PAOLO MANCOSU

VISUALIZATION IN LOGIC AND MATHEMATICS

In the last two decades there has been renewed interest in visualization in logic and mathematics. Visualization is usually understood in different ways but for the purposes of this article I will take a rather broad conception of visualization to include both visualization by means of mental images as well as visualizations by means of computer generated images or images drawn on paper, e.g. diagrams etc. These different types of visualization can differ substantially but I am interested in offering a characterization of visualization that is as broad as possible. The article describes and explains (1) the way in which visual thinking fell into disrepute, (2) the renaissance of visual thinking in mathematics over recent decades, (3) the ways in which visual thinking has been rehabilitated in epistemology of mathematics and logic.

This renaissance of interest in visualization in logic and mathematics has emerged as a consequence of developments in several different areas, including computer science, mathematics, mathematics education, cognitive psychology, and philosophy. When speaking of renaissance in visualization there is an obvious implication that visualization had been relegated to a secondary role in the past. One usually refers to the fact that the development of mathematics in the nineteenth century had shown that mathematical claims that seemed obvious on account of an intuitive and immediate visualization turned out to be incorrect on closer inspection. This went hand in hand with a downgrading of *Anschauung* and specifically visuo-spatial thinking from the exalted status it had in Kant's epistemology of mathematics. The effects were also felt in pedagogy with a shift of emphasis away from visualization (for instance, in Landau's diagram-free text on calculus).

1. DIAGRAMS AND IMAGES IN THE LATE NINETEENTH CENTURY

Some of the standard cases mentioned in this connection are the belief that every continuous function must be everywhere differentiable except at isolated points, or the tacit assumption in elementary geometry that the two circumferences drawn in the construction of the equilateral triangle over any given segment in Euclid's *Elements* I.1 meet in one point (the vertex of the equilateral triangle).[1]

In both cases the claim seems to be obvious from the visual situation (diagrams or mental imagery) but turns out to be unwarranted. In the former case this is the consequence of the discovery of continuous nowhere differentiable functions. In the latter case, this was due to the realization

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that only a continuity axiom can guarantee the existence of the intersection point of the two circles. Such results and many other concomitant factors, led mathematicians to formulate more rigorous approaches to mathematics that excluded the recourse to such treacherous tools as images and diagrams in favor of a linguistic development of mathematics. Of course, the use of images and diagrams was still allowed at a heuristic level. The careful mathematician was however supposed to resist the chant of the visual sirens when it came to the context of justification:

> For the appeal to a figure is, in general, not at all necessary. It does facilitate essentially the grasp of the relations stated in the theorem and the constructions applied in the proof. Moreover, it is a fruitful tool to discover such relationships and constructions. However, if one is not afraid of the sacrifice of time and effort involved, then one can omit the figure in the proof of any theorem; indeed, the theorem is only truly demonstrated if the proof is completely independent of the figure. (Pasch, 1882/1926, 43).

In short, visualization seemed to lose its force in the context of justification while being allowed in the context of discovery and as something that simplifies cognition (but cannot ground it). Pasch is well known for being one of the pioneers of a development of geometry characterized by the rejection of diagrams as relevant to geometrical foundations. In the *Foundations of Geometry* (1899) Hilbert is not explicit about the role of diagrams in geometry. However, in a number of unpublished lectures he raises the issue. In lectures on the foundations of geometry from 1894 we read:

> A system of points, lines, planes is called a diagram or figure [Figur]. The proof [of the theorem he is discussing] can indeed be given by calling on a suitable figure, but this appeal is not at all necessary. [It merely] makes the interpretation easier, and it [the appeal to diagrams] is a fruitful means of discovering new propositions. Nevertheless, be careful, since it [the use of figures] can easily be misleading. A theorem is only proved when the proof is completely independent of the diagram. The proof must call step by step on the preceding axioms. The making of figures is [equivalent to] the experimentation of the physicist, and experimental geometry is already over with the [laying down of the] axioms. (Hilbert, 1894, 11).

And in other lectures from 1898 and 1902 Hilbert provides examples of how one can be misled by diagrams by going through a proof of the claim that "every triangle is equilateral". In introducing the example he says:

> One could also avoid using figures, but we will not do this. Rather, we will use figures often. However, we will *never rely on them* [*niemals auf sie verlassen*]. In the use of figures one must be especially careful; we will always have care to make sure that the operations applied to a figure remain correct from a purely logical perspective. (Hilbert, 1902, 602).

These motivations, emerging from the foundational work in geometry and analysis, led to a conception of formal proof that has dominated logic in the past century (and it is usually attached to the names of Frege, Hilbert, and Russell). This conception of formal proof relies on a linguistic characterization of proofs as a sequence of sentences. We find the essential elements of such conception already in Pasch:

> We will acknowledge only those proofs in which one can appeal step by step to preceding propositions and definitions. If for the grasp of a proof the corresponding figure is indispensable then the proof does not satisfy the requirements that we imposed on it. These requirements are fulfillable; in any complete proof the figure is dispensable [...]. (Pasch, 1882/1926, 90).

These attitudes towards diagrammatic reasoning and visualization have thus a complex history, which still calls for a good historian. Certainly one would have to take into account the importance of the development of projective and non-Euclidean geometries in the nineteenth century and of the arithmetization of analysis.[2]

However, I am not convinced that we can tell a linear story where the heroes finally attained a level of rigor hitherto unprecedented, thus leaving the opposition in disarray. For instance, I think there is much to learn from taking a look at the opposition between Klein and the Weierstrass school or the debate that opposed the "rigorist" Segre to the "intuitionists" Severi and Enriques in algebraic geometry.[3]

I will limit myself to a remark on one of the main paradigmatic examples that were used to discredit the role of geometric intuition in analysis, e.g. Weierstrass' discovery of a continuous nowhere differentiable function. Weierstrass' result was announced in 1872 (and published by du Bois Reymond (1875, 29)). The function in question was given by the equation

$$f(x) = \sum b^n \cos(a^n x)\pi$$

with a odd, $b \in [0,1)$ and $ab > 1 + 3\pi/2$.

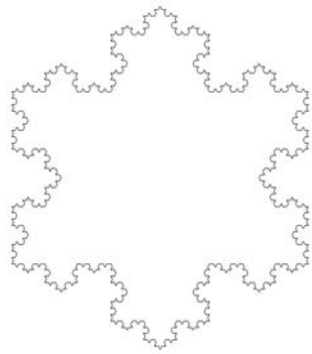

FIGURE 1. von Koch's snow-flake.

Weierstrass' result was given in a strictly analytic way (du Bois Reymond (1875, 29-31)) and it left obscure what the geometrical nature of the example might be. This was somehow characteristic of the school of Weierstrass, which – as Poincaré poignantly puts it – "ne cherche pas a voir mais à comprendre" (Poincaré, 1898, 16). However, there were mathematicians who did not accept this distinction between seeing and understanding. A case in point is Helge von Koch. Von Koch is now well known for his snowflake, one of the earliest examples of fractals and up to this day one of the paradigmatic examples of fractals.

What is not well known is the motivation that led von Koch to his discovery of the snowflake. In his 1906 von Koch begins by remarking that until Weierstrass came up with his example of a continuous nowhere differentiable curve it was a widespread opinion ("founded no doubt on the graphical representation of curves") that every continuous curve had a definite tangent (with the exception of singular points). But then he adds:

> Although Weierstrass' example has once and for all corrected this mistake, it is insufficient to satisfy our mind from the geometrical point of view; for the function in question is defined by an analytic expression which hides the geometrical nature of the corresponding curve so that one does not see, from this point of view, why the curve has no tangent; one should say rather that the appearance is here in *contradiction* with the reality of the fact established by Weierstrass in a purely analytic manner. (von Koch, 1906, 145-6).

Restoring geometrical meaning to the analytic examples was at the source of the work:

> This is why I have asked myself – and I believe that this question is of importance when teaching the fundamental principles of analysis and geometry – whether one could find a curve without tangent for which the geometrical appearance is in agreement with the fact in question. The curve which I found and which is the subject of this paper is defined by a geometrical construction, sufficiently simple, I believe, that anyone should be able to see [pressentir] through "naïve intuition" the impossibility of a determinate tangent. (von Koch, 1906, 146).

Von Koch's project must be seen against the background of the philosophical discussion among mathematicians on the demarcation between "visualizable" (or "intuitable") and "non-visualizable" curves. This discussion (see Volkert (1986)), to which Klein, du Bois Reymond, Köpke, Chr. Wiener and others contributed, should draw our attention to the fact that a detailed history of attitudes towards visualization in the twentieth century might reveal a more complex pattern than a simple and absolute predominance of a linguistic, non visual, notion of proof.

2. THE RETURN OF THE VISUAL AS A CHANGE IN MATHEMATICAL STYLE

But granting the predominance of a linguistic, non visual, notion of formal proof in mathematics – and examples such as Bourbaki make clear that this not a myth – let us now try to characterize the salient features of this 'return of the visual'.

One of the most important aspects is certainly the development of visualization techniques in **computer science** and its impact on mathematics. Here there has clearly been a two ways influence as mathematical techniques have helped shape techniques in computer science (including those leading to great progress in visualization techniques). Conversely, developments in visualization techniques developed by computer scientists have had important effects on mathematics. Computer graphics has allowed researchers to display information (say, analytic or numerical information) in ways that can be represented in the form of a graph, or a chart or in other forms but in any case in a form that allows for a quick visual grasp. Two areas are usually singled out as paradigmatic of the powerful role displayed by visualization in the mathematical arena. The first one is the area of chaos theory, and in particular fractal theory (see Evans (1991)). In "Visual theorems", Philip Davis emphasizes that "aspects of the figures can be read off (visual theorems) that cannot be concluded through non-computational mathematical

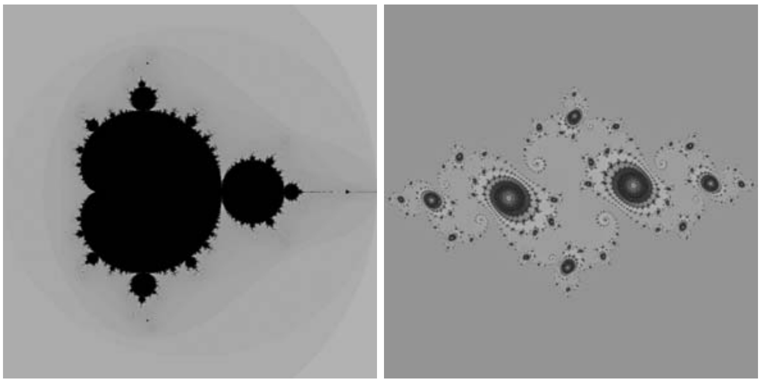

FIGURE 2. Mandelbrot and Julia sets.

devices" (1993, 339). For delightful examples of visual proofs see Roger B. Nelsen (1993, 2000).

Here one can point at the dramatic case of the relationship between the Mandelbrot set and all the Julia sets sitting inside it. It would have been impossible to recognize analytically, without the visual support offered by the computer, that the Julia sets are present inside the Mandelbrot sets. Moreover, the connectedness of the Mandelbrot set became apparent to Mandelbrot on the basis of its graphical appearance.

Another area where the benefits of computer graphics have been greatly exploited is differential geometry. The visual study of three-dimensional surfaces was pioneered in the late seventies by T. Banchoff and C. Strauss. Through the use of computer graphic animation they were able to construct surfaces and gain a better grasp of them by the application of transformations. However, the two most eventful results obtained in this way were the problem of everting the 2-sphere and the discovery of new minimal surfaces.[4] Palais aptly summarizes the situation:

> Two problems in mathematics have helped push the state of the art in mathematical visualization – namely, the problem of everting the 2-sphere and of constructing new, embedded, complete minimal surfaces, especially higher-genus examples. In the case of eversion, the goal was to illuminate a process so complex that very few people, even experts, could picture the full details mentally. In the case of minimal surfaces, the visualizations actually helped point the way to rigorous mathematical proofs. (Palais, 1999, 654).

In his account of the latter discovery David Hoffman says:

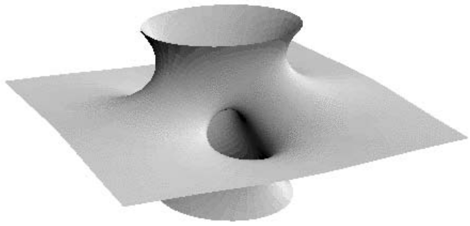

FIGURE 3. Costa's surface.

> In 1984, Bill Meeks and I established the existence of an infinite family of complete embedded minimal surfaces in R^3. For each $k > 0$, there exists an example which is homeomorphic to a surface of genus k from which three points have been removed. Figure [3] is a picture of genus-one example. The equations for this remarkable surface were established by Celsoe Costa in his thesis, but they were so complex that the underlying geometry was obscured. We used the computer to numerically approximate the surface and then construct an image of it. This gave us the clues to its essential properties, which we then established mathematically. (Hoffman, 1987, 8).

Hoffman emphasizes the importance of computer generated images as "part of the process of doing mathematics". However, in his paper he also emphasizes the importance of proving 'mathematically' the properties of the surface which can be 'seen' directly in the visualization:

> Also it [the surface] was highly symmetric. This turned out to be the key to getting a proof of embeddedness. Within a week, the way to prove embeddedness via symmetry was worked out. During that time we used computer graphics as a guide to "verify" certain conjectures about the geometry of the surface. We were able to go back and forth between the equations and the images. The pictures were extremely useful as a guide to the analysis. (Hoffman 1987, p.17)

We thus see that the 'return of the visual' has led to new mathematical discoveries, which, it might be argued, could not have been obtained without the application of computer generated images. Nonetheless, these 'images' are not taken at face value. The properties they display must then be verified 'mathematically'.[5]

The reaction against a purely symbolical conception of mathematics has also found its way in new presentations of certain mathematical subjects that emphasize the visual aspects of the discipline. Paradigmatic examples are Fomenko's 'Visual geometry and topology' (1994) and Needham's 'Visual complex analysis' (1997).

Both of them recognize the importance of the influence of computer science in the recent shift towards more visual methods but their call for a return to intuition and visualization runs deeper and it is rooted in an appreciation of the importance of visual intuition in areas such as geometry, topology, and complex analysis. Fomenko quotes Hilbert approvingly to the effect that notwithstanding the importance of analytical and abstract reasoning "visual perception [Anschauung] still plays the leading role in geometry". Fomenko however does not consider a visual presentation to be logically self-sufficient:

> Many modern fields of mathematics admit visual presentations which do not, of course, claim to be logically rigorous but, on the other hand, offer a prompt introduction into the subject matter. (Fomenko, 1994, preface p. vi).

And later:

> It happens rather frequently that the proof of one or another mathematical fact can at first be 'seen', and only after that (and following the visual idea) can we present a logically consistent formulation, which is sometimes a very difficult task requiring serious intellectual efforts. (Fomenko, 1994, preface p. vii)

Thus, Fomenko's emphasis is on the pedagogical and heuristic value of visual thinking and he does not seem to ascribe to results obtained by visual thinking a justificatory status comparable to that obtained by a 'logically consistent formulation'.

Needham is also very strongly critical of the tendency of modern mathematics to downplay the importance of visual arguments. In a 'parable' he compares the situation in contemporary mathematics to that of a society in which music can only be written and read but never be 'listened to or performed'. He says:

> In this parable, it was patently unfair and irrational to have a law forbidding would-be music students from experiencing and understanding the subject directly through 'sonic intuition'. But in our society of mathematicians we *have* such a law. It is not a written law, and those who flout it may yet prosper, but it says, *Mathematics must not be visualized*![6]

Just like Fomenko, Needham concedes that "many of the arguments [in the book] are not rigorous, at least as they stand" but that "an initial lack of rigor is a small price to pay if it allows the reader to see into this world more directly and pleasurably than would otherwise be possible" (p. xi).

In concluding this section then I would like to point out that many contemporary mathematicians are calling for a return to more visual approaches to mathematics. However, this return of the visual does not seem to upset the traditional criteria of rigor. In all the cases mentioned above all the authors remark on the cognitive importance of visual images in doing mathematics but also seem to recognize that images do not satisfy the criteria of rigor necessary to establish the results being investigated. In this sense this new trend towards visualization, while marking an important shift in style of research and mathematical education (on mathematics education see Zimmerman and Cunningham (1991)), does not seem to me to bring about a radically new position on the issue of the epistemic warrant which can be attributed to arguments which rely on visual steps. In any case, this problem is not addressed directly by any of the authors mentioned above.

At this point several problems could be raised. First, it would be interesting to know more about the cognitive visual roots of our mathematical reasoning and the exact role that mental imagery plays in our mathematical thinking. Second, a number of classical foundational and epistemological questions can still be raised about the warrant afforded by diagrammatic or visual reasoning. In the next section I will try to mention what seem to be the most promising directions of research in these areas at the moment.

3. NEW DIRECTIONS OF RESEARCH AND FOUNDATIONS OF MATHEMATICS

Despite the great revival of interest for visual imagery in **cognitive psychology** (Kosslyn, 1980, 1983; Shepard and Cooper, 1982; Denis, 1989) research in the specific field of mathematical visualization has still a long way to go. Some interesting results are emerging in the study of diagrammatic reasoning (Larkin and Simon, 1987; Glasgow et al., 1995) and problem solving (Kaufmann (1979); for a survey and extensive references see Antonietti et al. (1995)). However, the two most up to date treatments of how the brain does arithmetic (Butterworth, 1999; Deheane, 1997) contain very little on visualization in arithmetic. These investigations in cognitive psychology have influenced a number of researchers active in foundations of mathematics, who are interested in addressing the complex web of issues related to perception, imagery, diagrammatic reasoning and mathematical cognition.

Let me begin by mentioning the project "Géométrie et cognition" led by G. Longo, J. Petitot, and B. Teissier at the ENS in Paris. Their approach emphasizes the need to provide cognitive foundations for mathematics, in opposition to logical foundations à la Hilbert. Their research is strongly influenced by the dramatic developments in cognitive psychology, especially in the area of perception theory. And although mental imagery is not stressed in their 'manifesto' (where the emphasis is on perception)[7], it is obvious that their program calls for an account of the cognitive role of mental imagery in mathematics. This 'cognitive' approach to the foundations is less concerned with the traditional goals of logical foundation, as it had been pursued in the tradition of proof theory. Rather, it goes back to the tradition represented by Riemann, Poincaré, Helmholtz and Weyl. Moreover, they appeal to Husserl's phenomenological analyses and in particular the work on genesis of concepts. For this tradition the foundational task is essentially to give an epistemological analysis of the constitutive role of the mind in the construction of mathematics and geometry in particular.

Of great interest is also the epistemological work on visualization carried out by Marcus Giaquinto.[8] One can read Giaquinto's project as trying to account for the epistemological status of certain experiences of visualization which, he argues, are substantially different both from observation and from conceptual reasoning. There is obviously a "Kantian" flavor to the project. The thesis, as presented by Giaquinto, is that the epistemic function of visualization in mathematics can go beyond the merely heuristic one and be in fact a means of discovery. We are used to associate discovery with the heuristic context. But discovery is taken here in a technical sense according to which "one discovers a truth by coming to believe it independently in an epistemically acceptable way". The independence criterion is meant to exclude cases in which one comes to believe a proposition just by being told. One of the conditions on the requirement of epistemic acceptability is that the way in which one comes to believe a proposition is reliable. Giaquinto then proposes a case of visualization (a simple geometrical fact about squares) for which he claims that through that process of visualization one could have arrived at a discovery (in the sense above, which does not entail priority) of the result. However, the justification provided by the visualization need not be a demonstrable justification, i.e. a justification that can be checked by intersubjective standards of proof. And nonetheless this is, he concludes, a legitimate way to come to know a mathematical proposition. The upshot of his investigations is the claim that whereas in elementary arithmetic and geometry it is possible to discover truths by means of visualization this is not the case in elementary real analysis, except perhaps in extremely restricted cases. It is important to point out here some important features of Giaquinto's approach. In traditional philosophy of mathematics the emphasis is mostly on major theories, such as arithmetic, analysis, or set theory. The main question that has been pursued is whether these theories are true and, if so, how do we know them to be true. Since deduction can preserve knowledge, usually the question becomes that of the epistemology of the axioms of such theories. Giaquinto asks the analogous question for the case of the individual and his or her mathematical beliefs. How do people know their initial (uninferred) mathematical beliefs? And more generally, how do they acquire their beliefs, whether initial or derived? His strategy is then to investigate how people actually acquire their beliefs, and this is where cognitive psychology comes in. Once we have isolated the cognitive mechanisms of belief acquisition then we can subject them to epistemological analysis and ask whether they are in fact knowledge-yielding. While it is beyond the scope of this paper to address the argumentative line defended by Giaquinto, let me just grant him the thesis and see how it fits within the spectrum of positions in foundations of mathematics. Obviously, his project dovetails quite well with the issues raised by those philosophers of mathematics who insist that foundations of mathematics should address issues concerning the epistemology of mathematics. Giaquinto's last writings on this issue in fact (see this volume) put forth an account of the interaction between perception, visual imaging, concepts and belief formation in the realm of elementary geometry. But, as mentioned before, it is not central to Giaquinto's claims that the types of visualizations he discusses would count as proofs in the traditional sense. He thus moves away from the traditional concerns in philosophy of mathematics in two ways. First of all, he shifts the focus from the community to the individual. Second, since justification will ultimately depend on some unjustified premises that we must hold to be true, the question becomes how do we know these ultimate premises to be true. And that, Giaquinto argues, cannot be done by giving another justification (which would involve a regress) but rather by an epistemic evaluation of the way we come to believe those premises. But this is a question that even those who focus on major mathematical theories will have to address, as the starting point of those theories would have to be accounted for epistemologically. And how could that be done, without going back to the mechanisms of belief acquisition of the individual? In this way, Giaquinto's work shows its relevance also for traditional programs in the philosophy of mathematics.

By contrast, the work carried out by Barwise and Etchemendy on visual arguments in logic and mathematics is motivated in great part by the proof-theoretic foundational tradition.[9] While Giaquinto was mainly concerned with discovery (in the technical sense we have pointed out), Barwise and Etchemendy focus on proof.

Barwise and Etchemendy begin by acknowledging the important heuristic role of visual representations but want to go further:

> We claim that visual forms of representation can be important, not just as heuristic and pedagogical tools, but as legitimate elements of mathematical proofs. As logicians, we recognize that this is a heretical claim, running counter to centuries of logical and mathematical tradition. This tradition finds its roots in the use of diagrams in geometry. The modern attitude is that diagrams are at best a heuristic in aid of finding a real, formal proof of a theorem in geometry, and at worst a breeding ground for fallacious inferences. (Barwise and Etchemendy, 1996, 3).

Their position challenges the "dogma" 'that all valid reasoning is (or can be) cast in the form of a sequence of sentences in some language'. To this effect they aim at developing an information-based theory of deduction rich enough to assess the validity of heterogeneous proofs that use multiple forms of representation (both diagrammatic and verbal). The point is that language is just one of the many forms in which information can be couched. Visual images, whether in the form of geometrical diagrams, maps, graphs or visual scenes of real-world situations are other forms. The goal becomes then that of developing formal systems of reasoning in which diagrammatic elements play a central role. It is important here to keep two different claims in mind. The first claim is that "not all valid reasoning is (or can be cast) in the form of a sequence of sentences from some language." The second claim is that it is possible to construct heterogeneous systems of logic, which unequivocally show that it is possible to reason rigorously with diagrammatic elements. This requires extending to these new systems the analogue of notions of soundness, completeness etc., which are the adequacy conditions for formal systems of deductive inference. And in turn this requires a framework that 'does not presuppose that the information is presented linguistically'. Work along these lines has been done by Barwise, Etchemendy (see 1996) and their students (see Allwein and Barwise (1996); Shin (1994), among others).

What conclusions can one draw from the work that has been achieved in this area?

A far-reaching claim made by Barwise and Etchemendy was that "not all valid reasoning is (or can be cast) in the form of a sequence of sentences from some language". If what is meant is that there are forms of valid reasoning (visual or diagrammatic reasonings) which cannot be expressed in linguistic form, then I claim that the positive developments mentioned above do nothing at all to prove the point.[10] Indeed, even setting up the question in such a way is problematic, for there is very little clarity on what criteria one can appeal to in order to distinguish linguistic systems from visual systems. These issues are at the center of much recent work (Stenning, 2000).

However, the logical precision of these diagrammatic systems allows one to investigate a number of claims that were made for or against the use of visual elements in proofs. For instance, people have often noticed the lack of expressive power of diagrammatic systems. The setting up of diagrammatic systems has given us a better insight into the problem. Consider for instance Venn's idea of representing all relationships between an arbitrary number of classes by means of closed curves. It was obvious to Venn that if one only uses circles, there is no way to go beyond 3 classes, that is the addition of a fourth circle to the diagram will not be able to represent all the possible combinations between 4 classes.

In the case of Euler's diagrams there are also limitations, due to Helly's theorem, which shows that there are consistent sets of set intersection statements that cannot be represented by any diagram of convex curves. In short, it is essential to study how the geometrical and topological features of the representation system affects its expressivity.

Another advantage of setting up formal systems of diagrammatic reasoning is that one can give a logical analysis of the often made claim that diagrammatic systems are intrinsically more efficient. A recent article by Lemon and Pratt (1997) develops a computational complexity approach to the study of diagrammatic representations.

I would like to conclude with a reflection on how this work affects traditional foundational concerns. One claim made by Barwise, Etchemendy, Shin and others is concerned with the foundational issue of reasoning with diagrammatic representations, i.e. that it is possible to reason rigorously with diagrammatic elements. Thus, visual systems are not inherently deceptive, or no more than linguistic systems might be. Here I think that the work done by Barwise, Etchemendy, Shin and others proves the point. What they did was to show that to the traditional model of linguistic rigor we can now add rigorous forms of inference with diagrammatic elements.

However, there are several philosophers of mathematics who are opposed to this traditional approach and are interested in visual reasoning as an essential factor in providing a more 'realistic' philosophy of mathematics, sensitive to its practice and its cognitive roots. I believe that many of them would find the work by Barwise and Etchemendy on diagrammatic reasoning insufficient at best and misguided at worst. The problem, for many people in this tradition, is that exclusive attention to the goal of justification is unacceptable. There are many other important epistemic goals, such as discovery (in Giaquinto's sense), explanation, understanding, genesis of concepts etc., that philosophy of mathematics should account for.

In any case, the work on diagrammatic reasoning accounts for a very minimal part of our employment of visual tools in our logical and mathematical experience. Should the practicing mathematician feel more comfortable using visual or diagrammatic tools in his or her work? I think the work on diagrammatic reasoning does not do much to allay possible worries of being misled by the visual tools in research contexts but it does show that the reasons for why such tools are problematic is not necessarily on account of some intrinsic feature of the visual medium. It is rather that one must always check that the visual medium does not introduce constraints of its own on the representation of the target area. And I doubt this is an issue that can be settled a priori rather than by a detailed case by case analysis of such uses. But after all, mathematicians have been doing just that for more than two thousand years.

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Department of Philosophy
U.C. Berkeley
USA

NOTES

[1]See (Pasch, 1882/1926, 44).

[2]For diagrammatic reasoning in Greek mathematics see Netz (1999). For visualization in non-Euclidean geometry see Reichenbach (1956). For debates on visualization in the arithmetization of analysis see Volkert (1986).

[3]One should recall here Hadamard's distinction between visual and symbolic approaches to mathematical thinking. Hadamard himself claimed to think visually, and following Poincaré characterized mathematicians as falling into two broad classes, the analysts and the geometers. As for Hilbert, quoted by Hadamard: "I have given a simplified proof of part (a) of Jordan's theorem. Of course, my proof is completely arithmetizable (otherwise it would be considered non-existent); but, investigating it, I never ceased thinking of the diagram (only thinking of a very twisted curve), and so do I when remembering it" (Hadamard, 1949, 103).

[4]The problem of the eversion of the two sphere is that of turning it inside out without tearing. For reasons of space I will not give a detailed explanation of what the two problems are. Palais (1999) provides a very readable account of the two results. I trust that the main methodological point will be clear even for those who are not conversant with differential geometry.

[5]Indeed one should not make the mistake of underestimating the complexity of producing 'persuasive' images. For instance in his article Hoffman describes how much of the mathematical community admitted difficulty in understanding the images they were producing and that this forced them to produce more realistic images. See Hoffman, 1987, p.18.

[6]"More likely than not, when one opens a random modern mathematics text on a random subject, one is confronted by abstract symbolic reasoning that is divorced from one's sensory experience of the world, despite the fact that the very phenomena one is studying were often discovered by appealing to geometric (and perhaps physical) intuition.

This reflects the fact that steadily over the last hundred of years the honour of visual reasoning in mathematics has been bismirched. Although the great mathematicians have always been oblivious to such fashions, it is only recently that the "mathematician in the street" has picked up the gauntlet on behalf of geometry. The present book openly challenges the current dominance of purely symbolic logical reasoning by using new, visually accessible arguments to explain the truths of elementary complex analysis." (Needham, 1997, vii).

[7]See http://www.di.ens.fr/users/longo/geocogni.html#anchor1640003 for the program and further references.

[8]The epistemological investigations by Giaquinto (1992, 1994) also take their start from cognitive psychology. In particular, Giaquinto was influenced by Kosslyn, who in his book "Ghosts in the Mind's Machine" comments on the relationship between knowledge and imagery by making the point that previous knowledge constrains the images we come up with. By contrast, Brown (1997) takes its start from the work by Barwise and Etchemendy.

[9] I should immediately point out that the narrative contrast between Giaquinto's work and Barwise and Etchemendy has to be taken with caution. Barwise and Etchemendy focus on diagrammatic reasoning, which is a form of visual reasoning, but have nothing to say about the phenomenology of visualization, which constitutes the main contribution by Giaquinto.

[10] A proper discussion of this topic would quickly lead into cognitive psychology and to the 'imagery debate'. See Tye (1991).

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