Multibody dynamics in advanced education

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The use of multibody dynamics software in industrial firms is steadily increasing. Engineering curricula often do not include mandatory courses where multibody dynamics is taught. This paper discuss some relevant teaching issues in this specific cultural area. Noteworthy examples of the great variety of kinematic and dynamic formulations, available for teaching a basic course in multibody dynamics, are illustrated. The experience of the first author, when introducing a basic multibody dynamics course at an Italian university, is also reported.

1 Introduction

In 1999 the Education Ministers of 29 European countries signed in Bologna the document known as *Bologna Agreement*. The document was preceded by the *Sorbonne Agreement* signed in May 1998 and outlining cooperation between European countries in the field of higher education. The main goal of the Bologna Agreement is establishing, within 10 years, a standard European education system which would facilitate the implementation of clear and mutually recognized qualification standards. Thus, the purpose is to bring transparency to all national qualifications and to enable qualifications from one country to be recognised by another.

The proposed standard system of higher education would comprise 2 cycles:

- undergraduate bachelor degree (minimum 3 years), eligible also for professional employment;
- further studies, and complete higher education cycle with master and/or doctor degree.

The change is imposing a radical revision of the traditional program of studies in European countries. In many engineering faculties there is strong skepticism concerning the establishment of a bachelor's degree awarded after at least three years of study. In fact, it should be acknowledged that engineering teachers are complaining about the influence that this reform of studies is having on the level of technical skills of the European engineering graduates. We shall not discuss further on this topic, but

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the mention was necessary in order to introduce the cultural environnement in which multibody dynamics education is likely to be developed in Europe.

The organisation of education and research activities in the United States of America, when compared to Europe, is very different. Since most prominent universities are "graduate schools", the professors can concentrate their teaching on topics closely related to their research [1]. All European universities offer instead both undergraduate and graduate curricula. This involves

- courses with more general contents and rather decoupled from state-of-the-art research topics;
- heavy teaching load and administrative burden for the professors.

For these reasons the number of advanced multibody dynamics courses offered in the USA is higher than in Europe.

Software packages based on multibody dynamics techniques are becoming common design tools. The availability of software where the cycle of design and mechanical system simulation is made all within the same environment not only speeds up the design overall process, but is making a radical change both in the way engineers are approaching problems and in their required skills.

The situation calls for a revision of the current engineering curricula. However, there are several issues and constraints involved in such revision. In this paper we will focus and discuss on the most common ones, but without any claim of giving a definite answer.

Although the education system is adapting itself to the new design methodologies and tools it seems that there is not a general agreement on the most effective response. Regarding university courses dedicated to multibody dynamics common arguments of debate are:

- which should be the level of familiarity of the student with the fundamentals of analytical mechanics and machine dynamics before being exposed to commercial software?
- at university level should be given more emphasis on the theory behind the code or it is better to concentrate on the development of the modeling skills of the student?

Shortly, it seems that there is not a general consensus on the *best* approach for teaching multibody dynamics. However, it is generalized the need for the introduction of courses where the derivation of the equations of motion is presented in a computer oriented manner together with time integration algorithms.

In a recent past, multibody dynamics was more a research topic rather than an established subject for mandatory university courses. Although many distinguished researchers in the field published textbooks describing the details of their methods, few universities offered courses in modern multibody dynamics. The current situation has similarities with the pre-finite elements era. In the sixties few university curricula included a course in finite elements. Nowdays the situation is radically changed. There is not any university offering a degree in mechanical or civil engineering without a mandatory course in finite elements.

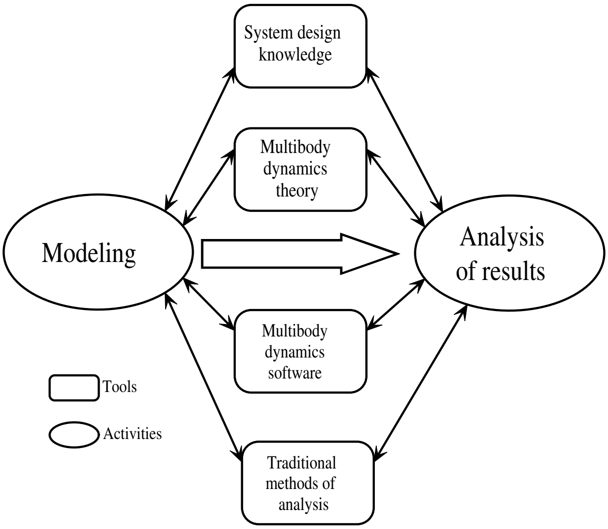

Fig. 1. Phases of system dynamics simulation

The content of textbooks traditionally used by undergraduate mechanical engineering students for learning machine dynamics (e.g. [2, 3]) focus on Newton-Euler approach and on simple planar inverse dynamics models. Forward dynamics is often limited to the classical one or two degrees-of-freedom linear models used in vibrations analysis. In some cases, advanced models of kinematic and dynamic systems, with several degrees-of-freedom, are based on clever formulations that take advantage of the specific problem to obtain a simplified form of equations of kinematics and dynamics.

When present, the computer coding of the problem is based on *ad hoc* models whose equations are deduced by hand.

In some instances the curricula reform which followed the Bologna agreement shrinked the number of credits allocated for mechanics courses. These time constraints stimulated a noteworthy attempt, due to Valasek [4], to introduce a multibody dynamics technique with a minimum or no prerequisite of analytical mechanics.

Thus, the teachers are forced to limit their lectures to the basics of machinery dynamics instead of extending them by introducing multibody dynamics techniques.

From colloquia with collegues it seems that the above situation is not uncommon at many European universities. This may slow down the introduction of multibody dynamics in engineering curricula.

As depicted in Fig. 1, a modern approach to modeling and analysis of results obtained from simulation involves knowledge and skills in different fields. In general the multibody dynamics approach to system simulation requires that the user is not just an analyst, but also a designer. This suggest that the student should have a basic knowledge of machinery dynamics before starting the study of multibody dynamics. In fact, he/she will have:

- a better appreciation of the power of systematic kinematic and dynamic formulations;
- the capability of a critical interpretation of results.

The widespread diffusion of multibody dynamics software calls for attention. Multibody dynamics modeling is applied on a greater extent within industrial companies or consulting firms. Thus, a significant portion of current European mechanical engineering graduates are potential users of such a software without having a minimal knowledge of its theoretical bases.

Moreover, many current users, although familiar with the graphic interface, lack the basic understanding of the theory behind the code. Unfortunately, software packages without a detailed theoretical manual are not rare and the courses organized by the companies, due also to time constraints, usually focus on the working features of the code and not on its theory.

A correct interpretation of the results obtained from a software requires a deep understanding of the theoretical bases used for its developement and of the numerical methods used. This is a widely accepted opinion in engineering. Errors and inaccuracies in data input, dynamic formulation limits, improper modeling, failure of numerical methods are common pitfalls for the user of multibody dynamics software. A formal education on multibody dynamics theory and software may reduce the probability of errors. However, considered the current status of mechanical engineering curricula, many users of multibody dynamics software may not be fully aware of the limits of their models. The potential danger of the above described situation requires some action from the multibody dynamics research community.

This action can take different forms according to the target of the instruction:

- students with no background in multibody dynamics and little or no experience of CAE software;
- engineers already using CAE software;

For this last cathegory of individuals, the continuing education programs offered by the universities or private companies should foresee lectures on theory and practice of multibody dynamics. Noteworthy attempts in this direction can be recorded.

Serious problems may arise by the use of simulation software, such as multibody dynamics codes, by CAD practitioners. Although extensive training is required to be expert in using a computer to create solid models, this does not makes the person an engineer [45]. Whoever is setting up the model not only must understand what kind of assumptions he/she is making, but should also be aware of the main pitfalls that can arise during simulation.

The difficulties of multibody dynamics modeling and simulation are very well expressed by Haug [14]

> "The insidious presence of nonlinearity in virtually all aspects of the kinematics and dynamics of machines leads to intricacies in mathematical and numerical analysis that are not easily solved as in the case in linear structural mechanics and associated finite element methods. Pathological forms of behavior, such as lock-up and branching of kinematic solutions, can best be

understoood and overcome by an engineer who develops both a firm mathematical foundation and a clear physical understanding of the behavior of mechanisms and machines."

Freeman [5] highlighted on some problems encountered by knowledgeable multibody analysts, with little background in vehicle dynamics, when modeling vehicles. The problems can be grouped as:

- Inappropriate application of models;
- Poor modeling assumptions;
- Incomplete model formulation or subsystem analysis;
- Methodology hides the completeness or lack thereof in the results.

Moreover, Freeman pointed out that

- it would be appropriate a direct interface of the numerical-multibody program with the CAD modeling program, making it easy the transition from model building to kinematic and dynamic simulation (much progress has been made in this direction);
- to simulate properly vehicle behavior the user must have knowledge of how different parameters and/or components affect system performance;
- only by understanding the relationship between the system components and system-level parameters one can unleash the power and flexibility of multibody dynamics software;
- ignorance of traditional analytical methods may lead easily to erroneous system behavior;
- these technical skills can only be achieved by education and experience.

The above conclusions are shared by most of the technical personnel that use multibody dynamics software within engineering firms.

Beside a broad theoretical and practical knowledge on mechanical system simulation to an engineer are also required many skills such as the ability:

- to interact with other team workers;
- to examine things critically and/or minutely, to separate the broad picture into its individual components;
- to write and speak clearly, to summarize and document information in a manner that other people can understand.

A modern education system must give the student the opportunity to develop and practice also the above skills. This require time, human and financial resources not always fully available.

The mechanical engineering community is currently benefitting from self-study systems and use of world wide web as a teaching tools [6]. Little is known on the effectiveness of such systems for learning multibody dynamics.

2 Multibody dynamics and mechanical systems modeling

At university level, different strategies can be adopted for teaching multibody dynamics and mechanical systems modeling.

One of the approaches, successfully experienced by Fisette and Samin [7], is based on a main course in classical mechanics which covers rigid body motion theory and 2D "academic" problems. In this course the students learn how to obtain analytical equations of motion. After that, with the active participation of a CAD course, the students are required to develop a multibody simulation of a realistic application.

The main steps of the project to be developed by a group of 6-8 undergraduate students are:

- Understanding the system to analyse
- Project planning
- Choice and practice of computer tools
- Formulation of relevant modeling hypotheses
- Symbolic multibody model deduction
- Data acquisition and/or computation
- Understanding and computation of environment forces
- Numerical program and first simulations
- System analysis and parameterization
- Project report, presentation and evaluation

According to Fisette and Samin [7] "this multi-disciplinary project really improves the student skills in the field of multibody system modeling". A somewhat different experience has been reported by Fanghella et al. [8] for teaching multibody dynamics

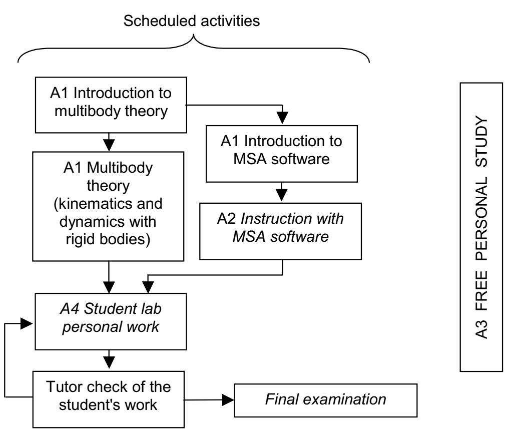

Fig. 2. Scheme of teaching and learning activities on multibody dynamics at University of Genoa [8]

to postgraduate students (i.e. attending the last year of a 3+2 engineering curriculum). In this case the lectures on multibody dynamics theory (32 hours) and hands on practice with commercial software (15 hours) are in parallel. At the end of these activities the student is required to solve an engineering analysis problem with numerical data defined. Fig. 2 depicts the flow of the described activities.

Industrial managers usually complain that most graduating engineers are very well qualified as far as a theoretical understanding is concerned, whereas few appear prepared for planning and implementing engineering projects which involve performance prediction for complex systems. In other terms, engineering graduates do not seem to have sufficient exposure to realistic modeling. Thus, it is important that the courses on multibody dynamics theory are followed or coupled with classes where are addressed realistic problems of mechanical system modeling. The structures of the courses can range from individual study to group projects, sometimes supervised jointly by faculty and industrial representatives.

The continuing education and training of engineers in system modeling by means of multibody dynamics software should be also of concern to industries. In this field it is auspicable a collaborative partnership between universities and industries.

3 Multibody dynamics in industry

Experienced engineers are challenged by technology moves beyond the levels that were current at the time of their formal education. Similarly, young engineers cannot achieve, through traditional mentoring and informal training, the levels of competence expected by a world-wide industrial competition.

In any case, gaining new skills and maintaining technical currency is a major concern for all professional engineers.

The introduction of powerful and versatile CAD programs signed a dramatic technological advancement. These tools allowed a significant reduction of the time required to move from initial design to full production. In this overall process dynamic simulation plays a fundamental role. The capability to predict the effects of design variables changes on system performance is necessary to optimize the mechanical system during the early phases of design. This allows also the reduction of costs of experimental analyses on physical prototypes. However, the mentioned capability rest on several factors such as:

- completeness and consistency of data;
- correcteness, sensitivity and reliability of the model;
- robustness of the numerical integration engine;
- expert interpretation of the results.

The discussed needs are all within the area of interest of multibody dynamics.

4 Preparation of the multibody dynamics course

An instructor willing to organize a multibody dynamics course will face different choices and options. The following items summarizes some of the main identified options.

- **Prerequisites**
  Multibody dynamics is an advanced topics. Students taking a course in this discipline should have a background in calculus, physics and machinery dynamics. Computer programming skills are also required. Knowledge of numerical analysis concepts would be preferable.
- **Kinematic formulation and dynamic principles**
  In multibody dynamics there is a wide choice of generalized coordinates (e.g. Euler angles, Cardan angles, Euler parameters, Denavit-Hartenberg, dual numbers,...) and methodologies for the systematic description of kinematic constraints (e.g. method of constraints, loop-closure equations,...)in mechanical systems.
  Moreover, the equation of dynamics may be deduced from different approaches (Newton-Euler, Lagrange, Gibbs-Appell, Jourdain, Gauss,...). Which kinematic formulation and dynamic principle is the most effective for teaching? Although this is a key question a definite answer cannot be given, and the final choice depends on the instructor's personal judgement and preferences. However, the teaching of dynamic formulations adopted in commercial software is recommended, at least in introductory courses. The student will have a better understanding of the theoretical bases and limitations of the software.
  In the bibliography a list of textbooks dedicated to multibody dynamics is included [10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30]. Likely the list is far to be exhaustive. However, it is a good starting point. A perspective on rigid multibody dynamics theory is presented in [16, 46].
- **Course length**
  How many hours are required to teach basic multibody dynamics course for graduate students?
  According to the authors' experience, 45 hours distributed in 15 lectures (2 lectures each week) should be enough. For experienced engineers, three days intensive course seems a reasonable length. In this case the instructor concentrates mostly on theory and numerical methods and less on programming.
- **Preparation of material to be distributed**
  Homework assignements and computer programming tasks should be clearly stated in handouts. Reference to the equations in textbooks will help the students. The reference to papers taken from technical literature may be useful when dealing with advanced topics.
- **Computer programming**
  The students must have normal skills of computer programming in a higher level language of their choice.

They should be required to develop software with the following modules: data input and output, automatic build up of equations and numerical solution. Through the computer programming of a multibody dynamics methodology the students usually reach a deep understanding of the subject. For this reason computer programming is strongly recommended in basic multibody dynamics classes. The instructor can make available pieces of code that the students may use during the developement of their own software. This will help the students not to get lost in coding the all software, but to concentrate on the overall structure of the program. Guidelines on the input/output of data and on the overall structure of the code should also be discussed by the instructor.
The students should also be encouraged to use professionally tailored linear algebra and numerical integration subroutines available in packages such as LAPACK, IMSL, NAG, ODEPACK, etc.

- **Use of commercial multibody dynamics software**
  At university level, how much emphasis should be given to the teaching of commercial software? Also in this case there are different answers. Some teachers argue that the students should be educated for a correct use of commercial software. Since user's interface and capabilities of a software changes almost every year, other teachers, during instruction, put more emphasis on the theoretical part of the methodology rather than on the practical use of the software.
  We believe that both aspects are relevant. The current trend is for the development of software with friendly user's interface. Thus an engineer can make himself familiar very quickly with the solid modeling and multibody dynamics software. Many companies offer specialized training on the specific software being used at their site. This training usually focus much more on the practical use of the software and less on its theoretical bases or limitations. The training offered at university level should involve a type of knowledge valid in the long term. Students are usually attracted by the use of simulation tools and less by the theory. Thus a teacher must search for a compromise between the instruction of the use of commercial software and the teaching of multibody dynamics theory.

5 The ingredients of a basic course in multibody dynamics

The research in analytical mechanics and computational dynamics provides many approaches for the systematic and computerized dynamic analysis of machine systems. None of the techniques available can be considered *a priori* the best one. Thus, the advantages and disadvantages of each technique should be evaluated by the instructor.

In the following sections a brief overview is offered.

5.1 The choice of coordinates

A multibody system is constituted of a number of parts, subject to interconnections and constraints of various kind[1]. There are several ways of representing a rigid body in space, however a system of coordinates must give at any time a unique representation of the configuration and displacements of the multibody system.

If $\delta q_1,\ \delta q_2, \ldots, \delta q_n$ are arbitrary infinitesimal increments of the coordinates in a dynamical system, these will define a possible displacement if the system is *holonomic*, while, for *non-holonomic* systems, a certain number, say m, of equations must be satisfied between them in order that they may correspond to a possible displacement.

The number

$$F = n - m \tag{1}$$

is the *number of degrees-of-freedom* of the system [31].

Hence, system of coordinates can be classified in

- *independent coordinates*, when $n = F$;
- *dependent coordinates*, when $n > F$.

Independent coordinates determine only the position of some parts. The positions of the remaining parts must be numerically computed solving a nonlinear system of equations with multiple solutions. Thus, independent coordinates are not suitable of unequivocally determine the position of the multibody system. Another problem involved with the use of independent coordinates is the variation of F during simulation. This is not a remote possibility and may happen, for instance, in linkages with particular dimensions or when modeling stiction in kinematic pairs.

The most common types of coordinates currently used to describe the motion of multibody systems are:

- *Relative coordinates.*

  The position of each element is defined with respect to the previous one (e.g. Fig. 3). These coordinates allow numerical efficiency due to their reduced number, but lead to small order and expensive to evaluate dense matrices. They are specially suited for open kinematic chain systems. The control of movement between adjacent parts is easy. The choice of variables requires a preprocessing, whereas a postprocessing is needed to determine the absolute motion of all the parts. The set of coordinates used by the well known Denavit-Hartenberg notation [35] belong to this cathegory. Noteworthy multibody dynamics formulations based on relative coordinates are reported in [17, 32].

  Often dual numbers algebra is used to express the relative motion between two adjacent links. Although unfamiliar to most engineers, dual algebra is a powerful tool for kinematic analysis. Dual numbers were proposed by Clifford (1873), but the first engineering publications were due to Denavit (1958) [35], Keler (1959) [36], Yang (1963) [37] and Yang and Freudenstein (1964) [38].

[1] For the sake of simplicity, only rigid parts and planar systems will be herein considered.

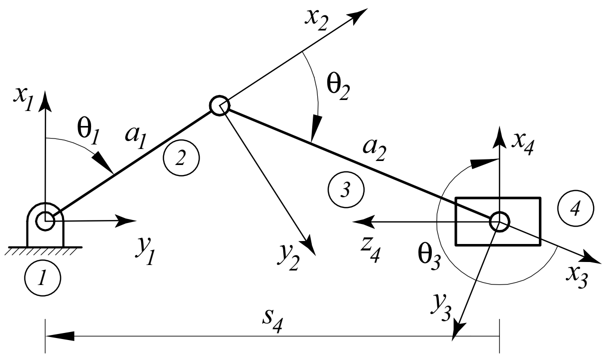

Fig. 3. Example of relative coordinates

At the textbook level, the works of Dimentberg [41], Beyer [40] and Fischer [34] are excellent introductory texts to the topic. Recent reviews of dual algebra kinematics are due to Wittenburg [42] and Angeles [43].

- *Cartesian generalized coordinates.*
  The absolute position of each body is independently located by a set of Cartesian generalized coordinates (3 for planar motion, 6 for spatial motion). Kinematic constraints between bodies are then introduced as algebraic equations among coordinates.

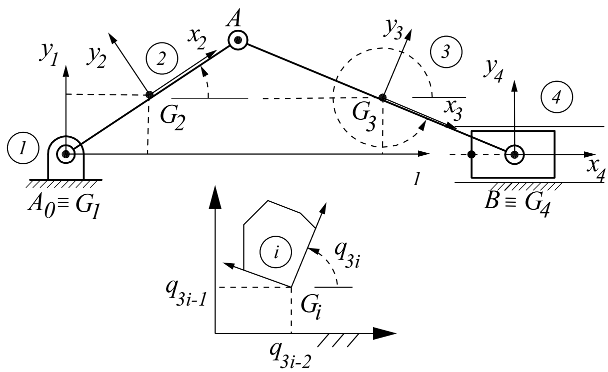

Fig. 4. Example of Cartesian generalized coordinates

Constraint expressions are numerous, but involve only the absolute coordinates of adjacent parts. (e.g. Fig. 4). With the purpose of avoiding singular configurations, some authors (*e.g.* [12, 14, 17, 24, 28]) prefer the definition of the spatial attitude of a body in terms of Euler parameters (4 coordinates) instead of Euler angles (3 coordinates).

Computer codes using these coordinates require only a minimal amount of pre and post processing. A substantial number of nonlinear constraint equations is involved. Coefficient matrices are large but sparse. It is advisable to take advantage of this condition in order to increase the numerical efficiency of the code. There are difficulties in prescribing the relative motion between adjacent bodies. The computer implementation is modular and library of components and standard joints can be defined and used in assembling a model. These coordinates are being used by general purpose dynamic simulation codes such as ADAMS [33] and DADS [14].

- *Natural coordinates*. Originally proposed by de Jalón and his coworkers, these coordinates are made by the Cartesian coordinates of a series of *basic points* of the mechanism. The points are chosen on the basis of the following criteria [18]:
  1. Each link must have at least two basic points.
  2. A basic point must be located at the center of revolute kinematic pairs. The point is shared by the two kinematic elements.
  3. In a prismatic pair one point is positioned on the line of action of the relative motion among the two kinematic elements. The other point is attached on the second kinematic element.

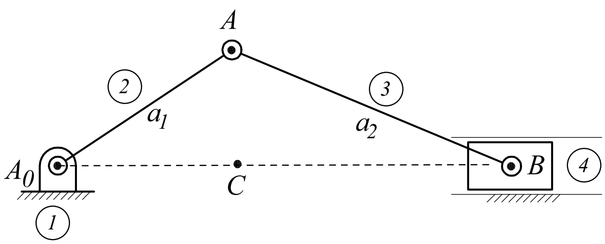

Fig. 5. Example of natural coordinates

  4. Other points whose motion needs to be monitored can be chosen as basic points.

  Fig. 5 shows the slider-crank mechanism modeled with natural coordinates.

Due to the elimination of angular coordinates, the number of natural coordinates required for the analysis is usually less than the number of Cartesian generalized coordinates.

de Jalón and Bayo [18] reported a thoughful discussion between these types of coordinates and their influence in the simulation process.

5.2 The modeling of kinematic constraints

The convenience of using sets of dependent coordinates has been already stated. These are related by the equations of constraints of the form

$$\Psi_i \left(q_1,\ q_2, \ldots, q_n\right) = 0,\ \left(i = 1, 2, \ldots, n\right) \tag{2}$$

$$\Psi_j \left(q_1,\ q_2, \ldots, q_n,\ t\right) = 0,\ \left(j = 1, 2, \ldots, n_t\right) \tag{3}$$

Equations (2) are the *spatial* or *scleronomic* constraints because only the space variables q appear as arguments. Equations (3) are said *driving* or *rheonomic* constraints because also the temporal variable t does appear explicitly.

Each multibody technique has a distinctive feature in the automatic generation and assembly of kinematic constraint equations. The algebraic structure of the constraint equations depends on the type of coordinates implemented.

The conditions used to generate the equations of constraints depend also on the type of coordinates used. For some set of coordinates previously discussed the constraint equations for the kinematic modeling of a slider-crank are reported in the following.

When relative coordinates $\{q\} = \left\{ \theta_1\ \theta_3\ s_4 \right\}^T$ are used, *loop closure conditions* are often imposed. For example, with reference to the nomenclature of Fig. 3, the following equations can be written[2]

$$\Psi_1 \equiv a_1 \sin\theta_1 - a_2 \sin\theta_3 - s_4 = 0 \tag{4}$$

$$\Psi_2 \equiv a_1 \cos\theta_1 + a_2 \cos\theta_3 = 0 \tag{5}$$

Let a_i, α_i, θ_i, s_i be the Denavit-Hartenberg parameters, and

$$\widehat{\alpha}_i = \alpha_i + \varepsilon a_i \tag{6}$$

$$\widehat{\theta}_i = \theta_i + \varepsilon s_i \tag{7}$$

their dual counterparts ($\varepsilon^2 = 0$).

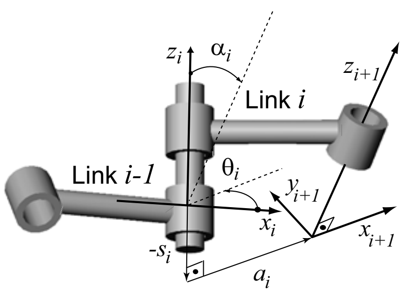

Fig. 6. Denavit-Hartenberg notation

With reference to Fig. 6, the links coordinate-transformation matrix takes the form [34]

[2] With this approach, the coordinate θ_2 is not involved.

$$\left[\widehat{T}\right]_{i+1}^{i} = \begin{bmatrix} \cos\widehat{\theta}_i & -\cos\widehat{\alpha}_i \sin\widehat{\theta}_i & \sin\widehat{\alpha}_i \sin\widehat{\theta}_i \\ \sin\widehat{\theta}_i & \cos\widehat{\alpha}_i \cos\widehat{\theta}_i & -\sin\widehat{\alpha}_i \cos\widehat{\theta}_i \\ 0 & \sin\widehat{\alpha}_i & \cos\widehat{\alpha}_i \end{bmatrix} \tag{8}$$

The closure condition of the slider-crank chain is expressed by the matrix product

$$\left[\widehat{T}\right]_{2}^{1} \left[\widehat{T}\right]_{3}^{2} \left[\widehat{T}\right]_{4}^{3} \left[\widehat{T}\right]_{1}^{4} = [I] \tag{9}$$

where $[I]$ is the unit matrix.

The constraint equations (4) and (5) follow by equating appropriate elements of the final matrix products.

Using Cartesian generalized coordinates, with reference to the nomenclature of Figure 4, the scleronomic constraints are expressed by the following equations

$$\Psi_1 \equiv X_{A_0}^{(1)} - X_{A_0}^{(4)} = 0 \tag{10}$$
$$\Psi_2 \equiv Y_{A_0}^{(1)} - Y_{A_0}^{(4)} = 0 \tag{11}$$
$$\Psi_3 \equiv X_{A}^{(2)} - X_{A}^{(3)} = 0 \tag{12}$$
$$\Psi_4 \equiv Y_{A}^{(2)} - Y_{A}^{(3)} = 0 \tag{13}$$
$$\Psi_5 \equiv X_{B}^{(3)} - X_{B}^{(4)} = 0 \tag{14}$$
$$\Psi_6 \equiv Y_{B}^{(3)} - Y_{B}^{(4)} = 0 \tag{15}$$
$$\Psi_7 \equiv Y_{B}^{(4)} = 0 \tag{16}$$

where $X_P^{(i)}, Y_P^{(i)}$ are the absolute coordinates of point P on the i^{th} body. Such coordinates are related to the generalized Cartesian coordinates by the transform

$$\begin{Bmatrix} X_P^{(i)} \\ Y_P^{(i)} \end{Bmatrix} = \begin{bmatrix} \cos q_{3i} & -\sin q_{3i} \\ \sin q_{3i} & \cos q_{3i} \end{bmatrix} \begin{Bmatrix} x_P^{(i)} \\ y_P^{(i)} \end{Bmatrix} + \begin{Bmatrix} q_{3i-2} \\ q_{3i-1} \end{Bmatrix} \tag{17}$$

Using the natural coordinates (see Fig. 5, A_0, A, B and C are basic points. The coordinates of A_0 and C are fixed and known. Thus, $\{q\} = \left\{ X_A \; Y_A \; X_B \; Y_B \right\}^T$ is the vector of variable coordinates. The constraints exquations are espressed by the following equations

$$\Psi_1 \equiv (X_A - X_{A_0})^2 + (Y_A - Y_{A_0})^2 - a_1^2 = 0 \tag{18}$$
$$\Psi_2 \equiv (X_A - X_B)^2 + (Y_A - Y_B)^2 - a_2^2 = 0 \tag{19}$$
$$\Psi_3 \equiv \det \begin{vmatrix} X_{A_0} & Y_{A_0} & 1 \\ X_B & Y_B & 1 \\ X_C & Y_C & 1 \end{vmatrix} = 0 \tag{20}$$

The constraints equations (2) e (3) are differentiated for velocity analysis[3]

[3] Dots denote differentiation w.r.t. time.

$$[\Psi_q]\{\dot{q}\} = -\{\Psi_t\} \tag{21}$$

and acceleration analysis,

$$[\Psi_q]\{\ddot{q}\} = \{\gamma\} \tag{22}$$

where $[\Psi_q]$ is the Jacobian of the constraint system and

$$\{\gamma\} = -\left([\Psi_q]\{\dot{q}\}\right)_q \{\dot{q}\} - 2[\Psi_{qt}]\{\dot{q}\} - \{\Psi_{tt}\} \tag{23}$$

5.3 Differential equation formulations

5.3.1 Newton-Euler equations

The Newton-Euler treatment is based on the consideration of a free rigid body, in the sense that, if constrained, the forces of constraint are included.

For each i^{th} $(i = 1, 2, \ldots, nb)$ body in the system, this treatment leads to:

- three translational equations of motion of the center of mass

$$m_i\{\ddot{r}_i\} = \{F_i\} \tag{24}$$

  where
  - m_i is the mass of the body;
  - $\{r_i\}$ is the vector which locate the absolute position of center of mass G_i of the body;
  - $\{F_i\}$ is the vector of the resultant of forces acting on the body;
- three equations which determine the rotational motion of the body

$$[J_i]\{\dot{\omega}_i\} + [\widetilde{\omega}_i][J_i]\{\omega_i\} = \{\tau_{G_i}\} \tag{25}$$

  where
  $[J_i]$ is the inertia matrix;
  $\{\omega_i\}$ is the angular velocity vector;
  $\{\tau_{G_i}\}$ is the vector of the resultant of torques computed w.r.t. center of mass G_i.

5.3.2 Principle of virtual work

The combination of the principle of virtual work[4] and d'Alembert principle is expressed by the equation

$$\{\delta r\}^T\left([M]\{\ddot{r}\} - \{F\}\right) + \{\delta\pi\}^T\left([J]\{\dot{\omega}\} + [\widetilde{\omega}][J]\{\omega\} - \{\tau_G\}\right) = 0 \tag{26}$$

where

[4] Lagrange in his *Mécanique Analitique* used the term *Principle of virtual velocities*.

$$\{\delta r\} = \left\{ \{\delta r_1\}^T, \{\delta r_2\}^T, \cdots, \{\delta r_{nb}\}^T \right\}^T \tag{27}$$

$$\{\delta \pi\} = \left\{ \{\delta \pi_1\}^T, \{\delta \pi_2\}^T, \cdots, \{\delta \pi_{nb}\}^T \right\}^T \tag{28}$$

$$[M] = \mathrm{diag}\left[m_1 [I_{3\times3}], m_2 [I_{3\times3}], \cdots, m_{nb} [I_{3\times3}] \right] \tag{29}$$

$$[J] = \mathrm{diag}\left[[J_1], [J_2], \cdots, [J_3] \right] \tag{30}$$

$$\{F\} = \left\{ \{F_1\}^T, \{F_2\}^T, \cdots, \{F_{nb}\}^T \right\}^T \tag{31}$$

$$\{\tau_G\} = \left\{ \{\delta \tau_{G_1}\}^T, \{\delta \tau_{G_2}\}^T, \cdots, \{\delta \tau_{G_{nb}}\}^T \right\}^T \tag{32}$$

$$\{\omega\} = \left\{ \{\omega_1\}^T, \{\omega_2\}^T, \cdots, \{\omega_{nb}\}^T \right\}^T \tag{33}$$

The expression (26) is also known as the variational Newton-Euler equation [14].

5.3.3 Lagrange equations

The Lagrange's equations of motion are expressed by

$$\frac{\mathrm{d}}{\mathrm{d}t} \left\{ \frac{\partial T}{\partial \{\dot{q}\}} \right\}^T - \left\{ \frac{\partial T}{\partial \{q\}} \right\}^T = \{Q\} \tag{34}$$

where

$$T = \frac{1}{2} \sum \left(m_i \{\dot{r}_i\}^T \{\dot{r}_i\} + \{\omega_i\}^T [J_i] \{\omega_i\} \right) \tag{35}$$

is the kinetic energy of the system, and $\{Q\}$ the vector of generalized forces.

The effect of constraints can be included by using the Lagrage's multiplier technique. In this case the equation are applied to the extended form of kinetic energy

$$T^\star = T - \{\lambda\}^T \{\Psi\} \tag{36}$$

where $\{\lambda\}$ is the vector of Lagrange's multipliers.

5.3.4 Gauss principle

Gauss's principle asserts that among all the accelerations $\{a\}$ that a system of particles of masses m_1, m_2,..., m_{nb} can have at time t which are compatible with the constraints, the actual ones $\{\ddot{r}\}$ are those that minimize the quantity

$$G(\ddot{r}) = \{\ddot{r} - a\}^T [M] \{\ddot{r} - a\} \tag{37}$$

where

$$\{a_i\} = \frac{\{F_i\}}{m_i} \quad (i = 1, 2, \ldots, nb) \tag{38}$$

are the accelerations of particles without constraints.

A modern treatment of Gauss principle is reported in [21].

5.3.5 Gibbs-Appell equations

The Jourdain principle, for a system of particles, can be written in the form

$$\sum_{j=1}^{n}\left[\sum_{k=1}^{nb}(F_k^e - m_k\ddot{r}_k)\frac{\partial \ddot{r}_k}{\partial \ddot{q}_j}\right]\delta\dot{q}_j = 0 \tag{39}$$

where F_k^e is the k^{th} external force.

If we let

$$S = \frac{1}{2}\sum_{k=1}^{nb} m_k\ddot{r}_k \cdot \ddot{r}_k \tag{40}$$

then, introduced the *quasi coordinates* u,

$$\frac{\partial S}{\partial \ddot{u}_i} = \sum_{k=1}^{nb} m_k\ddot{r}_k \cdot \frac{\partial \ddot{r}_k}{\partial \ddot{u}_i} \tag{41}$$

and (39) can be rewritten as follows

$$\sum_{j=1}^{n}\left[\frac{\partial S}{\partial \ddot{u}_i} - \left(\sum_{k=1}^{nb} F_k^e \cdot \frac{\partial \ddot{r}_k}{\partial \ddot{u}_i}\right)\right]\delta\dot{u}_i = 0 \tag{42}$$

More concisely, introduced the generalized forces

$$Q_i = \sum_{k=1}^{nb} F_k^e \cdot \frac{\partial \ddot{r}_k}{\partial \ddot{u}_i} = \sum_{k=1}^{N} F_k^e \cdot \frac{\partial \dot{r}_k}{\partial \dot{u}_i} \tag{43}$$

from (42) one obtains the Gibbs-Appell equations

$$\frac{\partial S}{\partial \ddot{u}_i} - Q_i = 0 \;\; (i = 1, 2, \ldots, n) \tag{44}$$

The extension of the Gibbs-Appell equations to systems of rigid bodies can be found in textbooks (e.g. [28]).

5.4 Computation of generalized forces

Libraries for the computation of generalized force elements due to external forces and torques are developed for a ready use when assembling the equations of motion. The most common element is the spring-damper-actuator.

5.5 Methodologies of numerical integration

Due to the use of redundant set of coordinates, a differential-algebraic equations (DAE) system of differential index 3 is composed of

$$\{\Psi(q,t)\} = 0 \tag{45}$$

together with Eqs. (22), (23) and

$$[M]\{\ddot{q}\} + [\Psi_q]^T \{\lambda\} = \{Q\} \tag{46}$$

often appear during the modeling process of multibody systems. The presence of actively controlled components may also require DAE for mathematical modeling.

There exists a large amount of literature on computational algorithms on DAE solving (e.g. [48, 49, 23]) and an exhaustive outline is not herein attempted.

The most straightforward approach requires the reduction of the original DAE to differential index 1. Thus only the simultaneous integration of Eqs (46) and (23) is herein considered. Since after numerical integration Eqs. (45) and (22) fail to be satisfied, the right side of the acceleration constraint is altered as follows

$$\{\bar{\gamma}\} = \{\gamma\} - 2\alpha\{\Psi\} - \beta\left\{\dot{\Psi}\right\} \tag{47}$$

where α and β have to be properly chosen. The DAE system to be integrated is thus transformed to

$$\begin{bmatrix} M & \Psi_q^T \\ \Psi_q & 0 \end{bmatrix} \begin{Bmatrix} \ddot{q} \\ \lambda \end{Bmatrix} = \begin{Bmatrix} Q \\ \bar{\gamma} \end{Bmatrix} \tag{48}$$

The coordinate partitioning method is an historically important and efficient computational scheme due to Wehage and Haug [47].

The set of coordinates $q \in \Re^n$ is partitioned into two sets $v \in \Re^F$ and $u \in \Re^m$ of independent and dependent coordinates, respectively. Thus, by definition, the sub-Jacobian $[\Psi_u]$ is non singular. Based on this partitioning and the DAE equations can be rewritten in the form[5]

$$[M^{vv}]\{\ddot{v}\} + [M^{vu}]\{\ddot{u}\} + [\Psi_v]^T\{\lambda\} = \{Q^v\} \tag{49}$$

$$[M^{uv}]\{\ddot{v}\} + [M^{uu}]\{\ddot{u}\} + [\Psi_u]^T\{\lambda\} = \{Q^u\} \tag{50}$$

$$\{\Psi(u,v)\} = 0 \tag{51}$$

$$[\Psi_u]\{\dot{u}\} + [\Psi_v]\{\dot{v}\} = 0 \tag{52}$$

$$[\Psi_u]\{\ddot{u}\} + [\Psi_v]\{\ddot{v}\} = \{\gamma\} \tag{53}$$

The non singularity of $[\Psi_u]$ and the implicit function theorem guarantee that $\{u\}$ con be locally computed as a function of $\{v\}$, i.e.

$$\{u\} = \{h(v)\} \tag{54}$$

With this the solution of the DAE system is reduced to a set of ODE system through the sequence of steps listed below

[5] It is assumed that the Jacobian $[\Psi_q]$ has full row rank.

1. Partition the vector $\{q\}$ of coordinates;
2. Determine $\{\dot{u}\}$ and $\{u\}$ at time t by means of Eqs. (52) and (54), respectively;
3. Solve Eqs. (49), (50, (53) w.r.t. $\{\ddot{u}\}$, $\{\ddot{v}\}$ and $\{\lambda\}$;
4. Integrate and compute $\{u\}$, $\{v\}$, $\{\dot{u}\}$, $\{\dot{v}\}$ at time $t + \Delta t$

A critical review of different dynamic formulations is offered in [50].

In order to reduce the DAE system to an ordinary differential equations (ODE) the elimination of the Jacobian matrix Ψ_q of the constraint equations from (46) is necessary. This approach offers the following advantages:

- The elimination of Lagrange's multipliers when solving equations:
- The possibility to partition the entire set of generalized coordinates into independent variables and dependent ones;
- The transform of the DAE system into a ODE gives the opportunity of a wider choice of numerical integration subroutines;
- Mechanical systems with a redundant number of constraints or with changing d.o.f. can be analysed.

For this purpose it is required to introduce a minimum set v of F independent coordinates. Let us append to the constraint vector $\{\Psi\}$ the equations $\{\Phi\}$ that can be established between v and q. Thus, we obtain

$$\{\Gamma\} = \left\{ \begin{array}{c} \Psi(q) \\ \Phi(v, q) \end{array} \right\} = 0 \tag{55}$$

The time derivative of (55) leads to

$$[\Gamma_v]\{\dot{p}\} + [\Gamma_q]\{\dot{q}\} = 0 \tag{56}$$

If we let

$$[V] = -[\Gamma_q]^{-1}[\Gamma_v] \tag{57}$$

one obtains

$$\{\dot{q}\} = [V]\{\dot{v}\} \tag{58}$$

When there is not any explicit dependence on time of constraints equations, the following *orthogonality condition* is deduced

$$[\Psi_q][V] = 0 \tag{59}$$

and the accelerations $\ddot{q}$ can be expressed in the form

$$\{\ddot{q}\} = [V]\{\ddot{v}\} + \left[\dot{V}\right]\{\dot{v}\} \tag{60}$$

Premultiplying both sides of the dynamic equation of system (46) and taking into account (59) and (60), the vector of Lagrange's multipliers is eliminated from the differential equations of equilibrium and the following ODE is obtained

$$[V]^T[M][V]\{\ddot{v}\} = [V]^T\{Q\} + [V]^T[M]\left[\dot{V}\right]\{\dot{v}\} \tag{61}$$

The matrix $[V]$ is not unique. The singular value decomposition algorithm [51] and the QR decomposition [52] are often used. In particular, when using the singular value decomposition, the Jacobian matrix is decomposed in the form

$$\begin{aligned}[\Psi_q]^T &= \left[[W_d] \; [W_i] \right] \begin{bmatrix} [\Lambda_1] \\ [0] \end{bmatrix} [U]^T \\ &= [W_d] \, [\Lambda_1] \, [U]^T \; , \end{aligned} \tag{62}$$

and

$$[V] = [W_i] \; . \tag{63}$$

When using the QR decomposition, the Jacobian matrix is decomposed in the form

$$[\Psi_q]^T = \left[[Q_1] \; [Q_2] \right] \begin{bmatrix} [R_1] \\ [0] \end{bmatrix} = [Q_1] \, [R_1] \; , \tag{64}$$

and

$$[V] = [Q_2] \; . \tag{65}$$

5.6 Interdisciplinary effects

The teaching of multibody dynamics involves disciplines like numerical analysis and computer graphics. Moreover, as previously mentioned, the knowledge of the particular field where multibody dynamics is being applied is required for a correct interpretation of results. The applications of multibody dynamics in human movement analysis and biomechanics are growing. This demonstrate the usefulness of multibody dynamics instruction in engineering curricula such as biomedical, control and computer science.

5.7 Computer programming and code organization

A moderate exposure of the student in computer programming of multibody dynamics algorithms seems appropriate for an effective learning. In order to reduce the burden of computer programming the instructor can make available to the students software modules for different main functions required to a multibody dynamics software. Thus the task of the student reduces to the correct assembly of parts and execution of the entire program. In some cases the instructor may encourage the introduction of improvements such as constraints stabilization or the testing of different integration algorithms settings.

The understanding of the overall structure of a multibody dynamics software (see Figure 7) and of basic functioning of its modules surely strength the knowledge of multibody dynamics theory and of the limits of the models developed.

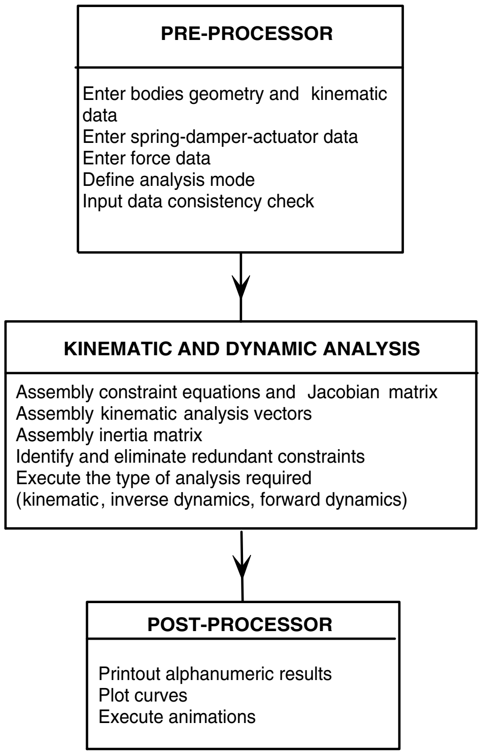

Fig. 7. Computational flow of a multibody dynamics simulation software

5.8 Home assignements and projects

In multibody dynamics instruction, the theoretical instruction and the practice can be variously combined. Practice in the form of homeworks and medium term projects are strongly recommended. At the conclusion of each theory topic, the instructor should require the students to work out autonomously by hand, or with the use of appropriate software tools, applications of the explained theory.

6 The author's experience

The first author (E.P.) recently introduced a course on multibody dynamics in the engineering curricula at Università di Roma Tor Vergata. The curricula were compliant with the Bologna agreement. The course is currently mandatory for fourth year mechanical and automatic control engineering students. This section describes some of the choices and may offer guidelines for similar initiatives.

6.1 Course syllabus

The method of constraints for planar kinematic analysis. Revolute, prismatic, gear and cam pairs are considered together with other 2 degrees-of-freedom types of constraints. The automatic assembly of the systems of equations for position, velocity and acceleration analysis. Iterative solution of systems of non linear equations. Geometry of masses. The principle of virtual work and Lagrange's equations. Dynamics of planar systems. Systematic computation and assembly of mass matrix. Computation of planar generalized forces for external forces and for actuator-spring-damper element. Simple applications of inverse and forward dynamic analysis. Numerical integration of first-order initial-value problems. The method of Baumgarte for the solution of mixed differential-algebraic equations of motion. The use of coordinates partitioning, QR and SVD decomposition for the orthogonalization of constraints. Kinematics of rigid bodies in space. Reference frames for the location of a body in space. Euler angles and Euler parameters. The formula of Rodrigues. Screw motion in space. Velocity, acceleration and angular velocity. Relationship between the angular velocity vector and the time derivatives of Euler parameters. Kinematic analysis of spatial systems. Basic kinematic constraints. Joint definition frames. The constraints required for the description in space of common kinematic pairs (revolute, prismatic, cylindrical, spherical). Equations of motion of constrained spatial systems. Computation of spatial generalized forces for external forces and for actuator-spring-damper element. Computation of reaction forces from Lagrange's multipliers.

Considered the introductory level of the course and the lack of funds for the renting of licenses, hands-on practice with multibody dynamics commercial software was not included in the course.

6.2 Homework and computer assignements

The homeworks requested are the development of computer codes for:

- the kinematic analysis of planar mechanisms with lower pairs
- the forward dynamic analysis of planar mechanisms with lower pairs, spring-damper-actuator elements, external forces;
- dynamic analysis of a 3D mechanical system composed of two rigid bodies with revolute, prismatic or spherical pairs.

The students were also requested to document their code to the best of their capabilities. The time allowed for each project was about three weeks. For this purpose, different pieces of software modules were made available to the students.

6.3 Course grading

The overall grading is based for 50% on the quality of the computer assignements handed during the course, 25% from written test and a 25% on a brief oral exam on different parts of the theory.

6.4 Response and comments from the students

The students seems enjoying the study of multibody dynamics. What they like most is the sistematicity of the approach. The computer oriented modeling of mechanical systems makes them more confident about the formal correctness of the governing equations deduced. Most of the complains arise from the limited time allowed for computer homework. In fact, they claim that the theory is easy to learn and understand, but debugging of the software takes most of their time. They find also useful the experience made with specialized software for linear algebra and numerical integration of differential equations. The software tools usually chosen for computer programming were Fortran90, C++, Matlab, Maple and Mathematica.

7 Needs

1. Increase the number of credits allocated for multibody dynamics courses in engineering curricula.
2. Multibody dynamics computer codes with open architecture.
3. Standardised input-output of data between multibody dynamics codes.
4. High-level programming languages geared toward multibody dynamics programming (mixed capabilities: numerical and symbolical).
5. Multibody dynamics software with a pre and post processing capabilities using web browsers only.
6. Centralized web resources for the exchange of informations between teachers, reserachers, students.

8 Conclusions

Different issues involved in multibody dynamics training of engineering students were discussed. On the basis of their experience, the authors recommend that multibody dynamics courses should be preferentially offered during or after the third year of an engineering curriculum. This will ensure a minimum of background. The question on the *most* effective syllabus is still open. It would be interesting to compare the proficiency in modeling mechanical systems of the students exposed to different multibody dynamics methodologies.

The inclusion of more advanced multibody dynamics topics in engineering curricula seems is not widespread in European universities. For instance flexible multibody dynamics is still perceived like a research topic rather than an established discipline. The scarsity of textbooks, usually very expensive, and of ready-to-use didactic material does not help the diffusion of courses. Due to the lack of funds, the renting and maintenance of commercial software licenses is also a problem. However, from the didactic point of view, the development of open source multibody dynamics software, made freely available to teachers and students or under a nominal fee, would greatly help the spread of the culture of multibody dynamic in the engineering curricula. Centralized web resources, where students and educators may find links to reports, tutorials, software on the different branches of multibody dynamics, are also useful for the above purposes.

The efforts in the developement of didactic tools and teaching methodologies in the field of multibody dynamics are worthwhile. Beside the already mentioned advantages of informed software users, the research in multibody dynamics will surely benefit of a large base of graduate students familiar with the basic techniques.

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