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THE INTERPLAY BETWEEN PROOF AND ALGORITHM IN 3RD CENTURY CHINA: THE OPERATION AS PRESCRIPTION OF COMPUTATION AND THE OPERATION AS ARGUMENT[0]

In the 1960s, historians of mathematics in China drew the attention of the international scholarly community to the fact that a Chinese text dating from the 3^{rd} century, in fact, contained mathematical proofs. Both their emphasis on the phenomenon and, in some respect, their way of analyzing it bore witness to the importance Western scholars attach to such facts.

As is well known, history of mathematics has regularly been used as a battlefield where nations, even civilizations, were competing and producing the evidence of their value. In this context, mathematical proof has played a dramatic part. As a last resort for some, it represents that by which the Western contribution to mathematics is deemed to be the most decisive. In some of my colleagues' opinion, it would be that which proves that only the West developed a speculative approach to mathematics.

It seems to me useful to recall this context, since it deeply influenced the way in which the proofs written by Liu Hui, our 3^{rd} century author, have regularly been analyzed. In the first place, they were compared with Greek geometrical texts of antiquity, or measured by a yardstick inspired by Aristotle's *Analytics*. This approach led to two opposite kinds of statement. Some scholars rejected the idea that these could be considered as proofs, since they did not emulate the axiomatico-deductive model: the fact that Liu Hui did not single out any axiom or definition ruined, for them, the contention that he proved anything. In opposition to the latter, others tried to elaborate ways in which one could consider some statements in Liu Hui's text as axioms, definitions and the like.

Despite the fact that they obtain opposite results, it seems to me important to stress that these two types of statement share the same basis. They all agree in taking a given practice of proof as an *a priori* norm, and they measure Chinese texts by this yardstick. I have argued elsewhere (Chemla, 1997) why I thought the procedure was questionable. Indeed, should history of proof be the history of proofs that mathematicians "should" have written, whatever the meaning of this "should" may be? In any case, if the question has to be: "Are there Greek proofs in Chinese texts?", we need not do research to guess that the answer will be: no! However, is this the question that should be asked? I do not think so. This has been the trick which blurred

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discussion on the history of proof – a trick which, incidentally, had the property of making what some call the "Western superiority in these matters" indisputable.

History of proof, I would argue, has very much been dominated by certain *a priori* ideas, normative ideas especially, concerning what a proof should look like, how it should be constructed and why it should be carried out. Liu Hui's text shows how a mathematician in a given tradition dealt with the problem of establishing why a given mathematical statement is correct, in a way that differs from what we can find in Greek geometrical texts of antiquity. This raises a couple of questions that seem to me worth addressing. Why did Liu Hui get interested in this problem? What were his motivations for it? What was he expecting from the answer? Which procedures did he think were adequate to this end? Since the way in which he tackles the problem would seem satisfactory to a mathematician today, I do not hesitate to call what Liu Hui wrote a proof. And, rather than worrying whether this title is rightly deserved, I choose to describe this practice for itself and pay attention to the way in which it inserts itself into the heart of mathematical activity, in a given historical context.

My thesis, which I shall not be able to substantiate here,[1] is that Liu Hui's proofs attest to a practice of mathematical proof as carried out in ancient China, which is sophisticated and which was produced through a process of elaboration. This practice differs from what the known Greek texts of antiquity attest to, and, apparently, it developed independently from them historically. In other words, we may well have here the testimony of another origin for mathematical proof.

Let me make clear that, in putting forward such a thesis, my intention is by no means to enter the battlefield on China's side and with new weapons. I believe that such ideological questions prevent us from thinking about mathematics. However, we'd better be aware of them, rather than let them surreptitiously creep into our assumptions. What is at stake here lies at another level.

A detour through China could help us analyze our categories, in this case, that of proof, in a critical way. These texts may attest to the elaboration of functions for a proof other than establishing the truth of a statement. Examining these Chinese texts can thereby provide us with tools for inquiring into some of the contemporary functions imparted to proving. This, in turn, raises questions relating to how the contemporary practices of proof were historically shaped.

To sum up, we may expect from our inquiry into such Chinese sources to understand better the nature of the activity of proving in mathematics as well as the processes through which the related cultural practices took shape in history. With these questions in mind, in what follows, I shall concentrate on how Liu Hui dealt with the measure of a circle. Examining how he handled a proof will put us in a position from which to analyze what is proved and how it was proved[2].

1. ELEMENTS OF CONTEXT

Before proceeding to dealing with our questions, we need to sketch the context within which Liu Hui operated. In fact, our third-century author, whose proofs we are to analyze, is a commentator. It is hence for the sake of exegesis that the first known mathematical proofs were composed in ancient China. The book that brought about such developments had been composed around the beginning of the Common Era and was entitled *The nine chapters on mathematical procedures*[3] – a title that, in what follows, I abbreviate into *The nine chapters*. This book, which carried out a compilation of mathematical knowledge available at the time, was to become the Canon *par excellence* for mathematics. Most of the mathematicians who worked in China up until the beginning of the 14^{th} century and whose writings came down to us demonstrate a knowledge of it, or refer to it.

Roughly three centuries after its completion, Liu Hui commented on *The nine chapters*, and its commentary was to be selected by the tradition to be transmitted, together with the Canon. The simultaneous use of the two texts appears to have become so systematic that, today, there is no surviving edition of *The nine chapters* that does not contain Liu Hui's commentary. The formation of such writings, composed of a Canon and commentaries selected by the written tradition, is typical of Chinese history, where most of the disciplines experienced prominence being bestowed on texts of this kind. It is thus within the framework of a commentary that Liu Hui was led to deal with the measure of the circle.

Here is, more precisely, the local context within which he inserts the development we are interested in. Problem 31 of chapter 1 of *The nine chapters* reads as follows[4]:

"SUPPOSE ONE HAS A CIRCULAR FIELD, WITH A CIRCUMFERENCE OF 30 *BU* , AND A DIAMETER OF 10 *BU*. ONE ASKS HOW LARGE THE FIELD IS.

"ANSWER: 75 *BU*.

(...[5])

"PROCEDURE: HALF OF THE CIRCUMFERENCE AND HALF OF THE DIAMETER BEING MULTIPLIED ONE BY THE OTHER, ONE OBTAINS THE *BU* OF THE PRODUCT (*JI*)."

It is, hence, in reaction to this piece of text that Liu Hui produces what I call a proof. Two remarks should be made on this excerpt. First, *The nine chapters* yield a procedure to compute the area of the circle that is exact. If we denote by *A*, the area of the circle, *C*, the circumference, and *D*, the diameter, the procedure can be represented as follows:

$$A = \frac{C}{2} \cdot \frac{D}{2}$$

Secondly, however, the terms of the problem supply both the diameter and the circumference as if they were independent of each other and both were needed. The ratio between them is that of 1 to 3. Yet, immediately afterwards, the Canon offers two other procedures, each of which uses only one of the data, and none of which is exact:[6]

"ANOTHER PROCEDURE: THE DIAMETER BEING MULTIPLIED BY ITSELF, MULTIPLY BY 3 AND DIVIDE BY 4."

"ANOTHER PROCEDURE: THE CIRCUMFERENCE BEING MULTIPLIED BY ITSELF, DIVIDE BY 12."

We shall analyze, in what follows, how Liu Hui comments on this set of problems and procedures. Let us, for the moment, stress that the passage of *The nine chapters* quoted gives a faithful idea of how the Canon is composed. It is constituted of problems, answers and algorithms, i.e., as it appears, lists of operations that rely on the data provided by the terms of the problems to yield the unknown sought. We can, however, question this reading, as will become clear below.

In echo with the composition of the Canon, Liu Hui's proofs systematically tackle how to establish the correctness of algorithms. Thus we are taken to a world different from, e.g., Euclid's *Elements*, where proofs mainly aimed at establishing the truth of propositions. Let us enter into it.

2. SKETCH OF THE PROOF

It will be useful, for developing our analysis, to start by outlining the proof Liu Hui presents for establishing the correctness of the first algorithm mentioned above.

The opening remarks of his commentary are devoted to making clear that the ratio between the circumference and the diameter, as 3 to 1, i.e., that between the two data provided by the terms of the problems devoted to computing the area of the circle, in fact holds true for the regular hexagon inscribed in the circle. The introduction of the figure of the hexagon initiates a development that Liu Hui concludes by the following statement:

" Therefore/This is why (*gu*), when one multiplies the half-circumference by the half-diameter, that makes the area (*mi*) of the circle."

In other words, depending on how one interprets the first word of the final statement, the commentator himself conceives of his development as *establishing the correctness* of the algorithm, or *bringing to light the reason why* the algorithm stated by the Canon yields the correct answer. Whichever interpretation one prefers, it remains true that, in Liu Hui's eyes, his commentary on the algorithm relates to establishing its correctness. This constitutes an additional reason why it seems to me adequate to refer to it as a "proof". This statement shows that it is not only from our perspective that the commentary may contain proofs. It appears to have been one of its function in the actors' own perspective. Therefore we must analyze how Liu Hui argues to reach such a conclusion. Let us follow the course of this reasoning.

In a first step, relying on the figure of the regular hexagon inscribed in the circle, Liu Hui brings to light that an exact relationship links the diameter, the circumference of this polygon and the area of the regular 12-gon inscribed in the circle. This relationship can be represented as follows:

the half-diameter multiplied by the half-circumference of the hexagon

=

Area of the 12-gon

This relationship can be easily grasped in Figure 1. The commentator appears to conceive of the hexagon, as well as the *n*-gons introduced in what follows, as a collection of quarters of polygons cut in sectors of the circle and assembled around its center[7]. Consider, on figure 1.a, one quarter composing the hexagon, OBD, cut along the radius OC, which goes through A, the middle of BD. If one introduces the two corresponding quarters of the 12-gon, as reproduced on figure 1.b, figure 1.c makes clear that multiplying AB by OC yields the area of these two quarters. Multiplying this by 6 yields the relationship sought-for. In Liu Hui's terms, cutting the quarter OBD along OC yields two quarters of a regular 12-gon inscribed in the circle.

The repetition of the operation (see figure 2) leads to a similar relationship between the 12-gon and the 24-gon, as follows:

the half-diameter multiplied by the half-circumference of the 12-gon

=

Area of the 24-gon

At this point, Liu Hui has introduced two elements that will prove fundamental in his reasoning: a sequence of *n*-gons, produced through cutting quarters of polygons within the body of the circle; and a relationship linking

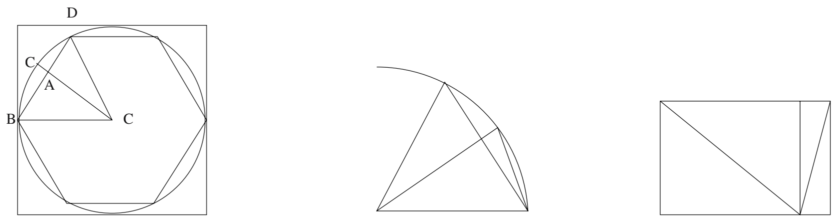

Fig. 1.a Fig. 1.b Fig. 1.c

FIGURE 1.

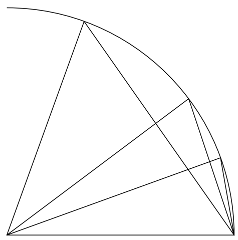

FIGURE 2.

the circumference of the n-gon, the area of the $2n$-gon and the half-diameter of the circle.

In what follows, he turns to considering these two elements separately, dealing first with the evolution of the relationship between the circle and the polygons generated by the successive cuts. He hence goes on:

"The finer one cuts, the smaller is that which is lost."

The quarters of the polygons yielded by the sequence of cuts, Liu Hui notices, are finer and finer. He then re-introduces the circle, through considering the evolution of the relation of the successive polygons to the circle. With respect to the unknown to be determined, i.e., the area of the circle, the area of the polygons formed, Liu Hui states, gets increasingly closer[8]. Having thus made explicit the relation of the figures first introduced to the problem considered, the commentator goes on:

"One cuts it and re-cuts it until one attains (*zhi*) what cannot be cut. Then its body (*ti*) makes but one (*he*) with the circumference of the circle, and there is nothing lost."

The statement consists in two parts, each corresponding to a member of the previous sentence.

The first part takes up again the cut introduced and prescribes to repeat it until "attaining what cannot be cut". What the commentator means exactly by this cannot be completely elucidated. Does he think of an actual infinity of steps? Or, does he prescribe to carry out the operation until the moment when our senses give us the quarters of the polygon as impossible to cut further? Or else, does he believe that magnitudes are composed of finite elementary constituents that can be reached after a given number of cuts? I have argued elsewhere in favor of the first of these interpretations, without being able to find evidence that would decisively rule out the other possibilities[9]. The main point, however, is that, according to Liu Hui's own terms, there is a moment when what cannot be cut any longer is *attained*. The term is not used by accident: the character expressing this nuance, *zhi,* occurs in two other contexts in which we would put into play an infinite number of steps[10].

Whichever interpretation one adopts, we find ourselves confronted with a direct reasoning, unlike the indirect arguments encountered in Euclid's or Archimedes' treatments of similar problems[11]. This feature evokes pre-Eudoxian fragments like Antiphon's, a point to which we shall come back.

Whatever the manner in which this attainment is realized, the second part of the sentence quoted formulates its consequences. First, the body of that which is produced is said to coincide with the circle, by virtue of the coincidence of their circumferences. Note that it is hence held to be different from the circle, but the contours "make but one". Secondly, from this, Liu Hui moves on to stating that the areas do not differ. The process would have thus yielded a polygon – the following sentences make clear that this is how the commentator conceives of the figure produced as a result of the process – the circumference and, hence, the area of which match those of the circle.

In what follows, Liu Hui offers an argument to establish this last statement. The first step consists in introducing a magnitude that will constitute the pivotal element in the reasoning: the so-called "diameter remaining". This expression refers to the part of the diameter that goes outside the n-gon, beyond the mid-point of one of its sides, which offers a kind of measure of the distance between the circumferences of the n-gon and the circle. On figure 1, it is measured by AC for the hexagon. The second step then introduces the sequence of rectangles, whose dimensions are respectively a side of an n-gon and the diameter remaining (see Figure 3). Their areas exceed that of the circular segments that represent the differences between the successive n-gons and the circle[12]. In other terms, the rectangles constitute an upper bound for "that which is lost". The point that will prove crucial is that their areas are expressed with respect to the so-called "diameter remaining".

Liu Hui has described the situation in general. In a third step, he focuses on the body produced at the point when one cannot cut the quarters any longer – this is where he still refers to it as "a polygon". Applying to

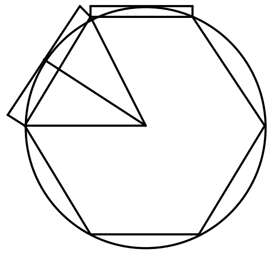

FIGURE 3.

it the previous device, he notices that the matching of the circumferences implies that the diameter remaining has vanished. As a consequence, the areas exceeding the difference between the areas of the n-gon and the circle, which depend on the diameter remaining, also vanish. Hence the equality of the areas of the "last" polygon and the circle.

This constitutes an essential feature in the structure of the reasoning, and one that distinguishes Liu Hui's argument from the pre-Eudoxian fragments evoked earlier, provided that we may judge them on the basis of the remaining evidence[13].

Liu Hui has thus exploited the evolution of the relation of the n-gons to the circle, by bringing to light a polygon whose body coincides with that of the circle and which shares the same area. In a final section of this part of his commentary, he turns to considering the transformation of the algorithm linking the circumference of the n-gon, the area of the $2n$-gon and the half-diameter of the circle, which he has introduced so far with respect to the hexagon and the dodecagon. His next point consists in highlighting the basic reason that grounds the correctness of this relationship for any n-gon. The central operation of multiplying a side of an n-gon by the half-diameter, he states, introduces two quarters of the $2n$-gon[14] and yields each of them twice. This amounts to stating that what figure 1 shows holds true with full generality.

Stating the relationship for the polygon yielded at the end of the process provides the algorithm that *The nine chapters* offered for computing the area of the circle. The circumference of the polygon fuses into the circumference of the circle. Multiplied by the half-diameter, it yields the area of the polygon, equal to that of the circle. Hence the conclusion, which completes Liu Hui's proof of the correctness of the algorithm provided by the Canon for the area of the circle.

3. FIRST REMARKS ON THE PROOF

Let us highlight some features of the proof just sketched.

It is interesting, in the first place, to notice that the proof proceeds through exact relationships, holding for a sequence of polygons. The final part of the commentary stands in contrast to this, since Liu Hui makes use of inequalities – the areas of the sets of rectangles constitute upper bounds of the areas left over, in which he is interested, and this is what makes the argument work. However, this last feature occurs seldom in Liu Hui's proofs, and this distinguishes them from, say, the arguments in Euclid's *Elements*, which mainly bring inequalities into play.

A second feature is worth mentioning. The proof actually embeds the circle in the set of all inscribed regular polygons, and it brings to light a common algorithm computing the areas of all these figures. The proof thus connects various realities, and it is through this extension that the reason for the correctness of the algorithm examined appears. This correlation between establishing the correctness of an algorithm and bringing to light more general operations underlying it is not an accident. On the contrary, it constitutes a characteristic feature of Liu Hui's proofs throughout his commentary. We can observe here how it manifests itself within the context of geometry[15]. In fact, this feature can be correlated with more general statements on mathematics made by the commentator[16]. In brief, he conceives of his commentary on a procedure as bringing to light its "source" (*yuan*), which, for him, appears to constitute the level at which it can be extended (*shi*) to deal with other categories (*lei*) of problems. Hence the connection between proving and bringing to light more general operations.

If we go back to the case of the circle and to the way in which the correctness of the algorithm is established, we notice that, once the algorithm has been described for the hexagon, then for the dodecagon, the proof considers the evolution of its terms – the circumference of the n-gon and the area of the $2n$-gon – and of the relationship between them, through considering the transformations of the underlying polygons. It thus brings to light *how* the algorithm for the circle is in continuity with those for the polygons. The proof shows by way of which variation the circle can be embedded in the set of all inscribed regular polygons.

A third point should be stressed. As already mentioned, the proof prescribes carrying out an operation until reaching the point at which it cannot be performed any longer. Such is the case also in the similar reasoning by which Liu Hui establishes the correctness of the algorithm given for computing the volume of the pyramid. The presence, in both cases, of this stage in which the decrease of the remainder is assessed – which we emphasized above – underlines the character of necessity that, in Liu Hui's view, it probably bore. The comparison of both reasonings shows that we are confronted here, not with an argument *ad hoc*, but with a stable mode of reasoning. This evokes the way in which the method of exhaustion manifests itself in Greek geometrical texts of antiquity[17]. As a matter of fact, the mathematical ideas used by Liu Hui and Euclid in the case of, both, the circle and the pyramid, are the same. The stable differences in their way of bringing these ideas into play in their proofs are all the more interesting and seem to refer to a stable difference in the practice of proof. However, I would not like to dwell here on such questions of comparison, all too frequently addressed. Nor am I going to ask whether Liu Hui's commentary outlined above offers a "real" proof, since, as I suggested, here lies the trap ready to open under the feet of the historian. Let us, instead go deeper in our analysis of the Chinese text.

4. THE OPERATION AS RELATION OF TRANSFORMATION

To this end, the first question I suggest to raise is very simple: what has Liu Hui proved? The answer seems to be very simple. We find it in Liu Hui's conclusion:

"Therefore/This is why (*gu*), when one multiplies the half-circumference by the half-diameter, that makes the area (*mi*) of the circle."

But, immediately after, Liu Hui adds a remark concerning the terms of "circumference" and "diameter" entering this statement:

"Here, by circumference and diameter, we designate the quantities that *attain (zhi*) what is so, what the *lü*'s of 3 for the circumference and 1 for the diameter are not." (My emphasis)

We need to sketch the meaning of the concept of *lü*, which appears in the statement, before commenting on it. Introduced in *The nine chapters* within the context of the rule of three, *lü* designates numbers that are defined only relatively to each other. For example, the numbers expressing a ratio between entities are referred to as *lü*. This is the case in the sentence quoted above. However, the extension of the concept goes beyond this case[18].

This implies that, for Liu Hui, the algorithm, the correctness of which was just established, bears on quantities different from those to be found in the terms of the problem of the Canon after which the algorithm is stated.

This remark extends even further. Elsewhere, Liu Hui speaks of the ratio between the diameter and the circumference of the circle as not possibly exactly expressible[19]. If the terms of the algorithm proved to be correct are "the quantities that *attain* what is so", they cannot be simultaneously expressed by actual values. As a consequence, we discover that Liu Hui has proved the correctness of an algorithm that, in his view, can lead to no computation. This algorithm expresses a relationship of transformation, or of production, between magnitudes, but cannot receive actual values for all its terms. This entails that, here, *the "algorithm" must be distinguished from the "prescription of a computation*". This conclusion incidentally highlights why the stand according to which mathematics in ancient China was only practical is indefensible.

If we now look up again at the Canon, we see that, there, the algorithm means, perhaps not only, but *also* computations, since an answer is provided for the problem. The procedure is what yields the value of the area of the circle, on the basis of the two data given for the circumference and the diameter in the terms of the problem. This aspect of the algorithm has not yet been addressed by the commentator, and Liu Hui will consider it in a second part of his commentary on the measure of the circle.

Before turning to this other aspect of his commentary, let us draw some general conclusions from what was just observed.

Such a case leads us to distinguishing the algorithm as producing a magnitude, expressed in terms of the situation of a given problem – in our case, the area of a circle –, from the algorithm as producing a value. We are to distinguish the algorithm as expressing a relation from the algorithm as prescribing a computation. In other terms, we are to dissociate the algorithm viewed from a *semantical* point of view and the algorithm considered from a *numerical* perspective. The same conclusions could derive from examining other parts of Liu Hui's commentary, the difference being that the field with respect to which the interpretation of the result of the operations is expressed is not always geometrical[20].

The two aspects of an algorithm can run in parallel. But there are cases when a discrepancy appears between the two, as is the case here. This situation results in having Liu Hui comment on the algorithm in two sections. He first deals with the algorithm semantically, establishing the correctness of the relation of transformation. It is only in a second section that he comments on the algorithm as pure computation, relating to the context of an actual problem, such as what can be found in the Canon.

What was first proved was thus the relation of transformation. But how was it proved?

If we look again at the proof, we discover retrospectively that it makes use of algorithms, as relations of transformation too, with no computations.

In fact, some of these algorithms are exact geometrically, semantically. It is as such that they are involved in the proof. But they can lead to no computation that would be exact from a numerical point of view: this is the case for the computation of the circumference of the sequence of polygons. If we start from a circle with 2 *chi* of diameter, as is suggested at the beginning of the commentary, attempts to yield the numerical values of the sides of the *n*-gons would soon lead us to introduce kinds of quantities that go beyond those considered by mathematicians in ancient China[21]. But, in fact, nothing is done in this part of the text to turn them into actual computations.

For other operations that are introduced for the sake of the proof, the point is in the value of the *algorithm as argument*, not in the computation it would prescribe. This aspect is clear with respect to the so-called "diameter remaining". It plays a crucial role in linking, through algorithms, the convergence of the areas to that of the circumferences of the *n*-gons. But its actual value does not matter. The important point is that an algorithm expresses how the convergence of areas towards the area of the circle depends on the nature of the evolution of the "diameter remaining", while the circumference of the *n*-gons approaches that of the circle. This algorithm constitutes what I would call an "operation-argument".

This analysis hence reveals a whole world of such algorithms, as relations of transformation, independently from algorithms as computations. The proof examined shows how they are articulated to one another, producing one another.

So much for now. Let us at this point turn to the second part of Liu Hui's commentary, in which he tackles the algorithm from a numerical point of view. And let us consider how it also reveals another characteristic of the practice of proof as carried out in ancient China.

5. THE ESSENTIAL LINK BETWEEN PROOF AND ALGORITHM

The point I want to make on the basis of the second part of Liu Hui's commentary consists in showing that, as his practice of proof highlights, proofs are not closed onto themselves. They do not only constitute an aim in themselves, as would be the case if they were understood as merely establishing the correctness of algorithms. On the contrary, they can also serve, for instance, as the basis for elaborating new algorithms. With this purpose in mind, let us follow how Liu Hui deals with the situation numerically[22].

To this end, the commentator puts forward an algorithm, the aim of which is to yield more precise values for the relationship between the circumference and the diameter. At each step, this algorithm makes the meaning of the computations explicit. Therefore, in the end, it is clear that the values produced are approximations, for what they are approximations and which kind of approximation they represent.

In fact, this algorithm appears to be derived from the proof outlined in the first part. Let us observe *how* this algorithm precisely relies on the operations of the previous demonstration.

The "procedure of the right-angled triangle (*gougushu*)", which, in the realm of algorithms, corresponds to the so-called Pythagorean theorem, is the object of the ninth of *The nine chapters*. Liu Hui brings it into play for transforming the previous geometrical argument into an algorithm.

Let us first sum up the main points of the text, before stressing other features of the practice of proof in ancient China.

The algorithm that Liu Hui now starts describing relies on an iteration. With respect to figure 1.a, we can summarize the procedure to be repeated as follows: in the right-angled triangle OAB, the hypotenuse OB is the half-diameter, and the base AB is half the side of the n-gon. Applying the "procedure of the right-angled triangle (*gougushu*)" yields the height OA. Furthermore, in the right-angled triangle ABC, the base is the difference between the half-diameter and OA, the height is half the side of the n-gon. Applying the "procedure of the right-angled triangle (*gougushu*)" yields CB, the corresponding hypotenuse, which turns out to be the side of the $2n$-gon. Here is how Liu Hui formulates the first application of this sub-procedure, to be thereafter iterated:

"Procedure consisting in cutting the hexagon in order to make a dodecagon:

Set up the diameter of the circle, 2 *chi*. Divide it by 2, that makes 1 *chi* and gives the side of the hexagon that is in the circle. Take half of the diameter, 1 *chi*, as hypotenuse, half of the side, 5 *cun*[23], as base, and look for the corresponding height. The square of the base, 25 *cun*, being subtracted from the square of the hypotenuse, there remains 75 *cun*. Extract the square root, descending to the *miao*, to the *hu*, then retrograde the divisor one more time[24], in order to find a digit from the decimal part (of the root). One takes as numerator the digit from the decimal part that has no name, and one takes 10 as denominator. By simplifying that makes two-fifths of *hu.* Consequently, one obtains 8 *cun* 6 *fen* 6 *li* 2 *miao* 5 and three-fifths *hu* for the height. Subtract this from the half-diameter, 1 *cun* 3 *fen* 3 *li* 9 *hao* 7 *miao* 4 and three-fifths *hu* remains, that one calls small base. Half of the polygon side then is called once again small height. Look for the corresponding hypotenuse. Its square is 267949193445 *hu*, the remaining fraction being abandoned. Extract the square root, that gives a side of the dodecagon."

This subprocedure is repeated, and, in the course of the first iterations, there is no computation carried out for determining either the area or the circumference of the successive n-gons. It is when he reaches the 48-gon that Liu Hui first actually computes the side of the 48-gon. Moreover, he uses the relation brought up during the previous demonstration to deduce from it the area of the 96-gon, obtaining 313 and 584/625 *cun*.

Again in the next iteration of the subprocedure, Liu Hui computes similarly the side of the 96-gon, and then the area of the 192-gon, obtaining 314 and 64/625 *cun*.

Having obtained values that are smaller than the area of the circle[25], Liu Hui turns to computing an upper bound for the value of the area of the circle, by bringing into play the same rectangles as those used for the proof (see Figure 3). The areas of the set of rectangles covering the segments of circle left over by the regular 96-gon inscribed in the circle is, however, computed with a new insight. The difference between the area of the 96-gon and that of the 192-gon is doubled, which yields the value sought-for. It is then added to the area of the 96-gon, providing the value of 314 and 169/625 *cun* as upper bound for the area of the circle. Since the lower and upper bounds found share the same integral part of 314 *cun*, it is kept to represent the *lü* of the area of the circle, with respect to the *lü* 400 for the square of the diameter. Thereby, new approximate values are offered for the area and circumference of the circle, in a way that clearly indicates their nature.

This sketch of the algorithm enables us to examine further some interesting features of the text. Notice, first, that the same figures as previously, based on the hexagon, are considered, and that the same central ideas are used. However, they are brought into play in different ways. The most striking example of this difference relates to the computation of the areas of the exceeding rectangles. When their areas were considered within the proof, they were computed so as to highlight their dependency with respect to the diameter remaining. However, when the operation-argument becomes the operation-computation, these areas are computed in another way. Yet the value of the diameter remaining (AC) is determined at each stage. This difference between the two contexts sheds more light on the essential part played by the circumference in the first part of the commentary we examined. It also highlights the process through which the proof is transformed into an algorithm.

Furthermore, with the example of this algorithm, we are in a position to observe another modality of the relation linking the proof and the algorithm. As already alluded to, the algorithm proceeds along parallel lines. It prescribes computations that are to be numerically performed. In addition, the *meaning* of the result is always made explicit in terms of the geometry of the situation. Look at, for instance, the concluding proposition of the passage quoted above: "..., that gives a side of the dodecagon".

More generally, the various algorithms corresponding to the Pythagorean theorem produce both a meaning and a result. In the latter example, one of these algorithms determines the interpretation of what is yielded as "hypotenuse", which is exact. It also produces a numerical value, which approximates that of the hypotenuse with an accuracy that is explicitly provided.

All in all, the algorithm aims at yielding actual numerical values. It can, in this sense, be compared to the algorithm as used by *The nine chapters* to determine a value for the area. However, at the same time, the algorithm shapes a semantical interpretation for each result, and is to be compared, in this sense, to the proof examined above, in the first part of the commentary.

The fact that *this algorithm also has this argumentative component* here can easily be deduced from the fact that some of the computations are stated *only* for the sake of the reasoning, but are not executed. This is for instance the case for the last computation of the paragraph quoted above, which gives a side of the dodecagon. The computation is prescribed, but not carried out. Instead, the square of the result is kept, since it is that which will be used at the beginning of the next sub-procedure, where the square of the half-side is needed.

This leads to another range of remark. In fact, one can prove that the argumentative function of the algorithm has prominence over the computational dimension. The first application of the subprocedure theoretically yielded a side of the dodecagon. This side was to be halved, and its half squared, to start the next application of the sub-procedure. Instead, the square is directly divided by 4. Rewriting the sequence of operations "searching the square root, halving and squaring" as "dividing by 4" is absolutely correct, at the algebraic level. This is also correct from a numerical point of view, if the result of the square root is given as "square root of N", when needed. This is, according to Liu Hui, the reason why quadratic irrationals were introduced in *The nine chapters*: he relates them to the requirement that squaring the result of a square root extraction should restore the number with which one started[26]. However, here, the results are given in an approximate way, and rounded off. This implies that applying the sequence of operations "searching the square root, halving and squaring" might *not* yield the same numerical result as "dividing by 4". Here, we have a point where the algorithm provided for shaping the interpretation of the result and proving the correctness of the computation diverges from the algorithm used for computing. However, they are stated in parallel. At the level of pure operations, extracting the square root and squaring, as relations of transformation, are useful for making sense of the flow of computation, but, in fact, they cancel each other. They do not at the level of the operations as carried out by Liu

Hui here. This remark reveals that the proof requires an algorithm that gives the meaning of the values computed. This algorithm is rewritten, at the level of pure operations, to yield the algorithm actually used for the computations. But we have clear evidence, here, that it is the level of the proof that guides the numerical execution of the algorithm.

In his commentary on the introduction of quadratic irrationals, Liu Hui compared them to fractions in that they allow cancelling the sequence of two inverse operations. This implies that the results of the two types of algorithm, that for proof and that for computation, remain identical. This property is put into play in a kind of "algebraic proof in an algorithmic context", as I call it[27]. It involves taking algorithms as lists of operations and rewriting them as such, without prescribing computations. This relates to what we just saw, regarding rewriting the algorithm. However, this also evokes the operations as practiced in the first part of the commentary, where Liu Hui addresses proving the correctness of the algorithm.

However, quadratic irrationals are not introduced in relation to the circle here. This results in having, right from the outset, a divergence between operations as relations of transformation and operations as prescriptions for computation.

If we go back to the algorithm analyzed, it is interesting to note that, in parallel to the fact that we had pure operations in the proof, we now discover argumentative operations in the algorithm. This reveals that this algorithm prescribes computations to produce values, *at the same time* as it produces the reasons for its correctness. *Proof is not to be expected to be always a text distinct from the text of what is proved.* Here *algorithm and proof have merged into a unique text*.

This text fulfils this double function simultaneously by making use of the double face of an operation, carrying it out both semantically and numerically.

This double face of an operation is reflected in two other features of Liu Hui's commentary. First, it corresponds, as we saw above, to the split of the commentary in two parts here. Secondly, it can be correlated to a specificity in the set of mathematical concepts to be found in the commentaries. In fact, the commentators make use of two concepts of area. *Ji* refers to the area as the number produced by the computation, which can be linked to the operation as numerical prescription. In contrast to it, *mi*, which is to be found only in the commentaries and not in *The nine chapters*, refers to the area as the spatial extension corresponding to the multiplication between two magnitudes. This concept may relate to the face of the operation as relation of transformation.

In the same vein, it is important to stress the part played by the problems for making possible the interpretation of the successive steps of an algorithm[28]. If we go back to the example mentioned above, asserting that a sub-procedure of the algorithm described by Liu Hui yields the hypotenuse of a right-angled triangle requires having identified that the terms to which it is applied are the base and the height of such a triangle. On this basis, one can recognize the problem, for the solution of which an algorithm has been shown to correctly yield the hypotenuse. As for any problem contained in the Canon, not only do we have the situation to which the subprocedure can be applied, but we also have numerical values for each of the term. Applying the algorithm yields both a value and a meaning for the number obtained, which are exactly the two tracks along which Liu Hui's text develops. In this way, problems are building blocks to write down a proof, in that they offer a field of interpretation with which to make explicit the meaning of the result of an operation.

One can interpret along the same lines the way in which, in the first part of the commentary, the meaning of the multiplication of the circumference of the n-gon by the half-diameter of the circle is brought to light. Liu Hui introduces a figure, that of the $2n$-gon, the area of which corresponds to the result obtained. It is the situation bringing together the circumference of the n-gon, the area of the $2n$-gon and the half-diameter of the circle that is rich enough to yield the interpretation of the operation. Interestingly enough, Liu Hui uses the same term *yi* to designate the meaning of an operation in both cases: when it is expressed in terms of the situation described in the terms of a problem and when its explicitation is made possible thanks to the introduction of visual auxiliaries[29].

6. CONCLUSION

At this point, let me gather the various threads that were followed, while attempting to describe this practice of proof for itself and observing how it was embedded in mathematical activity taken as a whole.

We stressed the fact that Liu Hui's commentaries bore on algorithms. Sometimes, they establish the validity of a relation of transformation, as in our first case. Sometimes, they establish that a value obtained is indeed the one sought-for, or in which ways it can stand for it, as in our second case.

In any case, Liu Hui's proofs present stable modes of reasoning.

In contrast to what we would expect if we took for granted that the sole aim for proving is to convince of the truth of a statement, we saw that proofs can serve as a basis for the production of new algorithms.

In fact, we met with two examples of this fact.

First, the algorithm constituting the second part of the commentary analyzed was produced on the basis of the proof delivered in the first part, and we observed modalities of the transformation of proof into algorithm.

Secondly, the proof of the relation yielding the area of the circle proceeded through an extension of the algorithm to be proved to a whole set of figures, thereby bringing to light the reason for its correctness as well as the connection of the circle to these other figures.

This suggests that mathematicians had other reasons to get interested in the question of knowing whether a statement was correct, besides establishing its correctness.

Obsessed as we have been by this latter function of a proof, haven't we failed to describe the general part played by proof in mathematics and its various articulations to other moments of mathematical activity? This failure seems to me to have had a lasting impact on the history of proof.

In any case, Chinese mathematicians like Liu Hui might have been interested in the correctness of algorithms for the mathematical productivity of the question or for the understanding it provides of that which has been proved.

Neither with respect to their nature, nor even with respect to the texts that give expression to them, did we observe a clear-cut opposition between what was proved and what proved it. The reason behind this is that proofs are constituted of operations – equalities, as we stress, and not inequalities. And, in order to describe the relationship between algorithm and proof in a more precise way, we were led to oppose operation-argument to operation-computation, with respect to the form of the operation. In another perspective, we opposed operation-relation of transformation to operation-prescription of computation, as regarded their nature.

On such a basis, the enunciation of an algorithm and the writing of a proof could interact in various ways with each other, a conclusion also supported by the analysis of other parts of Liu Hui's commentary[30]. The proofs thus open onto the production of new algorithms, whereas the algorithms can go along with a proof.

These Chinese authors experienced it: there is no antagonism between computation and reasoning. This remark sounds obvious to us, whose proofs proceed so often through computation, in contrast to what Euclid did. How did that happen? What are the consequences for the activity of proving? Liu Hui's text incites me to raise these questions. Perhaps it can help us answer them both conceptually and historically. Perhaps history of proof, too, will display a non-linear pattern.

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APPENDIX

"Commentary: half of the circumference makes the length and half of the diameter makes the width; consequently the width and the length being multiplied one by the other, this makes the *bu* of the product (*ji*).

"Let us suppose that the diameter of the circle is 2 *chi*; the values (*shu*) of a side of the hexagon inscribed in the circle and half of the circle's diameter are equal. Corresponding to the *lü* of diameter 1, the *lü* of the polygon's circumference is 3.

"Once again, relying upon the drawing, multiplying the half-diameter for a segment by half a side of the hexagon, that makes two pieces (*er*) of it and, multiplying this by six, one obtains the area (*mi*) of the dodecagon.

"If once again one cuts it, multiplying the half-diameter for a segment by a side of the dodecagon, that makes four pieces (*si*) of it and, multiplying this by 6, then one obtains the area (*mi*) of the 24-gon.

"The finer one cuts, the smaller is that which is lost.

"One cuts it and re-cuts it until one attains (*zhi*) what cannot be cut.

"Then its body (*ti*) makes but one (*he*) with the circumference of the circle, and there is nothing lost.

"If to the exterior of the sides of the polygon, there is still some diameter remaining, when one multiplies the remaining diameter by the sides, then the area (*mi*) extends to the exterior of the circular segments.

"In case this polygon attains a degree of fineness[31] such that its body (*ti*) coincides with the circle, then there is no diameter remaining to the exterior. If there is no diameter remaining to the exterior, then the area does not extend outside.

"When, with one side, one multiplies the half-diameter, this amounts to cutting the quarter of the polygon (*gu*) and each piece is obtained twice.

"Therefore/This is why (*gu*), when one multiplies the half-circumference by the half-diameter, that makes the area (*mi*) of the circle."

NOTES

[0]This paper was completed, while I was spending a week at the Fondation des Treilles, Tourtour. It is my pleasure to thank this institution for its hospitality. I am grateful to John McCleary for his help in the process of polishing the English.

[1]See (Chemla and Guo Shuchun, n.d.), especially chapter A.

[2]In the appendix, I give the translation of my critical edition of the proof by which Liu Hui establishes the correctness of an algorithm yielding the area of the circle. In Chemla (1996), I have discussed this passage of Liu Hui's writings extensively, especially regarding the philological problems that it raises. Here, I concentrate only on discussing questions relating to proof. The reader interested in the argumentation for establishing the critical edition, on which my analysis here is based, is referred to this former publication. There, he or she can find a more detailed discussion of the various received versions. This passage of Liu Hui's commentary has been previously discussed by several scholars, among whom: (Chen Liang-ts'o, 1986), (Guo Shuchun, 1983), (Lam Lay Yong and Ang Tian-Se, 1986), (Liu Dun, 1985), (Volkov, 1994).

[3](Chemla and Guo Shuchun, n.d.) provides a critical edition and a French translation of this book and the earliest extant commentaries. I am glad to acknowledge my debt towards Professor Guo Shuchun, with whom, since 1984, I have discussed each character of this text. This book also contains a glossary discussing the mathematical and philosophical terms of both *The nine chapters* and the commentaries, which I established. I shall refer below to it, as *Glossary*.

[4]I use capital letters for the text of the Canon, in opposition to lower case letters for the commentary.

[5]Here we skip the statement of a second problem, similar to the first one.

[6](Chemla, 1996) offers an interpretation of these facts. I refer the reader to it.

[7]The 7^{th} century commentator concretely describes how to produce the figure of the 6-gon by assembling 6 triangles around the center of the circle, see (Chemla and Guo Shuchun, n.d.). This is in agreement with the fact that the geometrical figures to which the commentators refer seem to have been material objects, the spatial extension of which appears to be their foremost feature. See (Chemla, 2001) and see *gu* "quarter of a polygon" in the *Glossary*.

[8]Interpreting in this way Liu Hui's statement is in agreement with several features underlined above: the polygon consists in a set of quarters; cutting the quarters of the n-gon yields the $2n$-gon; the first element attached to a geometrical figure in ancient China is its area.

[9]See (Chemla, 1996). (Volkov, 1994) offers a completely different interpretation, based on a numerical interpretation of the whole passage. In my view, the way in which he suggests to link the two parts of the commentary (for the second part, see the end of this paper) requires further examination.

[10]See the commentary on the area of the circular segment, after problem 36 of chapter 1, and the commentary of the volume of the pyramid, after problem 15 of chapter 5.

[11](Chemla, 1992, 1996) touch the comparison between these reasonings from different angles.

[12]The term *mi* used to designate their areas conveys both the idea of geometrical extension and measure, see *Glossary*. Multiplying the two lengths to produce such an area is an usual way of introducing a rectangle, through its length and width. One can find another such example at the beginning of the passage translated.

[13](Chemla, 1992) shows that this step characterizes all such reasonings by Liu Hui, and (Chemla, 1996) compares them to pre-Eudoxian fragments in a more detailed way.

[14]This sentence justifies our interpretation that "cutting a quarter of a n-gon" is meant to refer to the production of two pieces of the $2n$-gon.

[15](Chemla, 1992) discusses other manifestations of this feature in geometry. (Chemla, 1997) shows its relevance for interpreting the proofs in other cases, and discusses the correlations in terms of textual characteristics of Liu Hui's commentary.

[16]I have a paper in preparation on this topic, " Une conception du fondement des mathématiques chez les commentateurs chinois (1[er] au 13[e] siècle) des *Neuf chapitres sur les procédures mathématiques* ", presented at the Conference "Fondements des mathématiques", Nancy, September 2002. An abstract can be found in http://www.univnancy2.fr/ACERHP/colloques/symp02/PreliminaryProgram.htm

[17]This is the main point made in (Chemla, 1992).

[18]See (Li Jimin, 1982), (Guo Shuchun, 1984) and my *Glossary*, entry *lü*.

[19]See the discussion of this passage in (Chemla and Keller, 2002).

[20]Chapter A, in (Chemla and Guo Shuchun, n.d.), summarizes the argument that problems in ancient China offered fields of interpretation for the operations of the algorithm following them. More on this below.

[21](Chemla and Keller, 2002) discuss the quadratic irrationals introduced in ancient China and India, but, if we wanted to carry out the computations exactly, the iteration would soon break this framework and require the introduction of more complex quantities. In the second part of his commentary, devoted to computations, Liu Hui deals rather with decimal approximations of the quantities. See below.

[22](Chemla, 1996) deals with this part of the commentary in a more detailed way. I restrict myself here only to the points relating to the analysis of specific features of the practice of proof.

[23]10 *cun* = 1 *chi*. Other units appearing below in the text form a decimal sequence.

[24]This is a reference to the algorithm for extracting square roots as described in *The nine chapters*. The root is yielded digit by digit. Here, when one reaches the last unit available, the algorithm is carried out a last time, yielding a digit that is taken as numerator corresponding to the denominator 10. Concerning this aspect of the algorithm, see the corresponding introduction in (Chemla and Guo Shuchun, n.d.).

[25]In fact, the values are only smaller than the circle in the interpretation provided. They are interpreted to represent the areas of inscribed n-gons and their circumferences. As regards the actual values, Liu Hui seems to lose the control of the approximation. (Volkov, 1994) examines the conduct of the computation with great care.

[26]On this point, see (Chemla, 1997/98).

[27]For details about this, see (Chemla, 1997/98).

[28]This point is developed in (Chemla, 1997a) and (Chemla, 2002).

[29]See Chapter A, in (Chemla and Guo Shuchun, n.d.) and *yi* "meaning", in the *Glossary.* In a forthcoming paper, I shall address the way in which visual auxiliaries are used for determining the "meaning" of some algorithms.

[30]See for example (Chemla, 1997) and (Chemla, 1997/98).

[31]I.e., in case the quarters of the *n*-gon are the finest possible, those obtained at the point when one reaches "what cannot be cut".

REFERENCES

Chemla, K. (1992). Méthodes infinitésimales en Chine et en Grèce anciennes. In (Salanskis and Sinaceur, 1992, 31–46).

Chemla, K. (1996). Relations between procedure and demonstration. Measuring the circle in the *Nine Chapters on Mathematical Procedures* and their commentary by Liu Hui (3rd century). In (Jahnke et al., 1996, 69–112).

Chemla, K. (1997). What is at Stake in Mathematical Proofs from Third Century China?, *Science in Context* **10**(2): 227–51.

Chemla, K. (1997/98). Fractions and irrationals between algorithm and proof in ancient China, *Studies in History of Medicine and Science* **15**(1-2): 31–54. New Series, 1997/98.

Chemla, K. (1997a). Qu'est-ce qu'un problème dans la tradition mathématique de la Chine ancienne ? Quelques indices glanés dans les commentaires rédigés entre le 3^{ième} et le 7^{ième} siècles au classique Han *Les neuf chapitres sur les procédures mathématiques*, *Extrême-Orient, Extrême-Occident* **19**: 91–126.

Chemla, K. (2001). Variété des modes d'utilisation des *tu* dans les textes mathématiques des Song et des Yuan. Preprint for the conference "From Image to Action: The Function of Tu-Representations in East Asian Intellectual Culture", Paris September 2001, published on the website http://hal.ccsd.cnrs.fr/, section Philosophie, subsection "Histoire de la logique et des mathmatiques" . The final version is in preparation.

Chemla, K. (2002). What was a mathematical problem in ancient China?, in (Hart and Richards, n.d.). Preprint for the conference "The disunity of Chinese science", Chicago, May 2002, published on the website http://hal.ccsd.cnrs.fr/, section Philosophie, subsection "Histoire de la logique et des mathmatiques". The final version is in preparation.

Chemla, K. and Guo Shuchun (n.d.). *Les neuf chapitres. Edition critique, traduction et présentation des Neuf chapitres sur les procédures mathématiques (les débuts de l'ère commune) ainsi que des commentaires de Liu Hui (3^{ième} siècle) et de Li Chunfeng (7^{ième} siècle)*, Dunod. Forthcoming.

Chemla, K. and Keller, A. (2002). The Sanskrit *karanis*, and the Chinese *mian*, in (Dold-Samplonius et al., 2002).

Chen Liang-ts'o (1986). Research on Liu Hui's commentary on the procedure of the circular field in the *Nine Chapters on Mathematical Procedures*, *Sinological Research (Hanxue yanjiu)* **4**(1): 47–81. In Chinese.

Dold-Samplonius, Y., Dauben, J. W., Folkerts, M. and van Dalen, B. (eds) (2002). *From China to Paris: 2000 Years of Mathematical Transmission*, Steiner Verlag, Stuttgart.

Guo Shuchun (1983). Liu Hui's theory of areas, *Liaoning Shiyuan Xuebao Zirankexueban (Papers on the Natural Sciences from the Journal of the Normal University of Liaoning)* **1**: 85–96. In Chinese.

Guo Shuchun (1984). Analysis of the concept of *Lü* and its use in *The nine chapters on mathematical procedures* and in Liu Hui's commentary, *Kejishi Jikan (Journal of history of science and technology)* **11**: 21–36. In Chinese.

Hart, R. and Richards, B. (eds) (n.d.). *The disunity of Chinese science*.

Jahnke, H. N., Knoche, N. and Otte, M. (eds) (1996). *History of Mathematics and Education: Ideas and Experiences*, Vandenhoeck & Ruprecht, Göttingen.

Lam Lay Yong and Ang Tian-Se (1986). Circle Measurements in Ancient China, *Historia Mathematica* **13**: 325–340.

Li Jimin (1982). The theory of ratio in *The nine chapters on mathematical procedures*, in (Wu Wenjun, 1982, 228-45). In Chinese.

Liu Dun (1985). Comparison of Archimedes' and Liu Hui's research on the circle, *Ziran Bianzhengfa Tongxun (Journal of Dialectics of Nature)* **1**: 51–60. In Chinese.

Salanskis, J. M. and Sinaceur, H. (eds) (1992). *Le labyrinthe du continu*, Springer.

Volkov, A. K. (1994). Calculation of π in ancient China: from Liu Hui to Zu Chongzhi, *Historia Scientiarum* **4**: 139–57.

Wu Wenjun (ed.) (1982). *Jiuzhang suanshu yu Liu Hui, (The nine chapters on mathematical procedures and Liu Hui)*, Editions of Beijing Normal University.