J. HAFNER AND P. MANCOSU

# THE VARIETIES OF MATHEMATICAL EXPLANATION[0]

## 1. BACK TO THE FACTS THEMSELVES

When William James was faced with the task of writing in an encompassing way on religion he emphasized the variety of phenomena that fell under the topic and warned against the dangers of oversimplification:

> Most books on the philosophy of religion try to begin with a precise definition of what its essence consists of. Some of these would-be definitions may possibly come before us in later portions of this course, and I shall not be pedantic enough to enumerate any of them to you now. Meanwhile the very fact that they are so many and so different from one another is enough to prove that the word "religion" cannot stand for any principle or essence, but is rather a collective name. The theorizing mind tends always to the oversimplification of its materials. [...] Let us not fall immediately into a one-sided view of our subject, but let us rather admit freely at the outset that we may very likely find no one essence, but many characters which may alternately be very important to religion. (William James, The varieties of religious experience, 1902, p. 31)

If we substitute 'explanation' for 'religion' in the above quote the result captures our point of view about the philosophy of explanation. Contemporary work in scientific explanation has pursued to a great extent the project of a single unified account of the nature of explanation. Unfortunately the drive towards unification has also ignored an important number of phenomena. In particular, many theories of scientific explanation do not address mathematical explanation, either because they rule mathematical explanations out of court from the outset or because they hold that their account of explanation automatically takes care of mathematical explanation. Most of the time, mathematical explanation is simply not mentioned. This is a symptom, following James, of the dangers of the theorizing mind and like him we propose to begin "by addressing ourselves directly to the concrete facts". Of course, it is not our intention to downplay the importance of the work that has been pursued in the area of scientific explanation and which has yielded many remarkable insights. We do not even pass judgment on whether a more careful analysis of concrete scientific case studies of explanatory activity in the empirical sciences might have been beneficial for the subject as a whole.



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However, our topic demands a different approach. Indeed, in the case of mathematical explanation we cannot rely, as people do in the natural sciences, on well-entrenched intuitions concerning paradigmatic examples of explanations.

In this paper we will begin with some general methodological remarks about mathematical explanations. We will then point out that attention to mathematical practice reveals the presence of a great variety of mathematical explanations. This realization affects two important aspects of the discussion of the nature of mathematical explanation. First of all, most of the traditional debates (see Mancosu 1999, 2000, 2001) have focused on the opposition between explanatory and non-explanatory proofs. However, there are mathematical explanations that do not come in the form of proofs and this has in fact been recognized by several scholars. Second, the variety of mathematical explanations challenges the current philosophical accounts of mathematical explanation, i.e. those of Kitcher and Steiner. As detailed discussion of case studies is necessary to see the limitations of such accounts, in the second part of the paper we restrict our focus to Steiner's theory and to the discussion of an example of an explanatory proof which, we claim, Steiner's theory cannot account for.[1]

## 2. MATHEMATICAL EXPLANATION OR EXPLANATION IN MATHEMATICS?

In the above we have been using freely the expression "mathematical explanation". The use was intentionally ambiguous and we should now clarify the source of the ambiguity. "Mathematical explanation" could mean a) explanations as they are given in mathematics; or, b) explanations that make use of mathematics. The two definitions characterize different classes. In the first case we intend to refer to explanatory practices that take place within the realm of mathematics itself. In the second case, this would include, among other things, mathematical explanations of physical facts which clearly do not belong to the first class.

The second kind of explanation is part of a large problem area concerning mathematical applications. Shapiro recently remarked that "a scientific 'explanation' of a physical event often amounts to no more than a mathematical description of it." His favorite example is given in the form of an anecdote:

> The story relies on the unreliable memory of more than one person, but the situation is typical. A friend once told me

> that during an experiment in a physics lab he noticed a phenomenon that puzzled him. The class was looking at an oscilloscope and a funny shape kept forming at the end of the screen. Although it had nothing to do with the lesson that day, my friend asked for an explanation. The lab instructor wrote something on the board (probably a differential equation) and said that the funny shape occurs because a function solving the equation has a zero at a particular value. My friend told me that he became even more puzzled that the occurrence of a zero in a function should count as an explanation of a physical event, but he did not feel up to pursuing the issue further at the time. (Shapiro 2000, p. 34)

Shapiro's friend had all the rights to be puzzled. After all, it could be claimed that the explanation why the equation in question has a zero at a particular value rests on the physical situation and not vice versa. Of course, the equation has its zeros independently of any physical reality and thus the last remark makes sense only under the assumption that the equation "represents" the physical reality. But this only points to the fact that without a general account of how mathematics hooks on to reality the role of mathematical explanations in physics is bound to remain mysterious:

> Clearly, a mathematical structure, description, model, or theory cannot serve as an explanation of a non-mathematical event without some account of the relationship between mathematics per se and scientific reality. Lacking such an account, how can mathematical/scientific explanations succeed in removing any obscurity - especially if new, more troubling obscurities are introduced? (Shapiro 2000, p. 35; cf. p. 217)

This is a daunting problem indeed but fortunately we will not have to discuss it here, as our major aim is to investigate the first sense of mathematical explanation. Even with this restriction in place, things are far from easy. "Explanation" is a notoriously ambiguous word and this ambiguity shows up in mathematics just as much as in ordinary parlance. We can explain the rules of a certain calculus, the meaning of a symbol, how to carry out a construction, how to fix or set up a proof. These are all "instructions" on how to master the tools of the trade. There are however deeper uses of "explanation" in mathematics which call for an account of the mathematical facts themselves, the reason why. The distinction we just drew between "instructions" and deeper senses of "explanation" should be no more puzzling than the equivalent one in physics. While doing physics we might ask for an explanation of a certain notation or of how to describe a certain phenomenon by means of a new formalism. These uses of explanation are of a different category from that involved in explaining, for instance, why salt dissolves in water.

## 3. THE SEARCH FOR EXPLANATION WITHIN MATHEMATICS

In addition to "explanation" mathematicians and philosophers use a cluster of expressions to refer to this phenomenon. Here is an illustrative sample of expressions we found in the mathematical and philosophical literature in which the search for explanations is sometimes characterized as a search for:

(a) "the deep reasons"
(b) "an understanding of the essence"
(c) "a better understanding"
(d) "a satisfying reason"
(e) "the reason why"
(f) "the true reason"
(g) "an account of the fact"
(h) "the causes of"

Of course, we are not claiming that the above expressions have the same intension. However, we maintain that the cluster of notions we indicated is not accidentally related.

That mathematicians seek explanations in their ordinary practice and cherish different types of explanations is for us, after working on this topic for so long, so obvious as to require almost no proof. However, some of the philosophical literature on the topic has denied that there are mathematical explanations and thus it will be useful here to provide some examples of "explanatory" talk in mathematical practice.

First of all, the search for explanations is often the drive towards mathematical research. What motivates mathematicians to look for explanations? It is the old desire to know the reason why. This desire might be awakened by different factors, a sample of which is given by the following illustrative examples.

**1.** A number of mathematical phenomena are perceived as too complicated. A desire to bring order in the "realm of facts" will drive the mathematician to look for an explanation or a deeper explanation of what is going on.

**Example 1.** In the article "On the Kummer solutions of the hypergeometric equation" Reese T. Prosser describes his aim as follows:

> One of the oldest, and still one of the most interesting applications of group theory arises in the study of the transformations of an ordinary differential equation. If we know that a given differential equation admits a group of transformations, then we know that the solution set must admit that same group of transformations, and we can deduce properties of all the solutions from the properties of any one of them. A case in point is offered by the celebrated hypergeometric equation whose solutions include many of the most interesting special functions of mathematical physics [...] In 1836 Kummer published a set of six distinct solutions of the hypergeometric equation. [...] A glance at the list of these solutions reveals a rather complicated set of relationships which pleads for some simple explanation. We show here that the Kummer solutions are related by a finite group of transformations which serve to explain their relationships and to exemplify the use of transformation groups in the study of differential equations. (p. 535)

**2.** Sometimes it is a desire of explaining “resemblances”, mysterious or remarkable coincidences, as well as striking or deep analogies.

**Example 2a.** In “Eine Verbindung zwischen den arithmetischen Eigenschaften verallgemeinerter Bernoullizahlen”, Kurt Girstmair writes:

> Let $m \geq 1$, $n \geq 2$ be integers. There are two kinds of generalized Bernoulli numbers which occur in the arithmetic of Abelian number fields: on the one hand Leopoldt’s numbers [...], on the other hand, the cotangent numbers [...] For both kind of numbers theorems of the v. Staudt-Clausen type exist, which describe their (ideal) denominators. These theorems resemble each other in several respects, a fact that has not been explained so far. One aim of this paper is to supply this explanation. (p. 47)

**Example 2b.** In the article “On the Betti numbers of the moduli space of stable bundles of rank two on a curve” Bifet, Ghione and Letizia say:

> The aim of this paper is to begin exploring a new algebra-geometric approach to the study of the geometry of the moduli space of stable bundles on a curve $X$ over a field $k$. This approach establishes a bridge between the arithmetic approach of G. Harder and M.S. Narasimhan and the gauge

> group approach of M.F. Atiyah and R.H. Bott. In particular, it might help explain some of the mysterious analogies observed by Atiyah and Bott. (p. 92)

**3.** Very often the mathematical fact to be explained is understood from a certain point of view but one looks for alternative explanations. When mathematicians speak about explanations they often modify the phrase by specifying the nature of the explanation: analytical, algebraic, group-theoretical, combinatorial, categorical, geometric, function-theoretic, measure-theoretic, number-theoretic, probabilistic, cohomological, representation-theoretic, topological etc. In some cases several of these goals are pursued at once.

**Example 3.** Iku Nakamura in "On the equations $x^p + y^q + z^r - xyz = 0$" writes:

> We know two strange dualities - the duality of fourteen exceptional unimodular singularities and the duality of fourteen hyperbolic unimodular singularities. The first purpose of this article is to recall and compare them. The second is to give explanations for the second duality from various viewpoints. [...] In section 5 we give a number-theoretic explanation for the duality. We see that the duality is essentially the relationship between a complete module and its dual in a real quadratic field. In section 6 we provide a geometric explanation for the duality by means of general theory of surfaces of class $VII_0$. In section 7 we give a lattice-theoretic explanation for the duality. (pp. 281f)

**4.** However, most of the time explanations are provided for mathematical facts independently of whether a particular point of view is emphasized. While sometimes these facts might be "striking" or "curious" in many cases the explanation is sought whether or not the fact in question might be striking.

**Example 4a.** Kubo and Vakil in "On Conway's recursive sequence" say:

> The recurrence $a(n) = a(a(n-1)) + a(n-a(n-1))$, $a(1) = a(2) = 1$ defines an integer sequence, publicized by Conway and Mallows, with amazing combinatorial properties that cry out for explanation. We take a step towards unraveling this mystery by showing that $a(n)$ can (and should) be viewed as a simple 'compression' operation on finite sets. This gives a combinatorial characterization of $a(n)$ from which one can read off most of its properties. (p. 225)

**Example 4b.** Leyendekkers and others in "Analysis of Diophantine properties using modular rings with four and six classes" write:

> A modular ring $Z_A$ is described, and used together with a modular ring $Z_6$ and the Pythagorean triple grid, described earlier, to analyze various diophantine properties and explain why the area of a Pythagorean triangle can never be a square.

## 4. SOME METHODOLOGICAL COMMENTS ON THE GENERAL PROJECT

It should be obvious from the above that mathematicians seek explanations. But what form do these explanations take? It is here that two possibilities emerge. One can follow two alternative approaches: top-down or bottom-up. In the former approach one starts with a general model of explanation (perhaps because of its success in the natural sciences) and then tries to see how well it accounts for the practice. In the latter approach one begins by avoiding, as much as possible, any commitment to a particular theoretical/conceptual framework. We favor the second approach for the following reasons. As a rule contemporary accounts of explanation have been developed within the philosophy of natural science without addressing the specificity of mathematical explanations. Hence the conceptual resources of those accounts involving, e.g. the notions of causal connections or laws of nature seem inappropriate for capturing explanations in mathematics. Furthermore, even if some more abstract features of those accounts, e.g. construing the general form of explanations as answers to why-questions could perhaps be adopted for a theory of mathematical explanations[2] proceeding in this way would mean forcing the evidence from mathematical practice into a predefined mould, thereby narrowing the perspective from the outset and probably leading to distortions. The same holds for the few philosophical accounts of mathematical explanation found in the literature (Kitcher, Steiner). Making either theoretical unification or deformability (in Steiner's particular sense) the hallmark of mathematical explanations amounts to the imposition of a defining characteristic feature on what ought to be counted as "explanation" in mathematics.[3] Proofs, theories, methods etc. which do not satisfy that definition are then disregarded or discounted – regardless whether they are indeed taken to be explanatory by working mathematicians!

Thus, in our mind, a fruitful approach would consist in giving a taxonomy of recurrent types of mathematical explanation[4] and then trying to see whether these patterns are heterogeneous or can be subsumed under a general account. We maintain that mathematical explanations are heterogeneous. However, neither giving the taxonomy nor arguing for the previous claim is what we have set for ourselves in this paper. Rather, we would like to provide a single case study of how to use mathematical explanations as found in mathematical practice to test theories of mathematical explanation. This can be seen, as it were, as a case study of how to show that the variety of mathematical explanations cannot be easily reduced to a single model. In what follows we will thus look at Steiner's theory of explanation and discuss a counterexample to his theory.

## 5. MARK STEINER ON MATHEMATICAL EXPLANATION

In developing his own account of explanatory proofs in mathematics Mark Steiner discusses – and rejects – a number of initially plausible criteria for explanation, i.e. the (greater degree of) abstractness or generality of a proof, its visualizability, and its genetic aspect which would give rise to the *discovery* of the result. In contrast, Steiner takes up the idea "that to explain the behavior of an entity, one deduces the behavior from the essence or nature of the entity" (Steiner 1978, p. 143). In order to avoid the notorious difficulties in defining the concepts of *essence* and *essential* (or *necessary) property*, which, moreover, don't seem to be useful in mathematical contexts anyway since all mathematical truths are usually regarded as necessary, Steiner introduces the concept of *characterizing property*. By this he means "a property unique to a given entity or structure within a *family* or domain of such entities or structures" (*Ibid.*), where the notion of "family" is taken as undefined. Hence what distinguishes an explanatory proof from a non-explanatory one is that only the former involves such a characterizing property. In Steiner's words: "an explanatory proof makes reference to a characterizing property of an entity or structure mentioned in the theorem, such that from the proof it is evident that the result depends on the property" (*Ibid.*). Furthermore, an explanatory proof is *generalizable* in the following sense. Varying the relevant feature (and hence a certain characterizing property) in such a proof gives rise to an array of corresponding theorems, which are proved – and explained – by an array of "deformations" of the original proof. Thus Steiner arrives at two criteria for explanatory proofs, i.e. dependence on a *characterizing property* and *generalizability* through varying of that property (Steiner 1978, pp. 144, 147).

The following proof of the irrationality of $\sqrt{2}$ given by Steiner illustrates the two criteria.[5] Relying on the fact that each number has a unique prime power expansion (the Fundamental Theorem of Arithmetic) we can argue thus.

Assume that $2 = (\frac{a}{b})^2$, i.e. $a^2 = 2b^2$. The prime 2 has to appear with an even exponent in the prime power expansion of $a^2$. And since the same holds for the prime power expansion of $b^2$, the exponent of 2 in the expansion of $2b^2$ must be odd. Because of the uniqueness of prime power expansions it follows that $a^2 \neq 2b^2$ contradicting our assumption.

This proof is explanatory according to Steiner, because it uses – as a characterizing property of numbers – their prime power expansion. Also, the proof is generalizable to numbers different from 2, i.e. one can establish along the same lines the theorem that for any $n$, $\sqrt{n}$ is either a natural number or irrational. And generalizing further one can get the analogous result for the $p^{th}$ root in place of the square root of $n$.

Steiner's account has been criticized by Resnik and Kushner. They doubt the existence of explanatory proofs in general, denying an objective distinction between explanatory and non-explanatory proofs. But more concretely they also challenge Steiner's account by proposing counterexamples, i.e. a proof that meets his criteria but is not accepted as explanatory by Steiner himself. And on the other hand Resnik and Kushner claim there are proofs, namely a certain proof of the intermediate value theorem and Henkin's proof of the completeness of first-order logic, which seem to qualify as explanatory but apparently fail to meet Steiner's criteria. However, one may ask how well these instances really work as counterexamples. To begin with, what justifications are put forward by Resnik and Kushner for the classification of their examples as indeed (intuitively) explanatory? Besides simply claiming that these proofs "would seem to qualify as explanatory if any do" (Resnik & Kushner 1987, p. 147), it is contended with some – albeit rather vague – reference to mathematical/logical practice that Henkin's proof "is generally regarded as really showing what goes on in the completeness theorem and the proof-idea has been used again and again in obtaining results about other logical systems" (Resnik & Kushner 1987, p. 149). And with respect to the proof of the intermediate value theorem the authors "find it hard to see how someone could understand this proof and yet ask *why* the theorem is true (or what makes it true)" (*Ibid.*) and hence it has to be counted as explanatory. Yet we are not given any hint as to what exactly the explanatory feature(s) of this proof are supposed to consist in.

For counterexamples to Steiner's theory to carry real weight they would have to be much more closely related to mathematical practice. Contrary to what Resnik and Kushner claim (p. 151), mathematicians *often* describe themselves and other mathematicians as explaining. And their judgments concerning explanatory vs. non-explanatory proofs (and other varieties of explanation in mathematics as the case may be) has to figure as the basic evidence, however subjective or context dependent they may be. Claims to the effect that certain proofs are explanatory come from within mathematics not from philosophers of mathematics. Their sources, working mathematicians, are furthermore precisely identifiable, and the case for explanatoriness will be even stronger, if a certain proof is put forward explicitly with the aim to explain a "mathematical phenomenon", which has been acknowledged for a long time to be mysterious and puzzling by (a subgroup of) the mathematical community. A case of mathematical explanation rooted in this way in mathematical practice can justifiably serve as a test case for Steiner's account. It certainly cannot be dismissed easily if it should amount to a refutation of that account. And it is such a test case coming from the work of Alfred Pringsheim in the theory of infinite series which we want to present and discuss in the following.

## 6. KUMMER'S CONVERGENCE TEST

The following exposition is adapted from Pringsheim (1916). In order to make it more readable and clearly bring out the points which are relevant in our context we have simplified Pringsheim's account by stating some results in a slightly less general form than they could be formulated. But nothing essential is lost because of that (cf. footnote 7).

Let's start with some preliminary observations concerning infinite series. We will confine ourselves to infinite series $\sum_{n=1}^{\infty} a_n$ of positive terms,[6] i.e. $a_n > 0$ for $n = 1, 2, 3, \ldots$ and we will consider different convergence and divergence tests for them. Of fundamental importance are the following comparison tests.

(1) If $\sum c_n$ is a *convergent* series such that the terms of $\sum a_n$ satisfy $a_n \leq c_n$ for all $n$ (or at least for all values of $n$ greater than some fixed value $m$), then $\sum a_n$ is also convergent.

Similarly we have:

(2) If $\sum d_n$ is a *divergent* series such that the terms of $\sum a_n$ satisfy $d_n \leq a_n$ for all $n$ (or at least for all values of $n$ greater than some fixed value $m$), then $\sum a_n$ is also divergent.

It turns out that the comparison tests are often easier to work with in practice when they are stated in a slightly different form. In order to simplify the exposition we will for the remainder of this section adopt the convention to denote arbitrary infinite series by '$\sum a_n$', convergent ones by '$\sum c_n$', and divergent ones by '$\sum d_n$'. Let $C_n = \frac{1}{c_n}$, $D_n = \frac{1}{d_n}$ and let $g$ and $G$ be two positive numbers ($g$ can be thought of as arbitrarily small and $G$ as arbitrarily large). Now suppose for all $n$ (or at least for all $n \geq m$, for some $m$)

$$a_n \leq G \cdot c_n \qquad \text{or} \qquad a_n \geq g \cdot d_n.$$

Then we have

$$(3) \qquad \frac{1}{c_n} \cdot a_n = C_n \cdot a_n \leq G \implies \sum a_n \text{ converges.}$$

and

$$(4) \qquad \frac{1}{d_n} \cdot a_n = D_n \cdot a_n \geq g \implies \sum a_n \text{ diverges.}$$

Under the assumption that $\lim_{n\to\infty} C_n \cdot a_n$ and $\lim_{n\to\infty} D_n \cdot a_n$ exist,[7] hence $\lim_{n\to\infty} C_n \cdot a_n \leq G < \infty$ and $\lim_{n\to\infty} D_n \cdot a_n \geq g > 0$, we arrive finally at the following formulations.

$$(5) \qquad \lim_{n\to\infty} C_n \cdot a_n < \infty \implies \sum a_n \text{ converges.}$$

$$(6) \qquad \lim_{n\to\infty} D_n \cdot a_n > 0 \implies \sum a_n \text{ diverges.}$$

Tests (1) through (6) commonly also known as *comparison tests of the* $1^{st}$ *kind* arise from a direct comparison of the terms $a_n$ with $c_n$ or $d_n$. In contrast *comparison tests of the* $2^{nd}$ *kind* are based on quotients of two consecutive terms of the series and their comparison. This method is frequently very convenient since for many series of practical importance the quotient $\frac{a_n}{a_{n+1}}$ happens to be simpler than the general term $a_n$. With our conventions of denoting convergent and divergent series in place we can state these tests concisely as follows.

(7) If for all $n$ (or all $n \geq m$, for some $m$) $\frac{a_{n+1}}{a_n} \leq \frac{c_{n+1}}{c_n}$, then $\sum a_n$ converges.

And

(8) If for all $n$ (or all $n \geq m$, for some $m$) $\frac{a_{n+1}}{a_n} \geq \frac{d_{n+1}}{d_n}$, then $\sum a_n$ diverges.

They easily follow from the direct comparison of terms of the series involved[8] and again they can be reformulated in different ways. Simple transformations of the conditions in (7) and (8) yield $\frac{a_{n+1}}{a_n} \leq \frac{C_n}{C_{n+1}}$ and $\frac{a_{n+1}}{a_n} \geq \frac{D_n}{D_{n+1}}$ and then in turn we get

$$(9) \qquad \text{For all } n \geq m, \ (C_n \cdot \frac{a_n}{a_{n+1}} - C_{n+1}) \geq 0 \implies \sum a_n \text{ converges.}$$

$$(10) \qquad \text{For all } n \geq m, \ (D_n \cdot \frac{a_n}{a_{n+1}} - D_{n+1}) \leq 0 \implies \sum a_n \text{ diverges.}$$

And finally by assuming the existence of the involved limits,

$$(11) \qquad \lim_{n\to\infty} (C_n \cdot \frac{a_n}{a_{n+1}} - C_{n+1}) > 0 \implies \sum a_n \text{ converges.}$$

$$(12) \qquad \lim_{n\to\infty} (D_n \cdot \frac{a_n}{a_{n+1}} - D_{n+1}) < 0 \implies \sum a_n \text{ diverges.}$$

Having established the general form of comparison tests of the $1^{st}$ and $2^{nd}$ kind it now remains to determine concrete examples of convergent and divergent series, $\sum c_n$ and $\sum d_n$, which can be substituted in those tests in specific applications. For our purposes we don't need to proceed any further in this direction, instead we focus our attention on another formulation of a comparison test of the $2^{nd}$ kind due to Ernst Kummer.[9] Letting $(B_n)$ be an *arbitrary* sequence of positive numbers Kummer's test can be stated as follows.

$$(13) \qquad \lim_{n\to\infty} (B_n \cdot \frac{a_n}{a_{n+1}} - B_{n+1}) > 0 \implies \sum a_n \text{ converges.}$$

This test is rather striking because of the extreme generality or arbitrariness of the sequence $(B_n)$ occurring in it. Whereas the tests (11) and (12) above require the use of sequences $(C_n)$ and $(D_n)$ which derive, respectively, from convergent and divergent series, *any old sequence* $(B_n)$ will do in (13). Pringsheim calls it a "most remarkable criterion" (Pringsheim 1916, p. 379) of "indeed surprising generality" (p. VI) that stands in need of explanation or clarification [Aufklärung] (p. 379). Pringsheim's opinion was by no means exceptional, many mathematicians must have been similarly puzzled and left unsatisfied by Kummer's original proof of his criterion in 1835. As Knopp notes it wasn't until 1885 that O. Stolz gave an "extremely simple proof, by means of which the criterion was first rendered fully intelligible" (Knopp 1928, p. 311, fn. 52). Moreover, even after another 30 years had passed this criterion was apparently still viewed as an anomaly of sorts defying smooth integration into the theory of infinite series. Pringsheim notices (in 1916) that Kummer's criterion "appeared as totally erratic in other accounts [of convergence and divergence tests], seemingly lacking any analogue among the convergence criteria of the $1^{st}$ kind"[10] and he thus aimed at presenting it "freed from this mysterious isolation" (p. VI). Pringsheim gives two different proofs of it, one explanatory and another one which only "proves the correctness of the criterion *a posteriori* in a simpler way" (p. 379). Let's begin with the latter; it is essentially due to Stolz and it's indeed very simple.

If $\lim_{n\to\infty}(B_n \cdot \frac{a_n}{a_{n+1}} - B_{n+1}) > 0$, then there exists a $\rho$ such that from some stage $m$ on, $n \geq m$ implies

$$B_n \cdot \frac{a_n}{a_{n+1}} - B_{n+1} \geq \rho > 0$$

hence

$$B_n \cdot a_n - B_{n+1} \cdot a_{n+1} \geq \rho a_{n+1}. \tag{14}$$

Since the difference on the left hand side is thus positive it follows that the products $B_n \cdot a_n$ form a monotone decreasing sequence (of positive terms). So this sequence has a limit, say

$$\lim_{n\to\infty}(B_n \cdot a_n) = \alpha \geq 0.$$

If we now add, respectively, the left hand side and the right hand side terms in inequality (14) from stage $m$ to $k$ we get

$$(B_m \cdot a_m - B_{m+1} \cdot a_{m+1}) + (B_{m+1} \cdot a_{m+1} - B_{m+2} \cdot a_{m+2}) + \dots$$
$$+ (B_{k-1} \cdot a_{k-1} - B_k \cdot a_k) \geq \rho a_{m+1} + \dots + \rho a_k$$

which reduces to

$$(B_m \cdot a_m - B_k \cdot a_k) \geq \rho(a_{m+1} + \dots + a_k).$$

Consequently, for $k \to \infty$

$$(B_m \cdot a_m - \alpha) \geq \rho \cdot \sum_{j=m+1}^{\infty} a_j$$

Which shows that $\sum a_n$ is indeed convergent. This proof certainly establishes its result, i.e. it shows that Kummer's test works. But it fails to explain or even address the very aspect of this test which makes it so puzzling. – How come that the $C_n$ in (11) can be replaced by terms $B_n$ of a *completely arbitrary* sequence (as long as they are positive) and we still get a convergence test? Here is Pringsheim's explanation.

We first note elementary results concerning the representation of the terms $c_n$ and $d_n$ of convergent resp. divergent series.[11] For any $\sum c_n$ we can pick a strictly increasing sequence $(M_n)$ of positive numbers satisfying $\lim_{n\to\infty} M_n = +\infty$ such that

$$c_n = \frac{M_n - M_{n-1}}{M_n \cdot M_{n-1}}. \tag{15}$$

And conversely, every series whose terms are defined in this way is convergent.

In case of a divergent series $\sum d_n$ we can find a sequence $(M_n)$ as above such that

$$d_n = \frac{M_n - M_{n-1}}{M_{n-1}}. \tag{16}$$

And conversely, every series whose terms are defined in this way is divergent.

Now let's assume that for some sequence $(B_n)$ of positive numbers we have

$$\lim_{n\to\infty}(B_n \cdot \frac{a_n}{a_{n+1}} - B_{n+1}) > 0. \tag{17}$$

We have to show that $\sum a_n$ converges.

Considering $\sum b_n$, where $b_n = \frac{1}{B_n}$, there are only two cases possible. Either $\sum b_n$ converges, i.e. the sequence $(B_n)$ is of type $(C_n)$, then $\sum a_n$ converges because of criterion (11). Or, on the other hand, $\sum b_n$ is divergent, hence $(B_n)$ is of type $(D_n)$ and we can reformulate our assumption (17) thus

$$\lim_{n\to\infty}(D_n \cdot \frac{a_n}{a_{n+1}} - D_{n+1}) > 0.$$

This implies that there is a $\rho > 0$ such that for appropriate $m \geq 1$ we have for all $n \geq m$

$$D_n \cdot \frac{a_n}{a_{n+1}} - D_{n+1} \geq \rho$$

equivalently

$$\frac{1}{\rho} \cdot D_n \cdot \frac{a_n}{a_{n+1}} - \frac{1}{\rho} \cdot D_{n+1} \geq 1. \tag{18}$$

Now, clearly, if $\sum d_n$ is divergent then so is $\sum \rho \cdot d_n$. Hence the terms $\rho \cdot d_n$ can be expressed by means of a sequence $(M_n)$ according to (16) in the following way

$$\rho \cdot d_n = \frac{M_n - M_{n-1}}{M_{n-1}}$$

which yields

$$\frac{1}{\rho} \cdot D_n = \frac{1}{\rho \cdot d_n} = \frac{M_{n-1}}{M_n - M_{n-1}}.$$

By substitution for $\frac{1}{\rho} \cdot D_n$ in (18) we get

$$\frac{M_{n-1}}{M_n - M_{n-1}} \cdot \frac{a_n}{a_{n+1}} - \frac{M_n}{M_{n+1} - M_n} \geq 1.$$

Subtracting 1 and multiplying by $M_n$ gives

$$\frac{M_n \cdot M_{n-1}}{M_n - M_{n-1}} \cdot \frac{a_n}{a_{n+1}} - M_n \cdot (1 + \frac{M_n}{M_{n+1} - M_n}) \geq 0$$

that is

$$(19) \qquad \frac{M_n \cdot M_{n-1}}{M_n - M_{n-1}} \cdot \frac{a_n}{a_{n+1}} - \frac{M_{n+1} \cdot M_n}{M_{n+1} - M_n} \geq 0.$$

Yet according to the converse statement following (15) the terms $\frac{M_n \cdot M_{n-1}}{M_n - M_{n-1}}$ and $\frac{M_{n+1} \cdot M_n}{M_{n+1} - M_n}$ define terms $C_n = \frac{1}{c_n}$ and $C_{n+1} = \frac{1}{c_{n+1}}$ such that $\sum c_n$ converges. In other words, (19) can be written in the form

$$C_n \cdot \frac{a_n}{a_{n+1}} - C_{n+1} \geq 0$$

from which the convergence of $\sum a_n$ follows because of (9).
This finishes the proof of Kummer's test:

$$\lim_{n \to \infty} (B_n \cdot \frac{a_n}{a_{n+1}} - B_{n+1}) > 0 \quad \Longrightarrow \quad \sum a_n \text{ converges}.$$

According to Pringsheim this proof gives "the true reason why the $C_n$ which naturally occur in (5) can eventually be replaced by *completely arbitrary positive* numbers $B_n$" (Pringsheim 1916, p. 379).

Although Pringsheim's proof of Kummer's test explains why an arbitrary sequence $(B_n)$ occurs in it, it does not by itself solve a further mystery about Kummer's test, i.e. its apparent isolation within the general theory of convergence tests. According to Pringsheim (as already quoted above) Kummer's test seemed totally erratic because of its surprising generality and because it completely lacks, as a convergence test of the $2^{nd}$ kind, any analogue among the convergence tests of the $1^{st}$ kind. Pringsheim wants to free it from this (apparent) isolation and "show how it naturally fits into a systematically developed general theory". (Pringsheim 1916, p. VI)[12] To be sure, Pringsheim's explanatory proof already achieves something towards this goal of integration by making fully explicit how this test is connected with the basic form of comparison tests (9)-(12), but it doesn't relate it in any way to comparison tests of the $1^{st}$ kind. In order to do that and to remove the structural asymmetry Pringsheim supplies the missing analogue to Kummer's test by constructing a test of the $1^{st}$ kind exhibiting the same extreme generality. What Pringsheim is engaged in here is yet another explanatory project which goes beyond giving explanatory proofs. Rather, he aims at a "global" explanation of Kummer's test by embedding it in a reorganized theory. This kind of explanatory concern ties in very well with Pringsheim's approach to the foundations of complex analysis (cf. Mancosu 2001), and it also shows, again, that explanations in mathematical practice come in a wide variety. It certainly deserves to be analyzed in more detail and we refer the interested reader to part II of the appendix where we provide a derivation of Pringsheim's analogue to Kummer's test; however, since Steiner addresses almost exclusively *proofs* and their explanatoriness, we will focus in what follows on Pringsheim's proof of Kummer's test.

## 7. A TEST CASE FOR STEINER'S THEORY

How well can Steiner account for Pringsheim's explanation? An analysis of the explanatory nature of Pringsheim's proof would have to proceed from a characterizing property of some entity or structure in the result to be proved, i.e. in Kummer's convergence test (13). The proof counts as explanatory according to Steiner only if it makes it evident that the conclusion depends on this property. But here we already face a major difficulty. All "entities" in Kummer's test are *generic*, no concrete objects are mentioned in it (apart from the number 0 of course, but the proof is clearly not based on any characterizing property of 0). This generality makes it hard to come up with a property that uniquely determines some entity within a family of them. Indeed, the *complete arbitrariness* of the sequence $(B_n)$ in (13) makes Steiner's account come unstuck. It is obvious that this arbitrary sequence $(B_n)$ is the focus of Pringsheim's proof. After all, it is the very feature of Kummer's test that makes it so puzzling, thus prompting Pringsheim to provide an explanatory proof (different from Stolz's proof which verifies but doesn't explain the result). Yet, $(B_n)$ cannot be "characterized" in any way – the imposition of any constraining property would obviously result in *non*-arbitrariness! An arbitrary sequence simply cannot be distinguished – qua *arbitrary* sequence – within the family of all sequences by any property. That's just what it means to be arbitrary. Hence one couldn't base any proof on a characterizing property of $(B_n)$ (nor of $(a_n)$ for that matter, which are equally arbitrary), and so it's no surprise that no such property appears in Pringsheim's proof. Consequently, Steiner's account renders it non-explanatory because it fails to satisfy a necessary condition for explanatoriness. In other words, with respect to Pringsheim's proof Steiner finds himself plainly at odds with the *practice of explanation* in mathematics.

At this point one might object the following.[13] Although $(B_n)$ stands for an arbitrary sequence of positive terms, any such sequence has the property of giving rise to a series which is either convergent or divergent. And this in turn holds if and only if the terms $B_n$ can be represented according to the formulas (15) or (16) respectively. These representational facts are central to Pringsheim's proof. Exploiting them distinguishes it from Stolz's proof and constitutes a distinctive feature of it as an *explanatory* proof – as Pringsheim would argue. However, Steiner could maintain his account and make it work based on the following disjunctive property $C(x)$ *or* $D(x)$, which also incorporates the representation expressed by (16). Define $C(x)$ to be true of a sequence $(B_n)$ if and only if for all $n \geq 1,\ B_n > 0$ and $\Sigma\frac{1}{B_n}$ converges; and define $D(x)$ to be true of a sequence $(B_n)$ if and only if for any $\rho > 0$ there exists a strictly increasing sequence $(M_n)$, $n = 0, 1, 2, \ldots$, of positive numbers satisfying $\lim_{n\to\infty} M_n = +\infty$ such that for all $n \geq 1$, $\frac{1}{\rho} \cdot B_n = \frac{M_{n-1}}{M_n - M_{n-1}}$.

Pringsheim's proof clearly invokes and relies on the property $C(x)$ *or* $D(x)$, one can as it were "read it off" the proof structure directly.[14] Moreover this property is both necessary and sufficient for being an (arbitrary) sequence of positive numbers. Thus we have apparently managed to identify a characterizing property of $(B_n)$ after all.

This, however, is not the case. Steiner's account cannot be salvaged in this way. On closer inspection it turns out that $C(x)$ *or* $D(x)$ won't do as a characterizing property. To begin with, we should like to point out how the *problem of characterizing arbitrariness* recurs with respect to $C(x)$ *or* $D(x)$, which can be seen as the dual difficulty of the one mentioned above. Let's recall Steiner's definition of 'characterizing property'. It is defined as "a property unique to a given entity or structure within a *family* or domain of such entities or structures" (Steiner 1978, p. 142), i.e. such a property "picks out one from a *family*" (Steiner 1978, p. 147). One of Steiner's own paradigm examples, as mentioned already earlier, is "having a certain prime power expansion", which uniquely determines a number $n$ within the domain of all natural numbers. Now, it is obvious that the property $C(x)$ *or* $D(x)$ is *not* a characterizing property according to this definition, it *fails* to pick out any *particular* sequence of positive numbers. In this respect it is analogous for instance to the property "$n$ is even or $n$ is odd", which does not single out any particular element from the set of natural numbers. So $C(x)$ *or* $D(x)$ cannot be used by Steiner to account for the explanatoriness of Pringsheim's proof; as a (supposedly) *characterizing* property of sequences, being true of *every* sequence in the domain, it fails as badly as it is possible for a property to fail. We can now sum up Steiner's predicament as follows. No property which is indeed unique to a certain sequence. i.e. which in fact "picks out one from a family", can characterize arbitrary sequences in general. On the other hand, a property like $C(x)$ *or* $D(x)$ which holds true of *all* (and only) sequences of positive numbers fails to be characterizing in Steiner's sense.

This conclusion is based on the most straightforward understanding of the notion *characterizing property* in our context, namely as a property applying to an individual sequence. It might be tempting to think that the above predicament could be avoided by an appropriate reconstrual of that notion. So we have to explore in detail other options of interpreting 'characterizing property' and point out why none of them works. More precisely, we will show that neither construing $C(x)$ *or* $D(x)$ as characterizing a *set* of sequences (as opposed to an individual sequence), nor the weakening of *characterization* to *partial characterization* (of individual sequences) succeeds in the twofold task of (i) rendering $C(x)$ *or* $D(x)$ a characterizing property and, in turn, Pringsheim's proof explanatory; while (ii) remaining consistent with Steiner's theory in other respects especially concerning his own examples of characterizing properties and of explanatory as well as non-explanatory proofs. But before taking this up we need to address an even more basic problem, which is *completely independent* of how we construe the notion of characterizing property, yet whose solution is a *prerequisite* for a precise statement of Steiner's theory in the first place.

However 'characterizing property' might be defined in particular, it has to be first of all a property of "an entity or structure *mentioned in the theorem*" (Steiner 1978, p. 143 our italics, cf. also p. 147). And here we come up against a difficulty in Steiner's theory. Failing to provide any definitions of 'entity','structure', and most important 'mention in a theorem' Steiner left his theory vague or incomplete in crucial respects. In the absence of clear criteria to determine which, if any entities or structures are indeed mentioned in a theorem we may be unable in certain cases to even get started on applying Steiner's theory. What, for instance, is mentioned in Kummer's test? Certainly no object like *the* generic arbitrary sequence (whatever that may be); earlier we were speaking loosely when we said that apart from the number 0 all entities (or rather "entities") mentioned in Kummer's test (13) were generic. There are no singular terms in (13) referring to (particular or generic) sequences. The expression '$B_n$' is to be construed as a *variable* in the scope of a universal quantifier (and the same holds for the expression '$a_n$'). Hence unless we take (the elements in) the domain of discourse over which the quantifiers range as something which is "mentioned in a theorem" – and prima facie it is by no means clear whether this is the right way to go – there is *no explicit mention of sequences* in Kummer's test. Consequently, if we should have good reasons *not* to count quantifier ranges among what is mentioned in a theorem, then the whole issue as to whether or not $C(x)$ *or* $D(x)$ is a characterizing property would simply be preempted – there being no appropriate, i.e. mentioned, entity in Kummer's test which it could be the property of. In other words this attempt to make Steiner's account work vis à vis Pringsheim's proof would seem wrong-headed from the very start, and the same goes for any other attempt based on a supposedly characterizing property of sequences.

It is important to emphasize that we are dealing here not just with a marginal problem which comes up only with respect to quantifier ranges or in the context of Kummer's test. The problem is much more general. Take for instance a theorem containing the predicate '$x$ is prime'. Does this theorem mention the *property* (or the *concept*) of being prime, the *set* of all prime numbers, all the *individual prime numbers*, or none of the foregoing? Steiner remains silent on how to answer questions like this one in general; and some of the examples he provides rather than clarifying things add even further to the confusion – witness his remarks concerning explanatory proofs of the summation theorem

$$\text{(20)} \qquad \text{For all } n, \quad 1+2+\cdots+n = \frac{n(n+1)}{2}.$$

Steiner's remarks imply that he apparently takes the symmetry properties as well as the geometrical properties of the sum $1+2+\cdots+n$ as something – entities or structures? – *mentioned* in (20). This is very puzzling indeed and just highlights the need for precise definitions here. In the absence of such definitions, to repeat our point from above, we don't even have a clear enough grasp of Steiner's theory in order to apply and assess it in general.

Luckily, for our purpose of assessing Steiner's theory vis à vis Pringsheim's proof we don't need to solve the general problem. And concerning the question whether or not (the elements in) the range of quantifiers should be taken, on Steiner's view, to be indeed – explicitly or perhaps implicitly – mentioned in Kummer's test we don't have to resort to mere speculation either, since Steiner provides an answer to a question exactly parallel to ours when he discusses the inductive proof of theorem (20) above. Steiner argues that this proof is not explanatory because it lacks a characterizing property.

> "The proof by induction does not characterize anything mentioned in the theorem. Induction, it is true characterizes the *set* of all natural numbers; but this *set* is not mentioned in the theorem" (Steiner 1978, p. 145, emphasis in the original).

The set **N** of natural numbers is the range of the universal quantifier in (20) as the set **B** of sequences of positive numbers is the range of the universal quantifier in (13), Kummer's test, (once its quantificational structure has been made fully explicit). Moreover, although $C(x)$ *or* $D(x)$ clearly fails as a characterizing property of any particular sequence it can be argued, very much in line with one of Steiner's own examples,[15] that it does characterize the set **B** since we have for every sequence $s$ of real numbers

$$C(s) \text{ } or \text{ } D(s) \leftrightarrow s \in \mathbf{B}.$$

However, if according to Steiner the principle of induction does not characterize anything mentioned in (20), then by the same token neither $C(x)$ *or* $D(x)$ nor, for that matter, any other property true of all and only sequences of positive numbers characterizes anything mentioned in (13). To paraphrase Steiner: $C(x)$ *or* $D(x)$, it is true, characterizes the *set* **B** (within some family of sets of sequences); but this *set* is not mentioned in Kummer's test.[16]

This presents a real stumbling block for any attempt to account for the explanatoriness of Pringsheim's proof based on a property true of all and only sequences of positive numbers. Insisting that any such property is indeed a (characterizing) property of something mentioned in (13) implies, to repeat our point, by parity of reason, the rejection of Steiner's explicit claim that the principle of induction fails to be a property of anything mentioned in (20). Now, Steiner might be willing to give up his position here in order to account, in turn, for the explanatoriness of Pringsheim's proof (that is, pending its deformability into related proofs), since prima facie this concession might seem a relatively small price to pay.[17]

After all, it is *not* tantamount to pronouncing the inductive proof of (20) explanatory – which would indeed be very counterintuitive! More would be needed for that as Steiner himself emphasizes.

> "[...] a characterizing property is not enough to make an explanatory proof. One must be able to generate new, related proofs by varying the property and reasoning again. Inductive proofs usually do not allow deformation, since before one reasons one must have already conjectured the theorem" (Steiner 1978, p. 151 fn. 11).

Unfortunately for Steiner, though, the inductive proof of (20) *does* allow for deformation. Let us briefly sketch how it works. The property to be varied is the principle of induction which characterizes **N** within the family of, say, sets in the power-set of **N**. As a property of sets it contains a free set variable $X$.

$$1 \in X \ \& \ \forall x(x \in X \rightarrow (x+1) \in X) \ \&$$
$$\forall P[(P(1) \ \& \ \forall x(P(x) \rightarrow P(x+1))) \rightarrow (\forall x \in X,\ P(x))]$$

We'll use '$IND(1, x+1)$' as a convenient shorthand thus also clearly displaying its parameters. It should be obvious that $IND(1, x+1)$ besides characterizing **N** also passes Steiner's *dependence test* which is necessary to make a proof explanatory. This test requires

> "[...] that from the proof it is evident that the result depends on the [characterizing] property. It must be evident, that is, that if we substitute in the proof a different object of the same domain, the theorem collapses" (Steiner 1978, p. 143).

Trivially, theorem (20) could not be established by an inductive proof without $IND(1, x+1)$. In other words, if we substitute in the proof a different set of our domain, i.e. a proper subset of **N**, and a corresponding, different (restricted) induction principle, then we are clearly blocked from concluding (20).

Let's now turn to "deformations" of the principle of induction. Appropriate variation of $IND(1,x+1)$ yields characterizing properties of different sets in the given family. Below we list in pairs deformed induction principles and the respective sets characterized by them ($a$ and $b$ denote natural numbers).

| | |
|---|---|
| $IND(2,x+2)$ | $\mathbf{E} = \{2,4,6,8,\ldots\}$ |
| $IND(3,x+3)$ | $\mathbf{T} = \{3,6,9,12,\ldots\}$ |
| $IND(a,x+a)$ | $\mathbf{M}_a = \{a,2a,3a,4a,\ldots\}$ |
| $IND(2a,x+2a)$ | $\mathbf{E}_a = \{2a,4a,6a,8a,\ldots\}$ |
| $IND(1,x+2)$ | $\mathbf{O} = \{1,3,5,7,\ldots\}$ |
| $IND(a,x+2a)$ | $\mathbf{O}_a = \{a,3a,5a,7a,\ldots\}$ |
| $IND(a,x+b)$ | $\mathbf{L}_{a,b} = \{a,\ a+b,\ a+2b,\ a+3b,\ldots\}$ |
| $IND(2,x+1+\sqrt{4x+1})$ | $\mathbf{Q} = \{1\cdot 2,\ 2\cdot 3,\ 3\cdot 4,\ 4\cdot 5,\ldots\}$ |

We use lower case letters as variables ranging over the elements in the sets named by the respective upper case letters. For any variable '$v$' and its respective range $V$, we use '$v^+$' as notation for the successor of $v$ in $V$. The following rendering of theorem (20), which incorporates this successor notation, will be the basis for the array of related theorems obtained by a process of deformation.

$$\text{(21)} \qquad \text{For all } n, \quad 1+2+\cdots+n = \frac{n\cdot n^+}{2} = \frac{n\cdot n^+}{2(n^+-n)}.$$

And here are the related theorems.

$$\text{(22)} \qquad \text{For all } e, \quad 2+4+\cdots+e = \frac{e\cdot e^+}{2(e^+-e)} = \frac{e\cdot e^+}{4}.$$

$$\text{(23)} \qquad \text{For all } t, \quad 3+6+\cdots+t = \frac{t\cdot t^+}{2(t^+-t)} = \frac{t\cdot t^+}{6}.$$

$$\text{(24)} \qquad \text{For all } m_a, \quad a+2a+\cdots+m_a = \frac{m_a\cdot m_a^+}{2(m_a^+-m_a)} = \frac{m_a\cdot m_a^+}{2a}.$$

(25) For all $e_a$, $$2a+4a+\cdots+e_a = \frac{e_a \cdot e_a^+}{2(e_a^+ - e_a)} = \frac{e_a \cdot e_a^+}{4a}.$$

(26) For all $o$, $$1+3+\cdots+o = \frac{o \cdot o^+ + 1}{2(o^+ - o)} = \frac{o \cdot o^+ + 1}{4}.$$

(27) For all $o_a$, $$a+3a+\cdots+o_a = \frac{o_a \cdot o_a^+ + a^2}{2(o_a^+ - o_a)} = \frac{o_a \cdot o_a^+ + a^2}{4a}.$$

(28) For all $l_{a,b}$, $$a+(a+b)+\cdots+l_{a,b} = \frac{l_{a,b} \cdot l_{a,b}^+ + ab - a^2}{2(l_{a,b}^+ - l_{a,b})} = \frac{l_{a,b} \cdot l_{a,b}^+ + ab - a^2}{2b}.$$

(29) For all $q$, $$2+\cdots+q = \frac{q \cdot q^+}{\frac{3}{2}(q^+ - q)}.$$

A few comments are in order. Each of the theorems results from deforming the inductive proof of (20) by substituting a different subset of **N** together with its corresponding induction principle. Throughout the array of these proofs the "proof idea", induction (in various forms), is held constant. Although theorems (22), (23), (24), and (25) show in a straightforward way how theorem (20) changes in response to substituting in place of **N**, respectively, the set of even numbers, the set of multiples of 3, then more generally the set of multiples of $a$, and the set of even multiples of $a$; it has to be kept in mind that as a rule the process of "deformation" involves reworking, "not just mechanical substitution" (Steiner 1978, p. 147). In the case of (26) concerning the set **O** of odd numbers we need to observe that the recursive characterization of the members of **O** by $IND(1, x+2)$ yields $o = 1+2k$, $o^+ = (1+2k)+2$, for some $k \geq 0$. Hence $\frac{o \cdot o^+}{2(o^+ - o)} = \frac{4k^2+8k+3}{4}$. Each summand in the numerator, except 3, is divisible by 4, so in order to ensure getting an integer as a result we add 1 to the numerator and thus arrive at formula (26), which is then proved by induction according to $IND(1, x+2)$. (Of course, subtracting 3 may seem, prima facie, an equally plausible alternative here, but adding 1 is favored by staying closer to the original form of the summation theorem, i.e. by keeping the deformation minimal. Also the choice between between "adding 1" and "subtracting 3" can be decided by checking the resulting formulas against the summation of $1 + \cdots + o$ letting $o = 1$ (and $o^+ = 3$). It has to be stressed, however, that this slight element of trial and error can be completely avoided once the theorems (20) and (22) have been established.[18]) Deformations of a very similar kind[19] lead to further generalizations expressed in (27) and (28). The latter is a general theorem covering the summation of *arbitrary linear progressions* of natural numbers. Moreover, one can generalize even beyond linear progressions, as shown by (29), if one doesn't stick exclusively to deformations by means of additive terms involving only constants (and parameters).[20]

Although we could continue our list of generalizations of theorem (20) we stop here because the point should be clear by now. The inductive proof of theorem (20) meets all of Steiner's requirements to count as explanatory[21] – *provided*, that is, quantifier ranges are indeed taken to be entities which are *mentioned* in theorems. This puts Steiner in a dilemma. If he maintains that in general theorems make *no* mention of quantifier ranges, then $C(x)$ *or* $D(x)$ is ruled out out as a characterizing property. And since this is the most promising, perhaps even the only, candidate for such a property that could render Pringsheim's proof explanatory, Steiner's account seems bound to *undergenerate*, i.e. it seems thus blocked from fully capturing the intuitive notion of explanatory proof operative in mathematical practice. On the other hand, including quantifier ranges among the entities mentioned in theorems results in *overgeneration* by declaring, as we have just seen, the inductive proof of (20) explanatory, which it clearly isn't – neither by Steiner's own lights nor, as a rule, according to the understanding of working mathematicians (some mathematicians even take inductive proofs to be *paradigms of non-explanatory* proofs). So either way Steiner's theory runs counter to mathematical praxis.

Let us note, for the records, that this gives rise to an independent criticism of Steiner's account, since we can easily restate theorem (20), without any changes to its proof, avoiding sorted variables and making sure $\mathbf{N}$ is explicitly mentioned in it.

$$\text{For all } x \text{ in } \mathbf{N}, \quad 1 + 2 + \cdots + x = \frac{x(x+1)}{2}.$$

Now overgeneration is inevitable. Furthermore, it seems quite odd that Steiner's theory *qua* theory of the explanatoriness of *proofs* should turn out to be so overly sensitive to what appears to be a rather minor detail in the exact wording of a theorem which doesn't affect its proof.

Setting aside now the issue concerning quantifier ranges, let us investigate further how Steiner's account fares in the attempt to render Pringsheim's proof explanatory in terms of $C(x)$ *or* $D(x)$ as a characterizing property of the set **B**. After all, despite the fact that Steiner's account overgenerates there is still a question of independent interest as to whether or not it undergenerates as well. So let us grant that the set **B** of sequences of positive numbers is in some way or other indeed mentioned in Kummer's test. Then $C(x)$ *or* $D(x)$ does characterize **B**, and this property is also clearly exploited in Pringsheim's proof. But Steiner requires more, i.e. $C(x)$ *or* $D(x)$ has to pass Steiner's *dependence test*. In other words, it must be evident "that if we substitute in the proof a different object of the same domain, the theorem collapses" (Steiner 1978, p. 143). This raises the question, first of all, what the domain should be taken to consist of. When Pringsheim gives his proof of Kummer's test he is working exclusively with sequences of positive numbers, hence it appears most natural to take the power-set of **B** as the domain – from which **B** is then singled out by our characterizing property. However, this is already as far as we can get within Steiner's theory, since $C(x)$ *or* $D(x)$ obviously fails the dependence test. Once again it is the extreme generality of Kummer's test which creates a problem here. Since this convergence test works for *arbitrary* sequences $(B_n)$ of positive numbers, it clearly *won't collapse* no matter what subset of **B** gets substituted and its proof won't really be affected by it either! In order to make Kummer's test collapse we have to go outside of **B** and allow sequences to contain arbitrary real numbers $\neq 0$, positive and negative.[22] This constitutes already a deviation from Pringsheim's original setting yet even further adjustments are needed to make Steiner's theory work. Letting **S** be the set of arbitrary sequences of non-zero real numbers and $\mathcal{P}$ the power-set of **S**, we could, as a first try, take our domain $\mathcal{D}$ to contain the elements of $\mathcal{P}$ minus all the proper subsets of **B**. However, a closer look at Kummer's test, which is stated in terms of a *limit*, and at Pringsheim's proof reveals that neither of them demands $(B_n)$ to consist *exclusively* of positive numbers. Kummer's test still holds good and Pringsheim's proof goes through if we only require that *all but finitely many* terms of $(B_n)$ are positive, i.e. that there exists an $m$ such that for all $n \geq m$, $B_n > 0$; finite initial segments of $(B_n)$ don't matter. In other words, substituting for **B** the set $\mathbf{B}^*$, the superset of **B** which comprises all such "eventually positive" sequences, won't make the theorem (nor the proof of it) collapse. Hence $C(x)$ *or* $D(x)$ still fails the dependence test with respect to domain $\mathcal{D}$, that is, it does not fully capture – neither in the technical sense of Steiner's theory nor in the intuitive sense – what property of $(B_n)$ Kummer's test really depends on.

At this point Steiner has two options.[23] He could either further tailor the domain $\mathcal{D}$ to the purpose at hand by simply excluding $\mathbf{B}^*$ (and various other sets) from it, thus ensuring by brute force that $C(x)$ *or* $D(x)$ passes the dependence test. But this is unacceptable not only because it amounts to a completely artificial, *ad hoc* "immunization manoeuvre" to save his theory in the face of recalcitrant data. More importantly, such a move goes against the spirit of Steiner's theory. On his account the explanation provided by a proof consists (besides generalizability) in showing that and how the proved theorem depends on a certain characterizing property. In other words, an explanatory proof makes it evident that the characterizing property in question pinpoints the reason why, "essentially", the theorem is true. As we have seen, restricting quantification to elements of $\mathbf{B}$ is not essential for the truth of Kummer's test, it is a sufficient but not a necessary condition. So, pronouncing Pringsheim's proof explanatory in virtue of a spurious dependence of Kummer's test on the property $C(x)$ *or* $D(x)$ yields a correct result for a wrong reason.

Steiner's other option is to first *generalize* Kummer's test by explicitly turning it into a convergence test quantifying over sequences from $\mathbf{B}^*$, i.e. to get the dependence right, and then account with his theory for the explanatoriness of an – equally generalized – proof of it. This would then have to be done in terms of a correspondingly generalized property $C^*(x)$ *or* $D^*(x)$. But now the property $C(x)$ *or* $D(x)$ as well as Pringsheim's original proof are out of the picture, instead we are dealing with a *different* proof (and a *different* theorem), even though the difference consists merely in a slight generalization. Steiner can't claim that the two proofs are "essentially the same", since one turns out to be explanatory (if everything works out) while the other one doesn't. So we have to take them as in fact two distinct proofs. But in this case rendering one of them explanatory doesn't tell us anything about the explanatoriness of the other. Hence Pringsheim's proof still escapes Steiner's theory.

Let us finally look at the interpretation of $C(x)$ *or* $D(x)$ as a *partially characterizing* property of sequences – if only to point out why it won't help. Steiner concedes that in order to account for the explanatoriness of certain proofs the notion of *characterization* has to be weakened to that of *partial characterization*. It is quite common

> "to study domain $X$ by assigning a counterpart $Y$ to each object in $X$. The object in $Y$ need not uniquely characterize anything in $X$; examples are Galois theory and algebraic topology" (Steiner 1978, pp. 149f).

One worry one might have here at the outset concerning the introduction of partially characterizing properties is the danger of inflation. Although Steiner doesn't give us much to go on, presumably *any* property counts as partially characterizing unless, in the extreme cases, a property happens to be either empty or true of everything of the domain (which seem to be the only cases in which we can't plausibly claim that such a property characterizes anything at all even partially). Thus once partially characterizing properties are admitted to account for explanatoriness Steiner may find himself on the slippery slope to a vast overgeneration of his account.

Given the plausible restrictions on the notion of *partial characterization* just mentioned it follows that $C(x)$ *or* $D(x)$, being true of *every* sequence in Pringsheim's domain, won't even pass as a partially characterizing property of sequences. Which shouldn't come as a surprise since the arbitrariness of $(B_n)$ excludes even partial characterization within the domain **B**. Again we have to move beyond **B** and, in turn, the problem of passing the dependence test recurs.

So far all our attempts to get Steiner's theory off the ground vis à vis Pringsheim's proof have failed. Our best candidate for a (partially) characterizing property turned out either not to be (even partially) characterizing at all or still unable to do the job of rendering Pringsheim's proof explanatory. And in the absence of a characterizing property it doesn't even make sense to ask whether or not Steiner's second main criterion for explanatory proofs, generalizability, is satisfied. Because generalizability *presupposes* that there is in fact a characterizing property on which the theorem depends such that if we substitute (the characterizing property of) a different object of the domain we get a related "deformed" theorem. One has to be careful here, however, to distinguish generalizability from mere generality. It is well known that Kummer's test, because of its generality, is the source of many other convergence tests. By substituting specific sequences for $(B_n)$ one can obtain from it (or from Pringsheim's proof) as special cases, for instance, D'Alembert's test, Raabe's test, and Bertrand's test (cf. Tong 1994). But these tests are *special cases* of Kummer's test they are *not* gotten by generalizing it in the relevant sense of Steiner's theory. And Steiner himself is very clear about it. He states with respect to an analogous situation

> "The new result is contained within the old. The point is, however, that generalizability through varying a characterizing property is what makes a proof explanatory, not simple generality" (Steiner 1978, p. 146).

In other words, what has turned out again and again to be a difficult problem for Steiner's account, namely the generality of Kummer's test, cannot simply be declared a virtue which renders, by itself, Pringsheim's proof explanatory. There is no such "shortcut" in Steiner's theory from mere generality to explanatoriness.

Although there are many more features of Steiner's theory that deserve thorough reconstruction and critical assessment they are not of major importance in the given context so we conclude our discussion at this point. Enough has been said to bring out the substantial difficulties Steiner's theory has to account for the explanatoriness of Pringsheim's proof of Kummer's test – besides other problems of a general nature which came to light in the course of our investigations. It is our hope that this kind of testing theories of mathematical explanation against the practice of mathematical explanation will pave the the way to further studies in the same vein. This seems to us the most promising approach for making progress in this treacherous area.

*Department of Philosophy*
*U.C. Berkeley*
*USA*

## APPENDIX

*Part I.*

We show how to arrive at the equations (15) and (16) above, i.e. how to represent the terms $c_n$ and $d_n$ of convergent resp. divergent series by means of the positive terms $M_n$ of strictly increasing divergent sequences (cf. Pringsheim 1916, pp. 326ff and 332).

(A) Let $c_n$ be the terms of a convergent series, i. e. $\sum_{n=1}^{\infty} c_n = s$. We set $s_0 = 0$ and, for $n = 1, 2, 3, \ldots$, $s_n = c_1 + \ldots + c_n$. Notice that $s - s_n > 0$ for all $n$, since the $c_n$ are positive so $s > s_n$ for all $n$. We can therefore define, for $n = 0, 1, 2, \ldots$

$$M_n = \frac{1}{s - s_n}.$$

Since the sequence $(s - s_n)$ is strictly decreasing and converges to 0 it follows that $(M_n)$ is a strictly increasing sequence such that $\lim_{n\to\infty} M_n = +\infty$. Furthermore, we have

$$s - s_n = \frac{1}{M_n}$$

and, for $n = 1, 2, ...$

$$s - s_{n-1} = \frac{1}{M_{n-1}}$$

Now, for $n = 1, 2, ...$ it holds that $s_n = s_{n-1} + c_n$ and we can thus write

$$c_n = s_n - s_{n-1} = (s - s_{n-1}) - (s - s_n) = \frac{1}{M_{n-1}} - \frac{1}{M_n} = \frac{M_n - M_{n-1}}{M_n \cdot M_{n-1}}.$$

To show the converse, assume $(M_n)$ to be a strictly increasing sequence $(n = 0, 1, 2, \ldots)$ of positive numbers such that $\lim_{n\to\infty} M_n = +\infty$. Let $c_n = \frac{M_n - M_{n-1}}{M_n \cdot M_{n-1}} = \frac{1}{M_{n-1}} - \frac{1}{M_n}$. Then

$$\sum_{n=1}^{k} c_n = \sum_{n=1}^{k} \left(\frac{1}{M_{n-1}} - \frac{1}{M_n}\right) = \frac{1}{M_0} - \frac{1}{M_k}.$$

As $\lim_{k\to\infty} \frac{1}{M_k} = 0$,

$$\sum_{n=1}^{\infty} \left(\frac{1}{M_{n-1}} - \frac{1}{M_n}\right) = \frac{1}{M_0}$$

hence $\sum c_n$ is convergent.

(B) Let's now turn to the case of a divergent series $\sum_{n=1}^{\infty} d_n$ (such that $d_n > 0$). We first observe that

$$(1+d_1)(1+d_2)\cdots(1+d_k) \geq 1 + \sum_{n=1}^{k} d_n.$$

Since $\sum d_n$ is divergent, the left hand side also diverges as $k \to +\infty$. Furthermore, every factor $(1+d_i)$ in the product is $> 1$, hence the sequence $(M_n)$ as defined by

$$M_0 = 1$$
$$M_n = (1+d_1)\cdots(1+d_n)$$

is strictly increasing. For $n > 1$ we have $M_{n-1} = (1+d_1)\cdots(1+d_{n-1})$ and by division we get

$$\frac{M_n}{M_{n-1}} = 1 + d_n$$

hence

$$d_n = \frac{M_n}{M_{n-1}} - 1 = \frac{M_n - M_{n-1}}{M_{n-1}}.$$

This equation also holds for $n = 1$ by definition of $M_0$.

Conversely, let $(M_n)$ be a strictly increasing sequence $(n = 0, 1, 2, \ldots)$ of positive numbers such that $\lim_{n\to\infty} M_n = +\infty$. We have to show that $\sum d_n$ is divergent, where $d_n = \frac{M_n - M_{n-1}}{M_{n-1}}$. We start by noting that $\sum (M_n - M_{n-1})$ is divergent.[24] Because

$$\sum_{n=1}^{k} (M_n - M_{n-1}) = M_k - M_0$$

hence

$$\sum_{n=1}^{\infty} (M_n - M_{n-1}) = (\lim_{n\to\infty} M_n) - M_0 = +\infty.$$

Applying the logarithm function to the terms $M_n$ yields a divergent sequence $(\log M_n)$. Hence the previous result implies that also $\sum(\log M_n - \log M_{n-1})$ diverges. On the other hand, because of the equation

$$\log x < x - 1 \quad \text{for} \quad x > 0, \quad x \neq 1$$

and the fact that, for all $n$, $\frac{M_n}{M_{n-1}} > 1$ we have

$$\log M_n - \log M_{n-1} = \log \frac{M_n}{M_{n-1}} < \frac{M_n}{M_{n-1}} - 1 = \frac{M_n - M_{n-1}}{M_{n-1}}.$$

So by the comparison test (2) we conclude that a series $\sum d_n$ is indeed divergent if its terms satisfy

$$d_n = \frac{M_n - M_{n-1}}{M_{n-1}}.$$

*Part II.*

In the construction of a convergence test of the $1^{st}$ kind that exhibits the same kind of generality as Kummer's test Pringsheim proceeds as follows.[25]
As a special case of comparison test (3) we have

$$\text{If for all } n \geq m, \text{ for some } m, \ C_n \cdot a_n < 1 \quad \Longrightarrow \quad \sum a_n \text{ converges.}$$

Since all partial sums $s_n = \sum_{k=1}^{n} c_k$ are positive, this is equivalent to

$$\text{If for all } n \geq m, \text{ for some } m, \ (C_n \cdot a_n)^{\frac{1}{s_n}} < 1 \quad \Longrightarrow \quad \sum a_n \text{ converges.}$$

And by assuming the existence of the involved limit we get

(30) $$\lim_{n\to\infty} (C_n \cdot a_n)^{\frac{1}{s_n}} < 1 \quad \Longrightarrow \quad \sum a_n \text{ converges.}$$

On the other hand, letting $M_n$ as before denote the positive terms of a strictly increasing divergent sequence, we can show the following. (Its proof, though not difficult, is a bit more involved hence we postpone it for the sake of greater perspicuity of the main argument.)

(31) $$\lim_{n\to\infty} \left(\frac{a_n}{M_n - M_{n-1}}\right)^{\frac{1}{M_n}} < 1 \quad \Longrightarrow \quad \sum a_n \text{ converges.}$$

By setting

(32) $$d_1 = M_1 \quad \text{and} \quad d_n = M_n - M_{n-1} \ (n = 2, 3, 4, \ldots)$$

we obtain terms of a divergent series.[26] Furthermore we have

$$M_n = M_1 + \sum_{k=2}^{n} (M_k - M_{k-1}) = \sum_{k=1}^{n} d_k = s_n.$$

Thus by observing that $D_n = \frac{1}{M_n - M_{n-1}}$ convergence test (31) can be stated as follows

$$\lim_{n\to\infty} (D_n \cdot a_n)^{\frac{1}{s_n}} < 1 \quad \Longrightarrow \quad \sum a_n \quad \text{converges.} \tag{33}$$

The construction of the terms $d_n$ (resp. $D_n$) out of the given sequence $(M_n)$ does not - contrary to how it may appear - impose a constraint on the nature of the divergent sequence that can occur in (33), since the terms of *any* divergent sequence $\sum d_n$ admit of such a representation (32) by simply defining the required sequence $(M_n)$ thus

$$M_n = \sum_{k=1}^{n} d_k.$$

(In effect, what we have obtained here is another, simpler and more straightforward, representation of the terms $d_n$ than the one given by (16) above.) Now we are in a position to state a most general convergence test by combining (30) and (33). We only need to note that, obviously, any *arbitrary* positive sequence $(B_n)$ is either of type $(C_n)$ or type $(D_n)$. So we finally arrive at

$$\lim_{n\to\infty} (B_n \cdot a_n)^{\frac{1}{s_n}} < 1 \quad \Longrightarrow \quad \sum a_n \quad \text{converges}$$

where $s_n = \sum_{k=1}^{n} b_k$.

This is the most general convergence test of the $1^{st}$ kind and with regard to its surpassing generality it thus represents in Pringsheim's theory "the perfect analogue to Kummer's test" (Pringsheim 1916, p. 344).

To complete the foregoing proof it remains to establish proposition (31). We first show that if $\alpha > 1$, $q > 0$, and $(M_n)$ a strictly increasing divergent sequence of positive terms, then $\alpha^{M_n}$ eventually dominates $M_n^q$, i.e. there exists an $m$ such that for all $n \geq m$

$$\alpha^{M_n} > M_n^q.$$

To this end we start from the elementary inequality

$$e^x > x \quad (\text{for } \; x > 0).$$

Setting $x = \frac{p}{q+1} \cdot M_n$ for arbitrary but fixed $p > 0$, $q > 0$ yields

$$e^{\frac{p}{q+1} \cdot M_n} > \frac{p}{q+1} \cdot M_n \quad (\text{for each } \; n).$$

By raising both sides to the $(q+1)^{st}$ power we get

$$e^{p \cdot M_n} > (\frac{p}{q+1})^{q+1} \cdot M_n^{q+1}$$

hence

$$\frac{e^{p \cdot M_n}}{M_n^q} > \left(\frac{p}{q+1}\right)^{q+1} \cdot M_n.$$

Since $\lim_{n\to\infty} M_n = +\infty$, there is an $m$ such that for all $n \geq m$ the right hand side is greater than 1, thus

$$e^{p \cdot M_n} > M_n^q \quad (\text{for all} \;\; n \geq m).$$

If $\alpha > 1$, then $\log \alpha > 0$ so we can set $p = \log \alpha$ and conclude

$$\alpha^{M_n} = e^{\log \alpha \cdot M_n} > M_n^q \quad (\text{for all} \;\; n \geq m).$$

By letting now $q = 2$ and using the fact that for all $n$, $M_{n-1} < M_n$ we infer further

$$\alpha^{M_n} > M_n^2 > M_n \cdot M_{n-1} \quad (\text{for all} \;\; n \geq m)$$

hence

$$\frac{1}{\alpha^{M_n}} < \frac{1}{M_n \cdot M_{n-1}} \quad (\text{for all} \;\; n \geq m)$$

and by multiplying by the (positive) factor $M_n - M_{n-1}$

$$\frac{M_n - M_{n-1}}{\alpha^{M_n}} < \frac{M_n - M_{n-1}}{M_n \cdot M_{n-1}} \quad (\text{for all} \;\; n \geq m).$$

We know already (cf. *Part I* above) that the terms on the right hand side are terms $c_n$ of a convergent series, so comparison test (1) implies that also the terms on the left hand side are of type $c_n$. Substituting them in test (1) yields the following (for $\alpha > 1$)

$$\text{If for all} \;\; n \geq m, \;\; \text{for some} \;\; m, \;\; a_n \leq \frac{M_n - M_{n-1}}{\alpha^{M_n}} \quad \Longrightarrow \quad \sum a_n \quad \text{converges.}$$

Equivalently

$$\text{If for all} \;\; n \geq m, \;\; \text{for some} \;\; m, \;\; \left(\frac{a_n}{M_n - M_{n-1}}\right)^{\frac{1}{M_n}} \leq \frac{1}{\alpha} \quad \Longrightarrow \quad \sum a_n \quad \text{converges.}$$

Under the assumption that the limit below exists and observing that $\frac{1}{\alpha} < 1$ we eventually obtain proposition (31)

$$\lim_{n\to\infty} \left(\frac{a_n}{M_n - M_{n-1}}\right)^{\frac{1}{M_n}} < 1 \quad \Longrightarrow \quad \sum a_n \quad \text{converges.}$$

## NOTES

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[1]For a discussion of unificationist theories of explanation, such as Kitcher's, see Tappenden's contribution in this volume.

[2]This is far from obvious, see Sandborg 1998.

[3]Kitcher 1984 seems to accept the heterogeneity of mathematical explanations. In his book "The Nature of Mathematical Knowledge" (1984) he recognizes that mathematical explanations "appear heterogeneous": "Thus, at first sight, mathematical explanations, like scientific explanations, appear heterogeneous. Whether we shall some day achieve a single model which covers all cases of scientific explanation - or even of mathematical explanation - I do not know. However, we suggest that any adequate account of explanation in general should apply to the mathematical cases ("data") presented here." (p. 227) However, his later work seems to go against the grain of the previous approach and to imply that a unification account of scientific explanation will be able to account for mathematical explanation in general – "the fact that the unification approach provides an account of explanation, and explanatory asymmetries, in mathematics stands to its credit" (p. 437 of 1989).

[4]Sandborg 1997, chapter 3, developed a similar project but we envisage a different taxonomy.

[5]The proof is given by Steiner, the proof idea being due to G. Kreisel.

[6]Since in what follows we are dealing exclusively with series and sequences of positive real numbers, the qualification "of positive terms" will be omitted throughout.

[7]As a matter of fact this existence assumption is not really needed. The criteria in question can be stated, more generally, in terms of upper limit, in (5), and lower limit, in (6), in place of limits (cf. Pringsheim 1916, p. 318; Bromwich 1942, p. 30). However, the use of the weaker formulations allows us to simplify the exposition without losing anything important in our context. The same goes for (11), (12), and (13) below.

[8]In the case of the convergence test the condition $\frac{a_{n+1}}{a_n} \leq \frac{c_{n+1}}{c_n}$ implies $\frac{a_{n+1}}{c_{n+1}} \leq \frac{a_n}{c_n}$, hence the sequence $\frac{a_n}{c_n}$ decreases monotonically and there is some number $\gamma$ such that $\frac{a_n}{c_n} \leq \gamma$. Consequently $a_n \leq \gamma \cdot c_n$ for $n \geq m$, for some $m$, which implies the convergence of $\sum a_n$. The argument in case of the divergence test is analogous.

[9]Its history is somewhat entangled. When Kummer published it in 1835 he imposed the condition that $\lim(b_n a_n) = 0$, which was shown to be superfluous by U. Dini. According to Knopp this test was later "rediscovered several times and gave rise, as late as 1888, to violent contentions on questions of priority" (Knopp 1928, p. 311 fn. 52). Dini as well as Pringsheim improved it (cf. Bromwich 1942, p.37/38).

[10][... daß das Kummersche Konvergenzkriterium] bei der sonstigen Darstellungsweise völlig abseits stand und keinerlei Analogon unter den Kriterien *erster* Art zu besitzen schien (Pringsheim 1916, p. VI).

[11]Proofs of these results are presented in the appendix, part I.

[12][... das Kummersche Konvergenzkriterium] aus dieser rätselhaften Isolierung befreit als natürliches Glied einer folgerichtig aufgebauten allgemeinen Theorie erscheinen zu lassen. (Pringsheim 1916, p. VI)

[13]This possible objection was in fact suggested to us by Klaus Jørgensen.

[14]Another property, $L(x)$, which can be drawn from Kummer's test itself – as well as from Pringsheim's proof – and which appears to characterize perhaps even more precisely than $C(x)$ *or* $D(x)$ those sequences which Kummer's test and its proof are really about, is defined as follows. Let $L(x)$ hold of a sequence $(B_n)$ if and only if for all $n \geq 1$, $B_n > 0$ and there exists a sequence $(a_n)$ of positive terms such that $\lim_n(B_n \cdot \frac{a_n}{a_{n+1}} - B_{n+1}) > 0$. However, we can find for *any* $(B_n)$ a sequence $(a_n)$ such that $\lim_n(B_n \cdot \frac{a_n}{a_{n+1}} - B_{n+1}) = 1 > 0$ by setting $a_1 = 1$ and $a_{n+1} = a_n \cdot (\frac{B_n}{B_{n+1}+1})$. Hence $L(x)$ and $C(x)$ *or* $D(x)$ turn out to be co-extensional after all and any of our arguments below concerning the latter property equally applies to the former. So we will just focus on $C(x)$ *or* $D(x)$.

[15]With respect to an explanatory proof of the Pythagorean Theorem Steiner points out that a right-angled triangle is characterized by the property of being decomposable into two triangles similar to each other and to the whole (Steiner 1978, p. 144). Evidently, this property does not pick out any individual right-angled triangle, the only way to render it in fact *characterizing* seems by taking it as defining the *set* of right-angled triangles.

[16]Nothing hinges on the fact that the principle of induction as a property of sets picks out $\mathbf{N}$ "directly", i.e. characterizes it in a top-down way whereas $C(x)$ *or* $D(x)$ characterizes $S$ in a bottom-up way via its members. It is clear that $C(x)$ *or* $D(x)$ characterizes, if anything, the *set* of sequences of positive numbers but certainly not any particular such sequence.

[17]It should indeed be a rather small concession on Steiner's part, given that he has to make an analogous move in the context of the Pythagorean Theorem anyway. As pointed out in footnote 15 Steiner declares a property characterizing which picks out the set of right-angled triangles, and that set is no more explicitly mentioned in the Pythagorean Theorem than $\mathbf{N}$ is mentioned in (20).

[18]The deformation at the level of characterizing properties, or sets, which occurs in the move from $\mathit{IND}(1, x+1)$ to $\mathit{IND}(1, x+2)$, or from $\mathbf{N}$ to $\mathbf{O}$, which is thus evidently effected by *skipping the even numbers* (i.e. the members of $\mathbf{E}$) translates directly into the following *subtraction* at the level of summation formulas. $1 + \cdots + o = \frac{n \cdot n^+}{2} - \frac{e \cdot e^+}{4}$, where $n = o$, $e = o - 1$, $n^+ = e^+ = o + 1$. So we have

$$1 + \cdots + o = \frac{2o(o+1) - (o-1)(o+1)}{4} = \frac{o^2 + 2o + 1}{4} = \frac{o(o+2) + 1}{4} = \frac{o \cdot o^+ + 1}{4}.$$

In an analogous way we obtain formula (27) directly from (24) and (25).

[19]In case of (28) we can prove, on the one hand, that adding $(ab-a^2)$ effects a deformation which is *minimal* relative to a family of prima facie equally plausible alternatives. On the other hand one may proceed, again, at the level of sets from the deformation of $\mathbf{N}$ into $\mathbf{L}_{a,b}$, keeping track of which elements of $\mathbf{N}$ get skipped. This is then paralleled by a corresponding deformation of the summation formula in the following way. We start by applying formula (20) to $1+\cdots+l_{a,b}$ taking into account the necessary subtractions which correspond to the skipping of numbers in $\mathbf{L}_{a,b}$ (with respect to $\mathbf{N}$). To increase perspicuity we abbreviate '$l_{a,b}$' by '$l$', denote $1+\cdots+l$ by '$\Sigma$' and also make further use of (20). Thus we readily arrive at the equation

$$\Sigma = \frac{l(l+1)}{2} - [\frac{(a-1)a}{2} + (b-1)(\Sigma - l) + k\cdot\frac{(b-1)b}{2}].$$

The number $k$ appearing here is some natural number $\geq 0$ such that $l = a+kb$. From this equation we now work out the resulting deformation of the formula $\frac{l(l+1)}{2}$ step by step.

$$\Sigma + (b-1)\Sigma = \frac{l(l+1)-(a-1)a+2(b-1)l-k(b-1)b}{2}$$

$$\Sigma = \frac{l(l+1)+(b-1)l-a^2+a+(b-1)(l-kb)}{2b}$$

$$\Sigma = \frac{l(l+b)-a^2+a+(b-1)a}{2b} = \frac{l\cdot l^+ + ab - a^2}{2b}.$$

[20]Here is how we arrive at formula (29). $\frac{q\cdot q^+}{2(q^+-q)} = \frac{q(q+1+\sqrt{4q+1})}{2(\sqrt{4q+1}+1)} = \frac{q(\sqrt{4q+1}+3)}{8}$. Checking against one or two concrete summations of elements of $\mathbf{Q}$ indicates the change to $\frac{q(\sqrt{4q+1}+3)}{6}$. We try here, as before, to make do with just additive constants in order to keep the deformation minimal. Expressing this change in terms of $q$ and $q^+$ yields $\frac{q\cdot q^+}{\frac{3}{2}(q^+-q)}$, which is then proved by induction according to $IND(2, x+1+\sqrt{4x+1})$.

[21]It is important to note that nothing more is demanded of deformation (or generalizability) in Steiner's account than that it should lead to related theorems; other than that it is explicitly left undefined (cf. Steiner 1978, p. 147). In particular questions as to e.g. the efficiency (or "naturalness", whatever that may mean) either of the process of deformation or of the resulting proofs compared to other methods don't enter at all into Steiner's criteria for explanatoriness. Hence they need not concern us here.

[22]To construct a counterexample let e.g. $a_n = 1$ for all $n$, and $B_n = -n$.

[23]It should be clear that simply dropping the dependence requirement from the account is *not* an option for Steiner. Dispensing with it leads immediately to overgeneration, i.e. it allows the construction of easy recipes for churning out "explanatory" proofs. For instance, take any proof of a theorem of the form 'For all $x$, $\varphi(x)$', specialize to some element $a$ in the domain and add on as an idle element in the proof some characterizing property $\psi_a(x)$ of $a$. Thus we obtain an "explanatory"

proof of '$\varphi(a)$' (no matter how non-explanatory the proof may actually appear to us intuitively): it makes reference to a characterizing property and can also be deformed to yield related results '$\varphi(b)$', '$\varphi(c)$', ... by specializing to other elements $b$, $c$, ... in the domain, substituting for $\psi_a(x)$ equally idle characteristic properties $\psi_b(x)$, $\psi_c(x)$, ... .

[24]The divergence of $(M_n)$ is the only property that is needed here, neither monotonicity nor the positivity of its terms come in.

[25]We'll focus only on the main steps in the derivation and, as before, simplify matters slightly. For the strongest formulation of the results the reader is referred to Pringsheim 1916, pp. 337-334.

[26]For a proof of the divergence of $\sum d_n$ see part (B) of *Part I* above.

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