4

Solvability of the uniform output regulation problem

In this chapter we establish general conditions for solvability of the global and local uniform output regulation problem. First, we review some known ideas and results related to the conventional local output regulation problem. These results are based on the center manifold theorem. In order to extend these results to the uniform output regulation problem for both the local and global case, we present invariant manifold theorems, which serve as non-local counterparts of the center manifold theorem. In the formulation of these invariant manifold theorems, the notion of convergent systems, developed in Chapter 2, plays a central role. Based on these invariant manifold theorems, general necessary and sufficient conditions for the solvability of the global and local uniform output regulation problems are derived. These conditions also indicate what kind of properties a controller must have to solve the uniform output regulation problem. This information will be exploited at the stage of controller design in Chapter 5.

4.1 Analysis of the conventional local output regulation problem

The conventional local output regulation problem, which can also be called the local exponential output regulation problem, has been solved in [39] (see also [8, 38]). In that paper necessary and sufficient conditions for the solvability of this problem are obtained. We will review one of these results in order to motivate its extensions to the global uniform output regulation problem.

To understand the ideas and techniques used in the analysis of the conventional local output regulation problem, we investigate the dynamics of the closed-loop system (3.8), (3.9) corresponding to a controller (3.6), (3.7) solving the conventional local output regulation problem.

By $z := (x^T, \xi^T)^T$ denote the state of the closed-loop system (3.8), (3.9) and by $F(z, w)$, its right-hand side. With these new notations, the regulated output e equals $e = \bar{h}_r(z, w) := h_r(x, w)$. Therefore, the combination of the closed-loop system and the exosystem can be written as

$$\dot{z} = F(z, w), \tag{4.1}$$
$$\dot{w} = s(w), \tag{4.2}$$
$$e = \bar{h}_r(z, w).$$

As follows from the formulation of the conventional local output regulation problem, for any controller solving this problem the corresponding closed-loop system is such that $F(0,0) = 0$ and the function $F(z, w)$ has continuous partial derivatives of some high order. Moreover, the fact that for $w(t) \equiv 0$ the closed-loop system has an asymptotically stable linearization at the origin is equivalent to the Jacobian matrix $\partial F/\partial z(0,0)$ being Hurwitz. At the same time, the fact that the zero solution $w(t) \equiv 0$ of the exosystem is Lyapunov stable in forward and backward time (this is a consequence of the neutral stability assumption on the exosystem) implies that $\partial s/\partial w(0)$ has all its eigenvalues on the imaginary axis. These conditions allow us to apply the center manifold theorem (see, e.g., [10]), a particular case of which is formulated below.

Theorem 4.1. *Consider systems (4.1) and (4.2). Suppose $F(z, w)$ and $s(w)$ are C^2 vector-functions with $F(0,0) = 0$, $s(0) = 0$, and all eigenvalues of $\partial F/\partial z(0,0)$ have negative real parts, while all eigenvalues of $\partial s/\partial w(0)$ have zero real parts. Then there exist $\delta > 0$ and a C^1 function $\alpha(w)$ defined for all $|w| < \delta$ such that $\alpha(0) = 0$ and the graph $z = \alpha(w)$ is a locally invariant and locally exponentially attractive manifold for systems (4.1) and (4.2). The mapping $\alpha(w)$ satisfies the partial differential equation*

$$\frac{\partial \alpha}{\partial w}(w)s(w) = F(\alpha(w), w). \tag{4.3}$$

If a set $\mathcal{W} \subset \{w : \ |w| < \delta\}$ is (positively) invariant with respect to system (4.2), then the graph

$$\mathcal{M}(\mathcal{W}) := \{(z, w) : \ z = \alpha(w), \ w \in \mathcal{W}\}$$

is (positively) invariant with respect to systems (4.1) and (4.2), and for all solutions $(z(t), w(t))$ starting close enough to the origin $(0,0)$ it holds that

$$|z(t) - \alpha(w(t))| \leq C\mathrm{e}^{-\beta t}|z(0) - \alpha(w(0))| \tag{4.4}$$

for some $C > 0$ and $\beta > 0$.

The manifold $\mathcal{M}(\mathcal{W})$ is called *the center manifold*. As follows from (4.3), if $w(t)$ is a solution of system (4.2) satisfying $|w(t)| < \delta$ for all $t \in \mathbb{R}$, then $\bar{z}_w(t) := \alpha(w(t))$ is a solution of system (4.1) defined for all $t \in \mathbb{R}$. In general, the center manifold theorem is formulated for bidirectionally coupled systems, i.e., when the right-hand side of system (4.2) also depends on z. For the output regulation problem it is sufficient to formulate the center manifold theorem only for unidirectionally coupled systems (4.1) and (4.2).

Applying the center manifold theorem (Theorem 4.1) to systems (4.1) and (4.2), we conclude that there exists $\delta > 0$ and a C^1 mapping $\alpha(w)$ defined for all $|w| < \delta$ such that $\alpha(0) = 0$ and the graph $z = \alpha(w)$ is locally invariant and locally exponentially attractive with respect to systems (4.1) and (4.2). The mapping $\alpha(w)$ satisfies the partial differential equation

$$\frac{\partial \alpha}{\partial w}(w)s(w) = F(\alpha(w), w) \tag{4.5}$$

for all $w \in \mathcal{W}$. Moreover, since the zero solution $w(t) \equiv 0$ of the exosystem is Lyapunov stable in forward and backward time, there exists a neighborhood of the origin $\mathcal{W} \subset \{w : \ |w| < \delta\}$ that is invariant with respect to (4.2). Hence, the graph $\mathcal{M}(\mathcal{W}) := \{(z, w) : \ z = \alpha(w), \ w \in \mathcal{W}\}$ is invariant with respect to systems (4.1), (4.2) and for all solutions $z(t), w(t)$ starting close enough to the origin $(0, 0)$ it holds that

$$z(t) - \alpha(w(t)) \to 0 \quad \text{as} \quad t \to +\infty. \tag{4.6}$$

This fact shows that in some neighborhood of the origin the dynamics of the closed-loop system (4.1) coupled with the exosystem (4.2) reduce, after transients, to the dynamics on the center manifold $\mathcal{M}(\mathcal{W})$. Hence, the properties of this center manifold determine whether the regulated output $e(t)$ tends to zero along solutions of the closed-loop system or not. In particular, it can be shown (see, e.g., [8, 39]) that, under the neutral stability assumption on the exosystem, the fact that $e(t) = \bar{h}_r(z(t), w(t)) \to 0$ as $t \to +\infty$ for all solutions of the closed-loop system (4.1) and the exosystem (4.2) starting close enough to the origin is equivalent to

$$\bar{h}_r(\alpha(w), w) = 0 \tag{4.7}$$

for all w in some neighborhood of the origin $\widehat{\mathcal{W}} \subset \mathbb{R}^m$.

As follows from the analysis presented above, the question of whether a controller solves the conventional local output regulation problem reduces to the questions of whether for $w(t) \equiv 0$ the corresponding closed-loop system has an asymptotically stable linearization at the origin and whether there exists a locally defined C^1 mapping $\alpha(w)$, with $\alpha(0) = 0$, satisfying (4.5) and (4.7). If we denote $(\pi(w), \sigma(w)) := \alpha(w)$, where $\pi(w)$ and $\sigma(w)$ are the components of the mapping $\alpha(w)$ corresponding to the x- and ξ-coordinates of the closed-loop system, respectively, this statement can be summarized in the following theorem.

Theorem 4.2 ([8]). *Under the neutral stability assumption on the exosystem (3.4), a controller of the form (3.6), (3.7) solves the conventional local output regulation problem if and only if the following two conditions hold:*

(i) *For* $w(t) \equiv 0$ *the corresponding closed-loop system (3.8), (3.9) has an asymptotically stable linearization at the origin.*
(ii) *There exist* C^1 *mappings* $\pi(w)$ *and* $\sigma(w)$ *defined in some neighborhood of the origin* $\widehat{\mathcal{W}}$ *and satisfying* $\pi(0) = 0$, $\sigma(0) = 0$ *and*

$$\begin{aligned}
\frac{\partial \pi}{\partial w}(w)s(w) &= f(\pi(w), \theta(\sigma(w), h_m(\pi(w), w)), w), \\
\frac{\partial \sigma}{\partial w}(w)s(w) &= \eta(\sigma(w), h_m(\pi(w), w)), \\
0 &= h_r(\pi(w), w) \quad \forall \;\; w \in \widehat{\mathcal{W}}.
\end{aligned}$$

This theorem provides a characterization of all controllers solving the conventional local output regulation problem. It also forms a foundation for further results related to solvability and controller design for the conventional local output regulation problem, which can be found, for example, in [8].

Since in this book we also study global variants of the output regulation problem, we need to extend the result of Theorem 4.2 to the global case. An essential obstacle for such an extension is that the analysis in Theorem 4.2 is based on the center manifold theorem (Theorem 4.1), which is a local result. Existing extensions of this theorem to nonlocal cases (see, e.g., [22, 49, 53, 87]) are based on certain *quantitative* conditions on the dynamics of the coupled systems (the closed-loop system and the exosystem in the case of the output regulation problem). We would like to avoid such quantitative conditions and find nonlocal counterparts of the center manifold theorem based on certain *qualitative* conditions on the coupled systems. As a preliminary observation, notice that in the center manifold theorem the Jacobian $\partial F/\partial z(0,0)$ must be a Hurwitz matrix. As we know from Theorem 2.41, this condition is equivalent to the requirement that system (4.1) be locally exponentially convergent for the class of inputs $\overline{\mathbb{PC}}_m$. This observation shows that the requirement of some convergence property on system (4.1) may serve as a nonlocal counterpart of the condition on $\partial F/\partial z(0,0)$. In fact, as we will see in the next section, existence of a continuous invariant manifold of the form $z = \alpha(w)$ for systems (4.1) and (4.2) is, under certain assumptions, equivalent to some form of the uniform convergence property of system (4.1). The invariant manifold theorems presented in the next section will naturally lead us to necessary and sufficient conditions for the solvability of the global and local variants of the uniform output regulation problem. This fact, in turn, explains why we have based the uniform output regulation problem studied in this book on the notion of uniform convergence.

4.2 Invariant manifold theorems

In this section we present certain invariant manifold theorems that serve as counterparts of the center manifold theorem for studying the solvability of the global and local variants of the uniform output regulation problem. To this end, we consider coupled systems of the form

$$\dot{z} = F(z, w), \tag{4.8}$$
$$\dot{w} = s(w), \tag{4.9}$$

where $z \in \mathbb{R}^d$, $w \in \mathbb{R}^m$. The function $F(z, w)$ is locally Lipschitz in z and continuous in w; $s(w)$ is locally Lipschitz. In the analysis of the uniform output regulation problem, system (4.8) corresponds to a closed-loop system and system (4.9) corresponds to an exosystem.

First, we consider the case of system (4.9) with some open invariant set of initial conditions $\mathcal{W} \subset \mathbb{R}^m$. Recall that $\mathcal{I}_s(\mathcal{W})$ denotes the class of all solutions of system (4.9) starting in $\mathcal{W}$. The next technical lemma provides conditions for the existence of a continuous asymptotically stable invariant manifold of the form $z = \alpha(w)$. This lemma will serve as a foundation for further results on invariant manifolds presented in this section.

Lemma 4.3. *Consider system (4.8) and system (4.9) with an open invariant set of initial conditions $\mathcal{W} \subset \mathbb{R}^m$. Suppose*

(i) *System (4.8) is uniformly convergent in a set $\mathcal{Z} \subset \mathbb{R}^d$ for the class of inputs $\mathcal{I}_s(\mathcal{W})$, and for any compact set $\mathcal{K}_0 \subset \mathcal{W}$ there exists a compact set $\mathcal{K}_z \subset \mathcal{Z}$ such that for any $w(\cdot) \in \mathcal{I}_s(\mathcal{W})$ satisfying $w(0) \in \mathcal{K}_0$ the corresponding steady-state solution satisfies $\bar{z}_w(t) \in \mathcal{K}_z$ for all $t \in \mathbb{R}$.*

Then

(ii) *There exists a continuous mapping $\alpha : \mathcal{W} \to \mathcal{Z}$ such that the graph*

$$\mathcal{M}(\mathcal{W}) := \{(z, w) : \ z = \alpha(w), w \in \mathcal{W}\}$$

is invariant with respect to systems (4.8) and (4.9). Moreover, for every $w(\cdot) \in \mathcal{I}_s(\mathcal{W})$ the corresponding solution of system (4.8) $\bar{z}_w(t) = \alpha(w(t))$ is uniformly asymptotically stable in $\mathcal{Z}$.

In general, the mapping $\alpha(w)$ is not unique. But for any two such mappings $\alpha_1(w)$ and $\alpha_2(w)$ and for any $w(\cdot) \in \mathcal{I}_s(\mathcal{W})$, it holds that

$$\alpha_1(w(t)) - \alpha_2(w(t)) \to 0 \quad \text{as} \quad t \to +\infty \tag{4.10}$$

and $\alpha_1(w(t)) \equiv \alpha_2(w(t))$ for any $w(t)$ lying in some compact subset of $\mathcal{W}$ for all $t \in \mathbb{R}$.
If system (4.9) satisfies the boundedness assumption **A1** *in the set $\mathcal{W}$, then statements* (i) *and* (ii) *are equivalent and the mapping $\alpha(w)$ defined in* (ii) *is unique.*

Proof: See Appendix 9.10.

This lemma is a preliminary technical result that allows us to obtain further global and local results related to the existence of continuous invariant manifolds of the form $z = \alpha(w)$. The conditions in Lemma 4.3 seem rather complicated because this lemma covers the general case. In particular cases of this lemma, which are formulated below, the conditions will simplify significantly. In particular, under the boundedness assumption **A1** on system (4.9) for $\mathcal{Z} = \mathbb{R}^d$ we obtain the following global result.

Theorem 4.4. *Consider system (4.8) and system (4.9) satisfying the boundedness assumption* **A1** *in some open invariant set* $\mathcal{W} \subset \mathbb{R}^m$. *The following statements are equivalent:*

(ig) *System (4.8) is globally uniformly convergent with the UBSS property for the class of inputs* $\mathcal{I}_s(\mathcal{W})$.

(iig) *There exists a unique continuous mapping* $\alpha : \mathcal{W} \to \mathbb{R}^d$ *such that the graph*

$$\mathcal{M}(\mathcal{W}) := \{(z, w) : \ z = \alpha(w), w \in \mathcal{W}\}$$

is invariant with respect to systems (4.8) and (4.9). Moreover, for every $w(\cdot) \in \mathcal{I}_s(\mathcal{W})$ *the corresponding solution of system (4.8)* $\bar{z}_w(t) = \alpha(w(t))$ *is uniformly globally asymptotically stable.*

Proof: We only need to show that the conditions given in (ig) are equivalent to the conditions (i) in Lemma 4.3 for $\mathcal{Z} := \mathbb{R}^d$.
(ig)$\Rightarrow$(i). Consider a compact set $\mathcal{K}_0 \subset \mathcal{W}$. By the boundedness assumption **A1**, there exists a compact set $\mathcal{K}_w \subset \mathcal{W}$ such that if a solution $w(t)$ of system (4.9) starts in $w(0) \in \mathcal{K}_0$ then $w(t) \in \mathcal{K}_w$ for all $t \in \mathbb{R}$. At the same time, by the UBSS property, there exists a compact set $\mathcal{K}_z \subset \mathbb{R}^d$ such that the fact that $w(t) \in \mathcal{K}_w$ for all $t \in \mathbb{R}$ implies $\bar{z}_w(t) \in \mathcal{K}_z$ for all $t \in \mathbb{R}$. This implies (i).
(i)$\Rightarrow$(ig). Consider a compact set $\mathcal{K}_w \subset \mathcal{W}$ and a solution of system (4.9) satisfying $w(t) \in \mathcal{K}_w$ for all $t \in \mathbb{R}$. In particular, this solution satisfies $w(0) \in \mathcal{K}_0 := \mathcal{K}_w$. By the conditions given in (i), there exists a compact set $\mathcal{K}_z \subset \mathbb{R}^d$ such that for any solution $w(t)$ starting in $w(0) \in \mathcal{K}_0$ (hence, for any $w(t)$ satisfying $w(t) \in \mathcal{K}_w$ for all $t \in \mathbb{R}$) the corresponding steady-state solution $\bar{z}_w(t)$ lies in $\mathcal{K}_z$. Thus, we have shown that system (4.8) has the UBSS property for the class of inputs $\mathcal{I}_s(\mathcal{W})$, i.e. we have shown (ig). □

Under the boundedness assumption **A1** the class of inputs $\mathcal{I}_s(\mathcal{W})$ is contained in $\overline{\mathbb{PC}}(\mathcal{W})$, so we therefore obtain the following corollary to Theorem 4.4.

Corollary 4.5. *Consider system (4.8) and system (4.9) satisfying the boundedness assumption* **A1** *in some open invariant set* $\mathcal{W}$. *Suppose system (4.8) is globally uniformly convergent with the UBSS property for the class of inputs* $\overline{\mathbb{PC}}(\mathcal{W})$. *Then statement* (iig) *of Theorem 4.4 holds.*

In the global forward time uniform output regulation problem we deal with exosystems that do not need to satisfy the boundedness assumption **A1**, but they satisfy the assumption **A2**, i.e., their solutions start in some compact positively invariant set of initial conditions $\mathcal{W}_+ \subset \mathbb{R}^m$. For such systems we formulate the following result.

Theorem 4.6. *Consider systems (4.8) and (4.9). Let $\mathcal{W}_+$ be a compact positively invariant set of system (4.9) and $\mathcal{W}_\pm \subset \mathcal{W}_+$ be an invariant subset of $\mathcal{W}_+$. Suppose system (4.8) is globally uniformly convergent with the UBSS property for the class of inputs $\overline{\mathbb{PC}}(\widetilde{\mathcal{W}})$, where $\widetilde{\mathcal{W}}$ is some neighborhood of $\mathcal{W}_+$. Then there exists a continuous mapping $\alpha : \widetilde{\mathcal{W}} \to \mathbb{R}^d$ such that the set*

$$\mathcal{M}(\mathcal{W}_+) := \{(z, w) : \ z = \alpha(w), \ w \in \mathcal{W}_+\}$$

is positively invariant with respect to (4.8), (4.9), and for any solution of system (4.9) $w(t)$ starting in $w(0) \in \mathcal{W}_+$ the corresponding solution of system (4.8) $\bar{z}_w(t) = \alpha(w(t))$ is uniformly globally asymptotically stable. In general, the mapping $\alpha(w)$ is not unique. But for any two such mappings $\alpha_1(w)$ and $\alpha_2(w)$ and for any $w(t)$ starting in $w(0) \in \mathcal{W}_+$ it holds that

$$|\alpha_1(w(t)) - \alpha_2(w(t))| \to 0 \quad \text{as} \quad t \to +\infty, \tag{4.11}$$

and $\alpha_1(w) = \alpha_2(w)$ for all $w \in \mathcal{W}_\pm$.

Proof: See Appendix 9.11.

In Theorem 4.6 the mapping $\alpha(w)$ may be nonunique as can be seen from the following example, which is a modified example from [78].

Example 4.7. Consider two scalar systems

$$\dot{z} = -z, \tag{4.12}$$

$$\dot{w} = -\frac{w^3}{2}. \tag{4.13}$$

System (4.12) is globally uniformly convergent with the UBSS property for the class of inputs $\overline{\mathbb{PC}}_1$, since for every input $w(t)$ the steady-state solution equals $\bar{z}_w(t) \equiv 0$ and it is globally exponentially stable. For every $r > 0$ the set $\mathcal{W}_+(r) := \{w : |w| \leq r\}$ is compact and positively invariant with respect to (4.13). The set $\mathcal{W}_\pm$ contains only the origin, $\mathcal{W}_\pm = \{0\}$. It can be easily checked that for any constant c the mapping

$$\alpha_c(w) = \begin{cases} c\mathrm{e}^{-1/w^2}, & w \neq 0, \\ 0, & w = 0, \end{cases}$$

is continuous and the graph $z = \alpha_c(w)$ is invariant with respect to (4.12) and (4.13). The mappings $\alpha_c(w)$ for all parameters c coincide in the origin, which belongs to $\mathcal{W}_{\pm}$. For any initial condition $w(0) \in \mathbb{R}$ the solution $w(t)$ of system (4.13) tends to zero, which implies $\alpha_c(w(t)) \to 0$ as $t \to +\infty$. Thus for any c_1 and c_2 it holds that

$$\alpha_{c_1}(w(t)) - \alpha_{c_2}(w(t)) \to 0, \quad \text{as } \ t \to +\infty. \triangleleft$$

The next theorem provides a local variant of the invariant manifold theorems presented above.

Theorem 4.8. *Consider systems (4.8) and (4.9) with $F(0,0) = 0$, $s(0) = 0$ and with $F(z,w)$ being C^1 with respect to z and continuous with respect to w. Let the equilibrium $w = 0$ of system (4.9) be stable in forward and backward time. Then the following statements are equivalent:*

(il) *System (4.8) is locally uniformly convergent for the class of inputs $\mathcal{I}_s(\mathcal{W}_*)$, where $\mathcal{W}_* \subset \mathbb{R}^m$ is some invariant neighborhood of the origin.*
(iil) *There exist an invariant neighborhood of the origin $\mathcal{W}$ and a unique continuous mapping $\alpha : \mathcal{W} \to \mathbb{R}^d$ such that $\alpha(0) = 0$ and the graph*

$$\mathcal{M}(\mathcal{W}) := \{(z,w): \ z = \alpha(w), \ w \in \mathcal{W}\}$$

is invariant with respect to systems (4.8) and (4.9). Moreover, there exists a neighborhood of the origin $\mathcal{Z} \subset \mathbb{R}^d$ such that for every $w(\cdot) \in \mathcal{I}_s(\mathcal{W})$ the solution $\bar{z}_w(t) := \alpha(w(t))$ is uniformly asymptotically stable in $\mathcal{Z}$.

Proof: See Appendix 9.12.

In general, it is not a simple task to find an invariant manifold even if its existence is guaranteed by the invariant manifold theorems presented above. Yet, in some simple cases such a manifold can be found analytically. We will show this with a few examples.

Example 4.9. Consider a linear system

$$\dot{w} = Sw, \quad w \in \mathbb{R}^m, \tag{4.14}$$

with the matrix S having all its eigenvalues simple and lying on the imaginary axis. This system satisfies the boundedness assumption **A1** in the whole state space. Consider a system given by the equation

$$\dot{z} = Az + q(w), \tag{4.15}$$

where A is a Hurwitz matrix and $q(w)$ is a polynomial in w of some finite degree n. Notice that this system is globally exponentially convergent with the UBSS property for the class of inputs $\overline{\mathbb{PC}}_m$ (see, for example, Theorem 2.29). By Corollary 4.5, there exists a unique continuous function $\alpha(w)$ such that the graph $\mathcal{M} := \{(z,w): \ z = \alpha(w), w \in \mathbb{R}^m\}$ is invariant with respect to systems (4.15) and (4.14). As follows from [8] (Lemma 1.2), the mapping $\alpha(w)$

is a polynomial in w of the same degree as the degree of $q(w)$. It is a unique solution of the equation

$$\frac{\partial \alpha}{\partial w}(w) S w = A\alpha(w) + q(w).$$

The right- and left-hand sides of this equation are polynomials in w. Thus, by equating the corresponding components of these polynomials, we find the unique coefficients of the polynomial $\alpha(w)$.$\triangleleft$

Using ideas from [8], this example can be extended in the following way.

Example 4.10. Consider the nonlinear system

$$\begin{aligned} \dot z_1 &= A_1 z_1 + q_1(z_2, \dots z_k, w), \quad & z_1 \in \mathbb{R}^{d_1}, \\ \dot z_2 &= A_2 z_2 + q_2(z_3, \dots z_k, w), \quad & z_2 \in \mathbb{R}^{d_2}, \\ \dots & & \\ \dot z_k &= A_k z_k + q_k(w), & z_k \in \mathbb{R}^{d_k}, \end{aligned} \tag{4.16}$$

where the matrices A_i, $i = 1, \dots, k$, are Hurwitz and $q_i(\cdot)$, $i = 1, \dots, k$, are polynomials of their arguments. Every ith subsystem of system (4.16) with $z_{i+1}, \dots, z_k$ and w as inputs is input-to-state convergent (see Theorem 2.29). Therefore, system (4.16) is a series connection of input-to-state convergent systems. By Property 2.27, this system is input-to-state convergent. By Property 2.19, input-to-state convergence, in turn, implies that system (4.16) is globally uniformly convergent with the UBSS property for the class of inputs $\overline{\mathbb{PC}}_m$. By Corollary 4.5, there exists a unique continuous mapping $\alpha(w)$ such that the manifold $\mathcal{M} := \{(z, w) : \ z = \alpha(w), w \in \mathbb{R}^m\}$, where $z := (z_1^T, \dots, z_k^T)^T$, is invariant with respect to systems (4.16) and (4.14). Applying the results obtained for system (4.15) to the last equation in (4.16), we find the component of $\alpha(w)$ corresponding to z_k. This component $\alpha_k(w)$ is a polynomial. Substituting this $\alpha_k(w)$ in the $(k-1)$th equation, we again obtain an equation of the form (4.15), from which we can find $\alpha_{k-1}(w)$. Repeating this process, we find the remaining components of the mapping $\alpha(w)$.$\triangleleft$

These examples indicate that in some cases it is possible to find the invariant manifold, whose existence is guaranteed by the invariant manifold theorems presented in this section, analytically.

The invariant manifold theorems presented in this section state equivalence between the existence of a (globally) uniformly asymptotically stable invariant manifold of the form $z = \alpha(w)$ with a continuous function $\alpha(w)$ on the one hand, and certain convergence properties of system (4.8) on the other hand (under Assumptions **A1**, **A2**, or under the neutral stability assumption on system (4.9)). The sufficient conditions for various convergence properties presented in Section 2.2.4 allow us to determine whether systems (4.8) and (4.9) have such an invariant manifold.

As will be seen from the next sections, the invariant manifold theorems will naturally lead us to certain necessary and sufficient conditions for the solvability of different variants of the uniform output regulation problem.

4.3 ω-limit sets

Prior to deriving the conditions for solvability of the uniform output regulation problem, we recall the notion of ω-limit sets. This notion appears to be an important ingredient of the solvability analysis. Consider the system

$$\dot{w} = s(w), \quad w \in \mathbb{R}^m, \tag{4.17}$$

with a locally Lipschitz function $s(w)$. Let $w(t, w_0)$ denote the solution of system (4.17) starting in $w(0, w_0) = w_0$.

Definition 4.11 ([3]). *A point $w_* \in \mathbb{R}^m$ is called an ω-limit point of the trajectory $w(t, w_0)$ if for any $T > 0$ and any $\varepsilon > 0$ there exists $t_* > T$ such that $|w(t_*, w_0) - w_*| < \varepsilon$. The set of all ω-limit points of the trajectory $w(t, w_0)$ is called the ω-limit set and denoted by $\Omega(w_0)$. For trajectories starting in some set $\mathcal{W} \subset \mathbb{R}^m$, the notation $\Omega(\mathcal{W})$ denotes $\Omega(\mathcal{W}) := \bigcup_{w_0 \in \mathcal{W}} \Omega(w_0)$.*

The following statements reflect some standard facts on ω-limit sets, see, e.g., [3]. For a trajectory $w(t, w_0)$ that is bounded for $t \geq 0$ the ω-limit set $\Omega(w_0)$ is a bounded invariant set. If $\mathcal{W} \subset \mathbb{R}^m$ is a bounded positively invariant set, then $\Omega(\mathcal{W})$ is a bounded invariant set that attracts all trajectories $w(t, w_0)$ starting in $w_0 \in \mathcal{W}$, i.e., for any $w_0 \in \mathcal{W}$ it holds that $\mathrm{dist}(w(t, w_0), \Omega(\mathcal{W})) \to 0$ as $t \to +\infty$. Here, the distance $\mathrm{dist}(w, \mathcal{W})$ between a point $w \in \mathbb{R}^m$ and a set $\mathcal{W} \subset \mathbb{R}^m$ is defined as $\mathrm{dist}(w, \mathcal{W}) := \inf_{w_* \in \mathcal{W}} |w - w_*|$. If $\mathcal{W}$ is a compact positively invariant set, then $\Omega(\mathcal{W}) \subset \mathcal{W}$.

With these facts at hand, we can proceed with the solvability analysis of the uniform output regulation problem.

4.4 Solvability of the global (forward time) uniform output regulation problem

In this section we apply the invariant manifold theorems to study solvability of the global uniform output regulation problem. Since there are two variants of the global uniform output regulation problem, we will obtain solvability results for both. Moreover, we will present solvability results for the robust global uniform output regulation problem.

4.4.1 Solvability of the global uniform output regulation problem

The next theorem, which is based on Theorem 4.4, establishes necessary and sufficient conditions for a controller (3.6), (3.7) to solve the global uniform output regulation problem.

Theorem 4.12. *Consider system (3.1)–(3.3) and exosystem (3.4) satisfying the boundedness assumption* **A1** *in an open invariant set of initial conditions $\mathcal{W}$. The following statements are equivalent:*

(i) *Controller (3.6), (3.7) solves the global uniform output regulation problem.*
(ii) *The closed-loop system is globally uniformly convergent with the UBSS property for the class of inputs $\mathcal{I}_s(\mathcal{W})$ and there exist continuous mappings $\pi : \mathcal{W} \to \mathbb{R}^n$ and $\sigma : \mathcal{W} \to \mathbb{R}^q$ satisfying the equations*

$$\frac{d}{dt}\pi(w(t)) = f(\pi(w), \theta(\sigma(w), h_m(\pi(w), w)), w),$$
$$\frac{d}{dt}\sigma(w(t)) = \eta(\sigma(w), h_m(\pi(w), w)), \tag{4.18}$$
$$\forall \;\; w(t) = w(t, w_0) \in \mathcal{W},$$
$$0 = h_r(\pi(w), w) \quad \forall \;\; w \in \Omega(\mathcal{W}). \tag{4.19}$$

(iii) *There exist continuous mappings $\pi : \mathcal{W} \to \mathbb{R}^n$ and $\sigma : \mathcal{W} \to \mathbb{R}^q$ satisfying equations (4.18) and (4.19) and for every $w(\cdot) \in \mathcal{I}_s(\mathcal{W})$ the solution of the closed-loop system $(\bar{x}_w(t), \bar{\xi}_w(t)) = (\pi(w(t)), \sigma(w(t)))$ is globally uniformly asymptotically stable.*

Proof: We will prove the equivalence of (i), (ii) and (iii) in the following sequence: (i)⇒(ii)⇒(iii)⇒(i).

(i)⇒(ii). Suppose controller (3.6), (3.7) solves the global uniform output regulation problem. Then the closed-loop system (3.8), (3.9) is globally uniformly convergent with the UBSS property for the class of inputs $\mathcal{I}_s(\mathcal{W})$. By Theorem 4.4, this implies the existence of a continuous mapping $\alpha(w)$ such that the graph of this mapping

$$\mathcal{M}(\mathcal{W}) := \{(x, \xi, w)) : (x, \xi) = \alpha(w), w \in \mathcal{W}\}$$

is invariant with respect to the closed-loop system (3.8), (3.9) and the exosystem (3.4). Denote by $\pi(w)$ and $\sigma(w)$ the x- and ξ-components of the mapping $\alpha(w)$. Since the graph $\mathcal{M}(\mathcal{W})$ is invariant, for any solution of the exosystem $w(t)$ starting in $w(0) \in \mathcal{W}$, the pair of functions $(\pi(w(t)), \sigma(w(t)))$ represents a solution of the closed-loop system (3.8), (3.9). This implies that the functions $\pi(w(t))$ and $\sigma(w(t)))$ satisfy (4.18). Since the regulated output $e(t)$ tends to zero along any solution of the closed-loop system and the exosystem starting in $(x(0), \xi(0)) \in \mathbb{R}^{n+q}$ and $w(0) \in \mathcal{W}$, respectively, $e(t)$ also tends to zero along the solution $(\pi(w(t)), \sigma(w(t)), w(t))$, i.e.,

$$h_r(\pi(w(t)), w(t)) \to 0 \quad \text{as} \;\; t \to +\infty. \tag{4.20}$$

Let us show that this fact implies (4.19). Suppose there exists $w_* \in \Omega(\mathcal{W})$ such that $h_r(\pi(w_*), w_*) \neq 0$. By the definition of the ω-limit set $\Omega(\mathcal{W})$, there exists a solution $w(t)$ starting in $w(0) \in \mathcal{W}$ and a sequence $\{t_k\}_{k=1}^{+\infty}$ such that $t_k \to +\infty$ and $w(t_k) \to w_*$ as $k \to +\infty$. Since $h_r(\pi(w), w)$ is continuous in $\mathcal{W}$, we obtain

$$h_r(\pi(w(t_k)), w(t_k)) \to h_r(\pi(w_*), w_*) \neq 0, \quad \text{as} \;\; k \to +\infty.$$

This contradicts (4.20). Thus, indeed, the equality (4.19) holds. This completes the proof of this implication.

(ii)⇒(iii). Since the closed-loop system (3.8), (3.9) is globally uniformly convergent with the UBSS property for the class of inputs $\mathcal{I}_s(\mathcal{W})$, by Theorem 4.4 for every solution of the exosystem $w(t)$ starting in $\mathcal{W}$, the solution of the closed-loop system $(\bar{x}_w(t), \bar{\xi}_w(t)) := (\pi(w(t), \sigma(w(t)))$ lying on this manifold is uniformly globally asymptotically stable.

(iii)⇒(i). By Theorem 4.4, the existence of the continuous mappings $\pi(w)$ and $\sigma(w)$ given in (iii) implies that the closed-loop system (3.8), (3.9) is globally uniformly convergent with the UBSS property for the class of inputs $\mathcal{I}_s(\mathcal{W})$. We only need to show that for any solution of the closed-loop system and the exosystem starting in $(x(0), \xi(0)) \in \mathbb{R}^{n+q}$ and $w(0) \in \mathcal{W}$, the regulated output $e(t)$ tends to zero. Consider a solution of the exosystem $w(t)$ starting in $w(0) \in \mathcal{W}$ and the solution of the closed-loop system $(\bar{x}_w(t), \bar{\xi}_w(t)) := (\pi(w(t)), \sigma(w(t))$. Since the solution $(\bar{x}_w(t), \bar{\xi}_w(t))$ is globally uniformly asymptotically stable, for any other solution of the closed-loop system $(x(t), \xi(t))$ it holds that $x(t) - \pi(w(t)) \to 0$ and $\xi(t) - \sigma(w(t)) \to 0$ as $t \to +\infty$. Thus,

$$e(t) = h_r(x(t), w(t)) \to h_r(\pi(w(t)), w(t)) \quad \text{as} \quad t \to +\infty. \tag{4.21}$$

At the same time, $\mathrm{dist}(w(t), \Omega(\mathcal{W})) \to 0$ as $t \to +\infty$ (see Section 4.3). Since $w(t)$ is bounded, this implies

$$h_r(\pi(w(t)), w(t)) \to h_r(\pi(\Omega(\mathcal{W})), \Omega(\mathcal{W})) = \{0\} \quad \text{as} \quad t \to +\infty.$$

Together with (4.21), this implies $e(t) = h_r(x(t), w(t)) \to 0$ as $t \to +\infty$. This completes the proof of the theorem. □

Remark. In the literature, global variants of the output regulation problem are considered mostly for the case of exosystems being linear harmonic oscillators. Such exosystems satisfy the boundedness assumption **A1**. Many of the proposed controllers solving such variants of the global output regulation problem (see, e.g., [12, 58, 69, 79]) are designed in such a way that they guarantee existence and global uniform asymptotic stability of a sufficiently smooth invariant manifold $(x, \xi) = (\pi(w), \sigma(w))$, with $\pi(w)$ and $\sigma(w)$ satisfying (4.18), (4.19). As follows from Theorem 4.12, such controllers solve the global uniform output regulation problem.◁

Theorem 4.12 provides a criterion for checking whether a particular controller solves the global uniform output regulation problem. It can be used directly for controller design (we will address this problem in Chapter 5) in the following way: given *some* controller such that the corresponding closed-loop system satisfies the conditions (ii) or (iii) in Theorem 4.12, this theorem guarantees that this controller solves the global uniform output regulation problem. Alternatively, we can specifically design a controller such that the corresponding closed-loop system satisfies conditions (ii) or (iii). At the same time, Theorem 4.12 allows one to obtain certain controller-independent necessary conditions for the solvability of the global uniform output regulation problem as follows from the next lemma.

Lemma 4.13. *The global uniform output regulation problem is solvable* only if *there exist continuous mappings $\pi(w)$ and $c(w)$ defined in some neighborhood of $\Omega(\mathcal{W})$ satisfying the equations*

$$\frac{d}{dt}\pi(w(t)) = f(\pi(w(t)), c(w(t)), w(t)), \tag{4.22}$$

$$0 = h_r(\pi(w(t)), w(t)), \tag{4.23}$$

for all solutions of the exosystem $w(t)$ satisfying $w(t) \in \Omega(\mathcal{W})$, $t \in \mathbb{R}$.

Proof: The statement of the lemma is obtained from (4.18) and (4.19) by denoting $c(w) := \theta(\sigma(w), h_m(\pi(w), w))$. □

Equations (4.22) and (4.23) are the so-called *regulator equations*, see, e.g., [8, 38, 39]. Solvability of the regulator equations guarantees that for every solution of the exosystem lying in the ω-limit set $\Omega(\mathcal{W})$ there exists a control input $\bar{u}_w(t) := c(w(t))$ for which system (3.1) has the solution $\bar{x}_w(t) := \pi(w(t))$, and along this solution the regulated output equals zero. Notice that the ω-limit set $\Omega(\mathcal{W})$ can be treated, in a certain sense, as the steady-state dynamics of the exosystem, because this set is invariant and attracts all solutions of the exosystem starting in $\mathcal{W}$. From this point of view, solvability of the regulator equations can be interpreted in the following way: for any solution $w(t)$ of the exosystem from the steady-state dynamics set $\Omega(\mathcal{W})$, there exists at least one control input $\bar{u}_w(t)$ such that system (3.1) with these $w(t)$ and $\bar{u}_w(t)$ has a solution $\bar{x}_w(t)$ along which the regulated output $e(t)$ is identically zero.

Originally, solvability of the regulator equations in some neighborhood of the origin was obtained as a necessary condition for the solvability of the conventional local output regulation problem under the assumption that exosystem (3.4) is neutrally stable. Lemma 4.13 shows that solvability of the regulator equations (4.22) and (4.23) is also necessary for the solvability of the *global* uniform output regulation problem.

With the regulator equations at hand, we can obtain further necessary conditions for the solvability of the global uniform output regulation problem. As follows from (4.18), controller (3.6), (3.7) is such that if we excite it with the input $\bar{y}_w(t) := h_m(\pi(w(t)), w(t))$, for some solution of the exosystem $w(t) \in \Omega(\mathcal{W})$, it has a solution $\bar{\xi}_w(t) = \sigma(w(t))$, which is bounded on $\mathbb{R}$, and along this solution the output of the controller equals $\bar{u}_w(t) = c(w(t))$, where $\pi(w)$ and $c(w)$ are solutions of the regulator equations defined above. This property motivates the introduction of the following definition.

Definition 4.14. *Consider controller (3.6), (3.7). Let $\bar{y}(t)$ and $\bar{u}(t)$ be defined and bounded for all $t \in \mathbb{R}$. We say that the input $\bar{y}(t)$ induces the output $\bar{u}(t)$*

in controller (3.6), (3.7), if for this $\bar{y}(t)$ system (3.6), (3.7) has a solution $\bar{\xi}(t)$ defined and bounded on $\mathbb{R}$ and satisfying the equality $\bar{u}(t) = \theta(\bar{\xi}(t), \bar{y}(t))$ for all $t \in \mathbb{R}$.

We will say that controller (3.6), (3.7) has a *generalized internal model property* if for any solution of the exosystem $w(t)$ lying in the ω-limit set $\Omega(\mathcal{W})$ the input $\bar{y}_w(t) := h_m(\pi(w(t)), w(t))$ induces the output $\bar{u}_w(t) = c(w(t))$ in controller (3.6), (3.7). The generalized internal model property closely relates to the notions of immersion and internal models used in the output regulation theory (see [8, 40, 42] for further details on immersion and internal models).

With these definitions at hand, we obtain the following necessary condition for the solvability of the global uniform output regulation problem.

Lemma 4.15. *Suppose the global uniform output regulation problem is solvable. Then there exists a controller of the form (3.6), (3.7) such that for any solution of the exosystem $w(t)$ lying in the ω-limit set $\Omega(\mathcal{W})$ the input $\bar{y}_w(t) := h_m(\pi(w(t)), w(t))$ induces the output $\bar{u}_w(t) = c(w(t))$ in the controller (3.6), (3.7), where $c(w)$ and $\pi(w)$ are solutions to the regulator equations (4.22) and (4.23). In other words, there exists a controller with the generalized internal model property. Moreover, the closed-loop system corresponding to this controller is globally uniformly convergent with the UBSS property for the class of inputs $\mathcal{I}_s(\mathcal{W})$.*

The requirement that the controller makes the corresponding closed-loop system globally uniformly convergent with the UBSS property for the class of inputs $\mathcal{I}_s(\mathcal{W})$ is natural, since it comes from the problem statement. The generalized internal model property guarantees that controller (3.6), (3.7) is capable of generating the steady-state control $\bar{u}_w(t) = c(w(t))$ (see Lemma 4.13) based on the measured signal $y(t)$.

Lemmas 4.13 and 4.15 provide necessary conditions for the solvability of the global uniform output regulation problem. In fact, as follows from the next theorem, these conditions are not only necessary, but also sufficient for the solvability of the problem.

Theorem 4.16. *Consider system (3.1)–(3.3) and exosystem (3.4) satisfying the boundedness assumption* **A1** *in an open invariant set of initial conditions $\mathcal{W}$. The global uniform output regulation problem is solvable if and only if the following conditions are satisfied:*

(i) *There exist continuous mappings $\pi(w)$ and $c(w)$ defined in some neighborhood of $\Omega(\mathcal{W})$ and satisfying the regulator equations (4.22) and (4.23) for all solutions $w(t)$ of exosystem (3.4) satisfying $w(t) \in \Omega(\mathcal{W})$ for all $t \in \mathbb{R}$.*
(ii) *There exists a controller of the form (3.6), (3.7) such that for any solution of the exosystem $w(t)$ lying in the set $\Omega(\mathcal{W})$ the input $\bar{y}_w(t) := h_m(\pi(w(t)), w(t))$ induces the output $\bar{u}_w(t) = c(w(t))$ in controller (3.6),*

(3.7), and the closed-loop system corresponding to this controller is globally uniformly convergent with the UBSS property for the class of inputs $\mathcal{I}_s(\mathcal{W})$.

Under these conditions, a controller solves the global uniform output regulation problem if and only if it satisfies the conditions given in (ii).

Proof: The *only if* part of the theorem follows from Lemmas 4.13 and 4.15. We only need to show that if the condition (i) is satisfied then a controller satisfying the conditions given in (ii) solves the global uniform output regulation problem. We will do this by showing that if a controller satisfies the conditions given in (ii), then the corresponding closed-loop system satisfies condition (ii) in Theorem 4.12. Thus, by Theorem 4.12 this controller solves the global uniform output regulation problem.

Suppose controller (3.6), (3.7) satisfies the conditions given in (ii). Then by Theorem 4.4 there exist continuous functions $\tilde{\pi}(w)$ and $\tilde{\sigma}(w)$ such that the graph $(x, \xi) = (\tilde{\pi}(w), \tilde{\sigma}(w))$ for $w \in \mathcal{W}$ is invariant with respect to the closed-loop system (3.8), (3.9) and the exosystem (3.4). This implies that $\tilde{\pi}(w)$ and $\tilde{\sigma}(w)$ satisfy the following equations:

$$\begin{aligned} \frac{d}{dt}\tilde{\pi}(w(t)) &= f(\tilde{\pi}(w), \theta(\tilde{\sigma}(w), h_m(\tilde{\pi}(w), w)), w), \\ \frac{d}{dt}\tilde{\sigma}(w(t)) &= \eta(\tilde{\sigma}(w), h_m(\tilde{\pi}(w), w)), \end{aligned} \tag{4.24}$$

for all solutions of the exosystem $w(t)$ lying in the set $\mathcal{W}$. Moreover, for every $w(t)$ lying in $\mathcal{W}$, the solution of the closed-loop system $(\tilde{x}_w(t), \tilde{\xi}_w(t)) := (\tilde{\pi}(w(t)), \tilde{\sigma}(w(t)))$ is globally uniformly asymptotically stable. Let us show that the mapping $\tilde{\pi}(w)$ also satisfies the equation

$$h_r(\tilde{\pi}(w), w) = 0 \quad \forall \ \ w \in \Omega(\mathcal{W}). \tag{4.25}$$

Once this equality is proved, by Theorem 4.12 we obtain that controller (3.6), (3.7) solves the global uniform output regulation problem.

In order to prove (4.25), we will show that

$$\pi(w(t)) \equiv \tilde{\pi}(w(t)) \tag{4.26}$$

for any solution of the exosystem lying in $\Omega(\mathcal{W})$. Then equality (4.25) will follow from (4.23) and from the fact that $\Omega(\mathcal{W})$ is an invariant set with respect to system (3.1) (i.e., for any $w_* \in \Omega(\mathcal{W})$ there exists a solution $w(t)$ lying in $\Omega(\mathcal{W})$ for all $t \in \mathbb{R}$ and satisfying $w(0) = w_*$).

Let us first show that for every solution $w(t)$ lying in $\Omega(\mathcal{W})$ the closed-loop system (3.8), (3.9) has a solution $(\bar{x}_w(t), \bar{\xi}_w(t))$ which is defined and bounded for all $t \in \mathbb{R}$. This fact follows from the regulator equations (4.22) and from the property of the controller that for the input $\bar{y}_w(t) := h_m(\pi(w(t)), w(t))$ it has a solution $\bar{\xi}_w(t)$ which is defined and bounded for all $t \in \mathbb{R}$ and for which $\theta(\bar{\xi}_w(t), h_m(\pi(w(t)), w(t))) \equiv c(w(t))$ for all $t \in \mathbb{R}$. Substituting this $(\bar{x}_w(t), \bar{\xi}_w(t)) := (\pi(w(t)), \bar{\xi}_w(t))$ in the equations of the closed-loop system (3.8), (3.9), one can easily check that such a pair $(\bar{x}_w(t), \bar{\xi}_w(t))$ is indeed a solution of the closed-loop system. Since $w(t)$ lies in a compact subset of $\Omega(\mathcal{W})$ (due to assumption **A1**) and since $\pi(w)$ is continuous in some neighborhood of $\Omega(\mathcal{W})$, the function $\pi(w(t))$ and hence $(\bar{x}_w(t), \bar{\xi}_w(t))$ are bounded for all $t \in \mathbb{R}$.

Recall that the solution $(\tilde{x}_w(t), \tilde{\xi}_w(t)) := (\tilde{\pi}(w(t)), \tilde{\sigma}(w(t)))$ is defined and bounded for all $t \in \mathbb{R}$ and it is globally uniformly asymptotically stable. By Property 2.4, this implies that $(\tilde{x}_w(t), \tilde{\xi}_w(t)) \equiv (\bar{x}_w(t), \bar{\xi}_w(t))$ for $t \in \mathbb{R}$. This, in turn, implies (4.26), which completes the proof of the theorem. $\square$

Theorem 4.16 provides a way to solve the global uniform output regulation problem. First, one needs to solve the regulator equations (4.22) and (4.23) (or show that they are not solvable, which implies that the problem cannot be solved) and then to find a controller satisfying the conditions given in (ii). Particular ways of finding such controllers will be discussed in Chapter 5.

4.4.2 Solvability of the robust global uniform output regulation problem

In this section we provide solvability conditions for the robust global uniform output regulation problem. In this problem we consider systems of the form (3.10)–(3.12) depending on a vector of unknown, but constant, parameters p taken from an open set $\mathcal{P}$. The problem is to find a controller (independent of p) that solves the global uniform output regulation problem for all $p \in \mathcal{P}$. This problem can be reduced to a regular variant of the global uniform output regulation problem by extending the exosystem in the following way:

$$\begin{pmatrix} \dot{w} \\ \dot{p} \end{pmatrix} = \begin{pmatrix} s(w) \\ 0 \end{pmatrix} =: \hat{s}(w, p). \tag{4.27}$$

After such an extension the parameter p is considered to be a part of the exosignal. Notice that if the original exosystem (3.4) satisfies the boundedness assumption **A1** in a certain open set $\mathcal{W} \subset \mathbb{R}^m$, then the extended exosystem (4.27) satisfies assumption **A1** in the set $\mathcal{W} \times \mathcal{P}$. Therefore, controller (3.6), (3.7) solves the global uniform output regulation problem for all parameters p taken from the set $\mathcal{P}$ if it solves the global uniform output regulation problem for the extended exosystem (4.27), with (w, p) being a new state of the exosystem. The converse statement is not true, in general, because the UBSS property of the closed-loop system for the class of inputs $\mathcal{I}_s(\mathcal{W})$ for every parameter $p \in \mathcal{P}$, which is required in the problem formulation of the robust global uniform output regulation problem, does not imply the UBSS property of the closed-loop system for the class of extended inputs $\mathcal{I}_{\hat{s}}(\mathcal{W} \times \mathcal{P})$, where $\mathcal{I}_{\hat{s}}(\mathcal{W} \times \mathcal{P})$ denotes all solutions of the extended exosystem (4.27) starting in the open invariant set $\mathcal{W} \times \mathcal{P}$. In fact, solvability of the global uniform output regulation problem for the extended exosystem (4.27) is necessary for the solvability of the so-called *strong robust global uniform output regulation problem*, which is formulated in the following way.

> Controller (3.6), (3.7) solves the **strong robust global uniform output regulation problem** if it solves the global uniform output regulation problem for all $p \in \mathcal{P}$, and for any compact subsets $\mathcal{K}_w \subset \mathcal{W}$ and $\mathcal{K}_p \subset \mathcal{P}$ there exists a compact set $\mathcal{K}_z \subset \mathbb{R}^d$ such that for any solution of the exosystem $w(t)$ starting in $w(0) \in \mathcal{K}_w$ and any parameter $p \in \mathcal{K}_p$ the corresponding steady-state solution $\bar{z}_{wp}(t)$ of the closed-loop system lies in the set $\mathcal{K}_z$ for all $t \in \mathbb{R}$.

One can easily check that this strong robust global uniform output regulation problem is equivalent to the global uniform output regulation problem for system (3.10)–(3.12) and exosystem (4.27). Using this fact, we can apply the results obtained in the previous section to study solvability of the strong robust global uniform output regulation problem. Consequently, we can formulate the following results, which are counterparts of Theorems 4.12 and 4.16.

Theorem 4.17. *Consider system (3.10)–(3.12) with the parameter p taken from an open set $\mathcal{P}$ and exosystem (3.4) satisfying the boundedness assumption* **A1** *in an open invariant set of initial conditions $\mathcal{W}$. The following statements are equivalent:*

(i) *Controller (3.6), (3.7) solves the strong robust global uniform output regulation problem.*

(ii) *The closed-loop system is globally uniformly convergent with the UBSS property for the class of inputs $\mathcal{I}_{\hat{s}}(\mathcal{W} \times \mathcal{P})$ and there exist continuous mappings $\pi(\cdot,\cdot) : \mathcal{W} \times \mathcal{P} \to \mathbb{R}^n$ and $\sigma(\cdot,\cdot) : \mathcal{W} \times \mathcal{P} \to \mathbb{R}^q$ satisfying the equations*

$$
\begin{aligned}
\frac{d}{dt}\pi(w(t),p) &= f(\pi(w,p), \theta(\sigma(w,p), h_m(\pi(w,p),w,p)), w, p),\\
\frac{d}{dt}\sigma(w(t),p) &= \eta(\sigma(w,p), h_m(\pi(w,p),w,p)), \qquad (4.28)\\
&\forall \;\; w(t) = w(t,w_0) \in \mathcal{W}, \quad p \in \mathcal{P},\\
0 &= h_r(\pi(w,p),w,p) \quad \forall \;\; w \in \Omega(\mathcal{W}), \quad p \in \mathcal{P}. \qquad (4.29)
\end{aligned}
$$

(iii) *There exist continuous mappings $\pi(\cdot,\cdot) : \mathcal{W} \times \mathcal{P} \to \mathbb{R}^n$ and $\sigma(\cdot,\cdot) : \mathcal{W} \times \mathcal{P} \to \mathbb{R}^q$ satisfying (4.28) and (4.29) and for every $w(\cdot) \in \mathcal{I}_s(\mathcal{W})$ and every $p \in \mathcal{P}$ the solution $(\bar{x}_{wp}(t), \bar{\xi}_{wp}(t)) = (\pi(w(t),p), \sigma(w(t),p))$ is globally uniformly asymptotically stable.*

The next theorem provides solvability conditions for the strong robust global uniform output regulation problem. It directly follows from Theorem 4.16.

Theorem 4.18. *Consider system (3.10)–(3.12) with the parameter p taken from an open set $\mathcal{P}$ and exosystem (3.4) satisfying the boundedness assumption* **A1** *in an open invariant set of initial conditions $\mathcal{W}$. The strong robust global uniform output regulation problem is solvable if and only if the following conditions are satisfied:*

(i) *There exist continuous mappings $\pi(w,p)$ and $c(w,p)$ defined in some neighborhood of $\Omega(\mathcal{W}) \times \mathcal{P}$, satisfying the regulator equations*

$$\frac{d}{dt}\pi(w(t),p) = f(\pi(w(t),p), c(w(t),p), w(t), p), \tag{4.30}$$

$$0 = h_r(\pi(w(t),p), w(t), p), \tag{4.31}$$

for all solutions $w(t)$ of the exosystem (3.4) lying in the set $\Omega(\mathcal{W})$ and for all $p \in \mathcal{P}$.

(ii) *There exists a controller of the form (3.6), (3.7) such that for any solution of the exosystem $w(t)$ lying in the set $\Omega(\mathcal{W})$ and for any $p \in \mathcal{P}$ the input $\bar{y}_w(t) := h_m(\pi(w(t),p), w(t), p)$ induces the output $\bar{u}(t) = c(w(t),p)$ in controller (3.6), (3.7) and the closed-loop system corresponding to this controller is globally uniformly convergent with the UBSS property for the class of inputs $\mathcal{I}_{\hat{s}}(\mathcal{W} \times \mathcal{P})$.*

Under these conditions, a controller solves the strong robust global uniform output regulation problem if and only if it satisfies the conditions given in (ii).

4.4.3 Solvability of the global forward time uniform output regulation problem

Solvability of the global forward time uniform output regulation problem can be studied in a similar way as solvability of the global uniform output regulation problem. The main difference is that instead of Theorem 4.4, which forms the foundation for the analysis in the previous sections, the results in this section are based on Theorem 4.6. The proofs are identical to the proofs of Theorems 4.12 and 4.16 and are omitted here. The first theorem, which is a counterpart of Theorem 4.12, provides necessary and sufficient conditions under which a controller solves the global forward time uniform output regulation problem.

Theorem 4.19. *Consider system (3.1)–(3.3) and exosystem (3.4) with a compact positively invariant set of initial conditions $\mathcal{W}_+ \subset \mathbb{R}^m$. The following statements are equivalent:*

(i) *Controller (3.6), (3.7) solves the global forward time uniform output regulation problem.*

(ii) *There exist continuous mappings* $\pi : \widetilde{\mathcal{W}} \to \mathbb{R}^n$ *and* $\sigma : \widetilde{\mathcal{W}} \to \mathbb{R}^q$*, where* $\widetilde{\mathcal{W}} \subset \mathbb{R}^m$ *is some neighborhood of* $\mathcal{W}_+$*, satisfying*

$$\begin{aligned} \frac{d}{dt}\pi(w(t)) &= f(\pi(w), \theta(\sigma(w), h_m(\pi(w), w)), w), \\ \frac{d}{dt}\sigma(w(t)) &= \eta(\sigma(w), h_m(\pi(w), w)), \\ &\qquad \forall \;\; w(t) = w(t, w_0) \in \mathcal{W}_+, \quad \text{for } t \geq 0, \end{aligned} \tag{4.32}$$

$$0 = h_r(\pi(w), w) \quad \forall \;\; w \in \Omega(\mathcal{W}_+), \tag{4.33}$$

and the closed-loop system (3.8), (3.9) is globally uniformly convergent with the UBSS property for the class of inputs $\overline{\mathbb{PC}}(\widetilde{\mathcal{W}})$.

The next theorem is a counterpart of Theorem 4.16. It provides necessary and sufficient conditions for solvability of the global forward time uniform output regulation problem.

Theorem 4.20. *Consider system (3.1)–(3.3) and exosystem (3.4) with a compact positively invariant set of initial conditions* $\mathcal{W}_+ \subset \mathbb{R}^m$*. The global forward time uniform output regulation problem is solvable if and only if the following conditions are satisfied:*

(i) *There exist continuous mappings* $\pi(w)$ *and* $c(w)$ *defined in some neighborhood of* $\Omega(\mathcal{W}_+)$ *and satisfying the regulator equations*

$$\frac{d}{dt}\pi(w(t)) = f(\pi(w(t)), c(w(t)), w(t)), \tag{4.34}$$

$$0 = h_r(\pi(w(t)), w(t)), \tag{4.35}$$

for all solutions of exosystem (3.4) satisfying $w(t) \in \Omega(\mathcal{W}_+)$ *for* $t \in \mathbb{R}$.

(ii) *There exists a controller of the form (3.6), (3.7) such that for any solution of the exosystem* $w(t)$ *lying in the set* $\Omega(\mathcal{W}_+)$ *the input* $\bar{y}_w(t) := h_m(\pi(w(t)), w(t))$ *induces the output* $\bar{u}_w(t) = c(w(t))$ *in controller (3.6), (3.7), and the closed-loop system corresponding to this controller is globally uniformly convergent with the UBSS property for the class of inputs* $\overline{\mathbb{PC}}(\widetilde{\mathcal{W}})$*, where* $\widetilde{\mathcal{W}}$ *is some neighborhood of* $\mathcal{W}_+$.

Under these conditions, a controller solves the global forward time uniform output regulation problem if and only if it satisfies the conditions given in (ii).

Results related to solvability of the robust variant of the global forward time uniform output regulation problem can be obtained in the same way as in Section 4.4.2.

4.5 Solvability of the local uniform output regulation problem

Analysis of the solvability of the local uniform output regulation problem is very close to the analysis in the global case (see Section 4.4.1). Analysis in the local case is based on the local invariant manifold theorem (Theorem 4.8). We omit the proofs of the results presented in this section since they are nearly identical to the proofs of the results from Section 4.4.1.

The following theorem provides necessary and sufficient conditions for a controller of the form (3.6), (3.7) to solve the local uniform output regulation problem.

Theorem 4.21. *Consider system (3.1)–(3.3) and exosystem (3.4) satisfying the neutral stability assumption. The following statements are equivalent:*

(i) *Controller (3.6), (3.7) solves the local uniform output regulation problem.*
(ii) *There exist continuous mappings $\pi(w)$ and $\sigma(w)$ defined in some invariant neighborhood of the origin $\mathcal{W} \subset \mathbb{R}^m$, satisfying $\pi(0) = 0$, $\sigma(0) = 0$, and*

$$\begin{aligned} \frac{d}{dt}\pi(w(t)) &= f(\pi(w), \theta(\sigma(w), h_m(\pi(w), w)), w), \\ \frac{d}{dt}\sigma(w(t)) &= \eta(\sigma(w), h_m(\pi(w), w)), \\ &\forall \;\; w(t) = w(t, w_0) \in \mathcal{W}, \end{aligned} \tag{4.36}$$

$$0 = h_r(\pi(w), w), \quad \forall \; w \in \mathcal{W}, \tag{4.37}$$

for all $w(\cdot) \in \mathcal{I}_s(\mathcal{W})$, and the closed-loop system (3.8), (3.9) corresponding to this controller is locally uniformly convergent for the class of inputs $\mathcal{I}_s(\mathcal{W})$.

The main difference between Theorem 4.21 and Theorem 4.12 (if we do not take into account that in the first case the analysis is local and in the second it is global) is in (4.37) and (4.19). In (4.37), the equality

$$h_r(\pi(w), w) = 0 \tag{4.38}$$

is required for all $w \in \mathcal{W}$, while in (4.19) this equality is required only for the set $\Omega(\mathcal{W})$. This difference is explained by the fact that the exosystem is neutrally stable. By the definition (see Definition 3.1), neutral stability implies that for some neighborhood of the origin $\widehat{\mathcal{W}}$ it holds that $\widehat{\mathcal{W}} \subset \Omega(\widehat{\mathcal{W}})$. Thus, for a sufficiently small neighborhood $\mathcal{W}$ of the origin the equality $h_r(\pi(w), w) = 0$ for all $w \in \Omega(\mathcal{W})$ implies that this equality is satisfied for all $w \in \mathcal{W}$. The opposite is also true. If equality (4.38) is satisfied for all w in some invariant neighborhood of the origin $\mathcal{W}$, one can choose another invariant neighborhood of the origin $\widetilde{\mathcal{W}}$ such that equality (4.38) holds for all $w \in \Omega(\widetilde{\mathcal{W}})$. The proof of this statement is as follows. From the definition of the set $\Omega(\widetilde{\mathcal{W}})$ one can conclude that $\Omega(\widetilde{\mathcal{W}}) \subset \operatorname{clos}(\widetilde{\mathcal{W}})$, where $\operatorname{clos}(\widetilde{\mathcal{W}})$ is the closure of the set $\widetilde{\mathcal{W}}$. Hence, if we find an invariant neighborhood of the origin $\widetilde{\mathcal{W}}$ such that $\mathrm{clos}(\widetilde{\mathcal{W}}) \subset \mathcal{W}$, then equality (4.38) is satisfied for all $w \in \Omega(\widetilde{\mathcal{W}})$. Such a neighborhood $\widetilde{\mathcal{W}}$ exists, because the trivial solution $w(t) \equiv 0$ is stable in forward and backward time (see the proof of Theorem 4.8, where this statement is proved and used several times).

The next theorem provides a local counterpart of Theorem 4.16.

Theorem 4.22. *Consider system (3.1)–(3.3) and exosystem (3.4) satisfying the neutral stability assumption. The local uniform output regulation problem is solvable if and only if the following conditions are satisfied:*

(i) *There exist continuous mappings $\pi(w)$ and $c(w)$ defined in some invariant neighborhood of the origin $\mathcal{W} \subset \mathbb{R}^m$, such that $\pi(0) = 0$, $c(0) = 0$, and*

$$\frac{d}{dt}\pi(w(t)) = f(\pi(w(t)), c(w(t)), w(t)), \tag{4.39}$$

$$0 = h_r(\pi(w(t)), w(t)), \tag{4.40}$$

for all solutions of exosystem (3.4) satisfying $w(t) \in \mathcal{W}$ for all $t \in \mathbb{R}$.

(ii) *There exists a controller of the form (3.6), (3.7) satisfying the following conditions:* **a)** *there exists a continuous mapping $\sigma : \mathcal{W} \to \mathbb{R}^q$ satisfying $\sigma(0) = 0$ and*

$$\begin{aligned} \frac{d}{dt}\sigma(w(t)) &= \theta(\sigma(w), h_m(\pi(w), w)), \\ c(w(t)) &= \theta(\sigma(w(t)), h_m(\pi(w(t)), w(t))), \end{aligned} \tag{4.41}$$

for all $w(t) \in \mathcal{W}$, and **b)** *the closed-loop system corresponding to this controller is locally uniformly convergent for the class of inputs $\mathcal{I}_s(\mathcal{W})$.*

Under these conditions, a controller satisfying the conditions given in (ii) *solves the local uniform output regulation problem.*

Remark. The requirement that the controller satisfy (4.41) for some continuous $\sigma(w)$ guarantees that for any solution of the exosystem $w(t)$ lying in the set $\mathcal{W}$ for all $t \in \mathbb{R}$ the input $\bar{y}_w(t) := h_m(\pi(w(t)), w(t))$ induces the output $\bar{u}_w(t) = c(w(t))$ in controller (3.6) (3.7).$\lhd$

4.6 Applications of the invariant manifold theorems

All solvability results presented in this chapter are based on the invariant manifold theorems (Theorems 4.4, 4.6, and 4.8). Although these theorems were derived for studying the output regulation problem, they are interesting in their own respect. In this section we discuss how these invariant manifold theorems can be applied in the scope of so-called *generalized synchronization* and for the analysis of nonlinear systems excited by harmonic signals.

4.6.1 Generalized synchronization

In the field of master-slave synchronization one considers coupled systems of the form

$$\dot{z} = F(z, w), \tag{4.42}$$
$$\dot{w} = s(w). \tag{4.43}$$

System (4.43) can be treated as a *master* system that generates a driving signal for the *slave* system (4.42). One of the phenomena studied in the context of the master-slave synchronization is the so-called *generalized synchronization* [64, 65, 76]. Roughly speaking, generalized synchronization occurs if for some continuous function $\alpha(w)$ all solutions $z(t)$ of system (4.42) converge to the manifold $z = \alpha(w)$, i.e., $\lim_{t\to+\infty}(z(t) - \alpha(w(t))) = 0$. As follows from Theorem 4.6, if all solutions of system (4.43) start in a compact positively invariant set $\mathcal{W}_+$ and system (4.42) is globally uniformly convergent with the UBSS property for the class of inputs $\overline{\mathbb{PC}}(\widetilde{\mathcal{W}})$, where $\widetilde{\mathcal{W}}$ is some neighborhood of $\mathcal{W}_+$, then there exists a continuous function $\alpha(w)$ defined in $\widetilde{\mathcal{W}}$ such that for all initial conditions $z(0) \in \mathbb{R}^d$ and $w(0) \in \mathcal{W}_+$ it holds that $\lim_{t\to+\infty}(z(t) - \alpha(w(t))) = 0$. Since the ω-limit set $\Omega(\mathcal{W}_+)$ is an invariant set inside $\mathcal{W}_+$, Theorem 4.6 implies that the mapping $\alpha(w)$ is uniquely defined for all $w \in \Omega(\mathcal{W}_+)$. Therefore, we see that the result of Theorem 4.6 can be applied for studying generalized synchronization phenomena.

4.6.2 Nonlinear frequency response functions

A common way to analyze the behavior of a dynamical system is to investigate its responses to harmonic excitations at different frequencies. For linear systems, the information on responses to harmonic excitations, which is contained in frequency response functions and usually represented in the form of Bode plots, allows us to identify the system and analyze its properties such as performance and robustness. Moreover, it serves as a powerful tool for controller design. There exists a vast literature on frequency domain identification, analysis, and controller design methods for linear systems. Most (high-performance) industrial controllers, especially for motion systems, are designed and tuned based on these methods. The lack of such methods for nonlinear systems is one of the reasons why nonlinear systems and controllers are not popular in industry. Even if a (nonlinear) controller achieves a certain control goal (e.g., tracking or disturbance attenuation), which can be proved, for example, using Lyapunov stability methods, it is very difficult to say something about performance of the corresponding nonlinear closed-loop system, while performance is critical in many industrial applications. So, there is a need to extend the linear analysis and controller design methods based on harmonic excitations to nonlinear systems.

One of the first difficulties in such an extension is that a general nonlinear system being excited by a periodic (e.g., harmonic) signal can have none, one, or multiple periodic solutions and, if a periodic solution does exist, its period can differ from the period of the excitation signal. Moreover, if such periodic solutions exist, they essentially depend not only on the excitation frequency, but also on the amplitude of the excitation. As follows from Property 2.23, uniformly convergent systems, although nonlinear, have relatively simple dynamics and for any periodic excitation there exists a unique periodic solution that has the same period as the excitation. Such periodic solutions can be found numerically using, for example, shooting and path following methods [63]. These methods require significant computational efforts, since they are based on the integration of the corresponding differential equations. At the same time, if in addition to the uniform convergence property a system has the UBSS property for the class of bounded piecewise continuous inputs, periodic solutions corresponding to *all* harmonic excitations of the form $u(t) = \mathcal{A}\sin(\omega t)$ for *all* frequencies ω and *all* amplitudes $\mathcal{A}$ can be found from only *one* function. This statement follows from the next theorem.

Theorem 4.23. *Consider the system*

$$\dot{z} = F(z, u), \tag{4.44}$$

$$y = h(z), \tag{4.45}$$

with state $z \in \mathbb{R}^d$, input $u \in \mathbb{R}$ and output $y \in \mathbb{R}$; the function $F(z, u)$ is assumed to be locally Lipschitz with respect to z and continuous with respect to u. Suppose system (4.44) is globally uniformly convergent with the UBSS property for the class of inputs $\overline{\mathbb{PC}}_1$. Then there exists a unique continuous mapping $\alpha : \mathbb{R}^3 \to \mathbb{R}^d$ such that $\bar{z}_u(t) = \alpha(\mathcal{A}\sin(\omega t), \mathcal{A}\cos(\omega t), \omega)$ is a unique periodic solution of system (4.44) corresponding to the excitation input $u(t) = \mathcal{A}\sin(\omega t)$. Moreover, $\bar{z}_u(t)$ is uniformly globally asymptotically stable.

Proof: The proof of this theorem follows from the fact that harmonic signals of the form $u(t) = \mathcal{A}\sin(\omega t)$ for various amplitudes and frequencies are generated by the following system:

$$\begin{aligned} \dot{w}_1 &= \omega w_2, \\ \dot{w}_2 &= -\omega w_1, \\ \dot{\omega} &= 0, \\ u &= w_1. \end{aligned} \tag{4.46}$$

The initial conditions of this system determine the excitation amplitude $\mathcal{A}$ and frequency ω. Thus, we can treat system (4.44) excited by the input $u(t) = \mathcal{A}\sin(\omega t)$ as a system being coupled with exosystem (4.46). One can easily check that system (4.46) satisfies the boundedness assumption **A1**. Thus, by Corollary 4.5 there exists a unique continuous function $\alpha : \mathbb{R}^3 \to \mathbb{R}^d$ such that the steady-state solution corresponding to the solution of the exosystem $(w_1(t), w_2(t), \omega(t)) = (\mathcal{A}\sin(\omega t), \mathcal{A}\cos(\omega t), \omega)$ equals

$$\bar{z}_u(t) = \alpha(\mathcal{A}\sin(\omega t), \mathcal{A}\cos(\omega t), \omega).$$

Since system (4.44) is globally uniformly convergent for the class of inputs $\overline{\mathbb{PC}}_1$, by Property 2.23 we obtain that $\bar{z}_u(t)$ is a unique periodic solution of system (4.44) and, in addition, it is uniformly globally asymptotically stable. $\square$

As follows from Theorem 4.23, the function $\alpha(w_1, w_2, \omega)$ contains *all* information related to periodic solutions of system (4.44) corresponding to harmonic excitations, and the function $h(\alpha(w_1, w_2, \omega))$ contains all information on the periodic outputs corresponding to harmonic excitations. So, the function $h(\alpha(w_1, w_2, \omega))$ can be considered as a *nonlinear frequency response function.* Notice that this frequency response function depends, in the nonlinear case, not only on the frequency of the excitation, but also on its amplitude and phase. For the analysis of nonlinear systems it can be useful to introduce some kind of a magnitude plot for $h(\alpha(w_1, w_2, \omega))$. This can be done in the following way. Suppose we are interested in responses of system (4.44) to harmonic excitations at all frequencies $\omega \geq 0$ and all amplitudes not exceeding some $\mathcal{A}^* > 0$. Define

$$\Upsilon_{\mathcal{A}^*}(\omega) := \sup_{\mathcal{A}\in(0,\mathcal{A}^*]} \left(\sup_{w_1^2+w_2^2=\mathcal{A}^2} \frac{|h(\alpha(w_1, w_2, \omega))|}{\mathcal{A}} \right).$$

This function is a nonlinear analog of the Bode magnitude plot. The meaning of this function is the following. First, we take some $\mathcal{A} \in (0, \mathcal{A}^*]$ and compute the maximal absolute value of the periodic output corresponding to the excitation $u(t) = \mathcal{A}\sin(\omega t)$. Then we divide it by $\mathcal{A}$. Such normalized maximal value is a gain $k(\omega, \mathcal{A})$ with the following meaning: if the harmonic excitation with frequency ω has amplitude $\mathcal{A}$, then the maximal absolute value of the periodic output corresponding to this excitation equals $k(\omega, \mathcal{A})\mathcal{A}$. Finally, $\Upsilon_{\mathcal{A}_*}(\omega)$ is the maximal value of the gain $k(\omega, \mathcal{A})$ over all amplitudes from the set $\mathcal{A} \in (0, \mathcal{A}^*]$. For linear systems of the form $\dot{z} = Az + Bu$ with a Hurwitz matrix A and output $y = Cz$, the gain $k(\omega, \mathcal{A})$ is independent of the amplitude $\mathcal{A}$ and it equals $k(\omega) = |C(i\omega I - A)^{-1}B|$. Hence, $\Upsilon_{\mathcal{A}^*}(\omega)$ is independent of $\mathcal{A}^*$ and it equals $\Upsilon_{\mathcal{A}_*}(\omega) = |C(i\omega I - A)^{-1}B|$. Therefore, we see that for linear systems the graph of $\Upsilon_{\mathcal{A}_*}(\omega)$ versus the excitation frequency ω coincides with the Bode magnitude plot. The function $\Upsilon_{\mathcal{A}_*}(\omega)$ can be used further to study dynamical properties of uniformly convergent systems. Depending on the inputs and outputs that we choose for the nonlinear system (4.44), we can define nonlinear variants of the sensitivity and complementary sensitivity functions of controlled convergent systems.

As has been mentioned in Section 4.2, the problem of finding the mapping $\alpha(w_1, w_2, \omega)$ is, in general, not an easy task. But in certain cases it is possible to find this mapping analytically. Let us find $\alpha(w_1, w_2, \omega)$ for a particular example.

Example 4.24. Consider the system

$$\dot{z}_1 = -z_1 + z_2^2, \tag{4.47}$$

$$\dot{z}_2 = -z_2 + u, \tag{4.48}$$

$$y = z_1. \tag{4.49}$$

This system is a series connection of input-to-state convergent systems. Therefore, by Property 2.27, system (4.47), (4.48) is input-to-state convergent. This, by Property 2.19, implies that system (4.47), (4.48) is globally uniformly convergent with the UBSS property for the class of inputs $\overline{\mathbb{PC}}_1$. Consequently, by Theorem 4.23 the mapping $\alpha(w_1, w_2, \omega)$ exists and it is unique. Using the method described in Example 4.10, we will first find $\alpha_2(w_1, w_2, \omega)$ (the second component of α) from (4.48). In our case, $\alpha_2(w_1, w_2, \omega)$ is a polynomial function of degree 1 in the variables w_1 and w_2. So, we will seek α_2 in the form:

$$\alpha_2(w_1, w_2, \omega) = a_1(\omega)w_1 + a_2(\omega)w_2.$$

Substituting this expression in (4.48), we find

$$a_1(\omega) = \frac{1}{1+\omega^2}, \quad a_2(\omega) = \frac{-\omega}{1+\omega^2}.$$

Then, substituting the obtained α_2 for z_2 in (4.47), we compute $\alpha_1(w_1, w_2, \omega)$. In our case, it is a polynomial of w_1 and w_2 of the same degree as the polynomial $(\alpha_2(w_1, w_2, \omega))^2$. Thus, we will seek $\alpha_1(w_1, w_2, \omega)$ in the form

$$\alpha_1(w_1, w_2, \omega) = b_1(\omega)w_1^2 + 2b_2(\omega)w_1w_2 + b_3(\omega)w_2^2. \tag{4.50}$$

After the corresponding computations, we obtain

$$b_1(\omega) = \frac{2\omega^4 + 1}{(1+4\omega^2)(1+\omega^2)^2}, \quad b_2(\omega) = \frac{\omega^3 - 2\omega}{(1+4\omega^2)(1+\omega^2)^2},$$

$$b_3(\omega) = \frac{2\omega^4 + 5\omega^2}{(1+4\omega^2)(1+\omega^2)^2}.$$

After the function $\alpha(w_1, w_2, \omega)$ is found, one can numerically, though very efficiently, compute the magnitude characteristics $\Upsilon_{\mathcal{A}^*}(\omega)$ for some maximal excitation amplitude $\mathcal{A}^*$ and all frequencies over the band of interest. In Figure 4.1, $\Upsilon_{\mathcal{A}^*}(\omega)$ is computed for $\mathcal{A}^* = 1$. Since $\alpha_1(w_1, w_2, \omega)$ is a uniform polynomial function of degree 2 with respect to the variables w_1 and w_2 (see formula (4.50)), one can easily check that for arbitrary $\mathcal{A}^* > 0$ it holds that $\Upsilon_{\mathcal{A}^*}(\omega) = \mathcal{A}^*\Upsilon_1(\omega)$. Here we see the dependency of the amplification gain on the amplitude of the excitation. This is an essentially nonlinear phenomenon. $\lhd$

It is common knowledge that nonlinear systems may have very complex dynamics and that, in general, it is not possible to apply linear analysis and

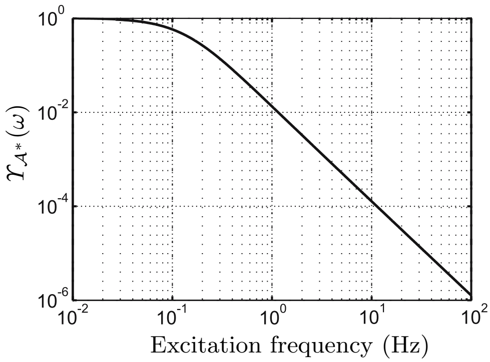

Fig. 4.1. The function $\Upsilon_{\mathcal{A}^*}(\omega)$ computed for $\mathcal{A}^* = 1$.

design methods to investigate nonlocal dynamical properties of nonlinear systems. At the same time, uniformly convergent systems, even when nonlinear, exhibit relatively simple dynamics. Moreover, for uniformly convergent systems with the UBSS property we can define a frequency response function and an analog of a well-known linear analysis tool such as the Bode plot, which can be used, for example, for studying attenuation properties at different excitation frequencies. It is still an open question whether such a nonlinear Bode plot contains enough information to fully identify the system or to design controllers based on this plot. Another open question is how to compute the function $\alpha(w_1, w_2, \omega)$. A standard solution would be to find it numerically. Yet, such numerical methods still need to be developed. As we have shown with an example, in certain cases $\alpha(w_1, w_2, \omega)$ can be found analytically. The results and open questions discussed in this section open an interesting direction in nonlinear systems and control analysis.

4.7 Summary

In this chapter we have presented several results related to solvability of the global, global robust, global forward time, and local uniform output regulation problems. Theorems 4.12, 4.17, 4.19, and 4.21 provide characterizations of all controllers solving the above-mentioned variants of the uniform output regulation problem. Theorems 4.16, 4.18, 4.20, and 4.22 provide necessary and sufficient conditions for the solvability of these problems. These solvability conditions consist of two ingredients: solvability of the regulator equations and existence of a controller which has the generalized internal model property and makes the closed-loop system uniformly convergent. Solvability of the regulator equations guarantees that for every solution of the exosystem lying in a certain ω-limit set it is possible to find at least one control input for which the controlled system has a solution along which the regulated output equals zero. The generalized internal model property of the controller guarantees that this controller is capable of generating this control input based on the information available from the measurements. The uniform convergence property guarantees that the above-mentioned solution, along which the regulated output equals zero, is (globally, locally) asymptotically stable.

All solvability results presented in this chapter are based on the invariant manifold theorems (Theorems 4.4, 4.6, and 4.8), which, in the context of the output regulation problem, serve as counterparts of the center manifold theorem. Although the invariant manifold theorems are derived in order to study solvability of the uniform output regulation problem, they are interesting in their own respect. As follows from the discussion in Section 4.6, these invariant manifold theorems can be used for checking the generalized synchronization property for coupled systems and for the computation of periodic solutions of uniformly convergent systems excited by harmonic inputs. Moreover, these theorems allow us to define nonlinear frequency response functions and a variant of the Bode plot for uniformly convergent nonlinear systems. This opens a new direction in the analysis of nonlinear systems.