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THE LIMITS OF TEXT IN GREEK MATHEMATICS

ABSTRACT

This article argues for a limited role of the text in Greek mathematics, in two senses of "text": the verbal as opposed to the visual; and the literate as opposed to the "oral" (understood in a wide sense). The Greek mathematical argument proceeds not within the confines of the verbal alone, but essentially relies upon diagrams. On the other hand, it does not use other specific techniques, such as those of the modern cross-reference, relying instead upon verbal echoes. The two, taken together, suggest a model of scientific writing radically different from what we associate with our own mathematics. In methodological terms, the article surveys its evidence in detail, and makes comments concerning the methodology of studying ancient texts through the evidence of those texts alone.

1. INTRODUCTION

The word "text" occurs in the title of this chapter, as presumably in many other titles in this book; yet I doubt if any two chapters share exactly the same definition of "text". Not that it is important to define such a concept. It is the one of the concepts we know too well, which we can only define by artificially limiting our use of it. So I do not intend to define what "text" is; I simply wish to explain what I mean by saying that the text had limits in Greek mathematics. What I mean is that the verbal, written aspect of Greek mathematics had limits. The verbal aspect had limits because of the central importance of the visual aspect, i.e. of diagrams; the written aspect had limits because of the central importance of the verbal, non-written aspect (which we may also call 'oral'—as long as this is not taken as the equivalent of 'illiterate'). So I discuss two limits, the limits of the verbal (measured against the visual), and the limits of the written (measured against the non-written or oral). I start with a relatively detailed discussion concerning Greek diagrams. The discussion of the limits of the written as opposed to the oral will be briefer.

In both cases, my intention is not merely to redress the balance, to reclaim the role of the non-textual elements; my intention is to argue that the textual and non-textual elements cannot be taken apart. This will be clarified in the following.

Before getting started, it is necessary to say something on the nature of the available evidence. An optimist would concentrate on the fact that we have got plenty of material: a few thousand ancient propositions, written by a few dozens of authors. Some of my colleagues in this book can only gasp at such richness of evidence. A pessimist, however,

K. Chemla (ed.), History of Science, History of Text, 161–176.

would stress the fact that practically nothing is available directly, and what we have got are manuscripts written at least a thousand years after the time we are most interested in. The pessimist could also point out that the mathematical works we do have are all first-order texts, none of them is a second-order Greek mathematician talking *about* Greek mathematics. Greek mathematicians are unlike Hilbert, say, in that they speak about triangles and circles but they never speak about mathematics.

Thus, to make a first important clarification, at the current stage of our knowledge it would be rash to offer wholesale conclusions on the precise shape diagrams take in our manuscripts. Secondly, it is a fact that whatever we may know about Greek mathematical attitudes must be deduced from the practice. We interrogate the Greek mathematicians, and the normal methods of interrogation yield nothing, because they keep their silence. We must invent new methods of interrogation to make them speak. In the following, part of the interest lies in what I have learned from Greek mathematicians, another part of the interest, I think, is in the methods of interrogation I have devised.

2. DIAGRAMS

The method of interrogation to be described has to do with what I call the "fixation of reference" or, the one word I will mostly use, "specification"[1]. My meaning is the following. The text of a Greek mathematical proposition always consists of expressions such as

". . . Alpha-Beta is therefore larger than Gamma-Delta"

or

"Since Alpha lies on Beta-Gamma"

and the like. In other words, a Greek mathematical text is, at face value, a discussion of Alpha, Beta, etc. These, of course, refer to other objects. Now, there must be some process of fixation of reference, whereby these letters are related to their objects. The process can be described, complementarily, either as the process in which letters are specified geometrically, or as the process in which geometrical objects are specified by letters. I shall use the two perspectives interchangeably, speaking either of "the specification of letters" or "the specification of objects".

In principle, the process can be verbal or visual; I will show that it is both.

The contrary assumption would take the following shape: that one knew what Alpha stood for by being told. For instance, one may have been told that "Let Alpha be the center of the circle". It should be noted that even here some extra information is called for. For instance, one needs to know that a circle has only one centre. I shall allow for the sake of the following discussion that such assumptions are taken for granted[2]. Even allowing this, I shall still claim that objects in diagrams, as a rule, are specified pragmatically, i.e. through inspection of the diagram.

We now move on to the main table of this article, showing the different ways in which letters are specified in Euclid's *Elements* XIII and Apollonius' *Conics* I. What I ask the reader to do is to look up the tables as she reads through my explanation of the significance of the different columns (this is my own textual modality, then!).

I shall say, beforehand, why I chose these two works—Apollonius' *Conics* Book I and Euclid's *Elements* Book XIII. The selection is meant to capture a certain middle position of Greek mathematics, between those works of Euclid which are elementary in an absolute sense, and the advanced studies of, say, Archimedes[3]. Furthermore, both works are works of "*Elements*", works which have, apparently, among other things, some pedagogic role; if they exhibit pragmatic specification, then A FORTIORI the same must be true for less self-conscious presentations (though, of course, I do not rely upon this A FORTIORI argument alone[4]).

I also want to add a brief explanation concerning my quantitative methodology. Let me first explain the methodology underlying my use of data such as the following tables. It would be a mistake to think of this methodology as statistical in any sense. First, as representatives of other treatises, the relationship between the two works studied here and the corpus of Greek mathematics taken as a whole is not a matter of statistical sampling but of historical understanding. Sampling would be irrelevant: the field of mathematical treatises is small and uneven; it reflects, more than anything else, the bias of survival whose nature is incompletely understood. Second, as representing the two treatises themselves, the methodology taken here is not statistical, as I survey the *entire* works. My approach, then, is not statistical in the technical sense, but is based on the traditional hermeneutic approach of the historian: I try to understand ancient practices by gaining an empathic understanding of the practice as it is present in a few ideal-type cases which, other things being equal, I take to be representative of the historical reality as a whole. The importance of the quantitative approach of the tables is not that they produce numbers, but rather that, by asking a quantitative question, I force myself to go through the entire treatise in reaching my empathic understanding. The table is an attempt to convey, in succinct form, the experience which underlies this understanding.[5]

Euclid's *Elements,* Book XIII

Prop.	Fully specified	Pre-specified	Under-specified	Un-Specified	Not used in text
1	ABEZ	ΓΔ	H	ΘΚΛΜΝΞΟ	
2	ΑΒΓΖΗ		Δ	ΕΘΜΝΞΚΛ	
3	ΑΒΔΕ	Γ		ΠΣΖΗΘΛΜ ΝΞΟΠ	
4	ΑΒΔΕ	Γ		ΚΘΗΖΛΜΝ	
5	ABE	Γ	Δ	ΘΚ	Λ
6	AB	Γ	Δ		
7	ΑΒΓΔΕ			Z	
8	ΑΒΓΔΕΘ				
9	E	ΒΓΔΑ			Z
10	ΖΘΛ	ΑΒΓΔΕ	HKM	N	
11	ZN	ΑΒΓΔΕ	ΗΘΚ	ΛΜ	
12	Δ	ΑΒΓ	E		

13	ΑΒΓΘ	ΔΕΖΗΚΛ			
13lemma	ΑΒΓΔΕ			Ζ	
14	ΑΒΓΕΖΗΘ	ΔΛΜ	Κ		
15	ΑΒΓΕΖΗΘ	ΔΚΛΜΝ			
16	ΑΒΓΛΜΝΞΟ ΠΡΣΤΨΦΑ′	ΔΕΖΗΘΚΩΨ		Χ	
17	ΑΒΓΔΕΖΗΘ ΚΛΜΝΞ	ΡΣΤΨΦΞΩ	Ψ	ΟΠ	
18	ΑΒΓΔΚ	ΕΗΝ	ΖΛΜ	Θ	
(Final lemma)	ΑΒΓΔΕΖ				
Total	102	55	16	42	2
% (rounded)	47	25	8	19	0

Apollonius' *Conics,* Book I

Prop.	Fully specified	Pre-specified	Under-specified	Un-Specified	Not used in text
1	ΑΒΓΖ	ΔΕ			
2	ΑΒΓΖΗΘΚ	ΔΕ			
3	Α	ΒΓ			
4	ΑΔΕΖΗΘΚ	ΒΓ			
5	ΑΛΖΜ	ΒΓΚΗΘ	ΔΕ		
6	ΑΜΝΔΕΚΖΗ	ΒΓΘ	Λ		
7	ΑΔΕΘΚ	ΒΓ	Ζ	Η	ΛΜ
8	ΑΔΕΗΘΞ	ΒΓ	ΖΚΛΜΝ		ΟΠΡΣ
9	ΑΘ	ΒΓ	ΔΚΕΖΗΛΝΞ	Μ	
10	ΑΗΘ	ΒΓ	ΔΕΖ		
11	ΑΔΕΚ	ΒΓΖΘ	ΗΛΜΝ		
12	ΑΔΕΘΜ	ΒΓΖΛΟΞ	ΗΚΝΠΡΣ		
13	ΑΖΗΛ	ΒΓΔΕΘΝ	ΚΜΞΟΠΡ		
14	Α	ΔΕΖΗΘΚΒΓ ΞΟΡΠ	ΛΥΣΤ	ΜΝ	
15	ΑΒΓΔΕΗΝΦ	ΖΛ	ΘΚΜΞΟΠΥΣΡ ΤΞΨ		
16	ΑΒΓΔΕΖΗΚΛ		ΘΜΝ	Ξ	
17	Γ	ΑΒ			
18	ΓΔΕ		ΑΖΒ		
19	ΓΔ	ΑΒ			
20	ΓΔΕΖΗ	ΑΒ			
21	Γ		ΑΒΔΕΖΗΘΚ		
22	ΓΔΕ	Β	Α	Ζ	
23	ΑΒΓΔΕΖΗ ΘΚΛ				

24	ΓΔΕΖ		ΑΒ		
25	ΑΒΓΔΕΖΗΘΚ				
26	ΔΖΗΘΚΛ	Ε	ΑΒΓ		
27	ΕΜ		ΑΒΓΔΗΖ	Κ	
28	ΑΒΕΖΗΛΝ		ΓΔΘΚΜ		
29	ΑΒΓΔΕΗ		Ζ		
30	ΑΒΓΖΗ		ΔΕ		
31	ΓΔΗΘ	Ε	ΑΒ	Ζ	
32	ΓΕΖ	Δ	ΑΒΘΛΚ	Η	
33	Ζ		ΑΒΓΔΗ	Ε	
34	ΓΔ	ΖΚ	ΑΒΕΗΘΛΜΞ	Ο	
35	ΓΖΔ		ΑΒΕ	Η	
36	ΕΖΘ		ΑΒΓΔΗ		
37	ΕΖ		ΑΒΓΔ		
38	ΓΔΜ	Ζ	ΑΗΒΕΛΘ		
39	ΖΕΗ		ΑΒΓΔ		
40	Κ	ΑΒ	ΖΓΔΕΘΛΗ		
41	ΕΖΗ		ΑΒΓΔΘ		
42	ΓΘ	Η	ΑΒΔΖΕ		
43	ΓΖΗΚΛ		ΑΒΔΕΗΘ	Μ	
44	ΓΟΝΘΞΔ	ΑΒΖ	ΕΗΛΚ	Μ	
45	Δ	ΑΒΓ	ΘΜΛΕΖΚΝ	Η	
46	Λ	ΒΘΜΔ	ΑΓΜΝΖΕΚΗ		
47	ΓΝΛ		ΑΒΔΕΘΟΗΞ ΖΜΚ		
48	ΑΒΓΝ		ΚΛΗΔ	ΟΕ	
49	ΒΗΚΞΜ	Γ	ΜΔΖΝΛΠ	Ε	Ο
50	ΓΚΗΛ	Θ	ΑΒΔΕΖΜΞΠ ΝΠΣΟ		
51	ΑΒΕΗΚΝ		ΓΔΛΜΞ	Ζ	
52	ΓΔΗΘΖΛ	ΑΕΚΞ	ΒΜΝ		
53	ΓΔΒΚ	ΘΑΕΛΜ	ΖΗ		
54	ΑΒΓΕΚΘ	ΔΗΠΡ	ΛΜΞ	ΝΟ	
55	ΑΒΓΔ	ΘΖ	ΗΛΚΜΝΞ	Ο	
56	ΑΒΓΔΕΗΚΜ	Λ	ΞΟΖΘΝ		
57	ΑΒΓΔΕΖΗ				
58	ΒΑΔΕΗΓ	Ζ	ΘΚΛΜΞ	Ν	
59	ΒΕΘΗΑΓΔΖ				
60	ΑΓΔΕΖΗΘΚ ΜΝΟΞ	Γ			
Total	268	100	231	24	7
% (rounded)	42	16	37	4	1

The table can be said to be motivated by the question: is the diagram reproducible from the text? Assume we have only the text (which is the case with a few ancient mathematical works, whose diagrams got lost in the process of transmission: e.g., Archimedes' *Arenarius*). Can we reconstruct the diagram?

The answer is generally positive. But then the question can become more specific: *How* do we reconstruct the diagram then? One way is the global way. In the global way, we read the mathematical proposition from beginning to end, forming a rough impression of what it is trying to say; from the general context, we know the kinds of problems of interest; we have expectations of mathematical relevance; and through the combination of these we gradually may reconstruct a diagram which fits the text and which makes what we see as the "correct mathematical sense". This is the global way, and it is very different from the local way. To reconstruct a diagram locally, what we do is to follow the text as it unfolds. The proposition demands "let a line be drawn"—so we draw that line—and so forth, till the end of the proposition.

Not all the assertions in the text can be used in this way. This is because some assertions clearly are not intended as specifications of objects, but rather assume those specifications. I mean the following:

Consider the following case. Given the following diagram (Fig. 1), the following assertion is made, at some stage of the proof: "and therefore AB is equal to BC". Suppose that nothing in the proposition so far vouchsafed for B being a centre of the circle. Is this assertion then a specification of B as the centre? Of course not, because of the "therefore" in the assertion. The assertion is meant to be a *derivation*, and making it into a specification would make it, effectively, a *definition*, and the derivation would become vacuous. Thus such assertions cannot constitute specifications. Roughly speaking, specifications occur in the imperative, not in the indicative. They are the "let the centre of the circle, B, be taken".

We can thus plot the sequence of moments of specification through a proposition[6]. Each moment of specification sets a locus. Saying "let the centre of the circle, B, be taken" sets a locus consisting of a single point; something like "Let D be taken, so that

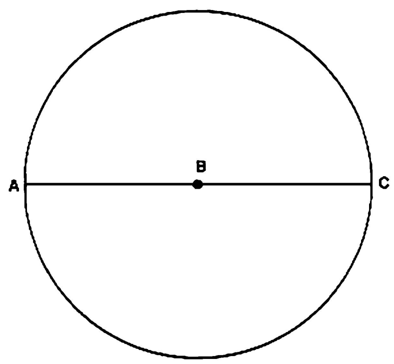


Figure 1.

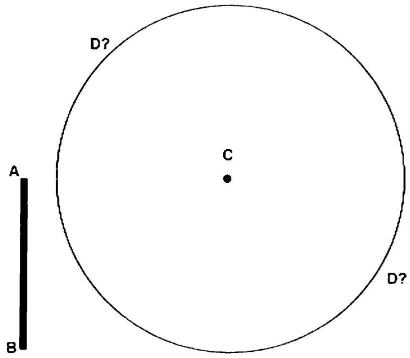


Figure 2.

CD is equal to AB", assuming A, B and C as given, sets up a locus consisting of a circle around C, whose diameter is equal to AB (Fig. 2).

Another locus is the locus which, mathematically speaking, is demanded by the proposition—the locus arrived at "globally". I insist on the concept "locus" here: as a rule, a point may have more than a single legitimate position, as far as the proposition is involved. For instance, there is an important class of true variable points, which, mathematically speaking, should be "any point" within a given domain. Thus, the mathematical locus for such points is the set of points in that domain. On the other hand, the mathematical locus of a point may be more limited. To repeat the example: "Let D be taken, so that CD is equal to AB". This may happen when the mathematical sense demands that D is on a certain given line (Fig. 3). In such a case, the mathematical locus consists of just two points, the intersection of the imaginary circle and the given line. So in such a case the two loci, that defined by the specification, and that demanded by the mathematical situation, diverge. Such divergences are what we are interested in now.

A final general note: it will be seen that the loci involved require, for their definition, letters distinct from the letter which is specified. It may happen that those letters

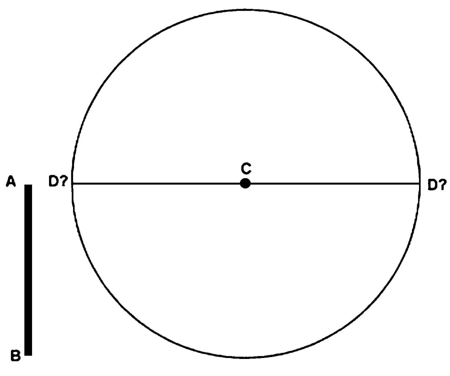


Figure 3.

are underspecified themselves. I have ignored this possibility. I have been like a very lenient teacher, who always gives his pupils a chance to reform. At any given moment, I have assumed that all the letters used in any act of specification were fully specified. I have concentrated on *relative* specification, specification of a letter relative to the preceding letters. This has obvious advantages, mainly in that the statistical results are more interesting: otherwise, practically all letters would turn out to be underspecified in some way.

So let us look at the evidence.

1. The first column, letters which are fully specified, is of course the simplest. The essence of this column is that the locus set up by the specification is identical with the locus demanded by the mathematical situation. A specification setting up a single-point locus, for instance, is bound to be a full specification—the mathematical locus simply cannot be smaller; so, for instance, centers of circles, e.g. E in Euclid's *Elements*, Proposition XIII.9, are indeed ideal. This is a case where specification must be complete because the locus set up by the specification is very small. When the locus demanded by the mathematical situation is very large, full specifications are to be expected, as well; for instance, where a "variable" point is taken, e.g. B in Apollonius' *Conics* I.1: "and let some point on the conical surface, B, be taken[7]" —the locus set up by the specification is the entire conic surface. Nothing more specific is demanded by the mathematical situation, and the point is therefore fully specified. Between these two extremes —the single-point locus, and the "variable" locus— there are many cases, and some of these are fully specified; but the general rule is that full specification is the result of some mathematical necessity — the mathematical locus has such a character which rules out anything short of complete specification.

2. The next column is titled "pre-specified". Pre-specified letters may exist because the same letter may be specified in more than one moment of specification. For instance, Apollonius' *Conics* I.14 (Fig. 4), and the letters B, Γ, Ξ, O. At (Heiberg 1891–1893, 1: 54, lines 7–9) these letters are specified as two pairs of indeterminate points on two respectively given circles. At (Heiberg 1891–1893, 1: 54, line 16) they are re-specified as the result of an intersection of these two circles with a given plane.

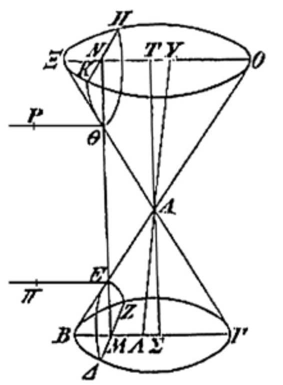


Figure 4. Apollonius' *Conics*, Proposition I.14.

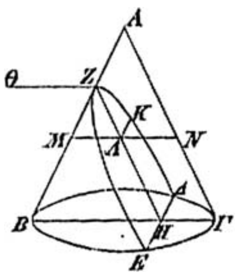


Figure 5. Apollonius' *Conics*, Proposition I.11.

What happened, on the cognitive level, between these two moments? Hypothetically, there are two options. One is the suspension of specification: i.e., the reader did not relate the letters to specific points. The other option is that the reader went beyond what was given in the text, and related the letters to the points as they appear in the configuration of the diagram.

Had this been the only class of letters failing full specification, the choice between these two options would have been real. However, given that there are other, clear cases, where the specification must have been aided by the diagram, I find it hard to believe that, in such cases alone, suspension of specification was exercised[8].

As explained in the table, I do not distinguish, in this column, between two possible cases: one, where the sum of all the moments of specification constitutes a full specification; another, when all the moments of specification taken together still constitute a mere under-specification. I now move on to explain the difference.

3. The third column is titled "under-specified".

Here the confessional mode may help to convert my readers. It took me a long time to realize how ubiquitous this form of specification is. The reason for my obtuseness was the fact that, as soon as one is even slightly acquainted with the Greek approach, one starts to fill the pragmatic gaps automatically. Reading a Greek proposition, one cannot but gaze constantly at the diagram; and visual information compels itself in an unobtrusive, almost unnoticed way. The following example came to me as a shock. It is, in fact, a very typical case.

Look at Λ in Apollonius' *Conics*, Proposition. I.11 (Fig. 5). It is specified at (Heiberg 1891–1893, 1: 38, line 26), where it is asserted to be on a parallel to ΔE, passing through K. This sets up the locus of a line; so how is one to know that Λ is in fact a very specific point on that line, the one intersecting with the line ZH? I know how I knew this: by looking at the diagram.

There is another way by which one could, theoretically, know this: by taking into account the entire proposition. Several later assertions in the proposition make sense only on the assumption that Z is where it is. To repeat: when we discover a partial manuscript, in which the diagrams are absent, we can still reconstruct them. But the process through which a diagram is thus reconstructed from the text is not a true reading of the text, for the following reason, already hinted above. In such a reconstruction, assertions which are meant to *derive*, are taken to *define*. To put this succinctly: un-interpreted by

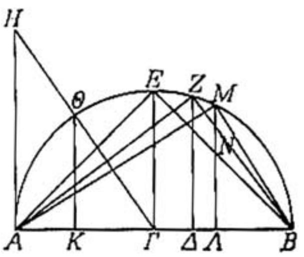


Figure 6. Euclid's *Elements*, Proposition XIII.18.

diagrams, propositions are invalid. Their invalidity consists not in contradictions, but in NON SEQUITURs. The NON SEQUITURs could be rectified, it is true, in principle: but why assume them to begin with?

4. The fourth column is titled "unspecified". This is when letters emerge out of thin air. They occur in assertions, without being anywhere specified. Take for instance Θ in Euclid's *Elements*, Proposition XIII.18. It is first mentioned in the following sentence: "let HΓ be joined, and let a perpendicular be drawn, from Θ to AB, <namely> ΘK" (Fig. 6). Clearly Θ is not specified here at all, it is taken as part of the specification of K. To specify Θ, one would have mentioned the semi-circle; this is not done here. So Θ appears out of thin air. More impressive still is Z in Apollonius' *Conics*, Proposition I.51. At (Heiberg 1891–1893, 1: 156, line 2) the line ΓE is drawn and produced; nothing further is said about it (Fig. 7). Then at (Heiberg 1891–1893, 1: 156, line 9) we are told that a certain property is obvious, "for ΓZ is twice ΓE". That is how Z is introduced into the proposition; very much like "for the Snark was a *Boojum*, you see".

Unspecified letters are relatively speaking rare. They are more common in Euclid's *Elements* Book XIII, but this is only because Euclid uses in propositions 1 to 5 the lazy formula "and let the figure be completed"[9] —an important practice, but not a representative one. Apollonius' *Conics* are more typical, and there unspecified letters occur occasionally, no more: a letter or two, every two or three propositions. This is not much; on the other hand, this is not negligible. It would be extremely rash to suggest that all of these unspecified letters are so many indications of lacunae in the text, though I would agree that some of them may result from such lacunae.

5. The last column is very hard to gauge, and it is included here merely for the sake of completeness. Very rarely, letters—apparently well attested in the manuscripts—occur

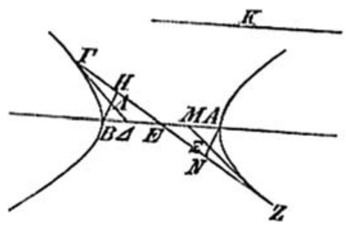


Figure 7. Apollonius' *Conics*, Proposition I.51.

in the diagram but not in the text. Who inserted them and why I do not profess to know: it must be said that most probably these started their lives as accidental marks of ink in some archetype.

The table shows that specification must be to a great extent pragmatic, almost certainly diagrammatic. This is the foundation on which all forms of verbal and visual interdependence are built. I will make two general comments on this, and then move on to the next limit.

A. Why are there so many cases falling short of full specification?

First, it should be quite clear, once again, that the tables have little quantitative significance. It is immediately clear that the way in which letters in Apollonius' *Conics* fail to get full specification is different from that of Euclid; and I suspect that there is a strong variability between works, even by the same author. The *way* in which letters are not fully specified depends upon mathematical situations. Euclid, for instance, may construct a circle, e.g. ABΓΔE, and then constructs a pentagon within the same circle, such that its vertices are the very same ABΓΔE; this is pre-specification, and is demanded by the subject matters dealt with by Euclid's book XIII. In Apollonius' *Conics*, parallel lines and ordinates are the common constructions, and letters on them are often underspecified.

What seems to be more stable is the percentage of fully specified letters: less than half the letters are fully specified —but not much less than a half. It is as if the authors were indifferent to the question whether a letter is specified or not, full specification being left as a random result[10].

This, I would claim, is exactly the case. In other words, I would say that nowhere in Greek mathematics do we find a moment of specification PER SE, a moment whose *purpose* is to make sure that the reference of letters in the text is fixed. Such moments are very common in modern mathematics, at least since Descartes[11]. But specifications in Greek mathematics are done, literally, AMBULANDO. The essence of the moments whose sequence I plot —the moments where the imperative mood is employed, "let a line be drawn . . . "— their essence is to do some job upon the geometric space, to get things moving there. When a line is drawn from one point to another, the letters corresponding to the start and end positions of movement ought to be mentioned; but they need not be carefully differentiated, one need not know precisely which is the start and which is the end, both would yield the same job, the same line (hence under specification); and points traversed through this movement may be left unmentioned (hence unspecification). So, to repeat, there is no specification PER SE, no moment at which the text sets out to take the reins of the proposition in its hands. The text does not try to subdue the diagram, to govern it. The text assumes the diagram. Rather than one of them governing the other, the text and diagram present, let us say, a *cohabitation*. For, indeed, not only is the diagram non-recoverable from text. The following is true as well:

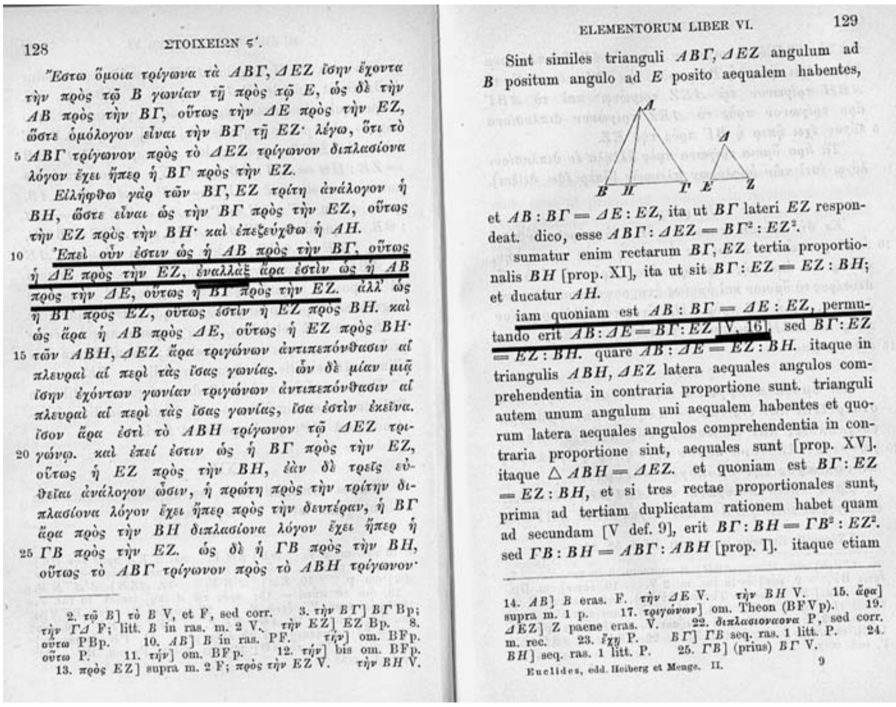
B. The text is not recoverable from the diagram.

Of course, the diagram does not tell us what the proposition asserts. It could do so, theoretically, by some symbolic apparatus; it does not. Further, the diagram does not specify all the objects on its own. There are many metrical properties —e.g. equality of lengths— that are not expressed diagrammatically in any real sense. When the diagram is "dense", a rich complex of lines and letters, even the attribution of letters to points may not be so obvious from the diagram, and modern readers, at least, using modern diagrams, use, to some extent, the text to elucidate the diagram. The stress of the discussion so far should be on *inter*-dependence. True to my promise, I did not try merely to offset the traditional balance between text and diagram; I have tried to show that they cannot be taken apart, none makes sense in the absence of the other.

3. CROSS-REFERENCES

I now move to the second limit, the limit of the written as opposed to the verbal (or the oral)[12]. There are various ways in which the oral plays a surprisingly important role in Greek mathematics —e.g., I would say that many arguments are mediated by the *verbal* structure of the assertions manipulated by those arguments. This is an important claim, which however I cannot support in detail here. I have chosen to concentrate here on a simpler practice, that of cross-references, and to show the role of the oral (or the limits of the written) concerning this practice.

The situation is easy to grasp by opening any edition of a Greek mathematical work. Heiberg —who edited practically all of them— had the useful practice of supplying a latin translation. Within this translation (Fig. 8), he also inserted, within square brackets, references. In these references he pointed out that a certain derivation or construction is validated by a certain prior result[13].



128 ΣΤΟΙΧΕΙΩΝ ϛʹ.

Ἔστω ὅμοια τρίγωνα τὰ ΑΒΓ, ΔΕΖ ἴσην ἔχοντα τὴν πρὸς τῷ Β γωνίαν τῇ πρὸς τῷ Ε, ὡς δὲ τὴν ΑΒ πρὸς τὴν ΒΓ, οὕτως τὴν ΔΕ πρὸς τὴν ΕΖ, ὥστε ὁμόλογον εἶναι τὴν ΒΓ τῇ ΕΖ· λέγω, ὅτι τὸ ΑΒΓ τρίγωνον πρὸς τὸ ΔΕΖ τρίγωνον διπλασίονα λόγον ἔχει ἤπερ ἡ ΒΓ πρὸς τὴν ΕΖ.

Εἰλήφθω γὰρ τῶν ΒΓ, ΕΖ τρίτη ἀνάλογον ἡ ΒΗ, ὥστε εἶναι ὡς τὴν ΒΓ πρὸς τὴν ΕΖ, οὕτως τὴν ΕΖ πρὸς τὴν ΒΗ· καὶ ἐπεζεύχθω ἡ ΑΗ.

Ἐπεὶ οὖν ἐστιν ὡς ἡ ΑΒ πρὸς τὴν ΒΓ, οὕτως ἡ ΔΕ πρὸς τὴν ΕΖ, ἐναλλὰξ ἄρα ἐστὶν ὡς ἡ ΑΒ πρὸς τὴν ΔΕ, οὕτως ἡ ΒΓ πρὸς τὴν ΕΖ. ἀλλ' ὡς ἡ ΒΓ πρὸς ΕΖ, οὕτως ἐστὶν ἡ ΕΖ πρὸς ΒΗ. καὶ ὡς ἄρα ἡ ΑΒ πρὸς ΔΕ, οὕτως ἡ ΕΖ πρὸς ΒΗ· τῶν ΑΒΗ, ΔΕΖ ἄρα τριγώνων ἀντιπεπόνθασιν αἱ πλευραὶ αἱ περὶ τὰς ἴσας γωνίας. ὧν δὲ μίαν μιᾷ ἴσην ἐχόντων γωνίαν τριγώνων ἀντιπεπόνθασιν αἱ πλευραὶ αἱ περὶ τὰς ἴσας γωνίας, ἴσα ἐστὶν ἐκεῖνα. ἴσον ἄρα ἐστὶ τὸ ΑΒΗ τρίγωνον τῷ ΔΕΖ τριγώνῳ. καὶ ἐπεί ἐστιν ὡς ἡ ΒΓ πρὸς τὴν ΕΖ, οὕτως ἡ ΕΖ πρὸς τὴν ΒΗ, ἐὰν δὲ τρεῖς εὐθεῖαι ἀνάλογον ὦσιν, ἡ πρώτη πρὸς τὴν τρίτην διπλασίονα λόγον ἔχει ἤπερ πρὸς τὴν δευτέραν, ἡ ΒΓ ἄρα πρὸς τὴν ΒΗ διπλασίονα λόγον ἔχει ἤπερ ἡ ΓΒ πρὸς τὴν ΕΖ. ὡς δὲ ἡ ΓΒ πρὸς τὴν ΒΗ, οὕτως τὸ ΑΒΓ τρίγωνον πρὸς τὸ ΑΒΗ τρίγωνον·

2. τῷ Β] τὸ Β V, et F, sed corr. 3. τὴν ΒΓ] ΒΓ Bp; τὴν ΓΔ F; litt. Β in ras. m. 2 V. τὴν ΕΖ] ΕΖ Bp. 8. οὕτω PBp. 10. ΑΒ] Β in ras. PF. τὴν] om. BFp. οὕτω P. 11. τὴν] om. BFp. 12. τὴν] bis om. BFp. 13. πρὸς ΕΖ] supra m. 2 F; πρὸς τὴν ΕΖ V. τὴν ΒΗ V.

ELEMENTORUM LIBER VI. 129

Sint similes trianguli ΑΒΓ, ΔΕΖ angulum ad Β positum angulo ad Ε posito aequalem habentes, et ΑΒ : ΒΓ = ΔΕ : ΕΖ, ita ut ΒΓ lateri ΕΖ respondeat. dico, esse ΑΒΓ : ΔΕΖ = $BΓ^2 : EZ^2$.

sumatur enim rectarum ΒΓ, ΕΖ tertia proportionalis ΒΗ [prop. XI], ita ut sit ΒΓ : ΕΖ = ΕΖ : ΒΗ; et ducatur ΑΗ.

iam quoniam est ΑΒ : ΒΓ = ΔΕ : ΕΖ, permutando erit ΑΒ : ΔΕ = ΒΓ : ΕΖ [V, 16], sed ΒΓ : ΕΖ = ΕΖ : ΒΗ. quare ΑΒ : ΔΕ = ΕΖ : ΒΗ. itaque in triangulis ΑΒΗ, ΔΕΖ latera aequales angulos comprehendentia in contraria proportione sunt. trianguli autem unum angulum uni aequalem habentes et quorum latera aequales angulos comprehendentia in contraria proportione sint, aequales sunt [prop. XV]. itaque △ ΑΒΗ = ΔΕΖ. et quoniam est ΒΓ : ΕΖ = ΕΖ : ΒΗ, et si tres rectae proportionales sunt, prima ad tertiam duplicatam rationem habet quam ad secundam [V def. 9], erit ΒΓ : ΒΗ = $ΓB^2 : EZ^2$. sed ΓΒ : ΒΗ = ΑΒΓ : ΑΒΗ [prop. I]. itaque etiam

14. ΑΒ] Β eras. F. τὴν ΔΕ V. τὴν ΒΗ V. 15. ἄρα] supra m. 1 p. 17. τριγώνων] om. Theon (BFVp). 19. ΔΕΖ] Ζ paene eras. V. 22. διπλασιονασονα P, sed corr. m. rec. 23. ἔχῃ P. ΒΓ] ΓΒ seq. ras. 1 litt. P. 24. ΒΗ] seq. ras. 1 litt. P. 25. ΓΒ] (prius) ΒΓ V.

Euclides, edd. Heiberg et Menge. II. 9

Figure 8. Heiberg's Edition of Euclid's *Elements*, Proposition VI.19.

References have wide applications in science, but in deductive mathematics they are crucial. In the deductive game, things may be asserted without proof only if they are known to have been proved earlier. So the knowledge of what was previously proved is an essential part of doing the actual deductive work. Without references, deduction collapses.

In the Latin side of an edition by Heiberg references are textual in a very precise sense. They are mediated through a written code. The question is, what happens in the Greek side. The answer is, first, that usually nothing happens explicitly.

Still working with the same two books, Apollonius' *Conics*, Book I and Euclid's *Elements*, Book XIII, we may see that Apollonius uses Euclid's *Elements* 195 times; Euclid's *Elements*, Book XIII uses Euclid's *Elements* (book XIII itself excluded) 126 times. For good measure, I add in Archimedes' *Spiral Lines*, where Euclid's *Elements* are used 36 times. So we have got a 357-strong population here; and I believe the reader would agree with me that the results here are not untypical of Greek mathematics as a whole.

What are these results? First, in the great majority of cases, the fact that the author uses a previously established result is not even registered. The derivation or the construction is simply made; it remained for Heiberg to note that some reference to a previous result was necessary (of course, Heiberg was following earlier editors and scholastic readers, perhaps from Late Antiquity onwards). Such completely tacit reference takes place 165 times in Apollonius, 107 times in Euclid and 33 times in Archimedes. The percentages are about 85% for both Apollonius and Euclid, and a (meaningless, given the size of the corpus selected) 92% for Archimedes.

All the remaining cases, except one, make a gesture, a nod, towards Euclid. And this gesture is not made through a written code, similar to Heiberg's. The gesture takes the form of a slight redundancy in the formulation of the construction or the derivation. Consider the following, typical example. There, the structure of the argument is:

since

a:b::c:d[14]

therefore

a:c::b:d

The shorter way to deal with such a derivation is simply to state it, in the abbreviated form above. But, quite often (for such references are especially common with this sort of move), a certain redundancy is made. Instead of saying

a:b::c:d, therefore a:c::b:d

The author will then say

a:b::c:d, therefore, *enallax*, a:c::b:d

The word ENALLAX is unmotivated, it is surprising in the context of the very economic language usually used by Greek mathematicians. What it does is to recall the formulation through which the relevant result was proved by Euclid, in *Elements* V.16[15].

This was an example of a minimum redundancy: a one-word redundancy. Sometimes a fuller evocation of the original formulation is made. This is not done often, but it does occur from time to time. For instance, Apollonius' *Conics* I.15 makes what is practically a quotation of Euclid's *Elements* I.5[16]. There are altogether about 10 such quotations in our sample; here especially it is interesting to note that Euclid's handling of quotations from Euclid is not different from Apollonius' handling of quotations from Euclid. So this is rare, but not absolutely rare.

What is absolutely rare is the single exception to the rule, the reference made by Euclid's *Elements* XIII.17 to Euclid's *Elements* XI.38[17]: "for this was proved in the penultimate theorem of the eleventh book". This is the only Heiberg-like reference to Euclid in our whole sample. And it is typical that this is a reference from Euclid to Euclid; for these rare references do occur elsewhere in Greek mathematics, but almost always in the context of references *within* the same work[18].

So we have seen that the form of reference is usually quite implicit, and when some explicit reference is made, it relies upon verbal formulations, almost never upon citations by books and proposition-numbers.

It should also be remembered that —as I have pointed out above— the sum total of the results required by Greek mathematicians for their proofs is, roughly, that contained in Euclid's *Elements*[19]. There are exceptions to that, to which I will return in a minute; but these exceptions are less important than the rule.

So I repeat:

A. Greek mathematicians refer through verbal echoes, not through reliance upon textual guides.

B. Greek mathematicians refer to Euclid's *Elements* and not to very much else besides.

The two are obviously connected. Euclid's *Elements* represent a pool, a set of results which Greek mathematicians would have internalized, and in a sense the internalization would have been oral. It would have relied upon the verbal formulation of propositions. So there is this verbal, oral pool, which one internalizes once and for all, and then uses as a mathematician. Euclid's *Elements* is the first floor of Greek mathematics, upon which the entire second floor must be built, and this is because the engineering principle which makes each floor rely upon the one beneath it is oral, not textual, and is therefore limited. A few very gifted mathematicians internalized the second floor as well, to some extent. Especially, they became very proficient in the *Conics* and developed, on the basis of the *Conics*, a few advanced theories. But this was the absolute limit of Greek mathematics. Greek mathematics was a three-level building, and you cannot build any taller building on an oral basis. Without (among other things) this emphasis on textual learning, science cannot explode exponentially, recursively, as it did since the invention of printing.

I wish to qualify my picture, however. Greek mathematics is of course not wholly oral. In some aspects, it is very written. The very use of diagrams in which *letters* are inscribed is —need I point out?— a feature of a written culture. While the principle by which Euclid was internalized was oral, the dissemination of Euclid was impossible without writing. Indeed, the numbers of mathematicians were always so small, that writing was a necessary medium of communication. The truth is, I think, one should try not to think of the written and the oral as two polar categories. They are two cognitive processes, which may occur side by side. To put this briefly, one may simultaneously write and speak. So I am not saying that Greek mathematics is an oral practice, to the exclusion of its written aspect, with the consequence that this written aspect is illusory. The written aspect is real. However, as is true of Hellenistic culture in general, Greek mathematics has a certain dual nature, both very oral and very written and, to repeat our preceding discussion, both very visual and very verbal. Hellenistic culture is shaped by certain dualities, certain tensions—tensions which were extremely fertile. The text had limits in Hellenistic culture. Or is it us who are limited[20]?

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NOTES

[1] Not to be confused with the technical term used by Morrow, G.R., in his translation of Proclus' Commentary on the First Book of Euclid's *Elements* (Princeton, 1970), to translate *diorismos* (p. 159).

[2] In other words, I assume that the contents of what we know as Euclid's *Elements* were a background to the very discourse of Greek mathematics; I shall return to this in the second part of this article.

[3] The tripartite scheme which is envisaged here —elementary, advanced, and another option, in the "middle"— will be clarified and defended in the following.

[4] The following is a list of some letters which are not fully specified, collected at haphazard; any Greek mathematical work will do. Archimedes' *Spiral Lines* 6, B; *Balancing Planes* I.13, I; *Sphere and Cylinder* I. 10, EBZ; II.4, ΑΒΓΔ; *Quadrature of Parabola* 16, Θ; *Method* 1, B; Euclid's *Elements* I.47, Z; XI.28, H; *Data* 70, Δ; *Optics* 40, E; Autolycus' *Moving Sphere* 10, ZH; Aristarchus' *On the Sizes and Distances of the Sun and the Moon* 7, Δ. I do not claim at all that these are the only letters in the relevant propositions which are not fully specified.

[5] The above paragraph is my only substantial addition to this chapter since its original writing in the 1990s. I ask the reader to take this paragraph as a note of explanation written as a response to a query I had often heard since publishing Netz (1999).

[6] This is not to say that these moments are *primarily* moments of specification. I will argue below that there are no moments of specification PER SE.

[7] Apollonius' *Conics*, (Heiberg 1891–1893, 1: 8, lines 26–27).

[8] Of course, some suspension of judgment is a feature of Greek—and English—syntax. In expressions such as, say, "and let the point A be taken on the line CD, so that AC is equal to AD", one might have said that there are two separate moments of specification, the "on the line CD" specification, and the "AC = AD" specification, so that we have got here a pre-specification. However, I ignore such bogus pre-specifications, where suspension of judgment is clearly demanded by the syntax. My pre-specifications are all genuine. For bogus pre-specifications, the verb *poiein* is a good guide; see e.g. Apollonius' *Conics*, Proposition I.55 (Heiberg 1891–1893, 1: 172, lines 9–12), the letters ZH.

[9] (Heiberg 1883, 4: 248, lines 12–13; 250, line 27 to 252, line 1; 254, line 6; 258, line 4; 260, line 10). The reference is to a specific type of figure, called by Taisbak 'gnomonic' (Taisbak 2003): the practice is lazy but not arbitrary.

[10] This is a qualitative, not a quantitative comment. The number "fifty percent" is highly suggestive, but it does not directly *prove* randomness: for this, we must have a measure of the "background" (for coins, we assume the "background" is 1/2, for dice, we assume it is 1/6)—and this we do not have here.

[11] In Descartes' *Geometry*, two sign systems are set side by side, the geometric and the algebraic, and a coordination between these two systems is required, hence a moment of specification PER SE.

[12] "Orality"—a concept inevitable in the human sciences, and yet impossibly misleading. So to clarify immediately, by "orality" I do not mean illiteracy. I mean a use of language in which the verbal properties of language (which I take to be essentially predicated upon the spoken and the heard) are seen as the central forms of communication.

[13] Some references, when relevant, point out the existence of a useful comment by Eutocius.

[14] This is a standard and useful way to encode—in a written symbolism, alien to the Greek and more oral approach—the assertion "a is to b as c is to d". Greeks would always say the full phrase—with names fuller than our 'a', 'b' etc.—a verbal formula, the non-written equivalent of our written formula.

[15] I leave aside for the moment an important question, namely the shape and diffusion of Euclid's *Elements* in antiquity. It may well be that Apollonius and Archimedes —and even Euclid!— used "*Elements*" different from those we know. "Euclid's *Elements*" is a label, no more. However, it is possible to reconstruct, as it were, the *Elements* used by Apollonius and Archimedes from their uses of basic theorems; and it becomes clear that, by and large, they had to use something very similar indeed, in content, to what we know as Euclid's *Elements*. The redundancies described here give as a rare glimpse into the *formulations* of the *Elements* they used; on the whole, though not always, these seem to be those of Euclid's *Elements* as we know them.

[16] (Heiberg 1883, 1: 60, lines 22–24). This quotation —and this is true in general of such quotations— is not VERBATIM. One is tempted to deduce that this is yet another manifestation of orality; however, it will be prudent to remember, as mentioned above, that our knowledge of the exact text of *Elements* used by Apollonius is conjectural.

[17] (Heiberg 1883, 4: 322, lines 19–20).

[18] I do not wish to explain away any evidence, but it must be remembered that, in general, such explicit references are the most natural glosses, and while many of them could be original, at least some must be later accretions. This is not a pedantic point; it hints at the important reflection, that the context within which Greek mathematical works were copied, Byzantine scholarship, was already a much more text-oriented context than that in which Greek mathematics originated.

[19] And in general not all of that: almost all of book X, for instance, a good quarter of the *Elements*, is a dead-end for most mathematical purposes.

[20] I wish to thank K. Chemla and B. Vitrac for many useful comments on an earlier version of this article.

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