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Introduction

1.1 The output regulation problem

The output regulation problem is one of the central problems in control theory. This problem deals with asymptotic tracking of prescribed reference signals and/or asymptotic rejection of undesired disturbances in the output of a dynamical system. The main feature that distinguishes the output regulation problem from conventional tracking and disturbance rejection problems is that, in the output regulation problem, the class of reference signals and disturbances consists of solutions of some autonomous system of differential equations. This system is called an *exosystem*. Reference signals and/or disturbances generated by the exosystem are called *exosignals*.

Many control problems can be formulated as a particular case of the output regulation problem. For example, in the set-point control problem the constant reference signals to be asymptotically tracked by the output of a system can be considered as outputs of an exosystem given by a differential equation with zero right-hand side. A particular value of the reference signal is, in this case, determined by the corresponding initial condition of the exosystem. In the same way, constant disturbances acting on a system can be considered as outputs of an exosystem with zero right-hand side. Therefore, the set-point control problem and the problem of asymptotic rejection of constant disturbances in the output of a system can be considered as particular cases of the output regulation problem. Similar to the case of constant exosignals, harmonic reference signals and disturbances can be considered as outputs of a linear harmonic oscillator. In this case, the parameters of the oscillator determine the frequency content of the exosignal, while the initial conditions of the oscillator determine particular amplitudes and phases of the exosignal. Here, we see that the problem of asymptotic tracking and disturbance rejection for the case of harmonic reference signals and disturbances can be considered as a particular case of the output regulation problem.

Examples of the output regulation problem with more complex exosystem dynamics can be found, for example, in the problem of controlled synchronization (see, e.g., [60, 61, 74]). In this problem one considers two systems of the same dimensions. The first system is autonomous and is called a "master" system. The master system usually has some complex dynamics, e.g., it may have a chaotic attractor. The second system can be controlled and is called a "slave" system. The controlled synchronization problem is to find a controller that, based on the measured signals from the master and slave systems, generates a control action such that the state of the slave system asymptotically tracks the state of the master system. In other words, the states of these two systems asymptotically synchronize. The fact that a controlled synchronization problem can be treated as a particular case of the output regulation problem was pointed out in [36]. From the formulation of the controlled synchronization problem, one can easily notice that this problem has a lot in common with the observer design problem for the autonomous master system. In fact, the slave system can be treated as an observer for the master system. Therefore, the problem of observer design for autonomous systems can also be considered as an output regulation problem.

For linear systems the output regulation problem was completely solved in the 1970s in the works of B.A. Francis, W.M. Wonham, E.J. Davison, and others [13, 21, 88]. This research resulted in the well-known *internal model principle* [21] and in the observation that solvability of the linear output regulation problem is related to the solvability of the so-called "regulator equations," which, in the linear case, are two linear matrix equations [20]. A different approach to the linear output regulation problem was pursued in the works of V.A. Yakubovich and his colleagues [56, 81, 90]. This approach is based on treating the output regulation problem as some kind of the linear-quadratic optimal control problem. Although controllers obtained within this approach do not guarantee that the regulated output converges to zero (it converges to small values depending on the chosen cost functional), they are less sensitive to variations in the exosystem parameters. The problem of output regulation for linear systems subject to constraints on the inputs and state variables was studied in a number of publications, see, e.g., [30, 77] and references therein.

Following the trend of developing nonlinear control systems theory (see, e.g., [38, 62] and references therein), in the 1980s several authors started studying the output regulation problem for nonlinear systems [16, 28, 29]. A breakthrough in the nonlinear output regulation problem was reported in the seminal paper [39] by A. Isidori and C.I. Byrnes. In that paper the authors showed that under the neutral stability assumption on the exosystem and some standard stabilizability/detectability assumptions on the system, the local output regulation problem is solvable if and only if certain mixed algebraic equations and partial differential equations are solvable. These equations are called the *regulator equations.* They are nonlinear counterparts of the regulator equations from the linear output regulation problem. An alternative solution to the local output regulation problem was proposed in [34]. These papers were followed by a number of publications dealing with various aspects of the local output regulation problem. For example, if it is difficult to solve the local output regulation problem (because it requires solving the regulator equations), then approximate (in some sense) solutions to the problem can be found, as reported in [8, 33, 35, 86]. The problem of structurally stable (i.e., when the system parameters are assumed to be close enough to their nominal values) output regulation was addressed in [8, 38]. The case when the system parameters are allowed to vary within a given compact set was considered in [8, 44, 50, 52]. The semiglobal output regulation problem with an adaptive internal model, which allowed for uncertainties in the exosystem, has been considered in [80]. Probably the most complete list of references to results on the output regulation problem can be found in [7, 8, 31, 42].

So far, the results on the output regulation problem mentioned above dealt either with the local or semiglobal (i.e., when initial conditions belong to some predefined compact set) case. Actually, the number of results on the global variant of the output regulation problem is very small compared to the number of results on the local and semiglobal cases. Only recently have more papers on the global output regulation problem started to appear. In [79] the global robust output regulation problem was solved for minimum-phase systems that are linear in the unmeasured variables. The same class of systems as in [79], but with unknown system and exosystem parameters, was considered in [17]. In that paper the global output regulation problem was solved using adaptive control techniques. In [12, 58] the global robust servomechanism problem for nonlinear systems in triangular form was considered. In [11] a problem formulation for the global robust output regulation problem was proposed and a possible conversion of this problem into a certain robust stabilization problem was suggested.

Careful examination of the global results mentioned above allows one to conclude two things. First, at the moment there is still no generally accepted problem statement for the global output regulation problem. Second, all these results start with the assumption that the regulator equations are solvable and that the corresponding solutions are defined either *globally* or in some predefined set. The only vague justification for this assumption is that in the *local* output regulation problem, the existence of *locally* defined solutions to the regulator equations is a necessary condition for the solvability of the problem. In fact, these two observations are, in a certain sense, coupled. Recall that in the local output regulation problem [8, 39], a properly chosen problem setting with a "right" set of standing assumptions allowed one to obtain necessary and sufficient conditions for the solvability of the problem and to build up a nice, complete theory for this problem. Our hypothesis, which is now confirmed by the results contained in this book, is that by choosing a proper problem setting for the *global* output regulation problem and a proper set of assumptions, one can build up a more or less complete theory for the *global* output regulation problem, just as was done for the local case in [8, 39]. Such a theory would include necessary and sufficient conditions for the solvability of the problem and would embrace the existing problem formulations and results on the global output regulation problem. Moreover, it would provide us with new solutions to the global output regulation problem for new classes of systems.

One possible way of defining such a new problem setting has been proposed in [40]. Although the approaches adopted in this book and in [40] are different, the corresponding final results are close to each other.

A cornerstone of such a new problem formulation for the global output regulation problem adopted in this book is the natural requirement that the closed-loop system must have some "convergence" property. Roughly speaking, this property means that all solutions of the closed-loop system "forget" their initial conditions and converge to some unique solution, which can be called a steady-state solution. This solution is determined only by the exosignal generated by the exosystem. This "convergence" property is discussed in the next section.

1.2 Convergent dynamics

In many control problems and, in particular, in the output regulation problem, it is required that controllers be designed in such a way that all solutions of the corresponding closed-loop system "forget" their initial conditions and converge to some steady-state solution, which is determined only by the input of the closed-loop system. This input can be, for example, a command signal or a signal generated by a feedforward part of the controller or, as in the output regulation problem, it can be the signal generated by the exosystem. For asymptotically stable linear systems excited by inputs, this is a natural property. Indeed, due to linearity of the system, every solution is globally asymptotically stable and, therefore, all solutions of such a system "forget" their initial conditions and converge to each other. After transients, the dynamics of the system are determined only by the input.

For nonlinear systems, in general, global asymptotic stability of a system with zero input does not guarantee that all solutions of this system with a *nonzero* input "forget" their initial conditions and converge to each other. There are many examples of nonlinear globally asymptotically stable systems that, being excited by a periodic input, have coexisting periodic solutions. These periodic solutions do not converge to each other. This fact indicates that for nonlinear systems the convergent dynamics property requires additional conditions.

The property that all solutions of a system "forget" their initial conditions and converge to some steady-state solution has been addressed in a number of papers. In [73] this property was investigated for systems of differential equations that are periodic in time. In that work systems with a unique periodic globally asymptotically stable solution were called *convergent*. Later, the definition of convergent systems given by V.A. Pliss in [73] was extended by B.P. Demidovich in [15] (see also [66]) to the case of systems that are not necessarily periodic in time. According to [15], a system is called convergent if there exists a unique globally asymptotically stable solution that is bounded on the whole time axis. Obviously, if such a solution does exist, all other solutions, regardless of their initial conditions, converge to it. This solution can be considered as a steady-state solution. In [14, 15] B.P. Demidovich presented a simple sufficient condition for such a convergence property (the English translation of this result can be found in [66]). With the development of absolute stability theory, V.A. Yakubovich showed in [89] that for a linear system with one scalar nonlinearity satisfying some incremental sector condition, the circle criterion guarantees the convergence property for this system with any nonlinearity satisfying this incremental sector condition.

In parallel with this Russian line of research, the property of solutions converging to each other was addressed in the works of T. Yoshizawa [91, 92] and J.P. LaSalle [54]. In [54] this property of a system was called *extreme stability.* In [91] T. Yoshizawa provided sufficient and, under certain assumptions, necessary conditions for this extreme stability. These conditions are formulated in terms of existence of a Lyapunov-type function satisfying certain conditions. Extremely stable systems with periodic and almost-periodic right-hand sides were studied in [92].

Several decades after these publications, the interest in stability properties of solutions with respect to one another revived. Incremental stability, incremental input-to-state stability, and contraction analysis are some of the terms related to such properties. In the mid-1990s, W. Lohmiller and J.-J.E. Slotine (see [57] and references therein) independently reobtained and extended the result of B.P. Demidovich. In particular, they pointed out that systems satisfying the (extended) Demidovich condition may enjoy certain properties of asymptotically stable linear systems that are not encountered in general asymptotically stable nonlinear systems. A different approach was pursued in the works by V. Fromion et al. [23–25]. In this approach a dynamical system is considered as an operator that maps some functional space of inputs to a functional space of outputs. If this operator is Lipschitz continuous (has a finite incremental gain or is incrementally stable), then, under certain observability and reachability conditions, all solutions of a state-space realization of this system converge to each other. The sufficient conditions for such Lipschitz continuity condition proposed in [25] are very close to the sufficient conditions for the convergence property obtained by Demidovich. In [2] D. Angeli developed a Lyapunov approach for studying both the global uniform asymptotic stability of all solutions of a system (in [2], this property is called incremental stability) and the so-called incremental input-to-state stability property, which is compatible with the input-to-state stability approach (see, e.g., [82]). As was pointed out in these recent papers, observer design and (controlled) synchronization problems are some of the possible applications of such stability properties.

In this book, for the property that all solutions of a system "forget" their initial conditions and converge to some steady-state solution, we will adopt the notion of *convergent systems* introduced by B.P. Demidovich. In comparison to the other notions mentioned above, the property of *convergence* has two main advantages: it is coordinate independent, while, for example, the notion of incremental stability and incremental input-to-state stability is not, and it allows us to define the steady-state solution in a *unique* way, which proves to be beneficial in further analysis and applications of convergent systems.

1.3 Book outline

In this book we systematically study the output regulation problem based on the notion of convergent systems. As a preliminary step, in Chapter 2 we extend the notion of convergent systems introduced by B.P. Demidovich, investigate various properties of such systems, and design certain tools for the analysis of convergent systems. All these results can be used not only in the context of the output regulation problem, but also in other problems in systems and control theory.

In Chapter 3 we formulate the so-called *uniform output regulation problem*. This is a new problem formulation for the output regulation problem based on the notion of convergent systems. We state global and local variants of the uniform output regulation problem as well as a robust variant of this problem for systems with uncertainties. This new problem formulation has several advantages over the existing problem formulations (see, e.g., [8, 11]). First, it allows one to deal with exosystems having complex dynamics, e.g., exosystems with a (chaotic) attractor. Up to now most of the results on the output regulation problem dealt only with exosystems having relatively simple dynamics, for example, with linear harmonic oscillators. The ability to deal with complex exosystem dynamics allows one to treat the problem of controlled synchronization (see, e.g., [36]) and the problem of observer design for autonomous systems with complex dynamics as particular cases of the uniform output regulation problem. The second advantage of this new problem setting is that, as will be discussed below, it allows one to treat the local and global variants of the uniform output regulation problem in a unified way regardless of the complexity of the exosystem dynamics. This new problem setting includes, as its particular cases, the output regulation problem for linear systems and the conventional local output regulation problem for nonlinear systems (see, e.g., [8]).

For the global, global robust, and local variants of the uniform output regulation problem, we provide necessary and sufficient conditions for the solvability of these problems as well as results on characterization of all controllers solving these problems. These results are presented in Chapter 4. For all these different variants of the problem, the obtained results on the solvability of the problem and controllers characterization look similar. Such a uniformity is a sign of the right choice of the problem setting. Moreover, we show that many of the existing controllers solving the global output regulation problem in other problem settings (see, e.g., [12, 58, 68, 69, 79]), which can be different from the global uniform output regulation problem, in fact solve the global uniform output regulation problem. The solvability analysis of the uniform output regulation problem is based on certain invariant manifold theorems. We demonstrate that these invariant manifold theorems can also be used for studying the so-called *generalized synchronization* of coupled systems, for the computation of periodic solutions of nonlinear systems excited by harmonic inputs and for the extension of frequency response functions and such a well-known analysis and design tool as the Bode magnitude plot from linear systems to nonlinear uniformly convergent systems. These nonlinear frequency response functions and the Bode plot can be used, for example, for nonlinear system performance analysis.

The solvability conditions for the global uniform output regulation problem do not provide direct recipes for finding controllers solving this problem. Therefore, in Chapter 5 we provide results on controller design for the global uniform output regulation problem for several classes of nonlinear systems. One of these controller designs is based on the notions of quadratic stabilizability and detectability. These notions extend the conventional notions of stabilizability and detectability from linear systems theory to the case of nonlinear systems. The controller design based on these notions extends the known controllers solving the linear and the local nonlinear output regulation problems to the case of the global uniform output regulation problem for nonlinear systems. For the case of a Lur'e system with a nonlinearity having a bounded derivative and an exosystem being a linear harmonic oscillator, feasibility conditions for this controller design are formulated in terms of linear matrix inequalities. Moreover, for this class of systems and exosystems we provide a robust controller design that copes not only with the uncertainties in the system parameters, but also with the uncertain nonlinearity from a class of nonlinearities with a given bound on their derivatives. All controller designs presented in Chapter 5 are based on certain general methods that allow us to design controllers making the corresponding closed-loop systems convergent. These methods can also be used for other control problems where the convergence property of the closed-loop system is required or desired.

If we cannot find a solution to the *global* uniform output regulation problem, it can still be possible to find a controller that solves the *local* output regulation problem. There are standard procedures for such controller designs (see, e.g., [8, 38]). The resulting controllers solve the output regulation problem for the initial conditions of the closed-loop system and the exosystem lying in *some* neighborhood of the origin. To enhance applicability of these controllers, in Chapter 6 we present estimation results that, given a controller solving the local output regulation problem, provide estimates of this neighborhood of initial conditions for which the controller works. Such estimation results are presented for both the exact and approximate variants of the local output regulation problem. These estimation results are also based on the notion of convergent systems.

The nonlinear output regulation problem has been studied from a theoretical point of view in a series of publications. At the same time, there are very few publications aiming at an experimental validation of solutions to the nonlinear output regulation problem [4, 55]. In Chapter 7 we address the nonlinear output regulation problem from an experimental point of view. We study a local output regulation problem for the so-called *TORA system*, which is a nonlinear mechanical benchmark system, see, e.g., [32, 45, 85]. A simple controller solving this problem is proposed. This controller is implemented in an experimental setup and its performance is investigated in experiments. The reason for this experimental study is twofold. The first reason is to check whether controllers from the nonlinear output regulation theory are applicable in an experimental setting in the presence of disturbances and modeling uncertainties, which are inevitable in practice. The second reason is to identify the factors that can deteriorate the controller performance and therefore require specific attention already at the stage of controller design. Successful results of this experimental study, which are presented in Chapter 7, show the applicability of the nonlinear output regulation theory in experiments and give new data for analysis and further developments in the field of nonlinear output regulation.

Finally, concluding remarks are presented in Chapter 8.