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PTOLEMY'S THEORIES OF THE LATITUDE OF THE PLANETS IN THE *ALMAGEST*, *HANDY TABLES*, AND *PLANETARY HYPOTHESES*

The theory of planetary latitude in Book 13 of the *Almagest* is known, if at all, for its complexity. This has the pleasant result that there is only a small literature on it and that literature is on a high level of technical competence. The same, by the way, is true of latitude theory in general. There are recent expositions by Pedersen and Neugebauer, earlier ones by Delambre and Herz, and a few briefer treatments. Paradoxically, the complexity of Ptolemy's theory is both its strength and its weakness, its strength because he reached it by doing everything right, at least in principle, its weakness because it is ultimately wrong, as was later recognized by Ptolemy himself, who went on to remedy its deficiencies. It is, as we may say, wrong for the right reasons. And since being wrong for the right reasons is more or less the subject of this collection – for is not most interesting older science wrong for the right reasons? – Ptolemy's latitude theory seems quite appropriate. Our object here is to explain the latitude theory, first its original form in the *Almagest*, then its later modifications in the *Handy Tables* and *Planetary Hypotheses*, each of which shows improvements, and to investigate its observational foundation, for it is the observations that are the cause of both its strength and its weakness. It is unusual to find any revisions in the work of an ancient scientist, but in the case of Ptolemy's latitude theory three distinct stages are known, which may be unique, showing that he himself knew something was wrong and twice set out to correct it.

The latitude theory of the *Almagest* is complex because it is so strictly empirical, which is true of all of Ptolemy's mathematical astronomy, and empiricism, we all agree, is a good thing. Every *hypothesis*, a technical term meaning 'model', is either derived or confirmed by observation, and every numerical parameter is derived directly and uniquely from observation. There is, however, a large range of precision in Ptolemy's observations, from positions and times measured to within a few minutes for the derivation of parameters, although their accuracy is more variable, to rough, qualitative observations for demonstrating the applicability of models. The observations upon which Ptolemy founds the theory of latitude fall somewhere in between these, and he uses them to derive both the model and its parameters. As crucial as these observations are, he gives no information about how they were made – he never mentions using an armillary, which could measure latitude – and many could be conventional estimates rather than his own measurements. It is this strict adherence to the requirements of the observations that makes the latitude theory complex, so complex that even

J.Z. Buchwald and A. Franklin (eds.), Wrong for the Right Reasons, 41–71.

Ptolemy remarks on it in a famous passage (13.2) on complexity and simplicity in astronomical hypotheses. He says, in essence, that we should seek the simpler hypotheses for the motions in the heavens, but failing that, any hypotheses that fit the phenomena. We must do the best we can with observation as our foundation and confirmation. And we must remember that our own ideas of complexity and simplicity may not be applicable to the heavens, which are eternal and unchanging in their motions, something not merely difficult, but impossible to us, meaning that *nature is not necessarily simple according to our way of thinking*, a lesson taught again and again by the science of every age including, or especially, our own. Further, the phenomena of latitude are distinctly different for the superior and inferior planets, and thus they require distinctly different models, another kind of complexity. We shall therefore consider the superior and inferior planets separately.

SUPERIOR PLANETS

The apparent motion in latitude of the superior planets (13.1) is as follows: (1) When the planet is near the apogee of the eccentric, it reaches its greatest northern latitude, and when near the perigee its greatest southern latitude, indicating that the eccentric is inclined to the north in the direction of the apogee and to the south in the direction of the perigee. (2) In the northern and southern limits, the latitudes are greater at opposition when the planet is at the perigee of the epicycle than near conjunction near the apogee – true conjunction itself is invisible – indicating that the epicycle is inclined with its perigee in the same direction, north or south, as the eccentric. In the case of Mars, the planet cannot be seen near conjunction because of its long period of invisibility, but the same conditions are assumed to hold. (3) When the center of the epicycle is a quadrant from the limits and the planet a quadrant from the apogee of the epicycle, it has no latitude, indicating that the epicycle then lies in the plane of the ecliptic. It is this condition that allows the direction of the nodal line, and thus of the limits, to be found from the computed longitude of the center of the epicycle when the planet has no latitude.

The model to account for these three conditions (13.2) is shown in Figure 1. (1) The earth is at O, through which passes the nodal line ☊☋ of the eccentric, which is inclined to the plane of the ecliptic at an angle i_1; N is the northern limit, near apogee, S is the southern limit, near perigee, and the midpoint of NS is M' at an eccentricity e' from O. M' and e' are the center of the eccentric and eccentricity projected into the line joining the limits. (2) When the center of the epicycle is at N or S, it is inclined to the plane of the eccentric in the line of sight, with the perigee to the north at N and to the south at S, so that the latitude β_o at opposition P_o is greater than the latitude β_c at conjunction P_c. It is found from observation that the difference between β_o and β_c is so large that the epicycle is also inclined to the plane of the ecliptic by i_2 and thus to the plane of the eccentric by $i_1 + i_2$. (3) When the center of the epicycle is at the ascending node ☊ or descending node ☋, it lies in the plane of the ecliptic so the planet has no latitude wherever it is located. Hence as the epicycle moves from

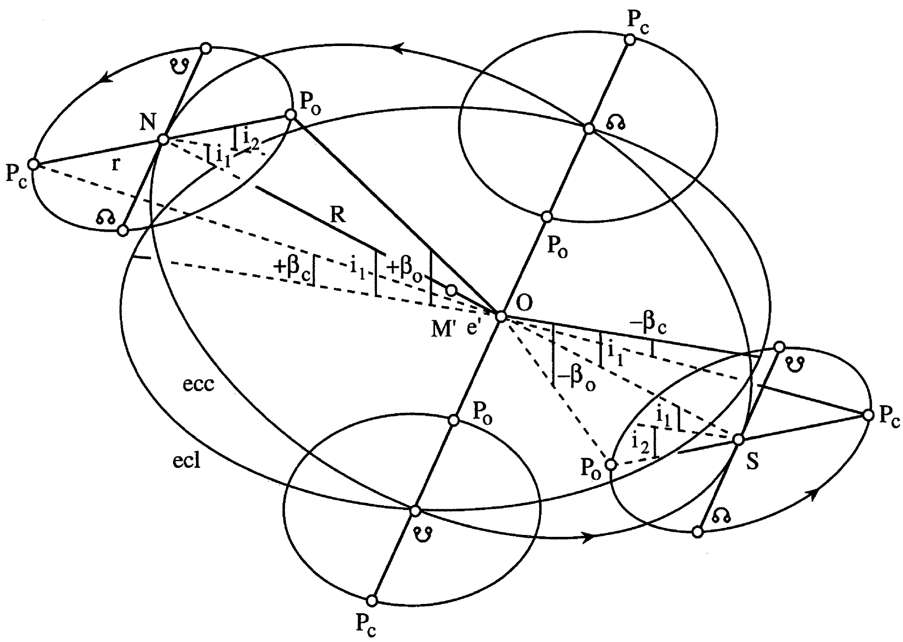

Figure 1. Superior planets.

the limit to the node, i_2 decreases from its maximum to zero, and as it moves to the next limit i_2 again increases to its maximum. Ptolemy treats $i_1 + i_2$ as a single inclination of the epicycle to the plane of the eccentric. But since i_1 may be taken as a fixed inclination, holding the epicycle parallel to the plane of the ecliptic, leaving i_2 alone variable, which we believe a clearer way of showing the variable inclination, we have divided the inclination of the epicycle into two components, the fixed i_1 and the variable i_2.

The derivation of the parameters (13.3) is rigorously empirical. Ptolemy derives i_1 and $i_1 + i_2$ from β_o and β_c using an ingenious method of interpolation in the correction tables for longitude, as though $i_1 + i_2$ were the anomaly on the epicycle measured from apogee or perigee and $\beta_o - i_1$ and $i_1 - \beta_c$ the equation of the anomaly. The derivation for Saturn and Jupiter is shown in Figure 2, in which the earth is at O, the center of the epicycle C is at either limit of latitude, as the eccentricity is neglected, and the planet is at opposition P_o with the larger latitude β_o and at conjunction P_c with the smaller latitude β_c where the difference $\beta_o - \beta_c = \delta$. The plane of the eccentric OC is inclined to the plane of the ecliptic by i_1 and the plane of the epicycle P_cCP_o is inclined to the plane of the eccentric by $i_1 + i_2$ and to a plane parallel to the ecliptic by i_2. We are given by observation β_o and β_c, and we wish to find i_1 and i_2. Now imagine the epicycle rotated into the plane perpendicular to the planes of the eccentric and the ecliptic. We may thus regard $i_1 + i_2$ as the 'anomaly' measured from the apogee

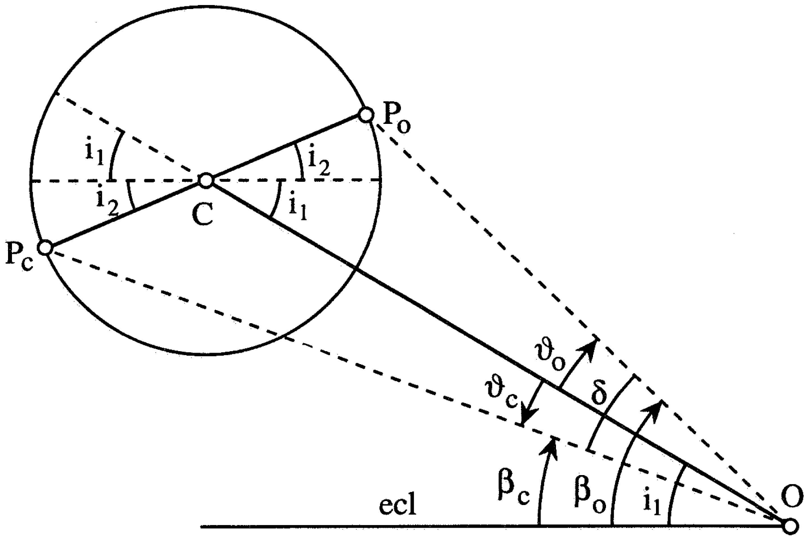

Figure 2. Inclinations i_1 and i_2 of Saturn and Jupiter.

or perigee of the epicycle – they are far smaller than they appear in the figure – and $\vartheta_o = \beta_o - i_1$ and $\vartheta_c = i_1 - \beta_c$ as proportional to the equations of the anomaly c_o and c_c at some small arc from apogee and perigee respectively. We now have the relations

$$\vartheta_o + \vartheta_c = \beta_o - \beta_c = \delta, \qquad \frac{\vartheta_c}{\vartheta_o} = \frac{c_c}{c_o},$$

from which,

$$\frac{c_o}{c_c}\vartheta_c + \vartheta_c = \vartheta_c\left(\frac{c_o + c_c}{c_c}\right) = \delta, \qquad \vartheta_c = \left(\frac{c_c}{c_o + c_c}\right)\delta,$$
$$\vartheta_o = \delta - \vartheta_c, \qquad i_1 = \beta_o - \vartheta_o = \beta_c + \vartheta_c.$$

And letting the 'anomaly' at opposition, angle $OCP_o = \alpha'$,

$$\frac{i_1 + i_2}{\alpha'} = \frac{\vartheta_o}{c_o}, \qquad i_1 + i_2 = \frac{\vartheta_o}{c_o}\alpha', \qquad i_2 = (i_1 + i_2) - i_1.$$

Since the extreme latitudes are the same at both limits, they are unaffected by the eccentricity. Hence, Ptolemy finds the equations of the anomaly c_c and c_o for mean distance, c_6 in the correction tables for longitude (11.11), which are described in the Appendix, taking c_c for 3° from apogee and c_o for 3° from perigee, and lets $\alpha' = 3°$. The latitudes β_c near conjunction and β_o at opposition, found by observation, simple integers, are obviously estimates. The computations of i_1 and i_2 are summarized as follows:

	$\pm\beta_c$	$\pm\beta_o$	δ	c_c	c_o	ϑ_c	ϑ_o	i_1	$i_1 + i_2$	i_2
Saturn	2°	3°	1°	0;18°	0;23°	0;26°	0;34°	$2;26° \approx 2\frac{1}{2}°$	$4;26° \approx 4\frac{1}{2}°$	2°
Jupiter	1	2	1	0;29	0;43	0;24	0;36	$1;24 \approx 1\frac{1}{2}$	$2;31 \approx 2\frac{1}{2}$	1

In the case of Mars, the period of invisibility near conjunction is so long, from 90 to more than 200 days, that a nearby latitude cannot be observed, so Ptolemy uses latitudes at opposition at the northern and southern limits, which differ greatly because the limits are exactly at apogee and perigee of the eccentric, which is very nearly true, and because of the large eccentricity, an effect not noticeable for Saturn and Jupiter. The principle of the derivation is nevertheless the same. In Figure 3 the earth is at O, the center of Mars's eccentric is M, the planet in the epicycle is at opposition in the perigee of the epicycle, P_n with latitude β_n at the apogee of the eccentric and northern limit N, P_s with latitude β_s at the perigee of the eccentric and southern limit S, and considering absolute values $\beta_s - \beta_n = \delta$. The plane of the eccentric is inclined to the plane of the ecliptic by i_1, and the plane of the epicycle is inclined to the plane of the eccentric by $i_1 + i_2$ and to a plane parallel to the ecliptic by i_2. Considering $i_1 + i_2$ as the 'anomaly' measured from the perigee of the epicycle, $\vartheta_n = \beta_n - i_1$ and $\vartheta_s = \beta_s - i_1$ are proportional to the equations of the anomaly c_n and c_s at a small arc from perigee. Consequently,

$$\vartheta_s - \vartheta_n = \beta_s - \beta_n = \delta, \qquad \frac{\vartheta_n}{\vartheta_s} = \frac{c_n}{c_s},$$

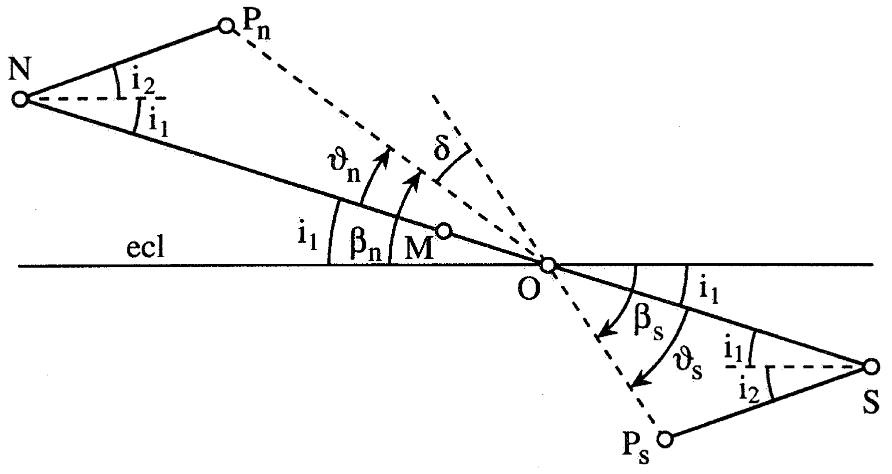

Figure 3. Inclinations i_1 and i_2 of Mars.

so that

$$\frac{c_s}{c_n}\vartheta_n - \vartheta_n = \vartheta_n\left(\frac{c_s - c_n}{c_n}\right) = \delta, \qquad \vartheta_n = \left(\frac{c_n}{c_s - c_n}\right)\delta,$$
$$\vartheta_s = \vartheta_n + \delta, \qquad i_1 = \beta_n - \vartheta_n = \beta_s - \vartheta_s.$$

Again letting the 'anomaly' at opposition at the northern limit, angle $ONP_n = \alpha'$,

$$\frac{i_1 + i_2}{\alpha'} = \frac{\vartheta_n}{c_n}, \qquad i_1 + i_2 = \frac{\vartheta_n}{c_n}\alpha', \qquad i_2 = (i_1 + i_2) - i_1.$$

For the computation Ptolemy takes the equation of the anomaly c_n at the apogee of the eccentric and c_s at the perigee, that is, from the equation tables for longitude, $c_n = c_6 - c_5$ and $c_s = c_6 + c_7$, each at 3° from the perigee of the epicycle, and likewise $\alpha' = 3°$. Hence, $c_n = 4;29°$ and $c_s = 8;5°$, from which he takes the ratio $\vartheta_n/\vartheta_s = c_n/c_s \approx 5/9$, which is quite accurate, so that $c_n/(c_s - c_n) \approx 5/4$. The observed latitudes at opposition, β_n and β_s, are again obviously estimates. The computation of i_1 and i_2, in which the roundings are very close, is summarized as follows:

	$+\beta_n$	$-\beta_s$	δ	c_n	c_s	ϑ_n	ϑ_s	i_1	$i_1 + i_2$	i_2
Mars	$4\frac{1}{3}°$	7°	$2\frac{2}{3}°$	4;29°	8;5°	$3\frac{1}{3}°$	6°	1°	$2\frac{1}{4}°$	$1\frac{1}{4}°$

With these inclinations, Ptolemy can compute the latitudes at conjunction at the limits, which could not be estimated from nearby observations, as was done for Saturn and Jupiter, because of Mars's long periods of invisibility on either side of conjunction. He places in the table of latitude (13.5) for 6° from apogee of the epicycle $\beta_n = +0;8°$ and $\beta_s = -0;4°$, which also apply very nearly to conjunction itself at apogee. These are virtually in the plane of the ecliptic and are erroneous, for as we shall see, correctly both latitudes are slightly over 1°.

The problem in Ptolemy's model is the variable inclination of the epicycles, which should always be parallel to the plane of the ecliptic, that is, correctly $i_2 = 0°$, for the epicycles of the superior planets are transformations of the heliocentric motion of the earth, which is always in the plane of the ecliptic. The variable inclinations in turn lead to yet more complications, as Ptolemy also describes (13.2) how they may be produced by small vertical circles upon which the inclining diameters move, with equant motion no less. These small circles, dismissed by Neugebauer with some justice as 'a feeble attempt' – they are a curious cross between a mathematical and mechanical model – have been described in detail for Venus and Mercury by Riddell. It is after describing the operation of these devices that Ptolemy writes his remarks on simplicity and complexity, and with good reason as the models for latitude are kinematically the most complex in the *Almagest*. Ptolemy himself obviously realized that there was something implausible about them. Our concern here, however, is not the complexity or implausibility of the models, but only to inquire into the reason for the variable inclinations of the epicycles, which is what makes them so complex.

It has been said that the reason is their equivalence to heliocentric models with the nodal line passing through, not the true sun, but the mean sun, the center of the earth's orbit, which is the principle of Copernicus's transformation of them to a heliocentric form, in which the plane of the eccentric has a variable inclination. However, the plane of the eccentric passing through the mean sun introduces a variation of inclination that is only a small fraction of that in Copernicus's model and takes place in different directions on either side of the ecliptic, as has been shown by Swerdlow and Neugebauer, and the same is true of the equivalent small variation of inclination of the epicyclic plane in Ptolemy's model. Thus, it is not the cause of the large variation of the inclination considered here. Rather, the cause is the very strength of Ptolemy's mathematical astronomy, its rigorous empiricism, for the variable inclinations are directly determined, indeed dictated, by the very observed extreme latitudes at opposition and near conjunction from which the inclinations are derived. Likewise, the inclination of the eccentric in Copernicus's model varies, not because its plane passes through the mean sun, but because the model is a transformation of Ptolemy's based upon the same extreme latitudes, although for Mars Copernicus was forced to make small adjustments because the models are not exactly equivalent.

Ptolemy's derivations require observations of the planet at opposition and as near as possible to conjunction, with the center of the epicycle at each of the limits of latitude, but these conditions occur simultaneously only rarely. There is an opposition or conjunction *near* each limit once in 30 years for Saturn, once in 12 years for Jupiter, once in 15 or 17 years for Mars, but the distance from the limit may be quite large for Jupiter and Mars, and finding any of these *at* each limit is much less frequent. And while observations at opposition may be made with the planet well above the horizon, with clearly visible reference stars if such were used in any way, observations near conjunction, thus shortly after first and before last visibility, must be made low on the horizon, possibly without suitable reference stars, and affected by refraction. As it turns out, none of the apparent latitudes Ptolemy cites, without specific information, without any information, about how they were found, is particularly accurate, and the latitudes at opposition are not really better than near conjunction. They were surely not derived strictly from these conditions, but were probably only conventional estimates in integer degrees, with a single simple fraction for Mars, and are insufficiently accurate to find the correct inclinations, which would show that always $i_2 = 0°$. Further, the method of deriving the inclinations, the computation itself, is so sensitive to small imprecisions and roundings that even with accurately observed latitudes, it would still be difficult to find the inclinations exactly.

The clearest way to show this is to begin with correct apparent latitudes according to modern theory and use them to compute the inclinations by Ptolemy's method with the rest of his parameters, the equations of the anomaly c at apogee and perigee of the epicycle, the same. We have mentioned that the simultaneous conditions for extreme latitudes occur rarely; by inspection of Tuckerman's tables for much of the second century, a rather long period, we find the following values for latitudes at conjunction, which strictly could not be observed, and opposition at the limits of latitude rounded to the nearest 0;3° = 0.05°, which we compare with Ptolemy's:

	$\pm\beta_c$	P. $\pm\beta_c$	$\pm\beta_o$	P. $\pm\beta_o$
Saturn	2;18°	2°	2;51°	3°
Jupiter	1;9	1	1;45	2
Mars	+1;9	–	+4;36	+4;20
Mars	−1; 6	–	−6;54	−7

There are differences in the tables of less than 0.05° in the positive and negative latitudes of Saturn and Jupiter. But even 0.05 = 0;3° is far less than anything Ptolemy could measure, which he seems to have believed was $\frac{1}{6}^{\circ}$ = 0;10°, so these results show that his assumption that β_c and β_o are the same on either side of the ecliptic was correct even if the errors in his integer values of β_c and β_o reach 0.3° = 0;18°. For Mars, however, although the difference in $\pm\beta_c$ is very small, the difference in $\pm\beta_o$ is very large, and thus for this reason as well as the invisibility of $\pm\beta_c$, it is also correct that he based his derivations solely on observations at opposition. We can also clearly see the error of the computation in his tables of β_c for Mars as +0;8° and −0;4°, for correctly both exceed 1°.

The recomputations have been done in two ways: (1) In keeping with a maximum precision of Ptolemy's observations of $\frac{1}{6}^{\circ}$, which we note is very optimistic for latitudes near conjunction, we round the modern computed values of $\pm\beta$ to the nearest 0;10°. (2) To show the extreme sensitivity of the computation, we also take $\pm\beta$ to the nearest 0;3°, as given above, although this is far beyond the precision of any observation possible to Ptolemy. We then repeat the computations carried out before, using the same values of the equations of the anomaly c at apogee and perigee of the epicycle.

	$\pm\beta_c$	$\pm\beta_o$	δ	c_c	c_o	ϑ_c	ϑ_o	i_1	$i_1 + i_2$	i_2
Saturn (1)	2;20°	2;50°	0;30°	0;18°	0;23°	0;13°	0;17°	2;33°	2;13°	−0; 20°
Saturn (2)	2;18	2;51	0;33	0;18°	0;23°	0;14	0;19	2;32	2;29	−0; 3
Jupiter (1)	1;10	1;50	0;40	0;29	0;43	0;16	0;24	1;26	1;40	0;14
Jupiter (2)	1;9	1;45	0;36	0;29	0;43	0;15	0;21	1;24	1;28	0;4

	$+\beta_n$	$-\beta_s$	δ	c_n	c_s	ϑ_n	ϑ_s	i_1	$i_1 + i_2$	i_2
Mars (1)	4;40°	6;50°	2;10°	4;29°	8;5°	2;42°	4;52°	1;58°	1;48°	−0;10°
Mars (2)	4;36	6;54	2;18	4;29	8;5	2;52	5;10	1;44	1;55	0;11

Note that for Saturn and Jupiter i_1 is nearly the same as found by Ptolemy, 2;30° and 1;30° – correctly Saturn is 2;33° and Jupiter 1;25°, very close to these calculations – but i_2 is reduced by nearly 1° for Jupiter and more than 2° for Saturn, for which it is here even slightly negative, meaning that the epicycle is inclined in the opposite direction. Since correctly $i_2 = 0°$, what this shows is that, although the method of derivation, including the use of the correction tables for longitude, is satisfactory for finding i_1, it is too sensitive to small errors in β and c to find i_2 with great accuracy. For Mars, i_1 is increased from 1° to nearly its correct value 1;52° and i_2 is reduced from 1;15° nearly to 0° and is both positive and negative; correctly both i_1 and i_2 are about midway between (1) and (2). In fact, as uncertain as these results may be for i_2, for all three planets i_1 is close to its correct value and i_2 is at least close to 0°. Hence, the problem in Ptolemy's latitude theory is not the model itself, in which the inclination of the epicycle is an independently derived parameter, nor the method of deriving the parameters, as sensitive as it is for i_2, but the observations, rough estimates of latitude at opposition and near conjunction, which, by Ptolemy's rigorously empirical method, require the variable inclination of the epicycle. We shall return to this subject in considering the revised latitude theory of the *Planetary Hypotheses*.

There is one further parameter in Ptolemy's latitude theory of the superior planets, the distance of the northern limit of latitude from the apogee, which is used to find the argument of latitude measured from the northern limit. He first locates the northern limits rather roughly (13.1) as near the beginning of Libra for Saturn and Jupiter and near the end of Cancer for Mars, almost exactly at the apogee. Then, taking the longitudes of the apogees, with slight rounding, he gives distances from the apogee to the northern limit ω_A (13.6): −50° for Saturn, +20° for Jupiter, 0° for Mars. (Correctly for A.D. 140 these are about: Saturn −42°, Jupiter +28°, Mars +11°. The distances from the aphelia, apsidal lines through the true sun, are about: Saturn −49°, Jupiter +7°, Mars +3°, two of which are, by coincidence, closer to Ptolemy's values.) This parameter is difficult to find with accuracy. At the limits the latitude is highly variable and the maximum latitude at opposition, which could in principle locate the limit, very seldom occurs exactly at a limit. Whenever the planet crosses the ecliptic, the center of the epicycle is in the nodal line ±90° from the limits. But this too is not easy to observe as the latitude of the planet changes most rapidly when crossing the ecliptic and the chance of catching it exactly in the ecliptic is slight. Still, some kind of interpolation between small latitudes on either side of the ecliptic is probably the most reasonable way of finding the longitude of the nodes and thus, by ±90°, of the limits. Whether this explains the errors in Ptolemy's locations of the limits, I do not know. It does not help in finding this parameter that the period of Saturn is nearly 30 years and of Jupiter nearly 12 years, so for long periods no useful observations can be made for finding either limits or nodes; and although the period of Mars is less than two years, its long periods of invisibility and rather large and irregular synodic arcs also make it difficult to observe exactly at a limit or crossing the ecliptic.

The tables for latitude of the superior planets (13.5) are very easy to use although at the cost of some precision. An example from the table for Mars is given here at intervals of 6°:

1	2	3	4	5
Argument		$+\beta(\alpha)$	$-\beta(\alpha)$	Inter. (ω_C)
6°	354°	0;8°	0;4°	0;59,36
12	348	0;9	0;4	0;58,36
18	342	0;11	0;5	0;57,0
...	...	...	...	...
84	276	0;46	0;42	0;6,24
90	270	0;52	0;49	0;0,0
96	264	0;55	0;52	0;6,24
...	...	...	...	...
168	192	4;0	5;53	0;58,36
174	186	4;14	6;36	0;59;36
180	180	4;21	7;7	1;0,0

Columns 1 and 2 are arguments of entry for 6°–180° and 180°–354° at intervals of 6° for 270°–90° and 3° for 90°–270°. Columns 3 and 4 are latitudes as a function of true anomaly α on the epicycle computed for the maximum inclination $i_1 + i_2$ and the center of the epicycle at the limits of latitude, column 3 northern, column 4 southern. The differences in the two columns are due to the different distances of the limits on the eccentric; these are large for Mars – at opposition 2;46°, at conjunction 0;4°, but note that $+\beta$ is twice $-\beta$ – since the eccentricity is large and the limits are in the apsidal line, but small for Jupiter and Saturn at opposition and conjunction – from 0;2° to 0;4°, each a very small fraction of β – since their eccentricities are smaller and their limits are removed from the apsidal line, for Jupiter by +20° and for Saturn by −50°. This is a rather crude way of handling the effect of distance, and Ptolemy developed a more accurate method in the *Handy Tables*. Column 5 is a coefficient of interpolation for locations of the center of the epicycle other than the limits as a function of the distance of the center of the epicycle from the northern limit, $\omega_C = \lambda_C - \lambda_N = \kappa - \omega_A$, where κ is the true eccentric anomaly. It is used as a cosine since both the latitude on the eccentric and the inclination i_2 of the epicycle vary nearly as cos ω_C, that is $c_5(\omega_C) = \cos\ \omega_C$; it is, however, computed by multiplying the tabulated lunar latitude for each entry, with a maximum of 5°, by 0;12. The computation of the latitude from the table is simply

$$+\beta = c_5(\omega_C) \cdot c_3(\alpha), \qquad \text{if } 270° \leq \omega_C \leq 90°,$$
$$-\beta = c_5(\omega_C) \cdot c_4(\alpha), \qquad \text{if } 90° \leq \omega_C \leq 270°.$$

INFERIOR PLANETS

The apparent motion in latitude of the inferior planets is entirely different and even more complex. Because their orbits are inside the heliocentric orbit of the earth, their motions in latitude can be seen in two ways, both in the line of sight and across the line of sight, and these appear to behave quite differently. It happens that the nodal lines of the inferior planets are rather close to the directions of the apsidal lines found by Ptolemy – $+5^\circ$ to the ascending node for Venus, $+16^\circ$ to the descending node for Mercury – and his description of the latitudes (13.1) follows from assuming that the apsidal and nodal lines coincide. (1) When the center of the epicycle is a quadrant from the apsidal line, the greatest differences in latitude occur, on the same side of the ecliptic, near superior and inferior conjunction, the larger near inferior conjunction, the smaller near superior conjunction, and when the planet is a quadrant from either conjunction it has no latitude. (2) When the center of the epicycle is in the apsidal line, the greatest differences in latitude occur, on opposite sides of the ecliptic, at opposite greatest elongations, differing by approximately equal amounts from the latitudes at apogee and perigee. (3) When the center of the epicycle is in the apsidal line and the planet is near superior or inferior conjunction, it has a small latitude, to the north for Venus and to the south for Mercury.

The model to account for these latitudes (13.2) is shown in Figure 4. The earth is at O, the center of the eccentric at M, and the epicycle is shown at apogee at 0°, and at 90°, 180°, and 270° from apogee. The eccentricity itself has a small effect on latitude for Mercury and virtually none for Venus, and is shown here to distinguish the direction of the apsidal line. Consider the configuration at 90° and 270°. (1) The epicycle is inclined in the line of sight at an angle i_1 so the greatest differences in latitude are on the same side of the ecliptic, β_a at the apogee of the epicycle P_a and β_b at the perigee P_b, and because the distance OP_b is less than OP_a, β_b is greater than β_a. This component of latitude, which we call β_1, in the line of sight, is called the 'inclination' (*enklisis*), the same term used for the latitude of the superior planets, which is also

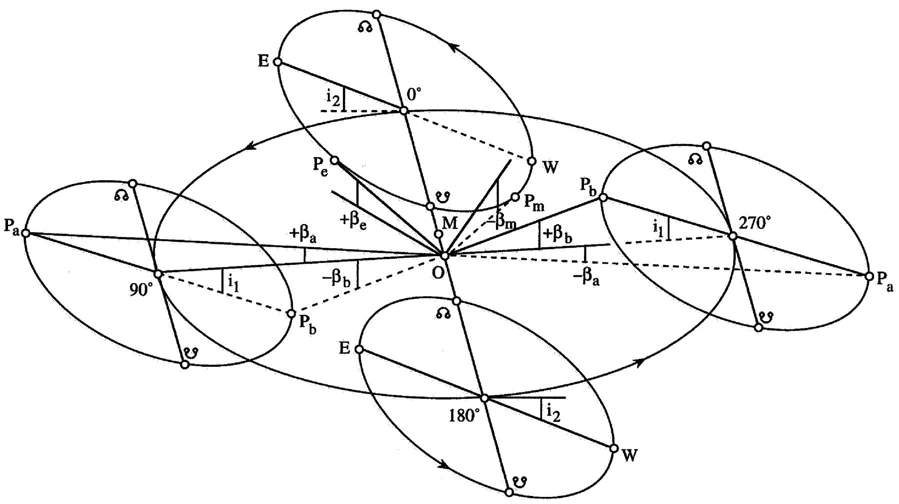

Figure 4. Inferior planets.

in the line of sight, and i_1 is the inclination of the epicycle. When the planet is in the nodal line across the line of sight a quadrant from P_a and P_b, it is in the plane of the ecliptic without latitude. As the epicycle moves from 270° to 0°, the inclination i_1 decreases to zero and the epicycle takes on a second inclination i_2 along a nodal line in the line of sight. (2) At 0° the epicycle is inclined across the line of sight and the greatest differences in latitude are on opposite sides of the ecliptic, β_e at greatest evening elongation P_e and β_m at greatest morning elongation P_m. This component of latitude, which we call β_2, across the line of sight, is called the 'slant' or 'obliquity' (*loxosis*), and i_2 the slant or obliquity of the epicycle. As the epicycle continues to 90°, i_2 goes to zero and i_1 increases to its maximum, and so on. (3) Finally, and this is not illustrated, to account for the small latitude β_3 near conjunction, when the center of the epicycle is at 0° and 180°, the plane of the eccentric has a small inclination i_3 on a nodal line perpendicular to the apsidal line, thus passing through 90° and 270°, moving the epicycle and the planet to the north for Venus and to the south for Mercury, an inclination that also goes to zero as the epicycle moves to 90° and 270°. Ptolemy's calls this the 'inclination of the eccentric' as it is also in the line of sight.

The inclination i_1 is found from β_1 using the correction table for longitude, in much the same way as for the superior planets (13.3). In Figure 5, the earth is at O and the plane of the epicycle is inclined to the plane of the eccentric OC, which lies in the plane of the ecliptic, by i_1 such that the planet P_a at superior conjunction at apogee has latitude β_a and P_b at inferior conjunction at perigee has latitude β_b. Neither location can be directly observed since the center of the epicycle lies *nearly* in the direction of the mean sun ($\bar{S}$), and thus the planet is too close to the true sun to be visible, so β must be inferred from nearby observations, a difficulty to which we shall return. Now imagine the epicycle rotated into a plane perpendicular to the plane of the ecliptic; then i_1 may be taken as proportional to the anomaly at a small arc α_a from apogee and α_b from perigee, and β_a and β_b as proportional to equations of the anomaly c_a and c_b corresponding to these arcs, that is, in each case $i_1/\alpha = \beta/c$. Since the maximum effect of i_1 takes place ±90° from the apsidal line, the equations for

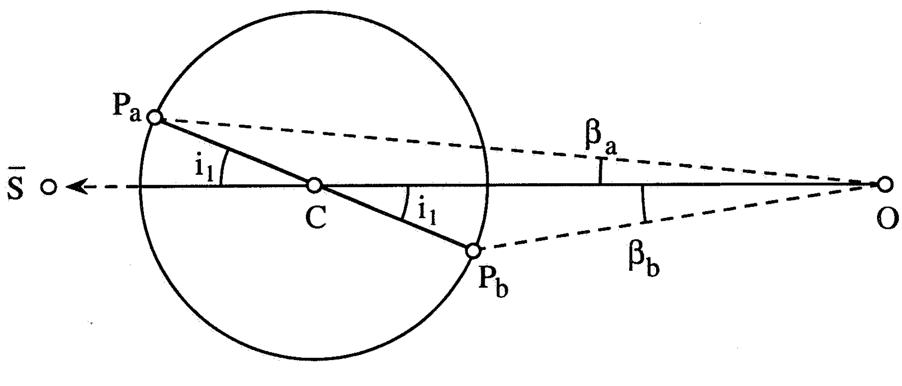

Figure 5. 'Inclination' i_1 of inferior planets.

Venus are taken for mean distance, $c = c_6$ in the correction tables, but for Mercury, which is not at mean distance, $c = c_6 + c_8 \cdot c_7$ for 90° of mean eccentric anomaly for which $c_8 \approx 0;40$. From the tables (11.11), letting $\alpha_a = 6°$ and $\alpha_b = 3°$, we find two values, i_{1a} and i_{1b}, from $i_1 = \beta(\alpha/c)$, which we compare with Ptolemy's i_1 and his β_a and β_b computed for confirmation from $\beta = i_1(c/\alpha)$, three of which have discrepancies of ±0;1° compared to an accurate calculation.

	β_a	β_b	c_a	c_b	i_{1a}	i_{1b}	P.i_1	P.β_a	P.β_b
Venus	1°	$6\frac{1}{3}°$	2;31°	7;38	2;23°	2;29°	$2\frac{1}{2}°$	1;2°	6;22°
Mercury	$1\frac{3}{4}$	4	1;41	1;57	6;14	6;9	$6\frac{1}{4}$	1;46	4;5

Ptolemy's i_1 is very close to i_{1b} for Venus and i_{1a} for Mercury, and his recomputations of β, which are also the extreme latitudes in his tables in 13.5, are sufficiently close to the observed latitudes that the selection and rounding of i_1 are reasonable. Still, there are problems with β_1 and i_1 that we shall take up after considering β_2 and i_2.

The slant of the epicycle i_2 is found (13.4) from the maximum apparent latitude β_2, which takes place at greatest elongation (from the mean sun) where Venus is always visible and Mercury usually visible, although near apogee of the eccentric only its morning elongation and opposite to apogee only its evening elongation are visible. (In fact, the maximum latitudes β_2 do not take place at greatest elongation from the mean sun, but at maximum equation of the anomaly, when the line from the earth to the planet is tangent to the epicycle. However, in the apsidal line the equation of center is zero and these are the same.) In Figure 6 the planet P_e is shown at greatest evening elongation, to the east of the mean sun $\bar{S}$, with latitude β_2. Since CP_eO and CGP_e are right angles, triangles CP_eO and CGP_e are similar so that $a/\rho = r/R'$ where $R' = OC$. It follows that

$$d = \rho \sin \beta_2 = a \sin i_2 = \rho \frac{r}{R'} \sin i_2, \qquad \sin i_2 = \frac{R'}{r} \sin \beta_2.$$

Ptolemy notes (13.1, 13.4) that the maximum opposite apparent latitudes, on either side of the ecliptic, at greatest elongations at apogee and perigee of the eccentric differ, for Venus by slightly under 5° at apogee and slightly over 5° at perigee, and for Mercury by about $5° - \frac{1}{2}°$ at apogee and about $5° + \frac{1}{2}°$ at perigee, the differences from 5° due to the effect of the greater distance at apogee and lesser distance at perigee. For the derivation of i_2, he ignores the variation and takes a difference of about 5°, and for the latitudes on each side of the ecliptic he takes the arithmetic mean so that $\beta_2 \approx \pm 2\frac{1}{2}°$ for both planets. This, as we shall see, is a considerable simplification although it leads to excellent results. The distance of the center of the epicycle of Venus in the apsidal line is $R + e$ at apogee and $R - e$ at perigee, and since the effect of change of distance is small, he takes $R' = R = 60$. The distance of the center of the epicycle of Mercury at apogee is $R + 3e = 69$ and at the opposite end of the apsidal line $R - e = 57$, which is not its least distance, and for the derivation he takes the

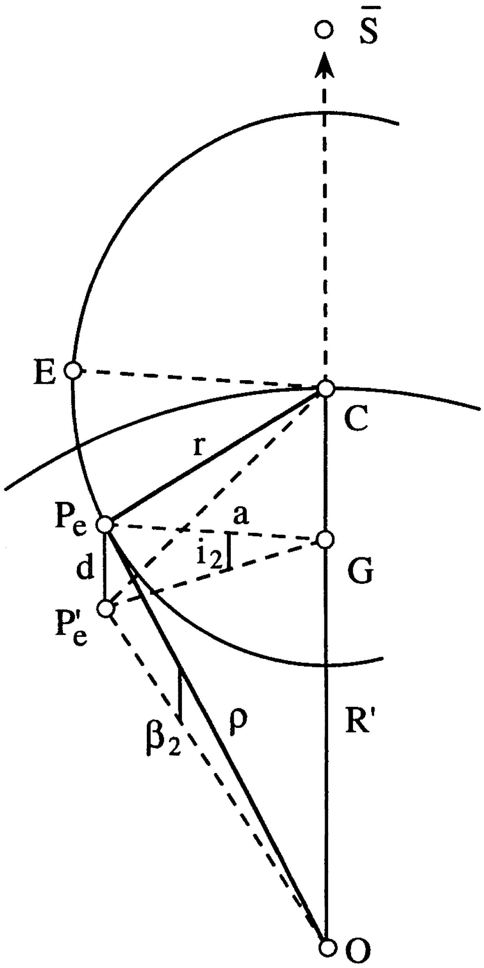

Figure 6. 'Slant' i_2 of inferior planets.

mean $R' = R + e = 63$. With these values, the computation, from $\sin i_2 = (R'/r) \sin \beta_2$, is straightforward.

	R'	r	β_2	i_2
Venus	60	43;10	2;30°	3;28° $\approx 3\frac{1}{2}°$
Mercury	63	22;30	2;30	7

In order to analyze empirically the 'inclination', latitude β_1 and inclination i_1, we find the apparent latitudes as close as possible to superior and inferior conjunction at the limits of latitude according to modern theory, using inspection in Tuckerman's tables for the second century. Since these change in the course of the century, particularly for Venus, as the conjunction points shift with respect to the limits, we cite them only to tenths of a degree. The inclination i_1 is found from β_a at superior conjunction and apogee of the epicycle or from β_b at inferior conjunction and perigee. These occur in the invisible arc and must be inferred from observations before last visibility and after first visibility when the planet is near the horizon, affected by refraction, and possibly with no reference stars. For Venus the period of invisibility around superior conjunction is quite long, from 55 to 69 days at Ptolemy's latitude, and around inferior conjunction, entirely within the retrograde arc, from 1 to 18 days. Mercury's period of invisibility at superior conjunction is from about 27 days to an entire invisible evening phase and at inferior conjunction, at least in part within the retrograde arc, from about 13 days to an invisible morning phase. The strictly invisible extreme values of β_a and β_b to 0;6° $= 0.1°$ are as follows along with the values cited by Ptolemy:

	$+\beta_a$	$-\beta_a$	P. $\pm\beta_a$	$+\beta_b$	$-\beta_b$	P. $\pm\beta_b$
Venus	1;30°	1;30°	1°	8;36°	8;48°	6;20°
Mercury	1;48	2;0	1;45	3;48	4;42	4

The more serious errors, more than $\pm 2°$ for β_b, with more serious consequences, are for Venus. (I do not believe that modern computations and graphs of the motion of the planet from before last to after first visibility give any idea of just how difficult it is to measure these latitudes accurately.) We may use a selection of these values of β_a and β_b to 0;6°, more precise than Ptolemy could reach, to compute i_1 by Ptolemy's method, in Figure 5, $i_1 = \beta(\alpha/c)$, taking c from the correction tables (11.11) and letting $\alpha_a = 6°$ and $\alpha_b = 3°$. In this way, we find i_{1a} and i_{1b} and compare them with i_1 from Ptolemy and modern theory.

	β_a	β_b	c_a	c_b	i_{1a}	i_{1b}	P. i_1	M. i_1
Venus	1;30°	8;36°	2;31°	7;38	3;34°	3;23°	$2\frac{1}{2}°$	3;22°
Mercury	1;48	4;42	1;41	1;57	6;25	7;14	$6\frac{1}{4}$	6;58

Ptolemy's i_1 for Venus is erroneous, following from his erroneous value of β_a and β_b, while for Mercury it is close to i_{1a} as his $\beta_a = 1;45°$ is close to $\beta_a = 1;48°$ here; the modern i_1 is close to i_{1b} for both planets. A different selection of β_a and β_b would produce different results – for example, for Mercury $\beta_a = 2°$ gives $i_{1a} = 7;8°$, close to i_{1b} – and more precise values would give better agreement with modern theory. So again we see that the problem in Ptolemy's derivation is the inadequate observations, particularly for Venus, as it is difficult to estimate the invisible latitudes at superior and inferior conjunction. The derivation itself of i_1 by this method is about as sensitive to inaccuracies as finding i_1 for the superior planets, that is, moderately sensitive.

The difficulties of the 'slant', latitude β_2 and inclination i_2, are entirely different, and there are difficulties even though Ptolemy's results are excellent. Latitude β_2 is observed at greatest elongation, where Venus is always visible and Mercury usually visible, as far above the horizon as they can be seen, which appears promising. But for i_2 to have its greatest effect, and be isolated from i_1, the heliocentric orbit of the planet must be seen across the line of sight, which occurs when the earth is in the planet's nodal line, meaning, in Ptolemy's theory, when the center of the epicycle is in the planet's apsidal line, which happens to be close to the heliocentric orbit's nodal line. The difficulty here, for Venus above all, is that the planet is seldom at greatest elongation when the center of the epicycle is in the apsidal line, and small departures from these two conditions can noticeably affect the apparent latitude. The same problem occurs in Ptolemy's determination of the parameters for longitude, the eccentricity and radius of the epicycle, which require the same strictly unobtainable conditions, greatest elongation with the center of the epicycle in the apsidal line and at other specified locations. For Mercury, as noted before, it happens that near apogee only morning elongation and opposite to apogee only evening elongation are visible.

Ptolemy was doubtless aware of these difficulties, although he does not mention them, for they may explain the way he describes the behavior of the slant (13.1, 13.4). He does not say directly that at greatest elongation in the apsidal line $\beta_2 = \pm 2\frac{1}{2}^\circ$, even though he uses that value for finding i_2 for both planets. Rather, he says that the *total* variation in latitude, north and south of the ecliptic, for Venus is slightly under 5° at apogee of the eccentric and slightly over 5° at perigee, for Mercury is about $5^\circ - \frac{1}{2}^\circ$ at apogee and $5^\circ + \frac{1}{2}^\circ$ at perigee, and that he will use $\beta_2 \approx \pm 2\frac{1}{2}^\circ$ as a *mean* value. There is good reason for his description, for if we use Tuckerman's tables to examine the apparent latitudes *near* greatest elongation with the earth or sun *near* each planet's nodal line, where the slant is isolated from the inclination, the apparent latitude is *almost never* $\pm 2\frac{1}{2}^\circ$, but varies quite widely. As close as we can come to these conditions in the period A.D. 130–146, we find for Venus positive latitudes restricted to about $+2.1^\circ$ and $+3.1^\circ$, the mean of which is $+2.6^\circ$, and a single negative latitude of -2.4°. For Mercury we find a positive range of about $+2.4^\circ$ to $+2.7^\circ$ with a mean of $+2.55^\circ$ and a negative range of -2.5° to -3.2° with a mean of -2.85°, although I am not certain how many of these are actually visible. Hence, Ptolemy's mean latitudes for both planets, $\beta_2 \approx \pm 2\frac{1}{2}^\circ$, even if theoretical, and his inclinations, for Venus $i_2 = 3\frac{1}{2}^\circ$ and for Mercury $i_2 = 7^\circ$, are better than can be reached from observations during this period. This is also true of some of his other parameters.

Ptolemy describes the third component of latitude β_3 (13.1, 13.3) as an *equal* latitude from the ecliptic, reaching $+\frac{1}{6}^\circ$ for Venus and $-\frac{3}{4}^\circ$ for Mercury, at *both* apogee and perigee of the epicycle, thus at superior and inferior conjunction, as inferred from nearby observations, when the center of the epicycle is in the apsidal line. Since the latitude is the same at apogee and perigee of the epicycle, it is not affected by distance on the epicycle. Ptolemy therefore attributes β_3 to a variable inclination i_3 of the eccentric on a nodal line passing through the earth perpendicular to the apsidal line, hence through 90° and 270° in Figure 4; i_3 is maximum when the center of the epicycle is in the apsidal line, goes to zero at a quadrant from the apsidal line, and then returns to maximum in the same direction, to the north for Venus, to the south for Mercury.

It is not at all obvious how Ptolemy found β_3, but it does seem to be his own discovery rather than a conventional value. I know of no correct explanation of just what it is that he observed, what accounts for β_3, and will not trouble the reader with my own attempts. There are discussions by Pedersen and Neugebauer. In any case, inferring a latitude of $+0;10^\circ$ for Venus or even $-0;45^\circ$ for Mercury at conjunction when the planet can only be observed many days before or after, the latitude is changing the most rapidly across the line of sight, and is strongly affected by refraction near the horizon, seems very insecure. And just as for observing greatest elongations with the center of the epicycle in the apsidal line, the simultaneous conditions for observing the planet near conjunction with the center of the epicycle in or near the apsidal line occur rarely, especially for Venus. Since the effect of i_3 is the same wherever the planet is on the epicycle, β_3 could be related to a variation in observed latitudes of β_2 near greatest elongation when the center of the epicycle is near the apsidal line and the effect of β_2 is greatest. Ptolemy does describe effects of this sort, opposite for Venus and Mercury, at greatest elongations (13.1), but he also describes the latitudes near apogee and perigee, and I would not doubt his report of the kind of observations by which he discovered it. As he changed i_3 to a fixed inclination in the *Handy Tables* and the *Planetary Hypotheses*, he himself must have come to doubt these observations.

The tables for latitude of the inferior planets (13.5) are more complex than those for the superior planets, and the computation is considerably more complex since three components, the inclination and slant of the epicycle and the inclination of the eccentric, must be computed separately and added together. An example from the table for Venus is given here at intervals of 6°:

1	2	3	4	5
Argument		$\pm\beta_1(\alpha)$	$\pm\beta_2(\alpha)$	Inter. (κ)
6°	354°	1;2°	0;8°	0;59,36
12	348	1;1	0;16	0;58,36
18	342	1;0	0;25	0;57,0
...	...	...	...	...
84	276	0;8	1;50	0;6,24
90	270	0;0	1;57	0;0,0
96	264	0;10	2;3	0;6,24
...	...	...	...	...
126	234	1;18	2;27	0;35,12
132	228	1;38	2;30	0;40,0
138	222	1;59	2;30	0;44,24
...	...	...	...	...
168	192	5;13	1;27	0;58,36
174	186	5;52	0;48	0;59;36
180	180	6;22	0;0	1;0,0

Columns 1 and 2 are arguments of entry for 6°–180° and 180°–354° at intervals of 6° for 270°–90° and 3° for 90°–270°. Column 3 is the inclination β_1 computed from i_1 and column 4 the slant β_2 computed from i_2, both functions of the true anomaly α on the epicycle. There is also a rather crude correction for distance in computing β_2 for Mercury, taking $\frac{9}{10}c_4(\alpha)$ in the semicircle of the eccentric around apogee and $\frac{11}{10}c_4(\alpha)$ in the semicircle around perigee. Since $\frac{1}{10} \cdot 2\frac{1}{2}^\circ = \frac{1}{4}^\circ$, this gives a range of β_2 of $5 - \frac{1}{2}^\circ$ at apogee and $5^\circ + \frac{1}{2}^\circ$ at perigee, as Ptolemy reported. Column 5 is the same coefficient of interpolation tabulated for the superior planets, used as a cosine, as a function of the true eccentric anomaly $\kappa = \lambda_C - \lambda_A$ since the variations of i_1 and i_2 are both functions of the distance κ of the center of the epicycle from the apogee of the eccentric; hence $c_5(\kappa) = \cos\ \kappa$. There are rather complex rules (13.6) for how $c_5(\kappa)$ is applied to β_1 and β_2 because the inclinations i_1 and i_2 are maximum and zero 90° apart, so that one uses both $c_5(\kappa)$ and $c_5(\kappa \pm 90^\circ)$, and the rules are reversed for Venus and Mercury since the inclinations of their epicycles are in opposite directions. Column 5 is also used to compute the latitude β_3 due to the inclination of the eccentric i_3, +0;10° for Venus and −0;45° for Mercury; it is applied as $c_5(\kappa)^2 = \cos^2\kappa$ in order to compute both the variation of i_3 and the change of latitude on the eccentric, each separately a function of $\cos\ \kappa$. A complete statement of the rules for the application of $c_5(\kappa)$ is given by Neugebauer. Here we note only that one forms, with the proper signs and the correction in $c_4(\alpha)$ for Mercury,

$$\pm\beta_1 = c_5(\kappa) \cdot c_3(\alpha), \quad \pm\beta_2 = c_5(\kappa) \cdot c_4(\alpha), \quad \pm\beta_3 = c_5(\kappa)^2 \cdot i_3,$$
$$\beta = \beta_1 + \beta_2 + \beta_3.$$

Because of the errors in i_1 and β_1 in particular, the computed latitudes of the inferior planets are not at all accurate, with errors for Venus reaching over 2° near inferior conjunction, as we have seen. Ptolemy must have become aware of these problems, for the latitude theory of the inferior planets receives notable correction in the *Handy Tables*, to which we now turn.

HANDY TABLES

It is possible that Ptolemy's latitude theory differs from the *Almagest* in the *Canobic Inscription*, which shows a still earlier stage of his work, but the numbers in the text appear so corrupt that no conclusions can be drawn. Hence, we can say nothing about this earliest latitude theory if it did in fact differ. There is, however, no doubt that the latitude theory later received important modifications, improvements, in the *Handy Tables*, although with a curious error not present in the *Almagest*. The tables for computing latitude, which are entirely different from those in the *Almagest*, are similar in form to the correction tables for longitude, described briefly in the Appendix, and the computation is now the same for superior and inferior planets although it is also somewhat more laborious. There is a detailed examination by Neugebauer, in part following an analysis by van der Waerden, showing how the tables are computed from the underlying model, which is nowhere explained by Ptolemy. The latitude is computed as the sum of two components, one due to the inclination of the eccentric, the other due to the inclination of the epicycle, and the most notable result is a better control over the effect of the distance of the center of the epicycle. This is done in the same way as in the correction tables for longitude, that is, there is a column for latitude at mean distance of the center of the epicycle, two additional columns, a subtraction for greatest distance and an addition for least distance, and a coefficient of interpolation for intermediate distances. An example from the table for Mars is given here at 6° intervals:

1	2	3 (κ)	4 (α)	5 (α)	6 (α)	7 (ω)
6°	354°	−0;60	0;3°	0;54°	0;3°	0;60
12	348	−0;59	0;3	0;55	0;3	0;59
18	342	−0;57	0;3	0;56	0;4	0;57
...	...	...	...	...	...	...
84	276	−0;4	0;7	1;7	0;9	0;6
90	270	+0;3	0;8	1;11	0;11	0;0
96	264	+0;8	0;9	1;15	0;13	0;6
...	...	...	...	...	...	...
168	192	+0;58	0;51	3;46	1;29	0;59
174	186	+0;59	0;56	4;6	1;39	0;60
180	180	+0;60	0;59	4;20	1;46	0;60

Columns 1 and 2 are arguments for 3°–180° and 180°–357° at intervals of 3°. Column 7 is a coefficient of interpolation, a cosine, $c_7 = \cos\ c_{1,2}$, rounded from c_5 in the *Almagest*, which is used in two ways. In the first, it is used to compute the latitude β_1 due to the inclination of the eccentric i_1 by $\pm\beta_1 = c_7(\omega_p)\cdot i_1$, where ω_p is the distance of the planet from the northern limit, $\omega_p = \lambda_p - \lambda_N$, which also determines the sign of β_1. For the superior planets, this is close to the heliocentric latitude of the planet, which does not appear independently in the computation from the tables in the *Almagest*. Column 5 is the latitude due to the epicycle, under the assumption that the planet is always at greatest distance from the plane of the eccentric, as though the epicycle is parallel to the eccentric and raised from the eccentric by $i_1 + i_2$, as shown in Figure 7A, in which P_o is the planet at opposition, P'_c the projection of the planet P_c at conjunction, and P' the projection of an arbitrary position P. Column 5 is a function of the true anomaly α measured from the apogee of the epicycle when the center of the epicycle is at mean distances. Column 4 is the subtraction for the epicycle at apogee of the eccentric, column 6 is the addition for the epicycle at perigee of the eccentric, both also functions of α. Column 3 is a coefficient of interpolation for the distance *OC* of the center of the epicycle on the eccentric, a function of the true eccentric anomaly $\kappa = \lambda_C - \lambda_A$. To combine the effect of the two variables α and κ, one computes either of

$$\beta'_2 = c_5(\alpha) + c_3(\kappa)\cdot c_4(\alpha), \qquad \text{if } c_3(\kappa) < 0,$$
$$\beta'_2 = c_5(\alpha) + c_3(\kappa)\cdot c_6(\alpha), \qquad \text{if } c_3(\kappa) > 0.$$

It is this calculation, the principal innovation of the latitude tables in the *Handy Tables* and a great improvement over the treatment of the effect of distance in the *Almagest*,

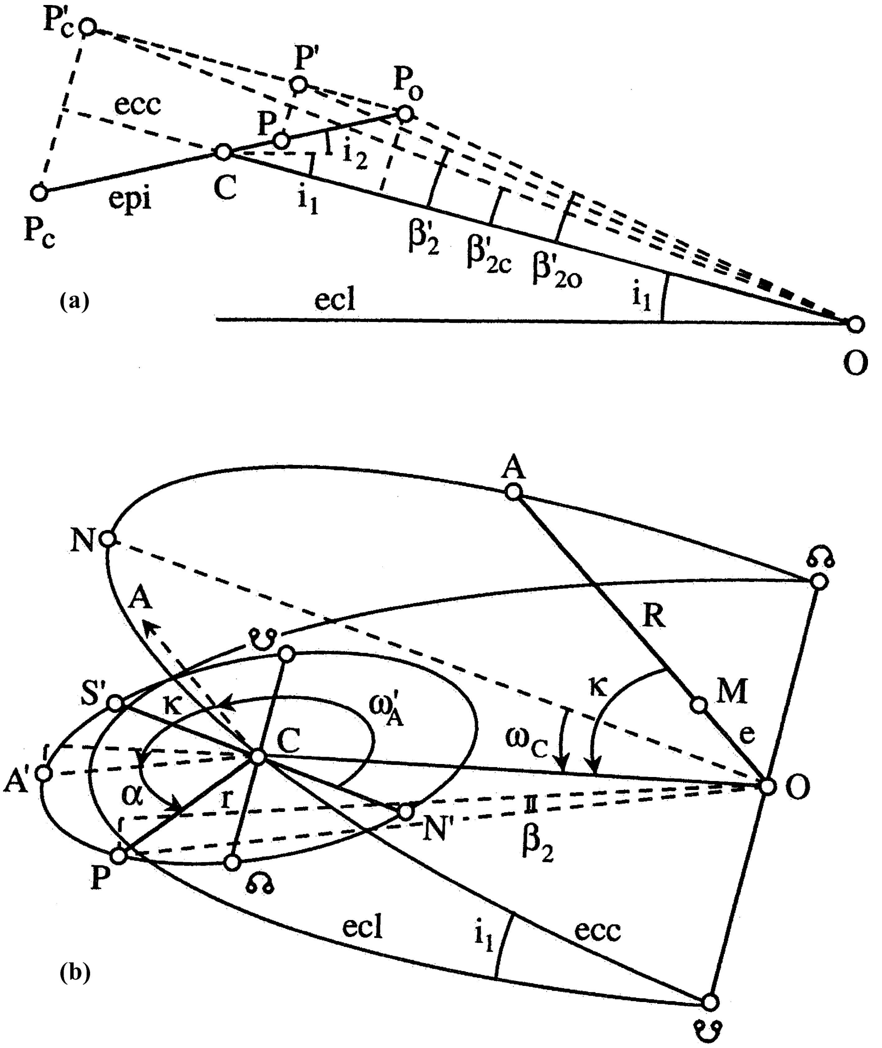

Figure 7. (A) Preliminary latitude β_2' in *Handy Tables*. (B) Inclination of epicycle and final latitude β_2 in *Handy Tables*.

that corresponds to the calculation of the second inequality in the correction tables for longitude. Finally, column 7 is again used as a coefficient of interpolation for the angular distance of the planet on the epicycle – now regarded as inclined to the eccentric *with a fixed inclination* – from the northern limit of the epicycle, $\omega_{\mathrm{p}}' = \omega_{\mathrm{A}}' + \kappa + \alpha$, as shown in Figure 7B. ω_{A}' is an invariable distance from the northern limit of the epicycle N', which holds a fixed direction, to a direction CA on the epicycle parallel to the direction OA of the apogee of the eccentric. Hence, one finds $\pm\beta_2 = c_7(\omega_{\mathrm{p}}') \cdot \beta_2'$, which also determines the sign of β_2, and then the final latitude $\beta = \beta_1 + \beta_2$. (The configuration and notation here differ slightly from Neugebauer's.)

The inclinations of the planes of the eccentric and the epicycle of the superior planets are the same as in the *Almagest*, and the latitudes at opposition and conjunction at the limits differ by not more than $\pm 0;1^\circ$. But there is no doubt that the inclination of the epicycle is now fixed, as has been noted by van der Waerden, and this seriously affects latitudes when the center of the epicycle is near the nodes, a problem that seems thus far to have gone unnoticed. Were the inclination variable as in the *Almagest*, for the superior planets from i_2 at its greatest value when the center of the epicycle is at the limits to $i_2 = 0^\circ$ at the nodes, one would have to multiply β_2' by a further coefficient using $c_7 = \cos c_{1,2}$, the distance of the center of the epicycle from the northern limit of the eccentric $\omega_C = \lambda_C - \lambda_N$, that is, $c_7(\omega_C) = \cos \omega_C$, the same coefficient used in the *Almagest*, so that $\beta_2 = c_7(\omega_C) \cdot c_7(\omega_p') \cdot \beta_2'$. But nothing of the kind is mentioned in Ptolemy's instructions, nor in Theon's instructions, in which β_2 is determined only by $\beta_2 = c_7(\omega_p') \cdot \beta_2'$ for a fixed inclination of the epicycle. Although this is correct for the inferior planets, for which the epicycle is always inclined to the ecliptic, it is an error for the superior planets, for it means that when the center of the epicycle is in the nodal line, the epicycle is still inclined to the ecliptic and the planet may have latitude. One may say that the superior planets now behave like the inferior planets, showing (1) an 'inclination' in the line of sight when the center of the epicycle is at the limits, $\pm 90^\circ$ from the nodal line, and the epicycle is inclined in the line of sight, and (2) a 'slant' across the line of sight when the center of the epicycle is in the nodal line and the epicycle is inclined across the line of sight. This is wrong because heliocentrically the center of the epicycle is the planet and the epicycle is the orbit of the earth, which is in the plane of the ecliptic; when the planet is in the nodal line, it can have no latitude no matter where the earth is in its orbit. Thus, geocentrically when the center of the epicycle is in the nodal line, the planet should have no latitude no matter where it is on the epicycle, which must therefore lie in the plane of the ecliptic. Ptolemy himself had stated this condition in the *Almagest*, and it is hard to know why he would wish to change it in the *Handy Tables*. Perhaps he meant to write also to use $c_7(\omega_C)$ for the superior planets in computing β_2, and just nodded – as we should prefer to believe – but there is no way of confirming this, and that Theon's instructions are the same as Ptolemy's probably rules out a textual error. Hence, as the tables and instructions have been transmitted, the inclination of the epicycle is fixed.

For the inferior planets, with the inclination of the epicycle fixed, the nodal line of the epicycle is always parallel to the apsidal line. Hence, (1) when the center of the epicycle is in the apsidal line, the nodal line of the epicycle coincides with the apsidal line in the line of sight, the fixed inclination of the epicycle is observed across the line of sight and one sees the 'slant'; (2) when the center of the epicycle is $\pm 90^\circ$ from the apsidal line, the nodal line of the epicycle is across the line of sight, the fixed inclination of the epicycle is observed in the line of sight and one sees the 'inclination'. Thus, both components are now a single latitude resulting from one fixed inclination seen in two ways. So the fixed inclination of the epicycle, which is incorrect for the superior planets, is correct for the inferior planets.

We may investigate the consequences of fixed inclinations by computing from the tables the maximum latitude when the center of the epicycle is at the ascending and descending node of the eccentric for the superior planets, and at apogee and the point opposite apogee, which for Mercury is not perigee, for the inferior planets. Distances from the apogee to the northern limit ω_A are now -40° for Saturn, which in the *Almagest* was -50°, $+20^\circ$ for Jupiter, and 0° for Mars, which are unchanged; hence the eccentric anomaly κ of the ascending node $\kappa_{☊} = \omega_A - 90^\circ$ and of the descending node $\kappa_{☋} = \omega_A + 90^\circ$. For the inferior planets $\kappa = (0^\circ, 180)$. The following table gives κ, the approximate range of β_2', of distances ω_p' from the northern limit of the epicycle at which the maximum latitude occurs – the true anomaly $\alpha = \omega_p' + 90^\circ$ – of the coefficient $c_7(\omega_p')$, and the resulting latitude β_2 computed from the tables by the procedure just explained. For the superior planets, the first computation is for the descending node and the second for the ascending node of the eccentric; for the inferior planets, the eccentric is taken to be in the plane of the ecliptic.

	κ	β_2'	ω_p'	$c_7(\omega_p')$	β_2	κ	β_2'	ω_p'	$c_7(\omega_p')$	β_2
Saturn	50°	0;30°	0–3°	1;0	0;30°	230°	0;32°	3–6°	1;0	0;32°
Jupiter	110	0;31	0–6	1;0	0;31	290	0;31	0–6	1;0	0;31
Mars	90	1;55–2;0	39–42	0;46–0;44	1;28	270	1;55–2;0	39–42	0;46–0;44	1;28
Venus	0	3;31–3;54	45–51	0;42–0;38	2;28	180	4;22	51	0;38	2;46
Mercury	0	2;7–2;9	12–15	0;59–0;58	2;5	180	2;52–2;56	24–27	0;55–0;54	2;38

In fact β_2 may stay within 0;1° of these maximum values for a considerable range of ω_p', and roundings to minutes in the tables introduce small irregularities by which the latitude can decrease and again increase. But the important point is that the superior planets may have latitude, a notable latitude, when the center of the epicycle is at a node, when the latitude should be zero. This is a significant error, an error not present in the *Almagest*. For the inferior planets, the maximum latitude is no longer ±2;30°, but varies with distance as it should, something treated only roughly by the correction to β_2 for Mercury of $\pm\frac{1}{10}c_4(\alpha)$ in the *Almagest*, and the maximum latitudes do not occur together with the maximum equations of the anomaly, as was true for the method of computing β_2 in the *Almagest*. (For this reason, one cannot compute the maximum latitude here as one computes the maximum slant in the *Almagest*, from $\sin\beta_2 = \sin i_2$ (r/R'), as in Figure 6, which is close for the inferior planets, but much smaller for the superior planets.) Neugebauer has computed 22 latitudes for Mars and 17 latitudes for Venus at 10-day intervals to show a more general comparison between the *Handy Tables* and the *Almagest*. The differences in these reach nearly 1° for Mars and are greatest near the nodal line for reasons just explained. The differences for Venus reach about 1;30°, are greatest where the 'inclination' has its greatest effect, ±90° from the apsidal line, and agree better with modern theory for reasons we shall now show.

For the inferior planets, as noted, the method of computation no longer distinguishes the inclination and slant, which nevertheless occur where and as they should, but gives a single latitude due to the inclination of the epicycle, which is added to the small latitude due to the inclination of the eccentric. The latitude just computed, when the center of the epicycle is in the apsidal line, is the 'slant'. The following table shows the inclination of the epicycle i_1 and the resulting latitude, the 'inclination', at superior conjunction at apogee of the epicycle β_a and at inferior conjunction at perigee β_b from the *Almagest*, *Handy Tables*, and a modern computation to tenths of a degree.

	Almagest			*Handy Tables*			Modern				
	i_1	$\pm\beta_a$	$\pm\beta_b$	i_1	$\pm\beta_a$	$\pm\beta_b$	i_1	$+\beta_a$	$-\beta_a$	$+\beta_b$	$-\beta_b$
Venus	2;30°	1;2°	6;22°	3;30°	1;29°	8;52°	3;22°	1;30°	1;30°	8;36°	8;48°
Mercury	6;15	1;45	4;5	6;30	1;50	4;14	6;58	1;48	2;0	3;48	4;42

The inclination of the epicycle of Venus in the *Handy Tables* is that of $i_2 = 3;30°$ in the *Almagest*, which is about correct and produces far better results for β_a and β_b. The inclination for Mercury, 6;30°, is closer to $i_1 = 6;15°$, although $i_2 = 7;0°$ in the *Almagest* is preferable; the resulting β_a and β_b differ only slightly from the *Almagest*. (In the smaller commentary to the *Handy Tables*, Theon gives β_b for Venus as 8;56° and for Mercury as 4;18°.)

The inclination of the eccentric i_3 is now a fixed inclination of 0;10° for both planets, with the northern limit at the apogee for Venus and the southern limit at the apogee for Mercury. Here too, as for the variable inclination in the *Almagest*, I know of no correct explanation of what Ptolemy actually observed that could account for i_3. The remarks made earlier about the difficulty of observing these latitudes at all still apply, and it is notable that each is at the limit of precision of Ptolemy's observations. They could perhaps have been derived from an effect seen near greatest elongation, as Ptolemy also mentioned in the *Almagest*, rather than near inferior or superior conjunction, but that too would be difficult and the *Handy Tables* contain no explanation.

The latitude tables of the *Handy Tables* seem to have been of little influence, even in tables otherwise based upon Ptolemy's models and parameters, which is of some interest. The correction tables for longitude of the planets in the *Handy Tables* are, directly or indirectly, the basis of many, perhaps most, later tables following Ptolemy, in Greek, Arabic, eventually Latin, differing for the most part only in textual errors or adjustments of interpolation for the intervals of 1° and one commonly altered parameter: Venus is given the equation of center of the sun, as is also found in tables based upon Indian models and parameters, although without adjustment for the effect of distance on the equation of the anomaly, which is so small as to be of no consequence. (The *Alfonsine Tables* also have a different equation of center for Jupiter, of unknown origin, likewise without adjustment of the equation of the anomaly.) However, later Ptolemaic tables for latitude are overwhelmingly based upon the tables in the *Almagest*, some with various modifications, mostly to facilitate computation, that have been described by van Dalen. There are two known partial and curious exceptions. Kennedy (2) lists maximum latitudes of the *Mumtaḥan Zīj*

(*Az-Zīj al-Ma'mūnī lī'l Mumtaḥan*) of the early ninth century, which agree with those given by Theon in his smaller commentary to the *Handy Tables*, although the latitude tables themselves are based upon nothing more than a sine function, which is quite primitive. The latitudes listed by Kennedy, some of which differ slightly from direct computation from the *Handy Tables*, are as follows:

	$+\beta_o$	$-\beta_o$		$\pm\beta_b$
Saturn	3;1°	3;6°	Venus	8;56°
Jupiter	2;3	2;9	Mercury	4;18
Mars	4;23	7;6		

These are maximum latitudes at opposition for the superior planets – Theon gives Saturn $+\beta_o = 3;2°$ – and at inferior conjunction for the inferior planets, although it is not clear just how they are applied in these tables. The same maximum latitudes are attributed by Ibn Hibintā of the mid-tenth century to either the *Zīj as-Sindhind* or the *Zīj ash-Shāh*, with $+\beta_o = 5;23°$ for Mars, doubtless a textual error; since the former is based on Indian models and parameters and the latter on a Pahlavi translation of an Indian original, latitudes from the *Handy Tables* seem out of place. Nevertheless, the tables and report do show that maximum latitudes from Theon's commentary were known and used, here it appears in rather crude adaptations.

The other example comes from a very different period and region, the *Zīj al-Muqtabis* of Ibn al-Kammād, who lived in Andalusia in the twelfth century, which survives in a Latin version made in Palermo in 1260 by one John of Dumpno, of which there is a study by Chabás and Goldstein. The latitude tables for the superior planets are those of the *Almagest*, but those for the inferior planets are from the *Handy Tables*, at 6° intervals and with a maximum for Venus in $c_5(\alpha)$ of 8;35° instead of 8;51°, presumably a textual error. The same tables are found in the *Tables of Barcelona* of the fourteenth century, published by Millás Vallicrosa and analyzed by Chabás; here the maximum for Venus in $c_5(\alpha)$ is 8;55°. These sources show that, in addition to Theon's commentary, the latitude tables themselves were known in Arabic, and it was the choice of Ibn al-Kammād, or of his source whatever that may have been, to use only those for the inferior planets. Why? Could someone have understood that the tables are in error for the superior planets, but not for the inferior planets, because the fixed inclination of the epicycle gives the planet latitude even when the center of the epicycle is in the nodal line? One would not need observations to understand this, just an understanding of the latitude theory in the *Almagest* and an ability to figure out that the computation in the *Handy Tables* implies a fixed inclination of the epicycle, which is not exactly obvious as the model is not explained. Still, the way astronomical tables were often haphazardly and inconsistently thrown together for hundreds of years, this is a higher level of understanding than one is accustomed to. It is obvious, for example, that Theon did not recognize a problem. Another possible explanation is that the latitude tables for the inferior planets in the *Almagest* were considered too complicated, which is true enough with all the rules for applying the coefficient $c_5(\kappa)$, although those in the *Handy Tables* are also complicated. Perhaps it is safer to confess that we have no idea why only the latitude tables for the inferior planets have been found, and someday a source containing the tables for the superior planets may be discovered. Benno van Dalen, who, in answer to my inquiry about latitude tables from the *Handy Tables* in Arabic referred me to Ibn al-Kammād's tables, informs me that thus far he knows of none for the superior planets, so if they exist at all they must be very uncommon.

PLANETARY HYPOTHESES

The *Planetary Hypotheses* may be considered Ptolemy's last word on latitude theory, in which he finally got nearly everything right. The models in the *Planetary Hypotheses* have the same inclinations as the *Handy Tables* for the inferior planets, but the superior planets differ from both the *Almagest* and the *Handy Tables*, and are definitely improved, one reason for believing that the *Hypotheses* is later than the *Handy Tables*. For the superior planets the epicycles now have fixed inclinations to the eccentric parallel to the ecliptic, that is, the inclination of the epicycle to the eccentric $i_1 + i_2 = i_1$, and thus correctly $i_2 = 0°$. The inclinations of the eccentric, for Saturn $i_1 = 2;30°$, for Jupiter $i_1 = 1;30°$, are unchanged, but for Mars $i_1 = 1;50°$, which is correct. It is easy to show that these inclinations produce quite accurate latitudes at conjunction and opposition at the limits. We have only to reverse Ptolemy's procedure for finding the inclinations from the correction tables for longitude in *Almagest* 11.11. Thus, in Figure 8, where i_1 is proportional to the 'anomaly' α measured from apogee or perigee of the epicycle, and ϑ is proportional to the equation of the anomaly c, so that $\vartheta/c = i_1/\alpha$, we have at both conjunction and opposition $\vartheta = (c/\alpha)i_1$, from

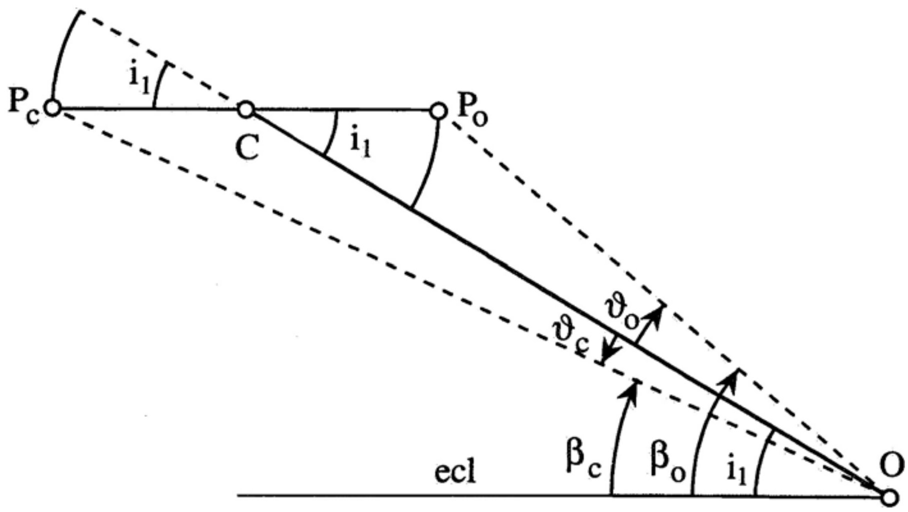

Figure 8. Latitudes β_c and β_o from inclination i_1 in *Planetary Hypotheses*.

which $\beta_c = i_1 - \vartheta_c$ and $\beta_o = i_1 + \vartheta_o$. In the computation we take the distance of the northern limit from apogee of the eccentric as: Saturn $-40°$, Jupiter $+20°$, Mars $0°$, and we take c from the correction tables for $\alpha = 3°$. The results are shown along with modern inclinations and latitudes to 0;3°.

	i_1	$+\beta_c$	$-\beta_c$	M. $\pm\beta_c$	$+\beta_o$	$-\beta_o$	M. $\pm\beta_o$	M. i_1
Saturn	2;30°	2;16°	2;14°	2;18°	2;48°	2;51°	2;51°	2;33°
Jupiter	1;30	1;16	1;15	1;9	1;50	1;53	1;45	1;25
Mars	1;50	1;9	1;3	1;9–1;6	4;34	6;46	4;36–6;54	1;52

The most notable improvement over Ptolemy's earlier theories is that β_c for Mars is now correctly just over 1° where before it was very close to 0°; every other value is also improved, with no error exceeding 0;8°, and of course when the center of the epicycle is at the nodal line the latitudes are correctly zero. (Had Ptolemy not rounded i_1 for Jupiter from 1;24° to 1;30° in the *Almagest*, its latitudes would be better, but the rounding of i_1 for Saturn from 2;26° to 2;30° is an improvement.) Intermediate latitudes, not at conjunction or opposition at the limits, or at the nodes with zero latitude, may not be as accurate as these, but the problems are due to errors in longitude theory and distance, not to the latitude theory itself. The latitudes of the inferior planets are the same as those of the *Handy Tables* tabulated earlier. One may now say that Ptolemy has corrected both the variable inclinations of the epicycles in the *Almagest* and, for the superior planets, the fixed inclinations not parallel to the ecliptic in the *Handy Tables*.

The subject of the first part Book I of the *Hypotheses*, where these parameters are given, is the construction of instruments, analogue computers consisting of graduated circles, eccentric and epicyclic, within a concentric graduated zodiac, for finding longitudes of the sun and longitudes and latitudes of the moon and planets without correction tables. They are the earliest known examples of what were later called 'equatoria'; later specimens, and there are many, generally do not include latitude. Kennedy (1) has described an instrument of the early fifteenth century for computing latitudes by the ingenious Jamshīd Ghiāth ad-Dīn al-Kāshī, based upon the latitude theory of the *Almagest*, in which the latitudes are projected into a plane. Ptolemy's instruments in the *Hypotheses*, however, actually have inclined rings. How one could make these things, presumably of metal, with tiny inclinations of about 2° to 6°, even 0;10° for the eccentrics of Venus and Mercury, is not a trivial question, and perhaps their use for latitude was only theoretical. One might suggest that the fixed inclinations of the epicycles in the *Hypotheses* were a simplification for making such instruments, as who could possibly construct the variable inclinations of the *Almagest* with their small vertical circles? But this cannot be so, first because the latitude theory of the fixed inclinations, with the new and correct inclination for Mars, really is superior to the variable inclinations in the *Almagest*, and to the fixed inclinations in the *Handy Tables*, surely deliberately so, and second because the spherical models described in

Book II, which are supposed to be the physical mechanisms in the heavens responsible for the motions of the planets, have the same fixed inclinations, for the superior planets parallel to the ecliptic. Of course historically the more important parts of the *Planetary Hypotheses* are the distances and sizes of the planets and the spherical models, the cosmology and physical astronomy, on which the principal studies are by Goldstein and Murschel, and the latitude theory, for all its ingenuity, seems to have been without influence, for it appears that no one made use of Ptolemy's final, corrected latitude theory.

How Ptolemy made these changes, he does not say except to remark that by continuous observations he has made corrections, compared to the *Almagest*, of the hypotheses themselves or of their proportions or periodic times, and in fact the *Planetary Hypotheses* contains various changes in the hypotheses and their parameters, including periods, of which the changes in the hypotheses and inclinations for latitude are the most significant. So one must conclude that he corrected the theory of latitude of both superior and inferior planets from his own observations, improving upon the rough values of extreme apparent latitudes, which we believe to have been conventional estimates, used earlier in the *Almagest* when they were all he had. Since it was not possible in the years of his observations to observe all the planets under the special conditions used to derive the inclinations in the *Almagest*, for the superior planets at opposition and near conjunction at the limits, he must have derived the inclinations from other sorts of observations. One possibility is a series of oppositions over several years with a large although not maximum latitude, showing by computation both the inclination i_1 and, from $i_1 + i_2 \approx i_1$, that $i_2 = 0^\circ$, that the epicycle remains parallel to the ecliptic, which would require two oppositions for each planet. In this way he could correct both the *Almagest* and the *Handy Tables*. But on such things one can only speculate, fully aware of the difficulty of finding these parameters even from accurate observations. However Ptolemy did it, he got it right.

It is of interest that John Bainbridge, Henry Savile's successor as professor of astronomy at Oxford, who edited and translated the part of Book I of the *Planetary Hypotheses* surviving in Greek in its first edition in 1620, specifically noticed the changes in the theory of latitude, which otherwise remained unknown until our own time. After stating that the three manuscripts of the *Hypotheses* he used were corrupt and incomplete, requiring much emendation by comparing the text with the *Almagest*, he remarks: 'In the hypotheses of latitudes I desired to change nothing. For wise (*prudens*) Ptolemy, if I judge this matter, departed from the hypotheses established in the *Syntaxis* and proposed other easy, convenient, and far truer (*longeque veriores*) hypotheses.' This observation is interesting, not only for noticing the differences in the hypotheses, but in recognizing that those in the *Planetary Hypotheses* are 'far truer' than those in the *Almagest*. The only way he could know this in 1620 was by having read, and approved, Kepler's *Astronomia nova* (1609), in which it is shown (13–14) using Tycho's observations that the inclination of the plane of Mars's eccentric to the ecliptic is about 1;50° and fixed, implying in a geocentric model a fixed inclination of the epicycle parallel to the ecliptic, just as Ptolemy found. And Kepler computes (65) the extreme latitudes: northern limit, opposition +4;31,45°, conjunction +1;8,30°;

southern limit, opposition −6;52,20°, conjunction −1;4,20°, all very close to the extreme latitudes from Ptolemy's theory. Remarkably, Ptolemy's latitude theory in the *Planetary Hypotheses* anticipates Kepler, as Bainbridge recognized. In a (lost) letter of 24 February 1625 Bainbridge brought the revised latitude theory to Kepler's attention, and later that year Kepler received Bainbridge's publication and noted the fixed inclination of 1;50° for Mars.

We conclude with a table of the inclinations in the *Almagest*, *Handy Tables*, and *Planetary Hypotheses* with modern values for A.D. 100 from P.V. Neugebauer, the same ones quoted earlier, which are also given by O. Neugebauer. In the *Almagest*, Ptolemy gives the inclination of the epicycle of the superior planets as $i_1 + i_2$, and the same inclination applies in the *Handy Tables* although it is not explicitly given. The inclination of the epicycle i_2, to a plane parallel to the plane of the ecliptic, is variable in the *Almagest* from its given value to 0° but fixed in the *Handy Tables*; in the *Planetary Hypotheses* the epicycle for the superior planets is parallel to the ecliptic so that $i_2 = 0°$. The two variable inclinations of the epicycle for the inferior planets i_1 and i_2 in the *Almagest* are replaced with a single fixed inclination we call i_1 in the *Handy Tables* and *Planetary Hypotheses*. The inclination of the eccentric for Venus and Mercury i_3 is variable in the *Almagest* from its given value to 0° but fixed in the *Handy Tables* and the *Planetary Hypotheses*. We have marked variable inclinations v; all others are fixed. Aside from the component i_3 for the inferior planets, it is obvious that in the *Planetary Hypotheses*, Ptolemy has reached something close to modern latitude theory, with a single defective inclination for Mercury, for which i_2 from the *Almagest* would have been correct. No one did better until Kepler in the *Epitome of Copernican Astronomy* and the *Rudolphine Tables*.

	Almagest			Handy Tables			Planetary Hypotheses		Modern
Planet	i_1	$i_2 v$	$i_3 v$	i_1	i_2	i_3	i_1	i_3	i_1
Saturn	2;30°	2°	–	2;30°	2°	–	2;30°	–	2;33°
Jupiter	1;30	1	–	1;30	1	–	1;30	–	1;25
Mars	1;0	1;15	–	1;0	1;15	–	1;50	–	1;52
Venus	2;30v	3;30	+0;10°	3;30	–	0;10°	3;30	0;10°	3;22
Mercury	6;15v	7;0	−0; 45	6;30	–	0;10	6;30	0;10	6;58

APPENDIX

CORRECTION TABLES FOR LONGITUDE IN THE *ALMAGEST* AND *HANDY TABLES*

The correction tables for longitude in the *Almagest* are used to derive the inclinations of the planes of the eccentric and epicycle in the theory of latitude, and the latitude tables in the *Handy Tables* control the effect of distance of the center of the epicycle in the same way as the correction tables for longitude. For these reasons, we include brief descriptions of the correction tables for longitude and of the formation of equations between mean and true motions. The reader is referred to Pedersen or Neugebauer for more detailed treatment of the tables in the *Almagest*, including the method of computation, and to Neugebauer for the *Handy Tables*. We show an excerpt from the table for Mars from *Almagest* 11.11.

1	2	3 ($\bar{\kappa}$)	4 ($\bar{\kappa}$)	5 ($\bar{\alpha}$)	6 ($\bar{\alpha}$)	7 ($\bar{\alpha}$)	8 ($\bar{\kappa}$)
6°	354°	1;0°	+0;5°	0;8°	2;24°	0;9°	−0; 59, 53
12	348	2;0	+0;10	0;16	4;46	0;18	−0;58,59
18	342	2;58	+0;15	0;24	7;8	0;28	−0; 57, 51
...	...	...	...	...	...	...	...
96	264	11;29	−0; 4	2;42	35;6	3;6	−0; 3, 3
99	261	11;32	−0; 8	2;49	35;56	3;15	+0;0,5
102	258	11;32	−0; 12	2;56	36;43	3;25	+0;3,13
...	...	...	...	...	...	...	...
174	186	1;30	−0; 10	2;27	11;15	4;26	+0;59,43
177	183	0;45	−0; 5	1;16	5;45	2;20	+0;59,52
180	180	0;0	−0; 0	0;0	0;0	0;0	+1;0,0

Columns 1 and 2 are arguments of entry for 6°–180° and 180°–354° at intervals of 6° for 270°–90° and 3° for 90°–270°. Column 3, a function of the mean eccentric anomaly $\bar{\kappa}$, the uniform motion of the center of the epicycle from apogee of the eccentric measured at the equant point, is the equation of center for an eccentric circle of eccentricity $2e$, from the earth to the equant point. Column 4, also a function of $\bar{\kappa}$, is a correction for the bisection of the eccentricity for an eccentric circle of eccentricity e, from the earth to the center of the eccentric. Columns 3 and 4 are added to find the equation of center $c_3' = c_3 + c_4$. The true eccentric anomaly κ is computed from $\bar{\kappa}$ by $\kappa = \bar{\kappa} + c_3'$, where $c_3' < 0°$ for $\bar{\kappa} < 180°$ and $c_3' > 0°$ for $\bar{\kappa} > 180°$. The true anomaly on the epicycle α is computed from the mean anomaly $\bar{\alpha}$ by $\alpha = \bar{\alpha} + c_3'$, where $c_3' > 0°$ for $\bar{\kappa} < 180°$ and $c_3' < 0°$ for $\bar{\kappa} > 180°$.

Columns 5–7, all functions of the true anomaly α, are used to compute the equation of the anomaly c due to motion of the planet on the epicycle: c_6 is the equation for the center of the epicycle at mean distance on the eccentric R, c_5 a subtraction for greatest distance $R + e$, and c_7 an addition for least distance $R - e$. For Mercury the distances are $R + 3e$ and $R- \sim \frac{3}{2}e$ respectively. Finally, column 8, a function of $\bar{\kappa}$, is a coefficient of interpolation for intermediate distances of the center of the epicycle, extending from -1 at greatest distance to $+1$ at least distance. The equation of the anomaly c is computed from either of

$$c = c_6(\alpha) + c_8(\bar{\kappa}) \cdot c_5(\alpha), \quad \text{if } c_8(\bar{\kappa}) < 0, \qquad c = c_6(\alpha) + c_8(\bar{\kappa}) \cdot c_7(\alpha), \quad \text{if } c_8(\bar{\kappa}) > 0.$$

In the *Handy Tables*, the entries in c_1 and c_2 are at intervals of 1°; the equation of center, $c_3 + c_4$ in the *Almagest*, is combined into a single c_3; c_4 is the coefficient of interpolation for computing the equation of the anomaly, computed as a function of the true eccentric anomaly $\kappa = \bar{\kappa} \pm c_3$ and rounded to one fractional place; columns c_5, c_6, c_7 for the equation of the anomaly are the same as in the *Almagest*. The contraction of intervals to 1°, which facilitates the use of the tables, is done by linear interpolation in the tables in the *Almagest* although there are small discrepancies of no consequence.

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