdots)$ so by definitions $a(\mathbf{k}|0>=0$ and $<0|a^\dagger(\mathbf{k})=0$. A little argument as before then gives

$\epsilon(u) =< 0|u|0 >$. The star structure in $S(V)$ involves $(\lambda 1)^* = \bar{\lambda}1,\ \phi(g)^* = \phi(\bar{g})$, and $(u \vee v)^* = v^* \vee u^* = u^* \vee v^*$. This makes $S(V)$ a Hopf star algebra since

$$\Delta(\phi(g)^*) = \Delta\phi(\bar{g}) = \phi(\bar{g})\otimes 1 + 1\otimes\phi(\bar{g}) = \phi(g)^*\otimes 1 + 1\otimes\phi(g)^* = (\Delta\phi(g))^* \tag{2.16}$$

Hence $\Delta u^* = \sum u_1^* \otimes u_2^*$ for any $u \in S(V)$ and

$$s(u^*) = (-1)^n u^*;\ s(s(u)^*)^* = (-1)^n s(u^*)^* = (u^*)^* = u \tag{2.17}$$

A little recapitulation here for illustrative purposes gives the following. The twisted product can be defined from the Laplace pairing

$$(\phi(f)|\phi(g))_+ = \int dxdy f(x)G_+(x-y)g(y) \tag{2.18}$$

(cf. Section 8.2). This is denoted by $\phi(f) \bullet \phi(g)$ and one has as in Section 8.2 $\phi(f)\bullet\phi(g) = \phi(f)\vee\phi(g) + (\phi(f)|\phi(g))_+$. On the other hand Wick's theorem gives $\phi(f)\cdot\phi(g) = \phi(f)\vee\phi(g) + < 0|\phi(f)\cdot\phi(g)|0 >$ and the last term can be written $< 0|\phi(f)\cdot\phi(g)|0 >= \int dxdy f(x)g(y) < 0|\phi(x)\cdot\phi(y)|0 >$ where $< 0|\phi(x)\cdot\phi(y)|0 >$ is the Wightman function $G_+(x-y)$. Thus $\phi(f)\bullet\phi(g) = \phi(f)\cdot\phi(g)$ as indicated earlier.

For the time ordered product one considers the Laplace pairing

$$(\phi(f)|\phi(g))_F = -i\int dxdy f(x)G_F(x-y)g(y) \tag{2.19}$$

We denote by $\circ$ the corresponding twisted product; for example $\phi(f)\circ\phi(g) = \phi(f)\vee\phi(g) + (\phi(f)|\phi(g))_F$. In particular it can be checked that

$$< 0|\phi(f)\circ\phi(g)|0 >= (\phi(f)|\phi(g))_F = -i\int dxdy f(x)G_F(x-y)g(y) \tag{2.20}$$

We see that the twisted product of two operators is equal to their time ordered product because $-iG_F(x-y) =< 0|T(\phi(x)\phi(y))|0 >$. One notes that $G_F(-x) = G_F(x)$ so the Laplace coupling $(\ |\ )$ is symmetric and the time ordered product is commutative (it is also associative). One notes also

$$< 0|T(\phi(f_1)\phi(f_2)\cdots\phi(f_n))|0 >= \epsilon(\phi(f_1)\circ\phi(f_2)\circ\cdots\circ\phi(f_n)) = \tag{2.21}$$
$$= t(\phi(f_1)\vee\phi(f_2)\vee\cdots\vee\phi(f_n))$$

Next we go to [**157**] where interacting quantum fields are considered. We summarize as follows. If C is a cocommutative coalgebra with coproduct Δ' and counit ϵ' the symmetric algebra $S(C) = \oplus_0^\infty S^n(C)$ can be equipped with the structure of a bialgebra. The product is denoted here by $\cdot$ and the coproduct is defined on $S^1(C) = C$ by $\Delta a = \Delta' z$ and extended to $S(C)$ via $\Delta(uv) = \sum u_1\cdot v_1 \otimes u_2\cdot v_2$. The counit ϵ of $S(C)$ is defined as ϵ' on $S^1(C) = C$ and extended via $\epsilon(1) = 1$ with $\epsilon(u\cdot v) = \epsilon(u)\epsilon(v)$. One checks that Δ is coassociative and cocommutative and that $\sum\epsilon(u_1)u_2 = \sum u_1\epsilon(u_2) = u$. Thus $S(C)$ is a commutative and cocommutative bialgebra graded as an algebra. We note that Δ^k is defined via $\Delta^0 a = a,\ \Delta^1 a = \Delta a$, and $\Delta^{k+1}a = (id\otimes\cdots\otimes id\otimes\Delta)\Delta^k a$ with

$\Delta^k a = \sum a_1 \otimes \cdots \otimes a_{k+1}$. A Laplace pairing is (again) defined as in (1.5) (with $\vee$ replaced by $\cdot$). Then from these definitions one gets for u^j, $v^j \in S(C)$

$$(u^1 \cdot \cdots \cdot u^k | v^1 \cdot \cdots \cdot v^\ell) = \sum \prod_{i=1}^{k} \prod_{j=1}^{\ell} (u^i_j | v^j_i) \tag{2.22}$$

For example $(u{\cdot}v{\cdot}w|s{\cdot}t) = \sum (u_1|s_1))u_2|t_1)(v_1|s_2)(v_2|t_2)(w_1|s_3)(w_2|t_3)$. A Laplace pairing is entirely determined by its value on C and it induces a twisted product $\circ$ on $S(C)$ as before (cf. (1.7)). Applying the counit to both sides in (1.7) yields $\epsilon(u \circ v) = (u|v)$. Following the proofs in [**160**] (sketched in Section 8.1) one gets formulas such as (1.10) - (1.11), etc. and in addition

$$\Delta(u^1 \circ \cdots \circ u^k) = \sum u^1_1 \circ \cdots \circ u^k_1 \otimes u^1_2 \cdot \cdots \cdot u^k_2 \tag{2.23}$$

This leads to the important relation

$$u^1 \circ \cdots \circ u^k = \sum \epsilon(u^1_1 \circ \cdots \circ u^k_1) u^1_2 \cdot \cdots \cdot u^k_2 \tag{2.24}$$

A second important identity in this context is also proved, namely

$$\epsilon(u^1 \circ \cdots \circ u^k) = \sum \prod_{i=1}^{k-1} \prod_{j=1+1}^{k} (u^i_{j-1}|u^j_i) = \sum \prod_{j>i} (u^i_{j-1}|u^j_i) \tag{2.25}$$

(the proof is in [**157**]).

Now one goes to interacting quantum fields. Recall the scalar fields are defined via (2.3) and interacting fields are products of fields at the same point. Thus define powers of fields $\phi^n(x)$ as the normal product of n fields at x (i.e. $\phi^n(x) =: \phi(x) \cdots \phi(x) :$). This is meaningful for $n > 0$ and $\phi^0(x) = 1$. Consider the coalgebra $\mathfrak{C}$ generated by $\phi^n(x)$ where x runs over space time and n goes from 0 to 3 for a ϕ^3 theory and from 0 to 4 for a ϕ^4 theory. The coproduct in $\mathfrak{C}$ is $\Delta\phi^n(x) = \sum_0^n \phi^k(x) \otimes \phi^{n-k}(x)$ and the counit is $\epsilon(\phi^n(x)) = \delta_{n,0}$. Scalar fields are bosons so we work with the symmetric algebra $S(\mathfrak{C})$ with product $=: uv :$ and $\epsilon(u) =< 0|u|0 >$. In $S(\mathfrak{C})$ the Laplace pairing is entirely determined by $(\phi^n(x)|\phi^m(x))$ which itself is determined by the value of $(\phi(x)|\phi(y) = G(x,y)$ if we consider $\phi^n(x)$ as a product of fields. More precisely $(\phi^n(x)|\phi^m(y)) = \delta_{m,n} G(x,y)^{(n)}$ where $G(x,y)^{(n)} = (1/n!)G(x,y)^n$. Here $G \sim G_+$ or G_F as before. Now noting that $\Delta^{k-1}\phi^n(x) = \sum \phi^{m_1}(x) \otimes \cdots \otimes \phi^{m_k}(x)$, with a sum over all nonnegative integers m_i such that $\sum_1^k m_i = n$ one can specialize (2.22) to $\mathfrak{C}$ as

$$(: \phi^{n_1}(x_1) \cdots \phi^{n_k}(x_k) : | : \phi^{p_1}(y_1) \cdots \phi^{p_\ell}(y_\ell) :) = \sum_M \prod_{i=1}^{k} \prod_{j=1}^{\ell} G(x_i, y_j)^{(m_{ij})} \tag{2.26}$$

where the sum is over all $k \times \ell$ matrices M of nonnegative integers m_{ij} such that $\sum_{j=1}^{\ell} m_{ij} = n_i$ and $\sum_{i=1}^{k} m_{ij} = p_j$. Note now that (2.24) applied to $\mathfrak{C}$ yields a classical result of QFT, namely

$$\phi^{(n_1)}(x_1) \circ \cdots \circ \phi^{(n_k)}(x_k) = \sum_{i_1=0}^{n_1} \cdots \sum_{i_k=0}^{n_k} < 0|\phi^{(i_1)}(x_1) \circ \cdots \circ \phi^{(i_k)}(x_k)|0 > \times \tag{2.27}$$

$$\times : \phi^{(n_1-i_1)}(x_1) \cdots \phi^{n_k-i_k)}(x_k) :$$

This equation appeared in [**339**] but (2.24) is clearly more compact and more general than (2.27). Finally specializing (2.25) to $\mathfrak{C}$ one obtains

$$< 0|\phi^{(n_1)}(x_1) \circ \cdots \circ \phi^{(n_k)}(x_k)|0 >= \sum_M \prod_{i=1}^{k-1} \prod_{j=1+1}^{k} G(x_i, x_j)^{(m_{ij}} \tag{2.28}$$

over all symmetric $k \times k$ matrices M of nonnegative integers m_{ij} with $\sum_{j=1}^k m_{ij} = n_j$ and $m_{ii} = 0$ for all i. When the twisted product is the operator product this expression appears in [**168**] but (2.28) is proved with a few lines of algebra whereas the QFT proof is long and combinatorial. When the twisted product is the time ordered product (2.28) has a diagrammatic interpretation in the spirit of Feynman (cf. [**157**] for details).

3. RENORMALIZATION AND ALGEBRA

We go now to [**157**] and start with the physics in order to motivate the algebra. Thus let $\mathfrak{B}$ be the bialgebra of scalar fields generated by $\phi^n(x)$ for $x \in \mathbf{R}^4$ and $n \geq 0$ (recall $\phi^n(x)$ is the normal product of n fields $\phi(x)$). The product of fields is the commutative normal product $\phi^n(x) \cdot \phi^m(x) = \phi^{m+n}(x)$ and the coproduct is

$$\delta_B \phi^n(x) = \sum_0^n \left(\begin{array}{c} n \\ k \end{array} \right) \phi^k(x) \otimes \phi^{n-k}(x) \tag{3.1}$$

The counit is $\epsilon_B(\phi^n(x)) = \delta_{n,0}$. One defines the symmetric algebra $S(\mathfrak{B})$ as the algebra generated by the unit operator 1 and normal products $: \phi^{n_1}(x_1) \cdots \phi^{n_m}(x_m) :$. The symmetric product is the normal product and the coproduct on $S(\mathfrak{B})$ is deduced from δ_B via

$$\delta : \phi^{n_1}(x_1) \cdots \phi^{n_m}(x_m) := \sum_{i_1=0}^{n_1} \cdots \sum_{i_m=0}^{n_m} \left(\begin{array}{c} n_1 \\ i_1 \end{array} \right) \cdots \left(\begin{array}{c} n_m \\ i_m \end{array} \right) \times \tag{3.2}$$

$$: \phi^{i_1}(x_1) \cdots \phi^{i_m}(x_m) : \otimes : \phi^{n_1-i_m}(x_1) \cdots \phi^{n_m-i_m}(x_m :$$

The counit of $S(\mathfrak{B})$ is given by the expectation value over the vacuum, i.e. $\epsilon(a) =< 0|a|0 >$ for $a \in S(\mathfrak{B})$.

Now for the time ordered product of $a \in S(\mathfrak{B})$ one takes $T(a) = \sum t(a_1)a_2$ where $t(a) = \epsilon(T(a))$. In expressions like $T(\phi^{n_1}(x_1) \cdots \phi^{n_m}(x_m))$ the product inside $T(\ )$ is usually considered to be the operator product. However since the fields $\phi^{n_i}(x_i)$ commute inside $T(\ )$ it is also possible to consider T as a map $S(\mathfrak{B}) \to S(\mathfrak{B})$. To stress this point one writes $T(: \phi^{n_1}(x_1) \cdots \phi^{n_m}(x_m) :)$. From $T(a) = \sum t(a_1)a_2$ and the definition of the coproduct and counit one recovers the standard formula (cf. [**168, 339**])

$$T(: \phi^{n_1}(x_1) \cdots \phi^{n_m}(x_m) :) = \sum_{i_1=0}^{n_1} \cdots \sum_{i_m=0}^{n_m} \left(\begin{array}{c} n_1 \\ i_1 \end{array} \right) \cdots \left(\begin{array}{c} n_m \\ i_m \end{array} \right) \times \tag{3.3}$$

$$< 0|T(: \phi^{i_1}(x_1) \cdots \phi^{i_m}(x_m) :)|0 >: \phi^{n_1-i_m}(x_1) \cdots \phi^{n_m-i_m} :$$

The relation between two time ordered products T and $\tilde{T}$ (cf. [**788**] and also Section 8.1) can be written (up to a convolution with test functions) as

$$\tilde{T}(a) = \sum T("O(b^1)\cdots O(b^k) :);\ a =: \phi^{n_1}(x_1 \cdots \phi^{n_m}(x_m) : \tag{3.4}$$

and the sum is over partitions of a (i.e. the different ways to write a as $: b^1 \cdots b^k :$ where $b^i \in S(\mathbf{B})$ and k is the number of blocks of the partition), while O is a map $S(\mathbf{B}) \to \mathbf{B}$ such that $O(: \phi^{n_1}(x_1)\cdots\phi^{n_m}(x_m) :)$ is supported on $x_1 = x_2 = \cdots = x_m$ (the notation Δ is used in [**768**] for the map O but is changed here to avoid confusion with the coproduct). For consistency one must put $T(1) = O(1) = 1$ and in standard QFT a single vertex is not renormalised so $\tilde{T}(a) = T(a) = a$ if $a \in \mathbf{B}$. This enables one to show that $O(a) = a$ if $a \in \mathbf{B}$ since if $a \in \mathbf{B}$ there is only one partition of a, namely a, and (3.4) becomes $\tilde{T}(a) = T(O(a))$. But $O(a) \in \mathbf{B}$ so $T(O(a)) = O(a)$; the fact that $\tilde{T}(a) = a$ implies that $O(a) = a$. If $a =: \phi^{n_1}(x_1)\cdots\phi^{n_m}(x_m) :$ with $m > 1$ then (3.4) can be rewritten

$$\tilde{(}a) = T(a) + T(O(a)) + \sum_u{}' T(: O(b^1)\cdots O(b^k) :) \tag{3.5}$$

where $\sum_u'$ is the sum over all partitions except $u = \{\phi^{n_1}(x_1), \cdots, \phi^{n_m}(x_m)\}$ and $u = a$. However $O(a) \in \mathfrak{B}$ and T acts as the identity on $\mathfrak{B}$ so

$$\tilde{T}(a) = T(a) + O(a) + \sum_u{}' T(: O(b^1)\cdots O(b^k) :) \tag{3.6}$$

In fact one proves in [**157**] that if $a =: \phi^{n_1}(x_1)\cdots\phi^{n_m}(x_m) :$ then $O(a) = \sum c(a_1)a_2$ where $c(a) = \epsilon(O(a))$ is a distribution supported on $x_1 = \cdots x_m$ which can be obtained recursively from $\tilde{t}$ and t via

$$c(a) = \tilde{t}(a) - t(a) - \sum_u{}' \sum c(b_1^1)\cdots c(b_1^k)t(: b_2^1 \cdots b_2^k :) \tag{3.7}$$

We will sketch the induction procedure. If $a =: \phi^{n_1}(x_1)\cdots\phi^{n_m}(x_m) :$ with $m = 2$ there are only two partitions of a, namely $u = \{: \phi^{n_1}(x_1)\phi^{n_2}(x_2) :\}$ and $u = \{\phi^{n_1}(x_1), \phi^{n_2}(x_2)\}$. Thus $\tilde{T}(a) = T(a) + O(a)$. We know that $T(a) = \sum t(a_1)a_2$ and $\tilde{T}(a) = \sum \tilde{t}(a_1)a_2$ so $O(a) = \sum c(a_1)a_2$ with $c = \tilde{t} - t$. Assume the proposition (3.7) is true up to $m - 1$ and take $a =: \phi^{n_1}\cdots\phi^{n_m}(x_m) :$. In (3.6) then the proposition is true for all $O(b^i)$ and thus

$$\tilde{T}(a) = T(a) + O(a) + \sum_u{}' T(: O(b^1)\cdots O(b^k) :) = T(a) + O(a) + \tag{3.8}$$

$$+ \sum_u{}' \sum c(b_1^1)\cdots c(b_1^k)T(: b_2^1 \cdots b_2^k :)$$

$T(a) = \sum t(a_1)a_2$ yields now

(3.9)

$$\sum \tilde{t}(a_1)a_2 = \sum t(a_1)a_2 + O(a) + \sum_u{}' \sum c(b_1^1)\cdots c(b_1^k)t(: b_2^1 \cdots b_2^k :) : b_3^1 \cdots b_3^k :$$

Since $a =: b^1 \cdots b^k :$ the factor $: b_3^1 \cdots b_3^k :$ can be written a_3 and the proposition is proved using the coassociativity of the coproduct. To prove the support properties of $c(a)$ use the fact that $O(a)$ is supported on $x_1 = \cdots = x_m$ so $c(a) = \epsilon(O(a))$ is supported on the same set and $c(a) = P(\partial)\delta(x_2 - x_1)\cdots\delta(x_m - x_1)$ where $P(\partial)$ is a polynomial in ∂_{x_i}.

From the proposition one has now

$$O(:\phi^{n_1}(x_1)\cdots\phi^{n_m}(x_m):) = \sum_{i_1=0}^{n_1}\cdots\sum_{i_m=0}^{n_m}\binom{n_1}{i_1}\cdots\binom{n_m}{i_m}\times \tag{3.10}$$

$$\times c(:\phi^{n_1}(x_1)\cdots\phi^{n_m}(x_m):):\phi^{n_1-i_m}(x_1)\cdots\phi^{n_m-i_m}(x_m):$$

Because of the support properties of c this can be rewritten as

$$O(:\phi^{n_1}(x_1)\cdots\phi^{n_m}(x_m):) = \sum_{i_1=0}^{n_1}\cdots\sum_{i_m=0}^{n_m}\binom{n_1}{i_1}\cdots\binom{n_m}{i_m}\times \tag{3.11}$$

$$\times c(:\phi^{n_1}(x_1)\cdots\phi^{n_m}(x_m):):\phi^{n_1-i_m}(x_1)\cdots\phi^{n_m-i_m}(x_1):=\sum_{i_1=0}^{n_1}\cdots\sum_{i_m=0}^{n_m}\times$$

$$\times\binom{n_1}{i_1}\cdots\binom{n_m}{i_m}c(:\phi^{n_1}(x_1)\cdots\phi^{n_m}(x_m):)\phi^{n_1-i_m}(x_1)\cdots\phi^{n_m-i_m}(x_1)$$

where the product in the last line is in $\mathfrak{B}$. Thus we can rewrite this as $O(a) = \sum c(a_1)\prod a_2$.

It remains to relate the last result with the renormalisation coproduct. By combining (3.4) and $O(a) = \sum c(a_1)\prod a_2$ one gets

$$\tilde{T}(a) = \sum_u T(:O(b^1)\cdots O(b^k):) = \sum_u \sum c(b_1^1)\cdots c(b_1^k)T(:\prod b_2^1\cdots\prod b_2^k:) \tag{3.12}$$

If we compare this with the commutative renormalisation coproduct

$$\Delta b = \sum_u b_1^1\cdots b_1^{\ell(u)}\otimes\{\prod b_2^1,\cdots,\prod b_2^{\ell(u)}\} \tag{3.13}$$

one sees that $\tilde{T}(a) = \sum C(a_{[1]})T(a_{[2]})$ (there are two coproducts here to be indicated below and the notation $[i]$ will be clarifed at that time); similarly $\tilde{t}(a) = \sum C(a_{[1]})t(a_{[2]})$. In any event this shows that the renormalisation cooproduct gives the same result as the Epstein-Glaser renormalisation. Further in order to deal with renormalisation in curved space time renormalisation at a point is needed and this is provided via

$$\tilde{\phi}^n(x) = \sum_0^n \binom{n}{k} c(\phi^k(x))\phi^{n-k}(x) \tag{3.14}$$

instead of $T(\phi^n(x)) = \phi^n(x)$.

We go now to the beginning of [**157**] and deal with the algebra. We will take $\mathfrak{B}$ as a (not necessarily unital) bialgebra with coproduct δ_B and counit ϵ_B. The product is denoted $x \cdot y$ and $T(\mathfrak{B})^+$ is the subalgebra $\oplus_{n\geq 1}T^n(\mathfrak{B})$. Here the generators $x_1 \otimes \cdots \otimes x_n$ are denoted by $(x_1, \cdots, x_n)$ and $x_i \in \mathfrak{B}$ and one uses $\circ$ to denote tensor product so that $(x_1, \cdots, x_n) \circ (x_{n+1}, \cdots, x_{m+n}) = (x_1, \cdots, x_{m+n})$. Finally write the product operation in $T(T(\mathfrak{B}^+)$ by juxtaposition so $T^k(T(\mathfrak{B})^+)$ is generated by $a_1 a_2 \cdots a_k$ where $a_i \in T(\mathfrak{B})^+$. The coproduct $\delta = \delta_B$ and counit $\epsilon = \epsilon_B$ extend uniquely to a coproduct and counit on $T(\mathfrak{B})$ compatible with the multiplication of $T(\mathfrak{B})$, making $T(\mathfrak{B})$ a bialgebra. This construction ignores completely the algebra structure of $\mathfrak{B}$; the bialgebra $T(\mathfrak{B})$ is the free bialgebra on the underlying coalgebra of $\mathfrak{B}$. Similarly the coproduct and counit of the nonunital bialgebra $T(\mathfrak{B})^+$ extend to define a free bialgebra structure on $T(T(\mathfrak{B})^+)$. The coproduct of both $T(\mathfrak{B})$ and $T(T(\mathfrak{B})^+)$ is denoted by δ. Hence if one uses the Sweedler notation $\delta(x) = \sum x_1 \otimes x_2$ for the coproduct in $\mathfrak{B}$ then for $a = (x^1, \cdots, x^n) \in T^b(\mathfrak{B})$ and $u = a^1 \cdots a^k \in T^k(T(\mathfrak{B})^+)$ we have

$$\delta(a) = \sum (x_1^1, \cdots, x_1^n) \otimes (x_2^1, \cdots, x_2^n); \ \delta(u) = \sum a_1^1 \cdots a_1^k \otimes a_2^1 \cdots a_2^k \tag{3.15}$$

The counit is defined by $\epsilon(a) = \epsilon_B(x^1) \cdots \epsilon_B(x^n)$ and $\epsilon(u) = \epsilon(a^1) \cdots \epsilon(a^k)$.

Before giving the new coalgebra structure on $T(T(\mathfrak{B})^+)$ some terminology arises. A composition ρ is a (possibly empty) finite sequence of positive integers (parts of ρ). The length is $\ell(\rho)$ (number of parts) and $|\rho|$ is the sum of the parts; ρ is called a composition of n if $|\rho| = n$. Let C_n be the set of all compositions of n and by C the set $\cup_{n\geq 0} C_n$ of all compositions of all nonnegative integers. For example $\rho = \{1, 3, 1, 2\}$ is a composition of 7 having length 4. The first four C_n are $C_0 = \{e\}$ where e is the empty composition, $C_1 = \{(1)\}$, $C_2 = \{(1,1), (2)\}$, and $C_3 = \{(1,1,1), (1,2), (2,1), (3)\}$. The total number of compositions of n is 2^{n-1}, the number of compositions of n of length k is $(n-1)!/(k-1)!(n-k)!$, and the number of compositions of n containing α_1 times the integer 1, α_2 times 2,$\cdots$, and α_n times n, with $\alpha_1 + 2\alpha_2 + \cdots + n\alpha_n = n$ is $(\alpha_1 + \cdots + \alpha_n)!/\alpha_1! \cdots \alpha_n!$. The set C is a monoid under the operation $\circ$ of concatenation of sequences, i.e. $(r_1, \cdots, r_n) \circ (r_{n+1}, \cdots, r_{m+n}) = (r_1, \cdots, r_{n+m})$ and the identity of C is the empty composition e.

Now the refinement relation goes as follows. If ρ and σ are compositions with $\sigma = (s_1, \cdots, s_k)$ then $\rho \leq \sigma \iff \rho$ factors in C as $\rho = (\rho|gs_1) \circ \cdots \circ (\rho|\sigma_k)$ where $(\rho|\sigma_i)$ is a composition of s_i for each $i \in \{1, \cdots, k\}$; $(\rho|\sigma_i)$ is called the restriction of ρ to the i^{th} part of σ. For example if $\sigma = (4, 5)$ and $\rho = (1, 2, 1, 2, 2, 1)$ then $\rho = (\rho|\sigma_1) \circ (\rho|\sigma_2)$ where $(\rho|\sigma_1) = (1, 2, 1)$ is a composition of 4 and $(\rho|\sigma_2) = (2, 2, 1)$ is a composition of 5. Thus $\rho \leq \sigma$ and on can say that ρ is a refinement of σ. Note that $|\rho| = |\sigma|$ and $\ell(\rho) \geq \ell(\sigma)$ if $\rho \leq \sigma$. If $\rho \leq \sigma$ one defines the quotient σ/ρ to be the composition of $\ell(\rho)$ given by $(t_1, \cdots, t_k)$ where $t_i = \ell((\rho|\sigma_i))$ for $1 \leq i \leq k$. In the example above $\sigma/\rho = (3, 3)$. Note that for $\sigma \in C_n$ with $\ell(\sigma) = k$ one has $(n)/\sigma = (k)$, $\sigma/(1, 1, \cdots, 1) = \sigma$, and $\sigma/\sigma = (1, 1, \cdots, 1) \in C_k$. Each of the sets C_n (as well as all of C) is partially ordered by refinement. Each C_n has unique minimal element $(1, \cdots, 1)$ and unique maximal element (n) and these are all the minimal and maximal elements in C. The partially ordered sets C_n are actually Boolean algebras.

One proves a lemma now needed to prove coassociativity of the coproduct. Thus if $\rho \leq \tau$ in C then the map $\sigma \to \sigma/\rho$ is a bijection from $\{\sigma : \rho \leq \sigma \leq \tau\}$ onto $\{\gamma : \gamma \leq \tau/\rho\}$. To see this suppose $\rho = (r_1, \cdots, r_k)$ and $\gamma = (s_1, \cdots, s_\ell) \leq \tau/\rho$. Define $\bar{\gamma} \in C$ via

$$\bar{\gamma} = (r_1 + \cdots + r_{s_1}, r_{s_1+1} + \cdots + r_{x_1+s_2}, \cdots r_{k-s_\ell+1} + \cdots + r_k) \tag{3.16}$$

Then evidently $\rho \leq \bar{\gamma} \leq \tau$ and the map $\gamma \to \bar{\gamma}$ is inverse to the map $\sigma \to \sigma/\rho$.

The monoid of compositions allows a grading on $T(T(\mathfrak{B})^+)$. For all $n \geq 0$ and $\rho = (r_1, \cdots, r_k)$ in C_n let $T^\rho(\mathfrak{B})$ be the subspace of $T^k(T(\mathfrak{B})^+)$ given by $T^{r_1}(\mathfrak{B}) \otimes \cdots \otimes T^{r_k}(\mathfrak{B})$. Then there is a direct sum decomposition $T(T(\mathfrak{B})^+) = \oplus_{\rho \in C} T^\rho(\mathfrak{B})$ where $T^\rho(\mathfrak{B}) \cdot T^\tau(\mathfrak{B}) \subseteq T^{\rho\circ\tau}(\mathfrak{B})$ for all ρ, $\tau \in C$ and $1_{T(T(\mathfrak{B}))} \in T^e(\mathfrak{B})$ (thus $T(T(\mathfrak{B})^+)$ is a C-graded algebra). One uses this grading to define operations on $T(T(\mathfrak{B})^+)$. Given $a = (x_1, \cdots, x_n) \in T^n(\mathfrak{B})$ and $\rho = (r_1, \cdots, r_k) \in C_n$ define $a|\rho \in T^\rho(\mathfrak{B})$ and $a/\rho \in T^k(\mathfrak{B})$ via

$$a|\rho = (x_1, \cdots, x_{r_1})(x_{r_1+1}, \cdots, x_{r_1+r_2}) \cdots (x_{n-r_k+1}, \cdots, x_n); \tag{3.17}$$

$$a/\rho = (x_1 \cdots x_{r_1}, x_{r_1+1} \cdots x_{r_1+r_2}, \cdots, x_{n-r_k+1} \cdots x_n)$$

where $x_1 \cdots x_j$ is the product in $\mathfrak{B}$. More generally for $u = a_1 \cdots a_\ell \in T^\sigma(\mathfrak{B})$ and $\rho \leq \sigma$ in C one defines $u|\rho \in T^\rho(\mathfrak{B})$ and $u/\rho \in T^{\sigma/\rho}(\mathfrak{B})$ via

$$u|\rho = a_1|(\rho|\sigma_1) \cdots a_\ell(\rho|\sigma_\ell);\ u/\rho = a_1/(\rho|\sigma_1) \cdots a_\ell/(\rho|\sigma_\ell) \tag{3.18}$$

EXAMPLE 3.1. Let $\rho = (1, 2, 1, 2, 2, 1)$, $\sigma = (3, 1, 2, 3)$, and $\tau = (4, 5)$ in C so that $\rho \leq \sigma \leq \tau$ in C. Then $\sigma/\rho = (2, 1, 1, 2)$, $\tau/\rho = (3, 3)$, and $\tau/\sigma = (2, 2)$. If $u = (x_1, x_2, x_3, x_4)(y_1, y_2, y_3, y_4, y_5)$ in $T^r(\mathfrak{B})$ then

$$u|\rho = (x_1)(x_2, x_3)(x_4)(y_1, y_2)(y_3, y_4)(y_5) \in T^\rho(\mathfrak{B}; \tag{3.19}$$

$$u|\sigma = (x_1, x_2, x_3)(x_4)(y_1, y_2)(y_3, y_4, y_5) \in T^\sigma(\mathfrak{B});$$

$$u/\rho = (x_1, x_2 \cdot x_3, x_4)(y_1 \cdot y_2, y_3 \cdot y_4, y_5) \in T^{\tau/\rho}(\mathfrak{B});$$

$$u/\sigma = (x_1 \cdot x_2 \cdot x_3, x_4)(y_1 \cdot y_2, y_3 \cdot y_4 \cdot y_5) \in T^{\tau/\sigma}(\mathfrak{B});$$

$$(u|\sigma)/\rho = (u/\rho)|(\sigma/\rho) = (x_1, x_2 \cdot x_3)(x_4)(y_1 \cdot y_2)(y_3 \cdot y_4, y_5) \in T^{\sigma/\rho}(\mathfrak{B})$$

The last equality illustrates the lemma that for all $\rho \leq \sigma \leq \tau$ in C and $u \in T^r(\mathfrak{B})$ the equalities

$$(u|\sigma)|\rho = u|\rho;\ (u/\rho)/(\sigma/\rho) = u/\sigma;\ (u|\sigma)/\rho = (u/\rho)|(\sigma/\rho) \tag{3.20}$$

hold in $T^{\sigma/\rho}(\mathfrak{B})$.

We refer to [**157**] for proof of

LEMMA 3.1. For all $\rho \leq \sigma$ and $u \in T^\sigma(\mathfrak{B})$ with free coproduct $\delta(u) = \sum u_1 \otimes u_2$ one has

$$\delta(u/\rho) = \sum u_1|\rho \otimes u_2|\rho;\ \delta(u/\rho) = \sum u_1/\rho \otimes u_2/\rho \tag{3.21}$$

Now define the coproduct Δ on $T(T(\mathfrak{B})^+)$ via

$$\Delta u = \sum_{\sigma \leq \tau} u_1|\sigma \otimes u_2/\sigma \tag{3.22}$$

for $u \in T^r(\mathfrak{B})$ with free coproduct $\delta(u) = \sum u_1 \otimes u_2$. Here Δ is called the renormalisation coproduct; it is an algebra map and hence is determined via $\Delta a = \sum_{\sigma \in C_n} a_1|\sigma \otimes a_2/\sigma$ for all $a \in T^n(\mathfrak{B})$ with $n \geq 1$. As an example consider (3.23)

$$\Delta x = \sum (x_1) \otimes (x_2);\ \Delta(x,y) = \sum (x_1)(y_1) \otimes (x_2, y_2) + \sum (x_1, y_1) \otimes (x_2 \cdot y_2);$$

$$\Delta(x,y,z) = \sum (x_1)(y_1)(z_1) \otimes (x_2, y_2, z_2) + \sum (x_1)(y_1, z_1) \otimes (x_2, y_2 \cdot z_2) +$$

$$+ \sum (x_1, y_1)(z_1) \otimes (x_2 \cdot y_2, z_2) + \sum (x_1, y_1, z_1) \otimes (x_2 \cdot y_2 \cdot z_2)$$

The counit ϵ of $T(T(\mathfrak{B})^+)$ is the algebra map to $\mathbf{C}$ whose restriction to $T(\mathfrak{B})^+$ is given by $\epsilon((x)) = \epsilon_B(x)$ for $x \in \mathfrak{B}$ and $\epsilon((x_1, \cdots, x_n)) = 0$ for $n \geq 1$. One proves then in [**157**]

THEOREM 3.1. The algebra $T(T(\mathfrak{B})^+)$ together with the structure maps Δ and ϵ defined above is a bialgebra called the renormalisation algebra.

One has now two coproducts on $T(T(\mathfrak{B})^+)$, namely δ and Δ defined via (3.22) and to avoid confusion one uses an alternate Sweedler notation for the new coproduct, namely $\Delta u = \sum u_{[1]} \otimes u_{[2]}$ for $u \in T(T(\mathfrak{B})^+)$. Note

$$\sum (x)_{[1]} \otimes (x)_{[2]} = \sum (x_1) \otimes (x_2) \tag{3.24}$$

for $x \in \mathfrak{B}$. A recursive definition of the coproduct can also be given. Thus the action of $\mathfrak{B}$ on itself by left multiplication extends to an action $\mathfrak{B} \otimes T(\mathfrak{B}) \to T(\mathfrak{B})$ denoted by $x \otimes a \to x \triangleright a$ via ($\bullet$) $x \triangleright a = (x \cdot x_1, \cdots, x_n)$ for $x \in \mathfrak{B}$ and $a = (x_1, \cdots, x_n)$. This action in turn extends to an action of $\mathfrak{B}$ on $T(T(\mathfrak{B})^+)$ denoted similarly by $x \otimes u \to x \triangleright u$ in the form ($\bullet\bullet$) $x \triangleright u = (x \triangleright a_1)a_2 \cdots a_k$ for $x \in \mathfrak{B}$ and $u = a_1 \cdots a_k \in T(T(\mathfrak{B})^+)$. The following proposition together with $\Delta 1 = 1 \otimes 1$ and $\Delta x = \sum (x_1) \otimes (x_2)$ for $x \in \mathfrak{B}$ determines Δ recursively on $T(\mathfrak{B})$ and hence by multiplicativity determines Δ on all $T(T(\mathfrak{B})^+)$.

PROPOSITION 3.1. For all $a \in T(\mathfrak{B})$ with $n \geq 1$ and $x \in \mathfrak{B}$

$$\Delta((x) \circ a) = \sum (x_1)a_{[1]} \otimes (x_2) \circ a_{[2]} + \sum (x_1) \circ a_{[1]} \otimes x_2 \triangleright a_{[2]} \tag{3.25}$$

We refer to [**157**] for proof. One may formulate (3.25) as follows. Corresponding to an element $x \in \mathfrak{B}$ there are 3 linear operators on $T(T(\mathfrak{B})^+)$, namely

$$A_x(u) = x \triangleright u;\ B_x(u) = ((x) \circ a_1)a_2 \cdots a_k;\ C_x(u) = (x)u \tag{3.26}$$

induced by left multiplication in $\mathfrak{B}$, $T(\mathfrak{B})$, and $T(T(\mathfrak{B})^+)$ respectively. With this notation (3.25) takes the form

$$\Delta(B_x(a)) = \sum (C_{x_1} \otimes B_{x_2} + B_{x_1} \otimes A_{x_2})\Delta a \tag{3.27}$$

As a third formulation let A, B, C be the mappings from $\mathfrak{B}$ to the set of linear operators on $T(T(\mathfrak{B})^+)$ respectively given by $x \to A_x$, $x \to B_x$, and $x \to C_x$. Then (3.25) takes the form

$$\Delta(B_x(a)) = (A \otimes B + B \otimes C)(\delta(x))(\Delta(a) \tag{3.28}$$

One also has

$$\Delta(A_x(a)) = (A \otimes A)(\delta(x))(\Delta a);\ \Delta(C_x(a)) = (C \otimes C)(\delta(x))(\Delta a) \tag{3.29}$$

Finally one has a more explicit expression for the coproduct which is useful in defining a coproduct on $S(S(\mathfrak{B})^+)$. Thus if $a = (x_1, \cdots, x_n)$ one has

$$\Delta a = \sum_u a_1^1 \cdots a_1^{\ell(u)} \otimes (\prod a_2^1, \cdots, \prod a_2^{\ell(u)}) \tag{3.30}$$

where the product $a_1^1 \cdots a_1^{\ell(u)}$ is in $T(T(\mathfrak{B})^+)$. This is a simple rewriting of (3.22) and u runs over the compositions of a. By a composition of a one means an element $u \in T(T(\mathfrak{B})^+)$ such that $u = (a|\rho)$ for some $\rho \in C_n$. If the length of ρ is k one can write $u = a^1 \cdots a^k$ whre $a^i \in T(\mathfrak{B})$ are called the blocks of u. Finally the length of u is $\ell(u) = \ell(\rho) = k$. To complete the definition of (3.30) one must still define a_1^i and $\prod a_2^i$. If $a^i = (y^1, \cdots, y^m)$ is a block then $a_1^i = (y_1^1, \cdots, y_1^m) \in T(\mathfrak{B})^+$ and $\prod a_2^i y_2^1 \cdots y_2^m \in \mathfrak{B}$.

Now assume $\mathfrak{B}$ is a graded bialgebra which is no loss of generality since one can always consider that all elements of $\mathfrak{B}$ have degree zero. The grading of $\mathfrak{B}$ will be used to define a grading on $T(T(\mathfrak{B})^+)$. Denote by $|x|$ the degree of a homogeneous element $x \in \mathfrak{B}$ and by $deg(a)$ the degree (to be defined) of a homogeneous $a \in T(T(\mathfrak{B})^+)$. First for $T(\mathfrak{B})$ the degree of 1 is zero, the degree of $(x) \in T^1(\mathfrak{B})$ is equal to the degree $|x|$ of $x \in \mathfrak{B}$. More generally the degree of $(x_1, \cdots, x_n) \in T(\mathfrak{B})$ is (**A82**) $deg((x_1, \cdots, x_n)) = |x_1| + \cdots + |x_n| + n - 1$. Finally if $a_1, \cdots, a_k$ are homogeneoous elements of $T(\mathfrak{B})^+$ the degree of their product in $T(T(\mathfrak{B})^+)$ is (**A83**) $deg(a_1 \cdots a_k) = deg(a_1) + \cdots + deg(a_k)$. Thus the degree is compatible with the multiplication in $T(T(\mathfrak{B})^+)$ and one shows that it is also compatible with the renormalisation coproduct (cf. [**157**] for details). For dealing with fermions one uses a $\mathbf{Z}_2$ graded algebra $\mathfrak{B}$ and we omit this for now (cf. [**157**]).

When the bialgebra $\mathfrak{B}$ is commutative one can work with the symmetric algebra $S(S(\mathfrak{B})^+)$ and to define $S(T(\mathfrak{B})^+)$ one takes quotients from $T(T(\mathfrak{B})^+)$ via the ideal $I = \{u(ab - ba)v;\ u, v \in T(T(\mathfrak{B})^+),\ a, b \in T(\mathfrak{B})^+\}$. Note that

$$\Delta u(ab - ba)v = \sum u_{[1]}a_{[1]}b_{[1]}v_{[1]} \otimes u_{[2]}a_{[2]}b_{[2]}v_{[2]} - \sum u_{[1]}b_{[1]}a_{[1]}v_{[1]} \otimes \tag{3.31}$$

$$\otimes u_{[2]}b_{[2]}a_{[2]}v_{[2]} = \sum u_{[1]}(a_{[1]}b_{[1]} - b_{[1]}a_{[1]})v_{[1]} \otimes u_{[2]}a_{[2]}b_{[2]}v_{[2]} +$$

$$+ \sum u_{[1]}b_{[1]}a_{[1]}v_{[1]} \otimes u_{[2]}(a_{[2]}b_{[2]} - b_{[2]}a_{[2]})v_{[2]}$$

Thus $\Delta I \subset I \otimes T(T(\mathfrak{B})^+) + T(T(\mathfrak{B})^+) \otimes I$. Moreover $\epsilon(I) = 0$ because $\epsilon(ab - ba) = 0$. Therefore I is a coideal and since it is also an ideal the quotient $S(T(\mathfrak{B})^+) =$

$T(T(\mathfrak{B})^+)/I$ is a bialgebra which is commutative. Now one defines $S(\mathfrak{B})^+$ as the subspace of $T(\mathfrak{B})^+$ generated as a vector space by elements

$$\{x_1, \cdots, x_n\} = \sum (x_{\sigma(1)}, \cdots, x_{\sigma(n)}) \tag{3.32}$$

where σ runs over the permutations of n elements. The symmetric product in $S(\mathfrak{B})^+$ is denoted by $\vee$ so that $\{x_1, \cdots, x_n\} \vee \{x_{n+1}, \cdots, x_{n+m}\} = \{x_1, \cdots, x_{m+n}\}$. If $\mathfrak{B}$ is commutative one shows that $S(S(\mathfrak{B})^+)$ is a subbialgebra of $S(T(\mathfrak{B})^+)$ with coproduct given by (3.13) (cf. [**157**] for the combinatorial proof). Some examples are given via

$$\Delta\{x\} = \sum \{x_1\} \otimes \{x_2\};\ \Delta\{x, y\} = \tag{3.33}$$

$$= \sum \{x_1\}\{y_1\} \otimes \{x_2, y_2\} + \sum \{x_1, y_1\} \otimes \{x_2 \cdot y_2\}$$

3.1. VARIOUS ALGEBRAS. When $\mathfrak{B}$ is the Hopf algebra of a commutative group there is a homomorphism between the gialgebra $S(S(\mathfrak{B})^+)$ and the Faa di Bruno algebra. As an algebra this is the polynomial algebra generated by u_n for $1 \leq n < \infty$. The coproduct of u_n is

$$\Delta u_n = \sum_{k=1}^{n} \sum_{\alpha} \frac{n!(u_1)^{\alpha_1} \cdots (u_n)^{\alpha_n}}{\alpha_1! \cdots \alpha_n!(1!)^{\alpha_1} \cdots (n!)^{\alpha_n}} \otimes u_k \tag{3.34}$$

The sum is over the n-tuples of nonnegative integers $\alpha = (\alpha_1, \cdots, \alpha_n)$ such that $\alpha + 1 + 2\alpha_2 + \cdots + n\alpha_n = n$ and $\alpha_1 + \cdots + \alpha_n = k$. For example

$$\Delta u_1 = u_1 \otimes u_1;\ \Delta u_2 = u_2 \otimes u_1 + u_1^2 \otimes u^2;\ \Delta u_3 = u_3 \otimes u_1 + 3u_1 u_2 \otimes u_2 + u_1^3 \otimes u_3 \tag{3.35}$$

It is a commutative noncocommutative bialgebra. To see the relation of this bialgebra with the composition of functions consider formal series $f(x) = \sum_1^\infty f_n(x^n/n!)$ and $g(x) = \sum_1^\infty g_n(x^n/n!)$. Define linear maps from the algebra of such formal series to $\mathbf{C}$ via $u_n(f) = f_n$. If one defines $\Delta u = \sum u_1 \otimes u_2$ then the terms of $f(g(x))$ are

$$f(g(x)) = \sum_1^\infty u_n(f \circ g)\frac{x^n}{n!} = \sum_1^\infty \sum u_{n(1)}(g) u_{n(2)}(f) \frac{x^n}{n!} \tag{3.36}$$

For example

$$\frac{df(g(x))}{dx} = g_1 f_1;\ \frac{d^2 f(g(x))}{dx^2} = g_2 f_1 + g_1^2 f_2;\ \frac{d^3 f(g(x))}{dx^3} = g_3 f_1 + 3 g_1 g_2 f_2 + g_1^3 f_3 \tag{3.37}$$

Following [**259**] it is possible to introduce a new (noncommutative) element X in the algebra such that $X, u_n] = u_{n+1}$ and then to generate the FdB coproduct via $\Delta u_1 = u_1 \otimes u_1$ and $\Delta X = X \otimes 1 + u_1 \otimes X$. Now take the bialgebra $\mathfrak{B}$ to be a commutative group Hopf algebra. If G is a commutative group the commutative algebra $\mathfrak{B}$ is the vector space generated by the elements of G and the algebra product is induced by the product in G. The coproduct is defined by $\delta_B x = x \otimes x$ for all $x \in G$. The definition (3.13) of coproduct becomes

$$\Delta b = \sum_u b^1 \cdots b^{\ell(u)} \otimes \{\prod b^1, \cdots, \prod b^{\ell(u)}\} \tag{3.38}$$

The homomorphism ϕ between $S(S(\mathfrak{B})^+)$ and the FdB bialgebra is given by $\phi(1) = 1$ and $\phi(a) = u_n$ for any $a \in S^n(\mathfrak{B})$ with $n > 0$. It can be established from (3.38) by noticing that the number of partitions of $\{x_1, \cdots, x_n\}$ with α_1 blocks of size $1, \cdots, \alpha_n$ blocks of size n is $n!/\alpha_1! \cdots \alpha_n!(1)^{\alpha_1} \cdots (n!)^{\alpha_n}$.

Next comes the Pinter Hopf algebra. The bialgebras $T(T(\mathfrak{B})^+)$ and $S(S(\mathfrak{B})^+)$ can be turned into Hopf algebras by quotienting with an ideal. The subspace $I = \{(x) - \epsilon_B(x)1;\ x \in \mathfrak{B}\}$ is a coideal because

$$\Delta((x) - \epsilon_B(x)1) = \sum (x_1) \otimes (x_2) - \epsilon_B(x) 1 \otimes 1 = \sum (x_1) \otimes (x_2) - \tag{3.39}$$

$$- \sum \epsilon_B(x_1)\epsilon_B(x_2) 1 \otimes 1 = \sum ((x_1) - \epsilon_B(x_1)1) \otimes (x_2) + \sum \epsilon_B(x_1) 1 \otimes ((x_2) - \epsilon_B(x_2)1)$$

Since $\epsilon(I) = 0$ the subspace I is a coideal. Therefore the space J of elements of the form uav where $u, v \in T(T(\mathfrak{B})^+)$ and $a \in I$ is an ideal and a coideal so $T(T(\mathfrak{B})^+)/J$ is a bialgebra. The action of the quotient is to replace all the (x) by $\epsilon(x)1$. For example

$$\Delta(x, y) = \sum 1 \otimes (x, y) + \sum (x, y) \otimes 1;\ \Delta(x, y, z) = \sum 1 \otimes (x, y, z) + \tag{3.40}$$

$$+ \sum (y_1, z_1) \otimes (x, y_2 \cdot z_2) + \sum (x_1, y_1) \otimes (x_2 \cdot y_2, z) + \sum (x, y, z) \otimes 1$$

More generally if $a = (x_1, \cdots, x_n)$ one writes

$$\Delta a = a \otimes 1 + 1 \otimes a + \sum{}' a_1 \otimes a_2 \tag{3.41}$$

where $\sum'$ involves elements a_1, a_2 of degrees strictly smaller that the degree of a. Hence the antipode can be defined as in [**861**] and $T(T(\mathfrak{B})^+)/J$ is a connected Hopf algebra. The same is true of $S(S(\mathfrak{B})^+)/J'$ where J' is the subspace of elements of the form uav with $u, \in S(S(\mathfrak{B})^+)$ and $a \in \{\{x\} - \epsilon_B(x)1;\ x \in \mathfrak{B}\}$.

Finally consider the Connes-Moscovici algebra. If we take the same quotient of the FdB bialgebra (i.e. by letting $u_1 = 1$) one obtains a Hopf algebra (FdB Hopf algebra). In [**259**] there is defined a Hopf algebra related to the FdB Hopf algebra as follows. If $\phi(x) = x + \sum_2^\infty u_n x^n/n!$ one defines δ_n for $n > 0$ via

$$-log\phi'(x) = \sum_1^\infty \delta_n (x^n/n!) \tag{3.42}$$

To calculate δ_n as a function of u_k write $\phi'(x) = 1 + \sum_1^\infty u_{n+1} x^n/n!$ and $-log(1 + z) = \sum_1^\infty (-1)^{k-1}(k-1)! z^k/k!$. Since the FdB formulas describe the composition of series one can use it to write

$$\delta_n = \sum_1^n (-1)^{k-1}(k-1)! \sum_\alpha \frac{n!(u_2)^{\alpha_2} \cdots (u_{n+1})^{\alpha_n}}{\alpha_1! \cdots \alpha_n!(1!)^{\alpha_1} \cdots (n!)^{\alpha_n}} \tag{3.43}$$

where the sum is over the n-tuples of nonnegative integers $\alpha = (\alpha_1, \cdots, \alpha_n)$ as before. For example $\delta_1 = u_2$, $\delta_2 = u_3 - u_2^2$, $\delta_3 = u_4 - 3u_3u_2 + 2u_2^3$, etc. Except for the shift the relation between u_n and δ_n is the same as the relation between the moments of a distribution and its cumulants, or between unconnected Greens functions and connected Greens functions. The inverse relation is obtained from $\phi(x) = \int_0^x dtexp(-\sum_1^\infty \delta_n(t^n/n!)$. Thus

$$u_{n+1} = \sum n!(\delta_1)^{\alpha_1} \cdots (\delta_n)^{\alpha_n})/(\alpha_1! \cdots \alpha_n!(1!)^{\alpha_1} \cdots (n!)^{\alpha_n}) \tag{3.44}$$

4. TAU FUNCTION AND FREE FERMIONS

We follow here [**191, 192, 205, 455, 737, 738, 739**]. For free fermions we adopt the notation of [**737**] for convenience (note in [**191**] $\psi \leftrightarrow \psi^*$). Thus

$$\psi_n|0>= 0 =< 0|\psi_n^* \ (n < 0); \ \psi_n^*|0>= 0 =< 0|\psi_n \ (n \geq 0) \tag{4.1}$$

Here the algebra of free fermions is a Clifford algebra A over $\mathbf{C}$ with

$$[\psi_m, \psi_n]_+ = [\psi_m^*, \psi_n^*]_+ = 0; \ [\psi_m, \psi_n^*] = \delta_{mn} \tag{4.2}$$

An element of $W = (\oplus_{m\in\mathbf{Z}}\mathbf{C}\psi_m) \oplus (\oplus_{m\in\mathbf{Z}}\mathbf{C}\psi_m^*)$ is called a free fermion. The Clifford algebra has a standard representation (Fock representation) as follows. Let

$$W_{an} = (\oplus_{m<0}\mathbf{C}\psi_m) \oplus (\oplus_{m\geq 0}\mathbf{C}\psi_m^*); \ W_{cr} = (\oplus_{m\geq 0}\mathbf{C}\psi_m) \oplus (\oplus_{m<0}\mathbf{C}\psi_m^*) \tag{4.3}$$

Then consider the left (resp. right) A-module $F = A/AW_{an}$ (resp. $F^* = W_{cr}A/A$). These are cyclic A- modules generated by the vectors $|0>= 1$ mod AW_{an} (resp. by $< 0| = 1$ mod $W_{cr}A$) with the properties given in (4.1). The Fock spaces F and F^* are dual with pairing defined via the vacuum expectation value $< 0| \cdot |0 >$ which has the properties

$$< 0|1|0 >= 1; \ < 0|\psi_m\psi_m^*|0 >= 1 \ (m < 0); \ < 0|\psi_m^*\psi_m|0 >= 1 \ (m \geq 0); \tag{4.4}$$
$$< 0|\psi_m\psi_n|0 >= (0|\psi_m^*\psi_n^*|0 >= 0; \ < 0|\psi_m\psi_n^*|0 >= 0 \ (m \neq n)$$

and the Wick property applies, namely

$$< 0|w_1 \cdots w_{2n+1}|0 >= 0; \tag{4.5}$$
$$< 0|w_1 \cdots w_{2n}|0 >= \sum_\sigma sgn(\sigma) < 0|w_{\sigma(1)}w_{\sigma(2)}|0 > \cdots < 0|w_{\sigma(2n-1)}w_{\sigma(2n)}|0 >$$

where $w_k \in W$ and σ runs over permutations such that $\sigma(1) < \sigma(2), \cdots, \sigma(2n-1) < \sigma(2n)$ and $\sigma(1) < \sigma(3) < \cdots < \sigma(2n-1)$. One considers also infinite matrices $(a_{ij})_{i,j\in\mathbf{Z}}$ satisfying the condition that there is an N such that $a_{ij} = 0$ for $|i - j| > N$; these are called generalized Jacobi matrices. They form a Lie algebra with bracket $[A, B] = AB - BA$. The quadratic elements $\sum a_{ij} : \psi_i\psi_j^* :$ are important where $: \psi\psi_j^* := \psi_i\psi_j^* - < 0|\psi_i\psi_j^*|0 >$. These elements (with 1) span an infinite dimensional Lie algebra $\widehat{\mathfrak{gl}}(\infty)$ where

$$[\sum a_{ij} : \psi_i\psi_j^* :, \sum b_{ij} : \psi_i\psi_j^* :] = \sum c_{ij} : \psi_i\psi_j^* : +c_0; \tag{4.6}$$
$$c_{ij} = \sum_k a_{ik}b_{kj} - \sum_k b_{ik}a_{kj}; \ c_0 = \sum_{i<0,j\geq 0} a_{ij}b_{ji} - \sum_{i\geq 0,j<0} a_{ij}b_{ji}$$

The last term c_0 commutes with each quadratic term. Thus the Lie algebra of quadratic elements $\sum a_{ij} : \psi_i\psi_j^* :$ is different from the algebra of generalized Jacobi matrices by virtue of the central extension c_0.

Now $g = exp(\sum a_{nm} : \psi_n\psi_m :)$ is an operator belong to the formal group corresponding to the Lie algebra $\widehat{\mathfrak{gl}}(\infty)$. Using (4.6) one obtains

$$g\psi_n = \sum_m \psi_m A_{mn} g;\ \psi_n^* g = g\sum_m A_{nm}\psi_m^* \tag{4.7}$$

where the coefficients A_{nm} are determined via a_{nm}. Then (4.7) implies

$$[\sum_{n\in Z} \psi_n \otimes \psi_n^*, g\otimes g] = 0 \tag{4.8}$$

This is very important and is in fact equivalent to the Hirota bilinear identity (cf. [**191, 664**]). One introduces now

$$H_n = \sum_{-\infty}^{\infty} \psi_k\psi_{k+n}^*\ (n\neq 0);\ H(t) = \sum_1^{\infty} t_n H_n;\ H^*(t^*) = \sum_1^{\infty} t_n^* H_{-n} \tag{4.9}$$

Here $H_n \in \widehat{\mathfrak{gl}}(\infty)$ while $H(t)$, $H^*(t^*)$ belong to $\widehat{\mathfrak{gl}}(\infty)$ if one restricts the number of non-vanishing parameters t_m, t_m^*. For H_n we have Heisenberg algebra commutation relations $[H_n, H_m] = n\delta_{m+n,0}$. One notes also $H_n|0>=0=<0|H_{-n}$ $(n>0)$. Next one introduces the fermions

$$\psi(z) = \sum_k \psi_k z^k;\ \psi^*(z) = \sum_k \psi_k^* z^{-k-1}dz \tag{4.10}$$

Using (4.2) and (4.9) there results ($\xi(t,z) = \sum_1^{\infty} t_n z^n$)

$$e^{H(t)}\psi(z)e^{-H(t)} = \psi(z)e^{\xi(t,z)};\ e^{H(t)}\psi^*(z)e^{H(t)} = \psi^*(z)e^{-\xi(t,z)}; \tag{4.11}$$

$$e^{-H^*(t^*)}\psi(z)e^{H^*(t^*)} = \psi(z)e^{-\xi(t^*,z^{-1})};\ e^{-H^*(t^*)}\psi^*(z)e^{H(t^*)} = \psi^*(z)e^{\xi(t^*,z^{-1})}$$

Using

$$\sum_{n\in\mathbf{Z}} \psi_n\otimes\psi_n^* = Res_{z=0}\psi(z)\otimes\psi^*(z) \tag{4.12}$$

and (4.6) one arrives at

$$[\sum_{n\in Z}\psi_n\otimes\psi_n^*, e^{H(t)}\otimes e^{H(t)}] = 0;\ [\sum_{n\in\mathbf{Z}}\psi_n\otimes\psi_n^*, e^{H^*(t^*)}\otimes e^{H^*(t^*)}] = 0 \tag{4.13}$$

Therefore if g solves (4.8) then $e^{H(t)}ge^{H^*(t^*)}$ also solves (4.8), i.e.

$$[Res_{z=0}\psi(z)\otimes\psi^*(z), e^{H(t)}ge^{H^*(t^*)}\otimes e^{H(t)}ge^{H^*(t^*)}] = 0 \tag{4.14}$$

Next to generate some generalized tau functions one defines vacuum vectors labelled by an integer. Thus write

$$<n| = <0|\Psi_n^*;\ |n> = \Psi_n|0>;\ \Psi_n = \psi_{n-1}\cdots\psi_1\psi_0\ (n>0);\ \Psi_n = \tag{4.15}$$

$$= \psi_n^*\cdots\psi_{-2}^*\psi_{-1}^*\ (n<0);\ \Psi_n^* = \psi_0^*\psi_1^*\cdots\psi_{n-1}^*\ (n>0);\ \Psi_n^* = \psi_{-1}\psi_{-2}\cdots\psi_n$$

$(n<0)$. Now given g satisfying the bilinear identity (4.8) one constructs Kadomtsev-Petviashvili (KP) and Toda lattice (TL) tau functions via

- $\tau_{KP}(n,t) = < n|exp(H(t))g|n >$
- $\tau_{TL}(n,t,t^*) = < n|exp(H(t))gexp(H^*(t^*))|n >$

If one fixes the variables n, t^* then τ_{TL} becomes τ_{KP}. Next one recalls the Schur functions via

$$s_\lambda(t) = det(h_{\lambda_i - i + j}(t))_{1\leq i,j\leq r} \tag{4.16}$$

$h_m(t)$ is the elementary Schur function defined by $exp(\xi(t,z)) = exp(\sum_1^\infty t_k z^k) = \sum_0^\infty z^n h_n(t)$. Another way of calculating $s_\lambda(t)$ follows from the formula
(4.17)

$$< (s-k)|e^{H(t)}\psi^*_{-j_1}\cdots\psi^*_{-j_k}\psi_{i_s}\cdots\psi_{i_1}|0 >= (-1)^{j_1+\cdots j_k+(k-s)(k-s+1)/2}s_\lambda(t)$$

where $-j_1 < \cdots < -j_k < 0 \leq i_s < \cdots < i_1$ and $s-k\geq 0$. The partition λ in (4.16) is defined in a complicated way and we prefer to use (4.17).

4.1. SYMMETRIC FUNCTIONS. A few words now on symmetric functions are in order. Thus one defines the power sums $(*)$ $\gamma_m = (1/m)\sum_i x_i^m$ which is the same as the Hirota-Miwa change of variables

$$mt_m = \sum x_i^m;\ mt_m^* = \sum y_i^m \tag{4.18}$$

The vector $\gamma = (\gamma_1,\gamma_2,\cdots)$ is an analogue of the higher times in KP theory. The complete symmetric functions h_n are given by the generating function

$$\prod_{i\geq 1}(1-x_i z)^{-1} = \sum_{n\geq 1} h_n(x)z^n \tag{4.19}$$

The symmetric polynomials with rational integer coefficients in n variables form a ring Λ_n; a ring of symmetric functions in countably many variables is denoted by Λ. If one defines $h_\lambda = h_{\lambda_1}h_{\lambda_2}\cdots$ for any λ then the h_λ form a $\mathbf{Z}$-basis of Λ. The complete symmetric functions are expressed in terms of power sums via

$$exp(\sum_1^\infty \gamma_m z^m) = \sum_{n\geq 0} h_n(\gamma)z^n \tag{4.20}$$

where $h_0 = 1$ (thus $h_n \sim$ Schur polynomial). Suppose now that n is finite and given a partition λ the Schur function s_λ is defined via

$$s_\lambda(x) = \frac{det(x_i^{\lambda_i+n-j})_{1\leq i,j\leq n}}{det(x_i^{n-j})_{1\leq i,j\leq n}};\ det(x_i^{n-j})_{1\leq i,j\leq n} = \prod_{1\leq i<j\leq n}(x_i - x_j) \tag{4.21}$$

The Schur functions $s_\lambda(x,\cdots,x_n)$ where $\ell(\lambda)\leq n$ form a $\mathbf{Z}$-basis of Λ_n. Note again that the Schur function s_λ can be expressed as in (4.16) as a polynomial in the complete symmetric functions where $n\geq \ell(\lambda)$ (namely $s_\lambda = det(h_{\lambda_i - 1 + j})1\leq i,j\leq n$). Next one defines a scalar product on Λ by requiring that for any pair of partitions λ and μ one has

$$< p_\lambda, p_\mu >= \delta_{\lambda,\mu}z_\lambda;\ z_\lambda = \prod_{i\geq 1} i^{m_i}\cdot m_i! \tag{4.22}$$

where $m_i = m_i(\lambda)$ is the number of parts of λ equal to i (here $p_n = \sum x_i^m$ and the $p_\lambda = p_{\lambda_1}p_{\lambda_2}\cdots$ form a basis of symmetric polynomial functions with rational coefficients). For any pair of partitions λ and μ one has $< s_\lambda, s_\mu >= \delta_{\lambda,\mu}$ so that the s_λ form an orthonormal basis of Λ and the s_λ such that $|\lambda| = n$ form an orthonormal basis for the symmetric polynomials of degree n.

Consider now a function r which depends on a single variable n (integer); given a partition λ define $r_\lambda(x) = \prod_{i,j\in\lambda} r(x+j-i)$. This is a product over all nodes of a Young diagram (i=row, j=column); we refer to [**205, 737**] for pictures). The value of $i-j$ is zero on the main diagonal and, listing the rows via length, the partition $(3,3,1)$ will have $j-i = 0,1,2,-1,0,1,-2$ so $r_\lambda(x) = r(x+2)(r(x+1))^2(r(x))^2 r(x-1)r(x-2)$. Given integer n and a function on the lattice $r(n)$, $n \in \mathbf{Z}$, define now a scalar product

$$< s_\lambda, s_\mu >_{r,n} = r_\lambda(n)\delta_{\lambda,\mu} \tag{4.23}$$

If $n_i \in \mathbf{Z}$ are the zeros of r and $k = min|n-n_i|$ then the product (4.23) is nondegenerate on Λ_k. Indeed if $k = n - n_i > 0$ the factor $r_\lambda(n)$ never vanishes for partitions of length no more than k. Hence the Schur functions of k variabes $\{s_\lambda(x^k),\ \ell(\lambda) \le k\}$ form a basis on Λ_k. If $n - n_i = -k < 0$ the factor $r_\lambda(n)$ never vanishes for the partitions $\lambda : \ell(\lambda') \le k\}$ where λ' is the conjugate partition (formed by reflecting the Young diagram in the main diagonal); then $\{s_\lambda(x^k),\ \ell(\lambda') \le k\}$ form a basis on Λ_k. If r is nonvanishing then the scalar product is nondegenerate on Λ_∞.

Take now $r(0) = 0$ and set $\hat{x} = (1/x)r(D)$ where $D = x(d/dx)$ so $\hat{x}\cdot x^n|_{x=0} = \delta_{m,n}r(1)r(2)\cdots r(n)$. Then it is proved in [**737**] that

$$< f(x), g(x) >_{r,n} = \frac{1}{n!}\Delta(\hat{x})f(\hat{x})\cdot\Delta(x)g(x)|_{x=0};\ \Delta(\hat{x} = \prod_{1\le i<j\le n}(\hat{x}_i - \hat{x}_j) \tag{4.24}$$

The proof follows from

$$\frac{1}{n!}det(\hat{x}^{\lambda_j+n-j})_{i,j=1,\cdots,n}\cdot det(x_i^{\mu_j+n-j})_{i,j=1,\cdots,n}|_{x=0} = \delta_{\lambda,\mu}r_\lambda(n) = < s_\lambda, s_\mu >_{r,n} \tag{4.25}$$

Next one recalls the Cauchy-Littlewood formula $\prod_{i,j}(1-x_iy_j)^{-1} = \sum s_\lambda(x)s_\lambda(y)$ with Schur functions defined as in (4.21). If now $\gamma = (\gamma_1, \gamma_2, \cdots)$ and $\gamma^* = (\gamma_1^*, \gamma_2^*, \cdots)$ then one proves in [**737**] that

$$exp(\sum_1^\infty m\gamma_m\gamma_m^*) = \sum_\lambda s_\lambda(\gamma)s_\lambda(\gamma^*) \tag{4.26}$$

where the s_λ are constructed as in (4.16) and (4.20). We will sketch the proof later. Consider now the pair of functions

$$exp(\sum_1^\infty m\gamma_m t_m);\ exp(\sum m\gamma_m t_m^*) \tag{4.27}$$

Using (4.23) and (4.26) one obtains

$$< exp(\sum_1^\infty m\gamma_m t_m), exp(\sum_1^\infty m\gamma_m t_m^*) >_{r,n} = \tag{4.28}$$

$$= \sum_{\ell(\lambda)\leq k} r_\lambda(n)s_\lambda(t)s_\lambda(t^*) = \sum_\lambda r_\lambda(n)s_\lambda(t)s_\lambda(t^*)$$

The series (4.28) is called a series of hypergeometric type; it is not necessarily convergent (cf. [**737**] for more on this).

Now given a partition $\lambda = (\lambda_i)$ with Young diagram $(\lambda_1, \lambda_2, \cdots)$ let there be r nodes (i,i). Set $\alpha_i = \lambda_i - i$ for the number of nodes in the ith row to the right of (i,i) and $\beta_i = \lambda'_i - i$ be the number of nodes in the ith column of λ below (i,i) $(1 \leq i \leq r)$. Then $\alpha_1 > \alpha_2 > \cdots > \alpha_r \geq 0$ and $\beta_1 > \beta_2 > \cdots > \beta_r \geq 0$; one writes

$$\lambda = (\alpha_1, \cdots, \alpha_r|\beta_1, \cdots, \beta_r) \tag{4.29}$$

(Frobenius notation). For example given $\lambda = (3,3,1)$ we get $\alpha_1 = 3-1 = 2$, $\alpha_2 = 3-2 = 1$, $\beta_1 = 3-1 = 2$, and $\beta_2 = 2-2 = 0$ so $\lambda = (2,1|2,0)$ in Frobenius notation. Now to prove (4.26) consider the vacuum TL tau function

$$< 0|exp(H(t))exp(H^*(t^*))|0 > \tag{4.30}$$

By the Heisenberg algebra commutation relations this is equal to $exp(\sum mt_mt_m^*)$. On the other hand one can develop $exp(H^*(t^*))$ in Taylor series and use the explicit form of H^* in terms of free fermions (cf. (4.9)) and then use (4.17) to get $\sum_\lambda s_\lambda(t)s_\lambda(t^*)$ (sum over all partitions including the zero partition). Now given $\lambda = (i_1, \cdots, i_s|j_1 - 1, \cdots, j_s - 1)$ write

$$|\lambda >= (-1)^{j_1+\cdots+j_s}\psi^*_{-j_1}\cdots\psi^*_{-j_s}\psi_{i_s}\cdots\psi_{i_1}|0>; \tag{4.31}$$
$$< \lambda| = (-1)^{j_1+\cdots+j_s} < 0|\psi^*_{i_1}\cdots\psi^*_{i_s}\psi_{-j_s}\cdots\psi_{-j_1}$$

Then

$$< \lambda|\mu >= \delta_{\lambda,\mu},\ s_\lambda(H^*)|0 >= |\lambda >;\ < 0|s_\lambda(H) =< \lambda| \tag{4.32}$$

where s_λ is defined via

$$s_\lambda(t) = det(h_{\lambda_i - i + j}(t))|_{1\leq i,j\leq \ell(\lambda)};\ exp(\sum_1^\infty z^m t_m) = \sum_0^\infty z^k h_k(t) \tag{4.33}$$

with t replaced by

$$H^* = \left(H_{-1}, \frac{H_{-2}}{2}, \cdots, \frac{H_{-m}}{m}, \cdots\right);\ H = \left(H_1, \frac{H_2}{2}, \cdots, \frac{H_m}{m}\cdots\right) \tag{4.34}$$

The proof of (4.32) follows from using (4.17) and (4.26) so that $exp(\sum_1^\infty H_m t_m) = \sum_\lambda s_\lambda(H)s_\lambda(t)$. Next one shows $s_\lambda(-A)|0 >= r_\lambda(0)|\lambda >$ where s_λ is defined via (4.33) with t replaced by

(4.35)

$$A = \left(A_1, \frac{A_2}{2}, \cdots, \frac{A_m}{m}, \cdots\right);\ A_k = \sum_{-\infty}^\infty \psi^*_{n-k}\psi_n r(n)r(n-1)\cdots r(n-k+1)$$

$(k = 1,2,\cdots)$. The proof goes as follows. First the components of A mutually commute (i.e. $[A_m, A_k] = 0$) so apply the development $exp(\sum_1^\infty A_m t_m) = \sum_\lambda s_\lambda(-A)s_\lambda(t)$ to both sides of the right vacuum vector, i.e. $exp(\sum_1^\infty A_m t_m)|0 >= \sum_\lambda s_\lambda(t)s_\lambda(-A)|0 >$. Then use a Taylor expansion of the exponential function and (4.35), (4.17), the definition of $|\lambda >$, and (4.32).

Recall now the definition of scalar product (4.22); it is known that one may present it directly via

$$\tilde{\partial} = (\partial_{\gamma_1}, (1/2)\partial_{\gamma_2}, \cdots, (1/n)\partial_{\gamma_n}, \cdots) \tag{4.36}$$

(cf. (*)). Then it is also known that the scalar product of polynomial functions may be written as

(4.37)
$$< f, g >= (f(\tilde{\partial}) \cdot g(\gamma))|_{\gamma=0};\ < \gamma_n, \gamma_m >= (1/n)\delta_{n,m};\ < p_n, p_m >= n\delta_{n,m}$$

(which is a particular case of (4.22). One proves then in [**737**] that with the scalar product (4.37) one has

$$< f, g >=< 0|f(H)g(H^*)|0 > \tag{4.38}$$

where H, H^* are as in (4.34). This follows from the fact that higher times and derivatives with respect to higher times give a realization of the Heisenberg algebra $H_k H_m - H_m H_k = k\delta_{k+m,0}$ and that

$$\partial_{\gamma_k} \cdot m\gamma_m - m\gamma_m \cdot \partial_{\gamma_k} = k\delta_{k+m,0};\ \partial_{\gamma_m} \cdot 1 = 0 \tag{4.39}$$

(i.e. the free term of ∂_{γ_m} is zero) and

$$H_m|n >= 0\ (m > 0);\ < n|H_m = 0\ (m < 0) \tag{4.40}$$

Now consider the deformed scalar product $< s_\mu, s_\lambda >_{r,n}= r_\lambda(n)\delta_{\mu,\lambda}$. Each polynomial function is a linear combination of Schur functions and one has the following realization of the deformed scalar product, namely

$$< f, g >_{r,n}=< n|f(H)g(-A)|N > \tag{4.41}$$

where H and A are determined as in (4.9), (4.34), (4.35). Now for the collection of independent variables $t^* = (t_1^*, t_2^*, \cdots)$ one writes $A(t^*) = \sum_1^\infty t_m^* A_m$ with A_m as in (4.33). Using the explicit form of A_m, (4.33), and (4.2) one obtains

(4.42)
$$e^{A(t^*)}\psi(z)e^{-A(t^*)} = e^{-\xi_r(t^*, z^{-1})} \cdot \psi(z);\ e^{A(t^*)}\psi^*(z)e^{-A(t^*)} = e^{\xi_{r'}(t^*, z^{-1})} \cdot \psi^*(z)$$

where the operators

$$\xi_r(t^*, z^{-1}) = \sum_1^\infty t_m \left(\frac{1}{z}r(D)\right)^m;\ D = z\frac{d}{dz};\ r'(D) = r(-D) \tag{4.43}$$

act on functions of z on the right side according to the rule $r(D) \cdot z^n = r(n)z^n$. The exponents in (4.42) - (4.43) are considered as their Taylor series. Next using (4.42) - (4.43) and the fact that inside Res_z the operator $(1/z)r(D)$ is the conjugate of $(1/z)r(-D) = (1/z)r'(D)$ one gets

$$[Res_{z=0}\psi(z) \otimes \psi^*(z), e^{A(t^*)} \otimes e^{A(t^*)}] = 0 \tag{4.44}$$

Now it is natural to consider the following tau function

$$\tau_r(n, t, t^*) =< n|e^{H(t)}e^{-A(t^*)}|n > \tag{4.45}$$

Using the fermionic realization of the scalar product (4.41) one gets

$$\tau_r(n,t,t^*) =< exp(\sum_1^\infty mt_m\gamma_m), exp(\sum_1^\infty mt^*_m\gamma_m) >_{r,n} \quad (4.46)$$

Then due to (4.28) one obtains

$$\tau_r(n,t,t^*) = \sum_\lambda r_\lambda(n)s_\lambda(t)s_\lambda(t^*) \quad (4.47)$$

Convergence questions are ignored here. The variables n, t play the role of higher KP times and t^* is a collection of group times for a commuting subalgebra of additional symmetries of KP (cf. [**191, 205**]). From another point of view (4.47) is a tau function of a two dimensional Toda lattice with two sets of continuous variables t, t^* and one discrete variable n. Defining $r'(n) = r(-n)$ one has properties

$$\tau_r(n,t,t^*) = \tau_r(n,t^*,t);\ \tau_{r'}(-n,-t,-t^*) = \tau_r(n,t,t^*) \quad (4.48)$$

Also $\tau_r(n,t,t^*)$ does not change if $t_m \to a^m t_m,\ t^*_m \to a^{-m}t^*_m,\ m = 1,2,\cdots$ and it does change (?) if $t_m \to a^m t_m,\ m = 1,2,\cdots$, and $r(n) \to a^{-1}r(n)$. (4.48) follows from the relations $r'_\lambda(n) = r_{\lambda'}(-n)$ and $s_\lambda(t) = (-1)^{|\lambda|}s_{\lambda'}(-t)$. One should note also that (4.47) can be viewed as a result of the action of additional symmetries on the vacuum tau function (cf. [**191, 205**] for terminology).

4.2. PSDO ON A CIRCLE. Given functions r, $\tilde{r}$ the operators

(4.49)

$$A_m = -\sum_{-\infty}^\infty r(n)\cdots r(n-m+1)\psi_n\psi^*_{n-m};\ \tilde{A}_m = \sum_{-\infty}^\infty \tilde{r}(n+1)\cdots\tilde{r}(n+m)\psi_n\psi^*_{n+m}$$

belong to the Lie algebra of pseudodifferential operators (PSDO) on a circle with central extension. These operators may be written in the form

(4.50)

$$A_m = \frac{1}{2\pi i}\oint : \psi^*(z)\left(\frac{1}{z}r(D)\right)^m \cdot \psi(z) :;\ \tilde{A}_m = -\frac{1}{2\pi i}\oint : \psi^*(z)(\tilde{r}(D)z)^m \cdot \psi(z) :$$

(recall $D = z(d/dz)$). The action is given via $r(D)\cdot z^n = r(n)z^n$ and one works in the punctured disc $0 < |z| < 1$. Central extensions of the Lie algebra of generators (4.49) are described by the formulae

$$\omega_n(A_m, A_k) = \delta_{mk}\tilde{r}(n+m-1)\cdots\tilde{r}(n)r(n)\cdots r(n-m+1) \quad (4.51)$$

with $\omega_n - \omega_{n'} \sim 0$. The extensions differing by the choice of n are cohomologically described. The choice corresponds to a choice of normal ordering : : which may be chosen in a different manner via $: a := a- < n|a|n >$ (n is an integer). When r, $\tilde{r}$ are polynomials the operators $[(1/z)r(D)]^m$ and $(\tilde{r}(D)z)^m$ belong to the W_∞ algebra while the fermionic operators (4.35) and (4.49) belong to the central extension $\widehat{W}_\infty$ (cf. [**579**]). Here one is interested in calculating the vacuum expectation values of different products of operators of the type $exp(\sum A_m t^*_m)$

and $exp(\sum \tilde{A}_m t_m)$ and for partitions $\lambda = (i_1, \cdots, i_s|j_1 - 1, \cdots, j_s - 1)$ and $\mu = (\tilde{i}_1, \cdots \tilde{i}_r|\tilde{j}_1 - 1, \cdots, \tilde{j}_r - 1)$ satisfying $\mu \subseteq \lambda$ one has

(4.52) $$< 0|\psi^*_{\tilde{i}_1} \cdots \psi^*_{\tilde{i}_r} \psi_{-\tilde{j}_1} \cdots \psi_{\tilde{j}_1} e^{A(t^*)} \psi^*_{j_1} \cdots \psi^*_{j_s} \psi_{i_s} \cdots \psi_{i_1}|0 >=$$
$$= (-1)^{\tilde{j}_1 + \cdots + \tilde{j}_r + j_1 + \cdots + j_s} s_{\lambda/\mu}(t^*) r_{\lambda/\mu}(0)$$

(4.53) $$< 0|\psi^*_{i_1} \cdots \psi_{i_s} \psi_{-j_s} \cdots \psi_{j_1} e^{\tilde{A}(t)} \psi^*_{\tilde{j}_1} \cdots \psi^*_{-\tilde{j}_r} \psi_{\tilde{i}_r} \cdots \psi_{\tilde{i}_1}|0 >=$$
$$= (-1)^{\tilde{j}_1 + \cdots + \tilde{j}_r + j_1 + \cdots + j_s} s_{\lambda/\mu}(t) \tilde{r}_{\lambda/\mu}(0)$$

where $s_{\lambda/\mu}(t)$ is a skew Schur function

(4.54) $$s_{\lambda/\mu}(t) = det(h_{\lambda_i - \mu_j - i + j}(t)); \; r_{\lambda/\mu} =$$
$$= \prod_{i,j \in \lambda/\mu} r(n + j - i); \; \tilde{r}_{\lambda/\mu}(n) = \prod_{i,j \in \lambda/\mu} \tilde{r}(n + j - 1)$$

The proof goes via a development of $exp(A) = 1 + A + \cdots$ and $exp(\tilde{A}) = 1 + \tilde{A} + \cdots$ and the direct evaluation of vacuum expectations (4.52) and (4.53) along with a determinant formula for the skew Schur function.

Now introduce vectors

(4.55) $$|\lambda, n >= (-1)^{j_1 + \cdots + j_s} \psi^*_{-j_1 + n}, \cdots, \psi^*_{-j_s + n} \psi_{i_s + n} \cdots \psi_{i_1 + n}|n >;$$
$$< \lambda, n| = (-1)^{j_1 + \cdots j_s} < n|\psi^*_{i_1 + n}, \cdots \psi^*_{i_s + n} \psi_{-j_s + n} \cdots \psi_{-j_1 + n}$$

We see that $< \lambda, n|\mu, m >= \delta_{mn}\delta_{\lambda\mu}$ One notes from (4.10) that

(4.56) $$\oint \cdots \oint \psi(z_1) \cdots \psi(z_n)|0 >< 0|\psi^*(z_n) \cdots \psi^*(z_1) = \sum_{\lambda, \ell(\lambda) \leq n} |\lambda, n >< \lambda, n|$$

which projects any state to the component F_n of the Fock space $F = \oplus_{n \in \mathbf{Z}} F_n$. Next using (4.36) one obtains

(4.57) $$|\lambda, n >= s_\lambda(H^*)|n >; \; < \lambda, n| =< n|s_\lambda(H);$$
$$s_\lambda(-A)|n >= r_\lambda(n) s_\lambda(H^*)|n >; \; < n|s_\lambda(\tilde{A}) = \tilde{r}(n) < n|s_\lambda(H); \; \tilde{A} = \left(\frac{\tilde{A}_n}{n}\right)$$

where $A_m, \; \tilde{A}_m$ are given via (4.31) and (4.49). One has then the following developments

(4.58) $$e^{-A(t^*)} = \sum_{n \in \mathbf{Z}} \sum_{\mu \subseteq \lambda} |\mu, n > s_{\lambda/\mu}(t^*) r_{\lambda/\mu}(n) < \lambda, n|;$$
$$e^{\tilde{A}(t)} = \sum_{n \in \mathbf{Z}} \sum_{\mu \subseteq \lambda} |\lambda, n > s_{\lambda/\mu}(t) \tilde{r}_{\lambda/\mu}(n) < \mu, n|$$

where $s_{\lambda/\mu}$ is the skew Schur function (4.54) and $r, \; \tilde{r}$ are also defined in (4.54). The proof follows from the relation $< s_{\lambda/\mu}, s_\nu >=< s_\lambda, s_\mu s_\nu >$. It follows from (4.58) that

(4.59) $$s_\nu(-A) = \sum_{n \in \mathbf{Z}} \sum_{\mu \subseteq \lambda} C^\lambda_{\mu\nu} r_{\lambda/\mu}(n)|\mu, n >< \lambda, n|;$$

$$s_\nu(\tilde{A}) = \sum_{n\in\mathbf{Z}}\sum_{\mu\subseteq\lambda} C^\lambda_{\mu\nu}\tilde{r}_{\lambda/\mu}(n)|\lambda,n><\mu,n|$$

Some calculation using $<\lambda,n|\mu,m>=\delta_{mn}\delta_{\lambda\mu}$ leads then to

$$\tau_g(n,t,t^*) =< n|exp(H(t))gexp(H(t^*))|n>=\sum_{\lambda,\mu} s_\lambda(t)g_{\lambda\mu}(n)s_\mu(t^*); \tag{4.60}$$

$$g_{\lambda\mu} =<\lambda,n|g|\mu,n>$$

Consequently one has the property

$$(g_1g_2)_{\lambda\mu}(n) = \sum_\nu (g_1)_{\lambda\nu}(n)(g_2)_{\nu\mu}(n) \tag{4.61}$$

Hence introduce the "integrable" scalar product $<s_\lambda, s_\mu>_{g,n}= g_{\lambda\mu}(n)$ (generally degenerate). This leads to

$$<\tau_{g_1}(n_1,t,\gamma),\tau_{g_2}(n_2,\gamma,t^*)>_{g,n}=\tau_{g_3}(0,t,t^*); \tag{4.62}$$

$$g_3(k) = g_1(n_1+k)g(n+k)g_2(n_2+k)$$

and one actually takes here $g_{\lambda\mu}(n) = \delta_{\lambda\mu}r_\lambda(n)$. There is much much more in [**737**] but we stop here for the moment.

5. INTERTWINING

We extract here from [**192**]. Intertwining is important in group representation theory and we indicate some aspects of this in quantum group situations. From [**191, 192, 287, 525**] consider classical KP/Toda. Vertex operators are ($\xi(t,z) = \sum_1^\infty t_kz^k$ and $\xi(\tilde{\partial},z^{-1}) = \sum_1^\infty z^{-k}\partial_k/k$)

$$X(z) = e^{\xi(t,z)}e^{-\xi(\tilde{\partial},z^{-1})};\ X^*(z) =^{-\xi(t,z)}\ e^{\xi(\tilde{\partial},z^{-1})}; \tag{5.1}$$

$$X^*(\lambda)X(\mu) = \frac{\lambda}{\lambda-\mu}X(\lambda,\mu,t);\ X(\lambda)X^*(\mu) = \frac{\lambda}{\lambda-\mu}X(\mu,\lambda,t)$$

$$X(z,\zeta,t) = e^{\sum_1^\infty(\zeta^k-z^k)}e^{\sum_1^\infty(z^{-k}-\zeta^{-k})\partial_k/k} = \sum_0^\infty\frac{(\zeta-z)^m}{m!}\sum_{p=-\infty}^\infty z^{-p-m}W_p^m$$

The wave functions are defined as usual in terms of the tau function via

$$\psi(t,z) = \frac{X(z)\tau(t)}{\tau(t)};\ \psi^*(t,z) = \frac{X^*(z)\tau(t)}{\tau(t)} \tag{5.2}$$

$$\psi^*(t,\lambda)\psi(t,\mu) = \frac{1}{\mu-\lambda}\partial\left(\frac{X(\lambda,\mu,t)\tau(t)}{\tau(t)}\right)$$

Now consider the free fermion operators

$$[\psi_n,\psi_m]_+ = 0 = [\psi_m^*,\psi_n^*]_+;\ [\psi_m,\psi_n^*]_+ = \delta_{mn} \tag{5.3}$$

with ($|vac>\sim|0>$ and $<0|vac\sim<0|$)

(1) $\psi_n|0>=0=<0|\psi_n^*\ \ (n<0)$
(2) $\psi_n^*|0>=0=<0|\psi_n\ \ (n\geq 0)$

Vacuum expectation values are defined via

$$<1>=1;\ <\psi_i\psi_j^*>=\delta_{ij}-<\psi_j^*\psi_i>=\begin{cases}=1 & (i=j<0)\\ 0 & otherwise\end{cases} \tag{5.4}$$

One expresses normal ordering via

$$:\psi_i\psi_j^*:=\psi_i\psi_j^*-<\psi_i\psi_j^*>;\ H(t)=\sum_1^{\infty}t_k\sum_{\mathbf{Z}}\psi_n\psi_{k+n}^* \tag{5.5}$$

and we see directly that $H(t)|0>=0$ while $<0|H(t)\neq 0$. The noncommutative algebra generated by ψ_n, ψ_m^* is denoted by $\mathfrak{A}$ and one writes $V=\oplus_{\mathbf{Z}}\mathbf{C}\psi_n$ and $V^*=\oplus_{\mathbf{Z}}\mathbf{C}\psi_n^*$ with $W=V\oplus V^*$. The left (resp. right) module with cyclic vector $|0>$ (resp. $<0|$) is called a left (resp. right) Fock space on which on has representations of $\mathfrak{A}$ via $\#1,2$ above. The vacuum expectation values give a $\mathbf{C}$ bilinear pairing

$$<0|\mathfrak{A}\otimes\mathfrak{A}|0>\to\mathbf{C};\ <0|a_1\otimes a_2|0>\mapsto<0|a_1a_2|0> \tag{5.6}$$

One denotes by $G(V,V^*)$ the Clifford group characterized via $g\psi_n=\sum\psi_m g a_{mn}$ and $\psi_n^*g=\sum g\psi_m^*a_{nm}$. Tau functions of KP are parametrized by $G(V,V^*)$ orbits of $|0>$ for example and such orbits (modulo constant multiples) can be identified with an infinite dimensional Grassmann manifold (UGM). The t-evolution of an operator $a\in\mathfrak{A}$ is defined as $a(t)=exp[H(t)]aexp[-H(t)]$.

We note that quadratic operators $\psi_m\psi_n^*$ satisfy

$$[\psi_m\psi_n^*,\psi_p\psi_q^*]=\delta_{np}\psi_m\psi_q^*-\delta_{mq}\psi_p\psi_n^* \tag{5.7}$$

and (with the element 1) these span an infinite dimensional Lie algebra $\mathfrak{g}(V,V^*)$ whose corresponding group is $G(V,V^*)$. Then $exp[H(t)]$ belongs to the formal completion of $G(V,V^*)$. One writes further $\Lambda\sim(\delta_{m+1,n})_{m,n\in\mathbf{Z}}$ and recalls the Schur polynomials are defined via

$$exp\left(\sum_1^{\infty}t_kz^k\right)=\sum_0^{\infty}s_\ell(t)z^\ell \tag{5.8}$$

It follows that (via $a(t)=exp[H(t)]aexp[-H(t)]$)

$$\psi_n(t)=\sum\psi_{n-\ell}s_\ell(t);\ \psi_n^*(t)=\sum\psi_{n+\ell}^*s_\ell(-t) \tag{5.9}$$

and for tau functions one has for $v=g|0>$ (recall $H(t)|0>=0$)

$$\tau(t,v)=<0|e^{H(t)}g|0> \tag{5.10}$$

This exhibits the context in which tau functions eventually can be regarded in terms of matrix elements in the representation theory of say $G(V,V^*)$ (some details below). To develop this one defines degree (or charge) via

$$deg(\psi_n)=1;\ deg(\psi_n^*)=-1 \tag{5.11}$$

so vectors $\psi^*_{m_1}\cdots\psi^*_{m_k}\psi_{n_k}\cdots\psi_{n_1}|0>$ with $m_1 < \cdots < m_k < 0 \leq n_k < \cdots n_1$ contribute a basis of $\mathfrak{A}(0)|0>$. Similarly for charge n one specifies vectors

$$<n| = \begin{cases} <0|\psi_{-1}\cdots\psi_{-n} & (n<0) \\ <0| & (n=0) \\ <0|\psi_0^*\cdots\psi^*_{n-1} & (n>0) \end{cases} \tag{5.12}$$

Putting in a bookkeeping parameter z one has an isomorphism

$$i: \mathfrak{A}|0> \to \mathbf{C}[t_1, t_2, \cdots; z, z^{-1}];\ a|0> \mapsto \sum_{\mathbf{Z}} <m|e^{H(t)}a|0> z^m \tag{5.13}$$

This leads to the action of $\mathfrak{A}$ on $\mathbf{C}[t_i; z, z^{-1}]$ via differential operators. Thus write

$$\psi(z) = \sum \psi_n z^n;\ \psi^*(z) = \sum \psi_n^* z^{-n} \tag{5.14}$$

and one checks that t-evolution is diagonalizable via

$$e^{H(t)}\psi(z)e^{-H(t)} = e^{\xi(t,z)}\psi(z);\ e^{H(t)}\psi^*(z)e^{H(t)} = e^{-\xi(t,z)}\psi^*(z) \tag{5.15}$$

It will then follow (nontrivially) that

$$<m|e^{H(t)}\psi(z) = z^{m-1}X(z) <m-1|e^{H(t)}; \tag{5.16}$$

$$<m|e^{H(t)}\psi^*(z) = z^{-m}X^*(z) <m+1|e^{H(t)}$$

This is actually a somewhat profound result whose clearest proof involves Wick's theorem and we refer to [**191, 205**] for details. This leads to the fundamental Hirota bilinear identity

$$\oint dz\psi(t,z)\psi^*(t',z) = 0 \tag{5.17}$$

via free fermion arguments which can be written out as

$$\sum_{j\in\mathbf{Z}} s_j(2y_i)s_{j+1}(-\tilde{\partial}_y)\tau^g(x_i+y_i)\tau^g(x_i-y_i) = 0 \tag{5.18}$$

in an obvious notation and this is KP theory.

Group theory underlies classical integrable systems and there are some different group structures for the same integrable system. Some of the groups act in the space of solutions to the integrable hierarchy and others can act on just the space time variables of the equations. In order to quantize an integrable system one needs only replace the first type of group structures by their quantum counterparts. However the groups acting in the space time still remain classical even for quantum systems. Now for KP elements of the form $g =: exp\{\sum_{mn} a_{mn}\psi_m^*\psi_n\}:$ are in 1-1 correspondence with solutions to the KP hierarchy and the group acting in the space of solutions is $GL(\infty)$. The same group acts in the Toda hierarchy with two infinite sets of times $\{t_k\}$ and $\{\tilde{t}_k\}$ with tau functions given by

$$\tau_n(t,\tilde{t}|g) = \frac{<n|e^{H(t)}ge^{\tilde{H}(\tilde{t})}|n>}{<n|g|n>} \tag{5.19}$$

where $\tilde{H}(\tilde{t}) = \sum H_{-k}\tilde{t}_k$. To extend KP/Toda one looks first at $GL(\infty)$ as a Hopf algebra with $\Delta(g) = g \otimes g$ with

$$g\psi_i g^{-1} = \sum R_{ik}\psi_k;\ g\psi_i^* g^{-1} = \sum \psi_k^* R_{ki}^{-1} \tag{5.20}$$

This means that the fermions are intertwining operators which intertwine the fundamental representations of $GL(\infty)$. For $\mathfrak{g}\ell(\infty)$ each vertex on the Dynkin diagram (∞ in both directions) corresponds to a fundamental representation F^n with arbitrary fixed origin $n = 0$; here $\psi_k^*|F^n >= 0$ for $k \geq n$ and $\psi_k|F^n >= 0$ for $k < n$. The relation (5.20) implies that $\sum \psi_i \otimes \psi_i^*$ commutes with g and leads to the Hirota bilinear identities. Now for the general case (following [**403, 404, 548, 549, 566, 656, 663, 664, 952**]) take a highest weight representation λ of a Lie algebra $\mathfrak{g}$ with UEA $U(\mathfrak{g})$. The tau function is defined as (cf. (5.10) - $\tau = \tau^\lambda(t, \tilde{t}|g)$)

$$\tau = < 0|\prod_k e^{t_k T_-^k} g \prod_i e^{\tilde{t}T_+^i}|0>_\lambda = \sum < n|g|m>_\lambda \prod \frac{t_i^{n_i}\tilde{t}_j^{m_j}}{n_i!m_j!} \tag{5.21}$$

where the vacuum state means the highest weight vector, $T_\pm^k$ are the generators of the corresonding Borel subalgebras of $\mathfrak{g}$, and the exponentials are supposed to be normal ordered in some fashion. One then proceeds as in the classical situation. For quantization one replaces the group by the corresponding quantum group and repeats the procedures but with the following new features:

- The tau function is no longer commutative. It is defined as in (5.21) with exponentials replaced by quantum exponentials.
- One will need to distinguish between left and right intertwiners.
- The counterpart of the group element g defined by $\Delta(g) = g \otimes g$ does not belong to the universal enveloping algebra (UEA) $U_q(\mathfrak{g})$ but rather to $U_q(\mathfrak{g}) \otimes U_q^*(\mathfrak{g})$ where $U_q^*(\mathfrak{g}) = \mathfrak{A}_q(G)$ is the algebra of functions on the quantum group.

Thus the tau function (5.21) is the average of an element from $U_q(\mathfrak{g}) \otimes \mathfrak{A}_q(G)$ over some representation of $U_q(\mathfrak{g})$ and hence belongs to $\mathfrak{A}_q(G)$ (and is consequently noncommutative).

More generally one considers a universal enveloping algebra (UEA) $U(\mathfrak{g})$ and a Verma module V of this algebra. A tau function is then defined to be a generating function of matrix elements $< k|g|n >_V$ of the form (we think of the q-deformed theory from the outset here)

$$\tau_V(t, \tilde{t}|g) = {}_V < 0|\prod_{\alpha>0} e_q(t_\alpha T_\alpha) g \prod_{\alpha>0} e_q(\tilde{t}_\alpha T_{-\alpha})|0>_V = \tag{5.22}$$

$$= \sum_{k_\alpha \geq 0, n_\alpha \geq 0} \prod_{\alpha>0} \frac{t_\alpha^{k_\alpha}\tilde{t}_\alpha^{n_\alpha}}{[k_\alpha]![n_\alpha]!} {}_V < k_\alpha|g|n_\alpha >_V$$

where $[n] = (q^n - q^{-n})/(q - q^{-1})$ with $[n]! = [n][n-1]\cdots[1]$ and $e_q(x) = \sum_{n\geq 0}(x^n/[n]!)$, etc. The $T_{\pm\alpha}$ are generators of $\pm$ maximal nilpotent subalgebras $N(\mathfrak{g})$ and $\tilde{N}(\mathfrak{g})$ with a suitable ordering of positive roots α while t_α and $\tilde{t}_\alpha \sim t_{-\alpha}$ are associated times. A vacuum state is annihilated by T_α for $\alpha > 0$

(i.e. $T_\alpha|0>=0$). The Verma module is $V=\{|n_\alpha>_V=\prod_{\alpha>0}T_{-\alpha}^{n_\alpha}|0>_V\}$. Except for special circumstances all $\alpha\in N(\mathfrak{g})$ are involved and since not all T_α are commutative the tau function has nothing a priori to do with Hamiltonian integrable systems. Sometimes (e.g. for fundamental representations of $\mathfrak{sl}(n)$) the system of bilinear equations obtained via intertwining can be reduced to one involving a smaller number of time variables (e.g. $rank(\mathfrak{g})$) and this returns one to the field of Hamiltonian integrable systems. One works with four Verma modules $V,\hat{V},V',\hat{V}'$; given V,V' every allowed choice of $\hat{V},\hat{V}'$ provides a separate set of bilinear identities. The starting point is **(A)**: Embed $\hat{V}$ into $V\otimes W$ where W is some irreducible finite dimensional representation of $\mathfrak{g}$ (there are only a finite number of possible $\hat{V}$). One defines a right vertex operator (or intertwining operator) of type W as a homomorphism of $\mathfrak{g}$ modules $E_R:\hat{V}\to V\otimes W$. This intertwining operator can be explicitly continued to the whole representation once it is constructed for its vacuum (highest weight) state

$$\hat{V}=\{|n_\alpha>_{\hat{V}}=\prod_{\alpha>0}(\Delta T_{-\alpha})^{n_\alpha}|0>_{\hat{V}}\}\tag{5.23}$$

where

$$|0>_{\hat{V}}=\left(\sum_{(p_\alpha,i_\alpha)}A(p_\alpha,i_\alpha)\left(\prod_{\alpha>0}(T_{-\alpha})^{p_\alpha}\otimes(T_{-\alpha})^{i_\alpha}\right)\right)|0>_V\otimes|0>_W\tag{5.24}$$

Then every $|n_\alpha>_{\hat{V}}$ is a finite sum of $|m_\alpha>_V$ with coefficients in W. Next **(B)**: Take another triple defining a left vertex operator $\tilde{E}_L':\hat{V}'\to W'\otimes V'$ such that $W\otimes W'$ contains a unit representation of $\mathfrak{g}$ with projection $\pi:W\otimes W'\to I$. Using π one can build a new intertwining operator

$$\Gamma:\hat{V}\otimes\hat{V}'\stackrel{E\otimes E';}{\to}V\otimes W\otimes W'\otimes V'\stackrel{I\otimes\pi\otimes I}{\to}V\otimes V'\tag{5.25}$$

such that (♦♦) $\Gamma(g\otimes g)=(g\otimes g)\Gamma$ for any group element g such that $\Delta(g)=g\otimes g$ ((♦♦) is in fact an algebraic form of the bilinear identities - one can use (5.22) to average (♦♦) with the evolution exponentials over the enveloping algebra and group elements can be constructed using the universal T operator). Finally **(C)**: One looks at a matrix element of (♦♦) between four states, namely

(5.26)
$$_{V'}<k'|\,_V<k|(g\otimes g)\Gamma|n>_{\hat{V}}\,|n'>_{\hat{V}'}=\,_{V'}<k'|\,_V<k|\Gamma(g\otimes g)|n>_{\hat{V}}\,|n'>_{\hat{V}'}$$

and rewrites this suitably.

In terms of functions the action of Γ can be presented via

$$\Gamma|n>_{\hat{\lambda}}\,|n'>_{\hat{\lambda}'}=\sum_{\ell,\ell'}|\ell>_\lambda\,\ell'>_{\lambda'}\Gamma(\ell,\ell',n,n')\tag{5.27}$$

($\otimes$ omitted) so (5.26) becomes

$$\sum_{m,m'}\Gamma(k,k'|m,m')\frac{\|k\|^2_\lambda\|k'\|^2_{\lambda'}}{\|m\|^2_{\hat{\lambda}}\|m'\|^2_{\hat{\lambda}'}}<m|g|n>_{\hat{\lambda}}<m'|g|n'>_{\hat{\lambda}'}=\tag{5.28}$$

$$= \sum_{\ell,\ell'} < k|g|\ell >_\lambda < k'|g|\ell' >_{\lambda'} \Gamma(\ell,\ell'|n,n')$$

To rewrite this as a differential or difference expression one needs to use formulas like ($\tau = \tau_V(t,\tilde{t}|g)$)

$$\tau = \sum_{m,\tilde{m}\in V} s^V_{m,\tilde{m}}(t,\tilde{t}) < m|g|\tilde{m} >_V;\ \tau_V(t,\tilde{t}|g) = < 0_V|U(t)g\tilde{U}(\tilde{t})|0_V > \quad (5.29)$$

which will give a generating function for identities (making use of explicit forms for $\Gamma(\ell,\ell'|n,n')$ that arise in a group theoretic framework - Clebsch-Gordon coefficients, etc.).

REMARK 8.5.1. Examples abound (see e.g. [**403, 404, 548, 549, 566, 663, 664**]) and the theory can be made very general and abstract. The fact that matrix elements for group representations along with intertwining is important is not new or surprising. What is interesting is the fact that group theory and intertwining leads to Hirota type formulas as in KP/Toda with their accompanying differential or q-difference equations involving tau functions as generating functions for the matrix elements. One knows also of course that special functions and q-special functions arise as matrix elements in the representation theory of groups and quantum groups with associated differential or q-difference operators whose intertwining corresponds to classical transmutation theory as in [**202, 205, 208, 209**]. However from (5.21) - (5.22) for example one sees that the t variables arise from Borel subalgebras of $\mathfrak{g}$ and it is the coefficients $< n|g|m >$ which give rise to variables $x \sim R$ for the radial part of a Casimir operator (cf. [**210**]).

With a view toward closer relations with operator transmutation we go to [**237**] and consider $\mathfrak{s}\ell(2)$ defined via $[e,f] = h$, $[h,e] = 2e$, $[h,f] = -2f$ with quadratic Casimir $C = (ef + fe) + (1/2)h^2 = 2fe + h + (1/2)h^2$. One looks at the principal series of representations with space V_λ and algebra actions

$$e = \partial_x,\ h = -2x\partial_x + \lambda,\ f = -x^2\partial_x + \lambda x \quad (5.30)$$

the vacua correspond to constants via highest weight vectors $e|0 >= 0$ with $h|0 >= \lambda|0 >$. A Whittaker vector $|w >^\mu_\lambda \in V_\lambda$ with $e|w >^\mu_\lambda = \mu|w >^\mu_\lambda$ is given via

$$|w >^\mu_\lambda = exp(\mu x) = \sum [\mu^n f^n/n!(\lambda,n)]|0 > \quad (5.31)$$

where $(\lambda,n) = \lambda(\lambda-1)\cdots(\lambda-n+1)$. A dual Whittaker vector is then

$$^\mu_\lambda < w| = \sum {}_\lambda < 0|[\mu^n e^n/n!(\lambda,n)] = x^{-\lambda-2} exp(-\mu x) \quad (5.32)$$

and one defines a Whittaker function $W = W^{\mu_L,\mu_R}_\lambda(\phi)$ via

$$W_\lambda = {}^{\mu_L}_\lambda < w|e^{\phi h}|w >^{\mu_R}_\lambda = W_\lambda(\phi) = 2e^{(\lambda+1)\phi}\left(\sqrt{\frac{\mu_L}{\mu_R}}\right)^{-(\lambda+1)} K_{\lambda+1}(2\sqrt{\mu_L\mu_R}e^{-\phi}) \quad (5.33)$$

(Macdonald function K_λ). One has then

$$\left(\frac{1}{2}\partial_\phi^2 + \partial_\phi - 2\mu_L\mu_R e^{-2\phi}\right) W_\lambda(\phi) = \left(\frac{1}{2}\lambda^2 + \lambda\right) W_\lambda(\phi) \tag{5.34}$$

Raising and lowering operators are obtained via intertwining of V_λ spaces

$$W_{\lambda+1} = e^\phi \frac{\partial_\phi + \lambda + 2}{2(\lambda+1)} W_\lambda;\ \mu_L W_{\lambda-1} = \frac{\lambda}{\mu_R} e^\phi \frac{\lambda - \partial_\phi}{2} W_\lambda \tag{5.35}$$

Similarly, using intertwiners $V_{\lambda+1} \otimes V_{\nu+1} \to V_\lambda \otimes V_\nu$ one gets bilinear relations leading to nonlinear Hirota type equations

$$\frac{\partial_\phi + \lambda + 2}{2(\lambda+1)} W_\lambda \frac{\mu_2^R(1-\partial_\phi)}{\nu+1} W_\nu + \frac{\mu_1^R}{\lambda+1} W_\lambda \frac{(\partial_\phi + \lambda + 2)}{\nu+1}(1+\partial_\phi) W_\nu = \tag{5.36}$$

$$= -W_{\lambda+1}\frac{\mu + 1 - \partial_\phi}{2\mu_2^L}\partial_\phi W_{\mu+1} + \frac{\lambda+1-\partial_\phi}{2\mu_1^L} W_{\lambda+1}\partial_\phi W_{\nu+1}$$

Product formulas are also obtained and all this extends to quantum group situations. Now, operator transmutation as envisioned in Section 1 would involve say $P \sim C_\lambda = (1/2)\partial_\phi^2 + \partial_\phi - 2\mu_L\mu_R exp(-2\phi)$ and $Q \sim$ some other operator in V_λ with eigenvalue $(1/2)\lambda^2 + \lambda$ (which could be arranged by scaling). The group theory makes no obvious entry and in fact one is much better off doing operator transmutation in a distribution context such as $\mathfrak{D}'_\phi$ where differential equations involving C_λ for W_λ would still prevail (a Hilbert space framework is restrictive). There may be some possible use of Hirota type formulas however in studying operator transmutation of "interesting" differential operators such as C_λ but this is not clear. We have mainly given in this Section a glimpse of quantum transmutations = intertwinings in the group theory context with many relations to integrable systems; this seems to emphasize a deep correspondence between group theory and integrability.

REMARK 8.5.2. Some perspective can perhaps be gained as follows. Extracting here from [**205**] let us specify and expand now some facts to be developed further. The classical Wick's theorem from say [**935**] is stated as: The time ordered product of n free fields A_i equals the sum of normal ordered products of all possible partial and complete contractions. Thus one stipulates that a contraction is $\widehat{A_1A_2} =< 0|T(A_1A_2)|0>$ satisfying the equation : $A_1\acute{A}_2A_3\grave{A}_4 := \widehat{A_2A_4} : A_1A_3 :$. Setting C_{ij} for the operator contracting A_i and A_j one has e.g.

$$T(A_1A_2A_3) =: A_1A_2A_3 : + : (C_{12} + C_{13} + C_{23})A_1A_2A_3 := \tag{5.37}$$

$$=: A_1A_2A_3 : + \widehat{A_1A_2}A_3 + \widehat{A_1A_3}A_2 + \widehat{A_2A_3}A_1$$

One finds also for $A_i = A_i^+ + A_i^-$ ($+ \sim$ creation)

$$\widehat{A_1A_2} = [A_1^-, A_2^+]\ (t_1 > t_2);\ \widehat{A_1A_2} = [A_2^-, A_1^+]\ (t_2 > t_1) \tag{5.38}$$

where $\widehat{A_1A_2} = T(A_1A_2) - : A_1A_2 :$ can serve as a definition. Comparing to Section 8.1 we have $(a \circ b \sim ab)$

$$a^\bullet b^\bullet\ (or\ a^\diamond b^\diamond) = (a|b) \sim \widehat{ab};\ a \circ b \sim T(ab)\ (or\ ab);\ a \vee b \sim: ab : \tag{5.39}$$

Note also from (4.5)
(5.40)
$$<0|w_1\cdots w_{2n}|0>=\sum_\sigma sgn(\sigma)<0|(w_{\sigma(j)}w_{\sigma(j+1)}|0>\cdots<0|w_{\sigma(2n-1)}w_{\sigma(2n)}|0>$$

where $\sigma(1)<\sigma(2),\cdots,\sigma(2n-1)<\sigma(2n)$ and $\sigma(1)<\sigma(3)<\cdots<\sigma(2n-1)$. Now the proof sketched in [**935**] (for $W(A_1,\cdots,A_n)$ the sum of all normal ordered partial and complete contractions) involved

$$T(A_1\cdots A_n)=A_1T(A_2\cdots A_n)=A_1W(A_2,\cdots,A_n)= \tag{5.41}$$
$$=A_1^+W+WA_1^-+[A_1^-,W]=W(A_1,\cdots,A_n)$$

since $[A_1^-,W]$ can be reduced to a sum of brackets as in (5.38). Next in (1.13) one looks at

$$:a_1\cdots a_n:b=:a_1\cdots a_nb:+\sum_{j=1}^n a_j^\bullet b^\bullet:a_1\cdots a_{j-1}a_{j+1}\cdots a_n: \tag{5.42}$$

which corresponds to $a_j^\bullet b^\bullet=(a_j|b)=\widehat{a_jb}$ and $ub=u\circ b$. Note
(5.43)
$$(a_1\vee\cdots\vee a_n)b=a_1\vee\cdots\vee a_n\vee b+\sum_1^n\widehat{a_jb}a_1\vee\cdots\vee a_{j-1}\vee a_{j+1}\vee\cdots\vee a_n$$
Here $u\circ b\sim T(ub)$. One compares also with (1.35), (1.38), and (1.39), Thus for $u=a_1\vee\cdots\vee a_n$ (cf. (1.4))
(5.44)
$$T(u)=\sum t(u_1)u_2;\ \Delta(u)=u\otimes 1+1\otimes u+\sum_{p=1}^{2n-1}\sum_\sigma a_{\sigma(1)}\vee\cdots a_{\sigma(p)}\otimes a_{\sigma(p+1)}\vee\cdots\vee a_{\sigma(2n)}$$

$$T(u)=t(u)+u+\sum\sum t(a_{\sigma(1)}\vee\cdots\vee a_{\sigma(p)})a_{\sigma(p+1)}\vee\cdots\vee a_{\sigma(2n)} \tag{5.45}$$

Recall $t(1)=1,\ t(a)=0$ for $a\in V$ and $t(u\vee v)=\sum t(u_1)t(v_1)(u_2|v_2)$. Hence in particular

$$t(a\vee b\vee c)=\sum t(a_1\vee b_1)t(c_1)(a_2\vee b_2|c_2)=0 \tag{5.46}$$

and we recall the equivalence of (1.38) and (1.39) (for (|) symmetric), namely

$$t(a_1\vee\cdots\vee a_{2n})=\sum_\sigma\prod_{j=1}^n(a_{\sigma(j)}|a_{\sigma(j+n)})\equiv \tag{5.47}$$
$$\equiv\frac{1}{2^nn!}\sum_\sigma(a_{\sigma(1)}|a_{\sigma(2)}\cdots(a_{\sigma(2n-1)}|a_{\sigma(2n)})$$

We recall also that Wick à la (5.39) implies (5.16) or

$$<m|e^{H(t)}\psi(z)=z^{m-1}X(z)<m-1|e^{H(t)}; \tag{5.48}$$
$$<m|e^{H(t)}\psi^*(z)=z^{-m}X^*(z)<m+1|e^{H(t)}$$

REMARK 8.5.3 Connections of tau functions to Hopf and Clifford algebras (and QFT) following work of Brouder, et al, Fauser et al, and Effros, et al should be worth exploring (cf. also [**45, 146, 278, 339, 356, 368, 369, 489, 582**] and Section 8.6 to follow).

REMARK 8.5.4. Tau functions may provide new perspective relative to QFT and we mention one possible direction. Thus try to write down a tau function for operator objects u, v based on $a_i \in W$ where $W \sim W^+ \oplus W^-$ say (e.g. $W^+ \sim W_{cr}$ and $W^- \sim W_{an}$ as in Section 8.4). One could look for analogues of $H(t)$ since $\tau(t,v) =< 0|exp(H(t))g|0 >$ with $v = g|0 >$ $(H(t) \sim \sum_1^\infty t_k H_k$ with $H_k \sim \sum_{-\infty}^{\infty} \psi_n \psi^*_{k+n})$, etc. or analogues of the vertex operators X, etc. of Section 8.6, and of the Hirota equations and one could also consider Hopf algebra ideas for free fermions. Consider normal order $a \vee b$ and operator product $\circ$ (no time order is needed momentarily) and $a^\diamond b^\diamond \sim \widehat{ab} \sim (a|b)$ and look at some formulas from preceeding sections

(5.49)

$$a \vee b \sim : ab :; \; W_{an} = \oplus_{m<0}\mathbf{C}\psi_m \oplus \oplus_{m\geq 0}\mathbf{C}\psi^*_m; W_{cr} = \oplus_{m\geq 0}\mathbf{C}\psi_m \oplus \oplus_{m<0}\mathbf{C}\psi^*_m;$$

$$\psi_n|0 >= 0 \; (n<0); \; \psi^*_n|0 >= 0 \; (n \geq 0); \; < 0|\psi_n = 0 \; (n \geq 0); \; < 0|\psi^*_n = 0 \; (n<0)$$

One could ask for Δ as before with $ab = a \circ b = a \vee b + (a|b)$; note

(5.50)

$$[\psi_n, \psi_m]_+ = [\psi^*_m, \psi^*_n]_+ = 0; \; [\psi_m, \psi^*_n]_+ = \delta_{mn}; \; \psi_m \psi^*_n =: \psi_m \psi^*_n : + (\psi_m|\psi^*_n);$$

$$: \psi_m \psi_n * := \psi_m \psi^*_n - < 0|\psi_m \psi^*_n|0 > \Rightarrow (\psi_m|\psi^*_n) =< 0|\psi_m \psi^*_n|0 >$$

Note also $\psi_m \in W_{an}$ for $m < 0$ so $\psi_m \psi^*_n$ has an annihilation operator on the left. Thus consider $: \psi_m \psi^*_n := \psi_m \psi^*_n - < 0|\psi_m \psi^*_n|0 >$. For $m \geq 0$ or $n \geq 0$ the bracket $< >= 0$; otherwise $m < 0$ and $n < 0$ which means $\psi_m \in W_{an}$ and $\psi^*_n \in W_{cr}$ so $< 0|\psi_m \psi^*_n|0 >= [\psi_m, \psi^*_n]_+$ and $: \psi_m \psi^*_n := -\psi^*_n \psi_m$. Recall we have a Clifford algebra here with $[\psi_m, \psi^*_n]_+ = \delta_{mn}$. This says that $< 0|\psi_m \psi^*_n|0 >$ is antisymmetric. Recall also $H(t) = \sum_1^\infty t_\ell \sum_{n\in\mathbf{Z}} \psi_n \psi^*_{n+\ell} = \sum t_\ell \sum_{n\in\mathbf{Z}} : \psi_n \psi^*_{n+\ell} :$. Also $H(t)|0 >= 0$ and $< 0|H(t) \neq 0$; to see this note for $n > 0$ one has $\psi^*_{n+\ell}|0 >= 0$ while for $n < 0$ one has $: \psi_n \psi^*_{n+\ell} := -\psi^*_{n+\ell}\psi_n$ and $\psi_n|0 >= 0$. For $< 0|H(t)$ consider for $n > 0$, $< 0|\psi_n \psi^*_{n+\ell} = - < 0|\psi^*_{n+\ell}\psi_n \neq 0$ when $n < 0$ and $n + \ell \geq 0$. Note also $< 0|H_{-n} = 0$ for $n > 0$ since $H_{-n} = \sum_{\mathbf{Z}} \psi_j \psi^*_{j-n}$ and $< 0|\psi_j \psi^*_{j-n} = - < 0|\psi^*_{j-n}\psi_j = 0$ for $j < 0$. Recall also

$$[\sum a_{ij} : \psi_i \psi^*_j :, \sum b_{ij} : \psi_i \psi^*_j :] = \sum c_{ij} : \psi_i \psi^*_j : + c_0; \tag{5.51}$$

$$c_{ij} = \sum_k a_{ik} b_{kj} - \sum_k b_{ik} a_{kj}; \; c_0 = \sum_{i<0, j\geq 0} a_{ij} b_{ji} - \sum_{i \geq 0, j<0} a_{ij} b_{ji}$$

One could ask now about the meaning in QFT (if any) of e.g.

$$\psi(z) = \sum \psi_n z^n; \; \psi^*(z); \; \psi(t,z) = (X(z)\tau(t)/\tau(t); \; \tau(t) =< 0|exp(H(t)g|0 >;$$

$$exp(H(t))\psi(z)exp(-H(t)) = exp(\xi(t,z))\psi(z);$$

$$< m|exp(H(t))\psi(z) = z^{m-1}X(z) < m-1|exp(H(t))$$

It seems clear however that some further input is needed if we are to properly express some putative "role" of tau functions in QFT (cf. [**191, 211, 662**]). The most obvious role seems to be in terms of the partition function of a matrix model for example. It seems that in order to gain any true insight into the tau function we need more perspective and one could begin by specifying the various known ways in which the tau function arises (beyond its emergence in the soliton mathematics of the Japanese school (Date, Hirota, Jimbo, Miwa, Sato, et. al. and more recently Shiota, Takasaki, Takebe, et. al.). We will refer to [**132, 142, 147, 191, 200, 205, 211, 564, 565, 566, 568, 569, 635, 652, 653, 996, 1009**] for tau functions and complex analysis, [**925**] for general ideas involving Riemann surfaces, [**20, 21, 137, 142, 340, 455, 456, 542, 569, 737, 738, 739, 1017**] for matrix models, [**567, 600**] for tau functions, Hurwitz spaces, and Frobenius manifolds, and [**638**] for twistor connections.

REMARK 8.5.5. We want to mention here the beautiful paper [**335**]. This gives an approach to q-Feynman diagrms using combinatorics and graph theory; it delves into q-Fock spaces, q-Wick formulas, q-Gaussians, etc. and is thoroughly lovely. Moreover the analysis of partitions, graphs, and symmetric functions leads one to the area of free probability theory and related topics for which we refer to [**45, 114, 146, 391, 406, 433, 818, 904, 908, 966**]. We wrote up a sketch of [**335**] with additional notes on free probability theory but it is too long to insert here.

6. VERTEX OPERATORS AND SYMMETRIC FUNCTIONS

For references here see [**92, 513, 516, 546**] and we begin wih [**516**]. One constructs first a family of symmetric functions generalizing the Hall-Littlewood symmetric functions and derives a raising operator formula. Thus let $\{a_n\}_{n\in\mathbf{N}}$ be a set of indeterminates and H an infinite dimensional Heisenberg algebra with infinitely many parameters a_i over $\mathbf{C}$ generated by h_n, $n \in \mathbf{Z}^\times = \mathbf{Z}/\{0\}$ and a central element c such that (★) $[h_m, h_n] = (m/a_{|m|})\delta_{m,-n}c$. The vector space V is the polynomial algebra generated by the h_{-n} ($n \in \mathbf{N}$) and H acts on V by the basic representation where $h_{-n} \sim$ multiplication operator and h_n acts as a differentiation operator subject to the commutation relations (★). One defines vertex operators on $V[z, z^{-1}]$ via

$$(6.1) \qquad X(z) = exp\left(\sum_1^\infty \frac{a_n}{n} h_{-n} z^n\right) exp\left(-\sum_1^\infty \frac{a_n}{n} h_n z^{-n}\right) = \sum_{n\in\mathbf{Z}} X_n z^{-n};$$

$$X^*(z) = exp\left(-\sum_1^\infty \frac{a_n}{n} h_{-n} z^n\right) exp\left(\sum_1^\infty \frac{a_n}{n} h_n z^{-n}\right) = \sum_{n\in\mathbf{Z}} X^*_{-n} z^n$$

The space V is a graded algebra with respect to the grading

$$(6.2) \quad V = \oplus_{n\in\mathbf{Z}} V_n, \ V_n = \{v \in V; deg(v) = n\}, \ deg(h_{-1}^{i_1} \cdots h_{-k}^{i_k}) = i_1 + \cdots + k i_k$$

The operators X_n, X_n^* act on the space via $X_n : V_m \to V_{m+n}$, $X_n^* : V_m \to V_{m-n}$. The normal product : : is defined as moving the elements h_n to the right of the elements h_{-n} and in order to calculate the product of vertex operators one introduces the analytic function

$$f(x) = exp\left(-\sum_1^\infty \frac{a_n}{n}x^n\right) = \prod_1^\infty exp\left(-\frac{a_n}{n}x^n\right) \tag{6.3}$$

As long as the variable $|x| < \overline{lim_{n\to\infty}}|a_n/a_{n+1}|$ the infinite product converges absolutely. Thus $f(x)$ will be analytic in the corresponding region. There are two cases of special interest when $a_n = q^n$ and $a_n = (1-t^n)/(1-q^n)$ (assuming $t, q > 0$). The convergence radius is q^{-1} in the first case and for the second case

- 1 for $t < 1$, $q < 1$
- q/t for $t > 1$, $q > 1$
- $1/t$ for $t > 1$, $q < 1$
- $1/q$ for $t < 1$, $q > 1$

One writes now $\lambda \vdash n$ to represent a partition of n of the form $\lambda = (\lambda_1, \lambda_2, \cdots)$ with $n = \lambda_1 + \lambda_2 + \cdots$. The q numbers and factorials follow the standard rules

$$(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1});\ (a;q)_\infty = \prod_{n\geq 0}(1-aq^n) \tag{6.4}$$

The space V can be identified with the ring of symmetric functions in infinitely many variables x_i by mapping h_n to the n^{th} power sum symmetric function $p_n = \sum x_i^n$. The ring of symmetric functions has various useful bases, e.g. the **Q** basis of power sum symmetric functions $p_\lambda = p_{\lambda_1}\cdots p_{\lambda_k}$; the **Z** basis of monomial symmetric functions $m_\lambda = \sum x_{i_1}^{\lambda_1}\cdots x_{i_k}^{\lambda_k}$; The **Z** basis of complete symmetric functions $c_\lambda = c_{\lambda_1}\cdots c_{\lambda_k}$ with $c_r = \sum_{|\lambda|=r} m_\lambda$; or the **Z** basis of Schur functions $s_\lambda = det(c_{\lambda_i - i + j} = m_\lambda + \cdots$ which can also be determined via

$$s_\lambda(x_1,\cdots,x_n) = \frac{\sum_{w\in S_n} sgn(w)x^{w(\lambda+\delta)}}{\prod_{i<j}(x_i - x_j)} \tag{6.5}$$

($\delta = (n-1, n-2, \cdots, 1, 0)$). Using Schur functions one can give an explicit expression for $f(x)$ and $f(x)^{-1}$. Thus

$$f(x) = \sum_{n\geq 0} f_n x^n = \sum_{n\geq 0}\left(\sum_{\lambda\vdash n}(-1)^{\ell(\lambda)}\frac{a_{\lambda_1}\cdots a_{\lambda_k}}{z_\lambda}\right)x^n \tag{6.6}$$

where

$$z_\lambda(a) = \prod_{i\geq 1} i^{m_i} m_i!(a_{\lambda_1}\cdots a_{\lambda_k})^{-1}\ (\lambda \sim (1^{m_1}2^{m_2}\cdots)) \tag{6.7}$$

Here the f_n are Schur functions s_{1^n} in terms of power sum variables a_n, e.g.

$$f_1 = -a_1;\ f_2 = -\frac{a_2}{2} + \frac{a_1^2}{2};\ f_3 = -\frac{a_2}{2} + \frac{a_1a_2}{2} - \frac{a_1^3}{6};\cdots \tag{6.8}$$

Similarly $f(x)^{-1} = \sum_{n\geq 0} g_n x^n$ with $g_n = s_n$, i.e.

$$g_n = \sum_{\lambda\vdash n}\frac{a_{\lambda_1}\cdots a_{\lambda_k}}{z_\lambda} \tag{6.9}$$

We will also need the symmetric function Q_n defined as the vector $X_{-n} \cdot 1$ in V, i.e.

$$\sum_0^\infty Q_n z^n = exp\left(\sum_1^\infty \frac{a_n}{n} h_{-n} z^n\right) \tag{6.10}$$

which is a generalized homogeneous symmetric function. One also writes $q_\lambda = Q_{\lambda_1} \cdots Q_{\lambda_k}$ and notes that $X(z)X(w) =: X(z)X(w) : f(w/z)$ where normal ordering moves the h_n to the right of the h_{-n} (roughly $h_n \sim \partial/\partial h_{-n}$).

One proves next that for any tuple $\mu = (\mu_1, \mu_2, \cdots, \mu_k)$

$$X_{-\mu} = X_{-\mu_1} \cdots X_{-\mu_k} \cdot 1 = \prod_{i<j} f(R_{ij}) Q_{\mu_1} \cdots Q_{\mu_k}; \tag{6.11}$$

$$R_{ij}(Q_{\mu_1} \cdots Q_{\mu_k}) = Q_{\mu_1} \cdots Q_{\mu_i+1} \cdots Q_{\mu_j-1} \cdots Q_{\mu_k}$$

REMARK 8.6.1. Here and in other places the first paper in [**516**] is sketchy and there are typos. However it does survey a lot of material so we continue even if some confusion arises. To review the whole subject of vertex operators and symmetric functions would be an enormous task and we only want to select a few areas for delectation under the possible illusion that upon typing this out I will learn what is going on.

The proof of (6.11) involves the assertion that $X_{-\lambda}$ is the coefficient of $z^\lambda = z_1^{\lambda_1} \cdots z_k^{\lambda_k}$ in the expression

$$\prod_{i<j} f(z_j/z_i) : X(z_1) \cdots X(z_k) : \tag{6.12}$$

where $|z_1| > \rho^{-1} z_2 > \cdots > \rho^{-k+1}|z_k|$ and ρ is the radius of convergence. In most cases one can assume $\rho = 1$ where the symmetric function $X_{-\lambda}$ can be expressed as a contour integral with the above function (6.12) as the integrand and the integration contours are concentric circles with radius $|z_1| > \cdots > |z_k|$. The symmetric functions $X_{-\lambda}$ form a basis in V and with various choices of the a_n they are or are closely related to some of the well known orthogonal symmetric functions.

- If $a_n = 1$ then $f(x) = 1 - x$ and $X_{-\lambda} \sim$ Schur symmetric functions associated to the partitions λ
- If $a_n = 1 - t^n$ then $f(x) = (1-x)/(1-tx)$ and $X_{-\lambda} \sim$ Hall-Littlewood symmetric functions for the partitions λ
- If $a_n = (1-t^n)/(1-q^n)$ then $f(x) = (x;q)_\infty/(tx;q)_\infty \sim$ Macdonald symmetric functions via basic hypergeometric functions (at least for two row partitions). This case contains all the above cases and especially the Jack polynomials corresponding to $a_n = \alpha$.
- Below one treats also $a_n = q^n$ which originates from the vertex representations of quantum affine algebras

Now define a Hermitian structure in v via $h_n^* = h_{-n}$ so that the power sum symmetric functions $h_{-\lambda} = h_{-\lambda_1} \cdots h_{-\lambda_k}$ are orthogonal in the sense that <

$h_{-\lambda}, h_{-\mu} >= \delta_{\lambda\mu} z_\lambda(a)$ (see above). The generalized orthogonal symmetric functions are defined by

$$P_\lambda(a,x) = m_\lambda(x) + \sum_{\lambda >' \mu} c_{\lambda\mu} m_\mu(x) \tag{6.13}$$

where $>'$ is a fixed total order (e.g. inverse lexicographic order) compatible to the dominance order on the partitions and $m_\lambda = x_1^{\lambda_1} \cdots x_k^{\lambda_k} + \cdots$ are the monomial symmetric functions (cf. [**613**]). The dominance order $>$ is defined by comparing two partitions via their partial sums of the respective parts; the compatibility means that if $\lambda\,'\mu$ then $\lambda > \mu$. This definition seems to depend on the total order but Macdonald polynomials have the triangularity relation for dominance order and thus they are unique up to any total (or partial) order compatible to the dominance order (somewhat mysterious). We are actually interested in the dual basis Q_λ i.e. $< P_\lambda, Q_\mu >= \delta_{\lambda\mu}$ which can be rephrased via

(6.14)
$$\prod_{i,j} f(x_i y_j) = \sum_\lambda \frac{1}{z_\lambda(a)} h_\lambda(x) h_\lambda(y) = \sum P_\lambda(a,x) Q_\lambda(a,y) = \sum_\lambda m_\lambda(x) q_\lambda(y)$$

One notes that the $X_{-\lambda}$ are not orthogonal in general although it is the case for the one row partition (i.e. $X_{-n} \cdot 1 = Q_n$). In fact the inner product $< X_{-\lambda}, X_{-\mu} >$ is the coefficient of $z^\lambda w^\mu$ in the series expansion

$$\prod_{i<j} f\left(\frac{z_i}{z_j}\right) \prod_{i<j} f\left(\frac{w_i}{w_j}\right) \prod_{i,j} f(z_i w_j)^{-1} \tag{6.15}$$

To see this one notes that from the product of vertex operators follows that $< X_{-\lambda} \cdot 1, X_{-\mu} \cdot 1 >$ is the coefficient of $z^\lambda w^\mu$ in the inner product

$$< X(z_1) \cdots X(z_k) \cdot 1, X(w_1) \cdots X(w_\ell) \cdot 1 >= \tag{6.16}$$

$$=<: X(z_1) \cdots X(z_k) : 1, : X(w_1, \cdots X(w_\ell) : 1 > \times$$

$$\times \prod_{ij} f\left(\frac{z_j}{z_i}\right) f\left(\frac{w_j}{w_i}\right) = \prod_{i<j} f\left(\frac{z_i}{z_j}\right) \prod_{i<j} f\left(\frac{w_i}{w_j}\right) \prod_{ij} f(z_i w_j)^{-1}$$

When $a_n = 1 - t^n$ the function $f(x) = (1-x)/(1-tx)$ and one has proved elsewhere that the matrix coefficients $< X_{-\lambda}, X_{-\mu} >$ are zero except when $\lambda = \mu$ which means that the $X_{-\lambda}$ are the Hall-Littlewood functions. In general one can calculate general matrix coefficients as follows. For an element $h_{-\lambda}$ in V we associate a sequence of nonnegative integers $m = m(\lambda) = (m_1, m_2, \cdots)$ where m_i is the multiplicity of i in the partition and write $|m >= h_{-\lambda} = h_{-1}^{m_1} h_{-2}^{m_2} \cdots$. Let $< m| = h_\lambda$ be the dual basis and then for any sequences r and s one has

$$< r|X(z)|s >= \prod_1^\infty z^{nr_n} (-z^{-1})^{ns_n} {}_2F_0\left(-r_n, -s_n; -; -\frac{n}{a_n}\right) \tag{6.17}$$

where ${}_2F_0\left(-n,-x;-;-\frac{1}{a}\right) = C_n^{(a)}(x)$ is a Charlier polynomial. More generally for real α_i one has

(6.18)

$$< r|X_{\alpha_1}(z_1)\cdots X_{\alpha_k}(z_k)|s >= \prod_{i,j} f\left(\frac{a_j}{a_i}\right)^{\alpha_i\alpha_j} \prod_1^{\infty}\left(\sum_i \alpha_i z^n\right)^{r_n}\left(-\sum_i \alpha_i z^{-n}\right)^{s_n} \times$$
$$\times {}_2F_0\left(-r_n,-s_n;-;-n\left(\sum_{i,j}\alpha_i\alpha_j\frac{z_i^n}{z_j^n}\right)/a_n\right)$$

As for classical symmetric note that using $< h_{-\lambda}, h_{-\nu} >= \delta_{\lambda\mu} z_\lambda(a)$ one can realize Schur and Hall-Littlewood functions in V. For practice take the case of Macdonald polynomials as an illustration. Thus introduce vertex operators

$$s(z) = exp\left(\sum_1^{\infty}\frac{1}{n}h_{-n}z^n\right) exp\left(-\sum_1^{\infty}\frac{a_n}{n}h_n z^{-n}\right) = \sum_{n\in\mathbf{Z}} s_n z^{-n}; \tag{6.19}$$
$$H(z) = exp\left(\sum_1^{\infty}\frac{1-t^n}{n}h_{-n}z^n\right) exp\left(-\sum_1^{\infty}\frac{a_n}{n}z^{-n}\right) = \sum_{n\in\mathbf{Z}} H_{-n}z^n;$$
$$S(z) = exp\left(\sum\frac{1-t^n}{n}h_{-n}z^n\right)\exp\left(-\sum_1^{\infty}\frac{a_n}{n(1-t^n)}h_n z^{-n}\right) = \sum_{n\in\mathbf{Z}} S_n z^{-n}$$

which realize the HL functions in variables x_i and the Schur functions in fictitious variables ξ_i via $\prod_i(1-\xi_i y) = \prod_i(1-tx_i y)/(1-x_i y)$. In order to find vertex operators giving the dual symmetric functions one looks at dual vertex operators defined via

$$\bar{s}(z) = exp\left(\sum_{n\geq 1}\frac{a_n}{n}h_{-n}z^n\right) exp\left(-\sum_{n\geq 1}\frac{1}{n}h_n z^{-n}\right); \tag{6.20}$$
$$\bar{H}(z) = exp\left(\sum_{n\geq 1}\frac{a_n}{n}h_{-n}z^n\right) exp\left(-\sum_{n\geq 1}\frac{1-t^n}{n}h_n z^{-n}\right);$$
$$\bar{S}(z) = exp\left(\sum_{n\geq 1}\frac{a_n}{(1-t^n)n}h_{-n}z^n\right) exp\left(-\sum_{\not\geq 1}\frac{1-t^n}{n}h_n z^{-n}\right)$$

Moreover $\bar{s}_{-\mu}$ is the Schur function associated to the function $\prod_i f(x_i)^{-1}$. To prove that these are dual operators for s, H, S one considers

$$Y(z) = exp\left(\sum_1^{\infty}\frac{a_n}{n}b_n h_{-n}z^n\right) exp\left(-\sum_1^{\infty}\frac{1-p^n}{nb_n}h_n z^{-n}\right); \tag{6.21}$$
$$\bar{Y}(z) = exp\left(\sum_1^{\infty}\frac{1-p^n}{nb_n}h_{-n}z^n\right) exp\left(-\sum_1^{\infty}\frac{a_n}{n}b_n h_n z^{-n}\right)$$

with b_n, p two independent scalars. One computes easily

$$Y(z)Y(w) =: Y(z)Y(w) : \frac{z-w}{z-pw};\ Y(z)\bar{Y}(w) = \tag{6.22}$$

$$=: Y(z)\bar{Y}(w) : \frac{z-pw}{z-w};\ Y_h^*(z) = \bar{Y}_h(z^{-1})$$

The contraction functions are exactly those for HL functions and it follows then from the orthogonality of the HL functions that $< Y_{-\lambda}, \bar{Y}_{-\mu} >= 0$ except when $\lambda = \mu$ so Y and $\bar{Y}$ are dual operators. The specific results (6.19) follow by taking (6.23)

$$(1)\ b_n = a_n^{-1},\ p = 0;\ (2)\ b_n = (1-t^n)/a_n,\ p = t;\ (3)\ b_n = (1-t^n)/a_n\ p = 0$$

REMARK 8.6.2. Other meaningful orthogonal operators may be obtaining by specializing b_n and p. For example consider $(A)\ b_n = a_n^{-1}$, $(B)\ b_n = 1 - q^n$, $(C)\ b_n = 1/a_n(1-q^n)$, $(4)\ b_n = 1$. For case (A) one has e.g.

$$Y(z) = exp\left(\sum_{n\geq 1}\frac{1}{n}h_{-n}z^n\right) exp\left(-\sum_{n\geq 1}\frac{(1-p^n)a_n}{n}h_n z^{-n}\right) \tag{6.24}$$

There are typos here in [**516**].

Going back to the usual Heisenberg algebra with $[h_m, h_n] = m\delta_{m,-n}c$ one takes V as the symmetric algebra generated by the elements h_{-n} ($n \in \mathbf{N}$). However there is is difference now in that one is changing the underlying inner product by taking Schur functions as the basic orthogonal symmetric functions instead of the generalized HL polynomials (usually Macdonald functions). The vertex operator approach is more suitable now in that some of the fundamental results beocme extremely simple. Define

$$s(z) = exp\left(\sum_1^\infty \frac{1}{n}h_{-n}z^n\right) exp\left(-\sum_1^\infty \frac{1}{n}h_n z^{-n}\right); \tag{6.25}$$

$$H(z) = exp\left(\sum_1^\infty \frac{1-t^n}{n}h_{-n}z^n\right) exp\left(-\sum_1^\infty \frac{1}{n}h_n z^{-n}\right);$$

$$Q(z) = exp\left(\sum_1^\infty \frac{1-t^n}{n(1-q^n)}h_{-n}z^n\right) exp\left(-\sum_1^\infty \frac{1}{n}h_n z^{-n}\right)$$

These are the vertex operators associated to the Schur functions, HL functions, and Macdonald functions respectively. Here the difference is that the elements h_n satisfy the new commutation relations above. More generally let

$$X(z) = exp\left(\sum_1^\infty \frac{a_n}{n}h_{-n}z^n\right) exp\left(-\sum_1^\infty \frac{1}{n}h_n z^{-n}\right); \tag{6.26}$$

$$X^*(z) = exp\left(-\sum_1^\infty \frac{1}{n}h_{-n}z^n\right) exp\left(\sum_1^\infty \frac{a_n}{n}h_n z^{-n}\right)$$

One has again (6.7) and for any tuple of integers $\mu = (\mu_1, \cdots, \mu_k)$
(6.27)
$$X_{-\mu} = X_{-\mu_1} \cdots X_{-\mu_k} = \prod_{i<j} F(R_{ij}) q_\mu; \ X^*_{-\mu} = X^*_{-\mu_1} \cdots X_{-\mu_k} = \prod_{i<j} F(R_{ij}) e_\mu$$
where $q_\lambda = Q_{\lambda_1} \cdots Q_{\lambda_k}$ are defined in (6.10) and e_n is the elementary symmetric function. It follows that the vertex operator $s(z)$ is a self dual operator under $h^*_n = h_{-n}$ and the dual of $H(z)$ is
(6.28)
$$\bar{H}(z) = exp\left(\sum_1^\infty \frac{1}{n} h_{-n} z^n\right) exp\left(-\sum_1^\infty \frac{1-t^n}{n} h_n z^{-n}\right); \ < H_{-\lambda}, \bar{H}_{-\mu} >= \delta_{\lambda\mu} b_\lambda(t)$$
where $b_\lambda(t) = \prod_{i\geq 1}(t;t)_{m_i(\lambda)}$. Consequently $< s_\lambda, \bar{H}_{-\mu} >$ are the Kostka-Foulkes polynomials $m_{\lambda,\mu}$.

The vertex operator representations of the quantum affine algebras are connected to the case $a_n = q^n$. To describe this we use again $[h_m, h_n] = m\delta_{m,-n}c$ $(m, n \in \mathbf{Z})$ and define
(6.29)
$$X^\pm(z) = exp\left(\pm\sum_1^\infty \frac{q^{\mp n}}{n} h_{-n} z^n\right) exp\left(\mp\sum_1^\infty \frac{q^{\mp n}}{n} h_n z^{-n}\right) = \sum_{n\in\mathbf{Z}^+} X^\pm_n z^{-n}$$
The commutation relations are

(6.30) $$X^\pm(z)X^\mp(w) =: X^\pm(z)X^\pm(w) : [1-(w/z)]^{-1}; \ X^\pm(z)X^\pm(w) =$$
$$=: X^\pm(z)X^\pm(w) : [1-(q^\mp(w/z)]; \ [X^\pm(z), X^\mp(w)] =: X^\pm(z)X^\mp(w) : \delta(w/z)$$
where as usual $\delta(x) = \sum_{n\in\mathbf{Z}} x^n$. It follows that the symmetric function $X^\pm_{-\lambda}$ is given by

(6.31) $$X^\pm_{-\lambda} = X^\pm_{-\lambda_1} \cdots X^\pm_{-\lambda_k} \cdot 1 = \prod_{i<j}(1 - q^\mp R_{ij}) Q_{\lambda_1} \cdots Q_{\lambda_k} q^{\mp|\lambda|/2}$$

where the Q_n is the complete symmetric function (c_n) or the Q_n defined with $a_n = 1$. Note that when $q = 1$ we obtain the Schur function so this is a deformation around $q = 1$ while the HL function is a deformation of the Schur function around $t = 0$. There are many other formulas which we omit. There are no earth shaking conclusions here; we have simply exhibited a number of formulas involving combinatorics, probability, QFT, tau functions, symmetric functions, vertex operators, etc. We believe that these will all be important in developing the program suggested in [**662**].

APPENDIX A

DeDONDER-WEYL THEORY

We extract here from [**586**] where a lovely discussion of Lagrangian systems can be found. Roughly let $L(x,z,v)$ is the Lagrangian density for a system with $x=(x^1,\cdots,x^m)$, $z=(z^1,\cdots,z^n)$, with $z^a=f^a(x)$ on the extremals and $v=(v_1^1\cdots,v_m^n)$ where $v_\mu^a=\partial_\mu f^a(x)$. Set $\pi_a^\mu=\partial L/\partial v_\mu^a$ and $H=\pi_a^\mu v_\mu^a-L$. Then the deDonder-Weyl Hamilton-Jacobi (DWHJ) equation is

$$\partial_\mu S^\mu(x,z)+H(x,z,\pi_a^\mu-\partial_a S^\mu)=0;\ \pi_a^\mu=\partial_a S^\mu(x,z)=\partial S^\mu/\partial z^a \tag{A.1}$$

Solutions $S^\mu(x,z)$, $z^b=f^b(x)$ lead to conserved currents $G^\mu=\partial S^\mu/\partial a(x,z=f(x))$ $(a=1,\cdots,n)$ In [**586**] one reformulates matters à la LePage [**599**] and we sketch only a few aspects here (standard differential geometry is mainly assumed to be known). One takes M^ℓ to be an ℓ-dimensional differential manifold with tangent vectors $Y(t)$ and local coordinates y so $Y\sim a^\lambda(y)\partial_\lambda$ with $Y_p\sim Y_y\in T_{y(p)}$. Integral curves of Y are given via $dy^\lambda/dt=a^\lambda(y)$ with $y^\lambda(0)=y_0^\lambda$ $(\lambda=1,\cdots,\ell)$. For M^{2n} with coordinates $y=(q^1,\cdots,q^n,p_1,\cdots,p_n)$ and Hamiltonian function $H(q,p)$ the integral curves ϕ_t of the vector field $Y_H=(\partial H/\partial p_i)(\partial/\partial q^i)-(\partial H/\partial q^i)(\partial/\partial p_i)$ define Hamiltonian flows. Each 1-parameter local transformation group $\phi_t:y\to\phi_t(y)$ induces a vector field on M^ℓ via

$$Yf(y)=lim_{t\to 0}\frac{1}{t}[f(\phi_t(y))-f(y)] \tag{A.2}$$

One forms (or Pfaffian forms) $\omega=b_\lambda(y)dy^\lambda$ are dual to vector fields and one writes $\omega(Y)=b_\lambda(y)a^\lambda(y)$. The wedge spaces $\wedge^nT^*(M^\ell)\sim\wedge^n$ are defined as usual and we recall

$$\omega_1\wedge\omega_2=(-1)^{pq}\omega_2\wedge\omega_1;\ (\omega_1\in\wedge^1,\,\omega_2\in\wedge^q);\ dim\wedge^p=\begin{pmatrix}\ell\\ p\end{pmatrix} \tag{A.3}$$

Given a map $\phi:\ M^\ell\to N^k$ one has maps $\phi_*:T(M^\ell)\to T(N^k)$ and $\phi^*:T^*(N^k)\to T^*(M^\ell)$ satisfying $\phi^*\omega(Y_1,\cdots,Y_p)=\omega(\phi_*(Y_1),\cdots,\phi_*(Y_p))$. Further $d:\wedge^p\to\wedge^{p+1}$ satisfies $d(\omega_1\wedge\omega_2)=d\omega_1\wedge\omega_2+(-1)^p\omega_1\wedge\omega_2$ for $\omega_1\in\wedge^p$ while $d(d\omega)=0$ (where $df=\partial_\lambda f dy^\lambda$). In addition $d\omega(Y_1,Y_2)=Y_1\omega(Y_2)-Y_2\omega(Y_1)-\omega([Y_1,Y_2])$ for $\omega\in\wedge^1$. Then $d\phi^*(\omega)=\phi^*(d\omega)$ and (Poincaré lemma) if $d\omega=0$ in a contractible region $(\omega\in\wedge^p)$ then there exists $\theta\in\wedge^{p-1}$ such that $\omega=d\theta$. Interior multiplication is defined via $i(Y):\wedge^p\to\wedge^{p-1}$ where $(\omega_1\in\wedge^p)$

$$i(Y)f=0;\ i(Y)df=Yf;\ i(Y)(\omega_1\wedge\omega_2)=(i(Y)\omega_1)\wedge\omega_2+ \tag{A.4}$$
$$+(-1)^p\omega_1\wedge(i(Y)\omega_2);\ (i(Y)\omega)(Y_1,\cdots,Y_{p-1})=\omega(Y,Y_1,\cdots,Y_{p-1})$$

The Lie derivative is defined via

$$L(Y) = i(Y)d + di(Y);\ L(Y)f = Yf;\ L(Y)df = d(Yf); \tag{A.5}$$

$$L(Y)\omega = lim_{t\to 0}\frac{1}{t}(\phi_t^*\omega - \omega)$$

where ϕ_t is generated by Y. One recalls also $\int_{\phi(G)} = \int_G \phi^*\omega$ and (Stokes' theorem) $\int_G d\omega = \int_{\partial G}\omega$ for suitable G. In particular if $\phi_\tau(y)$ is a 1-parameter "variation" on M^ℓ with $\phi_{\tau=0}(y) = y$ and V is the vector field generated by this variation then

$$A_\tau = \int_{\phi_\tau(G)} = \int_G \phi_\tau^*\omega;\ \frac{dA_\tau}{d\tau} = \lim_{t\to 0}\frac{1}{\tau}(A_\tau - A_0) = \tag{A.6}$$

$$= \int lim_{t\to 0}(\phi_\tau^*\omega - \omega) = \int_G L(V)\omega = \int_G i(V)d\omega + \int_{\partial G} i(V)\omega$$

One can treat many differential problems on manifolds via differential forms (Pfaffian systems - cf. [**214, 526**]). Thus linearly independent vector fields $X_1, \cdots, X_m$ on M^ℓ define a system of partial differential equations (PDE) whose solutions form an m-dimensional integral submanifold $I^m(X_1, \cdots, X_m)$ if certain integrability conditions hold, namely $[X_\mu, X_\nu] = 0$ (Frobenius integrability criterion). This can be restated in terms of differential forms as follows. Assume e.g. $X_\mu = \partial_\mu + \phi^a_\mu(x,z)\partial_a$ ($\mu = 1, \cdots, m$) where $x = (x^1, \cdots, x^m)$ and $z = (z^1, \cdots, z^n)$ with $m+n = \ell$. One wants 1-forms ω^a ($a = 1, \cdots, n$) such that $\omega^a(X_\mu) = 0$ (satisfied if $\omega^a = dz^a - \phi^a_\mu dx^\mu$). The forms ω^a generate an ideal $I[\omega^a]$ in $\wedge = \wedge^0 \oplus \wedge^1 \oplus \cdots \oplus \wedge^\ell$ (i.e. if ω^a vanishes on a set of vector fields X_μ so does $\omega \wedge \omega^a$ for any $\omega \in \wedge$). The integrability condition for the X_μ is equivalent to $d\omega \in I[\omega^a]$ if $\omega \in I[\omega^a]$. Next if $\omega \in \wedge^p$ the minimal number r of linearly independent 1-forms $\theta^\rho = f^\rho_\lambda(y)dy^\lambda$ by means of which ω can be expressed is called the rank of ω. If $r = p$ then ω is simple, i.e. $\omega = \theta^1 \wedge \cdots \wedge \theta^p$. At each point y the θ^ρ generate an r-dimensional subspace of T_y^* and the union of these subspaces generated by the θ^ρ is called $A^*(\omega)$. This determines an $\ell - r$ dimensional system $A(\omega) \subset T(M^\ell)$ of vector fields $Y = a^\lambda(y)\partial_\lambda$ satisfying $\theta^\rho(Y) = f^\rho_\lambda a^\lambda = 0$. Thus Y is associated with ω if and only if $i(Y)\omega = 0$. Note here that $A^*(\omega)$ is generated by the 1-forms $i(\partial_{\lambda_{p-1}})\cdots i(\partial_{\lambda_1})\omega$ where $\lambda_1, \cdots, \lambda_{p-1} = 1, \cdots, l$. Indeed, since $i(Y_1)i(Y_2) = -i(Y_2)i(Y_1)$ one has

$$i(Y)[i(\partial_{\lambda_{p-1}})\cdots i(\partial_{\lambda_1})\omega] = (-1)^{p-1} i(\partial_{\lambda_{p-1}})\cdots i(\partial_{\lambda_1})i(Y)\omega = 0$$

A completely integrable differential system $C(\omega) = A(\omega) \cap A(d\omega)$ is associated with ω and $C^*(\omega) = A^*(\omega) \cup A^*(d\omega)$ is the space of 1-forms annihilating $C(\omega)$ (characteristic Pfaffian system of ω). Evidently $Y \in C(\omega) \iff i(Y)\omega = 0 = i(Y)d\omega$ which is equivalent to $i(Y)\omega = 0 = L(Y)\omega$.

EXAMPLE A.1. Consider the form $\theta = -Hdt + p_j dq^j$ on $\mathbf{R}^{2n+1}$ with

$$d\theta = -dH(t,q,p) \wedge dt + ddp - j \wedge dq^j = -(\partial_j H dq^j + \tag{A.7}$$

$$+\frac{\partial H}{\partial p_j}dp_j + \partial_t H dt) \wedge dt + dp_j \wedge dq^j = (dp_j + \partial_j H dt) \wedge \left(dq^j - \frac{\partial H}{\partial p_j}dt\right)$$

Its associated system (= characteristic system since $d\theta$ is exact) is generated by the $2n$ Pfaffian forms

$$i(\partial_j)d\theta = -(dp_j + \partial_j H dt) = -\theta_j;\ i(\partial/\partial p_j) = dq^j - \frac{\partial H}{\partial p_j}dt = \omega^j \tag{A.8}$$

If there are no additional constraints the rank and class of $d\theta$ are $2n$ and the space $A(d\theta) = C(d\theta)$ of (characteristic) vector fields associated with $d\theta$ is 1-dimensional and can be generated via $X_H = \partial_j + (\partial H/\partial p_j)\partial_j - (\partial H/\partial q^j)(\partial/\partial p_j)$ (since $\theta_j(X_H) = 0 = \omega^j(X_H) = 0$. Thus the associated integral manifolds of $d\theta$ are the 1-dimensional solutions $\dot q^j = \partial H/\partial p_j$ and $\dot p_j = -\partial H/\partial q^j$.

We recall next the Legendre transformation $v^j \to p_j$ with $L \to H$. Thus the time evolution of a system involves a curve $C_0(t)$ based on $\dot q = dq/dt$ in say $[t_1, t_2] \times G^n = G^{n+1}$. For suitable L the curve is characterized by making the action integral $A[C] = \int_{t_1}^{t_2} L(t, q, \dot q)dt$ stationary, leading to the Euler-Lagrange (EL) equations $(d/dt)(\partial L/\partial \dot q^j) - \partial_j L = 0$ for $(j = 1, \cdots, n)$. This can be rephrased as follows (with a view toward the Lepage theory). Consider the Lagrangian $\hat L = L(t, q, v) - \lambda_j(v^j - \dot q^j)$ with EL equations $\partial \hat L/\partial v^j = \partial L/\partial v^j - \lambda_j = 0$ (since $\partial \hat L/\partial \dot v^j = 0$) leading to $\hat L(t, q, \dot q, v) = L(t, q, v) - v^j \partial L/\partial v^j + \dot q^j \partial L/\partial v^j$. Then introduce new variables $p_j = \partial L/\partial v^j$ and assume the matrix $(\partial p_j/\partial v^k) = (\partial^2 L/\partial v^k \partial v^j)$ to be regular. Solve the equations $p_j = \partial L/\partial v^j$ for $v^j = \hat\phi(t, q, p)$ and define $H(t, q, p) = \hat\phi(t, q, p)p_j - L[t, q, v = \hat\phi(t, q, p)]$. Then one obtains for the Lagrangian $\hat L$ the result $\hat L(t, q, \dot q, p) = -H(t, q, p) + \dot q^j p_j$ which contains $2n$ dependent variables q, p but depends only on the derivatives in q (not in p). Hence the EL equations are $-\partial \hat L/\partial p_j = \partial H/\partial p_j - \dot q^j = 0$ and $(d/dt)(\partial \hat L/\partial dotq^j) - \partial \hat L/\partial q^j) = \dot p_j + \partial_j H = 0$.

The Lepage method is now similar in spirit to this introduction of Hamilton's function $H(t, q, p)$ and the resulting derivation of the canonical equations of motion, but via the use of differential forms, it is more efficient. Thus let $\omega = L(t, q, \dot q)dt$ and the introduction of new variables v^j equal to $\dot q^j$ on the extremals is equivalent to introducing 1-forms $\omega^j = dq^j - v^j dt$ which vanish on the extremals $q(t)$ where $dq^j(t) = \dot q^j dt$, or where the tangent vectors $e_t = \partial_t + \dot q^j \partial_j$ are annihilated by each ω^j (i.e. $\omega^j(e_t) = \dot q^j - v^j = 0$). Thus as far as the extremals are concerned the form $\omega = Ldt$ is only one representative in an equivalence class of 1-forms, the most general of which is $\Omega = L(t, q, v)dt + h_j\omega^j$ (note the n Pfaffian forms ω^j generate an ideal $I[\omega^j]$ which vanishes on the extremals). The coefficients h_j (which correspond to Lagrange multipliers λ_j above) can be arbitrary functions of t, q, v and according to the Lepage theory they can be determined by the following condition. Thus note

$$d\Omega = \left(\frac{\partial L}{\partial v^j} - h_j\right) dv^j \wedge dt + (-\partial_j L dt + dh_j) \wedge \omega^j = \tag{A.9}$$

$$= \left(\frac{\partial L}{\partial v^j} - h_j\right) dv^j \wedge dt\ (mod\, I[\omega^j])$$

We therefore have $d\Omega = 0$ on the extremals where $\omega^j = 0$ if and only if $h_j = \partial L/\partial v^j = p_j$. Inserting this value for h_j into Ω one obtains

$$\Omega = Ldt + p_j\omega^j = Ldt + p_j(dq^j - v^j dt) = -Hdt + p_j dq^j = \theta \tag{A.10}$$

These equations provide an interpretation of the Legendre transformation via $v^j \to p_j = (\partial L/\partial v^j)(t,q,v)$ with $L \to H = v^j p_j - L$ (this can be implemented by a change of basis $dt \to dt$ and $\omega^j \to dq^j$) and this point of view is of importance in field theories.

Now one goes to the implications of generalizing Lepage's equivalence relation $\theta \equiv Ldt\ (mod\ I[\omega^j])$ and $d\theta \equiv 0\ (mod\ I[\omega^j])$ for $\theta = Ldt + p_j\omega^j = -Hdt + p_j dq^j$ to a field theory context. Thus working with 2 independent variables for simplicity let x^μ $\mu = 1,2$ be independent variables and z^a, $a = 1,\cdots,n$ those variables of $y = (x^1, x^2, z^1, \cdots, z^n)$ which become the dependent variables $z^a = f^a(x)$ on 2-dimensional submanifolds Σ^2. Further the variables v^a_μ become the derivatives $\partial_\mu z^a(x) = \partial_\mu f^a(x)$ on Σ^2 and especially on the extremals Σ^2_0 defined below. This last property says that the forms $\omega^a = dz^a - v^a_\mu dx^\mu$ vanish on the extremals Σ^2_0 where $v^a_\mu = \partial_\mu z^a(x)$. Consider the action integral $A[\Sigma^2] = \int_{G^2} L(x,z,\partial z)dx^1dx^2$ which is supposed to become stationary for the extremals Σ^2_0 when we vary the functions $z^a(x)$ and $\partial_\mu z^a(x)$ (see below). By using the forms $\omega^a = dz^a - v^a_\mu dx^\mu$ one can argue here as before with mechanics. As far as extremals go the Lagrangian 2-form $\omega = L(x,z,v)dx^1 \wedge dx^2$ is one representative in an equivalence class of 2-forms which can be written as

$$\Omega = \omega + h^1_a\omega^a \wedge dx^2 + h^2_a dxx^1 \wedge \omega^a + \frac{1}{2}h_{ab}\omega^a \wedge \omega^b \equiv \tag{A.11}$$

$$\omega\ (mod\ I[\omega^a]);\ \ h_{ba} = -h_{ab}$$

The term with h_{ab} is new compared to mechanics and is only possible because Ω is a 2-form and if $n \geq 2$. In mechanics the coefficients h_j in $\Omega = Ldt + h_j\omega^j$ were determined by the requirement $d\Omega = 0\ mod\ I[\omega^j]$ and for field theories we have

$$d\Omega = \left(\partial_a L dz^a + \frac{\partial L}{\partial v^a_\mu}dv^a_\mu\right) \wedge dx^1 \wedge dx^2 + dh^1_a \wedge \omega^a \wedge dx^2 - \tag{A.12}$$

$$-h^1_a dv^a_1 \wedge dx^1 \wedge dx^2 + dh^2_a \wedge dx^2 \wedge \omega^a + dh^2_a \wedge dx^1 \wedge \omega^a + h^2_a dx^1 \wedge dv^a_2 \wedge dx^2 +$$

$$+\frac{1}{2}d(h_{ab}\omega^a \wedge \omega^b) = \left(\frac{\partial}{\partial v^a_\mu} - h^\mu_a\right)dv^a_\mu \wedge dx^1 \wedge dx^2 + O(mod\, I[\omega^a])$$

where the equality $dz^a \wedge dx^1 \wedge dx^2 = \omega^a \wedge dx^1 \wedge dx^2$ has been used. Thus the condition $d\Omega = 0\ mod\, I[\omega^a]$ is equivalent to $h^\mu_a = \partial L/\partial v^a_\mu = \pi^\mu_a$ (as in mechanics). What is new however is that the condition $d\Omega = 0\ mod\, I[\omega^a]$ does not impose any restrictions on h_{ab} and this has far reaching consequences. Indeed inserting the formula for $h^\mu_a = \pi^a_\mu$ into (A.11) one gets

$$\Omega = Ldx^1 \wedge dx^2 + \pi^1_a\omega^a \wedge dx^2 + \pi^2_a dx^1 \wedge \omega^a + (1/2)h_{ab}\omega^a \wedge \omega^b \tag{A.13}$$

Each choice of h_{ab} defines a canonical form Ω_h and the implications will be seen. First introduce some notational simplifications via $a^\mu = Ldx^\mu + \pi^\mu_a\omega^a$ and $\epsilon_{\mu\nu} =$

$\epsilon^{\mu\nu}$, $\epsilon_{12}=1$, and $\epsilon_{\mu\nu}=-\epsilon_{\nu\mu}$. Then (A.13) can be written as

$$\Omega = Ldx^1\wedge dx^2+\pi^\mu_a\omega^a\wedge d\Sigma_\mu+(1/2)h_{ab}\omega^a\wedge\omega^b = \tag{A.14}$$

$$= a^\mu\wedge d\Sigma_\mu - Ldx^1\wedge dx^2+(1/2)h_{ab}\omega^a\wedge\omega^b;\ d\Sigma_\mu=\epsilon_{\mu\nu}dx^\nu$$

Now recall that the Legendre transformation $v^j\to p_j,\ L\to H$ was implemented via a change of basis $dt\to dt,\ \omega^j\to dq^j$ by inserting for ω^j the expression dq^j-v^jdt and subsequent identifications. Generalizing this procedure to (A.14) means that we replace ω^a by $dz^a-v^a_\mu dx^\mu$ and write Ω in terms of the basis $dx^1\wedge dx^2,\ dz^a\wedge dx^2,\ dx^1\wedge dz^a$, and $dz^a\wedge dz^b$ to get

$$\Omega=[L-\pi^\mu_a v^a_\mu+(1/2)h_{ab}(v^a_1v^b_2-v^a_2v^b_1)dx^1\wedge dx^2+ \tag{A.15}$$

$$+(\pi^1_a-h_{ab}v^b_2)dz^a\wedge dx^2+(\pi^2_a+h_{ab}v^b_1)dx^1\wedge dz^a+(1/2)h_{ab}dz^a\wedge dz^b$$

Define now $h^{\mu\nu}_{ab}=\epsilon^{\mu\nu}h_{ab}$ and rewrite the last equation as

$$\Omega=(L-\pi^\mu_a v^a_\mu+(1/2)h^{\mu\nu}_{ab}v^a_\mu v^b_\nu)dx^1\wedge dx^2+ \tag{A.16}$$

$$+(\pi^\mu_a-h^{\mu\nu}_{ab}v^b_\nu)dz^a\wedge d\Sigma_\mu+(1/2)h_{ab}dz^a\wedge dz^b$$

The generalized Legendre transformation (à la Lepage) defines the canonical momenta p^μ_a as $p^\mu_a=\pi^\mu_a-h^{\mu\nu}_{ab}v^b_\nu$ and the Hamiltonian as $H=\pi^\mu_a v^a_\mu-(1/2)h^{\mu\nu}_{ab}v^a_\mu v^b_\nu-L$ leading to

$$\Omega=-Hdx^1\wedge dx^2+p^\mu_a dz^a\wedge d\Sigma_\mu+(1/2)h_{ab}dz^a\wedge dz^b \tag{A.17}$$

In order to express H and h_{ab} as functions of p^μ_a the Legendre transformation $v^a_\mu\to p^\mu_a$ has to be regular, namely

$$\left|\frac{\partial p^\mu_a}{\partial v^b_\nu}\right|=\left|\left(\frac{\partial^2 L}{\partial v^b_\nu\partial v^a_\mu}-\frac{\partial h^{\mu\lambda}_{ac}}{\partial v^b_\nu}v^c_\lambda-h^{\mu\nu}_{ab}\right)\right|\neq 0 \tag{A.18}$$

The coefficients $h^{\mu\nu}_{ab}$ have been arbitrary up to now and an appropriate choice can always guarantee the inequality (A.18) which is an interesting possibility for defining a regular Legendre transformation if the conventional one $v^a_\mu\to\pi^\mu_a$ is singular as in gauge theories.

Assume now one can solve $p^\mu_a=\pi^\mu_a-h^{\mu\nu}_{ab}v^b_\nu$ for the v^a_μ, i.e. $v^a_\mu=\hat\phi^a_\mu(x,z,p)$ and putting this into $h^{\mu\nu}_a(x,z,v)$ one obtains $\eta^{\mu\nu}_{ab}(x,z,p)=h^{\mu\nu}_{ab}(x,z,v=\hat\phi(x,z,p))$ leading to

$$H(x,z,p)=\pi^\mu_a[x,z,v=\hat\phi(x,z,p)]\hat\phi^a_\mu(x,z,p)- \tag{A.19}$$

$$-(1/2)\eta^{\mu\nu}_{ab}\hat\phi^a_\mu\hat\phi^b_\nu-L(x,z,v=\hat\phi(x,z,p)]$$

This leads to

$$dH=v^a_\mu d\pi^\mu_a+\pi^\mu_a dv^a_\mu-(1/2)d(h^{\mu\nu}_{ab}v^a_\mu v^b_\nu)-dL; \tag{A.20}$$

$$d\pi^\mu_a=dp^\mu_a+d(h^{\mu\nu}_{ab}v^b_\nu)$$

Combining these equations yields

$$dH=v^a_\mu dp^\mu_a+\pi^\mu_a dv^a_\mu+(1/2)v^a_\mu v^b_\nu dh^{\mu\nu}_{ab}-dL \tag{A.21}$$

Inserting now into (A.21) $dH = \partial_\mu H dx^\mu + \partial_a H dz^a + (\partial H/\partial p_a^\mu)dp_a^\mu$ where it is specified that $\partial_\mu H = (\partial H/\partial x^\mu)|_{z,p\ fixed}$ along with

(A.22) $$dL = \partial_\mu L dx^\mu + \partial_a L dz^a + \pi_a^\mu dv_\mu^a;\ dh_{ab}^{\mu\nu} = d\eta_{ab}^{\mu\nu} = $$
$$= \partial_\lambda \eta_{ab}^{\mu\nu} dx^\lambda + \partial_c \eta_{ab}^{\mu\nu} dz^c + \frac{\partial \eta_{ab}^{\mu\nu}}{dp_c^\lambda} dp_c^\lambda$$

and comparing coefficients of $dx^\mu,\ dz^a,\ dp_a^\mu$ yields then

(A.23) $$\partial_\lambda H - \frac{1}{2}\hat{\phi}_\mu^a \hat{\phi}_\nu^b \partial_\lambda \eta_{ab}^{\mu\nu} = -\partial_\lambda L;$$
$$\partial_c H - \frac{1}{2}\hat{\phi}_\mu^a \hat{\phi}_\nu^b \partial_c \eta_{ab}^{\mu\nu} = -\partial_c L;\ \frac{\partial H}{\partial p_c^\lambda} - \frac{1}{2}\hat{\phi}_\mu^a \hat{\phi}_\nu^b \frac{\partial \eta_{ab}^{\mu\nu}}{\partial p_c^\lambda} = v_\lambda^c$$

Up to now the variables x^μ, z^a, and v_μ^a or p_a^μ have been treated as independent and this is no longer the case if one looks for the extremals $\hat{\Sigma}_0^2$, namely those 2-dimensional submanifolds $z^a = f^a(x)$, $p_a^\mu = g_a^\mu(x)$ for which the variational derivative $dA[\hat{\Sigma}^2]/d\tau$ vanishes at $\tau = 0$, i.e.

(A.24) $$A[\hat{\Sigma}^2] = \int_{\hat{\Sigma}^2} \Omega;\ \left.\frac{dA_\tau}{d\tau}\right|_{\tau=0} [\hat{\Sigma}_0^2] = \int_{\hat{\Sigma}_0^2} i(V)d\Omega + \int_{\partial\hat{\Sigma}_0^2} i(V)\Omega = 0$$
$$V = V_{(x)}^\mu \partial_\mu + V_{(z)}^a \partial_a + V_{(p)a}^\mu (\partial/\partial p_a^\mu)$$

Some calculations then lead to

(A.25) $$-\partial_a H - \partial_\mu p_a^\mu - (\partial_\mu \eta_{ab}^{\mu\nu})\partial_\nu z^b + (1/2)(\partial_a \eta_{bc}^{\mu\nu})\partial_\mu z^b \partial_\nu z^c -$$
$$-(\partial_c \eta_{ab}^{\mu\nu})\partial_\mu z^c \partial_\nu z^b - \frac{\partial \eta_{ac}^{\lambda\nu}}{\partial p_b^\mu} \partial_\lambda p_b^\mu \partial_\nu z^c = 0$$

which are the analogue of the canonical equations $\dot{p}_j + \partial_j H = 0$. If one defines now

(A.26) $$\frac{d}{dx^\mu} F(x,z,p) = \partial_\mu F + \partial_a F \partial_\mu z^a + \frac{\partial F}{\partial p_a^\nu} \partial_\mu p_a^\nu$$

(e.g. $dz^a(x)/dx^\mu = \partial_\mu z^a(x)$) one can rewrite (A.25) as

(A.27) $$\frac{dp_a^\mu}{dx^\mu} + \partial_a H = (1/2)\partial_a \eta_{bc}^{\mu\nu} \partial_\mu z^b \partial_\nu z^c - (d\eta_{ab}^{\mu\nu}/dx^\mu)\partial_\nu z^b$$

Further there are equations

(A.28) $$-\frac{\partial H}{\partial p_a^\mu} + \partial_\mu z^a + \frac{1}{2}\frac{\partial \eta_{bc}^{\lambda\nu}}{\partial p_a^\mu} \partial_\lambda z^b \partial_\nu z^c = 0$$

which are the same as the equations in (A.23), if $\partial_\mu z^a(x) = v_\mu^a$ (which is a consequence of (A.18)). The equations (A.27) and (A.28) represent a system of $n + 2n = 3n$ first order PDE for the 3n functions $z^a = f^a(x)$ and $p_a^\mu = g_a^\mu(x)$. By eliminating the canonical momenta p_a^μ one obtains a system of n second order PDE for the z^a, namely the EL equations $(d/dx^\mu)(\partial L/\partial v_\mu^a) - \partial_a L = 0$ (cf. [**586**] for details and note the h_{ab} have dropped out completely). One can also show that the inequality (A.18) implies $v_\mu^a = \partial_\mu z^a(x)$ on the extremals (at least in a neighborhood of $v_\mu^a = 0$ - recall $\pi_\mu^a = \partial L/\partial v_\mu^a$).

REMARK A.1. One notes an important structural difference between the variational system $\{i(\partial_a)d\Omega, i(\partial/\partial p^\mu_a)d\Omega\}$ of differential forms which determine the extremals here and the system $\{i(\partial_j)d\theta, i(\partial/\partial p_j)d\theta\}$ arising in mechanics. In the latter case the 1-forms $i(\partial_j)d\theta,\ i(\partial/\partial p_j)d\theta$ generate the characteristic Pfaffian system $C^*(d\theta)$ of $d\theta$; i.e. the extremals associated with the canonical form θ coincide with the characteristic integral submanifolds of the form $d\theta$ which means that the tangent vectors $\partial_t + \dot{q}^j p_j + \dot{p}_j(\partial/\partial p_j) = \partial_t + (\partial H/\partial p_j)\partial_j - (\partial H/\partial q^j)(\partial/\partial p_j)$ form a characteristic vector field associated with $d\theta$ (i.e. this vector field is annihilated by the forms $i(\partial_j)d\theta$ and $i(\partial/\partial p_j)d\theta$). The situation is different for field theories; here the tangent vectors $\hat{\Sigma}'_{(\mu)} = \partial_\mu + \partial_\mu z^a\partial_a + \partial_\mu p^\nu_a(\partial/\partial p^\nu_a)$ are annihilated in pairs - as a 2-vector $\hat{\Sigma}'_{(1)}\wedge\hat{\Sigma}'_{(2)}$ by the $3n+2$ two forms $i(\partial_a)d\Omega,\ i(\partial/\partial p^\mu_a)d\Omega$, and $i(\partial_\mu)d\Omega$. On the other hand the characteristic Pfaffian system $C^*(d\Omega)$ is generated by

$$i(\partial_\mu)i(\partial_\nu)d\Omega;\ i(\partial_\mu)i(\partial_a)d\Omega;\ i(\partial_a)i(\partial_b)d\Omega; \tag{A.29}$$

$$i(\partial_\mu)i(\partial/\partial p^\mu_a)d\Omega;\ i(\partial_a)i(\partial/\partial p^\nu_b)d\Omega$$

Each characteristic vector Y of the form $d\Omega$ (i.e. $i(Y)d\Omega = 0$) is also annihilated by the 2-forms $i(V)d\Omega$ where $V = \partial_\mu,\ \partial_a$, or $\partial/\partial p^\mu_a$. However if $i(Y)i(V)d\Omega = 0 = 0$ one cannot conclude that $i(Y)d\Omega = 0$. As a simple example take $n = 1$ and $L = T(v) - V(z)$ (a not unusual form in physics). Then $p^\mu = \pi^\mu$ and

$$\Omega = Ldx^1\wedge dx^2 + \pi^1\omega\wedge dx^2 + \pi^2 dx^1\wedge\omega = \tag{A.30}$$

$$-Hdx^1\wedge dx^2 + \pi^1 dz\wedge dx^2 + \pi^2 dx^1\wedge dz;\ \omega = dz - v_\mu dx^\mu;\ H = \pi^\mu v_\mu - L$$

Then $d\Omega = (\partial_z L dx^1\wedge dx^2 - d\pi^1\wedge dx^2 - dx^1\wedge d\pi^2)\wedge\omega$ is a 3-form in the 5 variables $x^\mu,\ z,\ v_\mu\ (\mu = 1, 2)$. Evidently $\omega\in C^*(d\Omega)$ but the factor

$$\rho = \partial_z L dx^1\wedge dx^2 - d\pi^1\wedge dx^2 - dx^1\wedge d\pi^2 = \tag{A.31}$$

$$= \partial_z L dx^1\wedge dx^2 - \frac{\partial^2 L}{\partial v_1\partial v_\nu}dv_\nu\wedge dx^2 - \frac{\partial^2 L}{\partial v_2\partial v_\nu}dx^1\wedge dv_\nu$$

is a 2-form in the 4 variables x^μ and v_μ with nontrivial rank 4 (cf. [**586**]). Hence $d\Omega$ has rank 5 and the integral submanifolds of the characteristic system $C^*(d\Omega)$ are 0-dimensional, whereas the integral submanifolds (the extremals) of the variational system $i(\partial_\mu d\Omega,\ i(\partial_a)d\Omega,\ i(\partial/\partial p^\mu_a)d\Omega$ are 2-dimensional in this example. Only if Ω is a 1-form will the variational system coincide with the characteristic system $C^*(d\Omega)$.

The essential new feature of the canonical framework above are the $h_{ab}(x,z,v) = \eta_{ab}(x,z,p)$. There are several examples: $h_{ab} = 0$ in which case $p^\mu_a = \pi^\mu_a$ and $H = \pi^\mu_a v^a_\mu - L$. If the Legendre transformation $v^a_\mu \to \pi^\mu_a$ is regular (i.e. $(\partial^2 L/\partial v^a_\mu\partial^b_\nu)$ is regular, one can express H as a function of x, z, π and the canonical equations (A.27) - (A.28) take the form $\partial_\mu\pi^\mu_a = -\partial_a H$ and $\partial_\mu z^a = \partial H/\partial\pi^\mu_a = \hat{\phi}^a_\mu(x,z,\pi)$. The choice $h_{ab} = 0$ is the conventiional one and is called the canonical theory for fields of deDonder and Weyl. For a system with just one real field this is the only possible one since there can be no nonvanishing h_{ab}. In more detail look at the variational system $I[i(V)d\Omega_0]$ which is generated by the 2-forms

$$i(\partial_a)d\Omega_0 = -\partial_a H dx^1\wedge dx^2 - d\pi^\mu_a\wedge d\Sigma_\mu = \lambda_a; \tag{A.32}$$

$$i(\partial/\partial\pi_a^\mu)d\Omega_0 = \omega^a \wedge d\Sigma_\mu = \omega_\mu^a;\ i(\partial_\mu)d\Omega_0 =$$

$$= \omega^a \wedge (\partial_a H d\Sigma_\mu + \epsilon_{\mu\rho} d\pi_a^\rho) - \hat{\phi}_\mu^a \lambda_a = (dh - \partial_\rho H dx^\rho) \wedge d\Sigma_\mu + \epsilon_{\mu\rho} dz^a \wedge d\pi_a^\rho = \omega_\mu$$

This differential system (A.32) has some peculiar properties

(1) One has

$$d\lambda_a = -d(\partial_a H) \wedge dx^1 \wedge dx^2;\ d\omega_\mu^a = -d(\hat{\phi}_\mu^a) \wedge dx^1 \wedge dx^2; \tag{A.33}$$

$$d\omega_\mu = -d(\partial_\mu H) \wedge dx^1 \wedge dx^2$$

and since $dx^\mu \wedge \lambda_a = d\pi_a^\mu \wedge dx^1 \wedge dx^2$ with $dx^\mu \wedge \omega_\nu^a = -\delta_\nu^\mu dz^a \wedge dx^1 \wedge dx^2$ we see that $d\lambda_a,\ d\omega_\mu^a$, and $d\omega_\mu$ belong to $I[\lambda_a, \omega_\mu^a, \omega_\mu]$.

(2) The $3n+2$ forms $\lambda_a,\ \omega_\mu^a$ and $d\omega_\mu$ are linearly independent.

(3) If $v = A^\mu \partial_\mu + B^a \partial_a + C_a^\mu (\partial/\partial\pi_a^\mu)$ is an arbitrary tangent vector then it is a 1-dimensional integral element of the system (A.32) (which contains no 1-forms) and its "polar" system $i(v)\lambda_a,\ i(v)\omega_\mu^a,\ i(v)\omega_\mu$ has at most the rank $2n+1$. This follows from the relations

$$A^\mu i(v)\omega_\mu^a = (B^a - A^\rho \hat{\phi}_\rho^a) A^\mu d\Sigma_\mu; \tag{A.34}$$

$$A^\mu i(v)\omega_\mu + B^a i(v)\lambda_a + C_a^\mu i(v)\omega_\mu^a = i(v)i(v)d\Omega_0 = 0$$

which shows that at most $2n+2$ of the 1-forms $i(v)\lambda_a,\ i(v)\omega_\mu^a,\ i(v)\omega_\mu$ are linearly independent. The maximal rank $2n+2$ is relized e.g. for the vector $v = \partial/\partial x^1$.

REMARK A.2. In more conventional language the terms involving h_{ab} may be interpreted as follows. Defining

$$\frac{d}{dx^\mu} = \partial_\mu + v_\mu^a \partial_a + v_{\mu\nu}^a \frac{\partial}{\partial v_\nu^a};\ v_{\nu\mu}^a = v_{\mu\nu}^a \tag{A.35}$$

one has, with $h_{ab}^{\mu\nu} = \epsilon^{\mu\nu} h_{ab}$, $h_{ab}^{\mu\nu} v_\mu^a v_\nu^b = h_{ab}^{\mu\nu} d(z^a v_\nu^b)/dx^\mu$. If in addition $dh_{ab}^{\mu\nu}/dx^\mu = 0$ (e.g. if the $h_{ab}^{\mu\nu}$ are constants, then $h_{ab}^{\mu\nu} v_\mu^a v_\nu^b = d(h_{ab}^{\mu\nu} z^a v_\nu^b)$ (i.e. the term $h_{ab}^{\mu\nu} v_\mu^a v_\nu^b$ is a total divergence. This implies that the Lagrangian $L^*(x,z,v) = L(x,z,v) - (1/2)h_{ab}^{\mu\nu} v_\mu^a v_\nu^b$ gives the same field equations as L itself, but the canonical momenta are $\partial L^*/\partial v_\mu^a = \pi_a^\mu - h_{ab}^{\mu\nu} v_\nu^b = p_a^\mu$.

EXAMPLE A.2. A nontrivial unique choice of the h_{ab} is obtained as follows. By means of 1-forms $a^\mu = L dx^\mu + \pi_a^\mu \omega^a$ and $\theta^\mu = -H dx^\mu + p_a^\mu dz^a$ one can construct forms as in (A.13) - (A.14), namely

(A.36)

$$\Omega_c = \tfrac{1}{L} a^1 \wedge a^2 = L dx^1 \wedge dx^2 + \pi_a^\mu \omega^a \wedge d\Sigma_\mu + \tfrac{1}{2L}(\pi_a^1 \pi_b^2 - \pi_a^2 \pi_b^1)\omega^a \wedge \omega^b =$$

$$= -\frac{1}{H}\theta^1 \wedge \theta^2 = -H dx^1 \wedge dx^2 + p_a^\mu dz^a \wedge d\Sigma^\mu - \frac{1}{2H}(p_a^1 p_b^2 - p_a^2 p_b^1) dz^a \wedge dz^b$$

This form Ω_c is unique among all possible forms (A.13) because it is the only one which has the minimal rank 2 and it defines Carathéodory's canonical theory for fields.

REMARK A.3. In mechanics if one replaces a given Lagrangian $L(t,q,\dot{q})$ by $L^* = L + df(q)/dt = L + \dot{q}^j \partial_j f$ then the EL equations of L and L^* are the same.

However for the canonical momenta one has $p_j^* = \partial L^*/\partial \dot{q}^j = \partial L/\partial \dot{q}^j + \partial_j f = p_j + \partial_j f$ so the canonical momenta are changed. Thus the canonical reformulation of n second order differential equations by a system of $2n$ first order differential equations is not unique. Further in field theories the corresponding freedom is even more substantial if $n \geq 2$. The most important classifying criterion will be the rank of the form Ω for a given set of coefficients h_{ab} since for $rank(\Omega) = r$ (with $d\Omega = 0$ on the extremals) the integral submanifolds associated with the canonical form Ω have dimension $n + 2 - r$ ($r \geq 2$). As examples consider:

- If $n = 1$ (i.e. only one dependent variable z) the rank of Ω is always 2.
- If all $h_{ab} = 0$ the deDonder-Weyl canonical form

$$\Omega_0 = a^\mu \wedge d\Sigma_\mu - L dx^1 \wedge dx^2 \tag{A.37}$$

  has rank 4 (for $n \geq 2$) so the integral submanifolds of Ω_0 have dimension $2 + n - 4 = n - 2$
- Carathéodory's canonical form (A.36) has rank 2; it is the only canonical form which allows for n-dimensional wave fronts.

We go now to the Hamilton-Jacobi (HJ) theory for deDonder-Weyl (DW) fields. One recalls that in mechanics the property $d\theta = 0$ with $\theta = -Hdt + p_j dq^j$ on the extremals, combined with the Poincaré lemma implies the basic HJ relation $dS(t,q) = -Hdt + p_j dq^j$. In the same manner one can conclude from $d\Omega = 0$ ($mod\ I[\omega^a]$) that Ω is locally an exact 2-form on the extremals Σ_0^2, which means that Ω is expressable by differentials $dS^1(x,z),\ dS^2(x,z)$, and $dx^1,\ dx^2, \cdots$ where the number of these differentials equals rank ω. For example in the Carathéodory theory where $\Omega = \Omega_c$ has rank 2 one has $dS^1 \wedge dS^2 = -(1/H)\theta^1 \wedge \theta^2$. For the deDonder-Weyl theory where $\Omega = \Omega_0$ has rank 4 one has

$$\Omega_0 = dS^1(x,z) \wedge dx^2 + dx^1 \wedge dS^2(x,z) = -H dx^1 \wedge dx^2 + \pi_a^\mu dz^a \wedge d\Sigma_\mu \tag{A.38}$$

which implies

$$\textbf{(a)}\ \ \pi_a^\mu = \partial_a S^\mu(x,z) = \psi_a^\mu(x,z); \tag{A.39}$$

$$\textbf{(b)}\ \ \partial_\mu S^\mu(x,z) = H(x,z,\pi = \psi(x,z))$$

These are simple generalizations of the relations $p_j = \partial_j S$ and $\partial_t S + H = 0$ in mechanics. Further according to (A.38) the wave fronts are given by $S^\mu(x,z) = \sigma^\mu = const.$ and $x^\mu = const.$ because

$$i(w)\Omega_0 = dS^1(w)dx^2 - w^{(2)}dS^1 + w^{(1)}dS^2 - dS^2(w)dx^1 = 0; \tag{A.40}$$

$$w = w^{(\mu)}\partial_{(\mu)} + w^a\partial_a;\ \partial_{(\mu)} = \partial/\partial x^\mu;\ dS^\mu(w) = w^{(\nu)}\partial_{(\nu)}S^\mu + w^a\partial_a S^\mu$$

implies $w^{(\mu)} = 0$ for $\mu = 1, 2$ and $w^a\partial_a S^\mu = w^a\pi_a^\mu = 0$ so the wave fronts associated with Ω_0 are $(n-2)$-dimensional and lie in the characteristic planes $x^\mu = const.$. Note also that (A.39)-b is a first order PDE for two functions S^μ so we can choose one, say S^2, with a large degree of arbitrariness and then solve for S^1. The main restriction on S^2 is the transversality condition that at each point $(x, z = f(x))$ the derivatives $\partial_a S^2(x,z)$ equal the canonical momenta $\pi_a^\mu(x)$.

Another important difference between HJ theories in mechanics and in field theories should be mentioned. Recall that any solution of the HJ equation in mechanics leads to a system $\dot{q}^j = \phi^j(t,q) = (\partial H/\partial_j)(t,q,p = \partial S(t,q))$ whose solutions $q^j(t) = f^j(t,u)$ $(u = (u^1,\cdots,u^n)$ constitute an n-parameter family of extremals which generate $S(t,q)$ provided $|\partial q^j/\partial u^k)| \neq 0$. Now calculate $S(t,q)$ by computing $\sigma(t,u) = \int^t d\tau L(\tau, f(\tau,u), \partial_\tau f(\tau,u))$, solving $q^j = f^j(t,u)$ for $u^k = \chi(t,q)$, and inserting the functions $\chi^k(t,q)$ into $\sigma(t,u)$ to get $S(t,q) = \sigma(t,\chi(t,q))$. This procedure is not possible in field theories. Indeed in the DW theory one has $v^a_\mu = \partial H/\partial\pi^\mu_a$ and given any solution $S^\mu(x,z)$ of the DWHJ equation (A.39)-b we define the slope functions

$$\phi^a_\mu(x,z) = \frac{\partial H}{\partial \pi^\mu_a}(x,z,\pi^\mu_a = \partial_a S^\mu(x,z)) \tag{A.41}$$

However the PDE $\partial_\mu z^a(x) = \phi^a_\mu(x,z)$ will only have solutions $z^a = f^a(x)$ under the integrability conditions

$$\frac{d}{dx^\nu}\phi^a_\mu(x,z(x)) = \partial_\nu\phi^a_\mu + \partial_b\phi^a_\mu\cdot\phi^b_\nu = \tag{A.42}$$

$$= \frac{d}{dx^\mu}\phi^a_\nu(x,z(x)) = \partial_\mu\phi^a_\nu + \partial_b\phi^a_\nu\cdot\phi^b_\mu$$

This involves rather stringent conditions (cf. [**586**] for examples). Suppose however that we have found solutions S^μ of the DWHJ equations (A.39)-b such that the slope functions (A.41) do obey the integrability conditions (A.42); then the solutions $z^a = f^a(x)$ of $\partial_\mu z^a = \phi^a_\mu$ are extremals and satisfy $d\pi^\mu_a = -\partial_a H$ as well. To see this recall $\pi^\mu_a(x) = \psi^\mu_a(x,z(x)) = \partial_a S^\mu(x,z(x))$ which implies

$$\frac{d\pi^\mu_a}{dx^\mu} = \partial_a\partial_\mu S^\mu(x,z(x)) + \partial_b\partial_a S^\mu(x,z(x))\partial_\mu z^b(x) \tag{A.43}$$

On the other hand one has $\partial_\mu S^\mu = -H[x,z(x),\pi = \partial S(x,z(x))$ and $(dH/dz^a) = D_a H = \partial_a H + (\partial H/\partial\pi^\nu_b)\partial_b\partial_a S^\nu$. Consequently

$$\frac{d\pi^\mu_a}{dx^\mu} = -\partial_a H + \left(\partial_\mu z^b(x) - \frac{\partial H}{\partial \pi^\mu_b}\right)\partial_a\partial_b S^\mu \tag{A.44}$$

which shows that the canonical equations $d\pi^\mu_a/dx^\mu = -\partial_a H$ are a consequence of (A.41).

Regarding currents note that if $S(t,q,a)$ is a solution of the HJ equation depending on a parameter a then $G(t,q,a) = \partial S(t,q,a)/\partial a$ is a constant of motion along any extremal where $p_j(t) = \partial_j S(t,q(t),a)$ holds. Thus $(d/dt)G(,q(t),a) = 0$ when $q^j(t) = f^j(t)$ is an extremal and to see this consider

$$dS(t,q,a) = \partial_t S dt + \partial_j S dq^j + (\partial S/\partial a)da = \tag{A.45}$$

$$= -H(t,q,p = \psi(t,q))dt + \psi_j(t,q,a)dq^j + Gda;\ \psi_j = \partial_j S(t,q,a)$$

Exterior differentiation of (A.45) gives

$$0 = -\left[-\partial - jHdq^j + \frac{|ppH}{\partial p_j}d\psi_j(t,q,a)\right]\wedge dt + d\psi_j(t,q,a)\wedge dq^j + dG\wedge da \tag{A.46}$$

Now suppose $q^j = q^j(t)$ is an arbitrary curve (not necessarily an extremal); then since $d\psi_j = \partial_t \psi_j + \partial_k \psi_j dq^k + (\partial \psi_j/\partial a)da$ with $dq^j = \dot{q}dt$ it follows from (A.46) that

(A.47) $$\left[\dot{q}^j - \frac{\partial H}{\partial p_j}(t, q, \psi(t,q))\right] \frac{\partial \psi_j}{\partial a} = \frac{d}{dt} G = 0$$

Thus if $\dot{q}^j = \partial H/\partial p_j$ then from (A.47) follows immediately $dG/dt = 0$ (Noether's theorem is a special case).

Recall that in mechanics if $S(t,q,a)$ is a solution of the HJ equaiton depending on n parameters a_j such that

(A.48) $$\left|\left(\frac{\partial^2 S}{\partial q^j \partial a_k} = \frac{\partial \psi_j}{\partial a_k}(t,q,a)\right)\right| \neq 0$$

then one can solve the equations $\partial S/\partial a_k = b^k = const.$ $(k = 1, \cdots, n)$ for the coordinates $q^j = f^j(t,a,b)$ and the curves $q = f(t,a,b)$ are extremals. This follows from (A.47) where $G = b^k = const.$ now and this implies $(\dot{q}^j - (\partial H/\partial p_j))(\partial \psi_j/\partial_k) = 0$. In view of (A.48) the coefficients $\dot{q}^j - (\partial H/\partial p_j)$ of this homogeneous system have to vanish which means that the functions $q^j = f^j(t,a,g)$ are solutions of the canonical equations $\dot{q}^j = (\partial H/\partial p_j)(t, q, p_j = \partial_j S(t,q))$. It then follows from previous discussion that they are also solutions of $\dot{p}_j = -\partial_j H$ as well and such a solution S of the HJ equation with the property (A.48) is called a complete integral; it provides a set of solutions $q^j = f^j(t,a,b)$ of the equations of motion which depends on the largest possible number $(2n)$ of constants of integration (i.e. the set is complete). There is a straightforward generalization to field theories. Suppose one has a solution $S^\mu(x,z)$ of (A.39)-b depending on $2n$ parameters a^ν_b such that $|(\partial^2 S^\mu/\partial z^c \partial a^\nu_b) = (\partial \psi^\mu_c/\partial a^\nu_b)| \neq 0$ and for which the equations (A.42) hold. Then each of the $2n$ parameters a^ν_b will generate a current $G^{a;b}_\nu = (\partial S^\mu/\partial a^\nu_b)(x,z,a)$. Suppose one has $4n$ functions $g^{\mu;b}_\nu(x)$, arbitrary up to the two properties that $(d/dx^\mu)g^{\mu;b}_\nu = 0$, such that

(A.49) $$G^{\mu;b}_\nu(x,z,a) = \frac{\partial S^\mu}{\partial a^\nu_b}(x,z,a) = g^{\mu;b}_\nu(x)$$

should be a solvable system of $4n$ equations for the n variables z^b. Thus $3n$ of the equations (A.49) cannot be independent of the other n and such functions satisfying $(d/dx^\mu)g^{\mu;b}_\nu(x) = 0$ are not difficult to find. Indeed one can take $2n$ arbitrary smooth functions $h^b_\nu(x)$ with $g^{\mu;b}_\nu(x) = \epsilon^{\mu\lambda}\partial_\lambda h^b_\nu(x)$. Then one needs appropriate solutions $S^\mu(x,z,a)$ and functions $g^{\mu;b}_\nu(x)$ such that the $4n$ equations (A.49) have n solutions $z^a = f^a(x)$; once found they will automatically be extremals since (cf. [**586**] for details)

(A.50) $$(\partial_\mu z^b(x) - (\partial H/\partial \pi^\mu_b))(\partial \psi^\mu_b/\partial a^\nu_c) = 0$$

Combined with the inequality above one concludes that $\partial^b_\mu(x) - (\partial H/\partial \pi^\mu_b) = 0$ leading to fulfillment of the integrability condition (A.42) with $(d\pi^\mu_a/dx^\mu) = -\partial_a H$. Thus for finding solutions of the field equations via the DWHJ equation (A.39)-b one can look for solutions S^μ which fulfill the integrability condition (A.42) and then either solve the equations $\partial_\mu z^a = \phi^a_\mu$, or, if the solution S^μ is a complete integral one can try to solve the algebraic equations (A.49). Another method is also indicated in [**586**].

For extension to $m > 2$ independent variables a few remarks are extracted here from [**586**]. Thus take $\omega = Ldx^1 \wedge \cdots \wedge dx^n$ as the Lagrangian m-form belonging to an equivalence class of m-forms Ω which consist of ω plus a linear combination of all m-forms obtained from $dx^1 \wedge dx^m$ by replacing the dx^μ by 1-forms $\omega^a = dz^a - v^a_\mu dx^\mu$ which vanish on the extremals where $v^a_\mu = \partial_\mu f^a(x)$ and generate an ideal $I[\omega^a]$. More specifically consider Minkowski space with coordinates $x^0, \cdots, x^3$ (with $c = 1$) and forms

(A.51) $$\Omega = Ldx^0 \wedge \cdots \wedge dx^3 + h^\mu_z \omega^a \wedge d^3\Sigma_\mu + (1/4)h^{\mu\nu}_{ab}\omega^a \wedge \omega^b \wedge d^2S_{\mu\nu} +$$

$$+\frac{1}{3!}h_{abc;\mu}\omega^a \wedge \omega^b \wedge \omega^c \wedge dx^\mu + \frac{1}{4!}h_{abcs}\omega^a \wedge \omega^b \wedge \omega^c \wedge \omega^s$$

$$d^3\Sigma_\mu = \frac{1}{3!}\epsilon_{\mu\alpha\beta\gamma}dx^\alpha \wedge dx^\beta \wedge dx^\gamma;\ d^2S_{\mu\nu} = (1/2)\epsilon_{\mu\nu\alpha\beta}dx^\alpha \wedge dx^\beta$$

Here $\epsilon_{\mu\nu\rho\sigma}$ is totally antisymmetric with $\epsilon_{0123} = 1$, $h^{\mu\nu}_{ab}$ are antisymmetric in $(\mu\nu)$ and (a, b) separately, $h_{abc;\mu}$ are completely antisymmetric in (a, b, c), etc. Thus the term h_{abcs} can occur only if $n \geq 4$ and the h-coefficients are arbitrary functions of x, z, v. As before h^μ_a are determined to be equal to $\pi^\mu_a = \partial L/\partial v^a_\mu$ by the requirement that $d\Omega = 0$ ($mod\ I[\omega^a]$). The Legendre transformation $v^a_\mu \to p^\mu_a$ and $L \to H$ is again implemented by inserting on the right in (A.51) for ω^a the expression $dz^a - v^a_\mu dx^\mu$ and identifying H with the resulting negative coefficient of $dx^0 \wedge \cdots \wedge dx^3$ while the canonical momenta p^μ_a are the coefficients of $dz^a \wedge d^3\Sigma_\mu$. If $h_{abc;\mu} = 0 = h_{abcs}$ one obtains

(A.52) $$p^\mu_a = \pi^\mu_a - h^{\mu\nu}_{ab}v^b_\nu;\ H = \pi^\mu_a v^a_\mu - (1/2)h^{\mu\nu}_{ab}v^a_\mu v^b_\nu - L$$

One notes the following identities (cf. [**657**])

(A.53) $$\epsilon^{\alpha\beta\gamma\mu}d^3\Sigma_\mu = dx^\alpha \wedge dx^\beta \wedge dx^\gamma;\ dx^\rho \wedge d^3\Sigma_\mu = \delta^\rho_\mu dx^0 \wedge \cdots \wedge dx^3;$$

$$dx^\rho \wedge d^2S_{\mu\nu} = \delta^\rho_\nu d^3\Sigma_\mu - \delta^\rho_\mu d^3\Sigma_\nu;\ dx^\sigma \wedge dx^\rho \wedge d^2S_{\mu\nu} = (\delta^\sigma_\mu\delta^\rho_\nu - \delta^\sigma_\nu\delta^\rho_\mu)dx^0 \wedge \cdots dx^3$$

If $d(h^{\mu\nu}_{ab})dx^\mu = 0$ where $(d/dx^\mu) = \partial_\mu + v^a_\mu\partial_a + v^a_{\mu\nu}\partial/\partial v^a_\nu$, and $v^a_{\mu\nu} = v^a_{\nu\mu}$, then

(A.54) $$h^{\mu\nu}_{ab}v^a_\mu v^b_\nu = \frac{d}{dx^\mu}(h^{\mu\nu}_{ab}z^a v^b_\nu)$$

For the DW canonical theory where all $h^{\mu\nu}_{ab} = 0$ (A.51) becomes

(A.55)

$$\Omega_0 = Ldx^0 \wedge \cdots \wedge dx^3 + \pi^\mu_a\omega^a \wedge d^3d\Sigma_\mu = a^\mu \wedge d^3d\Sigma_\mu - 3Ldx^0 \wedge \cdots \wedge dx^3$$

where $a^\mu = Ldx^\mu + \pi^\mu_a\omega^a = -T^\mu_\nu dx^\nu + \pi^\mu_a dz^a$. (A.55) shows that Ω_0 may be expressed by the 8 linearly independent 1-forms $dx^0, \cdots, dx^3, a^0, \cdots, a^3$ and therefore has rank 8 if $n \geq 2$ (for $n = 1$ it has rank 4 since one p-form in $p+1$ variables always has rank p). Replacing ω^a in Ω_0 by $dz^a - v^a_\mu dx^\mu$ results in

(A.56) $$\Omega_0 = -H_{DW}dx^0 \wedge \cdots \wedge dx^3 + \pi^\mu_a dz^0 \wedge d^3\Sigma_\mu;\ H_{DW} = \pi^\mu_a v^a_\mu - L$$

and the DWHJ equation is obtained from

$$dS^{\mu} \wedge d^3\Sigma_{\mu} = -H_{DW} dx^0 \wedge \cdots \wedge dx^3 + \pi_a^{\mu} dz^a \wedge d^3\Sigma_{\mu} \tag{A.57}$$

which implies

$$\partial_{\mu} S^{\mu}(x,z) + H_{DW}(x,z,\pi) = 0;\ \pi_a^{\mu} = \partial_a S^{\mu} \tag{A.58}$$

APPENDIX B

RELATIVITY AND ELECTROMAGNETISM

We extract first from [**12**], which still appears to be the best book ever written on classical general relativity, and will sketch some of the essential features. Four criteria for field equations are stated as:

(1) Physical laws do not distinguish between accelerated systems and inertial systems. This will hold if all laws are written in tensor form.
(2) Both gravitational forces and fictitious forces appear as Christoffel symbols (connection coefficients) in a mathematically similar form. This is desirable since they should be indistinguishable in the small.
(3) The gravitational equations should be phrased in covariant tensor form and should be of second order in the components of the metric tensor.
(4) For unique solutions one wishes the field equations to be quasi-linear (i.e. the second derivatives enter linearly).

Now the signature for a Lorentz metric is taken to be $(1, -1, -1, -1)$ and one writes $f_{|\alpha} = \partial f/\partial x^\alpha$ while $f_{||\alpha}$ is the covariant derivative (defined below). We assume known here the standard techniques of differential geometry as used in [**12**]. Then e.g. for a contravariant (resp. covariant) vector ξ^i (resp. η_m) one writes

$$\xi^i_{||k} = \xi^i_{|k} - \Gamma^i_{k\ell}\xi^\ell;\ \Gamma^i_{k\ell} = -\left\{ \begin{matrix} i \\ k\ \ell \end{matrix} \right\};\ \eta_{m||\ell} = \eta_{m|\ell} + \left\{ \begin{matrix} r \\ m\ \ell \end{matrix} \right\} \tag{B.1}$$

(the bracket notation is used for Christofel symbols which are connection coefficients). One defines the Riemann curvature tensor via

$$R^\alpha_{\ \eta\beta\gamma} = \left\{ \begin{matrix} \alpha \\ \beta\ \eta \end{matrix} \right\}_{|\gamma} - \left\{ \begin{matrix} \alpha \\ \eta\ \gamma \end{matrix} \right\}_{|\beta} + \left\{ \begin{matrix} \alpha \\ \tau\ \gamma \end{matrix} \right\}\left\{ \begin{matrix} \tau \\ \beta\ \eta \end{matrix} \right\} - \left\{ \begin{matrix} \alpha \\ \tau\ \beta \end{matrix} \right\}\left\{ \begin{matrix} \tau \\ \gamma\ \eta \end{matrix} \right\} \tag{B.2}$$

and a necessary (and sufficient) condition for a Riemann space to have a Lorentz metric is $R^\alpha_{\ \eta\beta\gamma} = 0$ (i.e. the space is flat); this equation is in fact a field equation for a flat and gravity free space (with Lorentz metric as a solution). Note here $\xi^\alpha_{\ ||\beta||\gamma} - \xi^\alpha_{\ ||\gamma||\beta} = R^\alpha_{\ \eta\beta\gamma}\xi^\eta$ which implies $\xi_{\alpha||\beta||\gamma} - \xi_{\alpha||\gamma||\beta} = R_{\alpha\rho\beta\gamma}\xi^\rho$ in a metric space (since the metric is used to lower indices, i.e. $T^\alpha_\gamma = g_{\gamma\beta}T^{\alpha\beta}$ etc.). Also one has generally

$$T^{\alpha\delta}_{\ \ ||\beta||\gamma} - T^{\alpha\delta}_{\ \ ||\gamma||\beta} = R^\alpha_{\ \tau\beta\gamma}T^{\tau\delta} + R^\delta_{\ \tau\beta\gamma}T^{\alpha\tau} \tag{B.3}$$

The notation $\{R_{\alpha\eta\beta\gamma}\xi^\eta\}_{(\alpha,\beta,\gamma)} = 0 = \{R_{\alpha\eta\beta\gamma}\}_{(\alpha,\beta,\gamma)}\xi^\eta$ involves an antisymmetrization in α, β, γ and since ξ^η is arbitrary this means $\{R_{\alpha\eta\beta\gamma}\}_{(\alpha,\beta,\gamma)} = 0$. Written out (with some relabeling and combination) this means that there are symmetries

$$\text{(B.4)} \qquad R_{\alpha\eta\beta\gamma} = -R_{\alpha\eta\gamma\beta},\ R_{\alpha\eta\beta\gamma} = -R_{\eta\alpha\beta\gamma},\ R_{\alpha\eta\beta\gamma} = R_{\beta\gamma\alpha\eta}$$

and $R_{1023} + R_{2031} + R_{3012} = 0$. The Bianchi identities are $\{R_{\alpha\eta\beta\gamma||\delta}\}_{(\beta,\gamma,\delta)} = 0$. Next via parallel transport one has $d\xi^\alpha = -\left\{ \begin{matrix} \alpha \\ \beta\ \gamma \end{matrix} \right\}\xi^\beta dx^\gamma$ and displacements along paths dx, $d\hat{x}$ and $d\hat{x}$, dx respectively leads to a vector transport difference $\Delta\xi^\alpha = R^\alpha_{\ \beta\eta\gamma}\xi^\beta dx^\eta d\hat{x}^\gamma$. Now the only meaningful contraction of $R_{\alpha\beta\gamma\delta}$ is given by $R_{\eta\gamma} = R^\alpha_{\ \eta\alpha\gamma} = g^{\alpha\beta}R_{\beta\eta\alpha\gamma} = g^{\alpha\beta}R_{\alpha\gamma\beta\eta} = R_{\gamma\eta}$ with 10 independent components. The equation $R_{\beta\delta} = 0$ satisfies conditions 1-4 above and has the Lorentz metric for one solution. Note here

(B.5)

$$R_{\beta\delta} = \left\{ \begin{matrix} \alpha \\ \beta\ \alpha \end{matrix} \right\}_{|\delta} - \left\{ \begin{matrix} \alpha \\ \beta\ \delta \end{matrix} \right\}_{|\alpha} + \left\{ \begin{matrix} \alpha \\ \tau\ \delta \end{matrix} \right\}\left\{ \begin{matrix} \tau \\ \beta\ \alpha \end{matrix} \right\} - \left\{ \begin{matrix} \alpha \\ \tau\ \alpha \end{matrix} \right\}\left\{ \begin{matrix} \tau \\ \beta\ \delta \end{matrix} \right\} = 0$$

can be written out in terms of the metric tensor $g_{\alpha\gamma}$ and this is the free space Einstein field equation. Some calculation shows that this can be written also in terms of the zero divergence Ricci tensor $G^{\beta\delta} = R^{\beta\delta} - (1/2)g^{\beta\delta}R = 0$ where $R = R^\eta_{\ \eta}$ is the Riemann scalar). Finally for a one parameter family $\Gamma(v)$ of geodesics given by $x^\mu = x^\mu(u,v)$ one has geodesic equations

$$\text{(B.6)} \qquad \frac{\partial^2 x^\mu}{\partial u^2} = -\left\{ \begin{matrix} \alpha \\ \beta\ \gamma \end{matrix} \right\}\frac{\partial x^\beta}{\partial u}\frac{\partial x^\gamma}{\partial u}$$

Now one can write out (B.5) in the form

$$\text{(B.7)} \qquad R_{\mu\nu} = \frac{1}{2}g^{\rho\sigma}\left[-g_{\mu\sigma|\nu|\rho} - g_{\nu\rho|\mu|\sigma} + g_{\mu\nu|\rho|\sigma} + g_{\rho\sigma|\mu|\nu}\right] + K_{\mu\nu}$$

where $K_{\mu\nu}$ contains only metric potentials and their first derivatives. One thinks of solving an initial value problem with data on a 3-dimensional hypersurface S described locally via $x^0 = 0$ (so $g_{00} > 0$). On S one gives $g_{\alpha\beta}$ and all first derivatives (only $g_{\mu\nu}$ and $g_{\mu\nu|0}$ need to be prescribed). Then one can calculate that

$$\text{(B.8)} \qquad R_{ij} = (1/2)g^{00}g_{ij|0|0} + M_{ij} = 0;\ R_{i0} = -(1/2)g^{0j}g_{ij|0|0} + M_{i0} = 0;$$

$$R_{00} = (1/2)g^{ij}g_{ij|0|0} + M_{00} = 0$$

where the $M_{\mu\nu}$ can be computed from data on S. A change of coordinates will make all $g_{\lambda 0|0|0} = 0$ on S (these are not contained in (B.8)) and thus (B.8) consists of 10 equations for six unknowns $g_{ij|0|0}$ on S which is overdetermined and leads to compatability conditions for the data $M_{\mu\nu}$ on S. This can be reduced to the form

$$\text{(B.9)} \qquad R_{ij} = 0;\ G^0_\lambda = 0$$

The first set of 6 equations determines the six unknowns $g_{ij|0|0}$ from initial data. The additional 4 equations in terms of the Ricci tensor G^0_λ represent necessary conditions on the initial data in order to insure a solution. There is much more material in [**12**] about the Cauchy problem which we omit here. For the Einstein equations in nonempty space one needs an energy momentum tensor for which a typical form is

(B.10) $$T^{\mu\nu} = \rho_0 u^\mu u^\nu + (p/c^2)(u^\mu u^\nu - g^{\mu\nu})$$

where ρ_0 is a density, p a pressure term, and u^μ a 4-velocity field. One assumes $T^{\mu\nu}$ has zero divergence or $T^{\mu\nu}_{\ \ ||\nu} = 0$ (which is a covariant formulation of fluid flow under the effect of its own internal pressure force).

REMARK B.1 One can also include EM fields in $T^{\mu\nu}$ by dealing with a Lorentz force $f^i = \sigma(\mathbf{E} + (\mathbf{v}/c) \times \mathbf{H})^i \sim -\sigma_0 F^{i\nu} u_\nu$ where $F^{i\nu}$ is the EM field tensor. However we will work in electromagnetism later in more elegant fashion via the Dirac-Weyl theory and omit this here (cf. [**12**] for more details).

In any event the Einstein field equations for nonempty space involve a zero divergence $T^{\mu\nu}$ so one uses the Ricci tensor $G_{\mu\nu} = R_{\mu\nu} - (1/2)g_{\mu\nu}R$ and notes that the most general second order tensor $B^{\alpha\gamma}$ of zero divergence can be written as $B^{\alpha\gamma} = G^{\alpha\gamma} + \Lambda g^{\alpha\gamma}$ (a result of E. Cartan). Hence one takes the Einstein field equations to be $G^{\alpha\gamma} + \Lambda g^{\alpha\gamma} = cT^{\alpha\gamma}$.

The exposition in Section 5.2 suggests the desirability of having a differential form discription of EM fields and we supply this via [**723**]. Thus one thinks of tensors $T = T^\sigma_{\mu\nu}\partial_\sigma \otimes dx^\mu \otimes dx^\nu$ with contractions of the form $T(dx^\sigma, \partial_\sigma) \sim T_\nu dx^\nu$. For $\eta = \eta_{\mu\nu}dx^\mu \otimes dx^\nu$ one has $\eta^{-1} = \eta^{\mu\nu}\partial_\mu \otimes \partial_\nu$ and $\eta\eta^{-1} = 1 \sim diag(\delta^\mu_\mu)$. Note also e.g.

(B.11) $$\eta_{\mu\nu}dx^\mu \otimes dx^\nu(\mathbf{u}, \mathbf{w}) = \eta_{\mu\nu}dx^\mu(\mathbf{u})dx^\nu(\mathbf{w}) =$$
$$= \eta_{\mu\nu}dx^\mu(u^\alpha\partial_\alpha)dx^\nu(w^\tau\partial_\tau) = \eta_{\mu\nu}u^\mu w^\nu$$

(B.12) $$\eta(\mathbf{u}) = \eta_{\mu\nu}dx^\mu \otimes dx^\nu(\mathbf{u}) = \eta_{\mu\nu}dx^\mu(\mathbf{u})dx^\nu =$$
$$= \eta_{\mu\nu}dx^\mu(u^\alpha\partial_\alpha)dx^\nu = \eta_{\mu\nu}u^\mu dx^\nu = u_\nu dx^\nu$$

for a metric η. Recall $\alpha \wedge \beta = \alpha \otimes \beta - \beta \otimes \alpha$ and

(B.13) $$\alpha \wedge \beta = \alpha_\mu dx^\mu \wedge \beta_\nu dx^\nu = (1/2)(\alpha_\mu\beta_\nu - \alpha_\nu\beta_\mu)dx^\mu \wedge dx^\nu$$

The EM field tensor is $F = (1/2)F_{\mu\nu}dx^\mu \wedge dx^\nu$ where

(B.14) $$F_{\mu\nu} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix};$$

$$F = E_x dx^0 \wedge dx^1 + E_y dx^0 \wedge dx^2 + E_z dx^0 \wedge dx^3 -$$
$$-B_z dx^1 \wedge dx^2 + B_y dx^1 \wedge dx^3 - B_x dx^2 \wedge dx^3$$

The equations of motion of an electric charge is then $d\mathbf{p}/d\tau = (e/m)\mathbf{F}(\mathbf{p})$ where $\mathbf{p} = p^\mu\partial_\mu$. There is only one 4-form, namely $\epsilon = dx^0 \wedge dx^1 \wedge dx^2 \wedge dx^3 = (1/4!)\epsilon_{\mu\nu\sigma\tau}dx^\mu \wedge dx^\nu \wedge dx^\sigma \wedge dx^\tau$ where $\epsilon_{\mu\nu\sigma\tau}$ is totally antisymmetric. Recall also for $\alpha = \alpha_{\mu\nu\cdots}dx^\mu \wedge dx^\nu \cdots$ one has $d\alpha = d\alpha_{\mu\nu\cdots} \wedge dx^\mu \wedge dx^\nu \cdots = \partial_\alpha\alpha_{\mu\nu\cdots}dx^\sigma \wedge dx^\mu \wedge dx^\nu \cdots$ and $dd\alpha = 0$. Define also the Hodge star operator on F and j via

${}^*F = (1/4)\epsilon_{\mu\nu\sigma\tau}F^{\sigma\tau}dx^\mu \wedge dx^\nu$ and ${}^*j = (1/3!)\epsilon_{\mu\nu\sigma\tau}j^\tau dx^\mu \wedge dx^\nu \wedge dx^\sigma$; these are called dual tensors. Now the Maxwell equations are

$$\partial_\mu F^{\mu\nu} = \frac{4\pi}{c}j^\nu;\ \partial^\alpha F^{\mu\nu} + \partial^\mu F^{\nu\alpha} + \partial^\nu F^{\alpha\mu} = 0 \tag{B.15}$$

and this can now be written in the form

$$dF = 0;\ d^*F = \frac{4\pi}{c} * j \tag{B.16}$$

and $0 = d^*j = 0$ is automatic. In terms of $A = A_\mu dx^\mu$ where $F = dA$ the relation $dF = 0$ is an identity $ddA = 0$.

A few remarks about the tensor nature of j^μ and $F^{\mu\nu}$ are in order and we write $n = n(x)$ and $\mathbf{v} = \mathbf{v}(x)$ for number density and velocity with charge density $\rho(x) = qn(x)$ and current density $\mathbf{j} = qn(x)\mathbf{v}(x)$. The conservation of particle number leads to $\nabla\cdot\mathbf{j}+\rho_t = 0$ and one writes $j^\nu = (c\rho, j_x, j_y, j_z) = (c\rho n, qnv_x, qnv_y, qnv_z)$ or equivalently $j^\nu = n_0 qu^\nu \equiv j^\nu = \rho_0 u^\nu$ where $n_0 = n\sqrt{1-(v^2/c^2)}$ and $\rho_0 = qn_0$ (ρ_0 here is charge density). Since j^ν consists of u^ν multiplied by a scalar it must have the transformation law of a 4-vector $j'^\beta = a^\beta_\nu j^\nu$ under Lorentz transformations. Then the conservation law can be written as $\partial_\nu j^\nu = 0$ with obvious Lorentz invariance. After some argument one shows also that $F^{\mu\nu} = a^\nu_\beta a^\mu_\alpha F'^{\alpha\beta}$ under Lorentz transformations so $F^{\mu\nu}$ is indeed a tensor. The equation of motion for a charged particle can be written now as $(d\mathbf{p}/dt) = q\mathbf{E} + (q/c)\mathbf{v}\times\mathbf{B}$ where $\mathbf{p} = m\mathbf{v}/\sqrt{1-(v^2/c^2)}$ is the relativistic momentum. This is equivalent to $dp^\mu/dt = (q/m)p_\nu F^{\mu\nu}$ with obvious Lorentz invariance. The energy momentum tensor of the EM field is

$$T^{\mu\nu} = -(1/4\pi)[F^{\mu\alpha}F^\nu_{\ \alpha} - (1/4)\eta^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}] \tag{B.17}$$

(cf. [**723**] for details) and in particular $T^{00} = (1/8\pi)(\mathbf{E}^2+\mathbf{B}^2)$ while the Poynting vector is $T^{0k} = (1/4\pi)(\mathbf{E}\times\mathbf{B})^k$.

One can equally well work in a curved space where e.g. covariant derivatives are defined via

$$\nabla_n T = lim_{d\lambda\to 0}[(T(\lambda+d\lambda) - T(\lambda) - \delta T]/d\lambda \tag{B.18}$$

where δT is the change in T produced by parallel transport. One has then the usual rules $\nabla_u(T\otimes R) = \nabla_u T\otimes R + T\otimes\nabla_u R$ and for $\mathbf{v} = v^\nu\partial_\nu$ one finds $\nabla_\mu\mathbf{v} = \partial_\mu v^\nu\partial_\nu + v^\nu\nabla_\mu\partial_\nu$. Now if $\mathbf{v}$ was constructed by parallel transport its covariant derivative is zero so, acting with the dual vector dx^α gives

$$\frac{\partial x^\nu}{\partial x^\mu}dx^\alpha(\partial_\nu) + v^\nu dx^\alpha(\nabla_\mu\partial_\nu) = 0 \equiv \partial_\mu v^\alpha + v^\nu dx^\alpha(\nabla_\mu\partial_\nu) = 0 \tag{B.19}$$

Comparing this with the standard $\partial_\mu v^\alpha + \Gamma^\alpha_{\mu\nu}v^\nu = 0$ gives $dx^\alpha(\nabla_\mu\partial_\nu) = \Gamma^\alpha_{\mu\nu}$. One can show also for vectors u, v, w (boldface omitted) and a 1-form α

(B.20)

$$(\nabla_u\nabla_v - \nabla_v\nabla_u - uv + vu)\alpha(w) = R(\alpha,u,v,w);\ R = F^\sigma_{\ \beta\mu\nu}\partial_\sigma\otimes dx^\beta\otimes dx^\mu\otimes dx^\nu$$

so R represents the Riemann tensor.

For the nonrelativistic theory first we go to [**650**] we define a transverse and longitudinal component of a field F via

$$F^{||}(r) = -\frac{1}{4\pi}\int d^3r' \frac{\nabla' \cdot F(r')}{|r-r'|};\ F^{\perp}(r) = \frac{1}{4\pi}\nabla\times\nabla\times\int d^3r' \frac{F(r')}{|r-r'|} \tag{B.21}$$

For a point particle of mass m and charge e in a field with potentials A and ϕ one has nonrelativistic equations $m\ddot{x} = eE + (e/c)v \times B$ (boldface is suppressed here) where one recalls $B = \nabla \times A$, $v = \dot{x}$, and $E = -\nabla\phi - (1/c)A_t$ with $H = (1/2m)(p - (e/c)A)^2 + e\phi$ leading to

$$\dot{x} = \frac{1}{2m}\left(p - \frac{e}{c}A\right);\ \dot{p} = \frac{e}{c}[v \times B + (v\cdot\nabla)A] - e\nabla\phi \tag{B.22}$$

Recall here also

$$B = \nabla\times A,\ \nabla\cdot E = 0,\ \nabla\cdot B = 0,\ \nabla\times E = -(1/c)B_t, \tag{B.23}$$
$$\nabla\times B = (1/c)E_t,\ E = -(1/c)A_t - \nabla\phi$$

(the Coulomb gauge $\nabla\cdot A = 0$ is used here). One has now $E = E^{\perp} + E^{||} \sim E^T + E^L$ with $\nabla\cdot E^{\perp} = 0$ and $\nabla\times E^{||} = 0$ and in Coulomb gauge $E^{\perp} = -(1/c)A_t$ and $E^{||} = -\nabla\phi$. Further

$$H \sim \frac{1}{2m}\left(p - \frac{e}{c}A\right)^2 + e\phi + \frac{1}{8\pi}\int d^3r((E^{\perp})^2 + B^2) \tag{B.24}$$

(covering time evolution of both particle and fields).

For the relativistic theory one goes to the Dirac equation $i(\partial_t + \alpha\cdot\nabla)\psi = \beta m\psi$ which, to satisfy $E^2 = \mathbf{p}^2 + m^2$ with $E \sim i\partial_t$ and $\mathbf{p} \sim -i\nabla$, implies $-\partial_t^2\psi = (-i\alpha\cdot\nabla + \beta m)^2\psi$ and ψ will satisfy the Klein-Gordon (KG) equation if $\beta^2 = 1$, $\alpha_i\beta + \beta\alpha_i \equiv \{\alpha_i, \beta\} = 0$, and $\{\alpha_i, \alpha_j\} = 2\delta_{ij}$ (note $c = \hbar = 1$ here with $\alpha\cdot\nabla \sim \sum \alpha_\mu\partial_\mu$ and cf. [**650, 632**] for notations and background). This leads to matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix};\ \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix};\ \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \tag{B.25}$$
$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix};\ \beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

where α_i and β are 4×4 matrices. Then for convenience take $\gamma^0 = \beta$ and $\gamma^i = \beta\alpha_i$ which satisfy $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$ (Lorentz metric) with $(\gamma^i)^\dagger = -\gamma^i$, $(\gamma^i)^2 = -1$, $(\gamma^0)^\dagger = \gamma^0$, and $(\gamma^0)^2 = 1$. The Dirac equation for a free particle can now be written

$$\left(i\gamma^\mu \frac{\partial}{\partial x^\mu} - m\right)\psi = 0 \equiv (i\partial\!\!\!/ - m)\psi = 0 \tag{B.26}$$

where $A\!\!\!/ = g_{\mu\nu}\gamma^\mu A^\nu = \gamma^\mu A_\mu$ and $\partial\!\!\!/ = \gamma^\mu\partial_\mu$. Taking Hermitian conjugates in, noting that α and β are Hermitian, one gets $\bar{\psi}(i\overleftarrow{\partial\!\!\!/} + m) = 0$ where $\bar{\psi} = \psi^\dagger\beta$. To define a conserved current one has an equation $\bar{\psi}\gamma^\mu\partial_\mu\psi + \gamma^\mu\bar{\psi}_\mu\psi = \partial_\mu(\bar{\psi}\gamma^\mu\psi) = 0$

leading to the conserved current $j^\mu = \bar{\psi}\gamma^\mu\psi = (\psi^\dagger\psi, \psi^\dagger\alpha\psi)$ (this means $\rho = \psi^\dagger\psi$ and $\mathbf{j} = \psi^\dagger\alpha\psi$ with $\partial_t\rho + \nabla\cdot\mathbf{j} = 0$). The Dirac equation has the Hamiltonian form

$$i\partial_t\psi = -i\alpha\cdot\nabla\psi + \beta m\psi = (\alpha\cdot\mathbf{p} + \beta m)\psi \equiv H\psi \quad \text{(B.27)}$$

($\alpha\cdot\mathbf{p} \sim \sum\alpha_\mu p_\mu$). To obtain a Dirac equation for an electron coupled to a prescribed external EM field with vector and scalar potentials A and ϕ one substitutes $p^\mu \to p^\mu - eA^\mu$, i.e. $\mathbf{p} \to \mathbf{p} - e\mathbf{A}$ and $p^0 = i\partial_t \to i\partial_t - e\Phi$, to obtain

$$i\partial_t\psi = [\alpha\cdot(\mathbf{p} - e\mathbf{A}) + e\Phi + \beta m]\psi \quad \text{(B.28)}$$

This identifies the Hamiltonian as $H = \alpha\cdot(\mathbf{p} - e\mathbf{A}) + e\Phi + \beta m = \alpha\cdot\mathbf{p} + \beta m + H_{int}$ where $H_{int} = -e\alpha\cdot\mathbf{A} + e\Phi$, suggesting α as the operator corresponding to the velocity v/c; this is strengthened by the Heisenberg equations of motion

$$\dot{\mathbf{r}} = \left(\frac{1}{i\hbar}\right)[\mathbf{r}, H] = \alpha;\ \dot{\pi} = \left(\frac{1}{i\hbar}\right)[\pi, H] = e(\mathbf{E} + \alpha\times\mathbf{B}) \quad \text{(B.29)}$$

Another bit of notation now from [**647**] is useful. Thus (again with $c = \hbar = 1$) one can define e.g. $\sigma_z = -i\alpha_x\alpha_y,\ \sigma_x = -i\alpha_y\alpha_z,\ \sigma_y = -i\alpha_z\alpha_x,\ \rho_3 = \beta,\ \rho_1 = \sigma_z\alpha_z = -i\alpha_x\alpha_y\alpha_z$, and $\rho_2 = i\rho_1\rho_3 = \beta\alpha_x\alpha_y\alpha_z$ so that $\beta = \rho_3$ and $\alpha^k = \rho_1\sigma^k$. Recall also that the angular momentum $\vec{\ell}$ of a particle is $\vec{\ell} = \mathbf{r}\times\mathbf{p}$ ($\sim (-i)\mathbf{r}\times\nabla$) with components ℓ_k satisfying $[\ell_x, \ell_y] = i\ell_z,\ [\ell_y, \ell_z] = i\ell_x$, and $[\ell_z, \ell_x] = i\ell_y$. Any vector operator L satisfying such relations is called an angular momentum. Next one defines $\sigma_{\mu\nu} = (1/2)i[\gamma_\mu, \gamma_\nu] = i\gamma_\mu\gamma_\nu$ ($\mu \neq \nu$) and $S_{\alpha\beta} = (1/2)\sigma_{\alpha\beta}$. Then the 6 components $S_{\alpha\beta}$ satisfy

$$S_{10} = (i/2)\alpha_x;\ S_{20} = (i/2)\alpha_y;\ S_{30} = (i/1)\alpha_z; \quad \text{(B.30)}$$
$$S_{23} = (1/2)\sigma_x;\ ,\ S_{31} = (1/2)\sigma_y;\ S_{12} = (1/2)\sigma_z$$

The $S_{\alpha\beta}$ arise in representing infinitesimal rotations for the orthochronous Lorentz group via matrices $I + i\epsilon S_{\alpha\beta}$. Further one can represent total angular momentum J in the form $J = L + S$ where $L = \mathbf{r}\times\mathbf{p}$ and $S = (1/2)\sigma$ (L is orbital angular momentum and S represents spin). We recall that the gamma matrices are given via $\gamma = \beta\alpha$. Now from

$$[(i\partial_t - e\phi) - \alpha\cdot(-i\nabla - e\mathbf{A}) - \beta m]\psi = 0 \quad \text{(B.31)}$$

one gets

$$[i\gamma^\mu D_\mu - m)\psi = [\gamma^\mu(i\partial_\mu - eA_\mu) - m]\psi = 0 \quad \text{(B.32)}$$

where $D_\mu = \partial_\mu + ieA_\mu \equiv (\partial_0 + ie\phi, \nabla - ie\mathbf{A})$. Working on the left with $(-i\gamma^\lambda D_\lambda - m)$ gives then $[\gamma^\lambda\gamma^\mu D_\lambda D_\mu + m^2]\psi = 0$ where $\gamma^\lambda\gamma^\mu = g^{\lambda\mu} + (1/2)[\gamma^\lambda, \gamma^\mu]$. By renaming the dummy indices one obtains

$$[\gamma^\lambda, \gamma^\mu]D_\lambda D_\mu = -[\gamma^\lambda, \gamma^\mu]D_\mu D_\lambda = (1/2)[\gamma^\lambda, \gamma^\mu][D_\lambda, D_\mu] \quad \text{(B.33)}$$

leading to

$$[D_\lambda, D_\mu] = ie[\partial_\lambda, A_\mu] + ie[A_\lambda, \partial_\mu] = ie(\partial_\lambda A_\mu - \partial_\mu A_\lambda) = ieF_{\lambda\mu} \quad \text{(B.34)}$$

This yields then

$$\gamma^\lambda\gamma^\mu D_\lambda D_\mu = D_\mu D_\lambda + eS^{\lambda\mu}F_{\lambda\mu} \quad \text{(B.35)}$$

where $S^{\lambda\mu}$ represents the spin of the particle. Therefore $[D_\mu D^\mu + eS^{\lambda\mu}F_{\lambda\mu} + m^2]\psi = 0$. Comparing with the standard form of the KG equation we see that this differs by the term $eS^{\lambda\mu}F_{\lambda\mu}$ which is the spin coupling of the particle to the EM field and has no classical analogue.

APPENDIX C

REMARKS ON QUANTUM GRAVITY

We refer here to [**55, 56, 57, 58, 60, 61, 62, 69, 70, 184, 303, 394, 495, 551, 630, 665, 819, 820, 896, 897, 898, 929, 930, 931, 932**] and will mainly follow [**69, 551**] for basic material. First (cf. [**69**] recall that upon assuming the spacetime manifold M to be diffeomorphic to $\mathbf{R}\times S$ where S is a 3-dimensional manifold one can choose spacelike slices $\Sigma\subset M$ with a space $Met(\Sigma)$ of Riemannian metrics. Writing $\phi:\ M\to\mathbf{R}\times S$ one defines a time coordinate via $\tau=\phi^*t$ and $\Sigma\subset M$ is determined via $\tau=constant$. The extrinsic curvature K of Σ provides Cauchy data $({}^3g,K)$ for the metric and regarding Einstein's 10 equations, 4 are constraint equations for the Cauchy data and 6 are evolution equations saying how the 3-metric changes in time. This is the Arnowitt-Deser-Misner (ADM) formulation. Now in more detail, one takes $g(v,v)>0$ for $v\in T\Sigma$ and $g(n,n)=-1$ (where n is the unit normal vector to Σ). One can write $v=-g(v,n)n+(v+g(v,n)n)$ in terms of orthogonal vectors and, for ∇ corresponding to the covariant derivative for the Levi-Civita connection, one can write

$$\nabla_uv=-g(\nabla_uv,n)+(\nabla_uv+g(\nabla_uv,n)n);\ K(u,v)n=-g(\nabla_uv,n)n \tag{C.1}$$

where K defines the extrinsic curvature. Since ∇ is torsion free one has $K(u,v)=K(v,u)$ and

$$K_{ij}u^iv^j=K(\partial_i,\partial_j)u^iv^j;\ K(u,v)=g(\nabla_un,v) \tag{C.2}$$

Write now $\partial_\tau=Nn+\vec{N}$ whee $\vec{N}$ is the shift field and N the lapse function. Then one has

$$N=-g(\partial_\tau,n);\ \vec{N}=\partial_\tau+g(\partial_\tau,n)n \tag{C.3}$$

Write the Christoffel symbols of the connection ∇ on Σ as ${}^3\Gamma^i{}_{jk}$ and the Riemann tensor of ${}^3g$ as ${}^3R^m_{ijk}$. Then some calculation (cf. [**69**]) gives the Gauss-Codazzi equations

$$R(\partial_i,\partial_j)\partial_k=({}^3\nabla_iK_{jk}-{}^3\nabla_jK_{ik})n+({}^3R^m_{ijk}+K_{jk}K^m_i-K_{ik}K^m_j)\partial_m \tag{C.4}$$

Now the Einstein tensor has the form $G_{\mu\nu}=R_{\mu\nu}-(1/2)g_{\mu\nu}R$ where $R_{\mu\nu}=R^\alpha_{\mu\alpha\nu}$ with $R=R^{\alpha\beta}_{\alpha\beta}$ and for vectors ∂_j tangent to Σ at a point in question this leads to (cf. [**69**]

$$G_{\mu\nu}n^\mu n^\nu=-\frac{1}{2}({}^3R+(TrK)^2-Tr(K^2));\ G_{\nu i}n^\mu={}^3\nabla_jK^j_i-{}^3\nabla_iK^j_j \tag{C.5}$$

The remaining 6 Einstein equations $G_{ij} = 0$ are dynamical in nature and describe the time evolution of ${}^3g$. Now looking only at the vacuum Einstein equations one writes $q_{ij} \sim {}^3g_{ij}$ with q for $det(q_{ij})$. It can be shown then (cf. [**69**]) that

$$K_{ij} = \frac{1}{2}N^{-1}(\dot{q}_{ij} - {}^3\nabla_i N_j - {}^3\nabla_j N_i) \tag{C.6}$$

The Lagrangian density for the Einstein-Hilbert action is $R\sqrt{-detg}d^4x$ and one writes here $\mathfrak{L} = R\sqrt{-g}$ which in terms of the 3-metric and lapse function becomes $\mathfrak{L} = q^{1/2}NR$. Discarding terms that give total divergences (and would integrate to zero for compact Σ at least) one has then

$$\mathfrak{L} = q^{1/2}N({}^3R + Tr(K^2) - (trK)^2) \tag{C.7}$$

Now the conjugate momenta are determined via

$$p^{ij} = \frac{\partial \mathfrak{L}}{\partial \dot{q}_{ij}} = q^{1/2}(K^{ij} - Tr(K)q^{ij}) \tag{C.8}$$

The Hamiltonian structure involves now

$$\mathfrak{H}(p^{ij}, q_{ij}) = p_{ij}\dot{q}^{ij} - \mathfrak{L};\ H = \int \mathfrak{H} d^3x;\ \mathfrak{H} = q^{1/2}(NC + N^iC_i) \tag{C.9}$$

where

$$C = -{}^3R + q^{-1}\left(Tr(p^2) - \frac{1}{2}Tr(p)^2\right);\ C_i = -2\,{}^3\nabla^j(q^{-1/2}p_{ij}) \tag{C.10}$$

Note one must specify the lapse and shift to know the meaning of time evolution. However one can compute that

$$C = -2G_{\mu\nu}n^\mu n^\nu;\ C_i = -2G_{\mu i}n^\mu \tag{C.11}$$

Now the equations $C = C_i = 0$ are precisely the constraint Einstein equations and hence $\mathfrak{H} = 0$ is a constraint on the phase space in $T^*Met(\Sigma)$ yet the dynamics is not trivial. To formulate Hamilton's equations one can define Poisson brackets on phase space via

$$\{f, g\} = \int_\Sigma \left\{\frac{\partial f}{\partial p^{ij}(x)}\frac{\partial g}{\partial q_{ij}(x)} - \frac{\partial f}{\partial q_{ij}(x)}\frac{\partial g}{\partial p^{ij}(x)}\right\} q^{1/2}d^3x \tag{C.12}$$

This is often written in a functional derivative notation but it simply amounts to defining say $\partial f/\partial q_{ij}(x)$ for example via

$$\int_\Sigma h_{ij}(x)\frac{\partial f}{\partial q_{ij}(x)}q^{1/2}d^3x = \frac{d}{ds}f(q + sh)|_{s=0} \tag{C.13}$$

for every symmetric $(0, 2)$ tensor field h (for $h \sim \delta g$ one writes the right side of (C.13) as δf. In this spirit one arrives at

$$\{p^{ij}(x), q_{k\ell}(y)\} = (\delta^i_k\delta^j_\ell + \delta^i_\ell\delta^j_k)\delta^3(x - y); \tag{C.14}$$
$$\{p^{ij}(x), p^{k\ell}(y)\} = 0;\ \{q_{ij}(x), q_{k\ell}(y)\} = 0$$

The equations $G_{ij} = 0$ in this disguise are now simply $\dot{q}^{ij} = \{H, q^{ij}\}$ and $\dot{p}_{ij} = \{H, p_{ij}\}$ which take the form

$$\dot{q}_{ij} = 2q^{-1/2}N\left(p_{ij} - \frac{1}{2}p^k_k q_{ij}\right) + 2\ {}^3\nabla_{[i}N_{j]}; \tag{C.15}$$

$$\dot{p}^{ij} = -Nq^{1/2}\left({}^3R^{ij} - \frac{1}{2}\,{}^3Rq^{ij}\right) + \frac{1}{2}Nq^{-1/2}q^{ij}\left(p_{ab}p^{ab} - -\frac{1}{2}(p^a_a)^2\right) -$$
$$-2Nq^{-1}\left(p^{ia}p^j_a - \frac{1}{2}p^a_a q^{ij}\right) + q^{1/2}(\nabla^i\nabla^j N - q^{ij}\ {}^3\nabla^{a\,3}\nabla_a N) +$$
$$+q^{1/2}\nabla_a(q^{-1/2}N^a p^{ij}) - 2p^{a[i3}\nabla_a N^{j]}$$

This is a horror story but it does show that the time evolution given by Hamilton's equations is nontrivial. One notes in passing that the lapse and shift measure how much the time evolution push the slice Σ in the normal or tangent direction respectively. For example for shift (resp. lapse) zero one has

$$C(N) = \int_\Sigma NCq^{1/2}d^3x \ \ (resp.\ C(\vec{N}) = \int_\Sigma N^i C_i q^{1/2} d^3x) \tag{C.16}$$

Here $C(\vec{N})$ or C_i (resp. $C(N)$ or C) is called the diffeomorphism (resp. Hamiltonian) constraint and one can calculate

$$\{C(\vec{N}), C(\vec{N}')\} = C([\vec{N}, \vec{N}']);\ \{C(\vec{N}), C(N')\} = C(\vec{N}N'); \tag{C.17}$$
$$\{C(N), C(N')\} = C((N\partial^i N' - N'\partial^i N)\partial_i)$$

where $\vec{N}N'$ is the derivative of N' in the direction $\vec{N}$ and $(N\partial^i N' - N'\partial^i N)\partial_i$ is the result of converting the 1-form $NdN' - N'dN$ into a vector field by raising indices. These formulas are known as the Dirac algebra and one notes that the constraints are closed under Poisson brackets.

Now for quantization one proceeds formally for various reasons (cf. [**69**]) and takes the operator corresponding to the 3-metric (and momentum) to be

$$(\hat{q}_{ij}(x)\psi)(q) = g_{ij}(x)\psi(q);\ (\hat{p}^{ij}(x)\psi)(q) = -i\frac{\partial}{\partial q_{ij}(x)}\psi(q) \tag{C.18}$$

where $g \in Met(\Sigma)$. These operators satisfy then

$$[\hat{p}^{ij}(x), \hat{q}_{k\ell}(y)] = -i(\delta^i_k\delta^j_\ell + \delta^i_\ell\delta^j_k)\delta^3(x,y); \tag{C.19}$$
$$[\hat{p}^{ij}(x), \hat{p}^{k\ell}(y)] = 0;\ [\hat{q}_{ij}(x), \hat{q}_{k\ell}(y)] = 0$$

In order to obtain quantum versions $\hat{C}$ and $\hat{C}_i$ one encounters operator ordering problems; ideally one would like

$$[\hat{C}(\vec{N}), \hat{C}(\vec{N}')] = -i\hat{C}(\vec{N}, \vec{N}']);\ [\hat{C}(\vec{N}, \hat{C}(N')] = -i\hat{C}(\vec{N}N'); \tag{C.20}$$
$$[\hat{C}(N), \hat{C}(N')] = -i\hat{C}((N\partial^i N' - N'\partial^i N)\partial_i)$$

but this seems virtually impossible to achieve. Suppose nevertheless that one obtained somehow satisfactory operators $\hat{C}$ and $\hat{C}_i$; then one could write

$$\hat{H} = \int_\Sigma (N\hat{C} + N^i\hat{C}_i)q^{1/2}d^3x \tag{C.21}$$

It could then be said that a vector $\psi \in L^2(Met(\Sigma))$ (whatever that may mean) is a physical state if it satisfies $\hat{C}(N)\psi = \hat{C}(\vec{N})\psi = 0$ or alternatively $\hat{H}\psi = 0$ for all choices of lapse and shift and this is the WDW equation. There are many problems here and even if one could find solutions there arises the problem of time, namely the Hamiltonian vanishes on the space of physical states so any operator A on $\mathfrak{H}_{phys}$ must satisfy $(d/dt)A_t = i[\hat{H}, A_t] = 0$ and the dynamics disappears. Recent developments using the Ashtekar variables have made some progress in this area and will be discussed briefly below.

We describe briefly now the Ashtekar variables following [**69**] which are based on a modification of the Palatini formalism. First for background the Palatini action is $S(g) = \int_M R \cdot vol$ rewritten so that it is not a function of the metric but rather a function of a connection and a frame field. Thus a trivialization of TM is a vector bundle isomorphism $e : M \times \mathbf{R}^n \to TM$ sending each fiber $\{p\} \times \mathbf{R}^n$ of the trivial bundle $M \times \mathbf{R}^n$ to the corresponding tangent space T_pM. A trivialization of TM is also called a frame field sending the standard basis of $\mathbf{R}^n$ to a basis of tangent vectors at p or a frame. If M is 3 (resp. 4) dimensional a frame field is called a triad (resp. tetrad). One goes back and forth now using the frame field e and its inverse $e^{-1} : TM \to M \times \mathbf{R}^n$. Given a basis of sections of $M \times \mathbf{R}^n$ of the form $\xi_i = (0, \cdots, 0, 1, 0, 0, \cdots)$ one writes any section as $s = s^I \xi_I$ where I denotes an internal index (whereas ∂_μ refers to coordinate vector fields on a chart). Thus $e(\xi_I) = e_I^\alpha \partial_\alpha$ and $e(\xi_I) \sim e_I$. Now given sections s, s' one can define $\eta(s, s') = \eta_{IJ} s^I s'^J$ where η_{IJ} is copied after a Minkowski metric $(-1, 1, \cdots, 1)$ (internal metric). One can raise and lower indices via η_{IJ} and set $g(v, v') = g_{\alpha\beta} v^\alpha v'^\beta$. The frame field is said to be orthonormal if $g(e_I, e_J) = \eta_{IJ}$; in this case one has $g(e(s), e(s')) = \eta(s, s')$ since

$$g(e(s), e(x')) = g(e(S^I \xi_I), e(s^J \xi_J)) = s^I s^J g(e_I, e_J) = \eta_{IJ} s^I s^J = \tag{C.22}$$

$$= \eta(s^I \xi_I, s^J \xi_J) = \eta(s, s')$$

Note that one can write $\eta_{IJ} = g(e_I, e_J) = g_{\alpha\beta} e_I^\alpha e_J^\beta$ and hence $\delta_J^I = e_\alpha^I e_J^\alpha$. Further $e^{-1}v = e_\alpha^I v^\alpha \xi_I$ since if $v = e(s)$ one has $e^{-1}v = e_\alpha^I v^\alpha \xi_I = e_\alpha^I e_J^\alpha s^J \xi_I = \delta_J^I s^J \xi_I = s^I \xi_I = s$ This leads to a formula for the metric g in terms of the coframe field e_α^I, namely

$$g_{\alpha\beta} = g(\partial_\alpha, \partial_\beta) = \eta(e^{-1}\partial_\alpha, e^{-1}\partial_\beta) = \eta(e_\alpha^I \xi_I, e_\beta^J \xi_J) = \eta_{IJ} e_\alpha^I e_\beta^J \tag{C.23}$$

The other ingredient in the Palatini formalism is a connection on the trivial bundle $M \times \mathbf{R}^n$. One says a connection D here is a Lorentz connection if $v\eta(x, s') = \eta(D_v s, s') + \eta(s, D_v s')$ (standard $S(O(n, 1)$ connection). Note torsion free is meaningless and there is no Levi-Civita connection on $M \times \mathbf{R}^n$; however the standard flat connection $D^0 s = v(s^I)\xi_I$ is nice and any connection can be written as $D = D^0 + A$ for some potential A, which is an $End(\mathbf{R}^n)$-valued 1-form on M. Thus

$$D_v s = (v(s^J) + A_{\mu I}^J v^\mu s^I)\xi_J;\ F_{\alpha\beta}^{IJ} = \partial_\alpha A_\beta^{IJ} - \partial_\beta A_\alpha^{IJ} + [A_\alpha, A_\beta]^{IJ} \tag{C.24}$$

where $F_{\alpha\beta}^{IJ}$ is the curvature of D. If A defines a Lorentz connection then $F_{\alpha\beta}^{IJ} = -F_{\beta\alpha}^{IJ} = -F_{\alpha\beta}^{JI}$. Next given a frame field e and a Lorentz connection D one can transfer the Lorentz connection from $M \times \mathbf{R}^N$ to TM to obtain a connection $\bar{\nabla}$ given by

$$\bar{\nabla}_\alpha \partial_\beta = \bar{\Gamma}^\gamma{}_{\alpha\beta}\partial_\gamma;\ \bar{\Gamma}^\gamma{}_{\alpha\beta} = A^J_{\alpha I} e^I_\beta e^\gamma_J \tag{C.25}$$

Here $\bar{\nabla}$ is called the imitation Levi-Civita connection and $\bar{\Gamma}^\gamma{}_{\alpha\beta}$ are the imitation Christoffel symbols, leading to an imitation Riemann tensor

$$\bar{R}_{\alpha\beta}{}^\delta_\gamma = F^{IJ}_{\alpha\beta} e^\gamma_I e^\delta_J;\ \bar{R}_{\alpha\beta} = \bar{R}^\gamma{}_{\alpha\gamma\beta};\ \bar{R} = \bar{R}^\alpha_\alpha \tag{C.26}$$

Now the Palatini action is basically the Einstein-Hilbert action in disguise, being a function of the frame field given by $g_{\alpha\beta} = \eta_{IJ} e^I_\alpha e^J_\beta$ in the form

$$S(A,e) = \int_M e^\alpha_I e^\beta_J F^{IJ}_{\alpha\beta} \cdot vol;\ \delta S = 2\int_M (\bar{R}_{\alpha\beta} - (1/2)\bar{R} g_{\alpha\beta}\eta^{IJ} e^\beta_J (\delta e^\alpha_I) \cdot vol \tag{C.27}$$

Thus $\delta S = 0$ for an arbitrary variation of the frame field when $\bar{R}_{\alpha\beta} - (1/2)\bar{R}g_{\alpha\beta} = 0$ which is of course Einstein's equation when $\bar{\nabla} = \nabla$. We refer to [**69**] for further computations, formulas, and discussion.

Now for the Ashtekar variables themselves define the complexified tangent bundle $\mathbf{C}TM$ to have fibers $\mathbf{C} \times T_pM$ and an imitation complexified tangent bundle $M \times \mathbf{C}^4$. A complex frame field is an isomorphism $e : M \times \mathbf{C}^4 \to \mathbf{C}TM$. A connection A on $M \times \mathbf{C}^4$ is an $End(\mathbf{C}^4)$ valued 1-form on M with components $A^J_{\alpha I}$ or A^{IJ}_α; it is Lorentz if $A^{IJ}_\alpha = -A^{JI}_\alpha$. Recall the Hodge $*$ operator maps 2-forms to 2-forms in 4 dimensions and define it here via

$$*T^{IJ} = (1/2)\epsilon^{IJ}{}_{KL}T^{KL};\ \epsilon^{i_1,\cdots,i_n} = \begin{cases} sgn(i_1,\cdots,i_n) & i_j\ distinct \\ 0 & otherwise \end{cases} \tag{C.28}$$

for any T with two antisymmetric raised internal indices. In particular

$$(*A)^{IJ}_\alpha = (1/2)\epsilon^{IJ}{}_{KL}A^{KL}_\alpha \tag{C.29}$$

Now write any Lorentz connection A as a sum of self-dual and anti-self-dual parts

$$A = {}^+A + {}^-A;\ *{}^\pm A = \pm i{}^\pm A;\ {}^\pm A = (1/2)(A \mp i * A) \tag{C.30}$$

In the self dual formulation of GR one of the two basic fields is a self-dual Lorentz connection, i.e. a Lorentz connection on $M \times \mathbf{C}^4$ with $*{}^+A = i{}^+A$. The other basic field is a complex frame field $e : M \times \mathbf{C}^4 \to \mathbf{C}TM$ and the action is built using the curvature ${}^+F$ of ${}^+A$ via

$${}^+F^{IJ}_{\alpha\beta} = \partial_\alpha {}^+A^{IJ}_\beta - \partial_\beta {}^+A^{IJ}_\alpha + [{}^+A_\alpha, {}^+A_\beta]^{IJ} \tag{C.31}$$

As in the Palatini formalism one writes

$$g_{\alpha\beta} = \eta_{IJ}e^I_\alpha e^J_\beta;\ e^{-1}\partial_\alpha = e^I_\alpha \xi_I \tag{C.32}$$

(note the metric is now complex). The self dual action is

$$S_{SD}({}^+A, e) = \int_M e^\alpha_I e^\beta_J {}^+F^{IJ}_{\alpha\beta} \cdot vol;\ vol = \sqrt{-g}d^4x \tag{C.33}$$

Now define the internal Hodge dual of the curvature of a connection F on $M \times \mathbf{C}^4$ via

$$(*F)^{IJ}_{\alpha\beta} = (1/2)\epsilon^{IJ}_{\ \ KL}F^{KL}_{\alpha\beta} \tag{C.34}$$

and call the curvature self dual if $*F = iF$. It turns out that the curvature of a self dual Lorentz connection is self dual (computation needed) and this has Lie algebraic meaning (cf. [**69**] for details). Next compute $\delta S_{SD} = 0$ and following [**69**] one obtains two equations. First by varying the self dual connection there is an equation saying that ^+A is the self dual part of a Lorentz connection A for which the self dual part of the Riemann tensor of g is related to ^+F, namely

$$^+R^{\alpha}_{\ \beta\gamma}{}^{\delta} = (1/2)(R^{\alpha}_{\ \beta\gamma}{}^{\delta} - (i/2)\epsilon^{\alpha\delta}_{\ \ \mu\nu}R^{\mu}_{\ \beta\gamma}{}^{\nu});\ ^+R^{\alpha}_{\ \beta\gamma}{}^{\delta} = {}^+F^{IJ}_{\beta\gamma}e^{\alpha}_{I}e^{\delta}_{J} \tag{C.35}$$

Second by varying the frame field there arises a self dual analogue of Einstein's equation

$$^+R_{\alpha\beta} - (1/2)g_{\alpha\beta}{}^+R = 0;\ ^+R_{\alpha\beta} = {}^+R^{\gamma}_{\ \alpha\gamma\beta};\ ^+R = {}^+R^{\alpha}_{\alpha} \tag{C.36}$$

Using symmetries of the Riemann tensor this is equivalent to the vacuum Einstein equation. Note however that we have complex metrics here; some reality conditions are needed and this is not exactly a trivial matter. Thus let Σ be a spacelike slice and work in coordinates such that ∂_0 is normal to Σ and ∂_i is tangent for spacelike indices. Given a self dual Lorentz connection $^+A^{IJ}_{\alpha}$ on $M \times \mathbf{C}^4$ one can restrict it to a connection A^{IJ}_i on $\Gamma \times \mathbf{C}^4$ with $A^{IJ}_i = -A^{JI}_i$ and $*A = iA$ (the + sign is gratuitously omitted here). Since $sl(2,\mathbf{C})$ has a basis in terms of Pauli matrices one can also write this as $-(i/2)A^a_i\sigma_a$ $(a = 1,2,3)$. The field playing the role analogous to position is the self dual Lorentz conection A^a_i with conjugate momentum $\tilde{E}^i_a = q^{1/2}e^i_a$ and one has Poisson brackets

$$\{\tilde{E}^i_a(x), A^b_j(y)\} = -i\delta^b_a\delta^i_j\delta^3(x,y);\ \{\tilde{E}^i_a(x), \tilde{E}^j_b(y)\} = 0 = \{A^a_i(x), A^b_j(y)\} \tag{C.37}$$

The Hamiltonian and diffeomorphism constraints are given via

$$\tilde{C} = \epsilon^{abc}\tilde{E}^i_a\tilde{E}^j_bF_{ijc};\ C_j = \tilde{E}^k_aF^a_{jk};\ G_a = D_i\tilde{E}^i_a \tag{C.38}$$

(the latter being a Gauss law constraint). The tilde appears because of densitation, i.e. $\tilde{C}$ is $q^{1/2}$ times the earlier C. To quantize now one writes

$$(\hat{A}^a_i(x)\psi)(A) = A^a_i(x)\psi(A);\ (\hat{E}^i_a(x)\psi)(A) = \frac{\partial}{\partial A^a_i(x)}\psi(A) \tag{C.39}$$

$$[\hat{E}^i_a(x), \hat{A}^b_j(y)] = \delta^b_a\delta^i_j\delta^3(x,y);\ [\hat{E}^i_a(x), \hat{E}^j_b(y)] = 0 = [\hat{A}^a_i(x), \hat{A}^b_j(y)] \tag{C.40}$$

A convenient choice of operator orderings is now

$$\hat{C} = \epsilon^{abc}\hat{E}^i_a\hat{E}^j_b\hat{F}_{ijc};\ \hat{C}_j = \hat{E}^k_a\hat{F}^a_{jk};\ \hat{G}_a = \hat{D}_i\hat{E}^i_a;\ (\hat{F}^a_{jk}(x)\psi)(A) = F^a_{jk}(x)\psi(A) \tag{C.41}$$

It seems that these operators satisfy commutation relations analogous to the Poisson brackets of the classical constraints. The physical state space $\mathfrak{H}_{phys}$ then consists of functions $\psi(A)$ satisfying the constraints in quantum form, i.e.

$$\mathfrak{H}_{phys} = \{\psi :\ \hat{C}\psi = \hat{C}_j\psi = \hat{G}_a\psi = 0\} \tag{C.42}$$

Here $\hat{C}_j\psi = 0$ means $\psi(A) = \psi(A')$ whenever A' is obtained from A by applying a diffeomorphism connected to the identity by a flow. Similarly $\hat{G}_a\psi = 0$ says that $\psi(A) = \psi(A')$ whenever A' is obtained from A by a small gauge transformation. The Hamiltonian constraint $\hat{C}\psi = 0$ contains the dynamics of the theory and finding solutions is difficult.

There is an interesting relation between Chern-Simons (CS) theory and quantum gravity however that provides some solutions. First if the cosmological constant is nonzero the Hamiltonian constraint becomes

$$\hat{C} = \epsilon^{abc}\hat{E}^i_a\hat{E}^j_b\hat{F}_{ijc} - \frac{\Lambda}{6}\epsilon_{ijk}\epsilon^{abc}\hat{E}^i_a\hat{E}^j_b\hat{E}^k_c \tag{C.43}$$

The CS state ψ_{CS} is defined as $\psi_{CS}(A) = exp[-(6/\hbar)S_{CS}(A)]$ where $S_{CS}(A) = \int_\Sigma Tr(A\wedge dA + (2/3)A\wedge A\wedge A)$. It is then shown in [**69**] that $\hat{C}_j\psi_{CS} = \hat{G}_a\psi_{CS} = \hat{C}\psi_{CS} = 0$ with some discussion. The book [**69**] was written in 1994 and there has since been enormous activity in loop quantum gravity for which we refer to [**53, 57, 60, 184, 303, 394, 551, 819, 820, 896, 929, 930, 931, 932**]. (cf. also [**714**] for connections between thermodynamics and gravity).

APPENDIX D

DIRAC ON WEYL GEOMETRY

First we give some background on Weyl geometry and Brans-Dicke theory following [**12**]; for differential geometry we use the tensor notation of [**12**] and refer to e.g. [**149, 358, 458, 723, 731, 972, 998**] for other notation (see also [**990**] for interesting variations). For general background see [**156, 179, 235, 350, 351, 361, 556, 601, 786, 801, 937, 938, 939, 1004, 1014**] and note that for our purposes the most important background features appear already in Sections 3.2, 3.2.2, and 4.1. One thinks of a differential manifold $M = \{U_i, \phi_i\}$ with $\phi : U_i \to \mathbf{R}^4$ and metric $g \sim g_{ij}dx^i dx^j$ satisfying $g(\partial_k, \partial_\ell) = g_{k\ell} =< \partial_k, \partial_\ell >= g_{\ell k}$. This is for the bare essentials; one can also imagine tangent vectors $X_i \sim \partial_i$ and dual cotangent vectors $\theta^i \sim dx^i$, etc. Given a coordinate change $\tilde{x}^i = \tilde{x}^i(x^j)$ a vector ξ^i transforming via $\tilde{\xi}^i = \sum \partial_i \tilde{x}^j \xi^j$ is called contravariant (e.g. $d\tilde{x}^i = \sum \partial_j \tilde{x}^i dx^j$). On the other hand $\partial\phi/\partial\tilde{x}^i = \sum(\partial\phi/\partial x^j)(\partial x^j/\partial\tilde{x}^i$ leads to the idea of covariant vectors $A_j \sim \partial\phi/\partial x^j$ transforming via $\tilde{A}_i = \sum(\partial x^j/\partial\tilde{x}^i)A_j$ (i.e. $\partial/\partial\tilde{x}^i \sim (\partial x^j/\partial\tilde{x}^j)\partial/\partial x^j$). Now define connection coefficients or Christoffel symbols via (strictly one writes $T^\gamma{}_\alpha = g_{\alpha\beta}T^{\gamma\beta}$ and $T_\alpha{}^\gamma = g_{\alpha\beta}T^{\beta\gamma}$ which are generally different; we use that notation here but it is not used in subsequent sections since it is unnecessary)

$$\text{(D.1)} \qquad \Gamma^r{}_{ki} = -\left\{ \begin{array}{c} r \\ k \;\; i \end{array} \right\} = -\frac{1}{2}\sum(\partial_i g_{k\ell} + \partial_k g_{\ell i} - \partial_\ell g_{ik})g^{\ell r} = \Gamma^r{}_{ik}$$

(note this differs by a minus sign from some other authors). Note also that (D.1) follows from equations

$$\text{(D.2)} \qquad \partial_\ell g_{ik} + g_{rk}\Gamma^r{}_{i\ell} + g_{ir}\Gamma^r{}_{\ell k} = 0$$

and cyclic permutation; the basic definition of $\Gamma^i{}_{mj}$ is found in the transplantation law

$$\text{(D.3)} \qquad d\xi^i = \Gamma^i{}_{mj}dx^m\xi^j$$

Next for tensors $T^\alpha_{\beta\gamma}$ define derivatives

$$\text{(D.4)} \qquad T^\alpha_{\beta\gamma|k} = \partial_k T^\alpha_{\beta\gamma};\; T^\alpha_{\beta\gamma||\ell} = \partial_\ell T^\alpha_{\beta\gamma} - \Gamma^\alpha_{\ell s}T^s_{\beta\gamma} + \Gamma^s{}_{\ell\beta}T^\alpha_{s\gamma} + \Gamma^s{}_{\ell\gamma}T^\alpha_{\beta s}$$

In particular covariant derivatives for contravariant and covariant vectors respectively are defined via

$$\text{(D.5)} \qquad \xi^i_{||k} = \partial_k\xi^i - \Gamma^i{}_{k\ell}\xi^\ell = \nabla_k\xi^i;\; \eta_{m||\ell} = \partial_\ell\eta_m + \Gamma^r{}_{m\ell}\eta_r = \nabla_\ell\eta_m$$

Now to describe Weyl geometry one notes first that for Riemannian geometry (D.3) holds along with

$$\ell^2 = \|\xi\|^2 = g_{\alpha\beta}\xi^\alpha\xi^\beta;\ \xi^\alpha\eta_\alpha = g_{\alpha\beta}\xi^\alpha\eta^\beta \tag{D.6}$$

However one does not demand conservation of lengths and scalar products under affine transplantation (D.3). Thus assume

$$d\ell = (\phi_\beta dx^\beta)\ell \tag{D.7}$$

where the covariant vector ϕ_β plays a role analogous to $\Gamma^\alpha_{\beta\gamma}$. Combining one obtains

$$d\ell^2 = 2\ell^2(\phi_\beta dx^\beta) = d(g_{\alpha\beta}\xi^\alpha\xi^\beta) = \tag{D.8}$$

$$= g_{\alpha\beta|\gamma}\xi^\alpha\xi^\beta dx^\gamma + g_{\alpha\beta}\Gamma^\alpha_{\rho\gamma}\xi^\rho\xi^\beta dx^\gamma + g_{\alpha\beta}\Gamma^\beta_{\rho\gamma}\xi^\alpha\xi^\rho dx^\gamma$$

Rearranging etc. and using (D.6) again gives

$$(g_{\alpha\beta|\gamma} - 2g_{\alpha\beta}\phi_\gamma) + g_{\sigma\beta}\Gamma^\sigma_{\alpha\gamma} + g_{\sigma\alpha}\Gamma^\sigma_{\beta\gamma} = 0 \tag{D.9}$$

leading to

$$\Gamma^\alpha_{\beta\gamma} = -\left\{ \begin{array}{c} \alpha \\ \beta\ \gamma \end{array} \right\} + g^{\sigma\alpha}[g_{\sigma\beta}\phi_\gamma + g_{\sigma\gamma}\phi_\beta - g_{\beta\gamma}\phi_\sigma] \tag{D.10}$$

Thus we can prescribe the metric $g_{\alpha\beta}$ and the covariant vector field ϕ_γ and determine by (D.10) the field of connection coefficients $\Gamma^\alpha_{\beta\gamma}$ which admits the affine transplantation law (D.3). If one takes $\phi_\gamma = 0$ the Weyl geometry reduces to Riemannian geometry. This leads one to consider new metric tensors via the gauge transformation $\hat{g}_{\alpha\beta} = f(x^\lambda)g_{\alpha\beta}$ and it turns out that $(1/2)\partial log(f)/\partial x^\lambda$ plays the role of ϕ_λ in (D.7). The ordinary connections $\left\{ \begin{array}{c} \alpha \\ \beta\ \gamma \end{array} \right\}$ constructed from $g_{\alpha\beta}$ are equal to the more general connections $\hat{\Gamma}^\alpha_{\beta\gamma}$ constructed according to (D.10) from $\hat{g}_{\alpha\beta}$ and $\hat{\phi}_\lambda = (1/2)\partial log(f)/\partial x^\lambda$. The generalized differential geometry is conformal in that the ratio

$$\frac{\xi^\alpha\eta_\alpha}{\|\xi\|\|\eta\|} = \frac{g_{\alpha\beta}\xi^\alpha\eta^\beta}{[(g_{\alpha\beta}\xi^\alpha\xi^\beta)(g_{\alpha\beta}\eta^\alpha\eta^\beta)]^{1/2}} \tag{D.11}$$

does not change under the gauge transformation above. Again if one has a Weyl geometry characterized by $g_{\alpha\beta}$ and ϕ_α with connections determined by (D.10) one may replace the geometric quantities by use of a scalar field f with

$$\hat{g}_{\alpha\beta} = f(x^\lambda)g_{\alpha\beta},\ \hat{\phi}_\alpha = \phi_\alpha + (1/2)(log(f)_{|\alpha};\ \hat{\Gamma}^\alpha_{\beta\gamma} = \Gamma^\alpha_{\beta\gamma} \tag{D.12}$$

without changing the intrinsic geometric properties of vector fields; the only change is that of local lengths of a vector via $\hat{\ell}^2 = f(x^\lambda)\ell^2$. Note that one can reduce $\hat{\phi}_\alpha$ to the zero vector field if and only if ϕ_α is a gradient field, namely $F_{\alpha\beta} = \phi_{\alpha|\beta} - \phi_{\beta|\alpha} = 0$ (i.e. $\phi_\alpha = (1/2)\partial_a log(f) \equiv \partial_\beta\phi_\alpha = \partial_\alpha\phi_\beta$). In this case one has length preservation after transplantation around an arbitrary closed curve and the vanishing of $F_{\alpha\beta}$ guarantees a choice of metric in which the Weyl geometry becomes

Riemannian; thus $F_{\alpha\beta}$ is an intrinsic geometric quantity for Weyl geometry - note $F_{\alpha\beta} = -F_{\beta\alpha}$ and

$$\{F_{\alpha\beta|\gamma}\} = 0;\ \{F_{\mu\nu|\lambda}\} = F_{\mu\nu|\lambda} + F_{\lambda\mu|\nu} + F_{\nu\lambda|\mu} \tag{D.13}$$

Similarly the concept of covariant differentiation depends only on the idea of vector transplantation. Indeed one can define

$$\xi^\alpha_{||\beta} = \xi^\alpha_{|\beta} - \Gamma^\alpha_{\ \beta\gamma}\xi^\gamma \tag{D.14}$$

In Riemann geometry the curvature tensor is

$$\xi^\alpha_{||\beta|\gamma} - \xi^\alpha_{||\gamma|\beta} = R^\alpha_{\eta\beta\gamma}\xi^\eta \tag{D.15}$$

Hence here we can write

$$R^\alpha_{\beta\gamma\delta} = -\Gamma^\alpha_{\ \beta\gamma|\delta} + \Gamma^\alpha_{\ \beta\delta|\gamma} + \Gamma^\alpha_{\ \tau\delta}\Gamma^\tau_{\ \beta\gamma} - \Gamma^\alpha_{\ \tau\gamma}\Gamma^\tau_{\ \beta\delta} \tag{D.16}$$

Using (D.11) one then can express this in terms of $g_{\alpha\beta}$ and ϕ_α but this is complicated. Equations for $R_{\beta\delta} = R^\alpha_{\beta\alpha\delta}$ and $R = g^{\beta\delta}R_{\beta\delta}$ are however given in [**12**]. One notes that in Weyl geometry if a vector ξ^α is given, independent of the metric, then $\xi_\alpha = g_{\alpha\beta}\xi^\beta$ will depend on the metric and under a gauge transformation one has $\hat{\xi}_\alpha = f(x^\lambda)\xi_\alpha$. Hence the covariant form of a gauge invariant contravariant vector becomes gauge dependent and one says that a tensor is of weight n if, under a gauge transformation $\hat{T}^{\alpha\cdots}_{\beta\cdots} = f(x^\lambda)^n T^{\alpha\cdots}_{\beta\cdots}$. Note ϕ_α plays a singular role in (D.12) and has no weight. Similarly $(\sqrt{-\hat{g}} = f^2\sqrt{-g}$ (weight 2) and $F^{\alpha\beta} = g^{\alpha\mu}g^{\beta\nu}F_{\mu\nu}$ has weight -2 while ,$\mathfrak{F}^{\alpha\beta} = F^{\alpha\beta}\sqrt{-g}$ has weight 0 and is gauge invariant; further $F_{\alpha\beta}F^{\alpha\beta}\sqrt{-g}$ is gauge invariant. Now for Weyl's theory of electromagnetism one wants to interpret ϕ_α as an EM potential and one has automatically the Maxwell equations $\{F_{\alpha\beta|\gamma}\} = 0$ along with a gauge invariant complementary set $\mathfrak{F}^{\alpha\beta}_{|\beta} = \mathfrak{s}^\alpha$ (source equations). These equations are gauge invariant as a natural consequence of the geometric interpretation of the EM field. For the interaction between the EM and gravitational fields one sets up some field equations as indicated in [**?**] and the interaction between the metric quantities and the EM fields is exhibited there.

REMARK 5.3.1. As indicated earlier in [**12**] $R^i_{\ jk}$ is defined with a minus sign compared with e.g. [**723, 998**]. There is also a difference in definition of the Ricci tensor which is taken to be $G^{\beta\delta} = R^{\beta\delta} - (1/2)g^{\beta\delta}R$ in [**12**] with $R = R^\delta_\delta$ so that $G_{\mu\gamma} = g_{\mu\beta}g_{\gamma\delta}G^{\beta\delta} = R_{\mu\gamma} - (1/2)g_{\mu\gamma}R$ with $G^\gamma_\eta = R^\eta_\eta - 2R \Rightarrow G^\eta_\eta = -R$ (recall $n = 4$). In [**723**] the Ricci tensor is simply $R_{\beta\mu} = R^\alpha_{\beta\mu\alpha}$ where $R^\alpha_{\beta\mu\nu}$ is the Riemann curvature tensor and $R = R^\eta_\eta$ again. This is similar to [**998**] where the Ricci tensor is defined as $\rho_{j\ell} = R^i_{ji\ell}$. To clarify all this we note that $R_{\eta\gamma} = R^\alpha_{\ \eta\alpha\gamma} = g^{\alpha\beta}R_{\beta\eta\alpha\gamma} = -g^{\alpha\beta}R_{\beta\eta\gamma\alpha} = -R^\alpha_{\ \eta\gamma\alpha}$ which confirms the minus sign difference.

For completeness it is worthwhile to reflect on the comments of a master craftsman and hence we refer her to [**302**] where first there are two papers on a new classical theory of the electron and in the third paper of [**302**] the original Dirac-Weyl action is developed (cf. also Sections 3.2.1 and 4.1) which we sketch here in some detail. The main point is to think of EM fields as a property of spacetime rather than something occuring in a gravity formed spacetime. This seems to be in the spirit of considering a microstructure of the vacuum (or an ether) and we find it attractive. The solution proposed by Weyl involved a length change $\delta\ell = \ell\kappa_\mu \delta x^\mu$ under parallel transport $x^\mu \to x^\mu + \delta x^\mu$. The κ_μ are field quantities occuring along with the $g_{\mu\nu}$ in a fundamental role. Suppose ℓ gets changed to $\ell' = \ell\lambda(x)$ and $\ell + \delta\ell$ becomes

$$\ell' + \delta\ell' = (\ell + \delta\ell)\lambda(x + \delta x) = (\ell + \delta\ell)\lambda(x) + \ell\lambda_{,\mu}\delta x^\mu \tag{D.17}$$

with neglect of second order terms (here $\lambda_{,\mu} \equiv \partial\lambda/\partial x^\mu$). Then

$$\delta\ell' = \lambda\delta\ell + \ell\lambda_{,\mu}\delta x^\mu = \lambda(\kappa_\mu + \phi_{,\mu})\delta x^\mu;\ \phi = log(\lambda) \tag{D.18}$$

Hence

$$\delta\ell' = \ell'\kappa'_\mu \delta x^\mu;\ \kappa'_\mu = \kappa_\mu + \phi_{,\mu} \tag{D.19}$$

If the vector is transported by parallel displacement around a small closed loop the total change in length is

$$\delta\ell = \ell F_{\mu\nu}\delta S^{\mu\nu};\ F_{\mu\nu} = \kappa_{\mu,\nu} - \kappa_{\nu,\mu} \tag{D.20}$$

and $\delta S^{\mu\nu}$ is the element of area enclosed by the small loop. this change is unaffected by (D.19). It will be seen that the field quantities κ_μ can be taken to be EM potentials, subject to the transformations (D.19) which correspond to no change in the geometry but a change only in the choice of artificial standards of length. The derived quantities $F_{\mu\nu}$ have a geometrical meaning independent of the length standard and correspond to the EM fields. Thus the Weyl geometry provides exactly what is needed for describing both gravitational and EM fields in geometric terms. There was at first some apparent conflict with atomic standards and the theory was rejected, leaving only the idea of gauge transformation for length standard changes.

Dirac's approach however helped to resurrect the Weyl theory; since we feel that this theory is not perhaps sufficiently appreciated a sketch is given here (cf. also [**121**]). Dirac first goes into a discussion of large numbers, e.g. e^2/GMm (proton and electron masses), e^2/mc^2 (age of universe), etc. and the Einsteinian theory requires that G be constant which seems in contradiction to $G \sim t^{-1}$ where t represents the epoch time, assumed to be increasing. Dirac reconciles this by assuming the large numbers hypothesis (all dimensionless large numbers are connected) and stipulating that the Einstein equations refer to an interval ds_E which is different from the interval ds_A measured by atomic clocks. Then the objections to Weyl's theory vanish and it is assumed to refer to ds_E. In this spirit then one deals with transformations of the metric gauge under which any length such as ds is multiplied by a factor $\lambda(x)$ depending on its position x, i.e. $ds' = \lambda ds$ and a localized quantity Y may get transformed according to $Y' = \lambda^n Y$, in which case Y is said to be of power n and is called a co-tensor. If $n = 0$ then Y is called an **in-tensor** and it is invariant under gauge transformations. The equation $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$ shows that $g_{\mu\nu}$ is a co-tensor of power 2, since the dx^μ are not affected by a gauge transformation. Hence $g^{\mu\nu}$ is a co-tensor of power -2 and one writes $\mathfrak{g}$ for $\sqrt{-g}$ with $T_{:\mu}$ denoting the covariant derivative ($\nabla_\mu T$ would be better). We see that the covariant derivative of a co-tensor is not generally a co-tensor. However there is a modifed covariant derivative $T_{*\mu}$ which is a co-tensor. Consider first a scalar S of power n; then $S_{:\mu} = S_{,\mu} \equiv S_\mu$; under a change of gauge it transforms to

$$S'_\mu = (\lambda^n S)_{,\mu} = \lambda^n S_\mu + n\lambda^{n-1}\lambda_\mu S = \lambda^n[S_\mu + n(\kappa'_\mu - \kappa_\mu)S] \tag{D.21}$$

(via (D.19)). Thus

$$(S_\mu - n\kappa_\mu S)' = \lambda^n(S_\mu - n\kappa_\mu S) \tag{D.22}$$

so $S_\mu - n\kappa_\mu S$ is a covector of power n and is defined to be the co-covariant derivative of S, i.e.

$$\S_{*\mu} = S_\mu - n\kappa_\mu S \tag{D.23}$$

To obtain the co-covariant derivative of co-vectors and co-tensors we need a modified Christoffel symbol

$${}^*\Gamma^\alpha_{\mu\nu} = \Gamma^\alpha_{\mu\nu} - g^\alpha_\mu \kappa_\nu - g^\alpha_\nu \kappa_\mu + g_{\mu\nu}\kappa^\alpha \tag{D.24}$$

(the notation $\Gamma^\alpha_{\mu\nu}$ for the more correct form $\Gamma^{\alpha}_{\ \mu\nu}$ is used in [**302**]). This is known to be invariant under gauge transformations. Let now A_μ be a co-vector of power n and form $A_{\mu,\nu} - {}^*\Gamma^\alpha_{\mu\nu}A_\alpha$ which is evidently a tensor since it differs from the covariant derivative $A_{\mu:\nu}$ by a tensor and under gauge transformations one has (cf. (D.19) where $\phi_{,\mu} = \kappa'_\mu - \kappa_\mu$)

$$\begin{aligned}(A_{\mu,\nu} - {}^*\Gamma^\alpha_{\mu\nu}A_\alpha)' &= \lambda^n A_{\mu,\nu} + n\lambda^{n-1}\lambda_\nu A_\mu - {}^*\Gamma^\alpha_{\mu\nu}\lambda^n A_\alpha = \\ &= \lambda^n[A_{\mu,\nu} + n(\kappa'_\nu - \kappa_\nu)A_\mu - {}^*\Gamma^\alpha_{\mu\nu}A_\alpha]\end{aligned} \tag{D.25}$$

Thus

$$(A_{\mu,\nu} - n\kappa_\nu A_\mu - {}^*\Gamma^\alpha_{\mu\nu}A_\alpha)' = \lambda^n[A_{\mu,\nu} - n\kappa_\nu A_\mu - {}^*\Gamma^\alpha_{\mu\nu}A_\alpha] \tag{D.26}$$

so take

$$A_{\mu*\nu} = A_{\mu,\nu} - n\kappa_\nu A_\mu - {}^*\Gamma^\alpha_{\mu\nu} \tag{D.27}$$

as the co-covariant derivative of A_α. this can be written via (D.24) as

$$A_{\mu*\nu} = A_{\mu:\nu} - (n-1)\kappa_\nu A_\mu + \kappa_\mu A_\nu - g_{\mu\nu}\kappa^\alpha A_\alpha \tag{D.28}$$

Similarly for a vector B^μ of power n one has

$$B^\mu_{*\nu} = B^\mu_{:\nu} - (n+1)\kappa_\nu B^\mu + \kappa^\mu B_\nu - g^\mu_\nu \kappa_\alpha B^\alpha \tag{D.29}$$

For a co-tensor with various suffixes up and down one can form the co-covariant derivative via the same rules; one notes that the co-covariant derivative always has the same power as the original. Next observe

$$(TU)_{*\sigma} = T_{*\sigma}U + TU_{*\sigma} \tag{D.30}$$

while

$$g_{\mu\nu*\sigma} = 0;\ G^{\mu\nu}_{*\sigma} = 0 \tag{D.31}$$

so one can raise and lower suffixes freely in a co-tensor before carrying out co-covariant differentiation. Thus one can raise the μ in (D.28) giving (D.29) with A^μ replacing B^μ and $n-2$ in place of n. The potentials κ_μ do not form a co-vector because of the wrong transformation laws (D.19) but the $F_{\mu\nu}$ defined by (D.19) are unaffected by gauge transformations so they form an in-tensor. One obtains the co-covariant divergence of a co-vector B^μ by putting $\nu = \mu$ in (D.29) to get

(D.32) $$B^\mu_{*\mu} = B^\mu_{:\mu} - (n+4)\kappa_\mu B^\mu$$

(for $n = -4$ this is the ordinary covariant divergence).

We list some formulas for second co-covariant derivatives now with a sketch of derivation. Thus for a scalar of power n

(D.33) $$S_{*\mu*\nu} = S_{*\mu:\nu} - (n-1)\kappa_\nu S_{*\mu} + \kappa_\mu S_{*\nu} - g_{\mu\nu}\kappa^\sigma S_{*\sigma}$$

Putting $S_{*\mu} = S_\mu - n\kappa_\mu S$ on gets

(D.34)
$$S_{*\mu*\nu} = S_{\mu:\nu} - n\kappa_{\mu:\nu} - n\kappa_\mu S_\nu - n\kappa_\nu(S_\mu - n\kappa_\mu S) + \kappa_\nu S_{*\mu} + \kappa_\mu S_{*\nu} - g_{\mu\nu}\kappa^\sigma S_\sigma$$

Now $S_{\mu:\nu} = S_{\nu:\mu}$ so

(D.35) $$S_{*\mu*\nu} - S_{*\nu*\mu} = -n(\kappa_{\mu:\nu} - \kappa_{\nu:\mu})S = -nF_{\mu\nu}S$$

This is tedious but instructive and we continue. Let A_μ be a co-vector of power n so

(D.36) $$A_{\mu*\nu*\sigma} = A_{\mu*\nu:\sigma} - n\kappa_\sigma A_{\mu*\nu} + (g^\rho_\mu\kappa_\sigma + g^\rho_\sigma\kappa_\mu - g_{\mu\sigma}\kappa^\rho)A_{\rho*\nu} + \\ +(g^\rho_\nu\kappa_\sigma + g^\rho_\sigma\kappa_\nu - g_{\sigma\nu}\kappa^\rho)A_{\mu*\rho}$$

A lengthy calculation then yields

(D.37) $$A_{\mu*\nu*\sigma} - A_{\mu*\sigma*\nu} = {}^*\mathfrak{B}_{\mu\nu\sigma\rho}A^\rho - (n-1)F_{\nu\sigma}A_\mu$$

where

(D.38)
$${}^*\mathfrak{B}_{\mu\nu\sigma\rho} = B_{\mu\nu\sigma\rho} + g_{\rho\nu}(\kappa_{\mu:\sigma} + \kappa_\mu\kappa_\sigma) + g_{\mu\sigma}(k_{\rho:\nu} + \kappa_\rho\kappa_\nu) - g_{\rho\sigma}(\kappa_{\mu:\nu} + \kappa_\mu\kappa_\nu) - \\ -g_{\mu\nu}(\kappa_{\rho:\sigma} + (\kappa_\rho\kappa_\sigma) + (g_{\rho\sigma}g_{\mu\nu} - g_{\rho\nu}g_{\mu\sigma})\kappa^\alpha\kappa_\alpha$$

One can consider ${}^*\mathfrak{B}$ as a generalized Riemann-Christoffel tensor but it does not have the usual symmetry properties for such a tensor; however one can write

(D.39) $${}^*\mathfrak{B}_{\mu\nu\sigma\rho} = {}^*B_{\mu\nu\sigma\rho} + (1/2)(g_{\rho\nu}F_{\mu\sigma} + g_{\mu\sigma}F_{\rho\nu} - g_{\rho\sigma}F_{\mu\nu} - g_{\mu\nu}F_{\rho\sigma})$$

and then ${}^*B_{\mu\nu\sigma\rho}$ has all the usual symmetries, namely

(D.40)
$${}^*B_{\mu\nu\sigma\rho} = -{}^*B_{\mu\sigma\nu\rho} = -{}^*B_{\rho\nu\sigma\mu} = {}^*B_{\nu\mu\rho\sigma};\ {}^*B_{\mu\nu\sigma\rho} + {}^*B_{\mu\sigma\rho\nu} + {}^*B_{\mu\rho\nu\sigma} = 0$$

Thus is is appropriate to call ${}^*B_{\mu\nu\sigma\rho}$ the Riemann-Christoffel (RC) tensor for Weyl space; it is a co-tensor of power 2. The contracted RC tensor is

(D.41) $${}^*R_{\mu\nu} = {}^*B^\sigma_{\mu\nu\sigma} = R_{\mu\nu} - \kappa_{\mu:\nu} - \kappa_{\nu:\mu} - g_{\mu\nu}\kappa^\sigma_{:\sigma} - 2\kappa_\mu\kappa_\nu + 2g_{\mu\nu}\kappa^\sigma\kappa_\sigma$$

and is an in-tensor. A further contraction gives the total curvature

(D.42) $${}^*R = {}^*R^\sigma_\sigma = R - 6\kappa^\sigma_{:\sigma} + 6\kappa^\sigma_\sigma$$

which is a co-scalar of power -2.

One gets field equations from an action principle with an in-invariant action, hence one of the form $I = \int \Omega \mathfrak{g} d^4x$ where Ω must be a co-scalar of power -4 to compensate $\mathfrak{g}$ having power 4. Ths usual contribution to Ω from the EM field is $(1/4)F_{\mu\nu}F^{\mu\nu}$ (of power -4 since it can be written as $F_{\mu\nu}F_{\rho\sigma}g^{\mu\rho}g^{\nu\sigma}$ with F factors of power zero and g factors of power -2). One also needs a gravitational term and the standard $-R$ could be ${}^*R$ but this has power -2 and will not do. Weyl proposed $({}^*R)^2$ which has the correct power but seems too complicated to be satisfactory. Here one takes ${}^*R = 0$ as a constraint and puts the constraint into the Lagrangian via γ^*R with γ a co-scalar field of power -2 in the form of a Lagrange multiplier. This leads to a scalar-tensor theory of gravitation and one can insert other terms involving γ. For convenience one takes $\gamma = -\beta^2$ with β as the basic field variable (co-scalar of power -1) and adds terms $k\beta^{*\sigma}\beta_{*\sigma}$ (co-scalar of power -4); terms $c\beta^4$ can also be added to get

$$I = \int [(1/4)F_{\mu\nu}F^{\mu\nu} - \beta^{2*}R + k\beta^{*\mu}\beta_{*\mu} + c\beta^4]\mathfrak{g}d^4x \tag{D.43}$$

as a vacuum action. Now $\beta^{*\mu}\beta_{*\mu} = (\beta^\mu + \beta\kappa^\mu)(\beta_\mu + \beta\kappa_\mu)$ and using (D.42) one obtains

$$-\beta^{2*}R + k\beta^{*\mu}\beta_{*\mu} = -\beta^2 R + k\beta^\mu\beta_\mu + (k-6)\beta^2\kappa^\mu\kappa_\mu + \\ +6(\beta^2\kappa^\mu)_{:\mu} + (2k-12)\beta\kappa^\mu\beta_\mu \tag{D.44}$$

The term involving $(\beta^2\kappa^\mu)_{:\mu}$ can be discarded since its contribution to the action density is a perfect differential, namely $(\beta^2\kappa^\mu)_{:\mu}\mathfrak{g} = (\beta^2\kappa^\mu\mathfrak{g})_{,\mu}$ and for the simplest vacuum equations one chooses $k = 6$ so that (D.43) becomes

$$I = \int [(1/4)F_{\mu\nu}F^{\mu\nu} - \beta^2 R + 6\beta^\mu\beta_\mu + c\beta^4]\mathfrak{g}d^4x \tag{D.45}$$

Thus I no longer involves the κ_μ directly but only via $F_{\mu\nu}$ and I is invariant under transformations $\kappa_\mu \to \kappa_\mu + \phi_{,\mu}$ so the equations of motion that follow from the action principle will be unaffected by such transformations (i.e. they have no physical significance). Now consider three kinds of transformation:

(1) Any transformation of coordinates.
(2) Any transformation of the metric gauge combined with the appropriate transformation of potentials $\kappa_\mu \to \kappa_\mu + \phi_{,\mu}$.
(3) In the vacuum one may make a transformation of potentials as above without changing the metric gauge or alternatively one may transform the metric gauge without changing the potentials. This works only where there is no matter.

For the field equations one makes small variations in all the field quantities $g_{\mu\nu}$, κ_μ, and β, calculates the change in I and sets it equal to zero. Thus write

$$\delta I = \int [(1/2)P^{\mu\nu}\delta g_{\mu\nu} + Q^\mu\delta\kappa_\mu + S\delta\beta)\mathfrak{g}d^4x \tag{D.46}$$

and drop the $c\beta^4\mathfrak{g}$ term since it is probably only of interest for cosmological purposes. One has

$$\delta[(1/4)F_{\mu\nu}F^{\mu\nu}\mathfrak{g}] = (1/2)E^{\mu\nu}\mathfrak{g}\delta g_{\mu\nu} - J^\mu\mathfrak{g}\delta\kappa_\mu \tag{D.47}$$

with neglect of a perfect differential. Here $E^{\mu\nu}$ is the EM stress tensor

(D.48)
$$E^{\mu\nu} = (1/4)g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta} - F^{\mu\alpha}F^{\nu}_{\alpha}$$

and J^{μ} is the charge current vector

(D.49)
$$F^{\mu} = F^{\mu\nu}_{:\nu} = \mathfrak{g}^{-1}(F^{\mu\nu}\mathfrak{g})_{,\nu}$$

Considerable calculation and neglect of perfect differentials leads finally to

(D.50)
$$P^{\mu\nu} = E^{\mu\nu} + \beta^2[2R^{\mu\nu} - g^{\mu\nu}R] - 4g^{\mu\nu}\beta\beta^{\rho}_{:\rho} + 4\beta\beta^{\mu:\nu} + 2g^{\mu\nu}\beta^{\sigma}\beta_{\sigma} - 8\beta^{\mu}\beta^{\nu};$$
$$Q^{\mu} = -J^{\mu};\ S = -2\beta R - 12\beta^{\mu}_{:\mu}$$

and the field equations for the vacuum are

(D.51)
$$P^{\mu\nu} = 0,\ Q^{\mu} = 0;\ S = 0$$

These are not all independent since

(D.52)
$$P^{\sigma}_{\sigma} = -2\beta^2 R - 12\beta\beta^{\sigma}_{:\sigma} = \beta S$$

so the S equation is a consequence of the P equations. If one omits the EM term from the action it becomes the same as the Brans-Dicke action except that the latter allows an arbitrary value for k; with $k \neq 6$ the vacuum equations are independent so the BD theory has one more vacuum field equation, namely $\Box(\beta^2) = 0$.

Now the action integral is invariant under transformations of the coordinate sysem and transformations of gauge; each of these leads to a conservation law connecting the quantities $P^{\mu\nu}$, Q^{μ}, S defined via (D.46). For coordinate transformations $x^{\mu} \to x^{\mu} + b^{\mu}$ one gets

(D.53)
$$-\delta g_{\mu\nu} = g_{\mu\sigma}b^{\sigma}_{,\nu} + g_{\nu\sigma}b^{\sigma}_{,\mu} + g_{\mu\nu,\sigma}b^{\sigma};\ -\delta\beta = \beta_{\sigma}b^{\sigma};\ -\delta\kappa_{\mu} = \kappa_{\sigma}b^{\sigma}_{,\mu} + \kappa_{\mu,\sigma}b^{\sigma}$$

Putting these variations in (D.46) yields

(D.54)
$$\delta I = -\int[(1/2)P^{\mu\nu}(g_{\mu\sigma}b^{\sigma}_{,\nu} + g_{\nu\sigma}b^{\sigma}_{,\mu} + g_{\mu\nu,\sigma}b^{\sigma}) +$$
$$+Q^{\mu}(\kappa_{\sigma}b^{\sigma}_{,\mu} + \kappa_{\mu,\sigma}b^{\sigma}) + S\beta_{\sigma}b^{\sigma}]\mathfrak{g}d^4x =$$
$$= \int[(P^{\mu}_{\sigma}\mathfrak{g})_{,\mu} - (1/2)P^{\mu\nu}g_{\mu\nu,\sigma}\mathfrak{g} + (Q^{\mu}\kappa_{\sigma}\mathfrak{g})_{,\mu} - Q^{\mu}\kappa_{\mu,\sigma}\mathfrak{g} - S\beta_{\sigma}\mathfrak{g}]b^{\sigma}d^4x$$

This δI vanishes for arbitrary b^{σ} so one puts the coefficient of b^{σ} equal to zero; using

(D.55)
$$(P^{\mu}_{\sigma}\mathfrak{g})_{,\mu} - (1/2)P^{\mu\nu}g_{\mu\nu,\sigma}\mathfrak{g} = P^{\mu}_{\sigma:\mu}\mathfrak{g};\ (Q^{\mu}\kappa_{\sigma}\mathfrak{g})_{,\mu} = \kappa_{\sigma}Q^{\mu}_{:\mu}\mathfrak{g} + \kappa_{\sigma,\mu}Q^{\mu}\mathfrak{g}$$

this reduces to

(D.56)
$$P^{\mu}_{\sigma:\mu} + \kappa_{\sigma}Q^{\mu}_{:\mu} + F_{\sigma\mu}Q^{\mu} - S\beta_{\sigma} = 0$$

Next consider a small transformation in gauge

(D.57)
$$\delta g_{\mu\nu} = 2\lambda g_{\mu\nu},\ \delta\beta = -\lambda\beta;\ \delta\kappa_{\mu} = [log(1+\lambda)]_{,\mu} = \lambda_{\mu}$$

Putting this in (D.46) yields

$$\text{(D.58)}\quad \delta I = \int [P^{\mu\nu}\lambda g_{\mu\nu} + Q^{\mu}\lambda_{\mu} - S\lambda\beta]\mathfrak{g}d^4x = \int [P^{\mu}_{\mu}\mathfrak{g} - (Q^{\mu}\mathfrak{g})_{,\mu} - S\beta\mathfrak{g}]\lambda d^4x$$

Putting the coefficient of λ equal to zero gives $P^{\mu}_{\mu} - Q^{\mu}_{:\mu} - S\beta = 0$ which with (D.56) comprise the conservation laws. For the vacuum one sees that (D.58) is the same as (D.52) since $Q^{\mu}_{:\mu} = 0$ from (D.50); also (D.56) reduces to

$$\text{(D.59)}\qquad P^{\mu}_{\sigma:\mu} + F_{\sigma\mu}Q^{\mu} - \beta^{-1}\beta_{\sigma}P^{\mu}_{\mu} = 0$$

which may be considered as a generalization of the Bianchi identities. The conservation laws (D.56) and (D.58) hold more generally than for the vacuum, namely whenever the action integral can be constructed from the field variables $g_{\mu\nu}$, κ_{μ}, β alone.

Now let the coordinates of a particle be z^{μ}, functions of the proper time s measured along its world line. Put $dz^{\mu}/ds = v^{\mu}$ for velocity so $v_{\mu}v^{\mu} = 1$ and v^{μ} is a co-vector of power -1. One adds to the action the further terms

$$\text{(D.60)}\qquad I_1 = -m\int \beta ds;\ I_2 = e\int \beta^{-1}\beta_{*\mu}v^{\mu}ds$$

(m and e being constants). Then these terms are in-invariants with

$$\text{(D.61)}\qquad I_2 = e\int(\beta^{-1}\beta_{\mu} + \kappa_{\mu})v^{\mu}ds = e\int[(d/ds)(log(\beta)) + \kappa_{\mu}v^{\mu}]ds$$

and the first term contributes nothing to the action principle. Thus $I_2 = e\int \kappa_{\mu}v^{\mu}ds$ which is unchanged when $\kappa_{\mu} \to \kappa_{\mu} + \phi_{,\mu}$ since the extra term is $e\int(d\phi/ds)ds$. Thus for a particle with action $I_1 + I_2$ the transformations (3) above are still possible. Now some calculation yields

$$\text{(D.62)}\qquad m[g_{\mu\sigma}d(\beta v^{\mu})/ds + \beta\Gamma_{\sigma\mu\nu}v^{\mu}v^{\nu} - \beta_{\sigma}] =$$
$$= -ev^{\mu}F_{\mu\sigma} \equiv m[d(\beta v^{\mu})/ds + \Gamma^{\mu}_{\rho\sigma}v^{\rho}v^{\sigma} - \beta^{\mu}] = eF^{\mu\nu}v_{\nu}$$

This is the equation of motion for a particle of mass m and charge e; if $e = 0$ it could be called an in-geodesic. If one works with the Einstein gauge then the case $e = 0$ gives the usual geodesic equation. Next one considers the influence the of particle on the field and this is done by generating a dust of particles and a continuous fluid leading to an equation

$$\text{(D.63)}\qquad \rho[(\beta v^{\mu})_{,\nu} + \Gamma^{\mu}_{\alpha\sigma}v^{\alpha}v^{\sigma} - \beta^{\mu}] = \sigma^{\mu\nu}$$

where ρ and σ refer to mass and charge density respectively.

APPENDIX E

BICONFORMAL GEOMETRY

We sketch here some beautiful work of J. Wheeler [**992, 994**] which has led to a number of important developments in mathematical physics (cf. also [**35, 36, 987, 980, 990, 991, 993**]). The paper [**992**] (not published) gives a very nice discussion of normal biconformal spaces and lays some of the foundation for some later work of Wheeler et al for which [**994**] is apparently the best starting point. We will therefore extract here from [**994**] and remark that this work alone transcends earlier approaches to unifying GR and EM. One works over an 8-D base space where in a flat situation the 4-D biconformal cospace to 4-D Minkowski space corresponds to a standard tangent space (or momentum space) with variables p_μ transforming via L^{-1} under Lorentz transformations L of x^μ (or dx^μ). The symplectic form given by the exterior derivative of the Weyl 1-form provides a typical Hamiltonian dynamical structure and one gives general necessary and sufficient conditions for curved 8-D geometry to be in 1-1 correspondence with 4-D Einstein-Maxwell spacetime; further a consistent unified geometrical theory of gravity and electromagnetism is obtained. This is very powerful stuff and we can only sketch a few items here. Some connections of biconformal geometry to QM are given in Section 3.5.1 and we refer to [**987, 994**] for history and philosophy.

The conformal group is the most general set of transformations preserving ratios of infinitesimal lengths. On a 4-D spacetime this group is 15 dimensional, with Lorentz transformations (6), translations (4), 4 inverse translations (special conformal transformations), and dilations (1). One concentrates on flat situations and develops biconformal structure as a conformal fiber bundle. This is constructed as the quotient $\mathcal{C}/\mathcal{C}_0$ of the conformal group by its isotropy subgroup (7-D homogeneous Weyl group) producing a conformal Cartan connection on an 8-D manifold; this is then generalized to a curved 8-D manifold with the 7-D homogeneous Weyl group as fiber by the addition of horizontal curvature 2-forms to the group structure equations. The resulting 8-D base manifold is called a biconformal space and the full 15-D fiber bundle the biconformal bundle. We will try to illustrate this via the equations. One uses the $O(4,2)$ representation of the conformal group for notation where $(A, B, \cdots) = (0, 1, \cdots, 5)$. Then the $O(4.2)$ metric is $\eta_{ab} = diag(1,1,1,-1)$ $(a, b = 1, \cdots, 4)$ with $\eta_{05} = \eta_{50} = 1$ and all other components are zero. Introducing a connection 1-form ω_B^A one has covariant constancy via

$$D\eta_{AB} = d\eta_{AB} - \eta_{CB}\omega_A^C - \eta_{AC}\omega_B^C = 0 \tag{E.1}$$

The conformal connection may be broken into 4 independent Weyl invariant parts, the spin connection ω_b^a, the solder form ω_0^a, the co-solder form ω_a^0, and the Weyl vector ω_0^0 where the spin connection satisfies $\omega_b^a = -\eta_{bc}\eta^{ad}\omega_d^c$ and the remaining components of ω_{AB} are related via

$$\omega_0^5 = \omega_5^0 = 0;\ \omega_5^5 = -\omega_0^0;\ \omega_5^a = -\eta^{ab}\omega_b^0;\ \omega_a^5 = -\eta_{ab}\omega_0^b \tag{E.2}$$

These constraints reduce the number of independent connection forms ω_B^A to the required 15 and one can run $A, B\cdots$ from 0 to 4 (with 5 implicit). The structure constants of the conformal Lie algebra now lead immediately to the Maurer-Cartan (MC) equations of the conformal group as $d\omega_B^A = \omega_B^C\omega_C^A$ (wedge product is asssumed) or written out

$$d\omega_b^a = \omega_b^c\omega_c^a + \omega_b^0\omega_0^a - \eta_{bc}\eta^{ad}\omega_d^0\omega_0^c; \tag{E.3}$$

$$d\omega_0^a = \omega_0^0\omega_0^a + \omega_0^b\omega_b^a;\ d\omega_a^0 = \omega_a^0\omega_0^0 + \omega_a^b\omega_b^0;\ d\omega_0^0 = \omega_0^g o_a^0$$

Note that d in (E.2) includes partial derivatives in all eight of the base space directions and when using coordinates one will write (x^μ, y_ν) corresponding to index positions on (ω_0^b, ω_b^0). Also $\partial_\mu\phi = \partial\phi/\partial x^\mu$ while $\partial^\mu\phi = \partial\phi/\partial y_\mu$. The generalization of (E.3) to a curved base space is obtained via

$$d\omega_b^a = \omega_b^c\omega_c^a + \omega_b^0\omega_0^a - \eta_{bc}\eta^{ad}\omega_d^0\omega_0^c + \Omega_b^a = \omega_b^c\omega_c^a + \Delta_{bc}^{da}\omega_d^0\omega_0^c + \Omega_b^a; \tag{E.4}$$

$$d\omega_0^a = \omega_0^0\omega_0^a + \omega_0^b\omega_b^a + \Omega_0^a;\ d\omega_a^0 = \omega_a^0\omega_0^0 + \omega_a^b\omega_b^0 + \Omega_a^0;\ d\omega_0^0 = \omega_0^a\omega_a^0 + \Omega_0^0$$

One calls the four types of curvature Ω_b^a, Ω_0^a, Ω_a^0, Ω_0^0 the Riemann curvature, torsion, co-torsion, and dilational curvature respectively. Horizontality requires each of the curvatures to take the form

$$\Omega_B^A = (1/2)\Omega_{Bcd}^A\omega_0^c\omega_0^d + \Omega_{Bd}^{Ac}\omega_0^d\omega_c^0 + (1/2)\Omega_B^{Acd}\omega_c^0\omega_d^0 \tag{E.5}$$

The connection of a flat biconformal space is in the standard flat form when written as

$$\omega_0^0 = \alpha_a(x)dx^a - y_a dx^a \equiv W_a dx^a;\ \omega_0^a = dx^a; \tag{E.6}$$

$$\omega_a^0 = dy_a - (\alpha_{a,b} + W_aW_b - (1/2)W^2\eta_{ab})dx^b;\ \omega_b^a = (\eta^{ac}\eta_{bd} - \delta_d^a\delta_b^c)W_c dx^d$$

Note that the Weyl vector $W_a = \alpha_a(x) - y_a$ depends on an arbitrary 4-vector α_a and also on the 4 coordinates y_a; the presence of α_a gives the generality required for the EM vector potential while the y_a keeps the dilational curvature zero. In general the dilational gauge vector of biconformal space is of the form

$$\omega_0^0 = \omega_{0\mu}^0(x,y)dx^\mu + \omega_0^{0\mu}(x,y)dy_\mu \tag{E.7}$$

(i.e. an 8-D vector field depending on 8 independent variables) but constraining the biconformal geometry to have vanishing curvatures forces $\omega_0^0 = (\alpha_\mu(x) - y_\mu)dx^\mu$ which is precisely the form required to give the Lorentz force law. Then one proves in [**994**] that when the curvatures of biconformal space $\Omega_B^A = 0$ there exist global coordinates $x^a, y_a)$ such that the connection takes the standard flat form.

Note in (E.6) if one holds the y coordinates fixed then equations 1,2,4 are the connection forms for a 4-D Weyl spacetime with conformally flat metric η_{ab}; the remaining equation 3 is then simply a 1-form constructed from the Weyl-Ricci tensor. However the dilational curvature of a 4-D Weyl geometry is given by the curl of the Weyl vector, equivalent here to the curl of the arbitrary α_a, and viewed from the Weylian 4-D perspective the solution gives unphysical size change; it is only with the inclusion of the additional momentum variables proportional to y_a that the dilational curvature can be seen to vanish. It is thus seen that the actual motion of a particle in biconformal space is 8 dimensional and one can in fact interpret biconformal space, and therefore conformal gauge theory, as a generalization of phase space (with symplectic structure). Indeed for a given Hamiltonian system one can specify a unique flat biconformal space by judicious choice of the solder and co-solder forms (see [**994**] for details); in particular the extra 4 dimensions are identified with momenta and the integral of the Weyl vector is identified with action. In order to make a full identification of a biconformal space with an Einstein-Maxwell spacetime one can proceed as follows. The idea is that the solder form should satisy the Einstein equations with arbitrary matter as source and the vector potential obtained via $\alpha_a = q(\phi, -A_i) = -qA_a$ should satisfy the Maxwell field equations with arbitrary EM currents. One assumes for example that the torsion is zero (but not the cotorsion) leading to constraints $\Omega^0_0 = 0$ and $\Omega^a_0 = 0$ and, setting $\omega^a_0 = e^a$, one wants a completion f_a to the e^a basis in which $\Omega^a_{bac} = 0$. Further one posits two field equations $*d*d\omega^0_0 = J = J_a(x)e^a$ and $\omega^0_a = T_a + \cdots$ where $T_a = -(1/2)(T_{ab} - (1/3)\eta_{ab}T)e^b$. This can all be achieved and leads to the general identification stated above (cf. [**994**]).

Now to set the stage for [**35**] and connections to QM we gather the material from the appendices to [**35**] (where the formulation is different). The conformal group generators include Lorentz transformations $M^a_b = -M_{ba}\eta_{ac}M^c_b$, translations P_a, special conformal transformations K^a, and dilatations D satisfying the commutation relations

$$[M^a_{\ b}, M^c_{\ d}] = -(\delta^c_b M^a_{\ d} + \eta_{df}\eta^{ac}M^f_{\ b} + \eta_{bd}\eta^{ae}M^c_{\ e} - \delta^a_d M^c_{\ b}); \tag{E.8}$$

$$[M^a_{\ b}, P_c] = -(\eta_{cb}\eta^{ad}P_d - \delta^a_c P_b);\ [M^a_{\ b}, K^d] = -(\delta^d_b\delta^a_c - \eta^{ad}\eta_{bc})D^c;$$

$$[P_a, K^b] = 2M^b_{\ a} - 2\delta^b_a D;\ [D, K^b] = K^b;\ [D, P_a] = -P_a$$

(note dilatation corresponds to dilation and both terms seem to be popular). The conformal Lie algebra has two independent involutive automorphisms; the first is

$$\sigma_1 :\ (M^a_b, P_a, K^a, D) \to (M^a_b, -P_a, -K^a, D) \tag{E.9}$$

and this identifies the invariant subgroup used as the isotropy subgroup in the bicomformal gauging. The second is

$$\sigma_2 :\ (M^a_b, P_a, K^a, D) \to (M^a_b, -\eta_{ab}K^b, -\eta^{ab}P_b, -D) \tag{E.10}$$

and may be chosen to be complex conjugation in order to define what are called σ_C representations of the algebra. Thus if we assume the generators to be complex, σ_C representations have P_a and K_a as complex conjugates, while M^a_b is real and D pure imaginary. As an illustration note that while both $so(3)$ and $su(2)$ have involutive automorphisms the existence of a σ_C representation singles out $su(2)$. Thus while

(E.11) $$[J_i, J_j] = \epsilon_{ijk} J_k;\ [\tau_i, \tau_j] = \epsilon_{ijk}\tau_k$$

are both invariant under

(E.12) $$\rho:\ (J_1, J_2, J_3) \to (-J_1, J_2, -J_3);\ (\tau_1, \tau_2, \tau_3) \to (-\tau_1, \tau_2, -\tau_3)$$

where $[J_j]_{ik} = \epsilon_{ijk}$ and $\tau_j = -(i/2)\sigma_j$ (σ_j being the Pauli matrices - cf. (B.25)) it is only with the complex representation that $\rho = \rho_C:\ \overline{(\tau_1, \tau_2, \tau_3)} = (-\tau_1, \tau_2, -\tau_3) = \rho(\tau_1, \tau_2, \tau_3)$. As examples of conformal representations with this property first consider the covering group $SU(2,2)$, whose Lie algebra is isomorphic to that of $O(4,2)$. Due to the local isomorphism between $Spin(4,2)$ and $SU(2,2)$ this algebra can be represented via spinors. Thus using 4×4 Dirac matrices γ^a one can write $su(2,2)$ via

(E.13) $$\{\gamma^a, \gamma^b\} = 2\eta^{ab} = 2diag(-1, 1, 1, 1)\ \ (a, b = 0, , 1, 2, 3)$$

One also defines

(E.14) $$\sigma^{ab} = -(1/8)[\gamma^a, \gamma^b];\ \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3$$

where the full Clifford algebra has the basis

(E.15) $$\Gamma = \{1, i1, \gamma^a, \sigma^{ab}, i\sigma^{ab}, \gamma_5\gamma^a, i\gamma_5\gamma^a, \gamma_5, i\gamma_5\}$$

The conformal Lie algebra may be obtained from this set be demanding invariance of a spinor metric Q given by $Q = i\gamma^0$. Then if one requires $Q\Gamma + \Gamma^\dagger Q = 0$ the generators of the conformal Lie algebra are (cf. [**35**])

(E.16)
$$M^a_{\ b} = \eta_{bc}\sigma^{ac};\ P_a = (1/2)\eta_{ab}(1+\gamma_5)\gamma^b;\ K^a = (1/2)(1-\gamma_5)\gamma^a;\ D = -(1/2)\gamma_5$$

Choosing any real representation for the Dirac matrices γ_5 is necessarily imaginary and it follows that

(E.17) $$\bar{M}^a_{\ b} = M^a_{\ b};\ \bar{P}_a = \eta_{ab}K^b;\ \bar{D} = -D$$

so the action of σ_C is realized. Alternatively one may consider a complex function space representation of the conformal algebra via

(E.18) $$M^\mu_\nu = -\frac{1}{2}\left(z^\mu\frac{\partial}{\partial z^\nu} + \bar{z}^\mu\frac{\partial}{\partial \bar{z}^\nu} - z_\nu\frac{\partial}{\partial z_\mu} - \bar{z}^\nu\frac{\partial}{\partial \bar{z}^\mu}\right);$$

$$D = z^\mu\frac{\partial}{\partial z^\mu} - \bar{z}^\nu\frac{\partial}{\partial \bar{z}^\nu};\ P_\mu = \frac{\partial}{\partial z^\mu} + \left(\bar{z}_\mu\bar{z}^\nu - \frac{1}{2}\bar{z}^2\delta^\nu_\mu\right)\frac{\partial}{\partial \bar{z}^\nu};$$

$$K_\mu = \frac{\partial}{\partial \bar{z}^\mu} + \left(z_\mu z^\nu - \frac{1}{2}z^2\delta^\nu_\mu\right)\frac{\partial}{\partial z^\nu}$$

Note the generators are complex but the group manifold is real. In either of these representations the Maurer-Cartan (MC) equations inherit the same symmetry under σ_C and in particular the gauge vector of dilatations (the Weyl vector) is imaginary. To clarify this one shows that the dilatations generated by an imaginary

D nevertheless give a real factor as expected. Thus first consider $su(2,2)$ with a basis of Dirac matrices in which

$$D = -\frac{1}{2}\gamma_5 = -\frac{1}{2}\begin{pmatrix} -\sigma_y & \\ & \sigma_y \end{pmatrix};\ \sigma_y = \begin{pmatrix} & -i \\ i & \end{pmatrix} \tag{E.19}$$

Define the definite conformal weight spinors χ^A, ψ^B via

$$D\chi = (1/2)\chi;\ D\psi = -(1/2)\psi;\ e^{\lambda D}\chi = e^{\lambda/2}\chi;\ e^{\lambda D}\psi = e^{-\lambda/2}\psi \tag{E.20}$$

For the complex function space representation of the conformal group (E.18) one has (for one variable $z = rexp(i\phi)$)

$$D \sim -i\frac{\partial}{\partial\phi} \tag{E.21}$$

so D measures the phase of a complex number. Homogeneous functions of z and $\bar{z}$ are then eigenfuncitons and D measures the degree of homogeneity. Thus if $f(z,\bar{z}) = z^a\bar{z}^b$ there results $exp(\lambda D)f(z,\bar{z}) = exp[(a-b)\lambda]f(z,\bar{z})$ so there are dilatations with the weight of the function encoded into the total phase. Similarly in multiple complex dimensions eigenfunctions can be built up from powers of the norms

$$f_{\alpha-\beta} = (\sqrt{z^2})^\alpha(\sqrt{\bar{z}^2})^\beta;\ Df_{\alpha-\beta} = D(z^2)^{\alpha/2}(\bar{z}^2)^{\beta/2} = (\alpha-\beta)f_{\alpha-\beta} \tag{E.22}$$

Note that $z^a\bar{z}_a$ is of weight zero with $D(z^a\bar{z}_a) = 0$.

Gauge transformations will remain real even though there is a complex valued connection. A local gauge transformation is given via

$$\Lambda = M^a_b\Lambda^b{}_a + D\Lambda^0 \tag{E.23}$$

Note Λ is complex, since Λ^b_a, Λ^0 are real parameters used to exponentiate the generators M (real) and D (imaginary), and it follows that a gauge transformation of the Weyl vector is $\delta\omega = -d\Lambda^0$ where Λ^0; one can then define a scale covariant derivative of a definite weight scalar field via

$$Df = df + k\omega f \tag{E.24}$$

where k is the conformal weight of f. To see that this is a gauge invariant expression one takes a dilatational gauge transformation

$$f' = fexp(k\Lambda^0);\ \omega' = \omega + \delta\omega = \omega - d\Lambda^0 \tag{E.25}$$

which implies

$$D'f' = d(fexp(k\Lambda^0)) + k(\omega - d\Omega^0)f = exp(k\Lambda^0)Df \tag{E.26}$$

Thus the equation is covariant and the MC structure equations are invariant under real scalings. Of course in generic gauges the Weyl vector is complex but the invariance fo the structure equations under gauge transformations guarantees consistency. Note also that whether the Weyl vector is complex or pure imaginary $exp(\oint\omega)$ remains a pure phase since the Weyl vector is pure imaginary in at least one gauge and the above expression is gauge invariant. Finally one writes the Cartan structure equations for flat σ_C biconformal space via

$$d\omega^a_b = \omega^c_b\omega^a_c + 2\omega_b\omega^a;\ d\omega^a = \omega^c\omega^a_c + \omega\omega^a;\ d\omega = 2\omega^a\omega_a \tag{E.27}$$

(and their conjugates); here ω^a corresponds to translation generators, ω is the Weyl vector, and ω^a_b is the spin connection. A first order perturbative solution is

$$\omega^a_b(\delta^a_e\eta_{cb} - \delta^a_c\eta_{eb})x^c dx^e + (\delta^a_e\eta_{cb} - \delta^a_c\eta_{eb})y^c dy^e; \tag{E.28}$$

$$\omega = i(y_a dx^a - x_a dy^a);\ \omega^a = \{dx^a + idy^a +$$

$$+\left(-\frac{1}{2}x^a x_e + \frac{i}{2}(\delta^a_e x_c y^c - x^a y_e) + \frac{1}{2}y^a y_e\right)(dx^e - idy^e)\}$$

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