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# THE AESTHETICS OF MATHEMATICS: A STUDY

To the memory of Heda Segvic

## 1. THE PROBLEM MOTIVATED

Let us start with a trivial example, which however already suggests the outlines of the problem at hand. Imagine I have collected my lunch at a self-service cafeteria so that now my tray holds, say, a paper plate with a sandwich on it, another one with fruit, and finally, a soda in a large cup (the kind known as “small”). Now, as I prepare to detach myself from the counter, I arrange the three objects on the tray. This can be approached through several theoretical perspectives.

First, there is the mathematical-physical perspective, employing the specific field of *statics* (pioneered, as we shall note again below, by Archimedes). The task is to arrange three objects on a plane, so that their individual centres of gravity, and the centre of gravity of the system as a whole, will ensure maximum stability. One should in particular consider the problem of the system’s robustness, i.e. how it may react with the disturbances it is likely to undergo as I move towards a table. This is a very complex problem, and the fact that we very often (not always) solve it in effective ways, may indicate our powers of unconscious computation.

The mention of the “unconscious” immediately brings to mind a further relevant theoretical perspective. It may be suggested that the desire to arrange objects in neat, ordered ways could reflect either an obsessive-compulsive disorder, or its more or less universal incipient form. Whatever one thinks of any particular form of *psychoanalysis*, it is clearly a possible way of explaining my acts as I rearrange the objects on the tray. While the mathematical perspective provides a possible functional role for the *arrangement obtained*, the psychoanalytic perspective provides a possible functional role for the *act of arrangement* itself – by uncovering the desires and needs which find their outlet in that act.

Finally, regardless of the desires that motivate the act of arrangement, and regardless of the physical function of that act, one can study the formal properties of the arrangement obtained, this time adopting the perspective of *aesthetics*. Merely as a visual pattern on the tray, the objects possess properties such as symmetry and composition. The driving force that makes *me* align my plates along a precise geometrical configuration is perhaps best understood by the psychoanalyst. However, some of the properties of the alignment I achieve are not psychological, but aesthetic: they belong, so to speak,



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not to the psychopathology of everyday life, but to the *aesthetics* of everyday life. Anxiety was momentarily warded off; but on the tray itself, we observe not the absence of anxiety, but the presence of, say, the golden section. In general, then, *statics* studies the function of the arrangement obtained; *psychoanalysis* studies the function of the act of arrangement; and *aesthetics* studies certain objective features of the arrangement obtained (which are of course difficult to characterize precisely). We may perhaps say (if only so as to have the word "function" available for our use on all occasions) that aesthetics studies the *aesthetic* function of an object. This is useful because now we can note that the various functions differ as to their dominance in different contexts. We may observe, for instance, how people rearrange their trays as they sit down on the table: the tray safely in place, the physical function lost its dominance and the aesthetic function is dominant instead.

I have introduced two theses. One is that the aesthetic function is ubiquitous; the second is that, in different contexts, it may be more or less dominant. Such theses have long been current (their best statement probably remains Mukarovsky (1970), translation of a work written in 1936), though perhaps interest in the aesthetic function of literary works has more recently waned in the English-speaking world. At any rate, the trivial example I have delineated is meant to introduce the idea of the aesthetic function of *mathematical texts*.

Thus, we should not be concerned about the fact that mathematical texts have obvious, overt functions (akin to the static features of the tray in my example), e.g. to obtain the truth of mathematical results for some possible mathematical or physical applications. This overt function can be separated, analytically, from the aesthetic features of a mathematical text. (Of course, there might be interesting interactions between such overt functions and the aesthetic function.) Further, we need not be concerned about another fact, that mathematical texts – like all texts – are motivated by all sorts of external forces, such as the sociological realities of publication and tenure, comparable to the psychological processes suggested to underlie my ordering of plates on the tray. (Once again, though, sociological factors may interestingly interact with the aesthetic factors.) Finally, I wish to stress – this is the main point of my trivial example – the mundane nature of the aesthetic. At least to begin with, in this article I do not intend to wax lyrical about the beauty of mathematics. Mathematical works are sometimes great works of art, sometimes (even when they are of considerable mathematical value) their presentation is boring and pedestrian. It is not my contention that mathematical texts are particularly beautiful, more so than other types of human expression. Rather, like all types of human expression, they possess, among other things, an aesthetic dimension.

Yet the question of mathematical beauty is of special urgency. Mathematicians – it seems, more than most other scientists – often claim to be motivated by the aesthetic dimension. To take the most famous example, Hardy insisted that "The mathematician's patterns, like the painter's or the poet's, must be *beautiful*... Beauty is the first test: there is no permanent place in the world for ugly mathematics" (Hardy, 1967, 85).[1] Hardy then went on to contrast the beauty of mathematics with its – as he claimed – inutility. To be precise, Hardy argued that the utility of a given piece of mathematics is inversely related to its beauty (so that, say, the multiplication table - perhaps the most 'useful' part of 'mathematics' – is so devoid of beauty as hardly to deserve the name 'mathematics'). In other words, Hardy considered the aesthetic dimension as dominant in mathematics. We should not follow judgements such as Hardy's blindly; what we need to do is to have some way to position them in the reality of mathematical experience. What is the objective feature that authors such as Hardy identify in mathematics, when they identify a 'beauty' within it? Only after we have answered such questions, we can return to answer usefully the question, why Hardy chose to value this feature above others. One purpose of this article is as a prolegomenon for such questions.

Another purpose lies within the history and philosophy of mathematics themselves. It appears that there is a certain difference between arguing that a certain piece of mathematics was created in order to get tenure, and arguing that it was created in order to produce beauty. My intuition is that, in the first case, we learn something about tenure, while in the second case we learn something about mathematics. A mathematician, QUA mathematician, may aim at truth, necessity, generality, and many other epistemic values – and at the same time, and still QUA mathematician, he or she may also aim at non-epistemic values such as beauty. Then again, he or she may aim at tenure – but there we may be inclined to drop the 'QUA mathematician' clause.

This then may be the contribution of the following discussion to the history and philosophy of mathematics. Working in this discipline, we naturally tend to concentrate on the *epistemic* values of mathematical activity – which were of course at the heart of the philosophy of mathematics from its inception with Plato onwards. If any non-epistemic values may be recognized, their role might be acknowledged, but then they are considered as extrinsic to mathematics itself. I would suggest that aesthetic value is a key example of a non-epistemic value that, however, is *intrinsic* to mathematics. The thrust of the articles collected at this volume is, I believe, to widen our picture of the field of mathematical practice as a rational activity: one that appeals to the visual and not merely to the symbolic, that aims at explanation and not merely at proof. It also appeals, I suggest, to the aesthetic. Among other things – and still as rational practitioners – mathematicians aim at beauty.

## 2. SOURCES OF BEAUTY IN MATHEMATICS

### *2.1. An Outline*

I propose in this section a typology of sources of mathematical beauty. However, I warn immediately of the simplifications I adopt, selecting from the complexity of the problem to focus on what is, I hope, a tractable and still significant domain.

To start with, the problem of mathematical beauty might be addressed at several levels, as beauty is encountered throughout the mathematical life. First, most mathematicians feel that there are aesthetic qualities to the mathematical pursuit itself. The states of mind accompanying the search for mathematical results are often felt as sublime; an aesthetic study seems warranted. This then is mathematical beauty as a property of states of mind. Second, beauty resides in the products of this pursuit – in mathematical theorems and treatises. This then is mathematical beauty as a property of texts. Finally, beauty resides in the entities studied by those theorems and treatises – in the many mathematical worlds – groups, spaces, numbers and sets... This then is mathematical beauty as a property of the ontological realm of mathematics. This ontological interpretation is perhaps the main context in which we think of "mathematical beauty".

In this article I focus on beauty as a property of mathematical texts. I do this for two extrinsic reasons and for one intrinsic reason. The first extrinsic reason is that texts are most readily available for our study: we have a clear and well-defined corpus for investigation. The second extrinsic reason – closely related to the first one – is that there is already a body of theory, in poetics, which I can take as suggestive for the study of beauty in mathematical texts.[2] Finally, and more intrinsically, I suggest that the study of beauty in mathematical texts may shed some light on the question of beauty in the mathematical pursuit and in the mathematical world. I shall try to argue for this in the conclusion, when we have completed the typology.

One simplification, then, is to focus on a single *layer* of mathematical beauty. Simultaneously, I limit myself to a single *field*. In recent years, it has been a tacit assumption of much of the work in the philosophy of mathematics that mathematical practice is heterogeneous. The nature of mathematics changes, depending on the discipline, the time and the place. In this investigation, we concentrate on properties of mathematical *texts*, which are even more obviously dependent on culturally specific settings than mathematical "ideas", say, are. Thus it seems prudent to start not from some global overview of the beauty in mathematical texts as such, but instead from a single genre of texts. In this article, I concentrate on ancient Greek mathematics, in particular geometry. This, once again, has extrinsic and intrinsic reasons. The extrinsic reason is that I am most familiar with this genre; the intrinsic reasons are that this genre is the foundation of western mathematics – and is often invoked as a model for the role of beauty in mathematics[3].

Briefly, then, this article offers a typology of the aesthetic issues in Greek mathematical texts. We now finally come to the subject matter itself, and let me explain how I intend to carve up this large field into a typology.

A very obvious initial distinction to be made is between the large scale and the small scale. On the one hand, beauty is felt at the level of whole treatises (or at the level of a proposition, taken as a whole). On the other hand, beauty is felt at the level of the mathematical text as it unfolds – in the immediacy of the texture of read words. The main difference suggested by this comparison, it seems to me, has to do not with scale itself as with the different kinds of experience it implies. At the large scale, beauty has to do more with the ways in which mathematical *contents* are arranged; at the small scale, the contents are less important, and the *form* of the arrangement becomes more important. To offer a rough analogue, one can liken the large-scale structure of a treatise to the *narrative* structure of prose works, e.g. novels – which of course is to a large extent an arrangement of the *contents* signified by the novels. On the other hand, one can liken the small-scale structure of a single mathematical statement to the *prosodic* structure of poems – which of course to a large extent has to do with phonological *form* independent of content. In the next two subsections, I shall discuss first the "narrative" properties of mathematical works (subsection 2.2) and then their "prosodic" properties (subsection 2.3). That this crude analogue is of service is part of what I need to show.

Narrative has to do with content; prosody has to do with form. Those are the two essential layers of any discourse, and it is often suggested that aesthetics has to do precisely with the *clashes* between layers of discourse.[4] If so, we should expect the relationship between form and content, itself, to be a source of beauty in mathematical texts. I try to show that this is the case in subsection 2.4 below. This area, of the relationship between form and content – signifier and signified – does not lend itself to an easy label but, for reasons which will be made clearer in subsection 2.4 itself, I title it "correspondence".

### 2.2. *"Narrative"*

The question of narrative often enters contemporary poetics in the form of narrative as a *process*: what may be called *narration*. Thus for instance one may note the distances between writers, authors and narrators, so as to follow the aesthetics of ironies and gaps (Booth, 1961). A defining feature of Greek mathematics is its implicit claim to transcend subjective perspectives: this approach is thus largely irrelevant for mathematics.[5] Narrative enters mathematics not as process, but as *structure*: ignoring the question of the identity of the narrator, something is being narrated, and we may note how elements are selected and combined along this narrative.[6]

Take for example Archimedes' first book on *Sphere and Cylinder*. This has for starting point a discursive introduction (addressed to Dositheus, a colleague), where the goal of the treatise is set out explicitly. Archimedes proudly says he had discovered fundamental results about the sphere, in particular that its surface is four times its great circle, and that its volume is two-thirds the cylinder enclosing it. Having said that, he moves on to offer a set of axioms or postulates (none of which is very closely related to the sphere), and plunges into the mathematical detail.

There is nothing about spheres or cylinders, their volumes or their surfaces. The main substantial sequence of results (propositions 2-6) deals with polygons and circles in proportion. Next, propositions 7-12 deal with surfaces of *pyramids*; propositions 13-20 – the surfaces of *cones* (and of various figures composed of segments of cones). Still no word of the sphere (though, with cones, we at least move into something resembling the *cylinder*). Then the following two propositions 21-22 move out to a totally new territory. Instead of having anything to do with three-dimensional figures, they return to the polygons of propositions 2-6 and state for them very complex and special results, having to do with proportions of lines drawn through the polygons. Those lines do not seem to have any relevance to anything – certainly not to spheres. (fig. 1).

Then, in proposition 23, we are asked to make a thought experiment. We rotate the circle, polygon and lines from fig. 1, and obtain in this way a sphere in which is enclosed a figure composed of segments of cones. It now becomes obvious that the results concerning polygons, and the results concerning cones, can be put together and (with the aid of the specific claims made about proportion, as well as about pyramids), can immediately give rise to the proportions determining the surface and volume of a sphere. The seemingly irrelevant and long preparation – just about half the book – is suddenly found to be directly relevant so that, indeed, the main line of reasoning

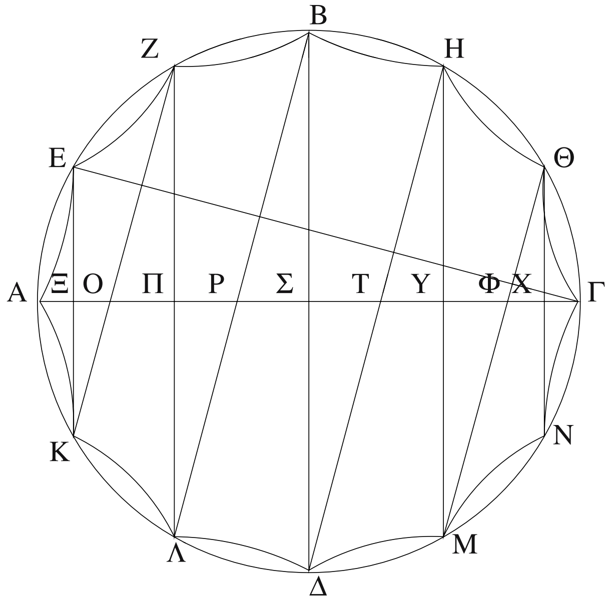


FIGURE 1.

can now proceed quickly to obtain Archimedes' main results in propositions 33-34.

As I promised already, it is not my intention to wax lyrical about mathematics. That Archimedes was a genius of narrative is what I subjectively feel. In more objective terms, however, all I am concerned with is that narrative structure is indeed a proper perspective through which to analyze Archimedes' performance. The simplest way to show this is by noting that he had *alternative ways of presenting his argument*. The most obvious one – and the one with potentially the greatest harm for his narrative achievement – would have been to *start* with the thought-experiment of proposition 23. Clearly then the sense of a brilliant master-stroke would have been completely eroded. Thus we notice a fundamental fact: the mathematical kernel of an argument – whatever we take *this* to be – only very weakly underdetermines the form it may take. The mathematician makes decisions for the form, decisions that are mathematically undetermined (in a traditional, narrow sense of mathematics) and therefore may well be dominated by the aesthetic function.

We should not think of Archimedes' *Sphere and Cylinder* as representing the only type of mathematical narrative structure. As one further example, let us now take Euclid's *Elements* book I. We may first note that Euclid does not use the device of setting out his goal explicitly. Instead, his work starts truly in medias res, with some definitions, postulates and common notions. The text has to start from nowhere, but it is quickly infused with momentum. A subject matter is implicitly defined – the triangle – and a string of ever-stronger results follows, very often in a neat sequence where one result leads on to the next. Thus we move almost imperceptibly from the state of nil knowledge, at which we start, to relatively remarkable results such as the congruities of triangles (propositions 4 and 8, to begin with). We also see the relationships between angles and sides (the famous proposition 5 – base angles in isosceles triangles are equal – and its converse 6). Problems, showing how to obtain a task, and theorems, showing the truth of a result, are neatly intertwined: problems 1-3, then theorems 4-8, then again problems 9-12. We thus establish a pattern: problems lead on to theorems, which in turn lead on to problems, and then again to theorems: theorems 13-21, and then again problems 22-23. At this stage we get to some very strong and general constructions: a triangle from any given appropriate three sides, an angle equal to any given angle. Now we press on again and, during the next phase of theorems, a variation on the theme of triangles-and-their-angles is offered, with the notion of parallel lines, introduced from proposition 27 onwards. This quickly leads to a problem, in 31, and then an application for the theme of triangles-and-angles (the famous "sum of angles equal to two right angles", in proposition 32), as we move on to widen our field to quadrilateral figures, from proposition 33 onwards. Parallelograms are studied, mainly through the perspective of triangles-and-parallels, until we reach again a sequence of problems on this set of issues, propositions 42-46. (45 is especially strong and general – to construct, in a given angle, a parallelogram equal to a given rectilineal figure – and Mueller (1981, 16) claims this is in some sense the goal of book I.) Finally the book ends with the coda of Pythagoras' theorem (understood through triangles and parallelograms) in proposition 47, with a quick converse, 48.

The movements of the text are all handled implicitly: the author never interferes, never speaks on behalf of the propositions. They do the narrative work on their own: pressing ahead with an even pace, moving from the absolute nothingness of the foundations of geometry and obtaining a full structure, reaching the capstone theorem of I.47[7]. (The architectonic metaphor is hard to avoid.) A few figures are elaborated throughout, gradually evolving. The evolution has a cyclical pattern – from problems to theorems and vice versa – and a linear pattern – from more elementary results, to stronger results based on them. Thus there is an overall structure of a widening spiral where every cycle of theorems and problems is capable of developing further the main themes. Finally we get to Pythagoras' theorem, obviously a most interesting result about triangles: triangles, the main character of the book, make the most remarkable journey, from nothingness to Pythagoras.

This does not work at the level of surprise and irony of Archimedes' *Sphere and Cylinder*, say, and the aesthetic principles are clearly different. Euclid does not aim to startle, in a quick stroke, but to impress, in a stately progression. Once again, Euclid could have made other choices, which would have given the work as a whole a different aspect. He could have introduced the circle at this stage, and so develop simultaneously the elementary results for all main plane figures (instead, he postponed the circle to book III). This would have made this book more comprehensive in scope, but lacking in narrative coherence. Or he could have ended this particular book with the results on parallelograms, for instance, leaving Pythagoras' theorem to another book. This, however, would be to miss on the sense of closure which this theorem provides in its great inherent interest, and in its reverting to the main character, triangles. In such ways, we can begin to substantiate one's immediate impression, that in Euclid's *Elements* I, narrative structure is a dominant organizing principle.

We have thus seen two special examples of narrative structure in Greek mathematical treatises. One can compare them, perhaps, to narrative structures in verbal art in general, for instance in the novel. Some novels are organized in complex structures of suspense and irony, which work by evoking expectations and then playfully subverting them; others are much more directly progressive, and create their sense of structure from a certain balance and directionality about the work as a whole. Archimedes' sudden revelation, (polygon)=(figure composed of conic segments), is perhaps comparable to, say, Charlotte Bronte's sudden revelation in *Jane Eyre*, (cries at night)=(mad wife). An earlier stage of the narrative is suddenly found to have a new, unanticipated meaning, by being retrospectively reinterpreted through a piece of information provided at a later stage of the narrative. Take, on the other hand, the stately progression of the triangle in *Elements* I, going through cycles of theorems and problems, in the process constructing a thick world. This is perhaps comparable to, say, the stately progression of the lives of Russian aristocrats in *Anna Karenin*, moving cyclically from the Anna plot to the Lyovin plot and finally leading to the fulfilment of the strands of narrative with Anna's suicide and Lyovin's family life. The examples are not meant with any great seriousness, and I certainly wouldn't like to suggest that, say, Euclid was a "realist" whereas Archimedes was a "romantic" etc.. I merely wish to point out that one can plausibly point to a variety of types of narrative structures which may be implemented to various aesthetic effects, and which can be found in mathematics just as they can be found in literature. In the above, I have suggested two possible types, and doubtless others can be observed as well.

It is immediately clear that narrative structures can be found in scales smaller than the treatise taken as a whole. In the genre of Greek mathematics, works are composed of a sequence of a few dozen smaller textual segments, today referred to as “propositions”. Each of those units has internal structure, and some aspects of it closely mirror the narrative structures suggested already. A proposition is a sequence of statements about objects. The pace in which objects are introduced, and the ways in which statements create expectations, fulfil or subvert them, may all be used for an overall aesthetic function. I have touched upon this topic, from a separate angle, in (Netz, 1999, 198-216), noting the Greek tendency to have smooth, linear progressions in their proofs. I have identified there what I still consider to be the dominant function: the desire to have the proof fit a certain model of persuasion. This may serve as an example of a more basic point. Persuasion, as such, is not an aesthetic function: but the practices of persuasion and of narrative are in fact closely implicated in each other. To persuade, the text must be perceived to have a certain unifying structure - and a structure that may be endowed with aesthetic properties. Furthermore, the very act of persuasion is about the structure of introducing objects, raising expectations about them, and fulfilling those expectations (or perhaps subverting them, e.g. in refutations). The structures that give rise to persuasion are precisely the structures that give rise to narrative structure. Thus, while the aesthetic may not be the dominant function of persuasive texts, it is an *inevitably* relevant function.[8]

I take a quick example. In (Netz, 1999, 213), I have suggested that Archimedes’ *Method* 1 is different from most other, “smoother” propositions. Instead of a clear linear structure, it has several hiatuses in the argument and, in particular, it has a very complex, quirky structure near its middle (I numbered the statements in sequence, so that the proof had 34 statements, and the complex passage is statements 13-18). I have suggested that there might have been a particular motive involved: Archimedes introduces here his surprising suggestion (to identify an area with a sum of lines). The structure is all designed to delay this suggestion, and then to bring it out in a startling way: exactly the same structure as we saw in larger scale in *Sphere and Cylinder I* as a whole. We may indeed have identified a feature of Archimedes’ *style*.

At any rate, it now seems plausible that narrative may sometimes serve in mathematics as a source of beauty. I shall now briefly suggest, in more metaphysical terms, why this, I think, may be the case.

As noted above, *narration* – the use of narrators' perspectives – does not play a role in Greek mathematics. The medium of truth, par excellence, is ordinary language. It is thus natural that verbal art – the art whose vehicle is language – should so often dramatize the issue of truth and belief, of objectivity and subjectivity. This however does not get dramatized inside *individual* Greek mathematical texts, precisely because Greek mathematical texts markedly dramatize this issue, *when they are taken as a genre*. In quite simple terms, I argue that Greek mathematics was read, partly, against the background of other forms of persuasion. Its claim to possess absolute objectivity and truth is reflected by a rigid form from which perspective-hood, so to speak, has been eliminated. Perhaps this basic decision may be read in aesthetic terms, so that the genre, as a whole, possesses beauty in its sublime impersonality.

However, another kind of narrative structure is allowed: the author may chose to reveal as much or as little of the plot as he or she pleases; he or she may structure this information in many possible sequences. Such choices may possess aesthetic value, and in this way mathematical texts may possess an aesthetic dimension. I now suggest that this aesthetic dimension reveals something fundamental about the relation between mathematics and beauty.

It might perhaps be considered strange that the author has so much choice in mathematics. After all, is not mathematics governed by necessity, so that mathematical truth simply unfolds as a matter of logic? In fact this image is deceptive. It is true that, in a valid argument, the conclusion does follow from the premises. If C follows from the combination of A and B, it is possible to argue "A, and B, therefore C", and C does not only appear to be inevitable: it is inevitable, in the sense that it cannot fail to be *true*. But it can easily fail to be *made*. In general, each mathematical text makes a double set of choices: which premises to assert, and which conclusions to draw *explicitly* from the premises. The fact that a premise is true, just as the fact that a conclusion follows from asserted premises, both do not constrain the mathematician, do not force the mathematician to make them. The mathematician works in absolute freedom – creating a fabric of text that is woven together by the ties of logical necessity.

This dialectic of freedom and necessity is, I suggest, often at the root of the beauty of mathematical narratives. What is, after all, a surprising result in mathematics? It is a result whose perception of inevitability is not determined by the text preceding it, so that it is perceived twice: once for the *freedom* of the author who uncovered it, and then for the *necessity* of logic the author has uncovered. Similarly, "smooth" structures work through the perception of effortless inevitability, which is striking both persuasively and aesthetically. Now, it has been frequently suggested that the dialectic of freedom and necessity is essential to art as such.[9] Narrative art, certainly, has for its protagonists individual persons, and for its form, structured plots. It thus cannot fail to dramatize the theme of freedom versus determinism. Among the many options open to persons, the author selects a single plot; and similarly, among the many options open to mathematical objects, the mathematician selects a single logical thread. Thus, mathematics cannot fail to dramatize the theme of freedom versus necessity. This is one way in which we see not only that mathematics possesses an aesthetic dimension, but also that this dimension is essential to it, and closely implicates it with other verbal forms that are more obviously "artistic".

### *2.3. "Prosody"*

The concept of "narrative" applies almost directly to mathematics, in that mathematical works – just like many other works of verbal art – tell a story: they have characters, and our information about the characters gradually evolves. The same, of course, cannot be said about "prosody". Literally speaking, the prosodic dimension is completely suppressed in Greek mathematics. The sequence of long and short syllables – the foundation of Greek poetic prosody – never seems to be an issue at all.[10] Here however I take the notion of "prosody" in a very metaphorical sense, referring to any compositional device that may be analyzed apart from the meaning of the text, referring purely to its form.

There are many compositional devices we can point out, some of them familiar, indeed, from literature. To begin with, let us return once more to the role of narrative. In literary theory, narrative has not only the global sense of "plot" and "subject" in the work as a whole, but also the local sense of a *narrative textual segment*, as opposed to other types of textual segments, most importantly *description*. One of the main literary compositional devices is this alternation of narrative and description. Some passages – descriptive – add detail to the fictional world, constructing its underpinning of reality; other passages – narrative – unfold the plot that takes place in that fictional world.[11] Description brings up things, narrative brings up the events which happen and which are true of those things. Things and events cross-determine each other, and in general narrative and descriptive passages may work together in interesting ways. The same is true of mathematics. In Greek mathematics, the two types of passages are technically known as *kataskeue*, "construction" and *apodeixis*, "proof". Construction is a descriptive passage where things are brought into existence, proof is a narrative passage where we are told what follows to those things. One should also

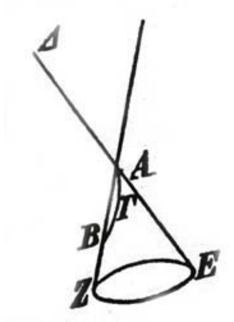


FIGURE 2.

add the (shorter) *ekthesis*, “setting out”, which is a descriptive passage, and the (shorter) *diorismos*, “definition of goal”, which is narrative.[12] The main difference between the “constructive” and the “argumentative” modes is that the constructive mode is hardly structured in the syntagmatic dimension[13]. The order of constructions is not set out as meaningful: they are merely a sequence of one observation after another, “and let”, “and let”. The syntagmatic dimension, however, is all-powerful in the “argumentative” mode, strongly structured by the sequence of “since – therefore”. (It is perhaps for this reason that the binary structure constructive/argumentative closely resembles the binary structure descriptive/narrative: literary narrative, too, is characterized by strong syntagmatic structure, absent from literary description).

Take for instance the first proposition in Apollonius’ *Conics* (I skip the *protasis* or enunciation whose function is separate): “Let there be a conic surface, whose vertex is the point A, and let some point – B – be taken on the conic surface, and let some line – AΓB – be joined”. So far we have the “setting out”, in mathematical terms a construction and in literary terms a description (or, more precisely, an ekphrasis of the accompanying diagram, fig. 2). “I say that the line AΓB is in the surface”. This is the “definition of goal”, formulaically employing the first person to introduce the narrative sequence. “For if possible, let it not be” (a meta-narrative statement, hard to classify in terms of “construction or “proof”), “and let the line drawing the surface be ΔE, and the circle, on which EΔ is carried – EZ”. (The “construction” proper: the ekphrasis of the diagram is now complete). “So if, the point A remaining in its place, the line DE is carried along the circumference of the circle EZ, it shall also pass through the point B” (the “proof”: we were now told a *story*, and here comes its *point*:) “and there will be the same limits

<shared> by two lines" (the end of the story proper) "which is impossible" (a final meta-narrative statement).

David Fowler uses to say that Greek mathematics is about "drawing a figure and telling a story" and Greek mathematical *texts* – to be more precise – are about "describing a figure and telling a story". This is their basic texture. In a miniature such as *Conics* I.1, the aesthetic effect derives from the modal variety itself – the very fact that there are both descriptive and narrative passages. In longer propositions, the alternation of description and narrative can be used for more precise stylistic effects. This is because there is a degree of freedom: the precise sequence of description and narrative is far from rigid. One can chose to have a complete ekphrasis of the diagram first, presented in great detail, then to move on to the proof (where no further constructions are being made). This is often the path taken by Apollonius: in *Conics* I.13, for instance, the "setting out" and "construction" take up (with the brief intervention of the "definition of goal" between them) 20 lines, followed by 30 lines of "proof". The very long and very static stage of the "setting out", in particular, has a certain ponderosity that is very characteristic of the style of Apollonius, and was clearly intentional. Compare this, say, to the third proposition of Aristarchus' *On the Sizes and Distances of the Sun and the Moon*. The text starts with a brief "setting out", immediately moving on to draw a conclusion ("proof" mode) from it, and then back to construction, and so on. With D for description, N for narrative, and the number of lines for each in brackets, the structure is

D (4) – N (1) – D (4) – N (1) – D (5) – N (15) – D (8)[14] – N (4)

Aristarchus, correspondingly, has a much more "lively", discursive style.

The binary structure of description and narrative is thus the chief compositional device of Greek mathematics. There are many other, more local compositional devices, all due to the fact that the mathematician actively selects from a variety of available modes. To begin with, mathematical arguments are characterized by their sources of validity. Some claims are based on visual considerations unpacking the diagram. Others are based on more formal, linguistic manipulations (e.g., that if A is to B as C is to D, then as A is to C as B is to D: proved in *Elements* V.16 and frequently used in Greek mathematics). In more general, there is a tool box of results the Greek mathematician knew well, and this tool box clearly has an internal structure: some results fall together to form clusters; (Netz, 1999, 216-235) (e.g., as we have seen for Book I of the *Elements*, elementary results about the triangle are more closely related to elementary results about parallels, than to elementary results about circles.) Thus the mathematical argument works by using sources of necessity of different *kinds*: a palette, from which the mathematician chooses and combines. This introduces the aesthetic dimension of *variety*. Consider for instance the capstone theorem to the last book of Euclid's *Elements* (which is an appendix to XIII.18), proving that there are only five regular solids. I quote a passage:

"For a solid angle cannot be constructed with two triangles, or indeed planes" (a direct visual intuition) "With three triangles the angle of the pyramid is constructed, with four the angle of the octahedron, and with five the angle of the icosahedron"; (we enumerate numerically, going through the ordinal sequence, certainty secured by the finite, inspectable nature of that sequence) "but a solid angle cannot be formed by six equilateral and equiangular triangles placed together at one point, for, the angle of the equilateral triangle being two thirds of a right angle, the six will be equal to four right angles": (this uses the properties of the triangle of Book I – together with a quick calculation, which is yet another source of necessity) "which is impossible, for any solid angle is contained by angles less than four right angles" (this is proved in book XI, and is thus a very distinct part of the tool box). This brief passage works then through visual intuition; through numbers perceived as ordinals and as an object of calculation; through results from book I and from book XI; all coming to function together organically. An obvious contrastive comparison would be propositions such as Euclid's *Elements* I.5, which work through an iterative application of a single source of necessity. *Elements* I.5 is often felt to be dull (it is the famous *pons asinoroum*), whereas the appendix to book XIII is obviously delightful. The difference is essentially that, by the time he has reached book XIII, Euclid has enormously widened his palette – he now has thirteen books to draw on whereas, in I.5, he had only a handful of basic presuppositions.

"Variety" has to do with *texture*, but it is "prosodic" only in a very metaphorical sense. My next example is nearly literally prosodic. For while the rhythm of long and short syllables is not itself a marked feature of Greek mathematical texts, the texts are marked by other rhythmic patterns, which are of clear aesthetic significance. The rhythmic pattern of verse represents the fact that verse is built from clearly defined units – lines – that participate in larger-scale structures – stanzas – and possess an internal structure – feet. Greek mathematical texts – perhaps more than any other prose style – are similarly built from clearly defined units, which allow a similar structural analysis. This is especially true of proofs which (as mentioned above) possess the strong syntagmatic structure of the "since – therefore" sequence. This sequence works on *assertions*, and combines them into *arguments*.

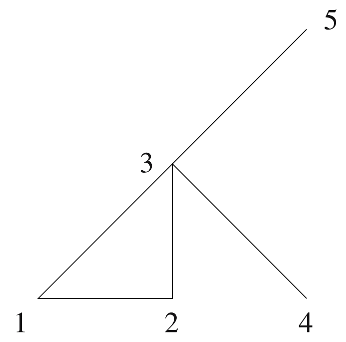


FIGURE 3.

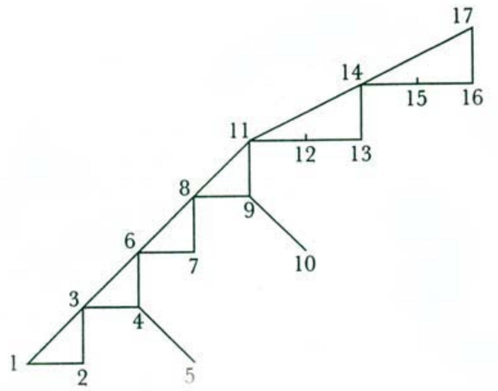


FIGURE 4.

When analyzing mathematical proofs, I number the assertions and draw the "trees" of the structure of the proof, e.g. representing an argument such as "(1) and (2), therefore (3) (for (4), too), hence (5), as well" by fig. 3

Thus we can compare the logical structure of *Elements* II.5 (fig. 4) – a prototypically "smooth" Euclidean proof – with that of *Method* 1 (fig. 5) – a complex proof I have mentioned above. To begin with, we can see how the notion of a "smooth" proof can be given concrete form. We may also begin to note further features of this, quasi-prosodic structure. First of all, the proof alternates between starting-points (assertions which are unargued for inside the proof itself and appear in the tree "on top of nothing" – in Euclid's *Elements* II.5 these are 1, 2, 5, 7, 10, 12, 13, 15, 16) and conclusions (assertions which follow from other assertions and which thus appear in the tree "on

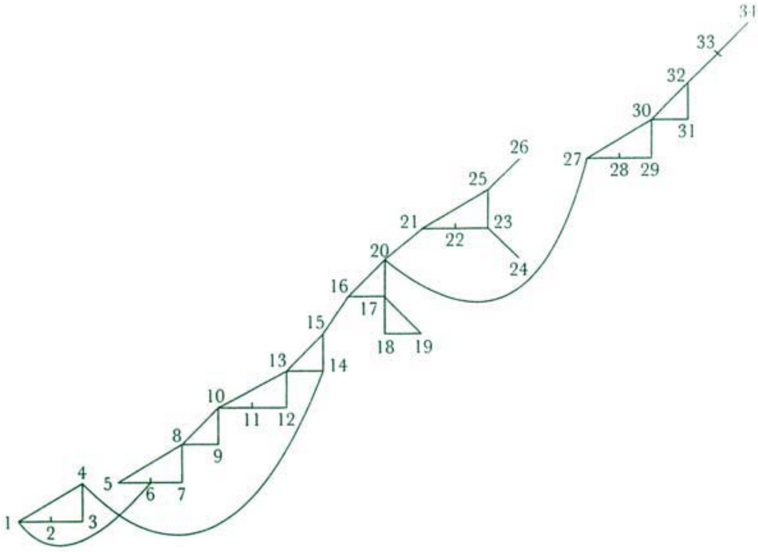


FIGURE 5.

top of" other assertions). Although one could, as a matter of logic, structure proofs so that all the starting-points are asserted first, followed by all the conclusions, this is in fact awkward both for the cognitive computation of the validity and for the aesthetic appreciation. The structure adopted instead is a constant interplay between starting-points and conclusions, which form together minimal units: *arguments*.[15] (Thus, in a tree, every triangle, or independent line, stands for an *argument*). Starting-points are the moments in which the proof is recharged, building its energy for the charge of the conclusion. I have offered a comparison of the mathematical assertion to a line of verse; I now suggest a further analogue – which I consider to be relevant almost in a literal way – comparing the structure of starting-points, conclusions and arguments, to the structure of unstressed syllables, stressed syllables and feet. It is indeed difficult not to think of structures of proof in terms of their "flow" or lack thereof (the presence or absence of regular meter) of becoming "quicker" or "slower" (shorter or longer feet). Consider figures 4 and 5: the precise pattern of figure 4 (iamb-troche-iamb, iamb-troche-iamb, anapest-anapest – if you see what I mean), the very complex structure of figure 5 (a radical free verse, where feet are consistently changed, though notice a gradual transition, from a very rough meter to start with, to a sequence without hiatuses in assertions 14-34, and in particular a sequence of three "iambs" right towards the end[16]). It is clear that some sort of rhythmic patterning is going on, and my intuition as a reader, at least, is that this pattern contributes to my appreciation of the text. Following a proof as it unfolds, you are *carried along* it; the structure of this intellectual motion has significance in both cognitive and aesthetic terms.

Moving back from rhythmic patterns as such to more general relations between signs, one finally notes the following. In Greek mathematical texts, signs are constantly being reformulated and co-related with each other, thus creating a rich texture. Consider for instance a passage from the construction in Euclid's *Elements* II.5:

"... [A]nd, through the <point> Δ, let a line, ΔH, be drawn parallel to either of the <lines> ΓE, BZ, and, through the <point> Θ, let again a line, <namely the line> KM, be drawn parallel to either of the <lines> AB, EZ, and again, through the <point> A, let a line, <namely the line> AK, be drawn parallel to either of the <lines> ΓΛ, BM".

It will be seen that we have here the same formula (for drawing a parallel) repeated three times, with the substitution of different letters and the addition of connecting particles. The effect in this case is probably one of sheer monotony, but the formulaic nature of the text is aesthetically significant. The Greek mathematical text is composed of nearly fixed expressions, such as the formula for drawing a parallel above. These stand to each other in several relations. First, they allow some freedom (for instance, the words "to either of", in the example above, are a local variation on the more standard formula which makes lines parallel only to a *single* other line). Second, formulae are subject to substitutions (as, in this example, that of letters). Because of these two factors, different tokens of the same formula type tend to be different (indeed, otherwise there would hardly be a mathematical point in repeating them). Third, such tokens are often related in meaningful way. This does not happen in the example above, as it is from the construction, where the syntagmatic arrangement is weak. A typical structure in which formulae figure inside proofs is, for instance, the result of *Elements* V.16, mentioned already:

"As A is to B so is C to D. Therefore, as A is to C so is B to D".

(In actual appearance, A, B, C and D would be spelled out as some geometrical object, providing the text with a richer pattern). We can now see that this is a sequence of two tokens of the same formula (that of proportion), arranged in the relation "since – therefore". This is a textbook case of a patterning of the syntagmatic and the paradeigmatic dimensions. This patterning is essential to Greek mathematics:[17] it is also "prosodic" in a rather direct sense, in that it is (at the abstract level suggested here) a form of *alliteration* ("cat, therefore mat"). Why is that *beautiful*? Partly, the answer has to do with the sheer presence of structure, but partly it has to do with the basic relation between sign and signified, and in this respect we shall return to discuss alliteration in the next subsection.

Before moving on, we need to point out the general moral of this subsection. The phenomena described are in a way heterogeneous. I have discussed several levels of formal structure – of selection and arrangement – the alternations of construction and proof, of various sources of necessity, of starting-point and conclusion, of different tokens of the same formula-type. They are all similar only in that they are all alternations; no deeper unity combines them all, and no generalization is possible at the level of their contents. But the very fact that mathematical texts support many layers of structure indicates an *essential* reason for the presence of an aesthetic dimension. Aesthetic appreciation is often based on the perception of structures inside the artistic object. A mere concatenation of objects, devoid of any structure, cannot function as a vehicle of communication, let alone a vehicle of beauty. Any perception is structural; by imposing structures on the sequence of objects, perception makes them meaningful – and opens up the possibility of aesthetic value.

Now mathematical perception, particularly in its Greek form, imposes not merely structure, but some very definite structure. Bringing in logical categories, its boundaries and markings are extraordinarily sharp. The assertion begins *here* and ends *there*; it is *definitely* a proof and not a construction; it is a conclusion from precisely *those* premises; it works through precisely *this* type of reasoning; it is made precisely of *this* sequence of formula-tokens that belong to precisely *the same* type as that used above.[18] Note further that it is in the nature of mathematics to make the signs themselves, taken formally, contribute to the logical sequence (A:B::C:D therefore A:C::B:D! - More on this below). Thus, the kinds of structural relations picked up by mathematical perception are directly relations of signs, and thus are directly of potential aesthetic significance. Nothing surprising, then, in that mathematical texts may display aesthetically pleasing forms: the more structured a text is, the more it is implicated in the pattern of objects and structures – that is, in an aesthetic pattern.

We saw above that mathematical texts are essentially implicated in the dialectic of freedom and necessity; now we see that they are essentially implicated in the dialectic of object and structure. I move on to discuss what I call here "correspondence", the locus of yet another dialectic.

### *2.4. "Correspondence"*

"Correspondence" is a thick jungle, and, to make some progress, I start with Jakobson's helpful terminology (which I have already used above without explanation).

Jakobson stressed the bipolar structure of language: selection and combination, similarity and contiguity, the paradeigmatic and the syntagmatic. For the notions of "selection" and "combination" note that, in a text, the speaker (a) *selects*, for each slot in the text, a unit of speech out of a large pool of available candidates, and also (b) *combines* the selected units in a certain order. Thus two kinds of structure are at work: *similarity* (of the various candidates for a single slot) and *contiguity* (of units which happen to lie next to each other). The "similarity" kind of structure is known as "*paradeigmatic*", the "contiguity" as "*syntagmatic*". Now, in even more general terms, we may say this. One possible textual device is to represent an object through its possible equivalents or near-equivalents, in other words through that to which it stands in the relation of similarity – and this is what Jakobson calls "*metaphor*". Another device would be to represent one object through that to which it stands in the relation of contiguity; this naturally would be Jakobsonian *metonym*.[19] While very typical of a certain theoretical approach, this set of notions is at bottom an analytical, indeed terminological exercise, in itself theory-free. Stripped of all jargon (and of the cognitive and linguistic assumptions which do make it stronger and more interesting) Jakobson's theory of metaphor and metonym is very simple indeed. Some signs are similar to each other, some are contiguous to each other; one of the devices of art is to put such similarities and contiguities on display. It is especially on the paradigmatic kind of correspondence that I concentrate here.[20]

Many mathematical signs stand to each other in close paradeigmatic relations. We already began to see this in the preceding subsection: several tokens of the same formulaic expression, say

A is to B as C is to D, C is to D as E is to F, A is to B as E is to F

are like several inflections of the same verbal root. In full form (where the A, B etc. are spelled out as full geometrical objects) this may be obscured at the level of performed text – just as, in ordinary language, the words sharing the same root may appear, in phonological form, rather distinct. Here, for instance, is a passage from Apollonius' *Conics* IV.46:

"Since it is: as the <square> on MΨ to the <square> on ΨI, so the <rectangle contained> by AΠB to the <rectangle contained> by ΔΠE, but as the <rectangle contained> by AΠB to the <rectangle contained> by ΔΠE, so the <square> on ΛT to the <square> on TI, therefore also: as the <square> on MΨ to the <square> on ΨI, so is the <square> on ΛT to the <square> on TI".

This is nothing more than a series of paradigmatically related signs, identical in some important ways. Now, the perception of hidden paradigmatic identity is a tool often used in poetic alliteration – say, the "hundred visions and revisions" of T.S. Eliot's *Prufrock* – and now we see that it is an essential feature of mathematical perception, as well. Quite simply, mathematics cannot work as a deductive exercise without the constant re-identification of signs. It would be difficult to show that this has a specifically aesthetic effect in mathematics, just because the deductive function is so central. As one reads, one constantly notes with satisfaction, "yes, it is indeed the same". This satisfaction of sameness recognized is sung most loudly by the bass section, exulting over the validity of the derivation; my own intuition is that, listening carefully, one can also discern, in the chorus of one's recognition, an alt voice rejoicing over the finding of sameness in difference.

This, at any rate, is an example where paradigmatically related signs are placed in syntagmatic order. We move a bit closer to "metaphor" when considering the structure induced on the text by the presence of paradigmatic relations that do *not* have syntagmatic meaning. To put this in simple terms, mathematical texts often return to speak about the very same topic, and thus they contain an element of repetition. Such repetitions may be handled in various ways, and thus we have an aesthetically significant choice. Parellalism is an extreme and therefore an illuminating case. As noted by Jakobson (in the same fundamental study mentioned above): "Rich material for the study of [metaphor and metonym] is to be found in verse patterns which require a compulsory parallelism between adjacent lines, for example in biblical poetry. . . This provides an objective criterion of what in a given community acts as a correspondence" (Jakobson, 1987, 110-111) . Greek mathematics possessed one such pattern of compulsory parallelism, namely the relation between general "enunciation" and particular "setting-out" and "definition of goal". So for instance the first proposition in Euclid's *Elements*:

[Enunciation] "On a given finite straight line to construct an equilateral triangle".

[Setting-out] "Let there be the finite straight line, AB".

[Definition of goal] "Thus it is required to construct an equilateral triangle on the straight line AB".

The two parts – enunciation on the one hand, setting-out and definition of goal on the other hand – are related paradigmatically. They are two inflections, general and particular, of the same root meaning; two signs differing in their relation to a single signified. They stand to each other, in this case, in very close explicit correspondence (although, of course, the "inflection" demands considerable rearrangement). This is the simplest, least ambitious type of metaphorical relationship, namely near-*synonymy* (compare, e.g. Psalms 2.1: "Why do *the nations* conspire / and *the peoples* plot in vain?"). Close parallelisms are a feature of Euclid's style, and while they seem to have a didactic motivation, they are also important for the fabric of the Euclidean text, contributing to its serenity and gravity.

In other authors, metaphor is often much more ambitious. I now quote from Archimedes' *Balancing Planes*, proposition 6 (this, incidentally, is the theorem we use when balancing plates on a tray):

[Enunciation] "Commensurable magnitudes balance at reciprocal distances having the same ratio as the weights".

[Setting-out] "Let there be commensurable magnitudes A, B whose centres are A, B, and let there be a certain distance, E Δ, and let it be: as A to B, so the distance Δ Γ to the distance Γ E";

[Definition of Goal] "it is to be proved that Γ is centre of the weight of the magnitude composed of both A, B."

Now the transformation requires an unpacking of the mathematical *meaning* of the enunciation. Instead of mere synonymy, we have two separate signs, whose only connection is their shared signified. Thus the text displays the paradigmatic structure of signs.

This relationship, between enunciation and the sequence of setting-out and definition of goal, is a special case of a very general feature of mathematical texts. They repeatedly need to speak about roughly the same objects, saying roughly the same things. Several propositions all use the same piece of construction; or they all rely on the same local argument (which did not get enshrined as a separate lemma, and therefore gets repeated from one proposition to another). Or, in the analysis-and-synthesis mode (used a few dozen times in extant Hellenistic mathematics), the proof goes through two parallel stages, inversely related, the analysis and the synthesis. In all such cases, the mathematician has several options. They range between two extreme positions that are, in themselves, aesthetically empty: exact repetition (where no paradigmatic distance opens up at all), and the explicit deletion of a passage by "as has been said in the previous proposition" (where paradigmatic distance becomes infinite, between expression and non-expression). In between lie various forms of variation that may or may not be intended to be perceived as such: in short, another avenue for aesthetic effect. (In more general, an aesthetic effect may be obtained by the overall pattern of decisions about which kind of repetition to employ – full, zero, or some metaphorical repetition).

We saw several cases where the problem of repetition arises from some mathematical functional constraint; the solution to this problem may then involve an aesthetic function. In other cases, the mathematical function of the repetition is much less obvious, and the aesthetic function may therefore be dominant. The clearest example is the phenomenon of alternative proofs. Inside a single book, say Archimedes' second book on *Sphere and Cylinder*, one may find the same theorem proved twice. Proposition 8 shows that, given two unequal segments of the sphere, the ratio of the greater volume to the smaller is smaller than the (what we would call) the square of the ratio of the surfaces, but greater than (what we would call) the 1.5 power of the same. Having proved this remarkable result, the text goes on to prove *the same result, once again*. Heiberg, the great editor of Greek mathematics, considered this alternative proof to be spurious, and of course he may have been right. (Heiberg, 1913, 217 n.1) It is always possible to argue that an alternative proof resulted not from the decision of a single author to produce more than a single proof, but from the decision of some later mathematician to try his or her hand at finding an alternative proof, and then from the decision of yet another later scribe, to put the two proofs together. Whether this is the case or not is significant in historical terms (and has some bearing on our aesthetic judgement), but it does not touch upon the basic aesthetic interpretation of alternative proofs. We merely need to transpose the locus of aesthetic judgement, from the original author to the later mathematicians (who went on to offer what is, in mathematically functional terms, a "redundant" proof), and the later scribes (who considered a juxtaposition of several proofs, whose result is identical, to be of interest). Now here we touch on a major theme of the history of mathematics. The development of mathematics is frequently motivated not by the desire to solve open problems, but by the desire to solve problems *that are solved already* - the most famous case in Greek mathematics is the duplication of the cube, for which see Eutocius' catalogue of solutions (Heiberg, 1915, 54–106). (In general, for the Greek accumulation of problems, see the fundamental study Knorr (1986).) Mathematicians went on proving the same enunciation, just as painters went on painting the same annunciation; in the first case as in the second, the desire to replicate sustained a perfection of styles and techniques.

Needless to say, the desire is not to replicate in a strict sense. Once again, we see that exact replication is aesthetically empty. The desire, instead, is to replicate-with-a-difference – to achieve the same result (or paint the same scene) through a different line of reasoning (or through a different mode of painting). I shall now try to explain why this may indeed be of such aesthetic significance. Before that, however, we need to widen even further our field, to further possible relations between signs and signifieds.

So far, we saw several ways in which the mathematical text contains a multitude of signs, all referring to the same signified. This was true at the level of the *text*, in the strict sense: we have dealt purely with the modality of written language. This is in some sense perhaps the main modality of the signs of Greek mathematics (it is at this level that Greek mathmatical texts are either true or false). But this is not the only modality of the signs of Greek mathematics, and an appreciation of its aesthetics must refer to this presence of many modalities. Greek mathematics relies essentially on at least two modalities, language and diagram. It thus involves simultaneously verbal and visual perception. Not only is it possible to have two written signs refer to the same object, then: we also have the possibility of having an object referred to simultaneously by both verbal and visual signs.

Greek mathematical texts, apart from their general enunciations, refer throughout to a diagram labeled by letters. The A, B, Γ of Greek mathematical proofs participate simultaneously in two semiotic systems, the text (where they are manipulated inside expressions), and the diagram (where they are spatially configured). I have argued in (Netz, 1999, Ch. 1), that, in logical terms, the two can not be understood separately: the text does not function unless we read it in the light of the diagram, the diagram is incomplete unless we interpret it through the text. We may now notice the aesthetic significance of this situation. Put simply: the diagram is *read*; the text is *visualized*. Greek mathematics relies, therefore, on a kind of synesthesia. In fact, the synesthetic structure is probably more complicated than that. The diagram is, in reality, statically present to the eye, but it is also discussed as if it were dynamically manipulated and constructed, in a language suggestive of motion in and through it. In other words, the verbal and the visual are also accompanied by the *kinesthetic*. An obvious case is the way in which parallel lines are mentioned so that they appear to "flow in the same direction" ("AB is parallel to CD", in fig. 6). Consider an even more beautiful example – an expression of a very common type. I quote from Apollonius' *Conics* I.41:


FIGURE 6.

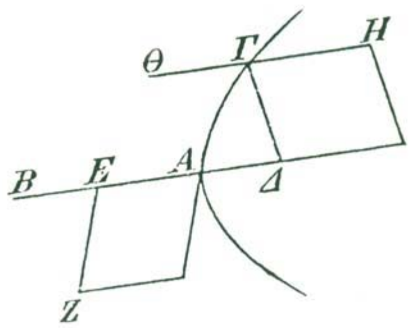


FIGURE 7.

“. . . ΔΓ has to ΓH the ratio composed of the <ratio> that ΔΓ has to ΓΘ, and the <ratio> that ΘΓ has to ΓH”. (Fig. 7).

Note how the three modalities interact: the statement works as a structural manipulation of verbal signs (it is a complex structured formulaic expression). It is also premised on the visual expression of the same signs – without which, indeed, the statement is hardly interpretable. But finally, note the very typical switch in *directionality*. ΓΘ is transformed into ΘΓ, in a kinesthetic metaphor (one can hardly avoid the term) for the canceling-out involved in the operation. The motion “in and out”, across the single line ΓΘ, introduces it and then cancels it.

Synesthesia is a key concept in romantic aesthetics. In its apparent irrationality, untrammeled sensuality, it answers to a certain kind of aesthetic temperament.[21] Parenthetically, I note that many modern mathematicians seem to have an enormous interest in synesthesia: in fact, I suspect it is fundamental to their reports of the mathematical process as beautiful. Friedberg (1968), for instance, in a very personal introduction to number theory, starts with a long and fascinating excursus about the author's polymodal synesthetic perception of numbers – colored, tactile, acoustic, what have you.[22] Now, it is of course more difficult to show that such aesthetic concerns motivated the Greeks. They definitely did not seek, Baudelaire-like, the expansion of perceptual experience for its own sake. A more modest claim, however, would be that we have here another kind of *variety*. The richness of modalities – that are *felt to be organically related* – is very different from the intended, jarring effect of the romantic juxtaposition of incommensurables. Greek mathematical synesthesia thus comes down to yet another form of the multiplicity of signs for a single signified, with the added complication that the different signs cone from separate modalities, and can only function as a whole: none stands on its own.

This mode of operation – simultaneously perceiving an object through several systems – occurs in mathematics in other, more technical ways. Even inside a given modality, two domains often interact. Most significantly, Greek mathematics translates the synesthesia of the linguistic and the visual into a dual set of *mathematical* domains: proportion theory, and geometry. Very often, a Greek mathematical proof operates by thinking of an object, simultaneously, through the more abstract properties revealed through proportion theory, and through the more concrete properties revealed through geometry: it is simultaneously a *line in space*, and a *magnitude in proportion*. Further global metaphors were offered in Greek mathematics, especially in its interface with physics: for instance, rays of vision are lines (optics), musical harmonies are numerical ratios (music), the motion of stars is a configuration of circles (astronomy). Archimedes, to mention one example, pioneered the science of statics with the global metaphor of balance as the (composite domain of) geometrical proportions. He first used this metaphor in *On Balancing Planes*, to obtain results on the physical balance. Then, he went on, in the *Method,* to use this metaphor in the reverse direction, now applying the results of *On Balancing Planes* to obtain new results about pure geometrical objects (seen now through the metaphor of the balance). It seems to me that this double metaphor is often considered Archimedes' most beautiful achievement.

The alignment of separate domains is mathematically functional: it allows different kinds of understanding to operate simultaneously and thus to generate results which would have been impossible with only one of the kinds. The whole theory of conic sections, for instance, would be impossible to develop on the basis of either geometry or proportion theory alone. At the same time, this global metaphorical structure is clearly perceived in aesthetic terms: the duality of the concrete and the abstract, in particular, seems to lie at the core of the Platonic fascination with mathematics.

In Greek mathematics, those bimodalities can be definitely mapped: between proportion theory and geometry, inside mathematics; between mathematics and physics, outside it. The same cannot be said for modern mathematics: just as modern mathematicians are fascinated by the specific notion of synesthesia, they also set out systematically, in the $19^{th}$ century and even more in the $20^{th}$ century, to seek out the global equivalences between domains. This quest shaped the content of modern mathematics, which is a tight network of poly-isomorphic disciplines.[23] It is also, I would suggest, at the core of the modern mathematical sense of beauty. This is very different from the much more static structure of Greek mathematics, where isomorphism is much less actively sought after: in this respect, it appears that different aesthetic temperaments are visible in ancient and modern mathematics.

We have made a long detour through synesthesia and global equivalences, but ultimately we return to the very basic phenomenon of the multiplicity of signs for a single signified: the paradigmatic dimension. The detour, however, may help us to perceive the special role this dimension has in mathematics. Why is it that it so obvious to us that several mathematical discourses are about 'the same thing'?

In fact, with mathematics we seem to stand in an easier position than with other verbal forms. The paradigmatic is especially easy to identify in mathematics. Why?

In literary texts we always face a central question about the paradigmatic: how is it to be defined? It would be natural to think of it in terms of a shared "core" between the two verbal segments standing in the paradigmatic relation – "cat" and "feline", or "cat" and "dog", or perhaps even "cat" and "mat" – but, as the examples show, "core", generally speaking, is a slippery concept. Equivalence is perhaps indefinable in the natural lexicon. But now it becomes clear that, in the mathematical context, the concept of "core" has a special relevance. Consider "alternative proofs": two proofs share a core meaning, if they prove the same result, i.e. if they are *mathematically equivalent*. Here is the concept organizing the paradigmatic dimension in mathematics. As we have mentioned already in the preceding subsection, the logical categories employed by mathematics make its structures much sharper than those of ordinary discourse, and the same goes for paradigmatic structure. "Equivalence" is a very clear term in mathematics, as perhaps nowhere else.

However, even mathematical equivalence is a complex object. Hence its aesthetic significance.

Let us try to approach in aesthetic terms the most basic phenomenon of mathematical equivalence, *derivation*. This is the major arranging principle of a mathematical text: P→Q.

We have in fact mentioned derivations already, in the context of "rhythm", where I have suggested that the metrical pattern of a mathematical proposition is marked by the sequence of "unstressed", argued assertions, alternating with "stressed", argued assertions. Thus, I suggested that the rhythmic pattern of the proposition is given by its pattern of derivations. I now suggest that this is also its main bind of correspondence. We thus see a certain dual level for derivations: at the large-scale level, they create the pattern governing the proposition; at the immediate, small scale, they are a strongly marked correspondence. This duality is very intriguing, since exactly the same holds with *rhyme*. Derivation, one may say, is the rhyme of mathematics. For rhyme, too, creates the strophic pattern of a (rhymed) poem - while being its most strongly felt correspondence.

This analogy has an even more direct application: derivations, like rhymes, are aesthetically effective where there is sufficient distance between the syllables/assertions – no "knight"/"night", P→P.[24] This is the principle that tautologies must be avoided. Nor of course should distance be too large: the rhyme must be heard, the derivation must be seen to be valid. Notice that the constraints on derivation are aesthetic rather than logical. A "trivial" derivation (P→P, or nearly so) is logically valid; while a logically valid derivation from P to Q, whose validity is, as stated, impossible to perceive, fails not as a matter of logic but as a matter of persuasion and pleasure. Briefly then, when following a derivation, we must see both that P and Q are distinct, and that they are identical. The two signs must both refer to the same signified, *without* being identical.

Consider e.g. the following type of derivation:[25]

"Triangle ABC is similar to triangle DEF. Therefore as AB to DE, so BC to EF".[26]

There is a beauty here, I feel: that two statements – one on a geometrical shape, another on a much more abstract proportion – are found to be closely related, indeed nearly "identical". (The first implies the second; the second does not fall much short from implying the first). Of course they are not identical: they say different things, in very different ways; they belong to different modalities; yet they are also nearly the same. Compare Larkin:

"Man hands on misery to man:
It deepens like a coastal shelf.
Get out as early as you can,
And don't have any kids yourself".

Once again: there is beauty in the startling, irrational apposition of "shelf" (metaphorical to begin with) and "self": the two, suddenly, the concrete and the abstract, are found to be somehow "nearly the same".

The relation between rhyme and derivation merits a closer look for it may, I think, offer a key to the more general relation between poetry and mathematics. Both rhyme and derivation share the same combination of difference and identity: they reveal that two entities, seemingly different, are at some level identical. In both cases, this can be done because the entities, to begin with, subsist at two separate levels – the sign and the signified. Thus we are shown that two sign/signified combinations are identical in one respect, different in another.

Rhyme and derivation are thus similar; but they are also different or, more precisely, *complementary*. Rhyme works by having two sign/signified combinations that are similar as signs and dissimilar as signifieds; derivation works by having two sign/signified combinations that are similar as signifieds and dissimilar as signs.

| | | |
|---|---|---|
| **Rhyme**: | sign $\rightarrow$ $\text{signified}_1$, $\text{signified}_2$ | ("shelf" / "self") |
| **Derivation**: | $\text{sign}_1$, $\text{sign}_2$ $\rightarrow$ signified | (similarity / proportion) |

The two sides of a mathematical derivation are very dissimilar in form; they approach each other at the level of content. In a complementary fashion, the two rhyming words are very dissimilar in meaning; they approach each other at the level of form. The mathematical relationship is anchored in meaning, marks the meaning; the poetic relationship is anchored in form, marks the form. In sum, then, mathematics and poetry both utilize the binary nature of the sign/signified relationship, to combine identity and difference; in mathematics, the identity is at the "signified" level, in poetry, it is at the "sign" level. The patterns of identity and difference are similar but complementary. Inasmuch as they are similar – merely as patterns of identity and difference – they both yield a pleasing aesthetic relation. But inasmuch as they are complementary – in the different levels they mark – they tend to have very different effects.

Let us see what – in a similar metaphysical level of abstraction – Jakobson had to say on the nature of poetry. I quote the conclusion of his article "What is Poetry?"[27]:

"Why is [poetry] necessary? Why is it necessary to make a special point of the fact that sign does not fall together with object? Because, besides the direct awareness of the identity between sign and object (A is $A_1$), there is a necessity for the direct awareness of the inadequacy of that identity (A is not $A_1$). The reason this antinomy is essential is that without contradiction there is no mobility of concepts, no mobility of signs, and the relationship between concept and sign becomes automatized. Activity comes to a halt, and the awareness of reality dies out".

Art, according to Jakobson, is about subverting our ordinary, automatic acceptance of reality – in this case our ordinary, automatic acceptance of the sign/signified relationship. Because poetry creates a web of relations that mark the *sign* aspect of the sign/signified combination, it subverts this very relationship. Why is that? Because in the ordinary, automatic acceptance of speech, we take it for granted that the relation is

$$\text{sign} \rightarrow \text{signified}$$

That is, the sign is there merely to mark the signified, and does not have a significance of its own. The sign is supposed to do no more than invoke the signified – that is, determine the signified.

But in poetry, this determination of signified by sign is subverted: it has the structure

$$\text{sign} \rightarrow \text{signified}_1, \text{signified}_2$$

I.e. similar signs yield very different signifieds and the determination fails. The very function

$$\text{sign} \rightarrow \text{signified}$$

Is thus being questioned: poetry, in this way, is a critique of language.

Mathematics, on the other hand, does nothing of the kind. It is fully anchored on the signified, and its structure

$$\text{sign}_1, \text{sign}_2 \rightarrow \text{signified}$$

Supports the intuition that signs are no more than entries into signifieds. The combination of sign/signified is not subverted, but supported. Mathematics is not a critique of language, but its affirmation.

Such considerations may seem perhaps rather removed from actual experience; perhaps they are. Yet this kind of metaphysical politics – the politics of abstract subversion, as it were – is central to contemporary literary theory. And certainly the sheer surprise of irrationality, of the breaking down of the relation between form and meaning, is part of aesthetic experience. This is especially true for a certain kind of romantic (or modernist) aesthetic temperament. Perhaps one might even suggest the following. If poetic correspondences *undermine* the notion of rational correspondence, while mathematical correspondences *affirm* the notion of rational correspondence, we should predict that, to some temperaments, poetry would seem suspect while mathematics would seem praiseworthy, indeed a model. Such may have been Plato's temperament.

With all such differences, however, the main result is this: that mathematics is shot through with the notion of correspondence. It fully partakes in the dialectic of identity and difference. Thus it creates a pattern, of potential aesthetic significance. Arguably, nowhere else is the dialectic of identity and difference so rich and visible as in mathematics. Perhaps the best evidence for this is, once again, the recent quotation from Jakobson:

"... Because, besides the direct awareness of the identity between sign and object (A is $A_1$), there is a necessity for the direct awareness of the inadequacy of that identity (A is not $A_1$)".

Jakobson, of course, was not above using quasi-mathematical notation to enhance the scientific credibility of his methodology. But could he really have chosen a better way to express the notions of identity and its absence? Nowhere else are those notions so central, so clear. Mathematicians keep affirming just that: that A equals B. No one else – not even poets – affirms such claims as often. The presence of the dialectic of identity and difference in mathematics is far from accidental: it is, quite simply, what mathematics is about.

## 3. CONCLUSION

I have offered a typology of possible sources of beauty in Greek mathematical texts. They fell into three main categories. The first, "narrative", is a consequence of the fact that mathematical texts are freely written, and yet display necessary connections. This allows mathematical texts to display all kinds of combinations of surprise, invention and retrospective inevitability. That is what I call the dialectic of freedom and necessity, a dialectic that often seems to speak to our sense of beauty.

The second, "prosody", is a consequence of the fact that mathematical perception organizes its reality in well-defined units that are strongly structured by a web of relations. This allows mathematical texts to display rich structures, in many interacting layers. That is what I call the dialectic of object and structure, which is at the heart of art and indeed communication in general.

Finally, "correspondence" is a consequence of the fact that mathematical texts constantly restate their contents in equivalent ways. Statements are subtly transformed and restated in derivations, and objects are perceived in sequence through several separate perspectives. This may be at the heart of mathematical beauty since this constant re-shuffling of equivalent statements is what allows mathematical texts to display, finally, both the combinations of surprise and necessity mentioned in the context of "narrative", and the rich structures mentioned in the context of "prosody". In a more narrow sense, the relations displayed in mathematical texts – true identities that bridge truly different objects – somehow pick up a kind of surprise and structure that is of special value. That is the dialectic of identity and difference, which is perhaps one of the major themes of the aesthetic experience. At any rate, in a sense, this dialectic is most perfectly instantiated in mathematics.

In this article, I narrowed the questions of mathematical beauty to the question of beauty in mathematical texts (concentrating on Greek mathematical texts). I have largely ignored the question of beauty as a property of mathematical states of mind, and of beauty as a property of mathematical objects.

I shall not try to offer here any generalizations across historical periods. I did make a few suggestions for possible historical *dis*continuities: the appearance of a personal voice in some early modern genres of mathematics; the valuation of synesthesia and metaphor as such in some fields of modern mathematics. I suspect the typology offered here has considerable continuity with many other genres inside the western tradition, if only because of their genetic dependence upon Greek mathematics. But it will be necessary to study each genre separately, uncovering its own internal aesthetic principles. In the study of experience there are no shortcuts.

Further, I have little to say on beauty as a feature of states of mind. Seen in an abstract light, such states of mind are "text", as well, but texts to which our only access is the mathematician's introspection. This I do not possess, and I can only salute Polya or Poincare, Hardy or Hadamard. The study of such mathematician's reports is important, and may, with caution, be used in a historical study (more on this below). I shall not try to pursue this here except noting that, once again, I suspect there are continuities between the texts of mathematics and "texts" of mathematical intuition. Perhaps surprise and inevitability, the concrete and the abstract, conspire first in the mathematician's imagination, bringing forth in his or her mind what will later on be enacted in writing.

Something similar may be said with greater confidence on the question of the beauty of mathematical objects. Here, I would suggest, we have a special or limiting case of the forces shaping the beauty of mathematical texts. This is most obvious with one major type of mathematical objects, namely mathematical *facts*. Mathematical facts (or results), such as Pythagoras' theorem, or that a sphere's surface is four times its greatest circle, are all clearly beautiful. They are also, simply, a limiting case of a narrative. A result is a narrative, stripped to its bare structure: instead of telling the elaborate story of the first book of Euclid's *Elements*, or of Archimedes' first book on *Sphere and Cylinder*, you reach directly for the punch line. But what makes this beautiful is the promise of a narrative. Any fool can tell you that a sphere's surface is four times its great circle – or indeed that it is three times its great circle (which sounds even nicer). The beauty resides in the statement's being demonstrably true. It is surprise and inevitability combined that make a mathematical statement beautiful: exactly the "narrative" mechanism we saw operating in the beauty of mathematical texts.

Moving on to "objects" in a more narrow sense – to the worlds of parabolas and perfect numbers – our observations on the mathematical perception in texts become relevant to the mathematical objects themselves. Mathematical perception is structural; aided by logic, it brings sharp contrasts, contours and connections. These may then be beautiful. This principle is true of derivations in the mathematical proof – and of objects in the mathematical world. We are able to perceive, through mathematics, that a parabola has infinitely many diameters, around which it is, in a clearly defined sense, symmetrical; we are able to perceive that a perfect number is precisely equal to the sum of its parts. I have said almost nothing so far about "symmetry", "harmony", "equality", "proportion", perhaps the notions that spring to mind most naturally when considering the beauty of mathematics. These are structural notions; "harmony" is perhaps nothing more than a structure we are able to perceive. At any rate, the continuity between the beauty of texts and the beauty of objects, based in both cases on structural perception, seems plausible. Finally, I would suggest the same for the final source of beauty in mathematical texts – the dialectic of identity and difference. This after all is a way in which texts refer to *objects*. It is the same parabola which is seen both as a geometrical cut in a cone, and as the site for abstract proportions; the same perfect number which is perceived both as a sum of numbers and their multiple. Mathematical perception is not only structural, but multi-layered. In particular, we repeatedly see things – when we see them mathematically – as both concrete and abstract. This is the Greek, and then western, bifocal vision of the mathematical object, simultaneously particular and general. And this is further repeated throughout the mathematical disciplines, whether "pure" (where a visual diagram is simultaneously an abstract, language-defined object), or "applied" (where the same object is simultaneously "mathematical" and "physical" – magnitudes in proportion that are plates on a tray). From Plato onwards, this coincidence of the concrete and the abstract seems to have informed the aesthetic appreciation of mathematical objects.

I now move on to a number of objections to the approach delineated in this article.

I can imagine a perplexed response, wondering how valid my account can be given its novelty. The very fact that mathematics is so little explicitly analyzed in aesthetic terms, while art is, seems to indicate a real distinction between the two. This response has to be qualified – in some ways, an aesthetic appreciation of mathematics is not novel, but commonplace – but it is essentially valid. A theoretical approach to the aesthetics of mathematics should also offer an account of its own absence.

Something has already been said in this respect at the end of the preceding section, where I pointed out the complementary nature of mathematics and poetry. In fact, mathematical texts do differ fundamentally from other forms of verbal art. Because their essential organizing principle is that of logical equivalence, they foreground a set of relations which is in principle independent of specific linguistic form. When the speaker and the audience share a large body of linguistic tools (as was true inside Greek mathematical communication), specific forms such as Greek mathematical formulaic expressions may be used to signal and support the logical relation of equivalence. But when the linguistic tools are no longer shared (as happens, for instance, when a Greek mathematical work gets translated by modern mathematicians), the specific linguistic tools are no longer of help at all. To perceive the logical equivalence, then, the modern mathematician must *substitute new forms for the old ones*. Typically, a modern rendering of an ancient mathematical text would transform it into modern algebraic notation. This would be done – here is a crucial realization – not to *suppress* its relevant aesthetic properties, but to *enhance* them. The modern algebraic notation is the restorer's paint, retouching a surface that became worn with time. There can be no aesthetic object where there is no perception, and the perception of mathematical relations is dependent upon using the tools of mathematical perception available in your own culture. Briefly, then, most aesthetic properties of mathematical texts become visible *only under translation*. This is true for most properties of "narrative" and "correspondence", and even to many properties of "prosody" (the rhythm of a mathematical proof, for instance, can only be perceived when the proof is perceived as a flowing sequence, i.e. translated to your own mathematical language). This is directly the opposite of verbal art par excellence – lyric poetry – where the dominant aesthetic properties reside in the specific linguistic form. It is for this reason that mathematics appears not at all to be a form of verbal art. Indeed it isn't. It is enacted in words, but its dominant aesthetics are located not in the verbal, but in the logical domain.

This, however, does not make it any less of an art. Logical relations are not any less interesting than verbal relations. In fact, they allow rich and yet precise structures, much more than verbal relations do. There should therefore be no surprise that poetics is applicable to science or to mathematics. In particular – just because of its great elegance – the glass slipper of structuralist poetics may, indeed, fit the foot of science even better than it fits that of art. Here I approach another possible reaction against my study. Perhaps one reason why many literary scholars are dissatisfied with the old structuralist model is precisely because of its precision, of its search for structural harmonies. In art, the ambiguous is sometimes as valuable as the precise, the jarring as much as the harmonious.[28] A theoretical model, where works of art are analyzed as elegant solutions to problems, cannot apply directly to works whose theme is failure and inresolution.[29] But – with the interesting exception of *paradoxes* - inresolution is not a theme in mathematics. Solutions are; as are structures and harmonies; hence there is also a poetics – and a rather straightforward one - of mathematics.

I am trying to reassure contemporary literary critics. No, I do not demand of them to find in literature quite the same elegance we find in mathematics. But then I know they can hardly feel reassured. In this article, I have argued for the presence of the aesthetic in mathematics, by arguing that mathematics has a quasi-literary structure. I am not so naïve as to fail to realize that, in contemporary literary theory, the notion of the aesthetic in literature has nearly become a taboo.[30] Nor am I so hypocritical as to deny that, in this article, part of my motivation was to challenge this taboo: I use the metaphor of mathematics as literature as beautiful so as, implicitly, to make more plausible the metaphor of literature as mathematics as beautiful. Nor, finally, am I so naïve as to believe it's as easy as that.

The issue goes to the heart of my suggestion at the introduction to this article, that while non-epistemic, beauty may be also a rational value: to put roughly, that there is some objective reality corresponding to the experience of mathematical beauty, which cannot be reduced to its historical construction. This is forcefully denied by contemporary literary theory, where the historicity of value is often seen as a proved fact.

I believe that, in this respect, contemporary literary scholars have an important and valid perception. Value should be historicized. But this, I believe, is not contradicted by the kind of methodology advocated in this article. To the contrary, I will argue that the approach of this article is a necessary – and currently absent – component of historicism. In conclusion, I shall now try to sketch this argument.

Let us start with the following methodological observation. The typology offered in this article was intended, in the first instance, not as a contribution to the pure metaphysics of beauty (an interesting field in its own right). My purpose was, indeed, historical: to find a way to describe a historical reality so that, among other things – armed with such typologies concerning the various possible perceptions of the aesthetic – we may be able to ask more specific historical questions. So, for instance, we could now finally turn to study, *as historians*, Hardy's aesthetic statements. This must be stressed: to be interested in the aesthetics of mathematics, does not at all entail a blind, uncritical trust of, say, Hardy. Exactly the opposite: it is only after we have built our own analytical tools for dealing with the mathematical aesthetic experience, that we are in a position to approach Hardy in a critical way. We may approach Hardy's mathematics, and the mathematics of his time, *independently* of Hardy: poetics, as it were, gives us a privileged access to texts and to the reality of mathematical experience, an access Hardy never had. We can now study the actual forms of aesthetic experience implicit in this particular genre of $20^{th}$ century mathematics, comparing it to other genres, from different times, places, and fields. And we may of course compare them to Hardy's words *about* the genre, in this way uncovering the ideological valuation of certain kinds of experience at the expense of others. All of this becomes possible *only after* we have made the aesthetic study.

Even without an aesthetic study, we could study, of course, the (rather obvious) ideological positions adopted by Hardy. But we wouldn't know how to situate those positions: as if we had a schematic map of a terrain we did not know. Hardy had a position where "aesthetic" is opposed to "utilitarian", and this is easy to find and to trace as schematic map. But what is the terrain to which this "aesthetic" refers? This terrain is at the level of experience. We just cannot read our map, then, without reference to an understanding of this level of experience. What was the thing Hardy was referring to when he was speaking about 'beauty'? An answer to this question does not require of us to share Hardy's valuation of the thing; but it does require us to try to analyze the reality of experienced texts underlying Hardy's statements. And therefore – as historians – we need to understand the whole range of phenomena described in this article – narrative and its flow, perception and its structure, the way reality is taken in. It is all very easy, to say that Hardy made an ideological valuation of *something*; the real challenge is to say what this something was.

Of course cultures create their patterns of value. How else could value become part of social existence? But they cannot make such patterns out of nothing, into nothing. They make such patterns out of something (out of the sheer facts of experience), into something (into another objectively felt experience, that of value). The historicism of value is an empty claim as long as it does not confront those – objective – realities, out of which and into which the subjective is made.

Having replied to my colleagues in literary theory, it is finally time for me to turn to my colleagues in the philosophy of mathematics. Indeed I need only glance behind my shoulder, to the preceding articles in this collection. In a sense they do not question my very enterprise in quite the same way the literary critics did. I imagined the literary critics questioning the very notion of 'mathematical beauty'. Not so the philosophers of mathematics. Their worry, it appears to me, is different and – I concede – justified. Having agreed on the existence of the category of 'the beautiful' in mathematics, how does it speak to the concerns of the philosophy of mathematics?

It is of course a contribution to the *aesthetics* of mathematics, and the main claim I now need to make is for the need for such a philosophical investigation, independent of the epistemic concerns that drive the other articles in the collection.

One could address the question, why a mathematician may wish, for instance, to have the narrative effect of surprise – in terms of the epistemic significance such an effect might possess. One may argue perhaps that a surprising result challenges us to uncover the links leading to its conclusion, in this way possessing a specific epistemic value. I do not believe Archimedes, for example, chose to have a surprising narrative in his *Sphere and Cylinder I* for this reason (a plausible historical argument can be made that Archimedes' aim was purely aesthetic), but in principle surprising narratives can definitely be epistemically motivated. Or one may argue the converse (which, in this case is, in my view, historically valid): one may argue that a stately, orderly progression of narrative such as that of Euclid's *Elements I* has a precise pedagogic effect, that helps the reader parse the text as it unfolds, in this way making it easier to follow not the results alone but also their pattern of logical interrelation, so that the text as a whole becomes richer, to the reader, in its explanatory meaning. I think it is likely that Euclid's choice to prefer this model of narrative, then, was pedagogically – that is epistemically – rather than aesthetically motivated.

In other words, one strategy I could have taken while presenting this paper to my colleagues in the philosophy of mathematics was to associate the aesthetic effects discussed to their epistemic correlates, so that my aesthetic styles would translate into epistemic styles in line with those discussed in the preceding articles. This would have been possible and, for certain purposes, valuable. Yet I have avoided, on purpose, doing so. And this was precisely because my philosophical point was to make the claim for the theoretical independence of an aesthetics of mathematics, as a philosophical domain existing apart from the epistemological one.

The issue is partly historiographical, partly philosophical. The historiographical point was already hinted at above: I suggested briefly that I believe Euclid's goal, in his choice of narrative form for *Elements I*, was primarily pedagogic, that is epistemic, while Archimedes' goal, in his choice of narrative form for *Sphere and Cylinder I*, was primarily aesthetic. Euclid wanted his readers to learn something; Archimedes wanted to elicit a gasp of pleasured surprise. I will not try and advance the historical argument here (admittedly certainty is hardly possible with such questions having to do with authorial intention). But the historiographical point is conceptually clear. The position I describe concerning Euclid and Archimedes is obviously a possibility. And, unless we allow the existence of a separate aesthetic realm independent from the epistemic, then we cannot even formulate such a position. For our writing of the history of mathematics, then, the aesthetic is a category we need in order to be able to state the full range of possible motivations of authors in their writings: as simple as that.

The philosophical point – related to the historiographical one – is more subtle. It has to do with the nature of mathematical experience itself.

It might appear strange for me to invoke, at this stage of the argument, mathematical experience. After all I have eschewed the difficult question of the mathematical states of mind, concentrating instead on the objective features of texts. My approach throughout was structuralist, looking at the semiotic properties of texts as vehicles for positively defined aesthetic effects. And yet my goal of course was to begin to stake a ground in this difficult terrain of experience. My approach throughout – here as in my other studies in the cognitive history of mathematics – is to concentrate on the surface details of texts, as offering us the best objective evidence to mathematics as it is actually experienced (as opposed to some logical analysis of its abstract content).

My philosopher colleagues will recognize my intention, if I stress my interest in the *phenomenology* of mathematics. I ask the question: How is mathematics present to the mind? And I concentrate on a more modest question (where there is some useful evidence to work with): how are mathematical texts present to the mind? My claim is that categories such as 'truth' or 'validity', even categories such as 'visual' and 'symbolic', cannot fully provide an answer to this question. Part of the answer has to bring in other categories of experience, such as 'delight', 'pleasure', 'surprise', or, indeed, 'beauty'. All those terms of value are important not so that we may judge mathematics, but so that we can describe it—in its phenomenal reality. A phenomenology that disavows the experiences of value offers an abstracted, synthetic vision of mathematics: as it were, a *disembeautied* vision, which therefore has to be also a *disembodied* vision. And so, to begin to outline the actual phenomenology of mathematics, we must start from its real phenomenal reality—the full range of its experience of value and cognition, perception and appetite. Which indeed reminds me that I should get back to my interrupted lunch.

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## NOTES

[1]Contemporary mathematicians often refer to Erdös' famous dictum on 'the book of beautiful proofs' (out of whose enumerable infinity mathematicians seek to find their proofs!) – see Aigner and Ziegler (1998).

[2]As is obvious by now, I am mainly influenced in my thinking by structuralist poetics, broadly construed - mostly because of its tangible, immediate applicability. I do not at all dismiss other possible approaches, and I would be happy to see my typology of sources of beauty enriched by further methodologies. Nor do I think "texts" exhaust the problem. The beauty inherent in states of mind seems to be a central theme in mathematicians' own reports, and thus deserves close study. The notion of the intrinsic beauty of the mathematical realm of being is one of the key issues of western philosophy from Plato onwards; see Burnyeat (1998). I shall briefly return to those general issues in the conclusion.

[3]See e.g. (Lang, 1985, 3): "The Greeks did mathematics for the beauty of it", or the whole of Artmann (1999) – a passionate reading of Euclid by a contemporary mathematician, arguing for the sources of many mathematical values, in particular beauty, in Greek mathematics.

[4]This is the main theme of Lotman (1976).

[5]One should note however the interesting complications of orders of reality, for instance in proof by contradiction, where an alternative reality is entertained "for the sake of the argument" – the perspective adopted and then finally discarded. I also ignore the role of the first person singular in some formulaic expressions such as "I say that", which, because formulaic, perhaps do not have much real force. I give an example of both phenomena at the beginning of the next subsection. Finally, note that in early modern mathematics, the authorial voice frequently interferes in the text, suggesting the line of discovery and playfully interweaving the subjective narrative of the implied author with the objective narrative of the proof. (For a celebrated example, see Descartes' *Geometry*; I thank Heda Segvic for suggesting this comparison.) The implicit claim of the absent perspective characterizes not the aesthetics of mathematics as such, but rather the aesthetics of Greek mathematics.

[6]My choice of the term 'narrative' for this process of selection and combination of information is not obvious. I could equally have called this 'rhetoric', which however I have refrained from doing, wishing to avoid the pointless debate of 'logic vs.

rhetoric'. Obviously the structures represented in mathematical texts are unlike the plots involving human agents which the term 'narrative' brings to mind: I hope to show that, even so, the term 'narrative' remains useful in mathematics as well. (For an interesting discussion of the applicability of the term 'narrative' to mathematics see Thomas (2002).

[7]The term 'capstone theorem' was suggested to me by Henry Mendell. Mendell has also suggested to me the further observation, that most books in Euclid's *Elements* seem to end with such a capstone theorem.

[8]As it were, the considerations of statics and of aesthetics, governing the arrangement of the plates on the tray, are both organized by the same principles (e.g. of simplicity of form and of proportion).

[9]The Kantian aesthetic program is to understand art through the dialectic of the subject's freedom and the world's lawlikeness: see e.g. Krukowski (1992) chapter 1.

[10]That prosody was suppressed in Greek mathematics can be seen from the fate of the Archimedean corpus: originally written in Archimedes' Doric dialect, it was at some point mostly transferred into the *Koine* dialect. Such dialect transformation impacts almost exclusively on prosody: but clearly readers did not consider that the text has lost any meaningful dimension.

[11]Sometimes "narrative" is given a wider sense, so that the contrast is *within* narrative structure, between description and *fabula*: see e.g. (Bal, 1997, 36-43) .

[12]The terms derive from Proclus' *In Eucl. I* 203.

[13]"Syntagmatic" is a technical term of structuralist poetics: I briefly explain a few of those terms at the start of subsection 2.4 below.

[14]The D (8) passage – Heath 362.12-364.4 – is an interesting complication: a mere unpacking of what the diagram stands for (description, then, in literary terms) is of immediate argumentative context (and therefore functions here as part of the proof, in the mathematical sense).

[15]On starting-points and arguments in general see (Netz, 1999, 169–198).

[16]I have noted the tendency to have a "smoother" movement towards the end of a proof, in (Netz, 1999, 206-207) , calling it "the cadenza effect". I have there stressed the possible rhetorical function of this effect; once again, the rhetorical and the aesthetic coincide.

[17]See (Netz, 1999, Ch. 4) for the role of formulaic expressions. I return to discuss these in greater detail below, when introducing the general notion of "correspondence".

[18]Note however that there might, in principle, be aesthetic value not in clarity, but in *ambiguity*. This is in fact exactly parallel to the case of *irony*, mentioned in the preceding subsection. The mathematical text largely forgoes the aesthetic possibilities of ambiguity, just as it largely forgoes the aesthetic possibilities of irony. This is a limitation on the aesthetic range of mathematics – though once again, a certain beauty resides in the limitation itself, providing mathematics with its sharp luminosity.

[19]For all this see especially Jakobson (1987) chapter 8, originally published in 1956.

[20]As noted by Jakobson himself (Jakobson, 1987, 113-114), metaphor is in general easier to understand than metonym. It is in fact difficult to think of examples of metonym in Greek mathematics. I note below the use of particular cases for general statements, which is perhaps metonym-like; probably the clearest example of metonym in mathematics in general is mathematical induction, where the argument relies on the ineritence of properties by objects contiguous to each other (this is a modern method: see Unguru (1991)). Perhaps even: when mathematicians say that proofs by mathematical induction are "strange" and do not reveal the "real reasons", they, among other things, display the typical human preference for metaphor over metonym?

[21]For a historical survey and interesting philosophical observations, see Dann (1998).

[22]On this very widespread experience see Seron et al. (1992).

[23]For the quest and its results, see, e.g., Corry (1996). Note further a special twist on this quest for global metaphor: the brilliant insight of some $20^{th}$ century mathematical works, that the signified can metaphorically act as sign – so that, for instance, *numbers* are equated with *statements about numbers*.

[24]Indeed the effectiveness of rhyme often has to do with a distance which is not only phonetic, but also semantic: compare e.g. (Wimsatt, 1954, 153–168).

[25]I concentrate on the simple form P→Q, ignoring the complication of the more common structures, P&Q→R, etc. The presence of several premises for a single conclusion is, among other things, a further device for creating a pleasing distance between set of premises and conclusion.

[26]Note that this is not the definition of similarity of triangles – defined by equal angles – but a result (Euclid's *Elements* VI.4). The derivation is thus a substantive equivalence between two different statements (rather than a disguised tautology).

[27](Jakobson, 1987, Ch. 19), translation of an article from 1933-4. Jakobson expresses here standard formalist positions, perhaps first articulated in Shklovsky (1919).

[28]The preference for the disharmonious may be related to the widespread contemporary interest in literature as "subversion" (it seems at any rate that such an interest in "subversion" was at the root of Bakhtin's own distancing from Jakobson's type of structuralist poetics which of course, through Kristeva (1980) and other routes, came to dominate contemporary literary theory).

[29]Of course, we will need the theoretical model to analyze how the effect of inresolution is obtained; but it is true that the basic tenor of structuralist poetics is alien to such an analysis.

[30]See Smith (1988) for a fascinating statement of the doubt in the aesthetic in contemporary literary theory.

## REFERENCES

Aigner, M. and Ziegler, G. M. (eds) (1998). *Proofs from the Book*, Springer, New York.

Artmann, B. (1999). *Euclid: the Creation of Mathematics*, Springer, New York.

Bal, M. (1997). *Narratology: Introduction to the Theory of Narrative*, University of Toronto Press, Toronto.

Booth, W. C. (1961). *The Rhetoric of Fiction*, University of Chicago Press, Chicago.

Burnyeat, M. F. (1998). Plato on why Mathematics is Good for the Soul, *Proceedings of the British Academy* **103**: 1–81.

Corry, L. (1996). *Modern Algebra and the Rise of Mathematical Structures*, Birkhäuser, Boston.

Dann, K. (1998). *Bright Colors Falsely Seen*, Yale University Press.

Friedberg, R. (1968). *An Adventurer's Guide to Number Theory*, McGraw-Hill, New York.

Hardy, G. H. (1967). *A Mathematician's Apology*, Cambridge University Press, Cambride. Reprint, 1*st* ed. 1940.

Heath, T. L. (1913). *Aristarchus of Samos, the Ancient Copernicus*, Clarendon Press, Oxford.

Heiberg, J. L. (1913). *Archimedes / Opera Vol. II*, Leipzig.

Heiberg, J. L. (1915). *Archimedes / Opera Vol. III*, Leipzig.

Jakobson, R. (1987). *Language in Literature*, Harvard, Cambridge MA.

Knorr, W. (1986). *The Ancient Tradition of Geometrical Problems*, Dover, Boston.

Kristeva, J. (1980). *Desire in Language: a Semiotic Approach to Literature and Art*, Columbia University Press, New York.

Krukowski, L. (1992). *Aesthetic Legacies*, Temple University Press, Philadelphia.

Lang, S. (1985). *The Beauty of Doing Mathematics*, Springer, New York.

Lotman, Y. M. (1976). *Analysis of the Poetic Text*, Ardis, Ann Arbor. Tr. of *analiz poeticheskovo teksta*, originally published 1972.

Mueller, I. (1981). *Philosophy of Mathematics and Deductive Structure in Euclid's Elements*, MIT Press, Cambridge MA.

Mukarovsky, V. (1970). *Aesthetic Function, Norm and Value as Social Facts*, University of Michigan, Ann Arbor. Tr. of estetika funkce, norma a hodnota jako socialni fakty, originally published 1936.

Netz, R. (1999). *The Shaping of Deduction in Greek Mathematics: a Study in Cognitive History*, Cambridge University Press, Cambridge.

Seron, X., Pesenti, M., Noel, M. P., Deloche, G. and Cornet, J. (1992). Images of number or “when 98 is upper left and 6 sky blue, *Cognition* **44**: 159–196.

Shklovsky, V. B. (1919). *Iskusstvo kak Priyom*, St. Petersburg.

Smith, B. H. (1988). *Contingencies of Value: Alternative Perspectives for Critical Theory*, Cambridge MA.

Thomas, R. D. (2002). Mathematics and Narrative, *Mathematical Intelligencer* **24**: 43–46.

Unguru, S. (1991). Greek Mathematics and Mathematical Induction, *Physis* **28**: 273–89.

Wimsatt, W. K. J. (1954). *The Verbal Icon*, The University of Kentucky Press.