EBERHARD KNOBLOCH

LEIBNIZ AND THE USE OF MANUSCRIPTS: *TEXT AS PROCESS*

ABSTRACT

Text played a crucial role in Leibniz's scientific thinking. This article describes four aspects of this interrelation. First of all, text served the art of invention. Tables, illustrations, and figures enabled Leibniz to find rules, laws, and regularities. This will be shown by means of examples taken from additive number theory and combinatorics. Secondly, text served the purpose of visualization of thoughts, theorems, and proofs. The examples concern the theory of prime numbers and of infinite series. Thirdly, Leibniz used text to fix certain results and insights, to detect errors, to elaborate treatises, and to generalize theories. These practices are illustrated by his studies on symmetric functions, on life annuities, on elimination theory, on conic sections, and on financial mathematics. Finally, Leibniz's texts reflect his monologues or dialogues with fictitious interlocutors, in other words his argumentation.

INTRODUCTION

"Those who know me on the basis of my publications, don't know me" said Leibniz about his scientific production. Indeed, the number of his handwritten items is close to 50,000, whereas the number of his publications is very small. Leibniz's posthumous writings provide a unique insight into his intellectual workshop and verify the above remark in a two-fold way: they not only reveal a tremendous amount of hitherto unknown scientific results and achievements but also show how Leibniz obtained them by thinking in writing. For him thinking was thinking in writing. Text was his instrument of thinking. I would like to describe this intellectual practice by dealing with the following four of its aspects:

1. Text serving the art of invention: the dynamic inductive role of his tables, illustrations, and figures
2. Text serving the visualization of his thoughts, theorems, and proofs
3. Text used to fix insights and to elaborate treatises
4. Text as discussion and argumentation: thinking by writing.

K. Chemla (ed.), History of Science, History of Text, 51–79.

1. TEXT SERVING THE ART OF INVENTION: THE DYNAMIC ROLE OF TABLES, ILLUSTRATIONS, AND FIGURES

Tables, illustrations, and figures play a crucial role in Leibniz's mathematical thinking. They enable him to find rules, laws, and regularities; in other words, they serve the art of invention, sometimes successfully, and sometimes not. Let us first consider two examples concerning additive number theory.

1.1 Tables

Leibniz looks for the number of additive partitions of a natural number into 2, 3, 4 or more terms, whose sum is equal to that natural number.

First example[1]

He aims at a universal rule believing that he will find it by generalizing the case of partitions into three terms. These partitions can be produced by means of partitions into two terms which are ordered according to the magnitude of the first term.

He exemplifies his method by using the number n = 8. There are n − 1 = 7 classes or partitions into two terms: 7 + 1, 6 + 2, 5 + 3, 4 + 4, 3 + 5, 2 + 6, 1 + 7. This first step implies repetition.

The partitions of their first term into two terms lead to the partitions into three terms. From class to class there is an increasing number of repetitions. Only the m-th partition of the m-th class provides a new partition into three terms. The first up to the (m−1)th partition is contained in the first up to the (m−1)th class respectively. Hence if the m-th class has (m−1) partitions, then it contains only repetitions. All m-th classes, which do not contain more than (m−1) partitions must also be cancelled. If one class must be cancelled, then all the following classes must also be cancelled. This is so because the first term continuously becomes smaller and admits fewer partitions.

In our case the first class, 7 + 1, provides 6 + 1 + 1, 5 + 2 + 1, 4 + 3 + 1, the second class, 6 + 2, provides 5 + 1 + 2, 4 + 2 + 2, 3 + 3 + 2. The first partition 5 + 1 + 2 of the second class is a repetition of 5 + 2 + 1 of the first class. Only the second partition 4 + 2 + 2 of the second class provides a new partition into three terms.

The third class, 5 + 3, has two partitions. Hence it contains only repetitions. The same applies to all the following classes.

In this way Leibniz deduces inductively a rule which is at once formulated for an arbitrary number n and an arbitrary number k of terms: Look for all partitions p_m^{k-1}, $1 \leq$ m $\leq$ n − 1, into $(k-1)$ terms of the numbers (n − 1) down to 1, but subtract from P_m^{k-1} at a time the number of the preceding numbers, that is, n − $(m+1)$.

Let us look again for all partitions of n = 8 into k = 3 terms. We must look for all partitions of
m = 7 into k −1 = 2 terms: we get 3 partitions;
m = 6 into two terms: we get 3 partitions;
m = 5 into two terms: we get 2 partitions.
We need not go further because we must subtract $8-(7+1)=0$ from 3, $8-(6+1)=1$ from 3, and $8-(5+1)=2$ from 2.

Figure 1. LH XXXV 12,1 sheet 15r; Leibniz 1976a, no. 40.

The rule itself has been cut off from the sheet of paper, that is, Leibniz separated the result from its origin, so that it is no longer contained in illustration 1 but on another sheet of paper.[2] The rule is indeed valid for partitions into three terms, but not for partitions into arbitrarily many terms:

$$P_8^3 = P_7^2 + \left(P_6^2 - 1\right) + \left(P_5^2 - 2\right) = 3 + (3-1) + (2-2) = 5.$$

But $P_8^4 = 5$, while the false recursion formula leads to

$$P_8^4 = P_7^3 + \left(P_6^3 - 1\right) = 4 + (3 - 1) = 6$$

In other words there are five partitions of n = 8 into k = 4 terms, while the formula provides the false result of six.

His added remark is very instructive and revealing: "Quaerere summam discerptionum sine Tabula, id ego aliis relinquo," "I leave it to others to look for partitions without table."[3]

Second example[4]

This time his aim is a recursion formula. He writes down all partitions of the numbers 1–12 into one, two, three, and four terms, whereby the terms are ordered according to their magnitude. Partitions which differ only by the order of the terms are counted just once. His list is distorted by some errors, so that it is all the more difficult for him to find any formation rule or regularity. Hence his first reaction is: "Ex his apparet progressionem Numeri discerptionum esse valde perplexam." "From this (that is, from this table of partition) it is clear that the progression of the numbers of partitions is very confused."[5]

However, by analyzing the table, he realizes that every number of partitions of a given number v into e terms is composed of the numbers of partitions of numbers smaller than v into (e − 1) terms. From a certain number onward partitions must be excluded because of the principle of order.

For example, if we are looking for the number of partitions of v = 10 into four terms, then we must add to the first term 7, 6, 5, 4 all partitions into three terms of the numbers x = 3, 4, 5, 6. If x = 7, then the partitions into three terms 5 + 1 + 1, 4 + 2 + 1 are useless, and that obviously because the sums of the last two terms 1 + 1, 2 + 1 (the partitions into two terms) behind the first terms 5 or 4 are too small: Leibniz tries to calculate the number of partitions into e terms by means of the number of partitions into (e − 1) and (e − 2) terms. In this way he deduces a recursion formula. But this formula is false.[6] He creates a new notation $\overline{\text{v}\,\boxed{\bar{\text{e}}\text{ sect.}}}$ for the number of partitions of the number v into e terms and remarks: "Mirabilia calculandi specimina", "Wonderful specimens of calculation".[7]

In terms of this notation his false equation reads as follows:

$$\overline{\text{v}\,\boxed{\bar{\text{e}}\text{ sect.}}} = \text{sum. } \overline{\text{v}-\text{x}\,\boxed{\overline{\text{e}-1}\text{ sect.}}}$$
$$-\text{sum. } \overline{\text{v}-2\text{x}-\text{y}\,\boxed{\overline{\text{e}-1}\text{ sect.}}}$$

Here "sum" means the sum of the numbers of all relevant partitions of v − x into e-1 terms or of v − 2x − y into e − 2 terms, respectively.

Further manipulations lead to an even more clumsy equation.

Leibniz ends by saying:

> "Sufficiet talia ad calculum revocasse ... Diligentius considerandum." "It suffices to have reduced such things to calculation ... this must be considered more diligently"[8].

Perhaps he suspected that something was wrong with his results. In other words, tables are a tool which should be replaced by rules of calculation. The same applies to the title of

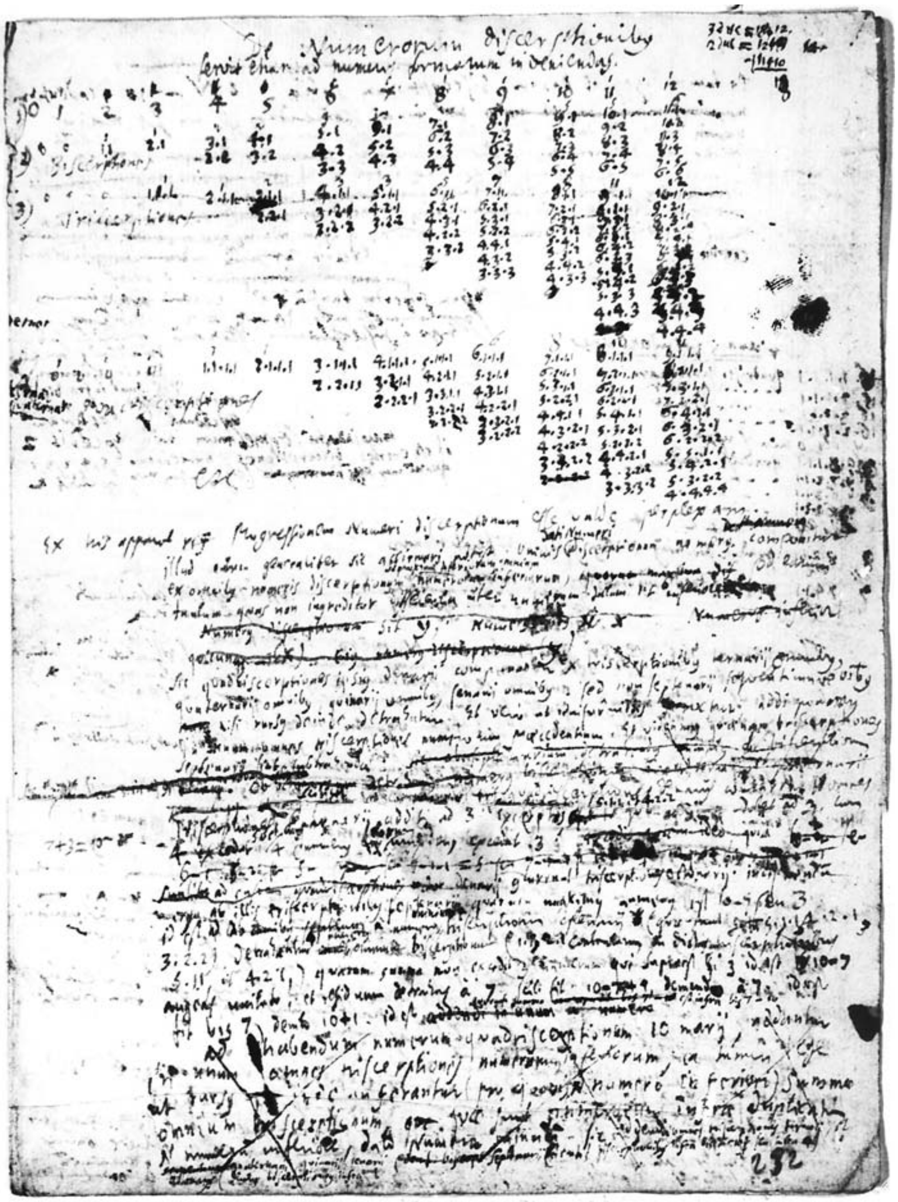


Figure 2. LH XXXV 12,1 sheet 232r; Leibniz 1976a, no. 46.

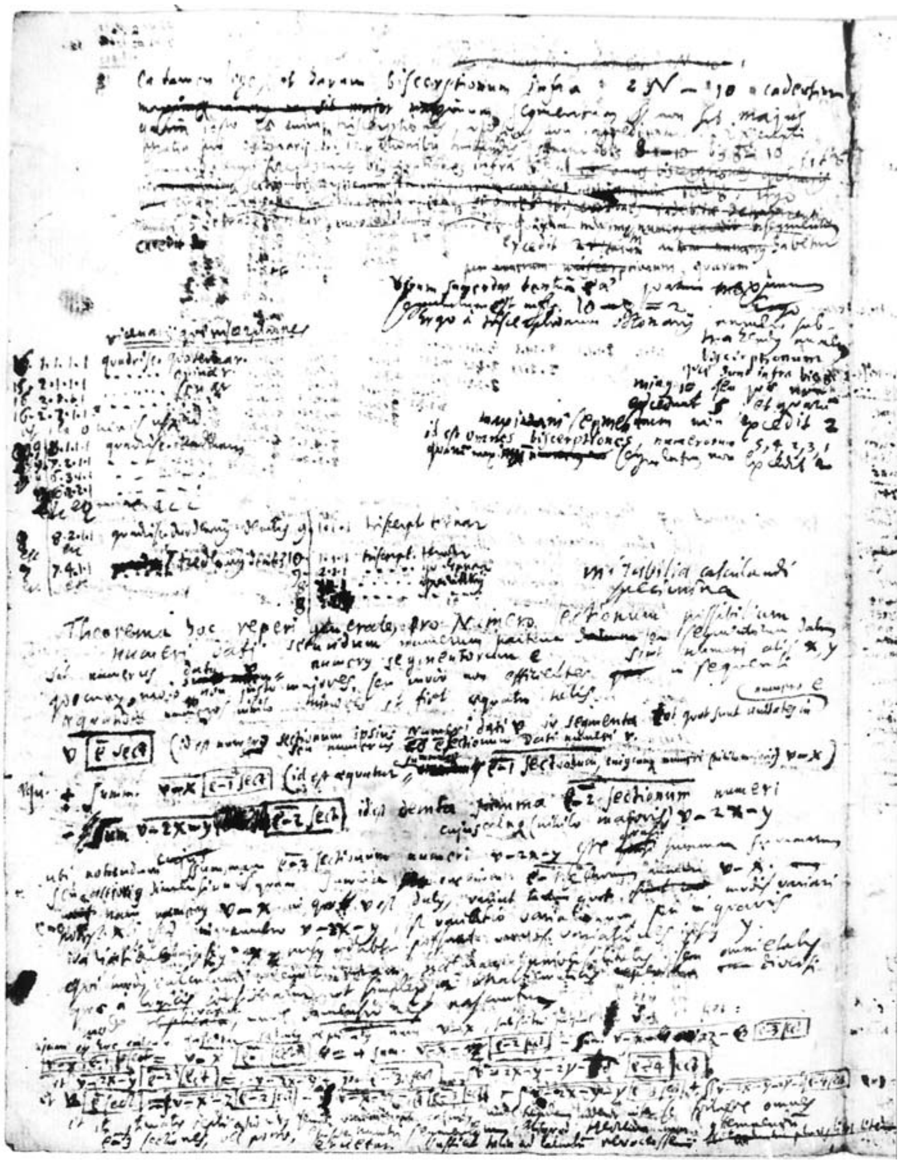


Figure 3. LH XXXV 12, 1 sheet 232v; Leibniz 1976a, no. 46.

his most important treatise on infinitesimal geometry written in 1675/76: “Arithmetical quadrature of the circle, the ellipse, and the hyperbola, which results in a trigonometry without tables.”[9] “Trigonometry without tables” means the use of certain infinite series. But while tables were necessary conditions for the text, the rule, or the recursion formula to come into being with regard to the number theoretical manuscript, tables play another role in this treatise. We will comment on this role somewhat later.

1.2. The role of illustrations or figures

The role of illustrations or figures in Leibniz's mathematical thinking becomes evident when he calculates the powers of polynomials such as $(a + b + c)^4$.[10] The result consists of terms such as a^4, a^3b, a^2bc etc. multiplied by special coefficients. These coefficients are, as he knows, the numbers of permutations of such expressions. For example, there are 12 different transpositions of the term a^2bc.

Third example[11]

Leibniz tries to illustrate these 12 permutations by means of a scheme, a figure. At first he draws four unsuitable figures. They are crossed out, because they do not reveal the symmetries underlying the problem.

The most satisfying figure provides a completely symmetrical figure which, as he says, makes evident to the observer, that there are as many possible transpositions as there are numbered ways of producing the same term a^2bc by means of which we can come from the supreme points to the lowest, touching those in between.

In other words, the completeness and symmetry of the figure enables him to check and to illustrate the solution of the permutation problem.

2. TEXT SERVING THE VISUALIZATION OF THOUGHTS, THEOREMS, AND PROOFS

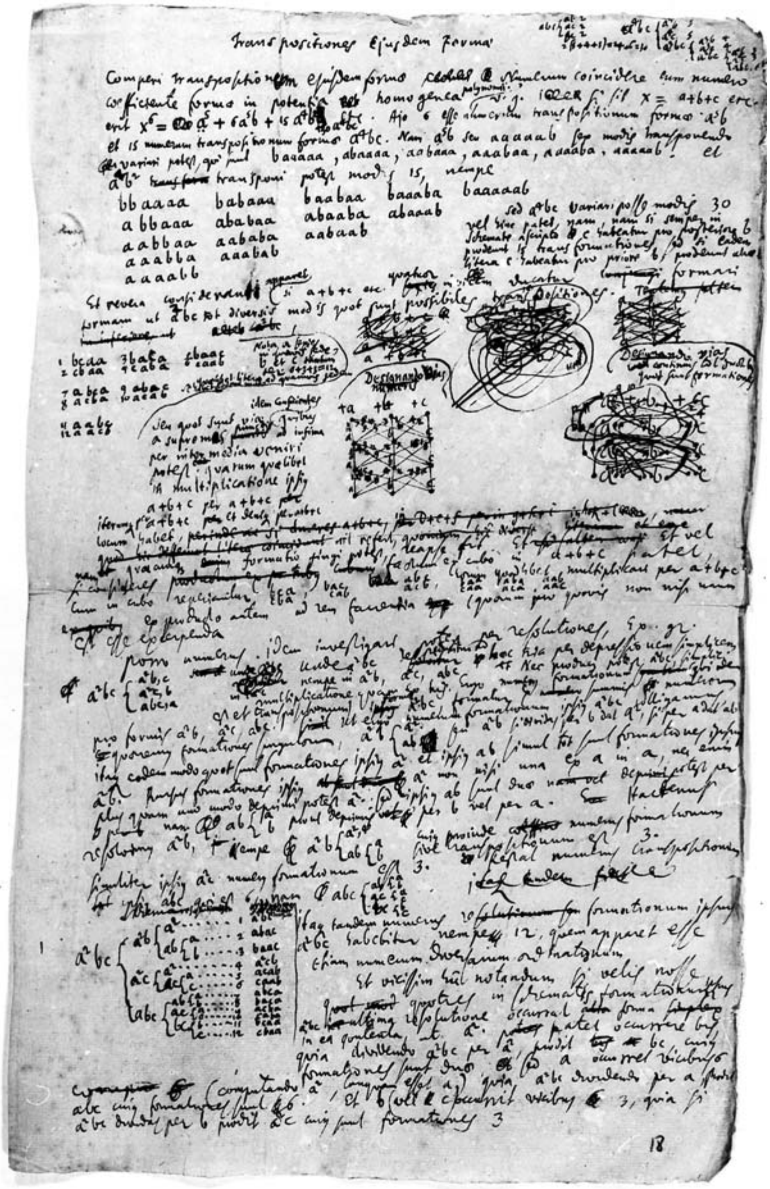
Thoughts and theorems cannot be heard or seen. But they can be made visible by certain representations. To this end, Leibniz uses special characters which lead to certain "apparitions".

2.1. Visualization as a tool for the art of invention

The following example is meant to demonstrate how such a visualization made possible certain mathematical insights in the true sense of the word. I would like to discuss Leibniz's inquiry into the law of distribution of prime numbers.[12] He aims at making visible the relations between prime and composite numbers by elaborating a figure. Thus, in this case, we might say, that visualization is a kind of geometrization of thoughts. He elaborates suitable figures in order to detect the law of distribution.

The first manuscript is entitled "Figura numerorum ordine dispositorum et punctatorum ut appareant qui multipli qui primitivi", "A figure of numbers arranged and punctuated in an order so that it becomes clear which are multiples and which are prime numbers." Leibniz constructs the figure as follows.

(a) He draws arbitrarily many dotted horizontal lines under the sequence of natural numbers. The first line begins below 2, the second below 3, the n-th below $(n + 1)$. In the first line every dot corresponds to a multiple of two, in the second to a multiple of three, and in the n-th line to a multiple of $(n + 1)$.


Transpositiones ejusdem formae

bbaaaa babaaa baabaa baaaba baaaab

abbaaa ababaa abaaba abaaab

aabbaa aababa aabaab

aaabba aaabab

aaaabb

Figure 4. LH XXXV 4,5 sheet 18r; Leibniz 1976a, no. 56.

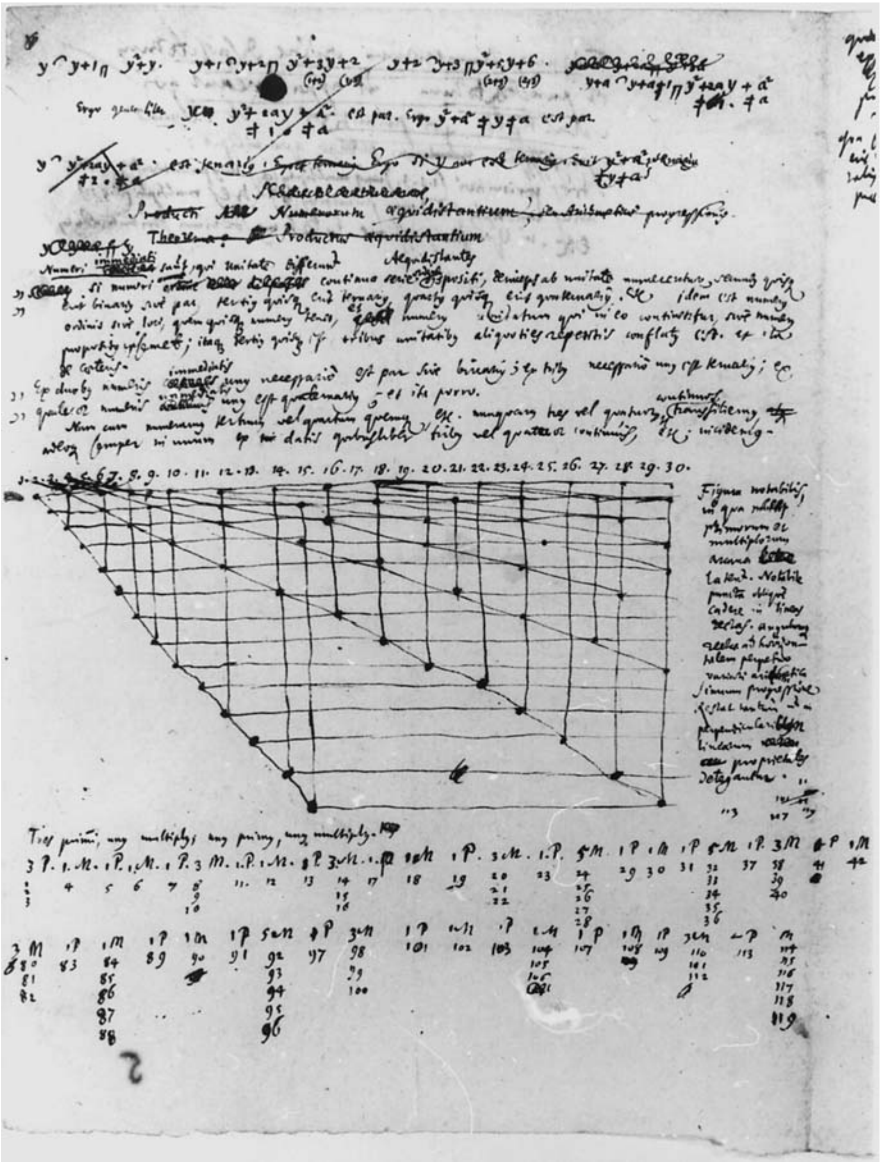


Figure 5a. LH XXXV 4,17 sheet 5v-6r; Leibniz 1990, no. 87.

(b) He draws inclined lines which connect the multiples of one, two, three, four, etc. belonging to different horizontal lines: Let us assume that a dot lies on the n-th inclined line and the k-th horizontal line. Then we have to take a step of n units in order to find the next dot on it which lies at the same time on the (k + 1)-th horizontal line.

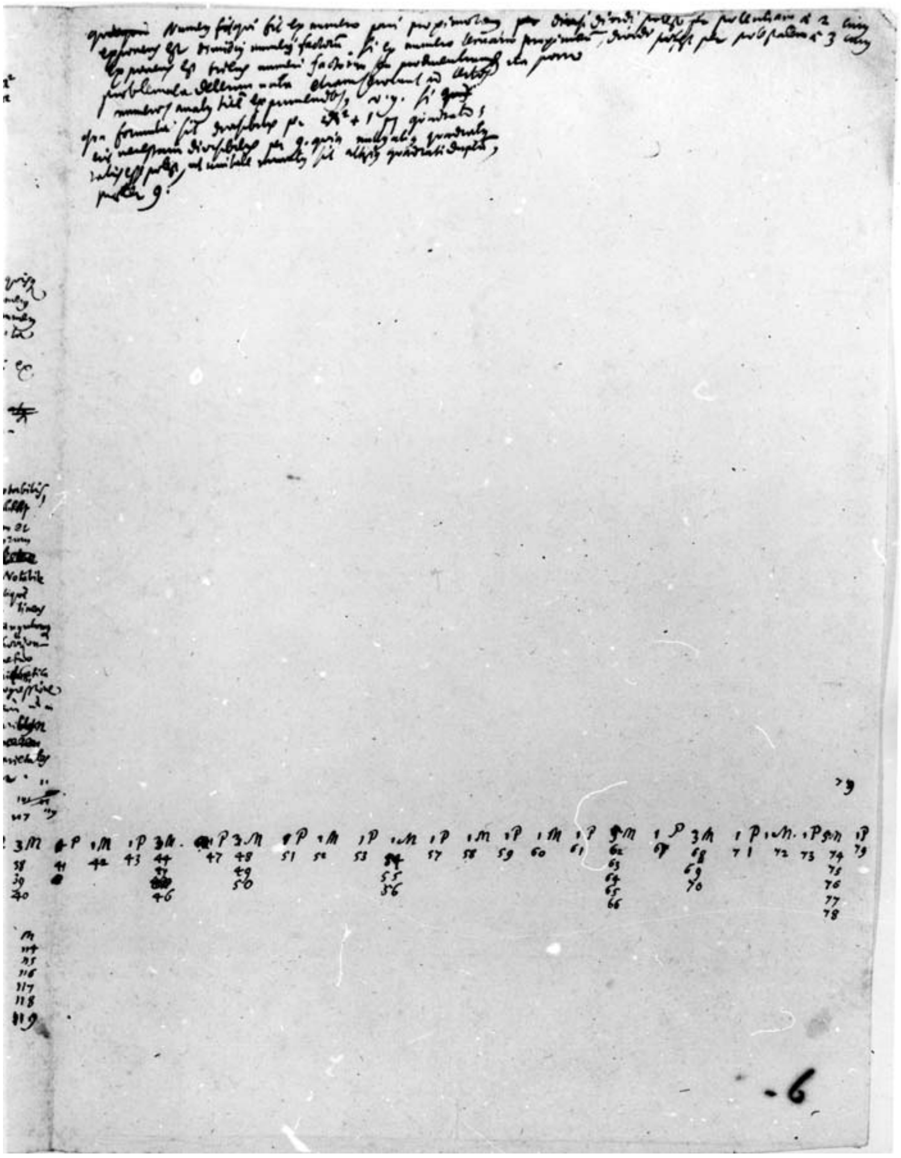


Figure 5b. (*Continued*)

(c) He draws perpendicular lines which connect dots lying one beneath the other and he comments: “Figura notabilis, in qua primorum et multiplorum arcana latent. . . . Restat tantum ut in perpendicularibus linearum proprietates detegantur”, “A notable figure, in which the secrets of primes and multiples lie hidden. It only remains to detect the properties of the lines on the perpendiculars.”[13]

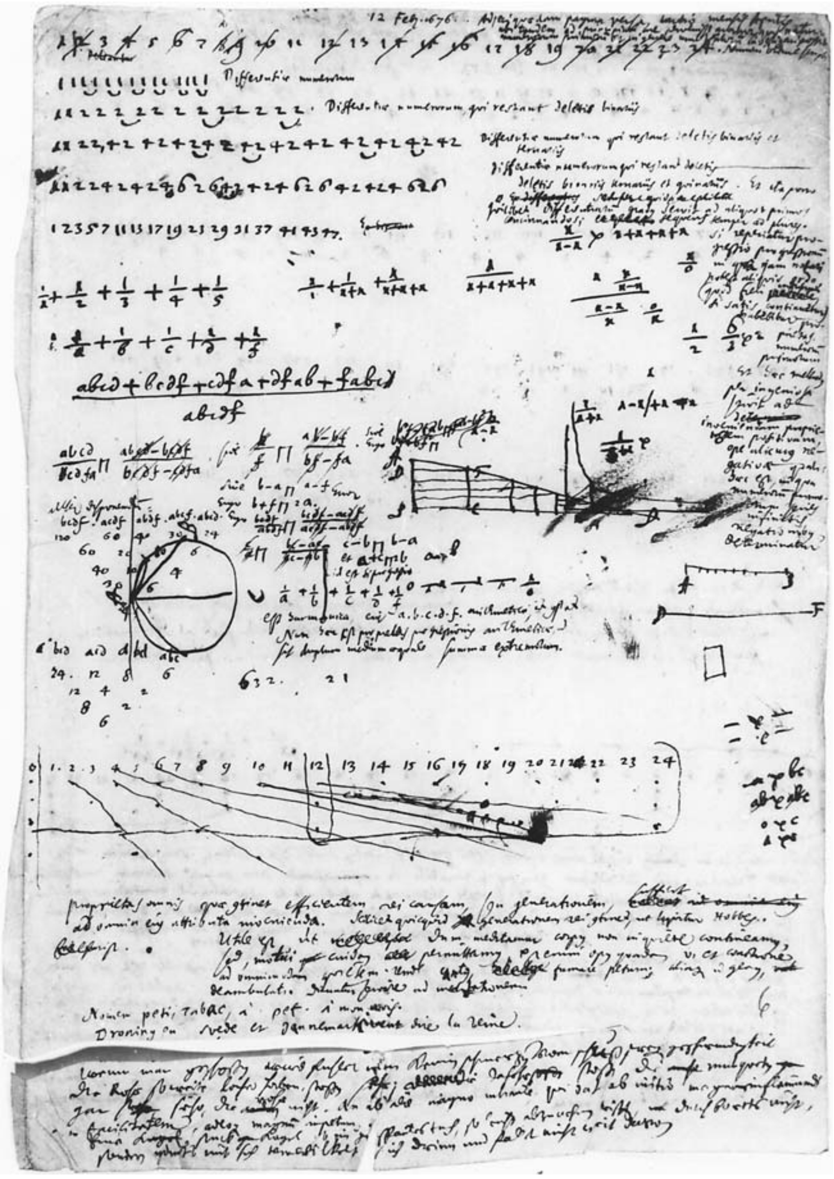


Figure 6. LH XXXV 3B, 15 sheet 6r; Leibniz 1990, no. 92.

Some weeks later he uses this approach once more.

This time only the inclined lines, together with their dots, are drawn. At the very top of this page he remarks that the addition of zero enabled him to see something miraculous, which he did not see before. By this only addition it shines forth at once, at first sight.[14]

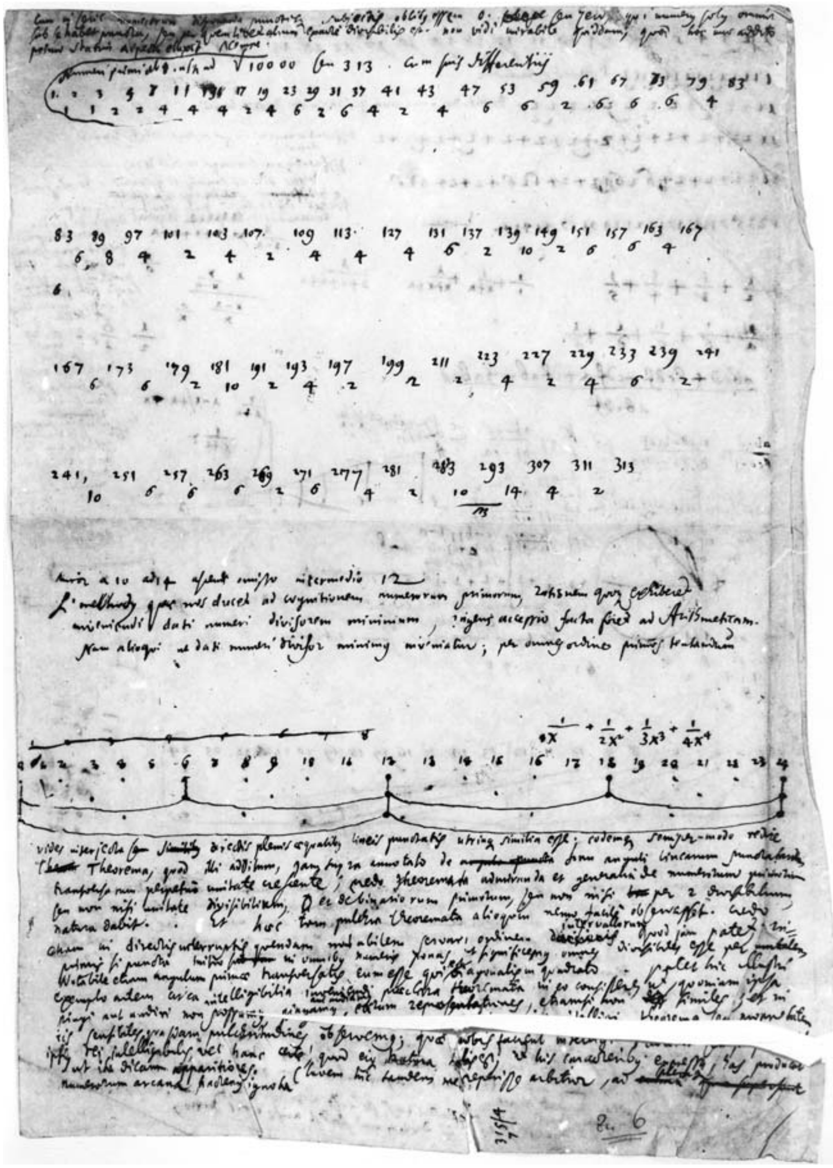


Figure 7. LH XXXV 3B, 15 sheet 6v, Leibniz 1990, no. 92.

He has in mind two observations (one true and one false): We see that the spaces between drawn perpendicular lines which are equal to the dotted perpendicular lines are similar on both sides. (This is true.) This series of the angles of inclinations of the inclined lines form an arithmetical progression. (This is false.) And he concludes: "Et haec tam pulchra theoremata alioquin nemo facile observasset. . . . Patet hic illustri exemplo artem circa intelligibilia inveniendi praeclara theoremata in eo consistere, ut, quoniam ipsa pingi aut audiri non possunt, pingamus aut audiamus earum repraesentationes . . . et in iis sensibiles quasdam pulchritudines observemus; quae nobis facient intelligi theorema seu proprietatem ipsius rei intelligibilis, vel hanc certe, quod eius natura talis est, ut his caracteribus expressa, has producat ut itam dicam apparitiones. Clavem hic tandem me reperisse arbitror, ad pleraque numerorum arcana, hactenus ignota." "And otherwise, nobody would have easily observed such beautiful theorems. It is evident here by an illustrious example that the art of inventing famous theorems, regarding the intelligible, consists in painting or hearing their representations, because they themselves cannot be painted or heard. . . . And in observing some sensible beauties in them. They will enable us to understand a theorem or the property of the intelligible thing, or at least that which is of such a nature that it produces, so to say, these apparitions if it is expressed by these characters. I believe, that I finally found here the key to most of the hitherto unknown secrets of numbers."[15]

Leibniz says, that we observe theorems. It goes without saying that he overestimated the capacity, the efficiency, of such geometrical means, of such devices. But we note once more that visualization was a necessary condition for Leibniz's text to come into being. It is a kind of self-generation of the text.

2.2. Visualization as a tool for illustrating known intellectual relations

Let us consider proposition 26 of his treatise mentioned above on the arithmetical quadrature of conic sections, which was written in 1675/76.[16] The sum s of a geometrical series which decreases to infinity is to the first term a as the first term is to the difference between the first and the second term aq, q being the quotient between two terms following one another:

$$s : a = a : (a - a.q) \text{ or } s = a.1/1\text{-}q$$

Leibniz uses a figure in order to prove this proposition by means of two similar triangles:

He says explicitly that he chose that proof among the many available which puts the problem before the eyes in a certain way ("quae rem quodammodo oculis subicit"). In other words, illustrative proofs are easier to understand than other kinds of proofs.

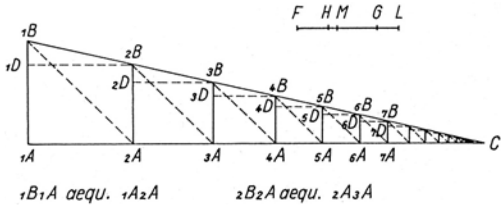


Figure 8. Leibniz 1993, 72.

3. TEXT AS FIXATION OF INSIGHTS AND ELABORATION OF TREATISES

I would like to distinguish six aspects:

3.1. The static, fixing role of tables

While the dynamic role of tables, discussed in section 1, consists in serving the art of invention, the static role of tables serves to avoid repeated calculations at future times. According to Leibniz, such a table is the best way of fixing the results which have been calculated once and for all. The tables can be used to find a needed value. While the purpose of the first use of tables, as we were told, is to reduce tables to calculations, the purpose of the second use of tables is to replace calculations by tables.

As a consequence, Leibniz elaborates, for example, multiplication tables for forms or symmetric functions[17].

Illustration 9 consists of three matrices (in modern terminology): the single terms a_i of the first column are multiplied by the single terms b_k of the first row. The element c_{ik} of the matrix is the product of a_i times b_k. Only some calculations of the products have been carried out.

Let us consider two examples:

First matrix:

$$a_1 = lmn,\ b_3 = l^3m^2.\ \text{Hence}\ a_1.b_3 = c_{13} = l^4m^3n$$

Third matrix:

$$a_9 = l^2m^2n,\ b_3 = l^4m^3n.\ \text{Hence}\ a_9.b_3 = c_{93} = l^6m^5n^2$$

If these terms are considered as terms which designate a symmetric function, then such multiplications become far more difficult.[18] In order to avoid unnecessary multiplications of terms we must know the number of terms which constitute a particular symmetric function. This number depends on the number of variables in the function. If, say, there are four variables l, m, n, o, then the function lm reads:
lm+ln+lo+mn+mn+no, giving six terms. Obviously, this is just a combinatorial question. Leibniz elaborated a table for the numbers of terms in an arbitrary form:

The different forms are enumerated in the first column. The numbers behind a form are the increasing numbers of terms of this form if there are 1, 2, 3 etc. variables.

Example:

Let there be three variables l, m, n and let l^2m be the form. We get:

$$l^2m = l^2m + lm^2 + l^2n + ln^2 + m^2n + mn^2,$$

that is, the form consists of six terms. We find this result in Leibniz's table (third row, third column).

These are the same kinds of tables about which Leibniz speaks in the title of his treatise on conic sections quoted earlier.

While this first aspect applied to the fixing of certain, reliable results, only such fixings of insights by means of schemes and figures enabled Leibniz to detect errors (aspect 3.2),

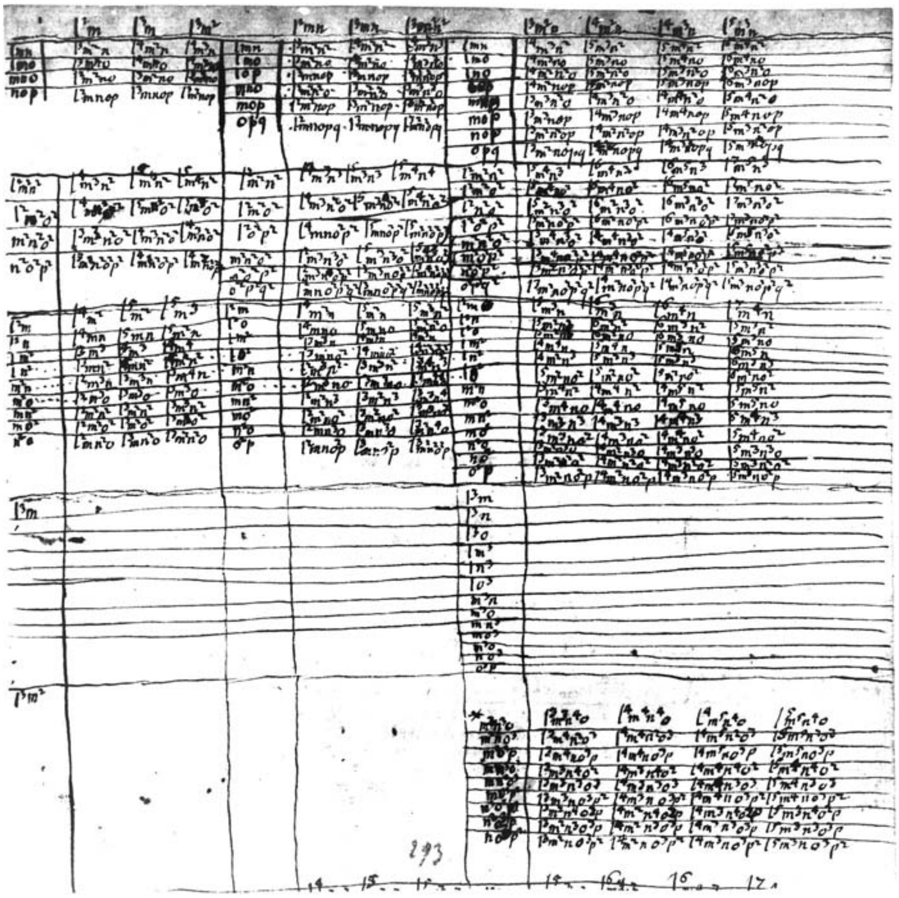


Figure 9. LH XXXV 14,1 sheet 293v; Leibniz 1976a, no. 24.

and to modify and improve chosen formulations. This implies a potentially infinite process (aspects 3.3 up to 3.6). I discuss the following examples in order to explain these five other aspects.

3.2. Clarifications or the detection of errors.[19]

Our example is related to Leibniz's inquiries into life annuities. He takes four living beings. They are supposed to belong to a species which dies after four years at the latest. He looks for the presumed life span of an arbitrary pair of such beings. It is obvious, he says, that they can be equally easily combined either when both live less than a year, or when one lives less than a year, the other 1, 2, or 3 years, or when one lives 1 year, the other one 2 or 3 years, as if we had two tetrahedra. He concludes: "Eruntque paria

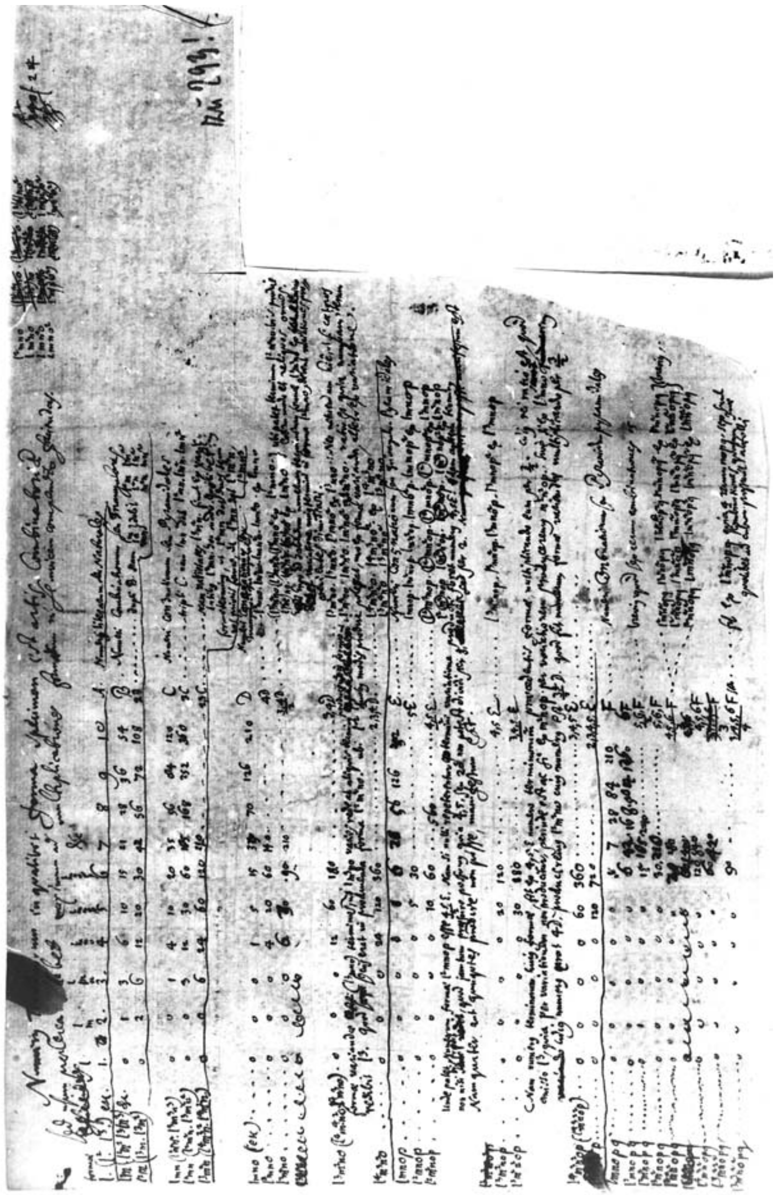


Figure 10. LH XXXV 14,1 sheet 293r; Knobloch 1973, folding sheet between pp. 248 and 249.

possibilia numero 16, ut apparet in schemate adjecto", "There will be 16 possible pairs, as becomes evident from the adjoined scheme"[20]:

Scheme of pairs

0.0	0.1	0.2	0.3
(0)	(1)	(2)	(3)
1.0	1.1	1.2	1.3
(1)	(1)	(2)	(3)
2.0	2.1	2.2	2.3
(2)	(2)	(2)	(3)
3.0	3.1	3.2	3.3
(3)	(3)	(3)	(3)

The life span of an association is the upper limit of the individual life spans of its members: an association survives until the death of its last member. In the foregoing example the associations consist of two persons (unarranged pairs) chosen at random in a group of four persons A, B, C, D. They are characterized by the individual life spans. According to the assumption, there are four such individual life spans, namely 0, 1, 2, or 3 years.

Leibniz considers arranged pairs: 2.1 means that the life span of the first person is two years, and the life span of the second person is one year. 1.2 means the opposite. In either case the life span of the pair is two years (2 is the upper limit of 1 and 2).

The numbers in brackets give the maximal life span of a pair. Leibniz's first explanation reads as follows: Unequal pairs (like 0.1, 1.2, etc.) occur always twice, because every number of years of life (0, 1, 2, 3) can be combined with every possible number of this kind.

His judgment is based on an analogy with a game with two unbiased six-faced dice where there are 36 possible combinations. If we look for the probability of throwing two numbers whose sum is two or three, then we must distinguish between (1, 2) and (2, 1). The probability of throwing the sum 3 is $2/36 = 1/18$, while the probability of throwing 2 is $1/36$, that is, it is twice as large. After finishing this argument Leibniz concludes that it cannot be transferred to the case of life spans, i.e., that he has detected an error. Hence he abandons the scheme of 16 pairs and replaces it by another triangular scheme adding another explanation:

Scheme of pairs

0.0	0.1	0.2	0.3
(0)	(1)	(2)	(3)
	1.1	1.2	1.3
	(1)	(2)	(3)
		2.2	2.3
		(2)	(3)
			3.3
			(3)

"Sophisma hic notabile", "Here is a notable sophism", he says, and continues at first: "Et facilitas tantum", "only the facility", still adhering to the language of probability: facility is the degree of probability. Then he corrects to "Et possibilitas tantum hedrarum fuit consideranda aeque enim facile cadere potuere hae duae, quam aliae duae", "Only the possibility of the faces had to be considered, because these two faces could be cast as easily as two other faces".[21]

Indeed, insofar as the maximal life span of a pair is concerned, it does not matter whether we consider the possible pair (n, m) or (m, n)—m and n can be unequal—because the maximal life span is an upper limit. Obviously, fixing a thought enabled Leibniz to modify this solution. But it matters, of course, if equally possible outcomes of two cast tetrahedra are considered. His "improved" explanation leads astray.[22]

3.3. Condensation—augmentation

For Leibniz, editing a text meant canceling or adding passages. Some of the passages involved were long. We will call such a transition from one status of the text to another a condensation or an augmentation.

Leibniz's treatise on conic sections might serve as an example.[23] Leibniz explains the limited results and achievements of his predecessors Fermat and Wallis. He mentions Roberval, who told him of their work. After lengthy, but very interesting, historical explanations he turns to the quadrature of the circle. While he at first continues the text by formulating proposition 27, he later on prefers to insert two corollaries. After reading once more his manuscript, he crosses out the whole historical passage and the two inserted corollaries as well. Thus he conceals valuable historical information. This example, however, must not lead to the conclusion that Leibniz usually concealed from the reader historical information about contemporary mathematical studies. On the contrary, somewhat later, in the same manuscript, he adds a very long and interesting scholium in which he explains Nicolaus Mercator's method of division and then says: "Sed haec clarissimum virum Isaacum Neutonum ingeniose ac feliciter prosecutum, nuper accepi, a quo praeclara multa theoremata expectari possunt", "But I learned recently that the most famous man Isaac Newton ingeniously and successfully achieved that, [Newton], from whom many excellent theorems can be expected."[24]

3.4. Generalization or a new treatment from a higher standpoint

Leibniz studied the elimination problem, that is, he tried to eliminate the common variable x from two algebraic equations, for example, cubic equations:[25]

$$\text{equ. 1} \quad 10x^3 + 11x^2 + 12x + 13 = 0$$
$$\text{equ. 2} \quad 20x^3 + 21x^2 + 22x + 23 = 0$$

His fictive numerical coefficients denote not natural numbers but double-indexed indeterminate coefficients. The first figures, 1 or 2, refer to the equations. The second figures, 0, 1, 2, 3, when added to the exponent of x, form a constant sum. In the case of the cubic equation the sum in question is 3.

In order to eliminate step by step the common variable x, Leibniz multiplies equation 2 by the first coefficient of equation 1, that is, by 10, and equation 1 by the first coefficient of equation 2, that is, by 20. Then the multiplied equation 1 is subtracted from the multiplied equation 2. Thereby the highest power of x is eliminated. The result is

$$\begin{gathered} 10\cdot 21x^2 + 10\cdot 22x + 10\cdot 23 \\ -11\cdot 20X^2 - 12\cdot 20x - 13\cdot 20 = 0 \end{gathered}$$

or

$$(20)x^2 + (21)x + (22) = 0$$

The new coefficients (20), (21), (22) can be described by means of the old coefficients, the old coefficients are, as Leibniz said, "unfolded" (explicati):

$$(20) = 10\cdot 21 - 11\cdot 20, (21) = 10\cdot 22 - 12\cdot 20, (22) = 10\cdot 23 - 13\cdot 20$$

This "unfolding" can be repeated again and again. According to the rule

$$2n = 10\cdot 2(n+1) - 1(n+1)\cdot 20, 0 \leq n \leq 8$$

Leibniz does not use the letter n nor does he discuss the question of what will happen to the right numerals if the unfolding should be repeated more often than nine times, or if the degree of the original equation is higher than 9. The mechanical unfolding (of the old) original coefficient 20 leads to a dichotomic table:

$$\begin{array}{cc} \multicolumn{2}{c}{20} \\ \overbrace{10\cdot 21 \qquad\qquad\qquad\qquad\qquad\qquad -11\cdot 20}^{} & \\ \overbrace{10\cdot 22 - 12\cdot 20} & \overbrace{10\cdot 21 - 11\cdot 20} \\ \overbrace{10\cdot 23 - 13\cdot 20}\ \overbrace{10\cdot 21 - 11\cdot 20} & \overbrace{10\cdot 22 - 12\cdot 20}\ \overbrace{10\cdot 21 - 11\cdot 20} \end{array}$$

etc.

If these substitutions are really carried out, then the third line reads:

$$\begin{aligned} &10^{3}\cdot 23 - 10^{2}\cdot 13\cdot 20 - 10^{2}\cdot 12\cdot 21 + 10\cdot 11\cdot 12\cdot 20 - 10^{2}\cdot 11\cdot 22 + 10\cdot 11\cdot 12\cdot 20 \\ &\quad + 10\cdot 11^{2}\cdot 21 - 11^{3}\cdot 20 \end{aligned}$$

It is this form of the dichotomic table which is calculated in a manuscript dating from 1693/94.[26]

The factors of an expression are vertically written one below the other. There is an obvious similarity between this table and a genealogical tree. Therefore Leibniz calls the terms of the tree father, first-born, second-born, etc. He begins to describe the rule of unfolding the terms by looking at his table. Then he cancels most of his explanations saying: "Sed rem omnem altius repetendo exponere placet". "But it is good to explain the whole problem by going further back." That is, he decides to elaborate a more general, systematical unfolding theory, based on the use of fictive numerical coefficients and comprising 34 rules (canones).

Let us consider a numerical example. In the example given above 20 is the "father", 10.21 is his first-born, −11.20 his second born. The left number of the first-born is 10.

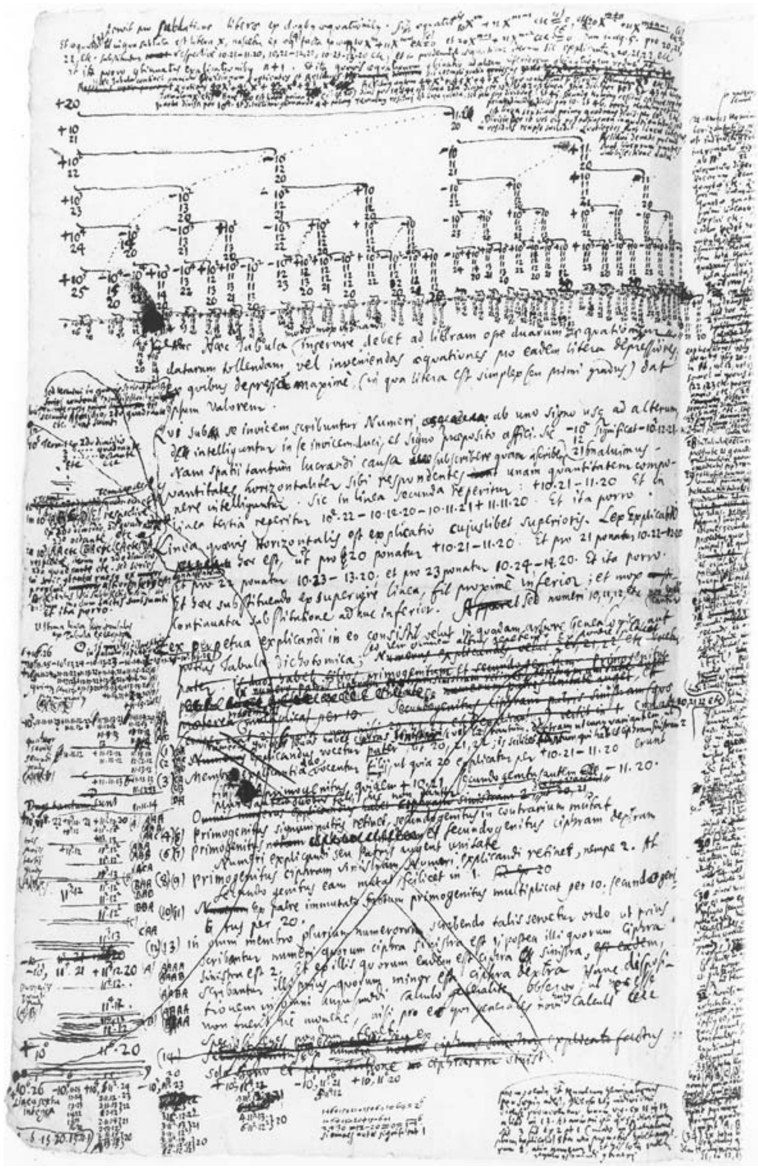


Figure 11. LH XXXV 14,1 sheet 167v; Knobloch 1974.

The right number 21 of the first-born is the number of the father enlarged by one, that is 21. The second born −11.20 permutes the right figures 0, 1 of the numbers of his brother.

The rules explain how the coefficients are formed, how to use the table in order to eliminate a common variable from two algebraic equations, the occurrence of the signs

Figure 12. LH XXXV 14,1 sheet 168r; Knobloch 1974.

of the terms, the relations between terms of the same and of different lines, etc. The many consequences make him continue the text in the left margin of the page. It becomes more and more difficult to read this text[27]. In a word, only after writing down his first ideas was Leibniz able to generalize and to systematize them.

3.5. Fixing of insights

The text reflects the process of recognition and of finding, and especially the elaboration of figures. In the first long version of his treatise on the arithmetical quadrature of the circle Leibniz outlines nine rough drafts of the figure relating to the parabola.[28] Again and again he is not content with his drawing. Sometimes it is the size of the drawing, and sometimes the curvatures, and sometimes something else fails to satisfy him.

He formulates the following problem: "Figuram analyticam simplicem quamlibet quadrare". "To square an arbitrary simple analytical figure." He calls a figure "simple"

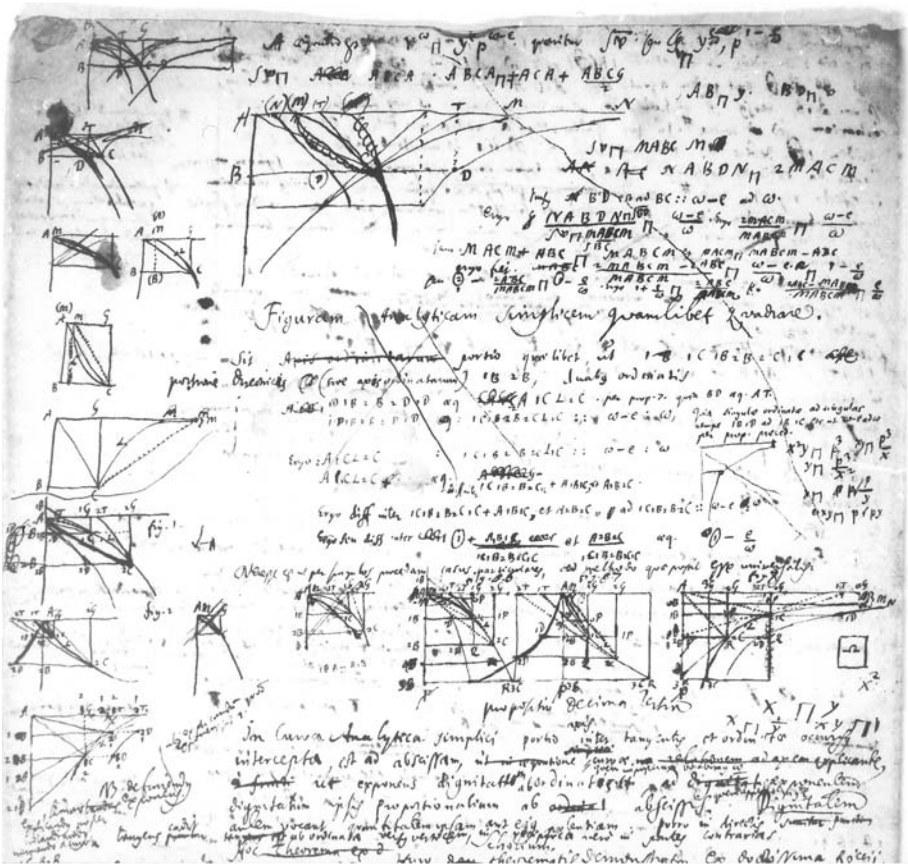


Figure 13. LH XXXV 2,1 sheet 155v.

if the relation between the ordinates and the abscissas can be expressed by an equation consisting of only two terms.

After introducing some notations relating to a (necessarily specialized) geometrical figure, he interrupts his considerations and concludes: "Necesse est ut per singulos procedamus casus particulares, sed methodo quae possit esse universalis", "It is necessary that we proceed by particular cases, but by a method which could be universal." This is at the same time an example of the dialogical or monological. At any rate, it exemplifies the discursive character of his text. Now he distinguishes between figures for the parabolic, elliptic, and hyperbolic cases, but formulates the general theorem 13 which relates to a simple analytical curve in general.

What is important here is Leibniz's methodological insight and scientific intention. He must consider particular cases but he is interested in a universal method. The fruitfulness of his method of indivisibles, in his interpretation of infinitely small quantities, led him to elaborate a treatise on the arithmetical quadrature not only of the circle but of conic sections. Consequently, we must say a few words about his elaboration of treatises.

3.6. The elaboration of treatises: Text production as a process

Whenever Leibniz published an important article, we can be sure that he elaborated many, more or less different, drafts. Whether he actually published one of them is another question. I would like to mention two illustrative examples:

(1) "Treatise on the arithmetical quadrature of the circle, the ellipse and hyperbola which results in a trigonometry without tables." We touched on his treatise several times. It was first published in 1993. The introduction to its edition describes the history of its origins.[29] By "arithmetical quadrature" Leibniz meant convergent infinite series of rational numbers. His arithmetical quadrature of the circle was his alternating series $1 - \frac{1}{3} + \ldots$ for $\frac{\pi}{4}$, of which he informed Christiaan Huygens in the summer of 1674. In October 1674 he worked out for him a first version of his treatise.[30] By the end of 1675 he prepared two other versions for Jean Paul de La Roque[31] and Jean Gallois but did not send them. A further, hitherto unpublished, extensive version "Arithmetical quadrature of the circle" was elaborated by Leibniz in 1675. By and large, its content is identical with that of the first 25 theorems of the last version, published in 1993. Originally, Leibniz wanted to write only a comprehensive treatise about the quadrature of the circle. But the generality of his proofs and the fecundity of his principles induced him to extend the treatise to all conic sections and to the logarithmic curve.

(2) "Meditatio juridico-mathematica de interusurio simplice", "A legal-mathematical memoir on the simple rebate". This memoir appeared in 1683[32].

We know of five preliminary versions which differ considerably in style, content, length, and title. None of them has been published. Leibniz published a completely altered version. The manuscript used by the printer does not exist any longer. What exists is an untitled manuscript whose text is nearly identical with that of the printed article.[33] The titles of these five preliminary versions read as follows:[34]

(2.1) "Meditatio juridico-mathematica quanto plus petere intelligatur qui plus tempore petit seu de resegmento anticipationis, vulgo Rabat", "Legal-mathematical memoir:

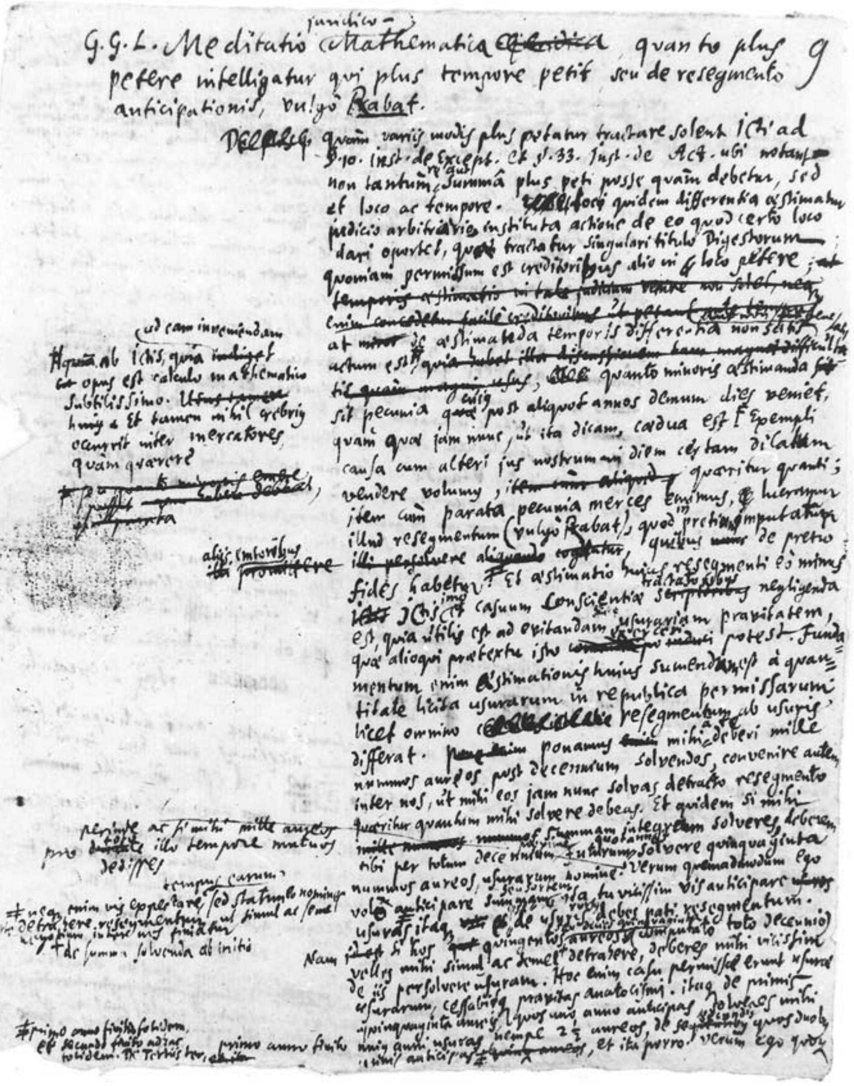


Figure 14. LH II 5,1 sheet 9r; Leibniz 2000, no. II.8.

By how much is somebody understood to claim more who claims more by time or on the allowance of advanced payment, popularly called rebate."[35] "More by time" means "prematurely."

(2.2) "Meditatio juridico-mathematica quanto plus petere intelligatur qui plus tempore petit seu de Resegmento Anticipationis vulgo Rabat, sive interusurio", "Legal-mathematical memoir: By how much is somebody understood to claim more who claims more by time or on the allowance of advanced payment, popularly called rebate, or on the interest of an intervening period."[36]

(2.3) "Meditatio juridico-mathematica de Interusurio sive Resegmento anticipationis vulgo Rabat", "Legal-mathematical memoir on the interest of an intervening period or on the allowance of advanced payment, popularly called rebate."[37]

(2.4) "Meditatio juridico-mathematica de interusurio seu de Resegmento Anticipationis vulgo Rabat, de aestimando jure percipiendi praestationes annuas certo annorum numero definitas, et Reditus ad vitam; ac de licitatione rei quae sub hasta distrahitur, oblata solutione particulari", "Legal-mathematical memoir on the interest of an intervening period or on the allowance of an advanced payment, popularly called rebate, on the estimation of the right to receive annual benefits assessed for a certain number of years and life annuities and on the bid for a thing which is sold by auction after presenting a particular solution."[38]

(2.5) "De interusurio seu Resegmento Anticipationis, vulgo Rabat, de aestimando jure percipiendi praestationes annuas certo Numero definitas, et Reditus ad vitam; ac de Licitatione rei quae sub hasta distrahitur, oblata solutione particulari", "On the interest of an intervening period or on reduction in the case of advanced payment popularly called rebate, on the estimation of the right to receive annual benefits, assessed for a certain number of years and life annuities, and on the bid for a thing which is sold by auction after presenting a particular solution."[39]

Already the titles allow us to form the following groups: versions (2.1) and (2.2); (2.3); (2.4); and (2.5). What is more, version (2.2) depends on version (2.1). Up to the word "differat" (line 31) it is a copy of version (2.1) (see above and below illustrations 14 and 15) for these reasons: Firstly, version (2.2) at once takes into account all additions and modifications of version (2.1) but leaves aside its cancelled passages. For example, version (2.1) adds "re aut" in line 3. These words are inserted into the text of version (2.2). Secondly, version (2.1) cancels some lines after "petere" which are left out in version (2.2) where the words "at de aestimanda" follow directly after "petere", etc. Thirdly, version (2.1) adds a paragraph "ab Jurisconsultis, quia ad eam inveniendam . . . " Version (2.2) inserts this paragraph into the text but leaves out the words "ab Jurisconsultis" and adds "accurate" between "eam" and "inveniendam". In the same way it can be proved that version 5 depends on version 4: sheets 19–20 are an improved, modified copy of the sheets $1 + 17 + 13 - 14$. It is very likely that version 3 precedes version 4, because it has a long paragraph at the very beginning which is not taken into account in version 4. But version 4 begins with a passage from version 3, which comes after this paragraph.

Leibniz himself says that the inquiry into the rebate problem led him to the question of life annuities. As a consequence, the titles of the later versions take this subject into account. There are four unpublished versions of an article on life annuities.[40]

4. TEXT AS A TRANSCRIBED DISCUSSION AND ARGUMENTATION

Leibniz's texts reflect his thinking as thinking while writing. Every idea, every question, every doubt, every access and optimism, every provisional result, every plan or intention is written down.

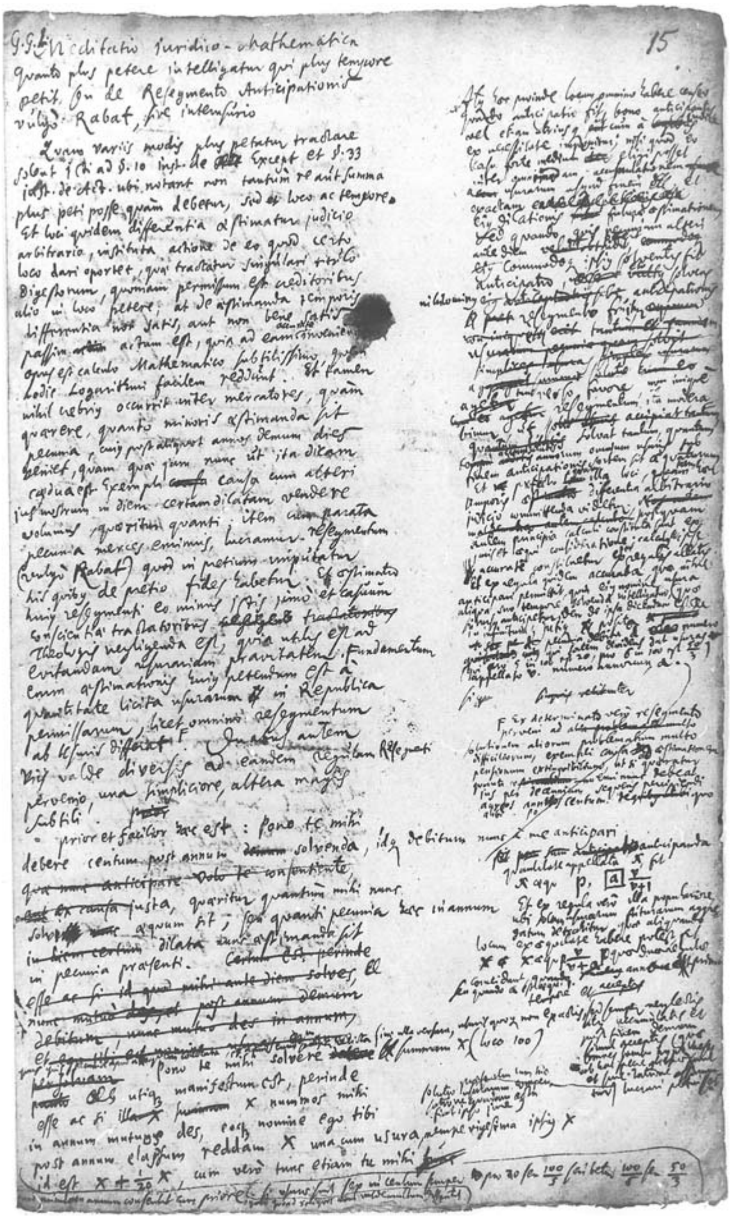

G.G.L. Meditatio Iuridico-Mathematica
Quanto plus petere intelligatur qui plus tempore petit, seu de Resegmento Anticipationis vulgo Rabat, sive interusurio

Figure 15. LH II 5,1 sheet 15r; Leibniz 2000, no. II.9.

The essential difference between his literary bequest and that of other mathematicians and scientists is based on this peculiarity. The text reflects his monologue or a dialogue with a fictitious interlocutor. At times, the interlocutor who utters doubts or objections is Leibniz himself, that is, Leibniz answers himself. Hence his style is very personal. He is accustomed to writing in the first person singular or plural:[41]

Investigavimus,	we shall investigate;
supponamus,	let us suppose;
investigemus,	let us investigate;
interea fateor,	in the meantime I confess;
puto idem sic solvi posse,	I think that the same can be solved in the following way;
possum ergo problema solvere hoc modo,	thus I can solve the problem in the following way;
ut inveniamus,	so that we find;
lucrati ergo sumus,	thus we have gained;
ut obiter dicam,	what I would like to say by the way
considerabimus,	we shall consider;
Ergo id dari posse non dubito,	hence I do not doubt that this can be given;
utemur,	we shall use;
videamus,	let us see
video,	I see;
quod ita ostendo,	what I demonstrate in the following way
poteram dividere,	I could have divided;
credo,	I believe.

Three examples illustrate his dialogical style.

1) He discusses a construction of curves and says that a special problem cannot be solved by the intersection of curves. The fictive interlocutor says: “At inquies hoc habet commodum constructio per intersectionem curvarum, ut per puncta describi possint curvae, saltem mechanice, cum hic opus sit motu.... Respondeo hic quoque designari posse quaesitum tentando.” “But you will say that the construction by means of the intersection of curves has the advantage that the curves can be described by means of points, at least mechanically, because here a motion is needed. ... I answer that here too what is looked for can be designated by trying.”[42]

2) He is looking for a solution of a number-theoretical problem and says that most of the superfluous quantities must be set equal to one. “At inquies, cum superfluae sint, poterant ab initio omitti. Ita est; si divinare possemus; nunc quando homines sumus; satis nobis esse debet artem reperisse, quae eas in progressu ostendat.” “But you will say: ‘Because they are superfluous, they could have been omitted from the beginning.’ ‘That’s the case, if we could prophesy. Because we are human beings, as things now stand, we must be content to have found an art which shows these quantities, if we get ahead.’ ”[43]

3) He recommends the test of nine in order to check calculations and refutes a whole series of potential objections:[44] “Dicet aliquis”, “Somebody will say”; “Dicetur”, “It will be said”; “At si inquiet ... Respondeo.” “But if somebody will say ... I answer.”

CONCLUSION

The examples discussed above bear overwhelming witness to the mathematical thinking of Leibniz. It is inseparably intertwined with the genesis of his mathematical texts. To a certain extent thinking and writing are for him two sides of the same coin, of his tremendously creative intellect.

T.U. Berlin

ACKNOWLEDGEMENT

I would like to thank Abe Shenitzer for polishing my English.

ABBREVIATION

LH Leibniz-Handschriften der Niedersächsischen Landesbibliothek Hannover

NOTES

[1] Leibniz 1976a, n. 40 (ca. August 1673).
[2] Leibniz 1976a, 261.
[3] Leibniz 1976a, 260.
[4] Leibniz 1976a, no. 46 (ca. 1678–1684).
[5] Knobloch 1973, 190; Leibniz 1976a, 275.
[6] Knobloch 1973, 190–94.
[7] Leibniz 1976a, 280.
[8] Leibniz 1976a, 281.
[9] Leibniz 1993.
[10] Knobloch 1973, 234.
[11] Leibniz 1976a, no. 56 (ca. 1676 or 1700).
[12] Leibniz 1990, no. 87 (ca. January 2, 1676), no. 92 (ca. February 12 and April 1976).
[13] Leibniz 1990, 580.
[14] Leibniz 1990, 597.
[15] Leibniz 1990, 598.
[16] Leibniz 1993, 71.
[17] Leibniz 1976, no. 24 (1677/78); Knobloch 1973, folding sheet between pp. 248 and 249.
[18] Knobloch 1973, 114–21.
[19] Leibniz 2000, no. III.11, part B (1680–1683).
[20] Leibniz 2000, 479.
[21] Leibniz 2000, 478.
[22] Leibniz 1995, 339–351.
[23] Leibniz 1993, 71 and 140f.
[24] Leibniz 1993, 77.
[25] Knobloch 1974.
[26] Knobloch 1974, 162f.
[27] Knobloch 1974.
[28] LH XXXV 2,1 sheet 155v.
[29] Leibniz 1993.
[30] Leibniz 1976b, no. 39.
[31] Leibniz 1976b, no. 72.
[32] Leibniz 1683; Leibniz 2000, no. II.22.

[33] LH II 5,1 sheets 33–34.
[34] Leibniz 2000, nos. II.8 – II.12.
[35] LH II 5,1 sheet 9.
[36] LH II 5,1 sheets 15–16.
[37] LH II 5,1 sheets $11 - 12 + 6 - 10$.
[38] LH II 5,1 sheets $1 + 17 + 13 - 14$.
[39] LH II 5,1 sheets $19 - 20$.
[40] Leibniz 2000, nos. III.3 – III.5, III.8.
[41] Leibniz 1990, 28; 38; 43, 67; 104; 285; 293; 294; 295; 297; 196; 177; 196; 196 etc. used constantly; 154, 193, 277, 280 etc.; 388; 387; 155, 277 etc.
[42] Leibniz 1990, 122.
[43] Leibniz 1990, 282.
[44] Leibniz 1990, 530.

REFERENCES

Knobloch, E. 1973. *Die mathematischen Studien von G.W. Leibniz zur Kombinatorik. Auf Grund fast ausschließlich handschriftlicher Aufzeichnungen dargelegt und kommentiert*. Wiesbaden: Steiner (Studia Leibnitiana Supplementa XI).

Knobloch, E. 1974. "Unbekannte Studien von Leibniz zur Eliminations—und Explikationstheorie." *Archive for History of Exact Sciences* 12: 142–73.

Leibniz, G.W. 2000. *Hauptschriften zur Versicherungs—und Finanzmathematik*. Ed. by E. Knobloch and J.-Matthias Graf von der Schulenburg. Berlin: Akademie Verlag.

Leibniz, G.W. 1683. "Meditatio juridico-mathematica de interusurio simplice." *Acta Eruditorum* October 1683: 425–32. Reprinted in: Leibniz, G.W. 1849–63. *Mathematische Schriften*, 7 vols. Ed. by C.I. Gerhardt. Berlin-Halle: Asher-Schmidt (Reprint Hildesheim 1962), vol. 7, 125–132.

Leibniz, G.W. 1976a. *Die mathematischen Studien von G.W. Leibniz zur Kombinatorik*, Textband. Ed. by E. Knobloch. Wiesbaden: Steiner. (Studia Leibnitiana Suppl. XVI).

Leibniz, G.W. 1976b. *Sämtliche Schriften und Briefe*, series III, vol. 1. Ed. by J.E. Hofmann. Berlin: Akademie Verlag.

Leibniz, G.W. 1990. *Sämtliche Schriften und Briefe*, series VII, vol. 1. Ed. by E. Knobloch and W.S. Contro. Berlin: Akademie Verlag.

Leibniz, G.W. 1993. *De quadratura arithmetica circuli ellipseos et hyperbolae cujus corollarium est trigonometria sine tabuli*s. Kritisch herausgegeben und kommentiert von E. Knobloch. Göttingen: Vandenhoeck & Ruprecht. (Abhandlungen der Akademie der Wissenschaften in Göttingen, Mathematisch-physikalische Klasse 3. Folge Nr. 43).

Leibniz, G.W. 1995. *L'estime des apparences, 21 manuscrits de Leibniz sur les probabilités, la théorie des jeux, l'espérance de vie*. Texte établi, traduit, introduit et annoté par M. Parmentier. Paris: Vrin.

Leibniz, G.W. 2004. *Quadrature arithmétique du cercle, de l'ellipse et de l'hyperbole et la trigonométrie sans tables trigonométriques qui en est le corollaire.* Introduction, traduction et notes de M. Parmentier. Texte latin édité par E. Knobloch. Paris: Vrin.